In this example the polynomial $x^3 - 3x^2 - x + 3$ has three factors, $(x + 1)$, $(x - 1)$ and $(x - 3)$. Each of these corresponds to an intersection with the x axis, and to a root of the equation $x^3 - 3x^2 - x + 3 = 0$. Clearly a cubic polynomial cannot have more than three factors of this type, since the highest power of x is 3. A cubic polynomial may, however, cross the x axis fewer than three times, as in the case of $f(x) = x^3 - x^2 - 4x + 6$ (see figure 2.30).

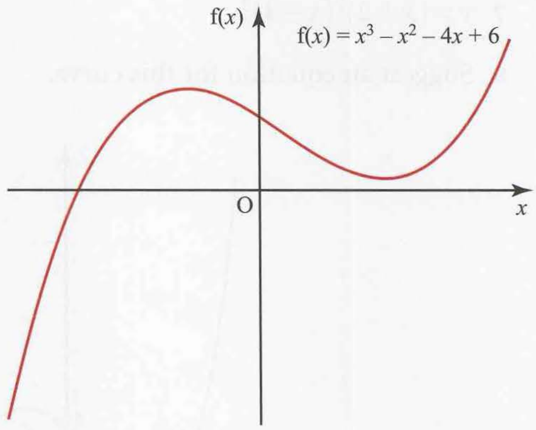
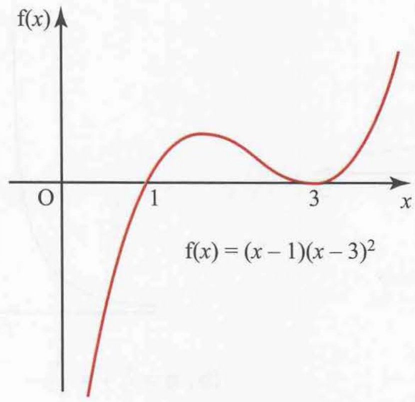


Figure 2.30

Note

This illustrates an important result. If $f(x)$ is a polynomial of degree n, the curve with equation $y = f(x)$ crosses the x axis at most n times, and the equation $f(x) = 0$ has at most n roots.

An important case occurs when the polynomial function has one or more repeated factors, as in figure 2.31. In such cases the curves touch the x axis at points corresponding to the repeated roots.



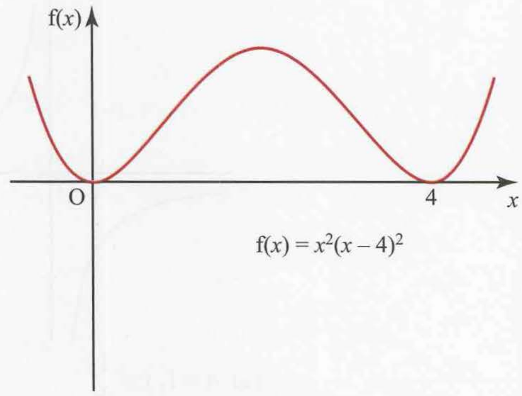


Figure 2.31

EXERCISE 2E

Sketch the following curves, marking clearly the values of x and y where they cross the co-ordinate axes.

1 $y = x(x-3)(x+4)$

2 $y = (x+1)(2x-5)(x-4)$

3 $y = (5-x)(x-1)(x+3)$

4 $y = x^2(x-3)$

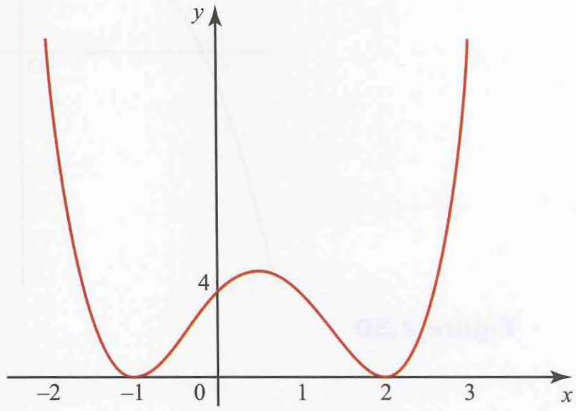
5 $y = (x+1)^2(2-x)$

6 $y = (3x-4)(4x-3)^2$

7 $y = (x+2)^2(x-4)^2$

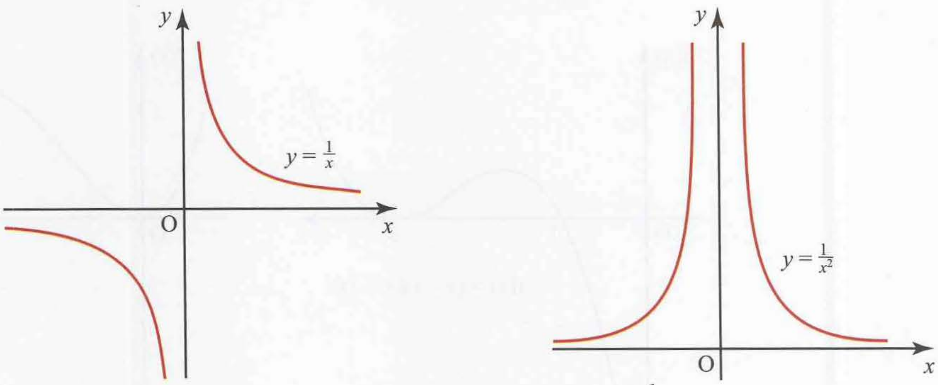
8 $y = (x-3)^2(4+x)^2$

9 Suggest an equation for this curve.



What happens to the curve of a polynomial if it has a factor of the form $(x-a)^3$? Or $(x-a)^4$?

Curves of the form $y = \frac{1}{x^n}$ (for $x \neq 0$)



(a) $n = 1, y = \frac{1}{x}$

(b) $n = 2, y = \frac{1}{x^2}$

Figure 2.32

The curves for $n = 3, 5, \ldots$ are not unlike that for $n = 1$, those for $n = 4, 6, \ldots$ are like that for $n = 2$. In all cases the point $x = 0$ is excluded because $\frac{1}{0}$ is undefined.

