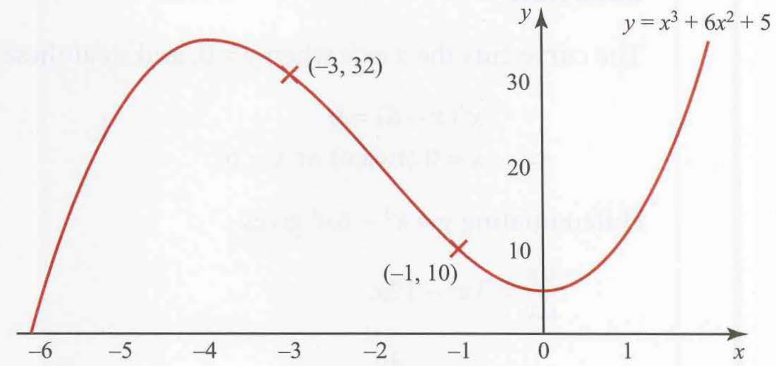


Figure 5.11

EXERCISE 5C

1 For each part of this question,

(a) find $\dfrac{dy}{dx}$

(b) find the gradient of the curve at the given point.

(i) $y = x^{-2}$; $(0.25, 16)$
(ii) $y = x^{-1} + x^{-4}$; $(-1, 0)$
(iii) $y = 4x^{-3} + 2x^{-5}$; $(1, 6)$
(iv) $y = 3x^4 - 4 - 8x^{-3}$; $(2, 43)$
(v) $y = \sqrt{x} + 3x$; $(4, 14)$
(vi) $y = 4x^{-\frac{1}{2}}$; $\left(9, 1\frac{1}{3}\right)$

2 **(i)** Sketch the curve $y = x^2 - 4$.
(ii) Write down the co-ordinates of the points where the curve crosses the x axis.
(iii) Differentiate $y = x^2 - 4$.
(iv) Find the gradient of the curve at the points where it crosses the x axis.

3 **(i)** Sketch the curve $y = x^2 - 6x$.
(ii) Differentiate $y = x^2 - 6x$.
(iii) Show that the point $(3, -9)$ lies on the curve $y = x^2 - 6x$ and find the gradient of the curve at this point.
(iv) Relate your answer to the shape of the curve.

4 **(i)** Sketch, on the same axes, the graphs with equations

$$y = 2x + 5 \quad \text{and} \quad y = 4 - x^2 \quad \text{for } -3 \leqslant x \leqslant 3.$$

(ii) Show that the point $(-1, 3)$ lies on both graphs.
(iii) Differentiate $y = 4 - x^2$ and so find its gradient at $(-1, 3)$.
(iv) Do you have sufficient evidence to decide whether the line $y = 2x + 5$ is a tangent to the curve $y = 4 - x^2$?
(v) Is the line joining $\left(2\frac{1}{2}, 0\right)$ to $(0, 5)$ a tangent to the curve $y = 4 - x^2$?

5 The curve $y = x^3 - 6x^2 + 11x - 6$ cuts the x axis at $x = 1$, $x = 2$ and $x = 3$.

(i) Sketch the curve.

(ii) Differentiate $y = x^3 - 6x^2 + 11x - 6$.

(iii) Show that the tangents to the curve at two of the points at which it cuts the x axis are parallel.

6 **(i)** Sketch the curve $y = x^2 + 3x - 1$.

(ii) Differentiate $y = x^2 + 3x - 1$.

(iii) Find the co-ordinates of the point on the curve $y = x^2 + 3x - 1$ at which it is parallel to the line $y = 5x - 1$.

(iv) Is the line $y = 5x - 1$ a tangent to the curve $y = x^2 + 3x - 1$? Give reasons for your answer.

7 **(i)** Sketch, on the same axes, the curves with equations

$$y = x^2 - 9 \quad \text{and} \quad y = 9 - x^2 \quad \text{for } -4 \leqslant x \leqslant 4.$$

(ii) Differentiate $y = x^2 - 9$.

(iii) Find the gradient of $y = x^2 - 9$ at the points $(2, -5)$ and $(-2, -5)$.

(iv) Find the gradient of the curve $y = 9 - x^2$ at the points $(2, 5)$ and $(-2, 5)$.

(v) The tangents to $y = x^2 - 9$ at $(2, -5)$ and $(-2, -5)$, and those to $y = 9 - x^2$ at $(2, 5)$ and $(-2, 5)$ are drawn to form a quadrilateral.
Describe this quadrilateral and give reasons for your answer.

8 **(i)** Sketch, on the same axes, the curves with equations

$$y = x^2 - 1 \quad \text{and} \quad y = x^2 + 3 \quad \text{for } -3 \leqslant x \leqslant 3.$$

(ii) Find the gradient of the curve $y = x^2 - 1$ at the point $(2, 3)$.

(iii) Give two explanations, one involving geometry and the other involving calculus, as to why the gradient at the point $(2, 7)$ on the curve $y = x^2 + 3$ should have the same value as your answer to part **(ii)**.

(iv) Give the equation of another curve with the same gradient function as $y = x^2 - 1$.

9 The function $f(x) = ax^3 + bx + 4$, where a and b are constants, goes through the point $(2, 14)$ with gradient 21.

(i) Using the fact that $(2, 14)$ lies on the curve, find an equation involving a and b.

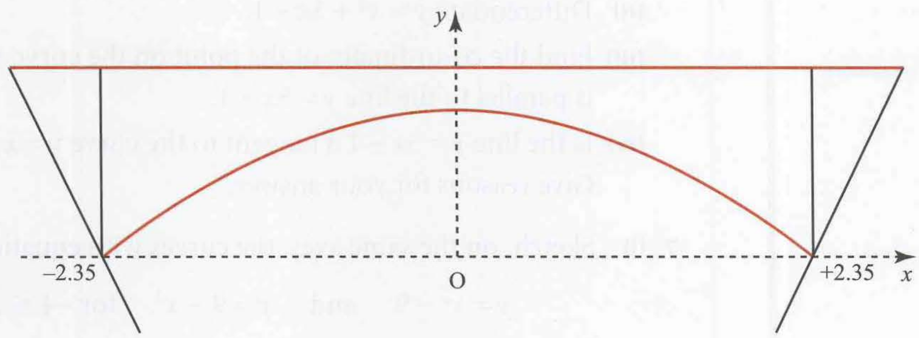
(ii) Differentiate $f(x)$ and, using the fact that the gradient is 21 when $x = 2$, form another equation involving a and b.

(iii) By solving these two equations simultaneously find the values of a and b.

10 In his book *Mathematician's Delight*, W.W. Sawyer observes that the arch of Victoria Falls Bridge appears to agree with the curve

$$y = \frac{116 - 21x^2}{120}$$

taking the origin as the point mid-way between the feet of the arch, and taking the distance between its feet as 4.7 units.



(i) Find $\frac{dy}{dx}$.

(ii) Evaluate $\frac{dy}{dx}$ when $x = -2.35$ and when $x = 2.35$.

(iii) Find the value of x for which $\frac{dy}{dx} = -0.5$.

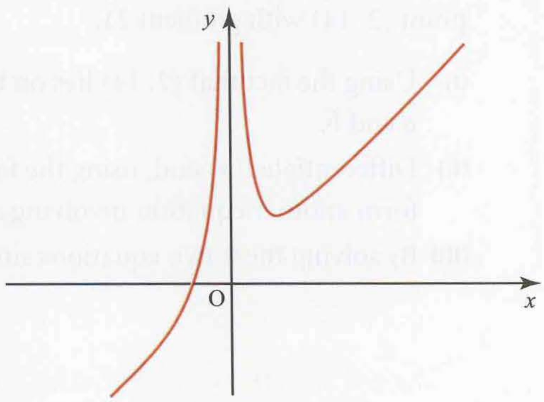
11 **(i)** Use your knowledge of the shape of the curve $y = \frac{1}{x}$ to sketch the curve $y = \frac{1}{x} + 2$.

(ii) Write down the co-ordinates of the point where the curve crosses the x axis.

(iii) Differentiate $y = \frac{1}{x} + 2$.

(iv) Find the gradient of the curve at the point where it crosses the x axis.

12 The sketch shows the graph of $y = \frac{4}{x^2} + x$.



(i) Differentiate $y = \dfrac{4}{x^2} + x$.

(ii) Show that the point $(-2, -1)$ lies on the curve.

(iii) Find the gradient of the curve at $(-2, -1)$.

(iv) Show that the point $(2, 3)$ lies on the curve.

(v) Find the gradient of the curve at $(2, 3)$.

(vi) Relate your answer to part **(v)** to the shape of the curve.

13 **(i)** Sketch, on the same axes, the graphs with equations

$$y = \frac{1}{x^2} + 1 \quad \text{and} \quad y = -16x + 13 \quad \text{for} \quad -3 \leqslant x \leqslant 3.$$

(ii) Show that the point $(0.5, 5)$ lies on both graphs.

(iii) Differentiate $y = \dfrac{1}{x^2} + 1$ and find its gradient at $(0.5, 5)$.

(iv) What can you deduce about the two graphs?

14 **(i)** Sketch the curve $y = \sqrt{x}$ for $0 \leqslant x \leqslant 10$.

(ii) Differentiate $y = \sqrt{x}$.

(iii) Find the gradient of the curve at the point $(9, 3)$.

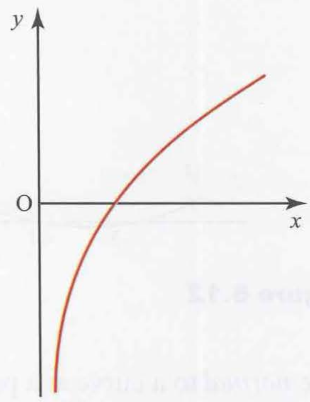
15 **(i)** Sketch the curve $y = \dfrac{4}{x^2}$ for $-3 \leqslant x \leqslant 3$.

(ii) Differentiate $y = \dfrac{4}{x^2}$.

(iii) Find the gradient of the curve at the point $(-2, 1)$.

(iv) Write down the gradient of the curve at the point $(2, 1)$.
Explain why your answer is $-1 \times$ your answer to part **(iii)**.

16 The sketch shows the curve $y = \dfrac{x}{2} - \dfrac{2}{x}$.



(i) Differentiate $y = \dfrac{x}{2} - \dfrac{2}{x}$.

