7 The function f is such that $f(x) = a - b\cos x$ for $0° \leqslant x \leqslant 360°$, where a and b are positive constants. The maximum value of f(x) is 10 and the minimum value is −2.

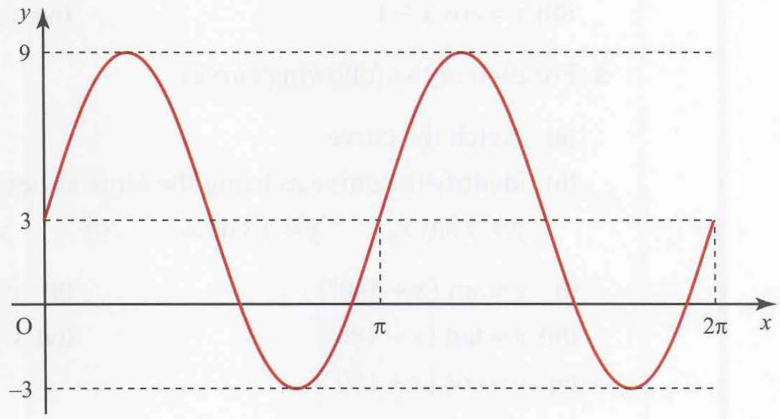
(i) Find the values of a and b.

(ii) Solve the equation $f(x) = 0$.

(iii) Sketch the graph of $y = f(x)$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 November 2008]

8 The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \leqslant x \leqslant 2\pi$.



(i) Find the values of a, b and c.

(ii) Find the smallest value of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $y = 0$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2009]

9 The function f is defined by $f: x \mapsto 5 - 3\sin 2x$ for $0 \leqslant x \leqslant \pi$.

(i) Find the range of f.

(ii) Sketch the graph of $y = f(x)$.

(iii) State, with a reason, whether f has an inverse.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q4 November 2009]

10 The function $f: x \mapsto 4 - 3\sin x$ is defined for the domain $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $f(x) = 2$.

(ii) Sketch the graph of $y = f(x)$.

(iii) Find the set of values of k for which the equation $f(x) = k$ has no solution.

The function $g: x \mapsto 4 - 3\sin x$ is defined for the domain $\frac{1}{2}\pi \leqslant x \leqslant A$.

(iv) State the largest value of A for which g has an inverse.

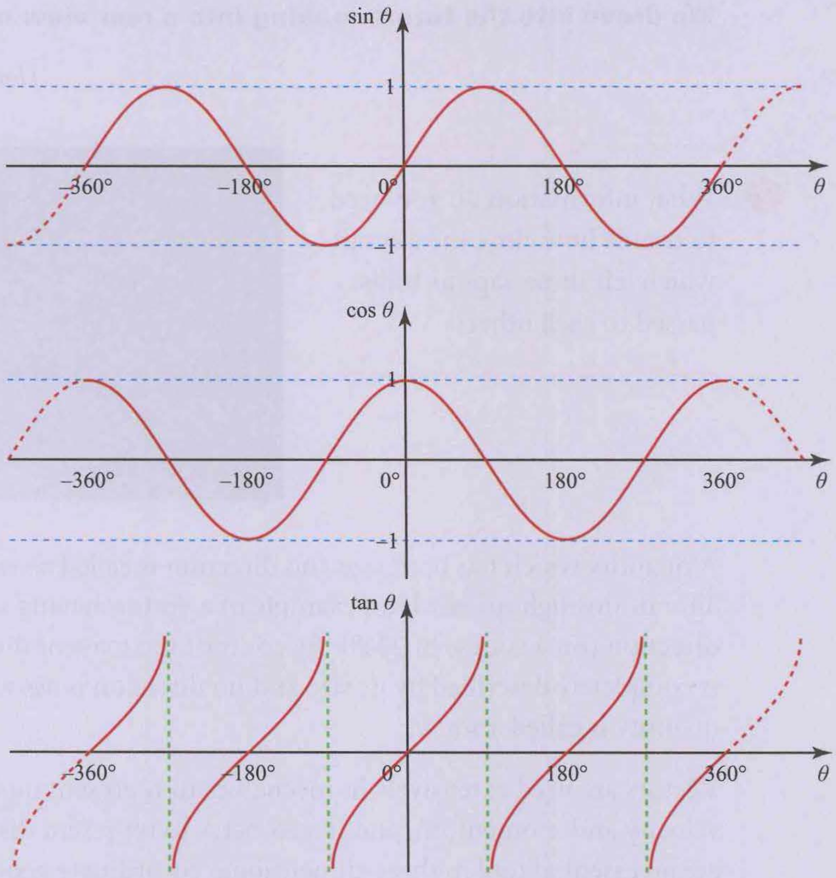
(v) For this value of A, find the value of $g^{-1}(3)$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q11 June 2010]

KEY POINTS

1 The point (x, y) at angle θ on the unit circle centre $(0, 0)$ has co-ordinates $(\cos\theta, \sin\theta)$ for all θ.

2 The graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are as shown below.



3 $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

4 $\sin^2\theta + \cos^2\theta \equiv 1$.

5 Angles can be measured in radians. π radians $= 180°$.

6 For a circle of radius r, $\left.\begin{array}{l}\text{arc length} = r\theta \\ \text{area of sector} = \frac{1}{2}r^2\theta\end{array}\right\}$ (θ in radians).

7 The graph of $y = \mathrm{f}(x) + s$ is a translation of the graph of $y = \mathrm{f}(x)$ by $\begin{pmatrix}0\\s\end{pmatrix}$.

8 The graph of $y = \mathrm{f}(x - t)$ is a translation of the graph of $y = \mathrm{f}(x)$ by $\begin{pmatrix}t\\0\end{pmatrix}$.

9 The graph of $y = -\mathrm{f}(x)$ is a reflection of the graph of $y = \mathrm{f}(x)$ in the x axis.

10 The graph of $y = a\mathrm{f}(x)$ is a one-way stretch of the graph of $y = \mathrm{f}(x)$ with scale factor a parallel to the y axis.

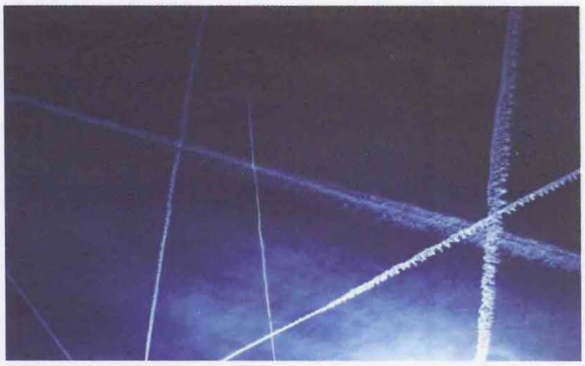
11 The graph of $y = \mathrm{f}(ax)$ is a one-way stretch of the graph of $y = \mathrm{f}(x)$ with scale factor $\dfrac{1}{a}$ parallel to the x axis.

8 Vectors

We drove into the future looking into a rear view mirror.

Herbert Marshall McLuhan

? What information do you need to decide how close the aircraft which left these vapour trails passed to each other?



A quantity which has both size and direction is called a *vector*. The velocity of an aircraft through the sky is an example of a vector, having size (e.g. 600 mph) and direction (on a course of 254°). By contrast the mass of the aircraft (100 tonnes) is completely described by its size and no direction is associated with it; such a quantity is called a *scalar*.

Vectors are used extensively in mechanics to represent quantities such as force, velocity and momentum, and in geometry to represent displacements. They are an essential tool in three-dimensional co-ordinate geometry and it is this application of vectors which is the subject of this chapter. However, before coming on to this, you need to be familiar with the associated vocabulary and notation, in two and three dimensions.

Vectors in two dimensions

Terminology

In two dimensions, it is common to represent a vector by a drawing of a straight line with an arrowhead. The length represents the size, or magnitude, of the vector and the direction is indicated by the line and the arrowhead. Direction is usually given as the angle the vector makes with the positive x axis, with the anticlockwise direction taken to be positive.

The vector in figure 8.1 has magnitude 5, direction +30°. This is written (5, 30°) and said to be in *magnitude–direction form* or in *polar form*. The general form of a vector written in this way is (r, θ) where r is its magnitude and θ its direction.

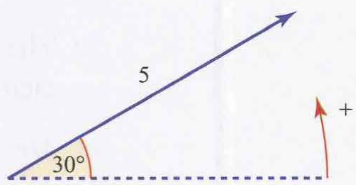


Figure 8.1

Note

In the special case when the vector is representing real travel, as in the case of the velocity of an aircraft, the direction may be described by a compass bearing with the angle measured from north, clockwise. However, this is not done in this chapter, where directions are all taken to be measured anticlockwise from the positive x direction.

An alternative way of describing a vector is in terms of *components* in given directions. The vector in figure 8.2 is 4 units in the x direction, and 2 in the y direction, and this is denoted by $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

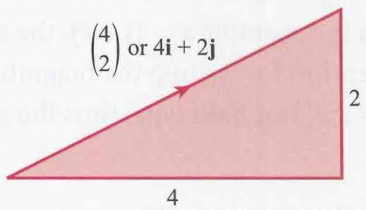


Figure 8.2

This may also be written as $4\mathbf{i} + 2\mathbf{j}$, where $\mathbf{i}$ is a vector of magnitude 1, a *unit vector*, in the x direction and $\mathbf{j}$ is a unit vector in the y direction (figure 8.3).

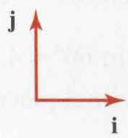


Figure 8.3

In a book, a vector may be printed in bold, for example **p** or **OP**, or as a line between two points with an arrow above it to indicate its direction, such as $\overrightarrow{\text{OP}}$. When you write a vector by hand, it is usual to underline it, for example, $\underline{p}$ or $\underline{OP}$, or to put an arrow above it, as in $\overrightarrow{OP}$.

To convert a vector from component form to magnitude–direction form, or vice versa, is just a matter of applying trigonometry to a right-angled triangle.

EXAMPLE 8.1

Write the vector $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$ in magnitude–direction form.

SOLUTION

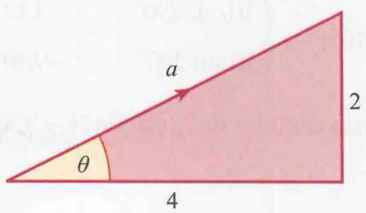


Figure 8.4

The magnitude of **a** is given by the length a in figure 8.4.

$$a = \sqrt{4^2 + 2^2} \quad \text{(using Pythagoras' theorem)}$$
$$= 4.47 \quad \text{(to 3 significant figures)}$$

The direction is given by the angle θ.

$$\tan\theta = \tfrac{2}{4} = 0.5$$
$$\theta = 26.6° \quad \text{(to 3 significant figures)}$$

The vector **a** is (4.47, 26.6°).

The magnitude of a vector is also called its *modulus* and denoted by the symbols | |. In the example $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$, the modulus of **a**, written | **a** |, is 4.47. Another convention for writing the magnitude of a vector is to use the same letter, but in italics and not bold type; thus the magnitude of **a** may be written a.

EXAMPLE 8.2

Write the vector (5, 60°) in component form.

SOLUTION

In the right-angled triangle OPX

$$\text{OX} = 5\cos 60° = 2.5$$

$$\text{XP} = 5\sin 60° = 4.33$$
(to 2 decimal places)

$$\overrightarrow{\text{OP}} \text{ is } \begin{pmatrix} 2.5 \\ 4.33 \end{pmatrix} \text{ or } 2.5\mathbf{i} + 4.33\mathbf{j}.$$

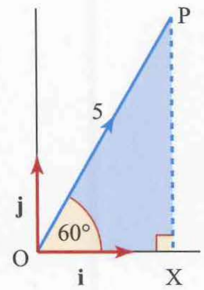


Figure 8.5

This technique can be written as a general rule, for all values of θ.

$$(r, \theta) \rightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j}$$

EXAMPLE 8.3

Write the vector (10, 290°) in component form.

SOLUTION

In this case $r = 10$ and $\theta = 290°$.

$$(10, 290°) \rightarrow \begin{pmatrix} 10\cos 290° \\ 10\sin 290° \end{pmatrix} = \begin{pmatrix} 3.42 \\ -9.40 \end{pmatrix} \text{ to 2 decimal places.}$$

This may also be written $3.42\mathbf{i} - 9.40\mathbf{j}$.

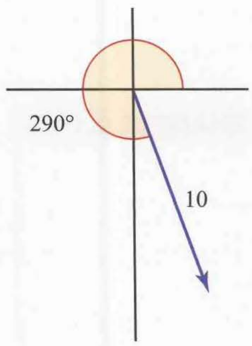


Figure 8.6

In Example 8.3 the signs looked after themselves. The component in the **i** direction came out positive, that in the **j** direction negative, as must be the case for a direction in the fourth quadrant ($270° < \theta < 360°$). This will always be the case when the conversion is from magnitude–direction form into component form.

The situation is not quite so straightforward when the conversion is carried out the other way, from component form to magnitude–direction form. In that case, it is best to draw a diagram and use it to see the approximate size of the angle required. This is shown in the next example.