An important feature of these curves is that they approach both the x and the y axes ever more closely but never actually reach them. These lines are described as *asymptotes* to the curves. Asymptotes may be vertical (e.g. the y axis), horizontal, or lie at an angle, when they are called oblique.

Asymptotes are usually marked on graphs as dotted lines but in the cases above the lines are already there, being co-ordinate axes. The curves have different branches which never meet. A curve with different branches is said to be *discontinuous*, whereas one with no breaks, like $y = x^2$, is *continuous*.

The circle

You are of course familiar with the circle, and have probably done calculations involving its area and circumference. In this section you are introduced to the *equation* of a circle.

The circle is defined as the *locus* of all the points in a plane which are at a fixed distance (the radius) from a given point (the centre). (Locus means path.)

As you have seen, the length of a line joining (x_1, y_1) to (x_2, y_2) is given by

$$\text{length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is used to derive the equation of a circle.

In the case of a circle of radius 3, with its centre at the origin, any point (x, y) on the circumference is distance 3 from the origin. Since the distance of (x, y) from $(0, 0)$ is given by $\sqrt{(x-0)^2 + (y-0)^2}$, this means that $\sqrt{(x-0)^2 + (y-0)^2} = 3$ or $x^2 + y^2 = 9$ and this is the equation of the circle.

This circle is shown in figure 2.33.

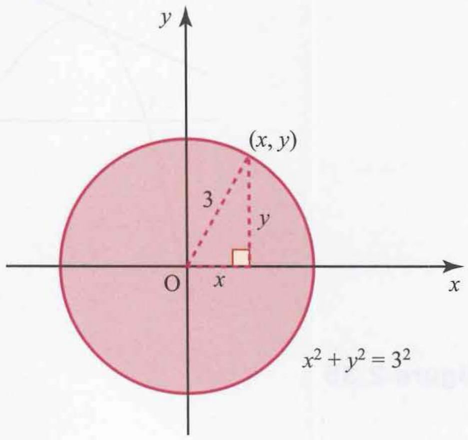


Figure 2.33

These results can be generalised to give the equation of a circle centre $(0, 0)$, radius r as follows:

$$x^2 + y^2 = r^2$$

The intersection of a line and a curve

When a line and a curve are in the same plane, there are three possible situations.

(i) *All points of intersection are distinct* (see figure 2.34).

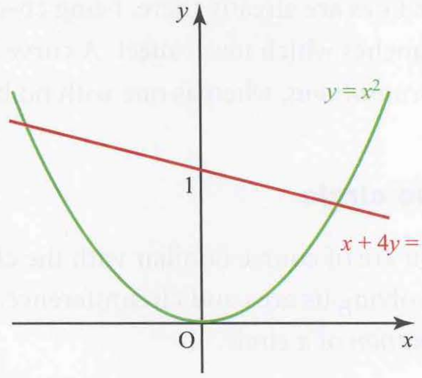


Figure 2.34

(ii) *The line is a tangent to the curve at one (or more) point(s)* (see figure 2.35).

In this case, each point of contact corresponds to two (or more) co-incident points of intersection. It is possible that the tangent will also intersect the curve somewhere else.

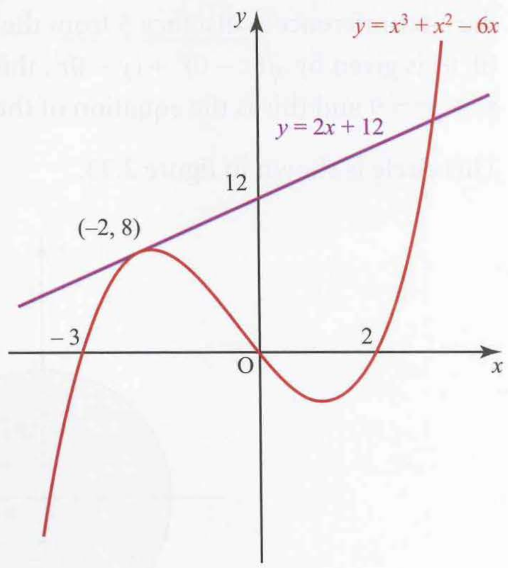


Figure 2.35

(iii) *The line and the curve do not meet* (see figure 2.36).

The co-ordinates of the point of intersection can be found by solving the two equations simultaneously. If you obtain an equation with no real roots, the conclusion is that there is no point of intersection.

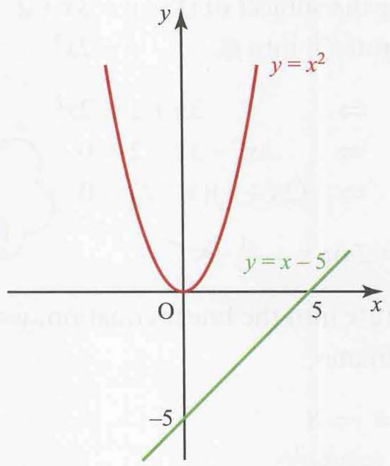


Figure 2.36

The equation of the straight line is, of course, linear and that of the curve non-linear. The examples which follow remind you how to solve such pairs of equations.

EXAMPLE 2.14

Find the co-ordinates of the two points where the line $y - 3x = 2$ intersects the curve $y = 2x^2$.

SOLUTION

First sketch the line and the curve.

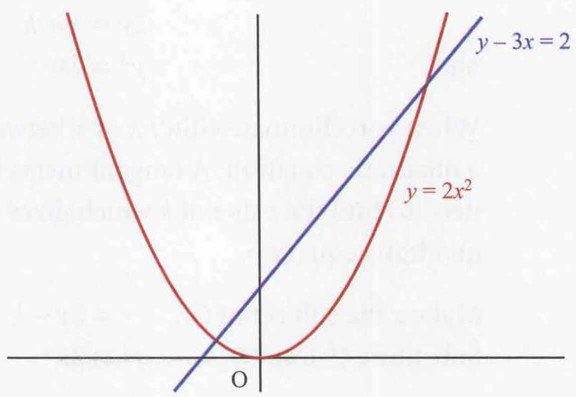


Figure 2.37

You can find where the line and curve intersect by solving the simultaneous equations:

$$y - 3x = 2 \quad ①$$

and $$y = 2x^2 \quad ②$$

Make y the subject of ①: $y = 3x + 2$ ③

Substitute ③ into ②: $y = 2x^2$

$$\Rightarrow \quad 3x + 2 = 2x^2$$
$$\Rightarrow \quad 2x^2 - 3x - 2 = 0$$
$$\Rightarrow \quad (2x + 1)(x - 2) = 0$$

$\Rightarrow \quad x = 2$ or $x = -\frac{1}{2}$

These are the x co-ordinates of the points of intersection.