(ii) Find the gradient of the curve at the point where it crosses the x axis.

17 The gradient of the curve $y = kx^{\frac{3}{2}}$ at the point $x = 9$ is 18. Find the value of k.

18 Find the gradient of the curve $y = \dfrac{x - 2}{\sqrt{x}}$ at the point where $x = 4$.

Tangents and normals

Now that you know how to find the gradient of a curve at any point you can use this to find the equation of the tangent at any specified point on the curve.

EXAMPLE 5.8

Find the equation of the tangent to the curve $y = x^2 + 3x + 2$ at the point (2, 12).

SOLUTION

Calculating $\frac{dy}{dx}$: $\frac{dy}{dx} = 2x + 3$.

Substituting $x = 2$ into the expression $\frac{dy}{dx}$ to find the gradient m of the tangent at that point:

$$\begin{aligned} m &= 2 \times 2 + 3 \\ &= 7. \end{aligned}$$

The equation of the tangent is given by

$$y - y_1 = m(x - x_1).$$

In this case $x_1 = 2$, $y_1 = 12$ so

$$y - 12 = 7(x - 2)$$

$$\Rightarrow \quad y = 7x - 2.$$

This is the equation of the tangent.

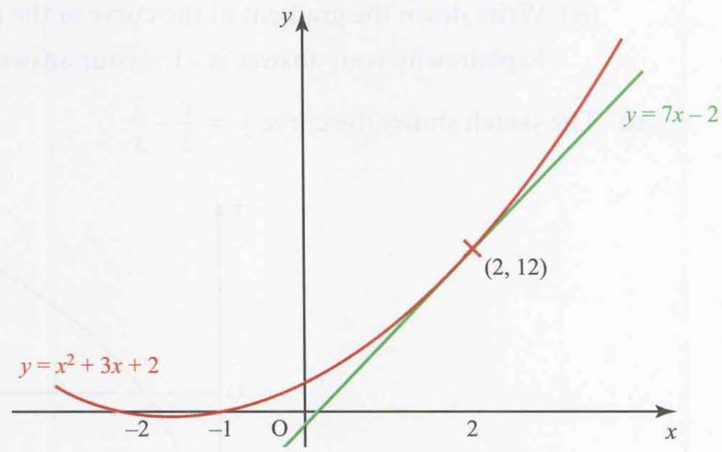


Figure 5.12

The *normal* to a curve at a particular point is the straight line which is at right angles to the tangent at that point (see figure 5.13). Remember that for perpendicular lines, $m_1 m_2 = -1$.

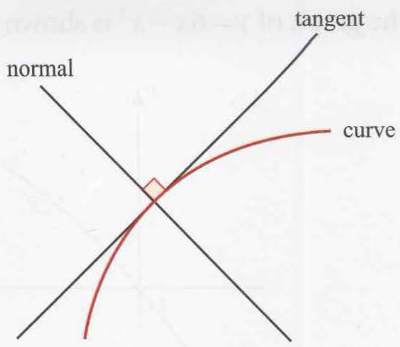


Figure 5.13

If the gradient of the tangent is m_1, the gradient, m_2, of the normal is given by

$$m_2 = -\frac{1}{m_1}.$$

This enables you to find the equation of the normal at any specified point on a curve.

EXAMPLE 5.9

A curve has equation $y = \frac{16}{x} - 4\sqrt{x}$. The normal to the curve at the point (4, –4) meets the y axis at the point P. Find the co-ordinates of P.

SOLUTION

You may find it easier to write $y = \frac{16}{x} - 4\sqrt{x}$ as $y = 16x^{-1} - 4x^{\frac{1}{2}}$.

Differentiating gives $\frac{dy}{dx} = -16x^{-2} - \frac{1}{2} \times 4x^{-\frac{1}{2}}$

$$= -\frac{16}{x^2} - \frac{2}{\sqrt{x}}$$

At the point (4, –4), $x = 4$ and

$$\frac{dy}{dx} = -\frac{16}{4^2} - \frac{2}{\sqrt{4}}$$
$$= -1 - 1 = -2$$

So at the point (4, –4) the gradient of the tangent is –2.

$$\text{Gradient of normal} = \frac{-1}{\text{gradient of tangent}} = \frac{1}{2}$$

The equation of the normal is given by

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = \tfrac{1}{2}(x - 4)$$
$$y = \tfrac{1}{2}x - 6$$

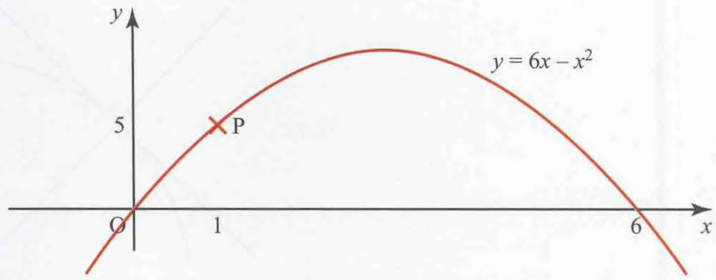
P is the point where the normal meets the y axis and so where $x = 0$.

Substituting $x = 0$ into $y = \frac{1}{2}x - 6$ gives $y = -6$.

So P is the point (0, –6).

EXERCISE 5D

1 The graph of $y = 6x - x^2$ is shown below.



The marked point, P, is (1, 5).

(i) Find the gradient function $\frac{dy}{dx}$.

(ii) Find the gradient of the curve at P.

(iii) Find the equation of the tangent at P.

2 (i) Sketch the curve $y = 4x - x^2$.

(i) Differentiate $y = 4x - x^2$.

(iii) Find the gradient of $y = 4x - x^2$ at the point (1, 3).

(iv) Find the equation of the tangent to the curve $y = 4x - x^2$ at the point (1, 3).

3 (i) Differentiate $y = x^3 - 4x^2$.

(ii) Find the gradient of $y = x^3 - 4x^2$ at the point (2, −8).

(iii) Find the equation of the tangent to the curve $y = x^3 - 4x^2$ at the point (2, −8).

(iv) Find the co-ordinates of the other point at which this tangent meets the curve.

4 (i) Sketch the curve $y = 6 - x^2$.

(ii) Find the gradient of the curve at the points (−1, 5) and (1, 5).

(iii) Find the equations of the tangents to the curve at these points.

(iv) Find the co-ordinates of the point of intersection of these two tangents.

5 (i) Sketch the curve $y = x^2 + 4$ and the straight line $y = 4x$ on the same axes.

(ii) Show that both $y = x^2 + 4$ and $y = 4x$ pass through the point (2, 8).

(iii) Show that $y = x^2 + 4$ and $y = 4x$ have the same gradient at (2, 8), and state what you conclude from this result and that in part **(ii)**.

6 (i) Find the equation of the tangent to the curve $y = 2x^3 - 15x^2 + 42x$ at (2, 40).

(ii) Using your expression for $\frac{dy}{dx}$, find the co-ordinates of another point on the curve at which the tangent is parallel to the one at (2, 40).

(iii) Find the equation of the normal at this point.

7 (i) Given that $y = x^3 - 4x^2 + 5x - 2$, find $\dfrac{dy}{dx}$.

The point P is on the curve and its x co-ordinate is 3.

(ii) Calculate the y co-ordinate of P.

(iii) Calculate the gradient at P.

(iv) Find the equation of the tangent at P.

(v) Find the equation of the normal at P.

(vi) Find the values of x for which the curve has a gradient of 5.

[MEI]

8 (i) Sketch the curve whose equation is $y = x^2 - 3x + 2$ and state the co-ordinates of the points A and B where it crosses the x axis.

(ii) Find the gradient of the curve at A and at B.

(iii) Find the equations of the tangent and normal to the curve at both A and B.

(iv) The tangent at A meets the tangent at B at the point P. The normal at A meets the normal at B at the point Q. What shape is the figure APBQ?

9 (i) Find the points of intersection of $y = 2x^2 - 9x$ and $y = x - 8$.

(ii) Find $\dfrac{dy}{dx}$ for the curve and hence find the equation of the tangent to the curve at each of the points in part **(i)**.

(iii) Find the point of intersection of the two tangents.

(iv) The two tangents from a point to a circle are always equal in length. Are the two tangents to the curve $y = 2x^2 - 9x$ (a parabola) from the point you found in part **(iii)** equal in length?

10 The equation of a curve is $y = \sqrt{x}$.

(i) Find the equation of the tangent to the curve at the point $(1, 1)$.

(ii) Find the equation of the normal to the curve at the point $(1, 1)$.

(iii) The tangent cuts the x axis at A and the normal cuts the x axis at B. Find the length of AB.

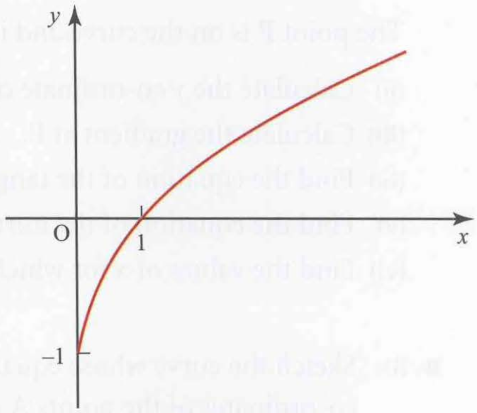
11 The equation of a curve is $y = \dfrac{1}{x}$.

(i) Find the equation of the tangent to the curve at the point $\left(2, \frac{1}{2}\right)$.

(ii) Find the equation of the normal to the curve at the point $\left(2, \frac{1}{2}\right)$.

(iii) Find the area of the triangle formed by the tangent, the normal and the y axis.

12 The sketch shows the graph of $y = \sqrt{x} - 1$.



(i) Differentiate $y = \sqrt{x} - 1$.

(ii) Find the co-ordinates of the point on the curve $y = \sqrt{x} - 1$ at which the tangent is parallel to the line $y = 2x - 1$.

(iii) Is the line $y = 2x - 1$ a tangent to the curve $y = \sqrt{x} - 1$? Give reasons for your answer.

13 The equation of a curve is $y = \sqrt{x} - \dfrac{1}{4x}$.

(i) Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}$.

(ii) Find the equation of the normal to the curve at the point where $x = \frac{1}{4}$.

(iii) Find the area of the triangle formed by the tangent, the normal and the x axis.