EXAMPLE 8.4

Write $-5\mathbf{i} + 4\mathbf{j}$ in magnitude–direction form.

SOLUTION

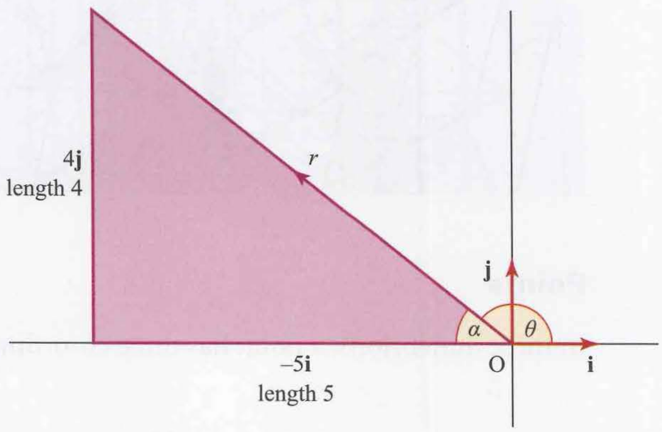


Figure 8.7

In this case, the magnitude $r = \sqrt{5^2 + 4^2}$

$= \sqrt{41}$

$= 6.40$ (to 2 decimal places).

The direction is given by the angle θ in figure 8.7, but first find the angle α.

$$\tan\alpha = \tfrac{4}{5} \quad\Rightarrow\quad \alpha = 38.7° \quad \text{(to nearest 0.1°)}$$

so $\quad \theta = 180 - \alpha = 141.3°$

The vector is (6.40, 141.3°) in magnitude–direction form.

Vectors in three dimensions

Points

In three dimensions, a point has three co-ordinates, usually called x, y and z.

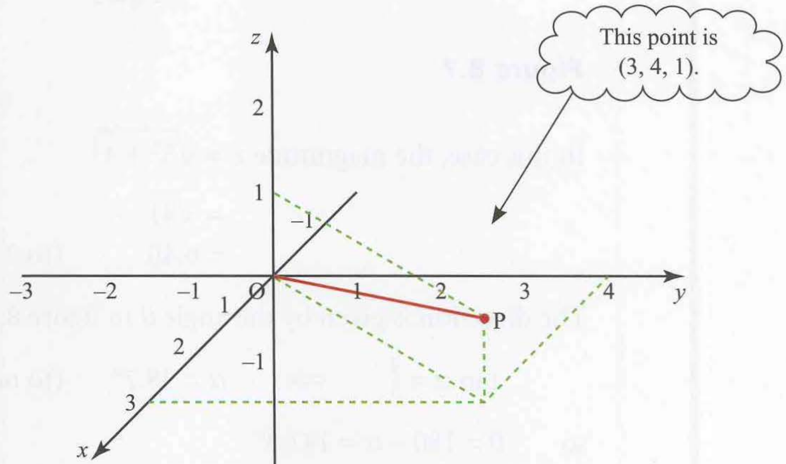


Figure 8.8

The axes are conventionally arranged as shown in figure 8.8, where the point P is (3, 4, 1). Even on correctly drawn three-dimensional grids, it is often hard to see the relationship between the points, lines and planes, so it is seldom worth your while trying to plot points accurately.

The unit vectors **i**, **j** and **k** are used to describe vectors in three dimensions.

Equal vectors

The statement that two vectors **a** and **b** are equal means two things.

- The direction of **a** is the same as the direction of **b**.
- The magnitude of **a** is the same as the magnitude of **b**.

If the vectors are given in component form, each component of **a** equals the corresponding component of **b**.

Position vectors

Saying the vector **a** is given by $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ tells you the components of the vector, or equivalently its magnitude and direction. It does not tell you where the vector is situated; indeed it could be anywhere.

All of the lines in figure 8.9 represent the vector **a**.

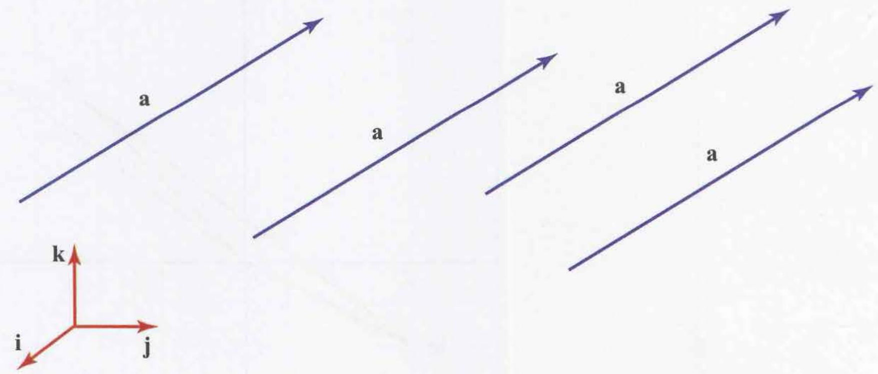


Figure 8.9

There is, however, one special case which is an exception to the rule, that of a vector which starts at the origin. This is called a *position vector*. Thus the line joining the origin to the point P(3, 4, 1) is the position vector $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ or $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

Another way of expressing this is to say that the point P(3, 4, 1) has the position vector $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$.

EXAMPLE 8.5

Points L, M and N have co-ordinates (4, 3), (−2, −1) and (2, 2).

(i) Write down, in component form, the position vector of L and the vector $\overrightarrow{MN}$.

(ii) What do your answers to part **(i)** tell you about the lines OL and MN?

SOLUTION

(i) The position vector of L is $\overrightarrow{OL} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The vector $\overrightarrow{MN}$ is also $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (see figure 8.10).

(ii) Since $\overrightarrow{OL} = \overrightarrow{MN}$, lines OL and MN are parallel and equal in length.

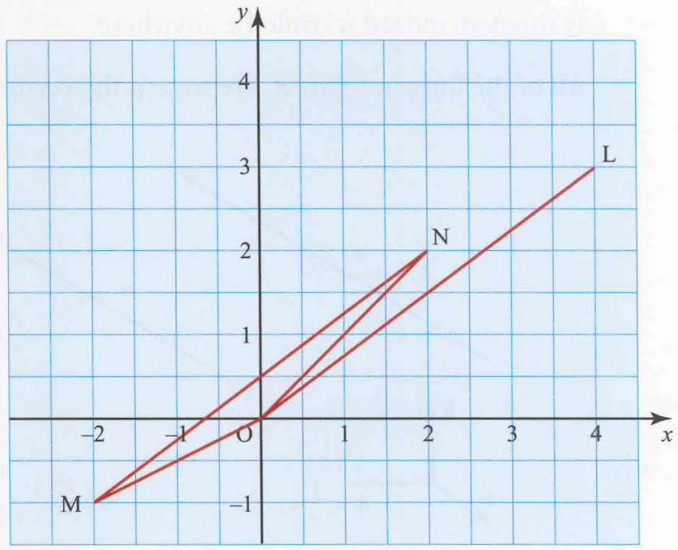


Figure 8.10

Note

A line joining two points, like MN in figure 8.10, is often called a *line segment*, meaning that it is just that particular part of the infinite straight line that passes through those two points.

The vector $\overrightarrow{MN}$ is an example of a displacement vector. Its length represents the magnitude of the displacement when you move from M to N.

The length of a vector

In two dimensions, the use of Pythagoras' theorem leads to the result that a vector $a_1\mathbf{i} + a_2\mathbf{j}$ has length $|\mathbf{a}|$ given by

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

Show that the length of the three-dimensional vector $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

EXAMPLE 8.6

Find the magnitude of the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$.

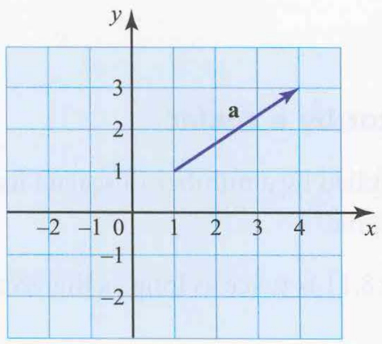
SOLUTION

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + (-5)^2 + 3^2} \\ &= \sqrt{4 + 25 + 9} \\ &= \sqrt{38} \\ &= 6.16 \text{ (to 2 d.p.)} \end{aligned}$$

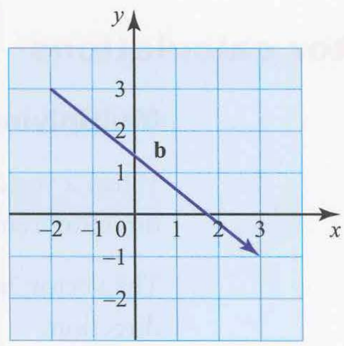
EXERCISE 8A

1 Express the following vectors in component form.

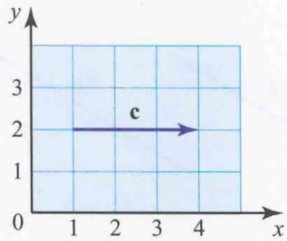
(i)



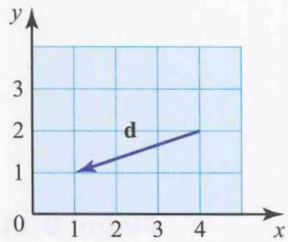
(ii)



(iii)



(iv)



2 Draw diagrams to show these vectors and then write them in magnitude–direction form.

(i) $2\mathbf{i} + 3\mathbf{j}$ **(ii)** $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **(iii)** $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$

(iv) $-\mathbf{i} + 2\mathbf{j}$ **(v)** $3\mathbf{i} - 4\mathbf{j}$

3 Find the magnitude of these vectors.

(i) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ **(iii)** $2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(iv) $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ **(v)** $\begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$ **(vi)** $\mathbf{i} - 2\mathbf{k}$

4 Write, in component form, the vectors represented by the line segments joining the following points.

(i) $(2, 3)$ to $(4, 1)$ (ii) $(4, 0)$ to $(6, 0)$

(iii) $(0, 0)$ to $(0, -4)$ (iv) $(0, -4)$ to $(0, 0)$

(v) $(0, 0, 0)$ to $(0, 0, 5)$ (vi) $(0, 0, 0)$ to $(-1, -2, 3)$

(vii) $(-1, -2, 3)$ to $(0, 0, 0)$ (viii) $(0, 2, 0)$ to $(4, 0, 4)$

(ix) $(1, 2, 3)$ to $(3, 2, 1)$ (x) $(4, -5, 0)$ to $(-4, 5, 1)$

5 The points A, B and C have co-ordinates $(2, 3)$, $(0, 4)$ and $(-2, 1)$.

(i) Write down the position vectors of A and C.

(ii) Write down the vectors of the line segments joining AB and CB.

(iii) What do your answers to parts (i) and (ii) tell you about

(a) AB and OC

(b) CB and OA?

(iv) Describe the quadrilateral OABC.

Vector calculations

Multiplying a vector by a scalar

When a vector is multiplied by a number (a scalar) its length is altered but its direction remains the same.

The vector $2\mathbf{a}$ in figure 8.11 is twice as long as the vector $\mathbf{a}$ but in the same direction.

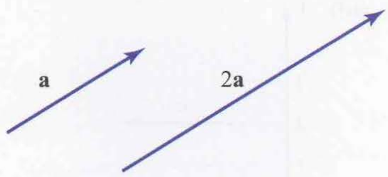


Figure 8.11

When the vector is in component form, each component is multiplied by the number. For example:

$$2 \times (3\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 6\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$$

$$2 \times \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 2 \end{pmatrix}.$$

The negative of a vector

In figure 8.12 the vector $-\mathbf{a}$ has the same length as the vector $\mathbf{a}$ but the opposite direction.

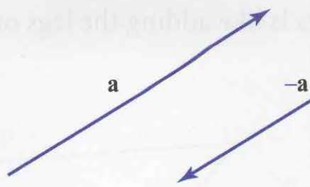


Figure 8.12

When **a** is given in component form, the components of −**a** are the same as those for **a** but with their signs reversed. So

$$-\begin{pmatrix} 23 \\ 0 \\ -11 \end{pmatrix} = \begin{pmatrix} -23 \\ 0 \\ +11 \end{pmatrix}$$

Adding vectors

When vectors are given in component form, they can be added component by component. This process can be seen geometrically by drawing them on graph paper, as in the example below.

EXAMPLE 8.7

Add the vectors $2\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$.