Substitute into the linear equation, $y = 3x + 2$, to find the corresponding y co-ordinates.

$x = 2 \Rightarrow y = 8$
$x = -\frac{1}{2} \Rightarrow y = \frac{1}{2}$

So the co-ordinates of the points of intersection are (2, 8) and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

EXAMPLE 2.15

(i) Find the value of k for which the line $2y = x + k$ forms a tangent to the curve $y^2 = 2x$.

(ii) Hence, for this value of k, find the co-ordinates of the point where the line $2y = x + k$ meets the curve.

SOLUTION

(i) You can find where the line forms a tangent to the curve by solving the simultaneous equations:

$$2y = x + k \quad ①$$

and $$y^2 = 2x \quad ②$$

When you eliminate either x or y between the equations you will be left with a quadratic equation. A tangent meets the curve at just one point and so you need to find the value of k which gives you just one repeated root for the quadratic equation.

Make x the subject of ①: $x = 2y - k$ ③

Substitute ③ into ②: $y^2 = 2x$

$$\Rightarrow \quad y^2 = 2(2y - k)$$
$$\Rightarrow \quad y^2 = 4y - 2k$$
$$\Rightarrow \quad y^2 - 4y + 2k = 0 \quad ④$$

You can use the discriminant, $b^2 - 4ac$, to find the value of k such that the equation has one repeated root. The condition is $b^2 - 4ac = 0$

$$y^2 - 4y + 2k = 0 \Rightarrow a = 1,\ b = -4 \text{ and } c = 2k$$
$$b^2 - 4ac = 0 \Rightarrow (-4)^2 - 4 \times 1 \times 2k = 0$$
$$\Rightarrow 16 - 8k = 0$$
$$\Rightarrow k = 2$$

So the line $2y = x + 2$ forms a tangent to the curve $y^2 = 2x$.

(ii) You have already started to solve the equations $2y = x + 2$ and $y^2 = 2x$ in part **(i)**. Look at equation ④ : $y^2 - 4y + 2k = 0$

You know from part **(i)** that $k = 2$ so you can solve the quadratic to find y.

$$y^2 - 4y + 4 = 0$$
$$\Rightarrow (y-2)(y-2) = 0$$
$$\Rightarrow y = 2$$

Notice that this is a repeated root so the line is a tangent to the curve.

Now substitute $y = 2$ into the equation of the line to find the x co-ordinate.

When $y = 2$: $2y = x + 2 \Rightarrow 4 = x + 2$

$x = 2$

So the tangent meets the curve at the point (2, 2).

EXERCISE 2F

1 Show that the line $y = 3x + 1$ crosses the curve $y = x^2 + 3$ at (1, 4) and find the co-ordinates of the other point of intersection.

2 **(i)** Find the co-ordinates of the points A and B where the line $y = 2x - 1$ cuts the curve $y = x^2 - 4$.

(ii) Find the distance AB.

3 **(i)** Find the co-ordinates of the points of intersection of the line $y = 2x$ and the curve $y = x^2 + 6x - 5$.

(ii) Show also that the line $y = 2x$ does not cross the curve $y = x^2 + 6x + 5$.

4 The line $3y = 5 - x$ intersects the curve $2y^2 = x$ at two points. Find the distance between the two points.

5 The equation of a curve is $xy = 8$ and the equation of a line is $2x + y = k$, where k is a constant. Find the values of k for which the line forms a tangent to the curve.

6 Find the value of the constant c for which the line $y = 4x + c$ is a tangent to the curve $y^2 = 4x$.

7 The equation of a curve is $xy = 10$ and the equation of a line l is $2x + y = q$, where q is a number.

(i) In the case where $q = 9$, find the co-ordinates of the points of intersection of l and the curve.

(ii) Find the set of values of q for which l does not intersect the curve.

8 The curve $y^2 = 12x$ intersects the line $3y = 4x + 6$ at two points. Find the distance between the two points.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 June 2006]

9 Determine the set of values of the constant k for which the line $y = 4x + k$ does not intersect the curve $y = x^2$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 November 2007]

10 Find the set of values of k for which the line $y = kx - 4$ intersects the curve $y = x^2 - 2x$ at two distinct points.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 June 2009]

KEY POINTS

1 The gradient of the straight line joining the points (x_1, y_1) and (x_2, y_2) is given by

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

when the same scale is used on both axes, $m = \tan\theta$.