14 The equation of a curve is $y = \dfrac{9}{\sqrt{x}}$.

The tangent to the curve at the point (9, 3) meets the x axis at A and the y axis at B.
Find the length of AB.

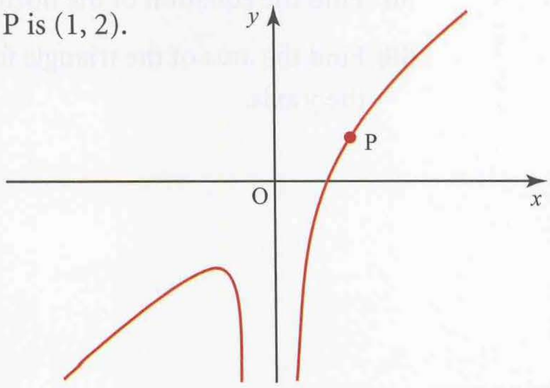
15 The equation of a curve is $y = 2 + \dfrac{8}{x^2}$.

(i) Find the equation of the normal to the curve at the point (2, 4).

(ii) Find the area of the triangle formed by the normal and the axes.

16 The graph of $y = 3x - \dfrac{1}{x^2}$ is shown below.

The point marked P is (1, 2).

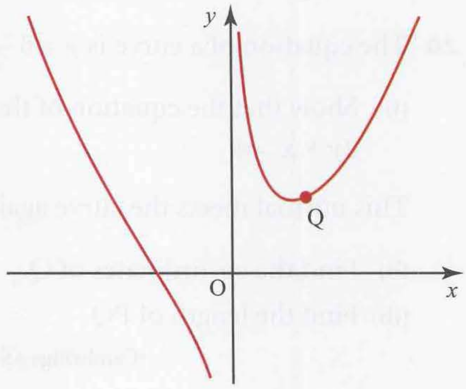


(i) Find the gradient function $\frac{dy}{dx}$.

(ii) Use your answer from part **(i)** to find the gradient of the curve at P.

(iii) Use your answer from part **(ii)**, and the fact that the gradient of the curve at P is the same as that of the tangent at P, to find the equation of the tangent at P in the form $y = mx + c$.

17 The graph of $y = x^2 + \frac{1}{x}$ is shown below. The point marked Q is (1, 2).



(i) Find the gradient function $\frac{dy}{dx}$.

(ii) Find the gradient of the tangent at Q.

(iii) Show that the equation of the normal to the curve at Q can be written as $x + y = 3$.

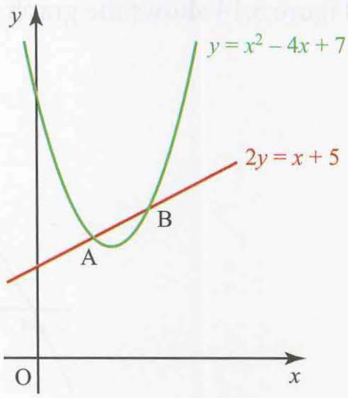
(iv) At what other points does the normal cut the curve?

18 The equation of a curve is $y = x^{\frac{3}{2}}$.

The tangent and normal to the curve at the point $x = 4$ intersect the x axis at A and B respectively.

Calculate the length of AB.

19 (i) The diagram shows the line $2y = x + 5$ and the curve $y = x^2 - 4x + 7$, which intersect at the points A and B.



Find

(a) the x co-ordinates of A and B,

(b) the equation of the tangent to the curve at B,

(c) the acute angle, in degrees correct to 1 decimal place, between this tangent and the line $2y = x + 5$.

(ii) Determine the set of values of k for which the line $2y = x + k$ does not intersect the curve $y = x^2 - 4x + 7$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 November 2009]

20 The equation of a curve is $y = 5 - \dfrac{8}{x}$.

(i) Show that the equation of the normal to the curve at the point P(2, 1) is $2y + x = 4$.

This normal meets the curve again at the point Q.

(ii) Find the co-ordinates of Q.

(iii) Find the length of PQ.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 November 2008]

Maximum and minimum points

ACTIVITY 5.6 Plot the graph of $y = x^4 - x^3 - 2x^2$, taking values of x from −2.5 to +2.5 in steps of 0.5, and answer these questions.

(i) How many stationary points has the graph?

(ii) What is the gradient at a stationary point?

(iii) One of the stationary points is a maximum and the others are minima. Which are of each type?

(iv) Is the maximum the highest point of the graph?

(v) Do the two minima occur exactly at the points you plotted?

(vi) Estimate the lowest value that y takes.

Gradient at a maximum or minimum point

Figure 5.14 shows the graph of $y = -x^2 + 16$. It has a *maximum point* at (0, 16).

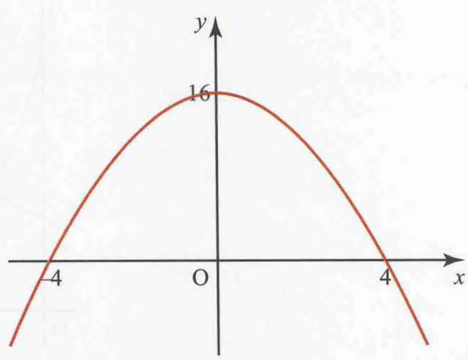


Figure 5.14

You will see that

- at the maximum point the gradient $\frac{dy}{dx}$ is zero
- the gradient is positive to the left of the maximum and negative to the right of it.

This is true for any maximum point (see figure 5.15).

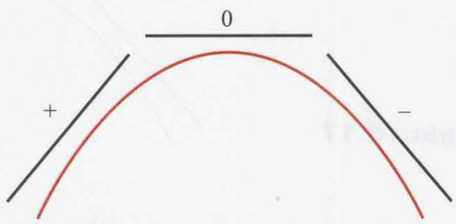


Figure 5.15

In the same way, for any minimum point (see figure 5.16):

- the gradient is zero at the minimum
- the gradient goes from negative to zero to positive.

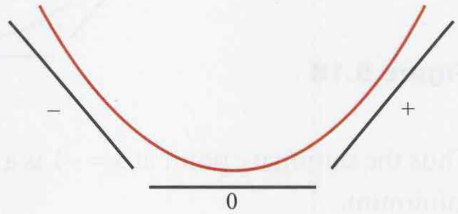


Figure 5.16

Maximum and minimum points are also known as stationary points as the gradient function is zero and so is neither increasing nor decreasing.

EXAMPLE 5.10

Find the stationary points on the curve of $y = x^3 - 3x + 1$, and sketch the curve.

SOLUTION

The gradient function for this curve is

$$\frac{dy}{dx} = 3x^2 - 3.$$

The x values for which $\frac{dy}{dx} = 0$ are given by

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ 3(x+1)(x-1) &= 0 \\ \Rightarrow \qquad x = -1 &\text{ or } x = 1. \end{aligned}$$

The signs of the gradient function just either side of these values tell you the nature of each stationary point.

For $x = -1$: $\quad x = -2 \Rightarrow \dfrac{dy}{dx} = 3(-2)^2 - 3 = +9$

$\quad\quad x = 0 \quad \Rightarrow \dfrac{dy}{dx} = 3(0)^2 - 3 = -3.$

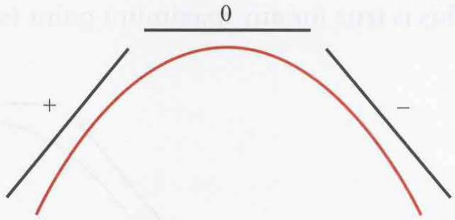


Figure 5.17

For $x = 1$: $\quad x = 0 \Rightarrow \dfrac{dy}{dx} = -3$

$\quad\quad x = 2 \Rightarrow \dfrac{dy}{dx} = 3(2)^2 - 3 = +9.$

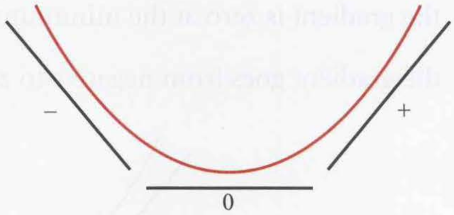


Figure 5.18

Thus the stationary point at $x = -1$ is a maximum and the one at $x = 1$ is a minimum.

Substituting the x values of the stationary points into the original equation, $y = x^3 - 3x + 1$, gives

when $x = -1$, $\quad y = (-1)^3 - 3(-1) + 1 = 3$
when $x = 1$, $\quad y = (1)^3 - 3(1) + 1 = -1.$

There is a maximum at $(-1, 3)$ and a minimum at $(1, -1)$. The sketch can now be drawn (see figure 5.19).

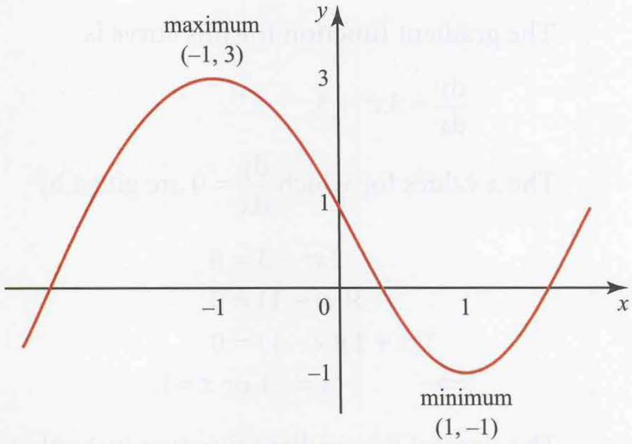


Figure 5.19

In this case you knew the general shape of the cubic curve and the positions of all of the maximum and minimum points, so it was easy to select values of x for which to test the sign of $\frac{dy}{dx}$. The curve of a more complicated function may have several maxima and minima close together, and even some points at which the gradient is undefined. To decide in such cases whether a particular stationary point is a maximum or a minimum, you must look at points which are *just* either side of it.

EXAMPLE 5.11

Find all the stationary points on the curve of $y = 2t^4 - t^2 + 1$ and sketch the curve.

SOLUTION

$$\frac{dy}{dt} = 8t^3 - 2t$$

At a stationary point, $\frac{dy}{dt} = 0$, so

$$8t^3 - 2t = 0$$
$$2t(4t^2 - 1) = 0$$
$$2t(2t - 1)(2t + 1) = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = 0 \text{ when } t = -0.5, 0 \text{ or } 0.5.$$

You may find it helpful to summarise your working in a table like the one below. You can find the various signs, + or –, by taking a test point in each interval, for example $t = 0.25$ in the interval $0 < t < 0.5$.