SOLUTION

$2\mathbf{i} - 3\mathbf{j} + 3\mathbf{i} + 5\mathbf{j} = 5\mathbf{i} + 2\mathbf{j}$

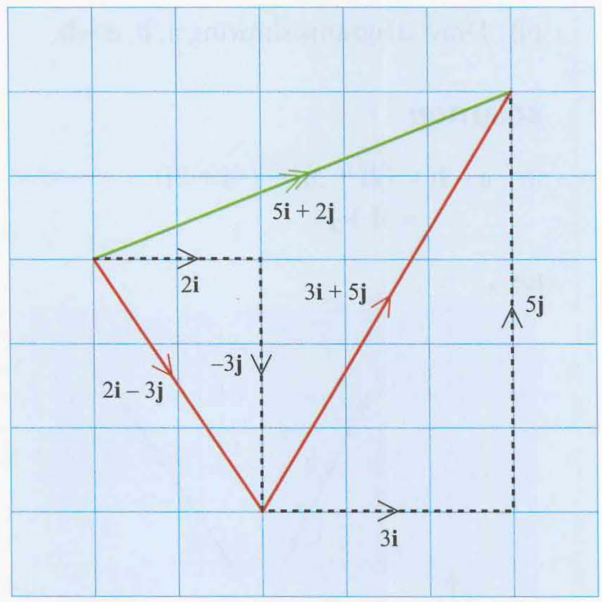


Figure 8.13

The sum of two (or more) vectors is called the *resultant* and is usually indicated by being marked with two arrowheads.

Adding vectors is like adding the legs of a journey to find its overall outcome (see figure 8.14).

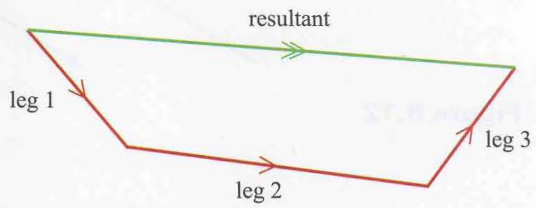


Figure 8.14

When vectors are given in magnitude–direction form, you can find their resultant by making a scale drawing, as in figure 8.14. If, however, you need to calculate their resultant, it is usually easiest to convert the vectors into component form, add component by component, and then convert the answer back to magnitude–direction form.

Subtracting vectors

Subtracting one vector from another is the same as adding the negative of the vector.

EXAMPLE 8.8

Two vectors **a** and **b** are given by

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} \qquad \mathbf{b} = -\mathbf{i} + 2\mathbf{j}.$$

(i) Find $\mathbf{a} - \mathbf{b}$.

(ii) Draw diagrams showing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} - \mathbf{b}$.

SOLUTION

(i) $\mathbf{a} - \mathbf{b} = (2\mathbf{i} + 3\mathbf{j}) - (-\mathbf{i} + 2\mathbf{j})$
$= 3\mathbf{i} + \mathbf{j}$

(ii)

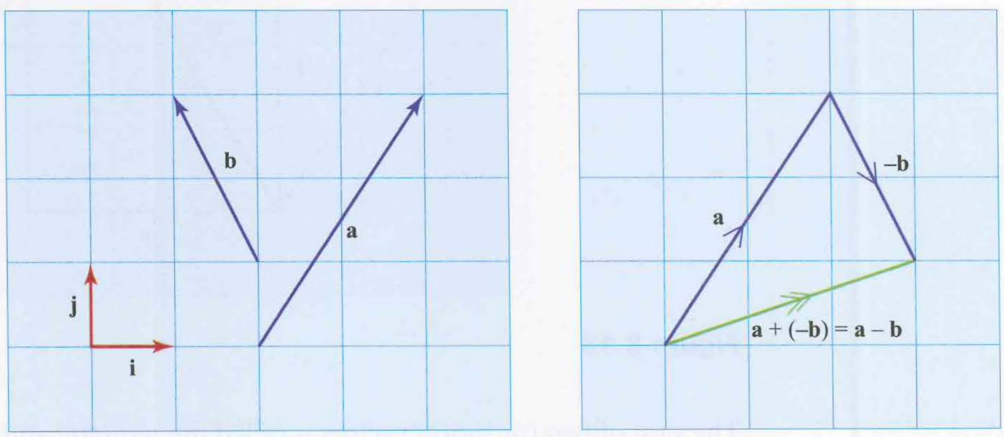


Figure 8.15

When you find the vector represented by the line segment joining two points, you are in effect subtracting their position vectors. If, for example,

P is the point (2, 1) and Q is the point (3, 5), $\overrightarrow{PQ}$ is $\begin{pmatrix}1\\4\end{pmatrix}$, as figure 8.16 shows.

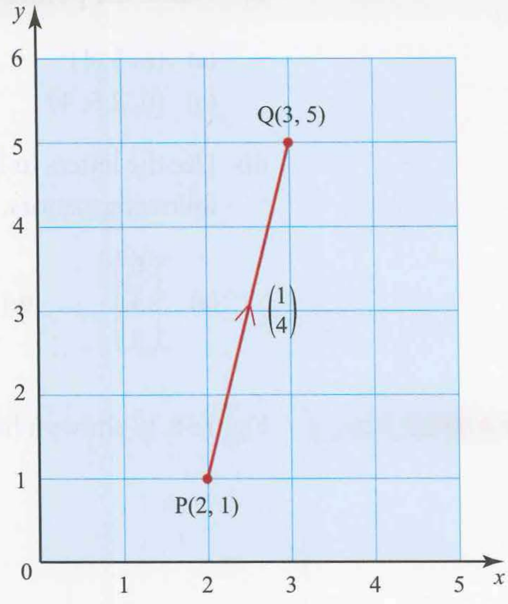


Figure 8.16

You find this by saying

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q}.$$

In this case, this gives

$$\overrightarrow{PQ} = -\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

as expected.

This is an important result:

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$$

where **p** and **q** are the position vectors of P and Q.

Geometrical figures

It is often useful to be able to express lines in a geometrical figure in terms of given vectors.

ACTIVITY 8.1 The diagram shows a cuboid OABCDEFG. P, Q, R, S and T are the mid-points of the edges they lie on. The origin is at O and the axes lie along OA, OC and OD, as shown in figure 8.17.

$$\overrightarrow{OA} = \begin{pmatrix}6\\0\\0\end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix}0\\5\\0\end{pmatrix}, \overrightarrow{OD} = \begin{pmatrix}0\\0\\4\end{pmatrix}$$

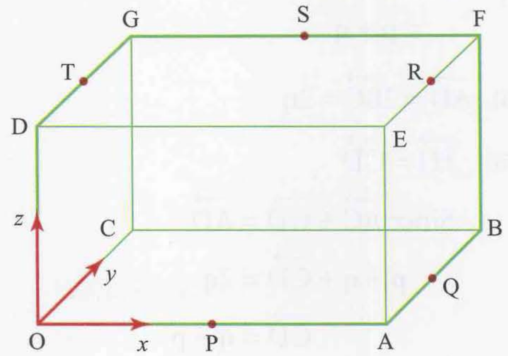


Figure 8.17

(i) Name the points with the following co-ordinates.

(a) (6, 5, 4) **(b)** (0, 5, 0) **(c)** (6, 2.5, 0)
(d) (0, 2.5, 4) **(e)** (3, 5, 4)

(ii) Use the letters in the diagram to give displacements which are equal to the following vectors. Give all possible answers; some of them have more than one.

(a) $\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix}$ **(d)** $\begin{pmatrix} -6 \\ -5 \\ 4 \end{pmatrix}$ **(e)** $\begin{pmatrix} -3 \\ 2.5 \\ 4 \end{pmatrix}$

EXAMPLE 8.9

Figure 8.18 shows a hexagonal prism.

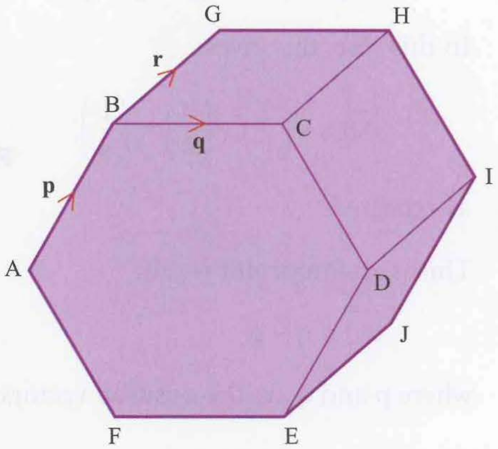


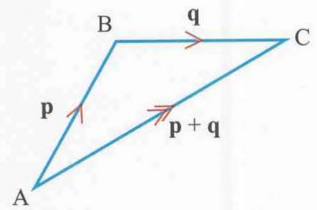
Figure 8.18

The hexagonal cross-section is regular and consequently $\overrightarrow{AD} = 2\overrightarrow{BC}$.

$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{BC} = \mathbf{q}$ and $\overrightarrow{BG} = \mathbf{r}$. Express the following in terms of **p**, **q** and **r**.

(i) $\overrightarrow{AC}$ **(ii)** $\overrightarrow{AD}$ **(iii)** $\overrightarrow{HI}$ **(iv)** $\overrightarrow{IJ}$
(v) $\overrightarrow{EF}$ **(vi)** $\overrightarrow{BE}$ **(vii)** $\overrightarrow{AH}$ **(viii)** $\overrightarrow{FI}$

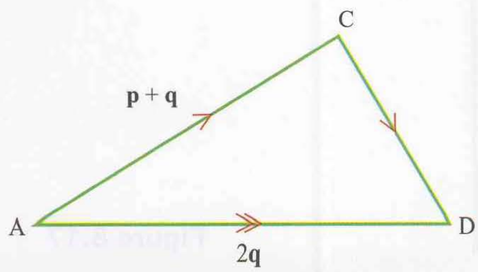
SOLUTION



(i) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= \mathbf{p} + \mathbf{q}$

(ii) $\overrightarrow{AD} = 2\overrightarrow{BC} = 2\mathbf{q}$

(iii) $\overrightarrow{HI} = \overrightarrow{CD}$



Since $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\mathbf{p} + \mathbf{q} + \overrightarrow{CD} = 2\mathbf{q}$

$\overrightarrow{CD} = \mathbf{q} - \mathbf{p}$

So $\overrightarrow{HI} = \mathbf{q} - \mathbf{p}$

(iv) $\overrightarrow{IJ} = \overrightarrow{DE}$
$= -\overrightarrow{AB}$
$= -\mathbf{p}$

(v) $\overrightarrow{EF} = -\overrightarrow{BC}$
$= -\mathbf{q}$

(vi) $\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$
$= \mathbf{q} + (\mathbf{q} - \mathbf{p}) + -\mathbf{p}$
$= 2\mathbf{q} - 2\mathbf{p}$

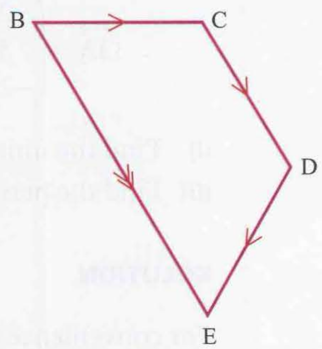


Figure 8.19

Notice that $\overrightarrow{BE} = 2\overrightarrow{CD}$.

(vii) $\overrightarrow{AH} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CH}$ ← $\overrightarrow{CH} = \overrightarrow{BG}$
$= \mathbf{p} + \mathbf{q} + \mathbf{r}$

(viii) $\overrightarrow{FI} = \overrightarrow{FE} + \overrightarrow{EJ} + \overrightarrow{JI}$ ← $\overrightarrow{FE} = \overrightarrow{BC}, \overrightarrow{EJ} = \overrightarrow{BG}, \overrightarrow{JI} = \overrightarrow{AB}$
$= \mathbf{q} + \mathbf{r} + \mathbf{p}$

Unit vectors

A unit vector is a vector with a magnitude of 1, like **i** and **j**. To find the unit vector in the same direction as a given vector, divide that vector by its magnitude.

Thus the vector $3\mathbf{i} + 5\mathbf{j}$ (in figure 8.20) has magnitude $\sqrt{3^2 + 5^2} = \sqrt{34}$, and so the vector $\frac{3}{\sqrt{34}}\mathbf{i} + \frac{5}{\sqrt{34}}\mathbf{j}$ is a unit vector. It has magnitude 1.

The unit vector in the direction of vector **a** is written as $\hat{\mathbf{a}}$ and read as 'a hat'.

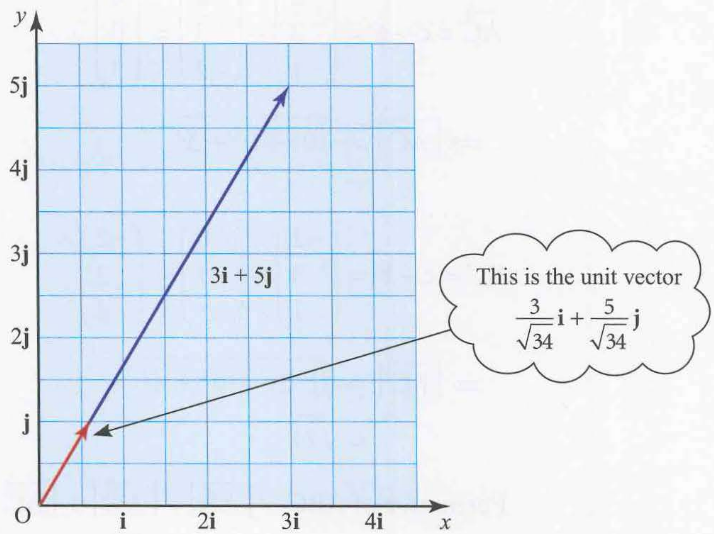


Figure 8.20

EXAMPLE 8.10

Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}, \ \overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

(i) Find the unit vector in the direction $\overrightarrow{AB}$.