2 Two lines are parallel when their gradients are equal.

3 Two lines are perpendicular when the product of their gradients is -1.

4 When the points A and B have co-ordinates (x_1, y_1) and (x_2, y_2) respectively, then

the distance AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

the mid-point of the line AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

5 The equation of a straight line may take any of the following forms:

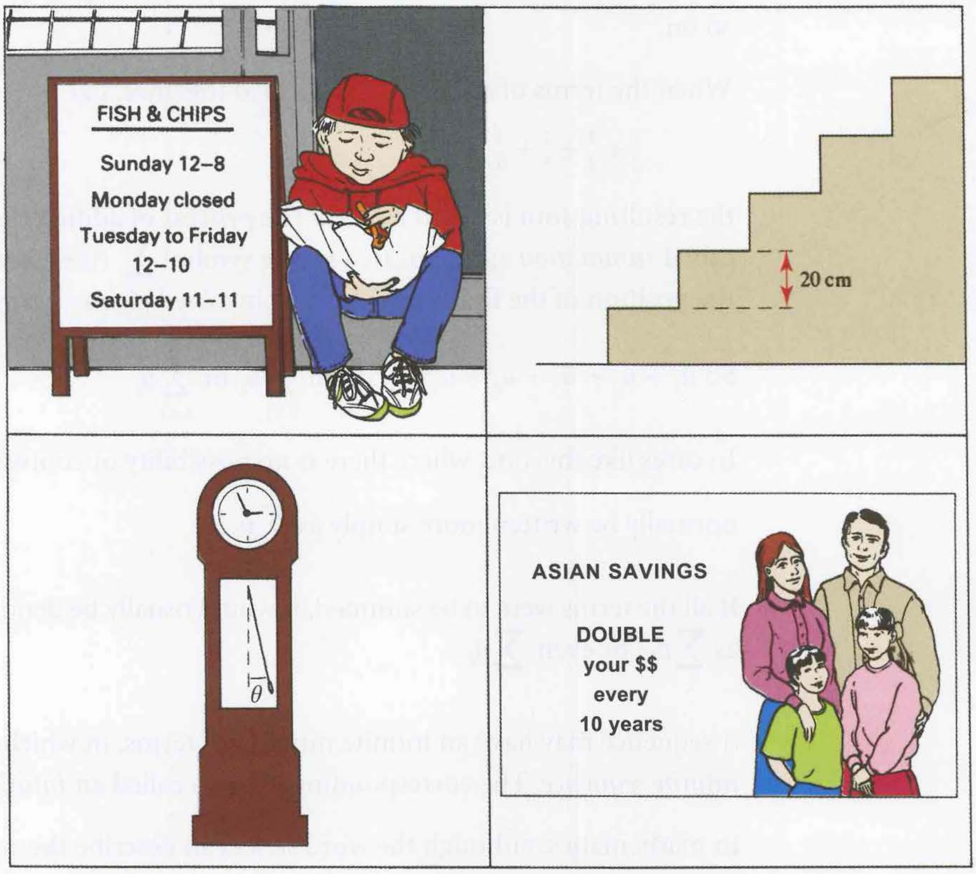
- line parallel to the y axis: $x = a$
- line parallel to the x axis: $y = b$
- line through the origin with gradient m: $y = mx$
- line through $(0, c)$ with gradient m: $y = mx + c$
- line through (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$
- line through (x_1, y_1) and (x_2, y_2):

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \text{or} \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

3 Sequences and series

Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second.

Thomas Malthus (1798)



Each of the following sequences is related to one of the pictures above.

(i) 5000, 10 000, 20 000, 40 000, … .
(ii) 8, 0, 10, 10, 10, 10, 12, 8, 0, … .
(iii) 5, 3.5, 0, –3.5, –5, –3.5, 0, 3.5, 5, 3.5, … .
(iv) 20, 40, 60, 80, 100, … .

(a) Identify which sequence goes with which picture.
(b) Give the next few numbers in each sequence.
(c) Describe the pattern of the numbers in each case.
(d) Decide whether the sequence will go on for ever, or come to a stop.

Definitions and notation

A *sequence* is a set of numbers in a given order, like

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Each of these numbers is called a *term* of the sequence. When writing the terms of a sequence algebraically, it is usual to denote the position of any term in the sequence by a subscript, so that a general sequence might be written:

$$u_1, u_2, u_3, \ldots, \text{ with general term } u_k.$$

For the sequence above, the first term is $u_1 = \frac{1}{2}$, the second term is $u_2 = \frac{1}{4}$, and so on.

When the terms of a sequence are added together, like

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

the resulting sum is called a *series*. The process of adding the terms together is called *summation* and indicated by the symbol $\sum$ (the Greek letter sigma), with the position of the first and last terms involved given as *limits*.

So $u_1 + u_2 + u_3 + u_4 + u_5$ is written $\sum\limits_{k=1}^{k=5} u_k$ or $\sum\limits_{k=1}^{5} u_k$.

In cases like this one, where there is no possibility of confusion, the sum would normally be written more simply as $\sum\limits_{1}^{5} u_k$.

If all the terms were to be summed, it would usually be denoted even more simply, as $\sum\limits_{k} u_k$, or even $\sum u_k$.

A sequence may have an infinite number of terms, in which case it is called an *infinite sequence*. The corresponding series is called an *infinite series*.

In mathematics, although the word *series* can describe the sum of the terms of any sequence, it is usually used only when summing the sequence provides some useful or interesting overall result.

For example:

This series has a finite number of terms (6).

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$$\pi = 2\sqrt{3}\left[1 + \left(\frac{-1}{3}\right) + 5\left(\frac{-1}{3}\right)^2 + 7\left(\frac{-1}{3}\right)^3 + \ldots\right]$$

This series has an infinite number of terms.

The phrase 'sum of a sequence' is often used to mean the sum of the terms of a sequence (i.e. the series).

Arithmetic progressions

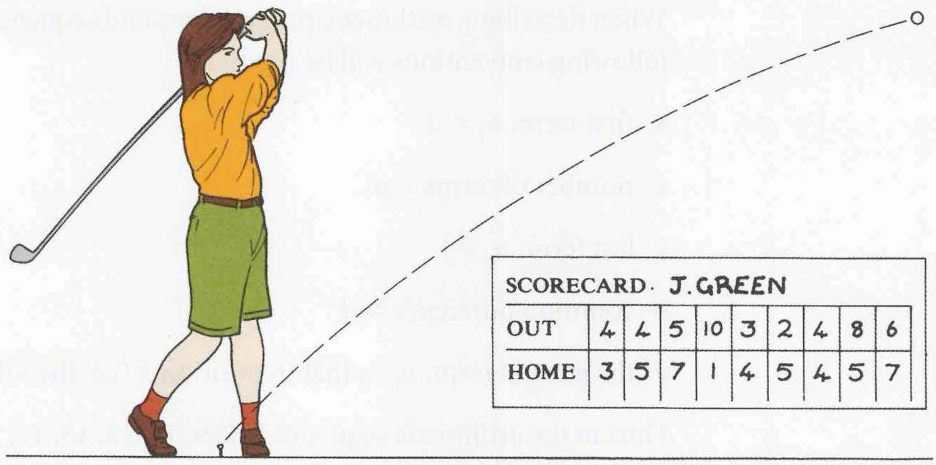


Figure 3.1

Any ordered set of numbers, like the scores of this golfer on an 18-hole round (see figure 3.1) form a sequence. In mathematics, we are particularly interested in those which have a well-defined pattern, often in the form of an algebraic formula linking the terms. The sequences you met at the start of this chapter show various types of pattern.