	$t < -0.5$	-0.5	$-0.5 < t < 0$	0	$0 < t < 0.5$	0.5	$t > 0.5$
Sign of $\frac{dy}{dt}$	–	0	+	0	–	0	+
Stationary point		min		max		min	

There is a maximum point when $t = 0$ and there are minimum points when $t = -0.5$ and $+0.5$.

When $t = 0$: $\quad y = 2(0)^4 - (0)^2 + 1 = 1$.
When $t = -0.5$: $\quad y = 2(-0.5)^4 - (-0.5)^2 + 1 = 0.875$.
When $t = 0.5$: $\quad y = 2(0.5)^4 - (0.5)^2 + 1 = 0.875$.

Therefore (0, 1) is a maximum point and (–0.5, 0.875) and (0.5, 0.875) are minima.

The graph of this function is shown in figure 5.20.

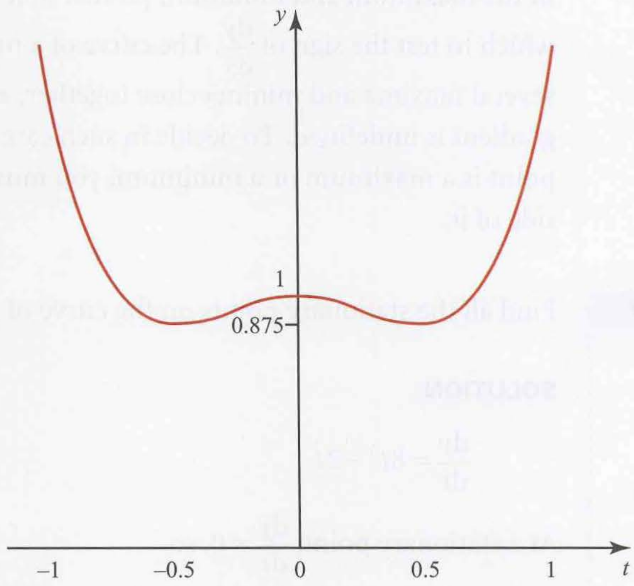


Figure 5.20

Increasing and decreasing functions

When the gradient is positive, the function is described as an increasing function. Similarly, when the gradient is negative, it is a decreasing function. These terms are often used for functions that are increasing or decreasing for all values of x.

EXAMPLE 5.12

Show that $y = x^3 + x$ is an increasing function.

SOLUTION

$y = x^3 + x \Rightarrow \dfrac{dy}{dx} = 3x^2 + 1.$

Since $x^2 \geqslant 0$ for all real values of x, $\dfrac{dy}{dx} \geqslant 1$

$\Rightarrow y = x^3 + x$ is an increasing function.

Figure 5.21 shows its graph.

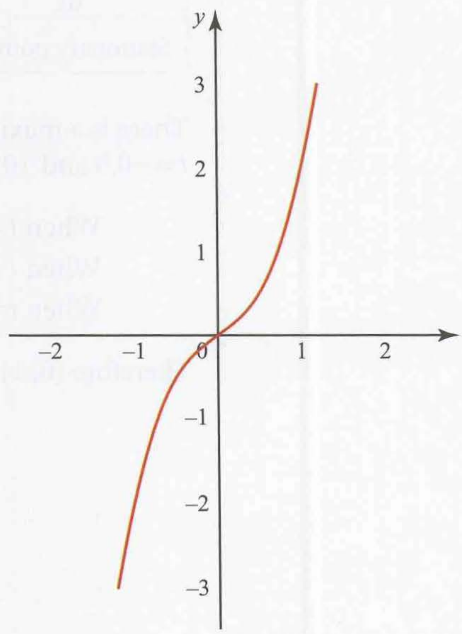


Figure 5.21

EXAMPLE 5.13

Find the range of values of x for which the function $y = x^2 - 6x$ is a decreasing function.

SOLUTION

$$y = x^2 - 6x \Rightarrow \frac{dy}{dx} = 2x - 6.$$

$$y \text{ decreasing} \Rightarrow \frac{dy}{dx} < 0$$

$$\Rightarrow \quad 2x - 6 < 0$$

$$\Rightarrow \quad x < 3.$$

Figure 5.22 shows the graph of $y = x^2 - 6x$.

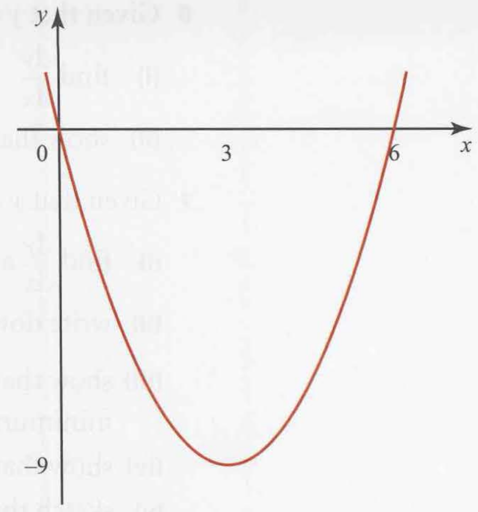


Figure 5.22

EXERCISE 5E

1 Given that $y = x^2 + 8x + 13$

(i) find $\frac{dy}{dx}$, and the value of x for which $\frac{dy}{dx} = 0$

(ii) showing your working clearly, decide whether the point corresponding to this x value is a maximum or a minimum by considering the gradient either side of it

(iii) show that the corresponding y value is -3

(iv) sketch the curve.

2 Given that $y = x^2 + 5x + 2$

(i) find $\frac{dy}{dx}$, and the value of x for which $\frac{dy}{dx} = 0$

(ii) classify the point that corresponds to this x value as a maximum or a minimum

(iii) find the corresponding y value

(iv) sketch the curve.

3 Given that $y = x^3 - 12x + 2$

(i) find $\frac{dy}{dx}$, and the values of x for which $\frac{dy}{dx} = 0$

(ii) classify the points that correspond to these x values

(iii) find the corresponding y values

(iv) sketch the curve.

4 **(i)** Find the co-ordinates of the stationary points of the curve $y = x^3 - 6x^2$, and determine whether each one is a maximum or a minimum.

(ii) Use this information to sketch the graph of $y = x^3 - 6x^2$.

5 Find $\frac{dy}{dx}$ when $y = x^3 - x$ and show that $y = x^3 - x$ is an increasing function for $x < -\frac{1}{\sqrt{3}}$ and $x > \frac{1}{\sqrt{3}}$.

6 Given that $y = x^3 + 4x$

(i) find $\frac{dy}{dx}$

(ii) show that $y = x^3 + 4x$ is an increasing function for all values of x.

7 Given that $y = x^3 + 3x^2 - 9x + 6$

(i) find $\frac{dy}{dx}$ and factorise the quadratic expression you obtain

(ii) write down the values of x for which $\frac{dy}{dx} = 0$

(iii) show that one of the points corresponding to these x values is a minimum and the other a maximum

(iv) show that the corresponding y values are 1 and 33 respectively

(v) sketch the curve.

8 Given that $y = 9x + 3x^2 - x^3$

(i) find $\frac{dy}{dx}$ and factorise the quadratic expression you obtain

(ii) find the values of x for which the curve has stationary points, and classify these stationary points

(iii) find the corresponding y values

(iv) sketch the curve.

9 **(i)** Find the co-ordinates and nature of each of the stationary points of $y = x^3 - 2x^2 - 4x + 3$.

(ii) Sketch the curve.

10 **(i)** Find the co-ordinates and nature of each of the stationary points of the curve with equation $y = x^4 + 4x^3 - 36x^2 + 300$.

(ii) Sketch the curve.

11 **(i)** Differentiate $y = x^3 + 3x$.

(ii) What does this tell you about the number of stationary points of the curve with equation $y = x^3 + 3x$?

(iii) Find the values of y corresponding to $x = -3, -2, -1, 0, 1, 2$ and 3.

(iv) Hence sketch the curve and explain your answer to part **(ii)**.

12 You are given that $y = 2x^3 + 3x^2 - 72x + 130$.

(i) Find $\frac{dy}{dx}$.

P is the point on the curve where $x = 4$.

(ii) Calculate the y co-ordinate of P.

(iii) Calculate the gradient at P and hence find the equation of the tangent to the curve at P.

(iv) There are two stationary points on the curve. Find their co-ordinates.

[MEI]

13 **(i)** Find the co-ordinates of the stationary points of the curve $f(x) = 4x + \frac{1}{x}$.

(ii) Find the set of values of x for which $f(x)$ is an increasing function.

14 The equation of a curve is $y = \frac{1}{6}(2x-3)^3 - 4x$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y axis.

(iii) Find the set of values of x for which $\frac{1}{6}(2x-3)^3 - 4x$ is an increasing function of x.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 June 2010]

15 The equation of a curve is $y = x^2 - 3x + 4$.

(i) Show that the whole of the curve lies above the x axis.

(ii) Find the set of values of x for which $x^2 - 3x + 4$ is a decreasing function of x.

The equation of a line is $y + 2x = k$, where k is a constant.

(iii) In the case where $k = 6$, find the co-ordinates of the points of intersection of the line and the curve.

(iv) Find the value of k for which the line is a tangent to the curve.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2005]

16 The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$.

(i) Find the x co-ordinates of the points of intersection of L and C.

(ii) Show that one of these points is also the stationary point of C.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2008]

e Points of inflection

It is possible for the value of $\frac{dy}{dx}$ to be zero at a point on a curve without it being a maximum or minimum. This is the case with the curve $y = x^3$, at the point (0, 0) (see figure 5.23).

$$y = x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2$$

and when $x = 0$, $\frac{dy}{dx} = 0$.