(ii) Find the perimeter of triangle ABC.

SOLUTION

For convenience call $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(i) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

To find the unit vector in the direction $\overrightarrow{AB}$, you need to divide $\overrightarrow{AB}$ by its magnitude.

$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{2^2 + (-2)^2 + (-1)^2} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

This is the magnitude of $\overrightarrow{AB}$.

So the unit vector in the direction $\overrightarrow{AB}$ is $\frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

(ii) The perimeter of the triangle is given by $|\overrightarrow{AB}| + |\overrightarrow{AC}| + |\overrightarrow{BC}|$.

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow |\overrightarrow{AC}| &= \sqrt{0^2 + 0^2 + 3^2} \\ &= 3 \end{aligned}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow |\overrightarrow{BC}| &= \sqrt{(-2)^2 + 2^2 + 4^2} \\ &= \sqrt{24} \end{aligned}$$

$$\begin{aligned} \text{Perimeter of ABC} &= |\overrightarrow{AB}| + |\overrightarrow{AC}| + |\overrightarrow{BC}| \\ &= 3 + 3 + \sqrt{24} \\ &= 10.9 \end{aligned}$$

EXERCISE 8B

1 Simplify the following.

(i) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}4\\5\end{pmatrix}$ (ii) $\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}$

(iii) $\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}-3\\-4\end{pmatrix}$ (iv) $3\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}1\\-2\end{pmatrix}$

(v) $6(3\mathbf{i}-2\mathbf{j})-9(2\mathbf{i}-\mathbf{j})$

2 The vectors **p**, **q** and **r** are given by

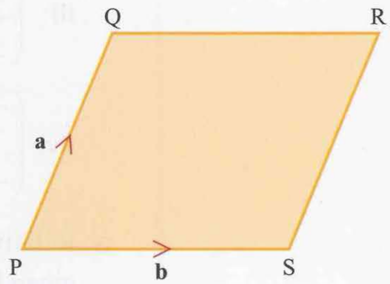
$$\mathbf{p}=3\mathbf{i}+2\mathbf{j}+\mathbf{k} \qquad \mathbf{q}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \qquad \mathbf{r}=-3\mathbf{i}-\mathbf{j}-2\mathbf{k}.$$

Find, in component form, the following vectors.

(i) $\mathbf{p}+\mathbf{q}+\mathbf{r}$ (ii) $\mathbf{p}-\mathbf{q}$ (iii) $\mathbf{p}+\mathbf{r}$

(iv) $3(\mathbf{p}-\mathbf{q})+2(\mathbf{p}+\mathbf{r})$ (v) $4\mathbf{p}-3\mathbf{q}+2\mathbf{r}$

3 In the diagram, PQRS is a parallelogram and $\overrightarrow{PQ}=\mathbf{a}$, $\overrightarrow{PS}=\mathbf{b}$.



(i) Write, in terms of **a** and **b**, the following vectors.

(a) $\overrightarrow{QR}$ (b) $\overrightarrow{PR}$

(c) $\overrightarrow{QS}$

(ii) The mid-point of PR is M. Find

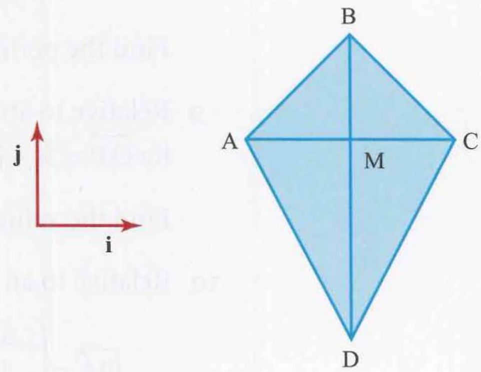
(a) $\overrightarrow{PM}$ (b) $\overrightarrow{QM}$.

(iii) Explain why this shows you that the diagonals of a parallelogram bisect each other.

4 In the diagram, ABCD is a kite. AC and BD meet at M.

$\overrightarrow{AB}=\mathbf{i}+\mathbf{j}$ and

$\overrightarrow{AD}=\mathbf{i}-2\mathbf{j}$



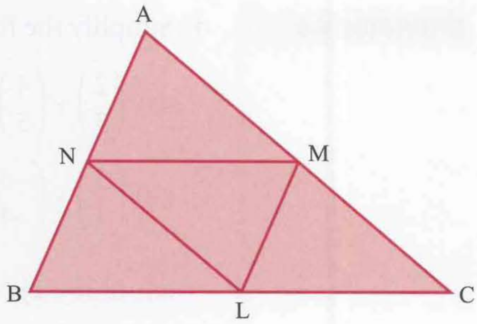
(i) Use the facts that the diagonals of a kite meet at right angles and that M is the mid-point of AC to find, in terms of **i** and **j**,

(a) $\overrightarrow{AM}$ (b) $\overrightarrow{AC}$

(c) $\overrightarrow{BC}$ (d) $\overrightarrow{CD}$.

(ii) Verify that $|\overrightarrow{AB}|=|\overrightarrow{BC}|$ and $|\overrightarrow{AD}|=|\overrightarrow{CD}|$.

5 In the diagram, ABC is a triangle. L, M and N are the mid-points of the sides BC, CA and AB.



$\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AC} = \mathbf{q}$

(i) Find, in terms of **p** and **q**, $\overrightarrow{BC}$, $\overrightarrow{MN}$, $\overrightarrow{LM}$ and $\overrightarrow{LN}$.

(ii) Explain how your results from part **(i)** show you that the sides of triangle LMN are parallel to those of triangle ABC, and half their lengths.

6 Find unit vectors in the same directions as the following vectors.

(i) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **(ii)** $3\mathbf{i} + 4\mathbf{j}$ **(iii)** $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ **(iv)** $5\mathbf{i} - 12\mathbf{j}$

7 Find unit vectors in the same direction as the following vectors.

(i) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **(ii)** $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ **(iii)** $3\mathbf{i} - 4\mathbf{k}$

(iv) $\begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$ **(v)** $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ **(vi)** $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

8 Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \ \overrightarrow{OB} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Find the perimeter of triangle ABC.

9 Relative to an origin O, the position vectors of the points P and Q are given by $\overrightarrow{OP} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OQ} = \mathbf{i} + x\mathbf{j} - 2\mathbf{k}$.

Find the values of x for which the magnitude of PQ is 7.

10 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}.$$

(i) Given that C is the point such that $\overrightarrow{AC} = 2\overrightarrow{AB}$, find the unit vector in the direction of $\overrightarrow{OC}$.

The position vector of the point D is given by $\overrightarrow{OD} = \begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix}$, where k is a constant, and it is given that $\overrightarrow{OD} = m\overrightarrow{OA} + n\overrightarrow{OB}$, where m and n are constants.

(ii) Find the values of m, n and k.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2007]

The angle between two vectors

p As you work through the proof in this section, make a list of all the results that you are assuming.

To find the angle θ between the two vectors

$$\overrightarrow{OA} = \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$$

and

$$\overrightarrow{OB} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$$

start by applying the cosine rule to triangle OAB in figure 8.21.

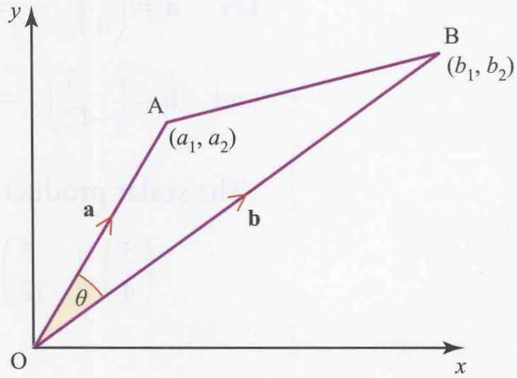


Figure 8.21

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2OA \times OB}$$

In this, OA, OB and AB are the lengths of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$, and so

$$OA = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \quad \text{and} \quad OB = |\mathbf{b}| = \sqrt{b_1^2 + b_2^2}.$$

The vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (b_1\mathbf{i} + b_2\mathbf{j}) - (a_1\mathbf{i} + a_2\mathbf{j})$

$$= (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j}$$

and so its length is given by

$$AB = |\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

Substituting for OA, OB and AB in the cosine rule gives

$$\cos\theta = \frac{(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - [(b_1 - a_1)^2 + (b_2 - a_2)^2]}{2\sqrt{a_1^2 + a_2^2} \times \sqrt{b_1^2 + b_2^2}}$$

$$= \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2 - \left(b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2\right)}{2|\mathbf{a}||\mathbf{b}|}$$

This simplifies to

$$\cos\theta = \frac{2a_1b_1 + 2a_2b_2}{2|\mathbf{a}||\mathbf{b}|} = \frac{a_1b_1 + a_2b_2}{|\mathbf{a}||\mathbf{b}|}$$

The expression on the top line, $a_1b_1 + a_2b_2$, is called the *scalar product* (or *dot product*) of the vectors **a** and **b** and is written **a.b**. Thus

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

This result is usually written in the form

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta.$$

The next example shows you how to use it to find the angle between two vectors given numerically.

EXAMPLE 8.11

Find the angle between the vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

SOLUTION

Let $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \Rightarrow \quad |\,\mathbf{a}\,| = \sqrt{3^2 + 4^2} = 5$

and $\mathbf{b} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad \Rightarrow \quad |\,\mathbf{b}\,| = \sqrt{5^2 + (-12)^2} = 13.$

The scalar product

$$\begin{aligned}\begin{pmatrix} 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 5 \\ -12 \end{pmatrix} &= 3 \times 5 + 4 \times (-12) \\ &= 15 - 48 \\ &= -33.\end{aligned}$$

Substituting in $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta$ gives

$$-33 = 5 \times 13 \times \cos\theta$$

$$\cos\theta = \frac{-33}{65}$$

$$\Rightarrow \qquad \theta = 120.5°.$$

Perpendicular vectors

Since $\cos 90° = 0$, it follows that if vectors **a** and **b** are perpendicular then $\mathbf{a}\,.\,\mathbf{b} = 0$.

Conversely, if the scalar product of two non-zero vectors is zero, they are perpendicular.

EXAMPLE 8.12

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are perpendicular.

SOLUTION

The scalar product of the vectors is

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} . \begin{pmatrix} 6 \\ -3 \end{pmatrix} \\ &= 2 \times 6 + 4 \times (-3) \\ &= 12 - 12 = 0.\end{aligned}$$

Therefore the vectors are perpendicular.

Further points concerning the scalar product

- You will notice that the scalar product of two vectors is an ordinary number. It has size but no direction and so is a scalar, rather than a vector. It is for this reason that it is called the scalar product. There is another way of multiplying vectors that gives a vector as the answer; it is called the *vector product.* This is beyond the scope of this book.

- The scalar product is calculated in the same way for three-dimensional vectors. For example:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = 2 \times 5 + 3 \times 6 + 4 \times 7 = 56.$$

  In general

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

- The scalar product of two vectors is commutative. It has the same value whichever of them is on the left-hand side or right-hand side. Thus $\mathbf{a} . \mathbf{b} = \mathbf{b} . \mathbf{a}$, as in the following example.

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 6 \\ 7 \end{pmatrix} = 2 \times 6 + 3 \times 7 = 33 \qquad \begin{pmatrix} 6 \\ 7 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 6 \times 2 + 7 \times 3 = 33.$$

---

p How would you prove this result?

The angle between two vectors

The angle θ between the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ in two dimensions is given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2} \times \sqrt{b_1^2 + b_2^2}} = \frac{\mathbf{a} . \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $\mathbf{a} . \mathbf{b}$ is the scalar product of $\mathbf{a}$ and $\mathbf{b}$. This result was proved by using the cosine rule on page 271.

Show that the angle between the three-dimensional vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is also given by

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

but that the scalar product $\mathbf{a}.\mathbf{b}$ is now

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Working in three dimensions

When working in two dimensions you found the angle between two lines by using the scalar product. As you have just proved, this method can be extended into three dimensions, and its use is shown in the following example.

EXAMPLE 8.13 The points P, Q and R are (1, 0, –1), (2, 4, 1) and (3, 5, 6). Find ∠QPR.