A sequence in which the terms increase by the addition of a fixed amount (or decrease by the subtraction of a fixed amount), is described as *arithmetic*. The increase from one term to the next is called the *common difference*.

Thus the sequence 5 8 11 14… is arithmetic with

+3 +3 +3

common difference 3.

This sequence can be written algebraically as

$u_k = 2 + 3k$ for $k = 1, 2, 3, \ldots$

When $k = 1$, $u_1 = 2 + 3 = 5$
$k = 2$, $u_2 = 2 + 6 = 8$
$k = 3$, $u_3 = 2 + 9 = 11$

This version has the advantage that the right-hand side begins with the first term of the sequence.

and so on.

(An equivalent way of writing this is $u_k = 5 + 3(k - 1)$ for $k = 1, 2, 3, \ldots$.)

As successive terms of an arithmetic sequence increase (or decrease) by a fixed amount called the common difference, d, you can define each term in the sequence in relation to the previous term:

$$u_{k+1} = u_k + d.$$

When the terms of an arithmetic sequence are added together, the sum is called an *arithmetic progression*, often abbreviated to A.P. An alternative name is an arithmetic series.

Notation

When describing arithmetic progressions and sequences in this book, the following conventions will be used:

- first term, $u_1 = a$
- number of terms $= n$
- last term, $u_n = l$
- common difference $= d$
- the general term, u_k, is that in position k (i.e. the kth term).

Thus in the arithmetic sequence 5, 7, 9, 11, 13, 15, 17,

$a = 5$, $l = 17$, $d = 2$ and $n = 7$.

The terms are formed as follows.

$$
\begin{aligned}
u_1 &= a &&= 5 \\
u_2 &= a + d &&= 5 + 2 &&= 7 \\
u_3 &= a + 2d &&= 5 + 2 \times 2 &&= 9 \\
u_4 &= a + 3d &&= 5 + 3 \times 2 &&= 11 \\
u_5 &= a + 4d &&= 5 + 4 \times 2 &&= 13 \\
u_6 &= a + 5d &&= 5 + 5 \times 2 &&= 15 \\
u_7 &= a + 6d &&= 5 + 6 \times 2 &&= 17
\end{aligned}
$$

The 7th term is the 1st term (5) plus six times the common difference (2).

You can see that any term is given by the first term plus a number of differences. The number of differences is, in each case, one less than the number of the term. You can express this mathematically as

$$u_k = a + (k - 1)d.$$

For the last term, this becomes

$$l = a + (n - 1)d.$$

These are both general formulae which apply to any arithmetic sequence.

EXAMPLE 3.1

Find the 17th term in the arithmetic sequence 12, 9, 6, ...

SOLUTION

In this case $a = 12$ and $d = -3$.

Using $\quad u_k = a + (k - 1)d$, you obtain

$$
\begin{aligned}
u_{17} &= 12 + (17 - 1) \times (-3) \\
&= 12 - 48 \\
&= -36.
\end{aligned}
$$

The 17th term is -36.

EXAMPLE 3.2 How many terms are there in the sequence 11, 15, 19, …, 643?

SOLUTION

This is an arithmetic sequence with first term $a = 11$, last term $l = 643$ and common difference $d = 4$.

Using the result $l = a + (n - 1)d$, you have

$$\begin{aligned} 643 &= 11 + 4(n - 1) \\ \Rightarrow \quad 4n &= 643 - 11 + 4 \\ \Rightarrow \quad n &= 159. \end{aligned}$$

There are 159 terms.

Note

The relationship $l = a + (n - 1)d$ may be rearranged to give

$$n = \frac{l - a}{d} + 1$$

This gives the number of terms in an A.P. directly if you know the first term, the last term and the common difference.

The sum of the terms of an arithmetic progression

When Carl Friederich Gauss (1777–1855) was at school he was always quick to answer mathematics questions. One day his teacher, hoping for half an hour of peace and quiet, told his class to add up all the whole numbers from 1 to 100. Almost at once the 10-year-old Gauss announced that he had done it and that the answer was 5050.

Gauss had not of course added the terms one by one. Instead he wrote the series down twice, once in the given order and once backwards, and added the two together:

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ S &= 100 + 99 + 98 + \ldots + 3 + 2 + 1. \end{aligned}$$

Adding, $2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101$.

Since there are 100 terms in the series,

$$\begin{aligned} 2S &= 101 \times 100 \\ S &= 5050. \end{aligned}$$

The numbers 1, 2, 3, … , 100 form an arithmetic sequence with common difference 1. Gauss' method can be used for finding the sum of any arithmetic series.

It is common to use the letter S to denote the sum of a series. When there is any doubt as to the number of terms that are being summed, this is indicated by a subscript: S_5 indicates five terms, S_n indicates n terms.

EXAMPLE 3.3

Find the value of $8 + 6 + 4 + \ldots + (-32)$.

SOLUTION

This is an arithmetic progression, with common difference -2. The number of terms, n, may be calculated using

$$n = \frac{l-a}{d} + 1$$

$$n = \frac{-32-8}{-2} + 1$$

$$= 21.$$

The sum S of the progression is then found as follows.

$$\begin{aligned} S &= 8 + 6 + \ldots - 30 - 32 \\ S &= -32 - 30 - \ldots + 6 + 8 \\ \hline 2S &= -24 - 24 - \ldots - 24 - 24 \end{aligned}$$

Since there are 21 terms, this gives $2S = -24 \times 21$, so $S = -12 \times 21 = -252$.