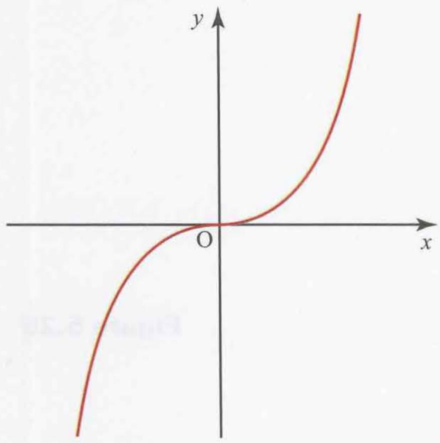


Figure 5.23

This is an example of a *point of inflection*. In general, a point of inflection occurs where the tangent to a curve crosses the curve. This can happen also when $\frac{dy}{dx} \neq 0$, as shown in figure 5.24.

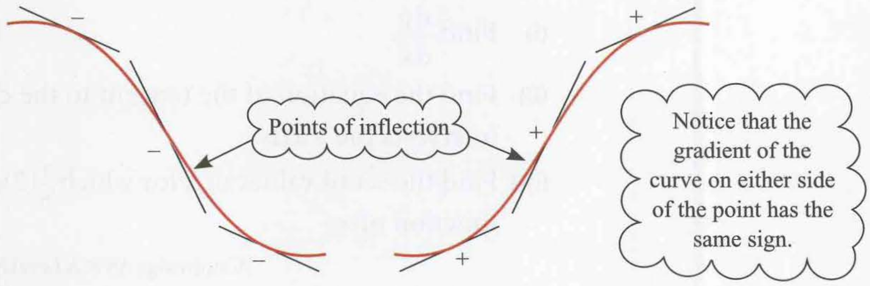


Figure 5.24

If you are a driver you may find it helpful to think of a point of inflection as the point at which you change from left lock to right lock, or vice versa. Another way of thinking about a point of inflection is to view the curve from one side and see it as the point where the curve changes from being concave to convex.

The second derivative

Figure 5.25 shows a sketch of a function $y = f(x)$, and beneath it a sketch of the corresponding gradient function $\frac{dy}{dx} = f'(x)$.

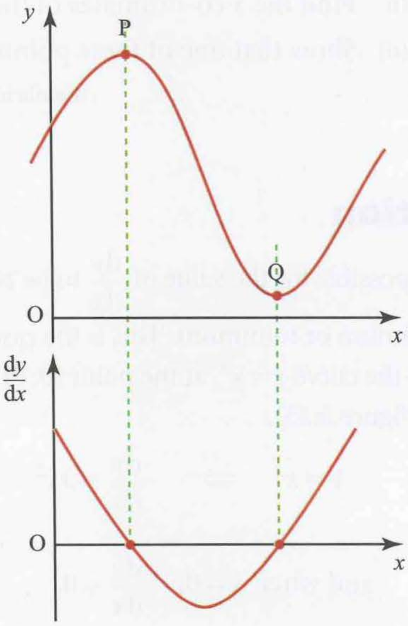


Figure 5.25

ACTIVITY 5.7 Sketch the graph of the gradient of $\frac{dy}{dx}$ against x for the function illustrated in figure 5.25. Do this by tracing the two graphs shown in figure 5.25, and extending the dashed lines downwards on to a third set of axes.

You can see that P is a maximum point and Q is a minimum point. What can you say about the gradient of $\frac{dy}{dx}$ at these points: is it positive, negative or zero?

The gradient of any point on the curve of $\frac{dy}{dx}$ is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. This is written as $\frac{d^2y}{dx^2}$ or $f''(x)$, and is called the *second derivative*. It is found by differentiating the function a second time.

The second derivative, $\frac{d^2y}{dx^2}$, is not the same as $\left(\frac{dy}{dx}\right)^2$.

EXAMPLE 5.14

Given that $y = x^5 + 2x$, find $\frac{d^2y}{dx^2}$.

SOLUTION

$$\frac{dy}{dx} = 5x^4 + 2$$

$$\frac{d^2y}{dx^2} = 20x^3.$$

Using the second derivative

You can use the second derivative to identify the nature of a stationary point, instead of looking at the sign of $\frac{dy}{dx}$ just either side of it.

Stationary points

Notice that at P, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$. This tells you that the gradient, $\frac{dy}{dx}$, is zero and decreasing. It must be going from positive to negative, so P is a maximum point (see figure 5.26).

At Q, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$. This tells you that the gradient, $\frac{dy}{dx}$, is zero and increasing. It must be going from negative to positive, so Q is a minimum point (see figure 5.27).

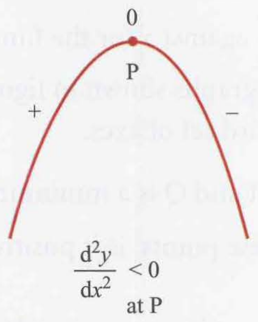


Figure 5.26

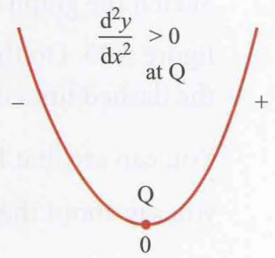


Figure 5.27

The next example illustrates the use of the second derivative to identify the nature of stationary points.

EXAMPLE 5.15

Given that $y = 2x^3 + 3x^2 - 12x$

(i) find $\frac{dy}{dx}$, and find the values of x for which $\frac{dy}{dx} = 0$

(ii) find the value of $\frac{d^2y}{dx^2}$ at each stationary point and hence determine its nature

(iii) find the y values of each of the stationary points

(iv) sketch the curve given by $y = 2x^3 + 3x^2 - 12x$.

SOLUTION

(i)
$$\begin{aligned}\frac{dy}{dx} &= 6x^2 + 6x - 12\\ &= 6(x^2 + x - 2)\\ &= 6(x+2)(x-1)\end{aligned}$$

$\frac{dy}{dx} = 0$ when $x = -2$ or $x = 1$.

(ii) $\frac{d^2y}{dx^2} = 12x + 6.$

When $x = -2$, $\frac{d^2y}{dx^2} = 12 \times (-2) + 6 = -18.$

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ a maximum.

When $x = 1$, $\frac{d^2y}{dx^2} = 6(2 \times 1 + 1) = 18.$

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ a minimum.

(iii) When $x = -2$,
$$\begin{aligned}y &= 2(-2)^3 + 3(-2)^2 - 12(-2)\\ &= 20\end{aligned}$$

so $(-2, 20)$ is a maximum point.

When $x = 1$,
$$\begin{aligned}y &= 2 + 3 - 12\\ &= -7\end{aligned}$$

so $(1, -7)$ is a minimum point.

(iv)

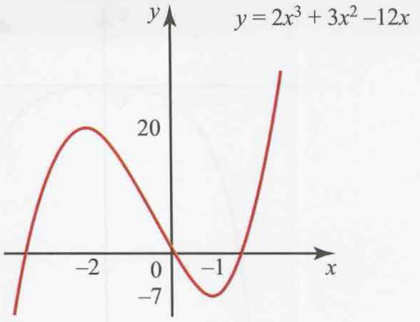


Figure 5.28

⚠ Remember that you are looking for the value of $\frac{d^2y}{dx^2}$ at the stationary point.

Note

On occasions when it is difficult or laborious to find $\frac{d^2y}{dx^2}$, remember that you can always determine the nature of a stationary point by looking at the sign of $\frac{dy}{dx}$ for points just either side of it.

⚠ Take care when $\frac{d^2y}{dx^2} = 0$. Look at these three graphs to see why.

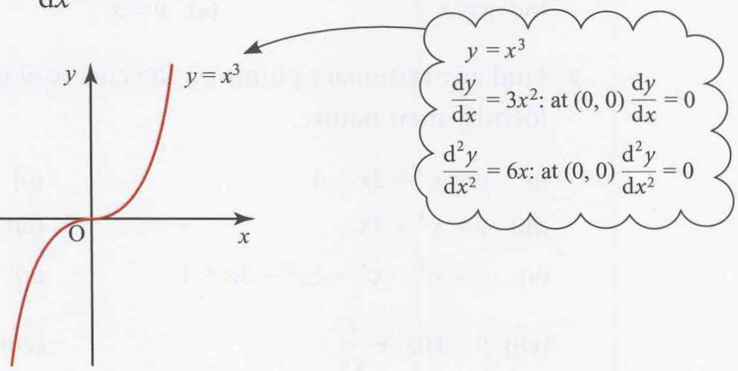


Figure 5.29

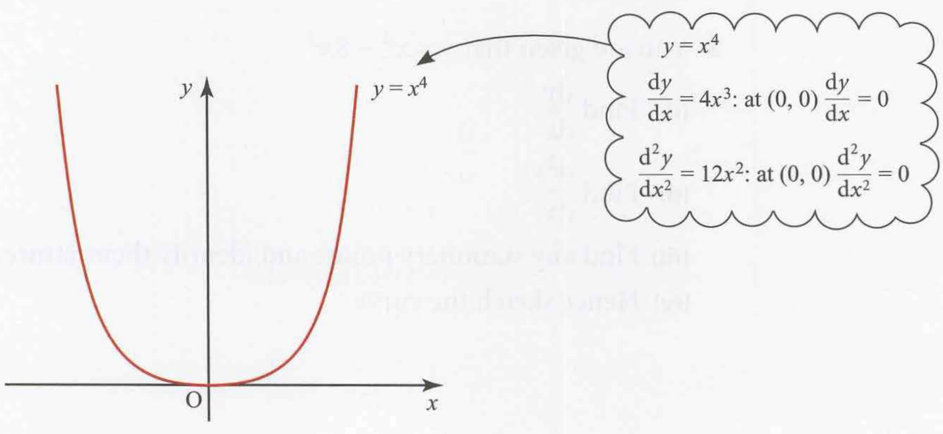


Figure 5.30

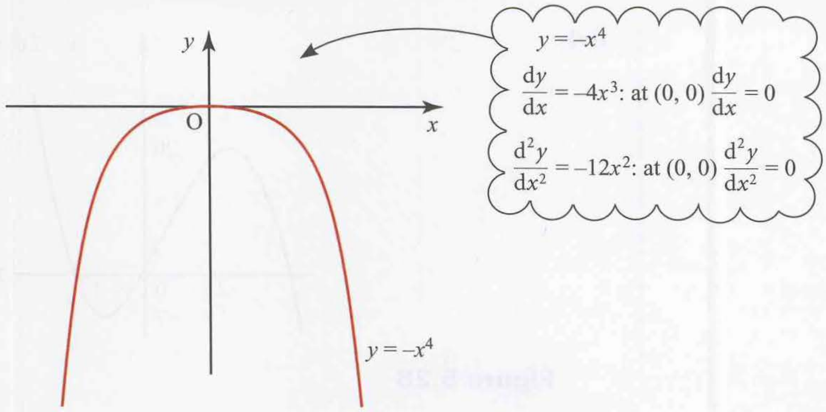


Figure 5.31

You can see that for all three of these functions both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are zero at the origin.