SOLUTION

The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by θ in

$$\cos\theta = \frac{\overrightarrow{PQ}.\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}$$

In this

$$\overrightarrow{PQ} = \begin{pmatrix}2\\4\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix} \qquad |\overrightarrow{PQ}| = \sqrt{1^2+4^2+2^2} = \sqrt{21}$$

Similarly

$$\overrightarrow{PR} = \begin{pmatrix}3\\5\\6\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}2\\5\\7\end{pmatrix} \qquad |\overrightarrow{PR}| = \sqrt{2^2+5^2+7^2} = \sqrt{78}$$

Therefore

$$\overrightarrow{PQ}.\overrightarrow{PR} = \begin{pmatrix}1\\4\\2\end{pmatrix}.\begin{pmatrix}2\\5\\7\end{pmatrix}$$

$$= 1\times2 + 4\times5 + 2\times7$$
$$= 36$$

Substituting gives

$$\cos\theta = \frac{36}{\sqrt{21}\times\sqrt{78}}$$

$$\Rightarrow \theta = 27.2°$$

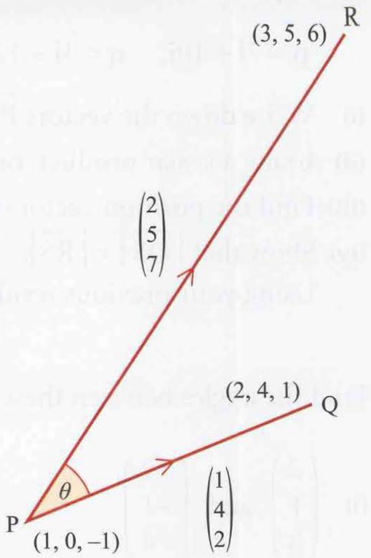


Figure 8.22

⚠ You must be careful to find the correct angle. To find ∠QPR (see figure 8.23), you need the scalar product $\overrightarrow{PQ} \cdot \overrightarrow{PR}$, as in the example above. If you take $\overrightarrow{QP} \cdot \overrightarrow{PR}$, you will obtain ∠Q′PR, which is (180° – ∠QPR).

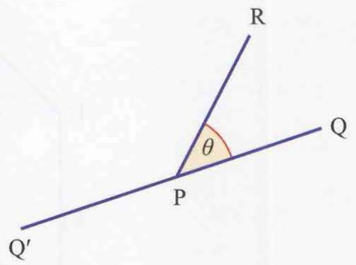


Figure 8.23

EXERCISE 8C

1 Find the angles between these vectors.

(i) $2\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{i} + \mathbf{j}$ **(ii)** $2\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$

(iii) $\begin{pmatrix}-1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\-2\end{pmatrix}$ **(iv)** $4\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$

(v) $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-6\\4\end{pmatrix}$ **(vi)** $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}-6\\2\end{pmatrix}$

2 The points A, B and C have co-ordinates (3, 2), (6, 3) and (5, 6), respectively.

(i) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(ii) Show that the angle ABC is 90°.

(iii) Show that $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.

(iv) The figure ABCD is a square.
Find the co-ordinates of the point D.

3 Three points P, Q and R have position vectors, **p**, **q** and **r** respectively, where

$$\mathbf{p} = 7\mathbf{i} + 10\mathbf{j}, \quad \mathbf{q} = 3\mathbf{i} + 12\mathbf{j}, \quad \mathbf{r} = -\mathbf{i} + 4\mathbf{j}.$$

(i) Write down the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$, and show that they are perpendicular.

(ii) Using a scalar product, or otherwise, find the angle PRQ.

(iii) Find the position vector of S, the mid-point of PR.

(iv) Show that $|\overrightarrow{QS}| = |\overrightarrow{RS}|$.

Using your previous results, or otherwise, find the angle PSQ.

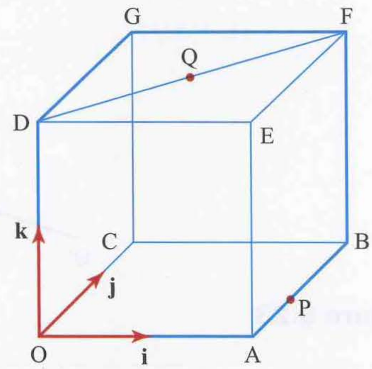
[MEI]

4 Find the angles between these pairs of vectors.

(i) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$

(iii) $3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $-4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

5 In the diagram, OABCDEFG is a cube in which each side has length 6. Unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The point P is such that $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$ and the point Q is the mid-point of DF.



(i) Express each of the vectors $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$ in terms of **i**, **j** and **k**.

(ii) Find the angle OQP.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q6 November 2009]

6 Relative to an origin O, the position vectors of points A and B are $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + p\mathbf{k}$ respectively.

(i) Find the value of p for which OA and OB are perpendicular.

(ii) In the case where $p = 6$, use a scalar product to find angle AOB, correct to the nearest degree.

(iii) Express the vector $\overrightarrow{AB}$ in terms of p and hence find the values of p for which the length of AB is 3.5 units.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2008]

7 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

(i) Find the value of $\overrightarrow{OA}\,.\,\overrightarrow{OB}$ and hence state whether angle AOB is acute, obtuse or a right angle.

(ii) The point X is such that $\overrightarrow{AX} = \frac{2}{5}\overrightarrow{AB}$. Find the unit vector in the direction of OX.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 June 2009]

8 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

(i) Use a scalar product to find angle AOB, correct to the nearest degree.

(ii) Find the unit vector in the direction of $\overrightarrow{AB}$.

(iii) The point C is such that $\overrightarrow{OC} = 6\mathbf{j} + p\mathbf{k}$, where p is a constant. Given that the lengths of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal, find the possible values of p.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2005]

9 Relative to an origin O, the position vectors of the points P and Q are given by

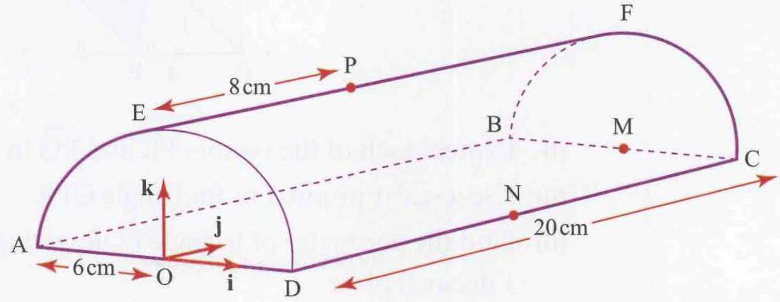
$$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ q \end{pmatrix}, \quad \text{where } q \text{ is a constant.}$$

(i) In the case where $q = 3$, use a scalar product to show that $\cos POQ = \frac{1}{7}$.

(ii) Find the values of q for which the length of $\overrightarrow{PQ}$ is 6 units.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 November 2005]

10 The diagram shows a semi-circular prism with a horizontal rectangular base ABCD. The vertical ends AED and BFC are semi-circles of radius 6 cm. The length of the prism is 20 cm. The mid-point of AD is the origin O, the mid-point of BC is M and the mid-point of DC is N. The points E and F are the highest points of the semi-circular ends of the prism. The point P lies on EF such that EP = 8 cm.



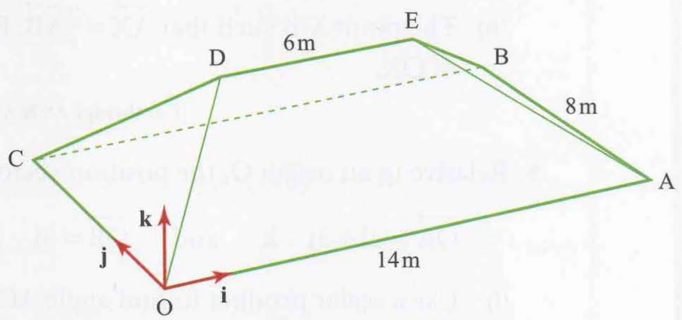
Unit vectors **i**, **j** and **k** are parallel to OD, OM and OE respectively.

(i) Express each of the vectors $\overrightarrow{PA}$ and $\overrightarrow{PN}$ in terms of **i**, **j** and **k**.

(ii) Use a scalar product to calculate angle APN.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 November 2008]

11 The diagram shows the roof of a house. The base of the roof, OABC, is rectangular and horizontal with OA = CB = 14 m and OC = AB = 8 m. The top of the roof DE is 5 m above the base and DE = 6 m. The sloping edges OD, CD, AE and BE are all equal in length.



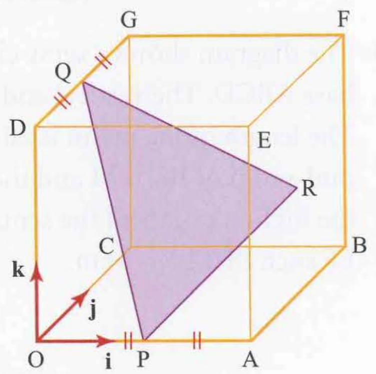
Unit vectors **i** and **j** are parallel to OA and OC respectively and the unit vector **k** is vertically upwards.

(i) Express the vector $\overrightarrow{OD}$ in terms of **i**, **j** and **k**, and find its magnitude.

(ii) Use a scalar product to find angle DOB.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2006]

12 The diagram shows a cube OABCDEFG in which the length of each side is 4 units. The unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The mid-points of OA and DG are P and Q respectively and R is the centre of the square face ABFE.



(i) Express each of the vectors $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of **i**, **j** and **k**.

(ii) Use a scalar product to find angle QPR.

(ii) Find the perimeter of triangle PQR, giving your answer correct to 1 decimal place.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2007]

KEY POINTS

1 A vector quantity has magnitude and direction.

2 A scalar quantity has magnitude only.

3 Vectors are typeset in bold, **a** or **OA**, or in the form $\overrightarrow{\mathrm{OA}}$. They are handwritten either in the underlined form $\underline{a}$, or as $\overrightarrow{OA}$.

4 The length (or modulus or magnitude) of the vector **a** is written as a or as $|\mathbf{a}|$.

5 Unit vectors in the x, y and z directions are denoted by **i**, **j** and **k**, respectively.

6 A vector may be specified in

- magnitude–direction form: (r, θ) (in two dimensions)
- component form: $x\mathbf{i} + y\mathbf{j}$ or $\begin{pmatrix} x \\ y \end{pmatrix}$ (in two dimensions)

  $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (in three dimensions).

7 The position vector $\overrightarrow{\mathrm{OP}}$ of a point P is the vector joining the origin to P.

8 The vector $\overrightarrow{\mathrm{AB}}$ is $\mathbf{b} - \mathbf{a}$, where **a** and **b** are the position vectors of A and B.

9 The angle between two vectors, **a** and **b**, is given by θ in

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$ (in two dimensions)

$= a_1b_1 + a_2b_2 + a_3b_3$ (in three dimensions).

Answers

Neither University of Cambridge International Examinations nor OCR bear any responsibility for the example answers to questions taken from their past question papers which are contained in this publication.

Chapter 1

? (Page 1)

Like terms have the same variable; unlike terms do not.

Note that the power of the variable must also be the same, for example $4x$ and $5x^2$ are unlike terms and cannot be collected.

Exercise 1A (Page 4)

1 **(i)** $9x$
(ii) $p-13$
(iii) $k-4m+4n$
(iv) 0
(v) $r+2s-15t$

2 **(i)** $4(x+2y)$
(ii) $3(4a+5b-6c)$
(iii) $12(6f-3g-4h)$
(iv) $p(p-q+r)$
(v) $12k(k+12m-6n)$

3 **(i)** $28(x+y)$
(ii) $7b+13c$
(iii) $-p+24q+33r$
(iv) $2(5l+3w-h)$
(v) $2(w+2v)$

4 **(i)** $2ab$
(ii) $n(k-m)$
(iii) $q(2p-s)$
(iv) $4(x+2)$
(v) -2

5 **(i)** $6x^3y^2$
(ii) $30a^3b^3c^4$
(iii) $k^2m^2n^2$
(iv) $162p^4q^4r^4$
(v) $24r^2s^2t^2u^2$

6 **(i)** $\dfrac{b}{c}$
(ii) $\dfrac{e}{2f}$
(iii) $\dfrac{x}{5}$
(iv) $2a$
(v) $\dfrac{2}{pr}$

7 **(i)** 1
(ii) 5
(iii) pq
(iv) $\dfrac{g^2h^3}{3f^2}$
(v) $\dfrac{m^3}{n^2}$

8 **(i)** $\dfrac{5x}{6}$
(ii) $\dfrac{49x}{60}$
(iii) $\dfrac{z}{3}$
(iv) $\dfrac{5x}{12}$
(v) $\dfrac{27y}{40}$

9 **(i)** $\dfrac{8}{x}$
(ii) $\dfrac{y+x}{xy}$
(iii) $\dfrac{4y+x^2}{xy}$
(iv) $\dfrac{p^2+q^2}{pq}$
(v) $\dfrac{bc-ac+ab}{abc}$

10 **(i)** $\dfrac{3x-1}{4}$
(ii) $\dfrac{7x+3}{15}$
(iii) $\dfrac{11x-29}{12}$
(iv) $\dfrac{76-23x}{10}$
(v) $\dfrac{26x-3}{24}$

11 **(i)** $\frac{1}{2}$
(ii) $\dfrac{2}{(2x+1)^3}$
(iii) $\dfrac{(y-3)^3}{4x}$
(iv) 6
(v) $\dfrac{x^3(3x+2)}{12}$

? (Page 6)

A variable is a quantity which can change its value. A constant always has the same value.

? (Page 6)

Starting from one vertex, the polygon can be divided into $n-2$ triangles, each with angle sum 180°.