Generalising this method by writing the series in the conventional notation gives:

$$\begin{array}{rcccccccc} S_n &=& [a] &+& [a+d] &+\ldots+& [a+(n-2)d] &+& [a+(n-1)d] \\ S_n &=& [a+(n-1)d] &+& [a+(n-2)d] &+\ldots+& [a+d] &+& [a] \\ \hline 2S_n &=& [2a+(n-1)d] &+& [2a+(n-1)d] &+\ldots+& [2a+(n-1)d] &+& [2a+(n-1)d] \end{array}$$

Since there are n terms, it follows that

$$S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$$

This result may also be written as

$$S_n = \frac{1}{2}n(a+l).$$

EXAMPLE 3.4

Find the sum of the first 100 terms of the progression

$$1, 1\tfrac{1}{4}, 1\tfrac{1}{2}, 1\tfrac{3}{4}, \ldots.$$

SOLUTION

In this arithmetic progression

$a = 1$, $d = \frac{1}{4}$ and $n = 100$.

Using $S_n = \frac{1}{2}n[2a + (n-1)d]$, you have

$$S_n = \tfrac{1}{2} \times 100\left(2 + 99 \times \tfrac{1}{4}\right)$$

$$= 1337\tfrac{1}{2}.$$

EXAMPLE 3.5 Jamila starts a part-time job on a salary of \$9000 per year, and this increases by an annual increment of \$1000. Assuming that, apart from the increment, Jamila's salary does not increase, find

(i) her salary in the 12th year

(ii) the length of time she has been working when her total earnings are \$100 000.

SOLUTION

Jamila's annual salaries (in dollars) form the arithmetic sequence

9000, 10 000, 11 000, … .

with first term $a = 9000$, and common difference $d = 1000$.

(i) Her salary in the 12th year is calculated using:

$$\begin{aligned} u_k &= a + (k-1)d \\ \Rightarrow \quad u_{12} &= 9000 + (12-1) \times 1000 \\ &= 20\,000. \end{aligned}$$

(ii) The number of years that have elapsed when her total earnings are \$100 000 is given by:

$$S = \tfrac{1}{2}n[2a + (n-1)d]$$

where $S = 100\,000$, $a = 9000$ and $d = 1000$.

This gives $\quad 100\,000 = \tfrac{1}{2}n[2 \times 9000 + 1000(n-1)]$.

This simplifies to the quadratic equation:

$$n^2 + 17n - 200 = 0.$$

Factorising,

$$(n-8)(n+25) = 0$$
$$\Rightarrow n = 8 \text{ or } n = -25.$$

The root $n = -25$ is irrelevant, so the answer is $n = 8$.

Jamila has earned a total of \$100 000 after eight years.

EXERCISE 3A

1 Are the following sequences arithmetic?
If so, state the common difference and the seventh term.

(i) 27, 29, 31, 33, … **(ii)** 1, 2, 3, 5, 8, … **(iii)** 2, 4, 8, 16, …

(iv) 3, 7, 11, 15, … **(v)** 8, 6, 4, 2, …

2 The first term of an arithmetic sequence is –8 and the common difference is 3.

(i) Find the seventh term of the sequence.

(ii) The last term of the sequence is 100.
How many terms are there in the sequence?

3 The first term of an arithmetic sequence is 12, the seventh term is 36 and the last term is 144.

(i) Find the common difference.

(ii) Find how many terms there are in the sequence.

4 There are 20 terms in an arithmetic progression.
The first term is -5 and the last term is 90.

(i) Find the common difference.

(ii) Find the sum of the terms in the progression.

5 The kth term of an arithmetic progression is given by

$$u_k = 14 + 2k.$$

(i) Write down the first three terms of the progression.

(ii) Calculate the sum of the first 12 terms of this progression.

6 Below is an arithmetic progression.

$$120 + 114 + \ldots + 36$$

(i) How many terms are there in the progression?

(ii) What is the sum of the terms in the progression?

7 The fifth term of an arithmetic progression is 28 and the tenth term is 58.

(i) Find the first term and the common difference.

(ii) The sum of all the terms in this progression is 444.
How many terms are there?

8 The sixth term of an arithmetic progression is twice the third term, and the first term is 3. The sequence has ten terms.

(i) Find the common difference.

(ii) Find the sum of all the terms in the progression.

9 **(i)** Find the sum of all the odd numbers between 50 and 150.

(ii) Find the sum of all the even numbers from 50 to 150, inclusive.

(iii) Find the sum of the terms of the arithmetic sequence with first term 50, common difference 1 and 101 terms.

(iv) Explain the relationship between your answers to parts **(i)**, **(ii)** and **(iii)**.

10 The first term of an arithmetic progression is 3000 and the tenth term is 1200.

(i) Find the sum of the first 20 terms of the progression.

(ii) After how many terms does the sum of the progression become negative?

11 An arithmetic progression has first term 7 and common difference 3.

(i) Write down a formula for the kth term of the progression.
Which term of the progression equals 73?

(ii) Write down a formula for the sum of the first n terms of the progression.
How many terms of the progression are required to give a sum equal to 6300? [MEI]

12 Paul's starting salary in a company is \$14 000 and during the time he stays with the company it increases by \$500 each year.

(i) What is his salary in his sixth year?

(ii) How many years has Paul been working for the company when his total earnings for all his years there are \$126 000?

13 A jogger is training for a 10 km charity run. He starts with a run of 400 m; then he increases the distance he runs by 200 m each day.

(i) How many days does it take the jogger to reach a distance of 10 km in training?

(ii) What total distance will he have run in training by then?

14 A piece of string 10 m long is to be cut into pieces, so that the lengths of the pieces form an arithmetic sequence.

(i) The lengths of the longest and shortest pieces are 1 m and 25 cm respectively; how many pieces are there?

(ii) If the same string had been cut into 20 pieces with lengths that formed an arithmetic sequence, and if the length of the second longest had been 92.5 cm, how long would the shortest piece have been?

15 The 11th term of an arithmetic progression is 25 and the sum of the first 4 terms is 49.

(i) Find the first term of the progression and the common difference.

The nth term of the progression is 49.

(ii) Find the value of n.

16 The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has n terms and the sum of all the terms is 90. Find the value of n.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 November 2008]

17 The training programme of a pilot requires him to fly 'circuits' of an airfield. Each day he flies 3 more circuits than the day before. On the fifth day he flew 14 circuits.