Consequently, if both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are zero at a point, you still need to check the values of $\frac{dy}{dx}$ either side of the point in order to determine its nature.

EXERCISE 5F

1 For each of the following functions, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(i) $y = x^3$ **(ii)** $y = x^5$ **(iii)** $y = 4x^2$

(iv) $y = x^{-2}$ **(v)** $y = x^{\frac{3}{2}}$ **(vi)** $y = x^4 - \frac{2}{x^3}$

2 Find any stationary points on the curves of the following functions and identify their nature.

(i) $y = x^2 + 2x + 4$ **(ii)** $y = 6x - x^2$

(iii) $y = x^3 - 3x$ **(iv)** $y = 4x^5 - 5x^4$

(v) $y = x^4 + x^3 - 2x^2 - 3x + 1$ **(vi)** $y = x + \frac{1}{x}$

(vii) $y = 16x + \frac{1}{x^2}$ **(viii)** $y = x^3 + \frac{12}{x}$

(ix) $y = 6x - x^{\frac{3}{2}}$

3 You are given that $y = x^4 - 8x^2$.

(i) Find $\frac{dy}{dx}$.

(ii) Find $\frac{d^2y}{dx^2}$.

(iii) Find any stationary points and identify their nature.

(iv) Hence sketch the curve.

4 Given that $y = (x - 1)^2(x - 3)$

(i) multiply out the right-hand side and find $\frac{dy}{dx}$

(ii) find the position and nature of any stationary points

(iii) sketch the curve.

5 Given that $y = x^2(x - 2)^2$

(i) multiply out the right-hand side and find $\frac{dy}{dx}$

(ii) find the position and nature of any stationary points

(iii) sketch the curve.

6 The function $y = px^3 + qx^2$, where p and q are constants, has a stationary point at $(1, -1)$.

(i) Using the fact that $(1, -1)$ lies on the curve, form an equation involving p and q.

(ii) Differentiate y and, using the fact that $(1, -1)$ is a stationary point, form another equation involving p and q.

(iii) Solve these two equations simultaneously to find the values of p and q.

7 You are given $f(x) = 4x^2 + \frac{1}{x}$.

(i) Find $f'(x)$ and $f''(x)$.

(ii) Find the position and nature of any stationary points.

8 For the function $y = x - 4\sqrt{x}$,

(i) find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(ii) find the co-ordinates of the stationary point and determine its nature.

9 The equation of a curve is $y = 6\sqrt{x} - x\sqrt{x}$.

Find the x co-ordinate of the stationary point and show that the turning point is a maximum.

10 For the curve $x^{\frac{5}{2}} - 10x^{\frac{3}{2}}$,

(i) find the values of x for which $y = 0$

(ii) show that there is a minimum turning point of the curve when $x = 6$ and calculate the y value of this minimum, giving the answer correct to 1 decimal place.

Applications

There are many situations in which you need to find the maximum or minimum value of an expression. The examples which follow, and those in Exercise 5G, illustrate a few of these.

EXAMPLE 5.16

Kelly's father has agreed to let her have part of his garden as a vegetable plot. He says that she can have a rectangular plot with one side against an old wall. He hands her a piece of rope 5 m long, and invites her to mark out the part she wants. Kelly wants to enclose the largest area possible.

What dimensions would you advise her to use?

SOLUTION

Let the dimensions of the bed be x m × y m as shown in figure 5.32.

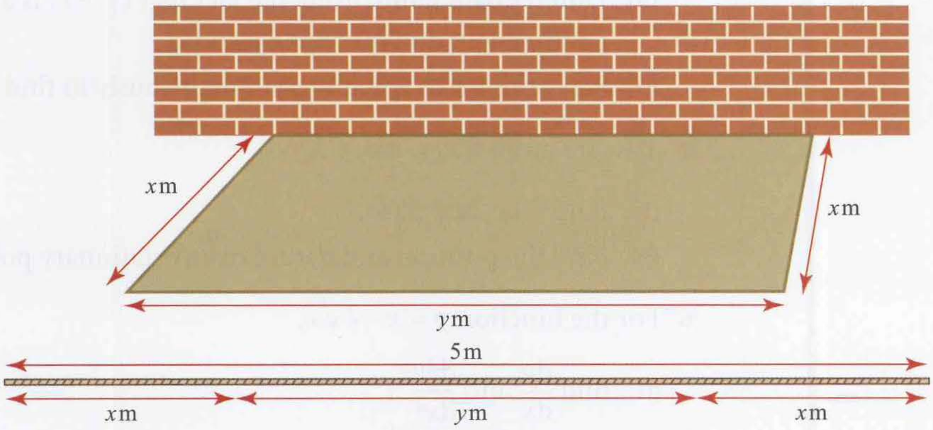


Figure 5.32

The area, A m^2, to be enclosed is given by $A = xy$.

Since the rope is 5 m long, $2x + y = 5$ or $y = 5 - 2x$.

Writing A in terms of x only $A = x(5 - 2x) = 5x - 2x^2$.

To maximise A, which is now written as a function of x, you differentiate A with respect to x

$$\frac{dA}{dx} = 5 - 4x.$$

At a stationary point, $\frac{dA}{dx} = 0$, so

$$5 - 4x = 0$$

$$x = \tfrac{5}{4} = 1.25.$$

$$\frac{d^2A}{dx^2} = -4 \Rightarrow \text{the turning point is a maximum.}$$

The corresponding value of y is $5 - 2(1.25) = 2.5$. Kelly should mark out a rectangle 1.25 m wide and 2.5 m long.

EXAMPLE 5.17

A stone is projected vertically upwards with a speed of $30\,\mathrm{m\,s^{-1}}$. Its height, h m, above the ground after t seconds ($t < 6$) is given by:

$$h = 30t - 5t^2.$$

(i) Find $\dfrac{dh}{dt}$ and $\dfrac{d^2h}{dt^2}$.

(ii) Find the maximum height reached.

(iii) Sketch the graph of h against t.

SOLUTION

(i) $\dfrac{dh}{dt} = 30 - 10t.$

$\dfrac{d^2h}{dt^2} = -10.$

(ii) For a stationary point, $\dfrac{dh}{dt} = 0$

$$30 - 10t = 0$$
$$\Rightarrow \quad 10(3 - t) = 0$$
$$\Rightarrow \quad t = 3.$$

$\dfrac{d^2h}{dt^2} < 0 \Rightarrow$ the stationary point is a maximum.

The maximum height is

$$h = 30(3) - 5(3)^2 = 45\,\mathrm{m}.$$

(iii)

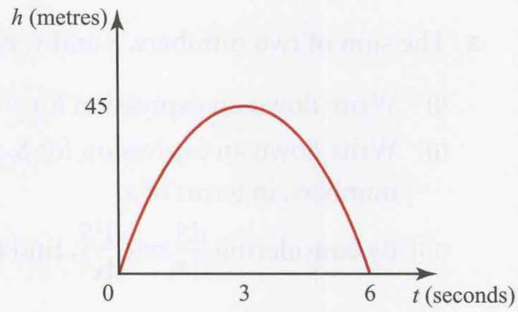


Figure 5.33

Note

For a position–time graph, such as this one, the gradient, $\dfrac{dh}{dt}$, is the velocity and $\dfrac{d^2h}{dt^2}$ is the acceleration.

EXERCISE 5G

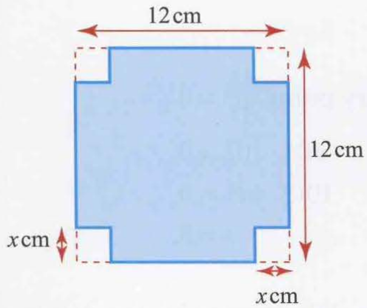
1 A farmer wants to construct a temporary rectangular enclosure of length x m and width y m for his prize bull while he works in the field. He has 120 m of fencing and wants to give the bull as much room to graze as possible.

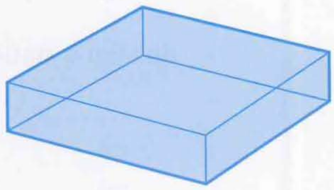
(i) Write down an expression for y in terms of x.

(ii) Write down an expression in terms of x for the area, A, to be enclosed.

(iii) Find $\frac{dA}{dx}$ and $\frac{d^2A}{dx^2}$, and so find the dimensions of the enclosure that give the bull the maximum area in which to graze. State this maximum area.

2 A square sheet of card of side 12 cm has four equal squares of side x cm cut from the corners. The sides are then turned up to make an open rectangular box to hold drawing pins as shown in the diagram.





(i) Form an expression for the volume, V, of the box in terms of x.

(ii) Find $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$, and show that the volume is a maximum when the depth is 2 cm.

3 The sum of two numbers, x and y, is 8.

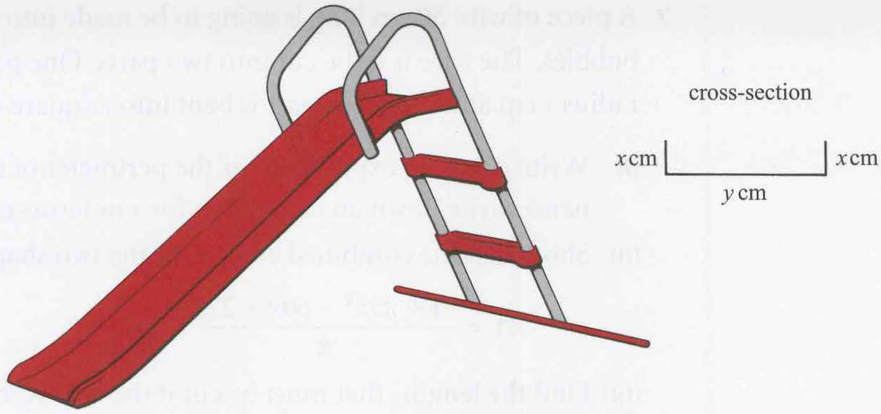
(i) Write down an expression for y in terms of x.