The angles of the triangles form the angles of the polygon.

? (Page 7)

You get $0=0$.

Exercise 1B (Page 9)

1 **(i)** $a=20$
(ii) $b=8$
(iii) $c=0$
(iv) $d=2$
(v) $e=-5$
(vi) $f=1.5$
(vii) $g=14$
(viii) $h=0$
(ix) $k=48$
(x) $l=9$
(xi) $m=1$
(xii) $n=0$

2 (i) $a+6a+75=180$

(ii) 15°, 75°, 90°

3 (i) $2(r-2)+r=32$

(ii) 10, 10, 12

4 (i) $2d+2(d-40)=400$

(ii) $d=120$, area $=9600\text{ m}^2$

5 (i) $3x+49=5x+15$

(ii) \$1

6 (i) $6c-q-25$

(ii) $6c-47=55:17$ correct

7 (i) $22m+36(18-m)$

(ii) 6 kg

8 (i) $a+18=5(a-2)$

(ii) 7

Exercise 1C (Page 12)

1 (i) $a=\dfrac{v-u}{t}$

(ii) $t=\dfrac{v-u}{a}$

2 $h=\dfrac{V}{lw}$

3 $r=\sqrt{\dfrac{A}{\pi}}$

4 (i) $s=\dfrac{v^2-u^2}{2a}$

(ii) $u=\pm\sqrt{v^2-2as}$

5 $h=\dfrac{A-2\pi r^2}{2\pi r}$

6 $a=\dfrac{2(s-ut)}{t^2}$

7 $b=\pm\sqrt{h^2-a^2}$

8 $g=\dfrac{4\pi^2 l}{T^2}$

9 $m=\dfrac{2E}{2gh+v^2}$

10 $R=\dfrac{R_1R_2}{R_1+R_2}$

11 $h=\dfrac{2A}{a+b}$

12 $u=\dfrac{fv}{v-f}$

13 $d=\dfrac{u^2}{u-f}$

14 $V=\dfrac{mRT}{M(p_1-p_2)}$

? (Page 12)

1 Constant acceleration formula

2 Volume of a cuboid

3 Area of a circle

4 Constant acceleration formula

5 Surface area of a closed cylinder

6 Constant acceleration formula

7 Pythagoras' theorem

8 Period of a simple pendulum

9 Energy formula

10 Resistances

11 Area of a trapezium

12 Focal length

13 Focal length

14 Pressure formula

? (Page 17)

100 m

Exercise 1D (Page 18)

1 (i) $(a+b)(l+m)$

(ii) $(p-q)(x+y)$

(iii) $(u-v)(r+s)$

(iv) $(m+p)(m+n)$

(v) $(x+2)(x-3)$

(vi) $(y+7)(y+3)$

(vii) $(z+5)(z-5)$

(viii) $(q-3)(q-3)=(q-3)^2$

(ix) $(2x+3)(x+1)$

(x) $(3v-10)(2v+1)$

2 (i) a^2+5a+6

(ii) $b^2+12b+35$

(iii) c^2-6c+8

(iv) $d^2-9d+20$

(v) e^2+5e-6

(vi) g^2-9

(vii) $h^2+10h+25$

(viii) $4i^2-12i+9$

(ix) $ac+ad+bc+bd$

(x) x^2-y^2

3 (i) $(x+2)(x+4)$

(ii) $(x-2)(x-4)$

(iii) $(y+4)(y+5)$

(iv) $(r+5)(r-3)$

(v) $(r-5)(r+3)$

(vi) $(s-2)^2$

(vii) $(x-6)(x+1)$

(viii) $(x+1)^2$

(ix) $(a+3)(a-3)$

(x) $x(x+6)$

4 (i) $(2x+1)(x+2)$

(ii) $(2x-1)(x-2)$

(iii) $(5x+1)(x+2)$

(iv) $(5x-1)(x-2)$

(v) $2(x+3)(x+4)$

(vi) $(2x+7)(2x-7)$

(vii) $(3x+2)(2x-3)$

(viii) $(3x-1)^2$

(ix) $(t_1+t_2)(t_1-t_2)$

(x) $(2x-y)(x-5y)$

5 (i) $x=8$ or $x=3$

(ii) $x=-8$ or $x=-3$

(iii) $x=2$ or $x=9$

(iv) $x=3$ (repeated)

(v) $x=-8$ or $x=8$

6 (i) $x=\frac{2}{3}$ or $x=1$

(ii) $x=-\frac{2}{3}$ or $x=-1$

(iii) $x=-\frac{1}{3}$ or $x=2$

(iv) $x=-\frac{4}{5}$ or $x=\frac{4}{5}$

(v) $x=\frac{2}{3}$ (repeated)

7 (i) $x=-4$ or $x=5$

(ii) $x=-3$ or $x=\frac{4}{3}$

(iii) $x=2$ (repeated)

(iv) $x=-3$ or $x=\frac{5}{2}$

(v) $x=-2$ or $x=3$

(vi) $x=4$ or $x=\frac{2}{3}$

8 **(i)** $x = \pm 1$ or $x = \pm 2$

(ii) $x = \pm 1$ or $x = \pm 3$

(iii) $x = \pm \frac{2}{3}$ or $x = \pm 1$

(iv) $x = \pm 1.5$ or $x = \pm 2$

(v) $x = 0$ or $x = \pm 0.4$

(vi) $x = 1$ or $x = 25$

(vii) $x = 1$ or $x = 2$

(viii) $x = 9$ (Note: $\sqrt{4}$ means $+2$)

9 **(i)** $x = \pm 1$

(ii) $x = \pm 2$

(iii) $x = \pm 3$

(iv) $x = \pm 2$

(v) $x = \pm 1$ or $x = \pm 1.5$

(vi) $x = 1$ or $x = \sqrt[3]{2}$

(vii) $x = 4$ or $x = 16$

(viii) $x = \frac{1}{4}$ or $x = 9$

10 $x = \pm 3$

11 **(i)** $w(w + 30)$

(ii) 80 m, 380 m

12 **(i)** $A = 2\pi rh + 2\pi r^2$

(ii) 3 cm

(iii) 5 cm

13 **(ii)** 14

(iii) 45

14 $x^2 + (x + 1)^2 = 29^2$;
20 cm, 21 cm, 29 cm

? (Page 22)

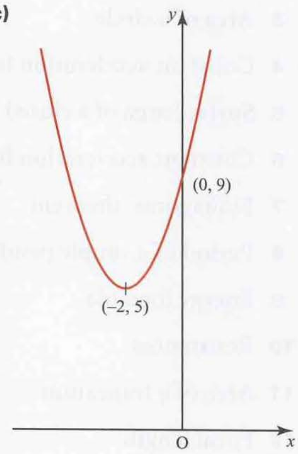
Since $\left(x + \frac{a}{2}\right)^2 = x^2 + ax + \frac{a^2}{4}$, it follows that to make $x^2 + ax$ into a perfect square you must add $\frac{a^2}{4}$ or $\left(\frac{a}{2}\right)^2$ to it.

Exercise 1E (Page 24)

1 **(i)** **(a)** $(x + 2)^2 + 5$

(b) $x = -2$; $(-2, 5)$

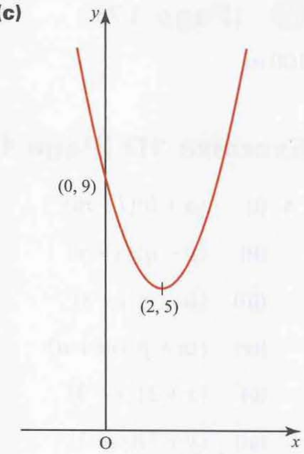
(c)



(ii) **(a)** $(x - 2)^2 + 5$

(b) $x = 2$; $(2, 5)$

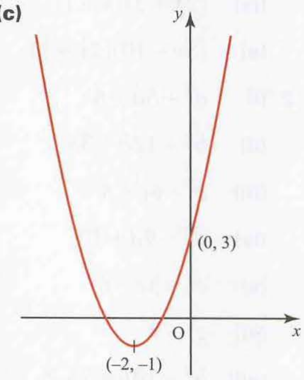
(c)



(iii) **(a)** $(x + 2)^2 - 1$

(b) $x = -2$; $(-2, -1)$

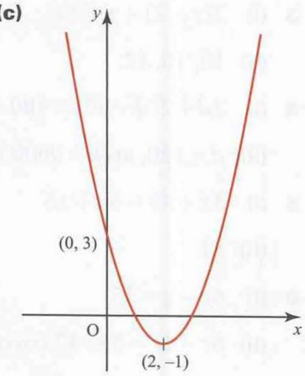
(c)



(iv) **(a)** $(x - 2)^2 - 1$

(b) $x = 2$; $(2, -1)$

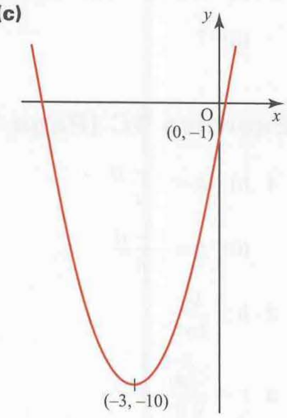
(c)



(v) **(a)** $(x + 3)^2 - 10$

(b) $x = -3$; $(-3, -10)$

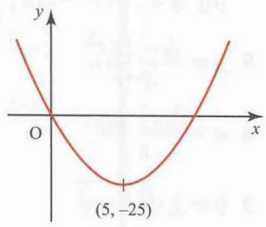
(c)



(vi) **(a)** $(x - 5)^2 - 25$

(b) $x = 5$; $(5, -25)$

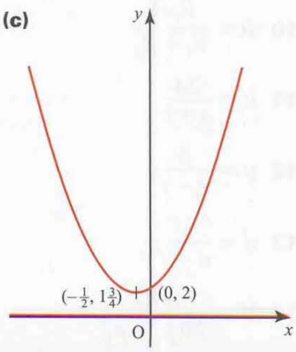
(c)



(vii) **(a)** $\left(x + \frac{1}{2}\right)^2 + 1\frac{3}{4}$

(b) $x = -\frac{1}{2}$; $\left(-\frac{1}{2}, 1\frac{3}{4}\right)$

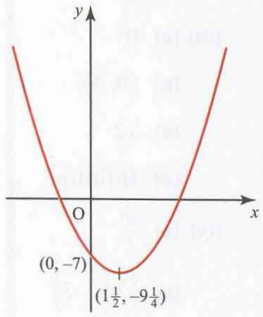
(c)



(viii) (a) $\left(x-1\frac{1}{2}\right)^2-9\frac{1}{4}$

(b) $x=1\frac{1}{2}$; $\left(1\frac{1}{2},-9\frac{1}{4}\right)$

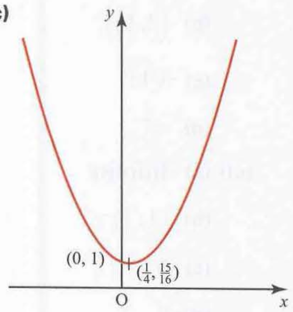
(c)



(ix) (a) $\left(x-\frac{1}{4}\right)^2+\frac{15}{16}$

(b) $x=\frac{1}{4}$; $\left(\frac{1}{4},\frac{15}{16}\right)$

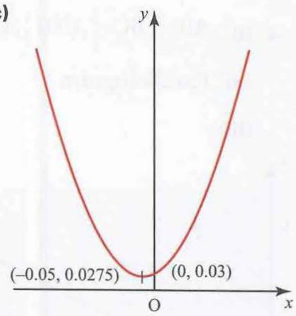
(c)



(x) (a) $(x+0.05)^2+0.0275$

(b) $x=-0.05$; $(-0.05, 0.0275)$

(c)



2 **(i)** x^2+4x+1

(ii) $x^2+8x+12$

(iii) x^2-2x+3

(iv) $x^2-20x+112$

(v) x^2-x+1

(vi) $x^2+0.2x+1$

3 **(i)** $2(x+1)^2+4$

(ii) $3(x-3)^2-54$

(iii) $-(x+1)^2+6$

(iv) $-2\left(x+\frac{1}{2}\right)^2-1\frac{1}{2}$

(v) $5(x-1)^2+2$

(vi) $4\left(x-\frac{1}{2}\right)^2-5$

(vii) $-3(x+2)^2+12$

(viii) $8\left(x+1\frac{1}{2}\right)^2-20$

4 **(i)** $b=-6$, $c=10$

(ii) $b=2$, $c=0$

(iii) $b=-8$, $c=16$

(iv) $b=6$, $c=11$

5 **(i)** $x=3\pm\sqrt{6}$; $x=5.449$ or $x=0.551$ to 3 d.p.

(ii) $x=4\pm\sqrt{17}$; $x=8.123$ or $x=-0.123$ to 3 d.p.

(iii) $x=1.5\pm\sqrt{1.25}$; $x=2.618$ or $x=0.382$ to 3 d.p.

(iv) $x=1.5\pm\sqrt{1.75}$; $x=2.823$ or $x=0.177$ to 3 d.p.