Calculate how many circuits he flew:

(i) on the first day

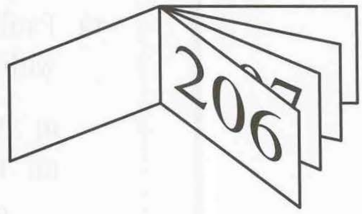
(ii) in total by the end of the fifth day

(iii) in total by the end of the nth day

(iv) in total from the end of the nth day to the end of the $2n$th day. Simplify your answer.

[MEI]

18 As part of a fund-raising campaign, I have been given some books of raffle tickets to sell. Each book has the same number of tickets and all the tickets I have been given are numbered in sequence. The number of the ticket on the front of the 5th book is 205 and that on the front of the 19th book is 373.



(i) By writing the number of the ticket on the front of the first book as a and the number of tickets in each book as d, write down two equations involving a and d.

(ii) From these two equations find how many tickets are in each book and the number on the front of the first book I have been given.

(iii) The last ticket I have been given is numbered 492. How many books have I been given?

[MEI]

Geometric progressions

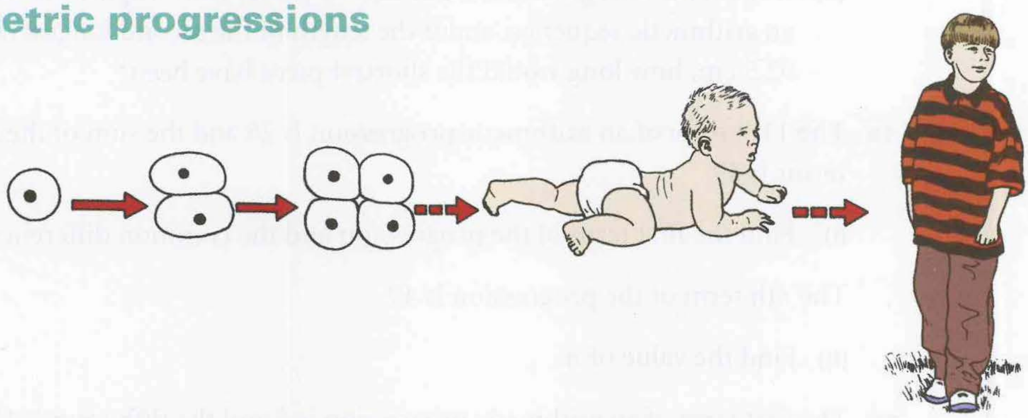


Figure 3.2

A human being begins life as one cell, which divides into two, then four… .

The terms of a geometric sequence are formed by multiplying one term by a fixed number, the common ratio, to obtain the next. This can be written inductively as:

$$u_{k+1} = ru_k \text{ with first term } u_1.$$

The sum of the terms of a geometric sequence is called a *geometric progression*, shortened to G.P. An alternative name is a *geometric series.*

Notation

When describing geometric sequences in this book, the following conventions are used:

- first term $u_1 = a$
- common ratio $= r$

- number of terms = n
- the general term u_k is that in position k (i.e. the kth term).

Thus in the geometric sequence 3, 6, 12, 24, 48,

$$a = 3,\ r = 2 \text{ and } n = 5.$$

The terms of this sequence are formed as follows.

$$\begin{aligned} u_1 &= a &&= 3 \\ u_2 &= a \times r &&= 3 \times 2 = 6 \\ u_3 &= a \times r^2 &&= 3 \times 2^2 = 12 \\ u_4 &= a \times r^3 &&= 3 \times 2^3 = 24 \\ u_5 &= a \times r^4 &&= 3 \times 2^4 = 48 \end{aligned}$$

You will see that in each case the power of r is one less than the number of the term: $u_5 = ar^4$ and 4 is one less than 5. This can be written deductively as

$$u_k = ar^{k-1},$$

and the last term is

$$u_n = ar^{n-1}.$$

These are both general formulae which apply to any geometric sequence.

Given two consecutive terms of a geometric sequence, you can always find the common ratio by dividing the later term by the earlier. For example, the geometric sequence … 5, 8, … has common ratio $r = \frac{8}{5}$.

EXAMPLE 3.6

Find the seventh term in the geometric sequence 8, 24, 72, 216, … .

SOLUTION

In the sequence, the first term $a = 8$ and the common ratio $r = 3$.

The kth term of a geometric sequence is given by $u_k = ar^{k-1}$,

and so $u_7 = 8 \times 3^6$
$= 5832.$

EXAMPLE 3.7

How many terms are there in the geometric sequence 4, 12, 36, … , 708 588?

SOLUTION

Since it is a geometric sequence and the first two terms are 4 and 12, you can immediately write down

First term: $a = 4$

Common ratio: $r = 3$ $\frac{12}{4} = 3$

The third term allows you to check you are right.

$12 \times 3 = 36$ ✓

The nth term of a geometric sequence is ar^{n-1}, so in this case

$4 \times 3^{n-1} = 708\,588$

Dividing through by 4 gives

$3^{n-1} = 177\,147$

You will learn about these in P2 and P3.

You can use logarithms to solve an equation like this, but since you know that n is a whole number it is just as easy to work out the powers of 3 until you come to 177 147.

They go $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, …

and before long you come to $3^{11} = 177\,147$.

You can do this by hand or you can use your calculator.

So $n - 1 = 11$ and $n = 12$.

There are 12 terms in the sequence.

? How would you use a spreadsheet to solve the equation $3^{n-1} = 177\,147$?

The sum of the terms of a geometric progression

The origins of chess are obscure, with several countries claiming the credit for its invention. One story is that it came from China. It is said that its inventor presented the game to the Emperor, who was so impressed that he asked the inventor what he would like as a reward.

'One grain of rice for the first square on the board, two for the second, four for the third, eight for the fourth, and so on up to the last square', came the answer.

The Emperor agreed, but it soon became clear that there was not enough rice in the whole of China to give the inventor his reward.

How many grains of rice was the inventor actually asking for?

The answer is the geometric series with 64 terms and common ratio 2:

$$1 + 2 + 4 + 8 + \ldots + 2^{63}.$$

This can be summed as follows.