(ii) Write down an expression for S, the sum of the squares of these two numbers, in terms of x.

(iii) By considering $\frac{dS}{dx}$ and $\frac{d^2S}{dx^2}$, find the least value of the sum of their squares.

4 A new children's slide is to be built with a cross-section as shown in the diagram. A long strip of metal 80 cm wide is available for the shute and will be bent to form the base and two sides.

The designer thinks that for maximum safety the area of the cross-section should be as large as possible.

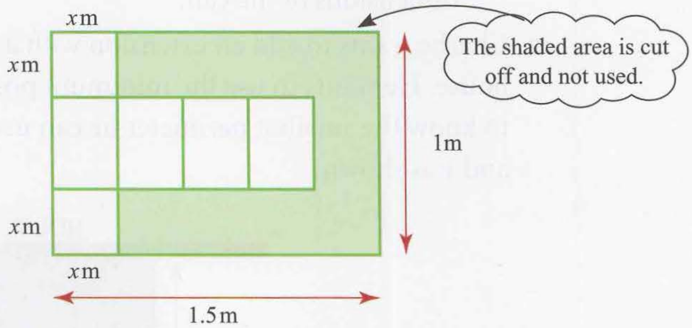


(i) Write down an equation linking x and y.

(ii) Using your answer to part **(i)**, form an expression for the cross-sectional area, A, in terms of x.

(iii) By considering $\frac{dA}{dx}$ and $\frac{d^2A}{dx^2}$, find the dimensions which make the slide as safe as possible.

5 A carpenter wants to make a box to hold toys. The box is to be made so that its volume is as large as possible. A rectangular sheet of thin plywood measuring 1.5 m by 1 m is available to cut into pieces as shown.



(i) Write down the dimensions of one of the four rectangular faces in terms of x.

(ii) Form an expression for the volume, V, of the made-up box, in terms of x.

(iii) Find $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$.

(iv) Hence find the dimensions of a box with maximum volume, and the corresponding volume.

6 A piece of wire 16 cm long is cut into two pieces. One piece is $8x$ cm long and is bent to form a rectangle measuring $3x$ cm by x cm. The other piece is bent to form a square.

(i) Find in terms of x:

(a) the length of a side of the square

(b) the area of the square.

(ii) Show that the combined area of the rectangle and the square is A cm^2 where $A = 7x^2 - 16x + 16$.

(iii) Find the value of x for which A has its minimum value.

(iv) Find the minimum value of A.

[MEI]

7 A piece of wire 30 cm long is going to be made into two frames for blowing bubbles. The wire is to be cut into two parts. One part is bent into a circle of radius r cm and the other part is bent into a square of side x cm.

(i) Write down an expression for the perimeter of the circle in terms of r, and hence write down an expression for r in terms of x.

(ii) Show that the combined area, A, of the two shapes can be written as

$$A = \frac{(4+\pi)x^2 - 60x + 225}{\pi}.$$

(iii) Find the lengths that must be cut if the area is to be a minimum.

8 A cylindrical can with a lid is to be made from a thin sheet of metal. Its height is to be h cm and its radius r cm. The surface area is to be 250π cm^2.

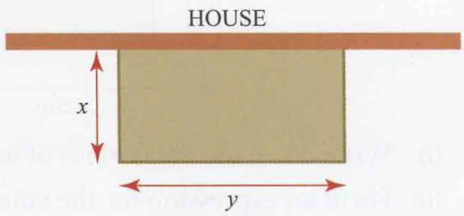
(i) Find h in terms of r.

(ii) Write down an expression for the volume, V, of the can in terms of r.

(iii) Find $\frac{dV}{dr}$ and $\frac{d^2V}{dr^2}$.

(iv) Use your answers to part **(iii)** to show that the can's maximum possible volume is 1690 cm^3 (to 3 significant figures), and find the corresponding dimensions of the can.

9 Charlie wants to add an extension with a floor area of 18 m^2 to the back of his house. He wants to use the minimum possible number of bricks, so he wants to know the smallest perimeter he can use. The dimensions, in metres, are x and y as shown.



(i) Write down an expression for the area in terms of x and y.

(ii) Write down an expression, in terms of x and y, for the total length, T, of the outside walls.

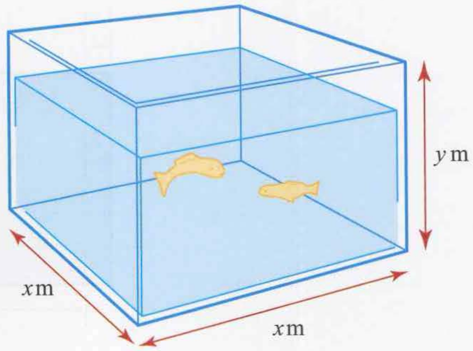
(iii) Show that

$$T = 2x + \frac{18}{x}.$$

(iv) Find $\frac{dT}{dx}$ and $\frac{d^2T}{dx^2}$.

(v) Find the dimensions of the extension that give a minimum value of T, and confirm that it is a minimum.

10 A fish tank with a square base and no top is to be made from a thin sheet of toughened glass. The dimensions are as shown.



(i) Write down an expression for the volume V in terms of x and y.

(ii) Write down an expression for the total surface area A in terms of x and y.

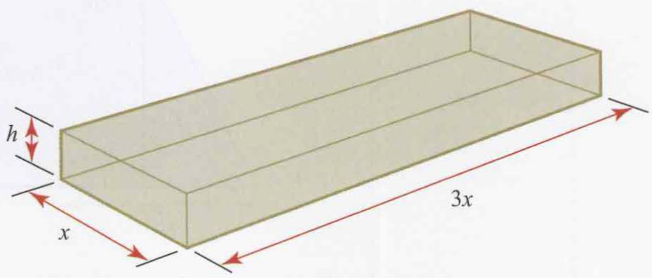
The tank needs a capacity of $0.5\,\text{m}^3$ and the manufacturer wishes to use the minimum possible amount of glass.

(iii) Deduce an expression for A in terms of x only.

(iv) Find $\frac{\mathrm{d}A}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$.

(v) Find the values of x and y that use the smallest amount of glass and confirm that these give the minimum value.

11 A closed rectangular box is made of thin card, and its length is three times its width. The height is h cm and the width is x cm.



(i) The volume of the box is $972\,\text{cm}^3$.
Use this to write down an expression for h in terms of x.

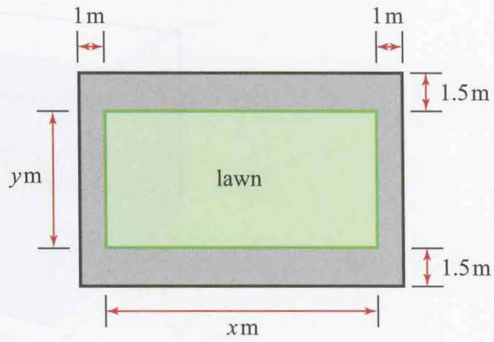
(ii) Show that the surface area, A, can be written as $A = 6x^2 + \frac{2592}{x}$.

(iii) Find $\frac{\mathrm{d}A}{\mathrm{d}x}$ and use it to find a stationary point.

Find $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$ and use it to verify that the stationary point gives the minimum value of A.

(iv) Hence find the minimum surface area and the corresponding dimensions of the box.

12 A garden is planned with a lawn area of $24\,\text{m}^2$ and a path around the edge. The dimensions of the lawn and path are as shown in the diagram.

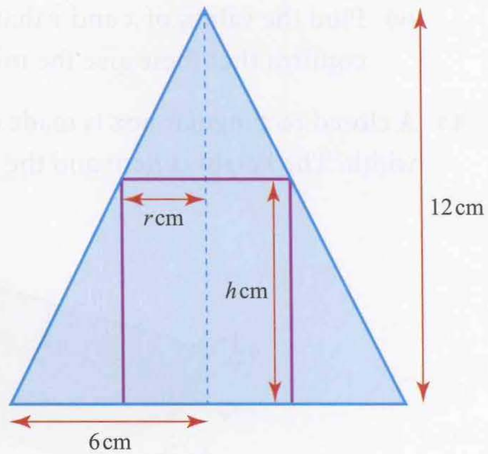


(i) Write down an expression for y in terms of x.

(ii) Find an expression for the overall area of the garden, A, in terms of x.

(iii) Find the smallest possible overall area for the garden.

13 The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.



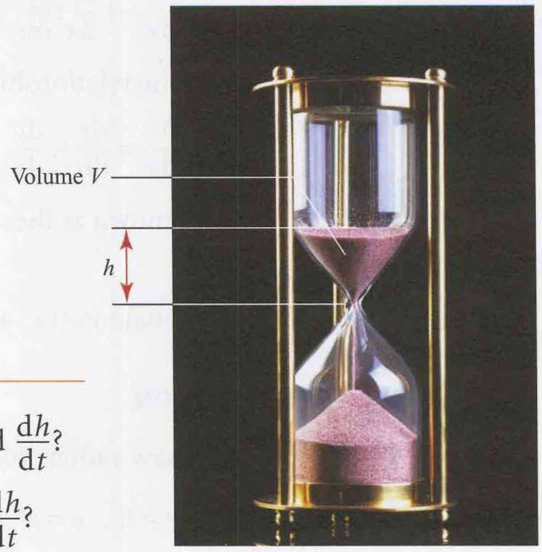
(i) Express h in terms of r and hence show that the volume, $V\,\text{cm}^3$, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3$$

(ii) Given that r varies, find the stationary value of V.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 November 2005]

The chain rule



? What information is given by $\dfrac{dV}{dh}$ and $\dfrac{dh}{dt}$?

What information is given by $\dfrac{dV}{dh} \times \dfrac{dh}{dt}$?

How would you differentiate an expression like

$$y = \sqrt{x^2 + 1}?$$

Your first thought may be to write it as $y = (x^2 + 1)^{\frac{1}{2}}$ and then get rid of the brackets, but that is not possible in this case because the power $\frac{1}{2}$ is not a positive integer. Instead you need to think of the expression as a composite function, a 'function of a function'.

You have already met composite functions in Chapter 4, using the notation $g[f(x)]$ or $gf(x)$.

In this chapter we call the first function to be applied $u(x)$, or just u, rather than $f(x)$.

In this case, $u = x^2 + 1$
and $y = \sqrt{u} = u^{\frac{1}{2}}$.