(v) $x=-0.4\pm\sqrt{0.56}$; $x=0.348$ or $x=-1.148$ to 3 d.p.

Exercise 1F (Page 29)

1 **(i)** $x=-0.683$ or $x=-7.317$

(ii) No real roots

(iii) $x=7.525$ or $x=-2.525$

(iv) No real roots

(v) $x=0.869$ or $x=-1.535$

(vi) $x=3.464$ or $x=-3.464$

2 **(i)** -7, no real roots

(ii) 25, two real roots

(iii) 9, two real roots

(iv) -96, no real roots

(v) 4, two real roots

(vi) 0, one repeated root

3 Discriminant $=b^2+4a^2$; a^2 and b^2 can never be negative so the discriminant is greater than zero for all values of a and b and hence the equation has real roots.

4 **(i)** $k=1$

(ii) $k=3$

(iii) $k=-\frac{9}{16}$

(iv) $k=\pm 8$

(v) $k=0$ or $k=-9$

5 **(i)** $t=1$ and 2

(ii) $t=3.065$

(iii) 12.25 m

Exercise 1G (Page 33)

1 **(i)** $x=1, y=2$

(ii) $x=0, y=4$

(iii) $x=2, y=1$

(iv) $x=1, y=1$

(v) $x=3, y=1$

(vi) $x=4, y=0$

(vii) $x=\frac{1}{2}, y=1$

(viii) $u=5, v=-1$

(ix) $l=-1, m=-2$

2 **(i)** $5p+8h=10$, $10p+6h=10$

(ii) Paperbacks 40c, hardbacks \$1

3 **(i)** $p=a+5$, $8a+9p=164$

(ii) Apples 7c, pears 12c

4 **(i)** $t_1+t_2=4$; $110t_1+70t_2=380$

(ii) 275 km motorway, 105 km country roads

5 **(i)** $x=3, y=1$ or $x=1, y=3$

(ii) $x=4, y=2$ or $x=-20, y=14$

(iii) $x=-3, y=-2$ or $x=1\frac{1}{2}, y=2\frac{1}{2}$

(iv) $k=-1, m=-7$ or $k=4, m=-2$

(v) $t_1=-10, t_2=-5$ or $t_1=10, t_2=5$

(vi) $p=-3, q=-2$

(vii) $k=-6, m=-4$ or $k=6, m=4$

(viii) $p_1=1, p_2=1$

6 **(i)** $h + 4r = 100,$
$2\pi rh + 2\pi r^2 = 1400\pi$

(ii) 6000π or $\dfrac{98000\pi}{27}$ cm^3

7 **(i)** $(3x + 2y)(2x + y)$ m^2

(iii) $x = \frac{1}{2}, y = \frac{1}{4}$

Exercise 1H (Page 37)

1 **(i)** $a > 6$

(ii) $b \leqslant 2$

(iii) $c > -2$

(iv) $d \leqslant -\frac{4}{3}$

(v) $e > 7$

(vi) $f > -1$

(vii) $g \leqslant 1.4$

(viii) $h < 0$

2 **(i)** $1 < p < 4$

(ii) $p \leqslant 1$ or $p \geqslant 4$

(iii) $-2 \leqslant x \leqslant -1$

(iv) $x < -2$ or $x > -1$

(v) $y < -1$ or $y > 3$

(vi) $-4 \leqslant z \leqslant 5$

(vii) $q \neq 2$

(viii) $y < -2$ or $y > 4$

(ix) $-2 < x < \frac{1}{3}$

(x) $y \leqslant -\frac{1}{2}$ or $y \geqslant 6$

(xi) $1 \leqslant x \leqslant 3$

(xii) $y < -\frac{1}{2}$ or $y > \frac{3}{5}$

3 **(i)** $k < \frac{9}{8}$

(ii) $k > -4$

(iii) $k > 10$ or $k < -10$

(iv) $k < 0$ or $k > 3$

4 **(i)** $k > 9$

(ii) $k < -\frac{1}{8}$

(iii) $-8 < k < 8$

(iv) $0 < k < 8$

Chapter 2

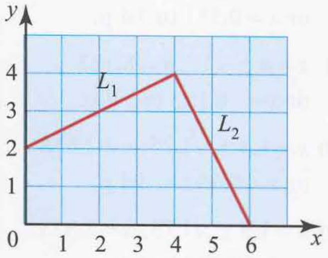
Activity 2.1 (Page 40)

A: $\frac{1}{2}$; B: -1; C: 0; D: ∞

? (Page 40)

No, the numerator and denominator of the gradient formula would have the same magnitude but the opposite sign, so m would be unchanged.

Activity 2.2 (Page 41)



An example of L_2 is the line joining (4, 4) to (6, 0).

$m_1 = \frac{1}{2}, m_2 = -2 \Rightarrow m_1 m_2 = -1.$

Activity 2.3 (Page 41)

$\triangle$ABE $\quad \triangle$BCD

AB = BC

$\angle$AEB = $\angle$BDC

$\angle$BAE = $\angle$CBD

$\Rightarrow$ Triangles ABE and BCD are congruent so BE = CD and AE = BD.

$m_1 = \dfrac{\text{BE}}{\text{AE}}; \ m_2 = -\dfrac{\text{BD}}{\text{CD}}$

$\Rightarrow m_1 m_2 = -\dfrac{\text{BE}}{\text{AE}} \times \dfrac{\text{BD}}{\text{CD}} = -1$

Exercise 2A (Page 44)

1 **(i)** **(a)** -2

(b) $(1, -1)$

(c) $\sqrt{20}$

(d) $\frac{1}{2}$

(ii) **(a)** -3

(b) $\left(3\frac{1}{2}, \frac{1}{2}\right)$

(c) $\sqrt{10}$

(d) $\frac{1}{3}$

(iii) **(a)** 0

(b) $(0, 3)$

(c) 12

(d) Infinite

(iv) **(a)** $\frac{10}{3}$

(b) $\left(3\frac{1}{2}, -3\right)$

(c) $\sqrt{109}$

(d) $-\frac{3}{10}$

(v) **(a)** $\frac{3}{2}$

(b) $\left(3, 1\frac{1}{2}\right)$

(c) $\sqrt{13}$

(d) $-\frac{2}{3}$

(vi) **(a)** Infinite

(b) $(1, 1)$

(c) 6

(d) 0

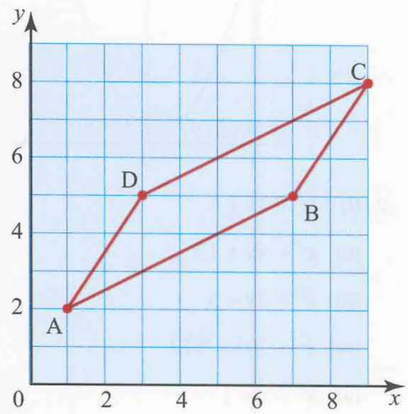
2 5

3 1

4 **(i)** AB: $\frac{1}{2}$, BC: $\frac{3}{2}$, CD: $\frac{1}{2}$, DA: $\frac{3}{2}$

(ii) Parallelogram

(iii)



5 **(i)** 6

(ii) AB = $\sqrt{20}$, BC = $\sqrt{5}$

(iii) 5 square units

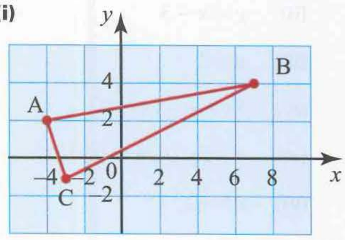
6 (i) 18

(ii) -2

(iii) 0 or 8

(iv) 8

7 (i)



(ii) $AB = BC = \sqrt{125}$

(iii) $\left(-3\frac{1}{2}, \frac{1}{2}\right)$

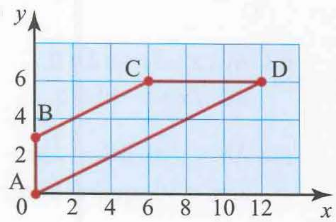
(iv) 17.5 square units

8 (i) $\frac{2y}{x}$

(ii) $(2x, 3y)$

(iii) $\sqrt{4x^2 + 16y^2}$

9 (i)



(ii) gradient BC = gradient AD

$= \frac{1}{2}$

(iii) (6, 3)

10 (i) 1 or 5

(ii) 7

(iii) 9

(iv) 1

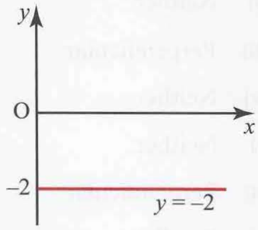
11 Diagonals have gradients $\frac{2}{3}$ and $-\frac{3}{2}$ so are perpendicular.

Mid-points of both diagonals are (4, 4) so they bisect each other.

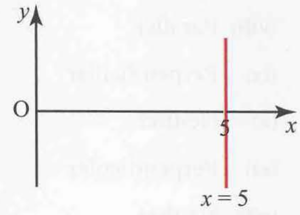
52 square units

Exercise 2B (Page 49)

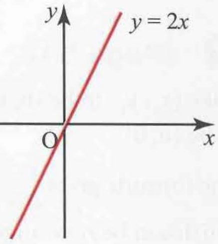
1 (i)



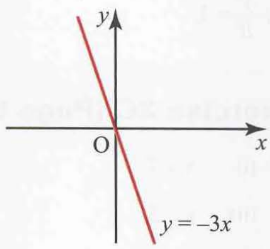
(ii)



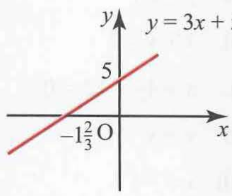
(iii)



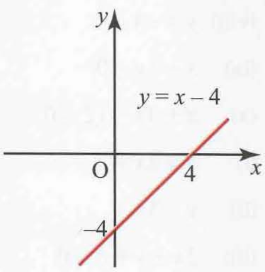
(iv)



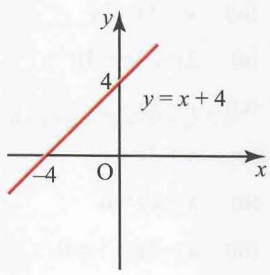
(v)



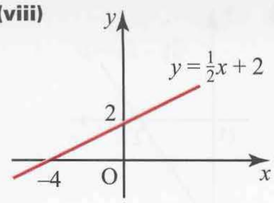
(vi)



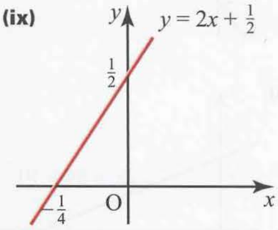
(vii)



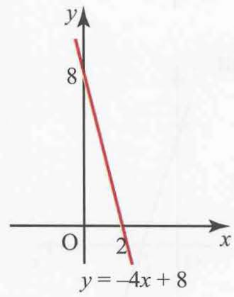
(viii)



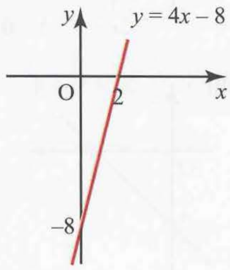
(ix)



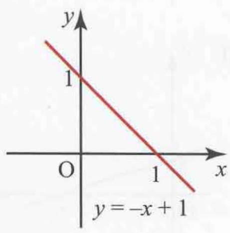
(x)



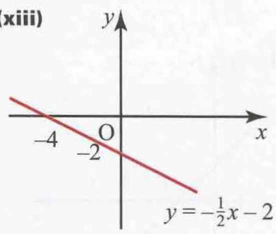
(xi)



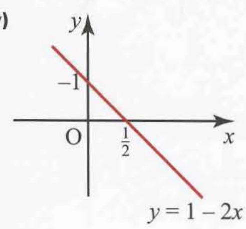
(xii)



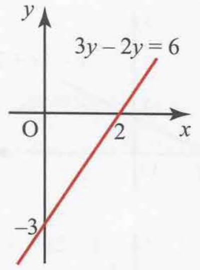
(xiii)



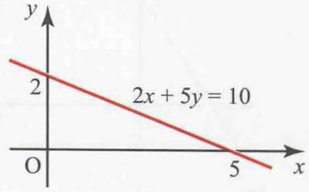
(xiv)



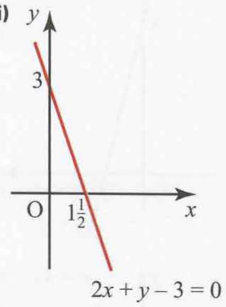
(xv)



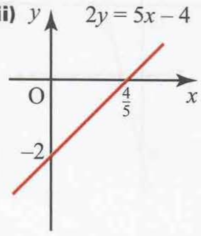
(xvi)



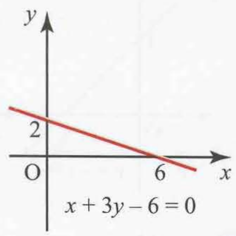
(xvii)



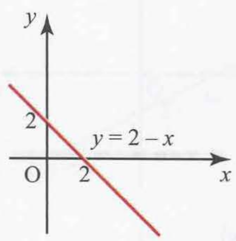
(xviii)



(xix)



(xx)



2 **(i)** Perpendicular
(ii) Neither
(iii) Perpendicular
(iv) Neither
(v) Neither
(vi) Perpendicular
(vii) Parallel
(viii) Parallel
(ix) Perpendicular
(x) Neither
(xi) Perpendicular
(xii) Neither

? (Page 51)

Take (x_1, y_1) to be $(0, b)$ and (x_2, y_2) to be $(a, 0)$.

The formula gives $\dfrac{y-b}{0-b} = \dfrac{x-0}{a-0}$ which can be rearranged to give $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Exercise 2C (Page 54)

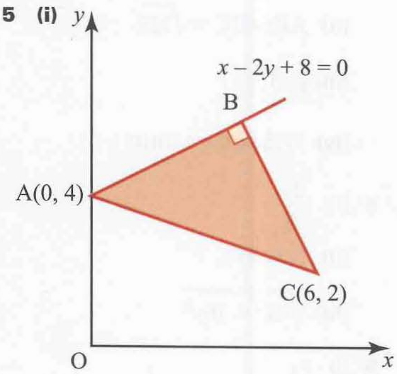
1 **(i)** $x = 7$
(ii) $y = 5$
(iii) $y = 2x$
(iv) $x + y = 2$
(v) $x + 4y + 12 = 0$
(vi) $y = x$
(vii) $x = -4$
(viii) $y = -4$
(ix) $x + 2y = 0$
(x) $x + 3y - 12 = 0$

2 **(i)** $y = 2x + 3$
(ii) $y = 3x$
(iii) $2x + y + 3 = 0$
(iv) $y = 3x - 14$
(v) $2x + 3y = 10$
(vi) $y = 2x - 3$

3 **(i)** $x + 3y = 0$
(ii) $x + 2y = 0$
(iii) $x - 2y - 1 = 0$
(iv) $2x + y - 2 = 0$
(v) $3x - 2y - 17 = 0$
(vi) $x + 4y - 24 = 0$

4 **(i)** $3x - 4y = 0$
(ii) $y = x - 3$
(iii) $x = 2$
(iv) $3x + y - 14 = 0$
(v) $x + 7y - 26 = 0$
(vi) $y = -2$

5 **(i)**

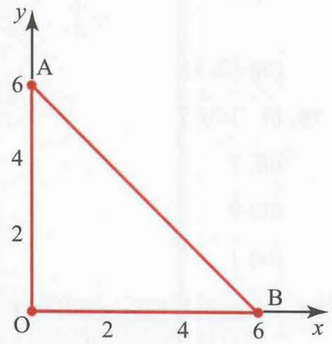


(ii) AC: $x + 3y - 12 = 0$, BC: $2x + y - 14 = 0$

(iii) AB = $\sqrt{20}$, BC = $\sqrt{20}$, area = 10 square units

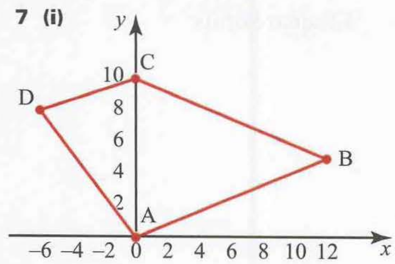
(iv) $\sqrt{10}$

6 **(i)**



(ii) $y = x$; $x + 2y - 6 = 0$; $2x + y - 6 = 0$

7 **(i)**



(ii) AB: $\frac{5}{12}$, BC: $-\frac{5}{12}$, CD: $\frac{1}{3}$,
AD: $-\frac{4}{3}$

(iii) AB = 13; BC = 13; CD = $\sqrt{40}$;
AD = 10

(iv) AB: $5x - 12y = 0$;
BC: $5x + 12y - 120 = 0$;
CD: $x - 3y + 30 = 0$;
AD: $4x + 3y = 0$

(v) 90 square units

? (Page 58)

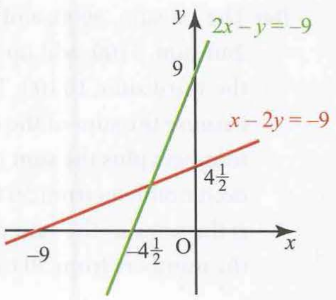
Attempting to solve the equations simultaneously gives 3 = 4 which is clearly false so there is no point of intersection. The lines are parallel.

Exercise 2D (Page 58)

1 (i) A(1, 1); B(5, 3); C(−1, 10)

(ii) BC = AC = $\sqrt{85}$

2 (i)



(ii) (−3, 3)

(iii) $2x - y = 3$; $x - 2y = 0$

(iv) (−6, −3); (5, 7)

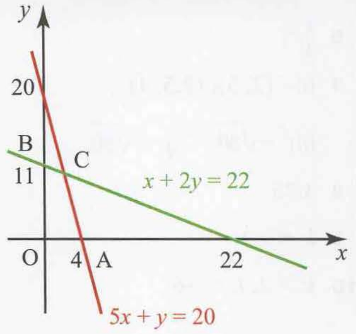
3 (i) $y = \frac{1}{2}x + 1$, $y = -2x + 6$

(ii) Gradients = $\frac{1}{2}$ and $-2 \Rightarrow$ AC and BD are perpendicular. Intersection = (2, 2) = mid-point of both AC and BD.

(iii) AC = BD = $\sqrt{20}$

(iv) Square

4 (i)



(ii) A: (4, 0), B: (0, 11), C: (2, 10)

(iii) 11 square units

(iv) (−2, 21)

5 (i) (2, 4)

(ii) (0, 3)

6 (i) $-\frac{1}{2}, \frac{3}{4}, -\frac{1}{2}, \frac{3}{4}$ parallelogram

(ii) 10

(iii) $-\frac{4}{3}$, $4x + 3y = 20$

(iv) (4.4, 0.8)

(v) 40 square units

7 (i) −3

(ii) $x - 3y + 5 = 0$

(iii) $x = 1$

(iv) (1, 2)

(vi) 3.75 square units

8 (i) $\frac{1}{2}(-2 + 14) = 6$

(ii) gradient of AD = $\frac{8}{h}$

gradient of CD = $\frac{8}{12 - h}$

(iii) x co-ordinate of D = 16

x co-ordinate of B = −4

(iv) 160 square units

9 M(4, 6), A(−8, 0), C(16, 12)

10 (i) $3x + 2y = 31$

(ii) (7, 5)

11 (i) $2x + 3y = 20$

(ii) C(10, 0), D(14, 6)

12 (6.2, 9.6)

13 (i) (4, 6)

(ii) (6, 10)

(iii) 40.9 units

14 B(6, 5), C(12, 8)

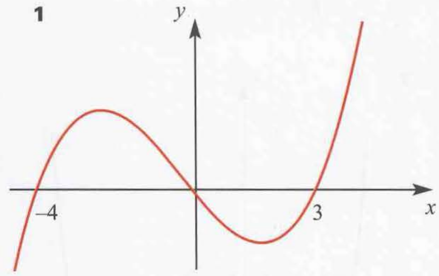
? (Page 63)

Even values of n: all values of y are positive; y axis is a line of symmetry.

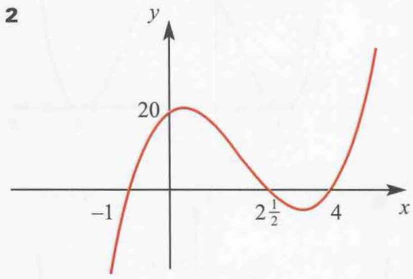
Odd values of n: origin is the centre of rotational symmetry of order 2.

Exercise 2E (Page 68)

1



2



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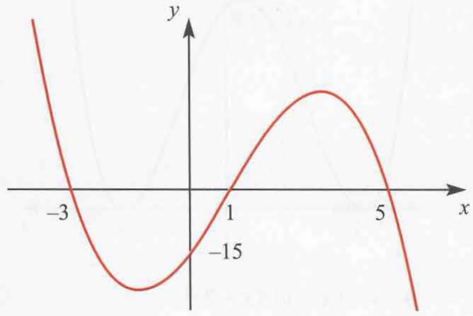

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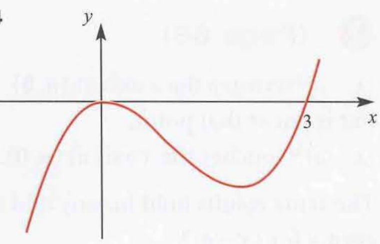

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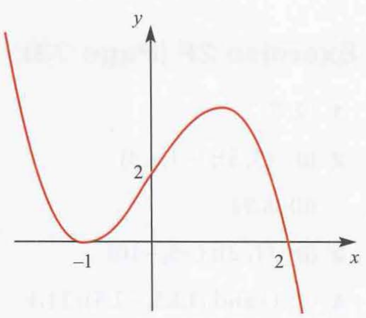

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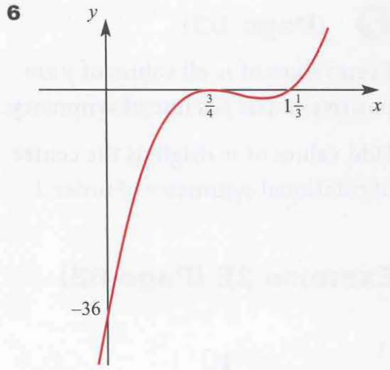

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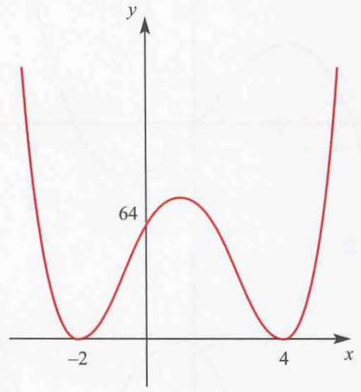

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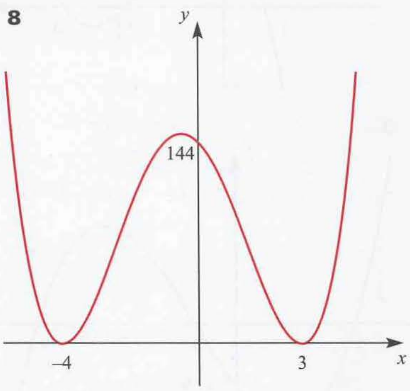

9 $y = (x+1)^2(x-2)^2$

(Page 68)

$(x-a)^3$: crosses the x axis at $(a, 0)$ but is flat at that point.
$(x-a)^4$: touches the x axis at $(a, 0)$.

The same results hold for any odd or even n for $(x-a)^n$.

Exercise 2F (Page 73)

1 $(2, 7)$

2 (i) $(3, 5); (-1, -3)$

(ii) 8.94

3 (i) $(1, 2); (-5, -10)$

4 $(2, 1)$ and $(12.5, -2.5)$; 11.1

5 $k = \pm 8$

6 $\frac{1}{4}$

7 (i) $(2, 5), (2.5, 4)$

(ii) $-\sqrt{80} < q < \sqrt{80}$

8 3.75

9 $k < -4$

10 $k > 2, k < -6$

Chapter 3

(Page 75)

(i) (a) Asian Savings

(b) 80 000, 160 000, 320 000, …

(c) Exponential geometric sequence

(d) The sequence could go on but the family will not live forever

(ii) (a) Fish & Chips opening hours

(b) 10, 10, 10, 10, 12, …

(c) They go in a cycle, repeating every 7

(d) Go on forever (or a long time)

(iii) (a) Clock

(b) 0, –3.5, –5, –3.5, 0, 3.5, …

(c) A regular pattern, repeating every 8

(d) Forever

(iv) (a) Steps

(b) 120, 140, 160, …

(c) Increasing by a fixed amount (arithmetic sequence)

(d) The steps won't go on forever

Exercise 3A (Page 81)

1 (i) Yes: $d = 2$, $u_7 = 39$

(ii) No

(iii) No

(iv) Yes: $d = 4$, $u_7 = 27$

(v) Yes: $d = -2$, $u_7 = -4$

2 (i) 10

(ii) 37

3 (i) 4

(ii) 34

4 (i) 5

(ii) 850

5 (i) 16, 18, 20

(ii) 324

6 (i) 15

(ii) 1170

7 (i) First term 4, common difference 6

(ii) 12

8 (i) 3

(ii) 165

9 (i) 5000

(ii) 5100

(iii) 10 100

(iv) The 1st sum, 5000, and the 2nd sum, 5100, add up to the third sum, 10 100. This is because the sum of the odd numbers plus the sum of the even numbers from 50 to 150 is the same as the sum of all the numbers from 50 to 150.

10 (i) 22 000

(ii) The sum becomes negative after the 31st term, i.e. from the 32nd term on.

11 (i) $u_k = 3k + 4$; 23rd term

(ii) $\frac{n}{2}(11 + 3n)$; 63 terms

12 (i) \$16 500

(ii) 8

13 (i) 49

(ii) 254.8 km

14 (i) 16

(ii) 2.5 cm

15 (i) $a = 10$, $d = 1.5$

(ii) $n = 27$

16 8