This is now in a form which you can differentiate using the *chain rule*.

Differentiating a composite function

To find $\dfrac{dy}{dx}$ for a function of a function, you consider the effect of a small change in x on the two variables, y and u, as follows. A small change δx in x leads to a small change δu in u and a corresponding small change δy in y, and by simple algebra,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}.$$

In the limit, as $\delta x \to 0$,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta y}{\delta u} \to \frac{dy}{du} \text{ and } \frac{\delta u}{\delta x} \to \frac{du}{dx}$$

and so the relationship above becomes

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This is known as the chain rule.

EXAMPLE 5.18

Differentiate $y = (x^2 + 1)^{\frac{1}{2}}$.

SOLUTION

As you saw earlier, you can break down this expression as follows.

$$y = u^{\frac{1}{2}}, \quad u = x^2 + 1$$

Differentiating these gives

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{x^2+1}}$$

and

$$\frac{du}{dx} = 2x.$$

By the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2\sqrt{x^2+1}} \times 2x \\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

Notice that the answer must be given in terms of the same variables as the question, in this case x and y. The variable u was your invention and so should not appear in the answer.

You can see that effectively you have made a substitution, in this case $u = x^2 + 1$. This transformed the problem into one that could easily be solved.

Note

Notice that the substitution gave you two functions that you could differentiate. Some substitutions would not have worked. For example, the substitution $u = x^2$, would give you

$$y = (u + 1)^{\frac{1}{2}} \text{ and } u = x^2.$$

You would still not be able to differentiate y, so you would have gained nothing.

EXAMPLE 5.19 Use the chain rule to find $\frac{dy}{dx}$ when $y = (x^2 - 2)^4$.

SOLUTION

Let $u = x^2 - 2$, then $y = u^4$.

$$\frac{du}{dx} = 2x$$

and

$$\frac{dy}{du} = 4u^3$$
$$= 4(x^2 - 2)^3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4(x^2 - 2)^3 \times 2x$$
$$= 8x(x^2 - 2)^3.$$

A student does this question by first multiplying out $(x^2 - 2)^4$ to get a polynomial of order 8. Prove that this heavy-handed method gives the same result.

With practice you may find that you can do some stages of questions like this in your head, and just write down the answer. If you have any doubt, however, you should write down the full method.

Differentiation with respect to different variables

The chain rule makes it possible to differentiate with respect to a variable which does not feature in the original expression. For example, the volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. Differentiating this with respect to r gives the rate of change of volume with radius, $\frac{dV}{dr} = 4\pi r^2$. However you might be more interested in finding $\frac{dV}{dt}$, the rate of change of volume with time, t. To find this, you would use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Notice that the expression for $\frac{dV}{dt}$ includes $\frac{dr}{dt}$, the rate of increase of radius with time.

You have now differentiated V with respect to t.

The use of the chain rule in this way widens the scope of differentiation and this means that you have to be careful how you describe the process.

⚠ 'Differentiate $y = x^2$' could mean differentiation with respect to x, or t, or any other variable. In this book, and others in this series, we have adopted the convention that, unless otherwise stated, differentiation is with respect to the variable on the right-hand side of the expression. So when we write 'differentiate $y = x^2$' or simply 'differentiate x^2', it is to be understood that the differentiation is with respect to x.

⚠ The expression 'increasing at a rate of' is generally understood to imply differentation with respect to time, t.

EXAMPLE 5.20

The radius r cm of a circular ripple made by dropping a stone into a pond is increasing at a rate of $8\,\text{cm}\,\text{s}^{-1}$. At what rate is the area $A\,\text{cm}^2$ enclosed by the ripple increasing when the radius is 25 cm?

SOLUTION

$$A = \pi r^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

The question is asking for $\frac{\mathrm{d}A}{\mathrm{d}t}$, the rate of change of area with respect to time.

$$\text{Now}\quad \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}.$$

When $r = 25$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 8$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi \times 25 \times 8$$

$$\approx 1260\,\text{cm}^2\,\text{s}^{-1}.$$

EXERCISE 5H

1 Use the chain rule to differentiate the following functions.

(i) $y = (x+2)^3$ **(ii)** $y = (2x+3)^4$ **(iii)** $y = (x^2-5)^3$

(iv) $y = (x^3+4)^5$ **(v)** $y = (3x+2)^{-1}$ **(vi)** $y = \dfrac{1}{(x^2-3)^3}$

(vii) $y = (x^2-1)^{\frac{3}{2}}$ **(viii)** $y = \left(\dfrac{1}{x}+x\right)^3$ **(ix)** $y = \left(\sqrt{x}-1\right)^4$

2 Given that $y = (3x-5)^3$

(i) find $\dfrac{dy}{dx}$

(ii) find the equation of the tangent to the curve at (2, 1)

(iii) show that the equation of the normal to the curve at (1, −8) can be written in the form

$$36y + x + 287 = 0.$$

3 Given that $y = (2x-1)^4$

(i) find $\dfrac{dy}{dx}$

(ii) find the co-ordinates of any stationary points and determine their nature

(iii) sketch the curve.

4 Given that $y = (x^2-x-2)^4$

(i) find $\dfrac{dy}{dx}$

(ii) find the co-ordinates of any stationary points and determine their nature

(iii) sketch the curve.

5 The length of a side of a square is increasing at a rate of $0.2\,\text{cm}\,\text{s}^{-1}$. At what rate is the area increasing when the length of the side is 10 cm?

6 The force F newtons between two magnetic poles is given by the formula $F = \dfrac{1}{500r^2}$, where r m is their distance apart.
Find the rate of change of the force when the poles are 0.2 m apart and the distance between them is increasing at a rate of $0.03\,\text{m}\,\text{s}^{-1}$.

7 The radius of a circular fungus is increasing at a uniform rate of 5 cm per day. At what rate is the area increasing when the radius is 1 m?

KEY POINTS

1 $y = kx^n \quad \Rightarrow \quad \dfrac{dy}{dx} = knx^{n-1}$

$y = c \quad \Rightarrow \quad \dfrac{dy}{dx} = 0$

Where k, n and c are constants.

2 $y = f(x) + g(x) \quad \Rightarrow \quad \dfrac{dy}{dx} = f'(x) + g'(x)$.

3 Tangent and normal at (x_1, y_1)

Gradient of tangent, $m_1 =$ value of $\dfrac{dy}{dx}$ when $x = x_1$.

Gradient of normal, $m_2 = -\dfrac{1}{m_1}$.

Equation of tangent is

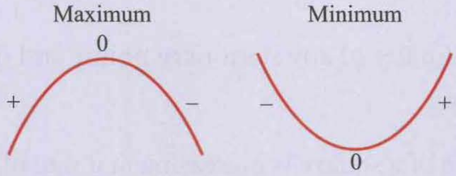
$$y - y_1 = m_1(x - x_1).$$

Equation of normal is

$$y - y_1 = m_2(x - x_1).$$

4 At a stationary point, $\dfrac{dy}{dx} = 0$.

The nature of a stationary point can be determined by looking at the sign of the gradient just either side of it.



5 The nature of a stationary point can also be determined by considering the sign of $\dfrac{d^2y}{dx^2}$.

- If $\dfrac{d^2y}{dx^2} < 0$, the point is a maximum.
- If $\dfrac{d^2y}{dx^2} > 0$, the point is a minimum.

6 If $\dfrac{d^2y}{dx^2} = 0$, check the values of $\dfrac{dy}{dx}$ on either side of the point to determine its nature.

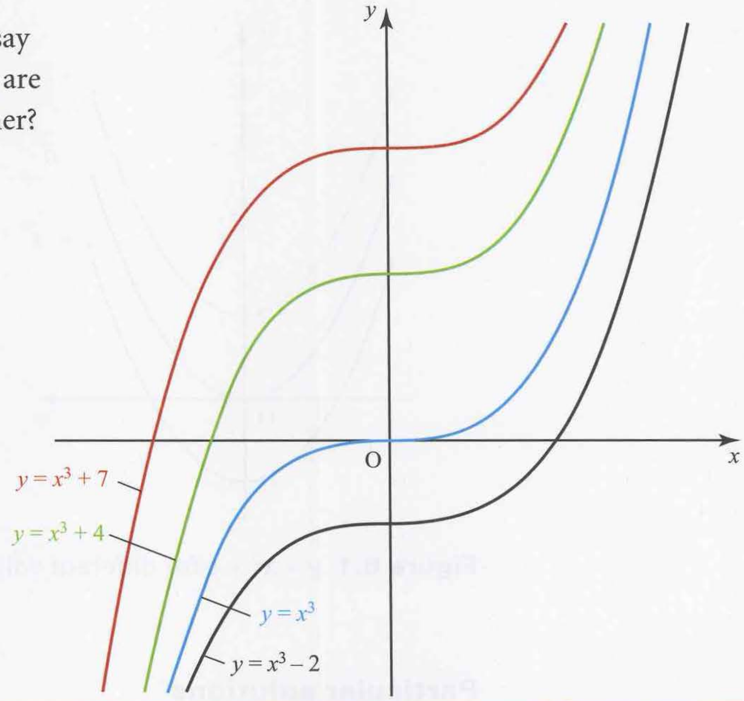
7 Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

6 Integration

Many small make a great.

Chaucer

In what way can you say that these four curves are all parallel to each other?



Reversing differentiation

In some situations you know the gradient function, $\frac{dy}{dx}$, and want to find the function itself, y. For example, you might know that $\frac{dy}{dx} = 2x$ and want to find y.

You know from the previous chapter that if $y = x^2$ then $\frac{dy}{dx} = 2x$, but $y = x^2 + 1$, $y = x^2 - 2$ and many other functions also give $\frac{dy}{dx} = 2x$.

Suppose that $f(x)$ is a function with $f'(x) = 2x$. Let $g(x) = f(x) - x^2$.
Then $g'(x) = f'(x) - 2x = 2x - 2x = 0$ for all x. So the graph of $y = g(x)$ has zero gradient everywhere, i.e. the graph is a horizontal straight line.
Thus $g(x) = c$ (a constant). Therefore $f(x) = x^2 + c$.

All that you can say at this point is that if $\frac{dy}{dx} = 2x$ then $y = x^2 + c$ where c is described as an *arbitrary constant*. An arbitrary constant may take any value.