Call the series S:

$$S = 1 + 2 + 4 + 8 + \ldots + 2^{63}. \qquad ①$$

Now multiply it by the common ratio, 2:

$$2S = 2 + 4 + 8 + 16 + \ldots + 2^{64}. \qquad ②$$

Then subtract ① from ②

$$② \quad 2S = \quad 2 + 4 + 8 + 16 + \ldots + 2^{63} + 2^{64}$$

$$① \quad S = 1 + 2 + 4 + 8 \quad + \ldots + 2^{63}$$

subtracting: $S = -1 + 0 + 0 + 0 \quad + \ldots + 2^{64}.$

The total number of rice grains requested was therefore $2^{64} - 1$ (which is about 1.85×10^{19}).

? How many tonnes of rice is this, and how many tonnes would you expect there to be in China at any time?

(One hundred grains of rice weigh about 2 grammes. The world annual production of all cereals is about 1.8×10^9 tonnes.)

Note

The method shown above can be used to sum any geometric progression.

EXAMPLE 3.8

Find the value of $0.2 + 1 + 5 + \ldots + 390\,625$.

SOLUTION

This is a geometric progression with common ratio 5.

Let $\quad S = 0.2 + 1 + 5 + \ldots + 390\,625. \qquad ①$

Multiplying by the common ratio, 5, gives:

$$5S = 1 + 5 + 25 + \ldots + 390\,625 + 1\,953\,125. \qquad ②$$

Subtracting ① from ②:

$$\begin{aligned} 5S &= \qquad 1 + 5 + 25 + \ldots + 390\,625 + 1\,953\,125 \\ S &= 0.2 + 1 + 5 + 25 + \ldots + 390\,625 \\ \hline 4S &= -0.2 + 0 + \ldots \qquad + 0 \qquad + 1\,953\,125 \end{aligned}$$

This gives $\quad 4S = 1\,953\,124.8$

$\Rightarrow \quad S = 488\,281.2.$

The same method can be applied to the general geometric progression to give a formula for its value:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad ①$$

Multiplying by the common ratio, r, gives:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n. \qquad ②$$

Subtracting ① from ②, as before, gives:

$$\begin{aligned}(r-1)S_n &= -a + ar^n \\ &= a(r^n - 1)\end{aligned}$$

so $$S_n = \frac{a(r^n - 1)}{(r-1)}.$$

This can also be written as:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}.$$

Infinite geometric progressions

The progression $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is geometric, with common ratio $\frac{1}{2}$.

Summing the terms one by one gives $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16} \ldots$.

Clearly the more terms you take, the nearer the sum gets to 2. In the limit, as the number of terms tends to infinity, the sum tends to 2.

As $n \to \infty$, $S_n \to 2$.

This is an example of a *convergent* series. The sum to infinity is a finite number.

You can see this by substituting $a = 1$ and $r = \frac{1}{2}$ in the formula for the sum of the series:

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

giving $$S_n = \frac{1 \times \left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

$$= 2 \times \left(1 - \left(\frac{1}{2}\right)^n\right).$$

The larger the number of terms, n, the smaller $\left(\frac{1}{2}\right)^n$ becomes and so the nearer S_n is to the limiting value of 2 (see figure 3.3). Notice that $\left(\frac{1}{2}\right)^n$ can never be negative, however large n becomes; so S_n can never exceed 2.

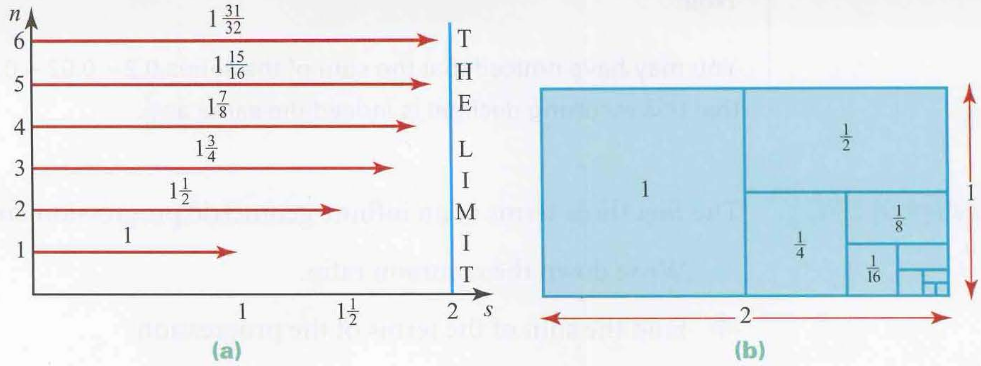


Figure 3.3

In the general geometric series $a + ar + ar^2 + \ldots$ the terms become progressively smaller in size if the common ratio r is between -1 and 1. This was the case above: r had the value $\frac{1}{2}$. In such cases, the geometric series is *convergent.*

If, on the other hand, the value of r is greater than 1 (or less than -1) the terms in the series become larger and larger in size and so the series is described as *divergent.*

A series corresponding to a value of r of exactly $+1$ consists of the first term a repeated over and over again. A sequence corresponding to a value of r of exactly -1 oscillates between $+a$ and $-a$. Neither of these is convergent.

It only makes sense to talk about the sum of an infinite series if it is convergent. Otherwise the sum is undefined.

The condition for a geometric series to converge, $-1 < r < 1$, ensures that as $n \to \infty$, $r^n \to 0$, and so the formula for the sum of a geometric series:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

may be rewritten for an infinite series as:

$$S_\infty = \frac{a}{1 - r}.$$

EXAMPLE 3.9

Find the sum of the terms of the infinite progression 0.2, 0.02, 0.002, … .

SOLUTION

This is a geometric progression with $a = 0.2$ and $r = 0.1$.

Its sum is given by

$$\begin{aligned} S_\infty &= \frac{a}{1 - r} \\ &= \frac{0.2}{1 - 0.1} \\ &= \frac{0.2}{0.9} \\ &= \frac{2}{9}. \end{aligned}$$

Note

You may have noticed that the sum of the series 0.2 + 0.02 + 0.002 + … is $0.\dot{2}$, and that this recurring decimal is indeed the same as $\frac{2}{9}$.

EXAMPLE 3.10

The first three terms of an infinite geometric progression are 16, 12 and 9.

(i) Write down the common ratio.

(ii) Find the sum of the terms of the progression.

SOLUTION

(i) The common ratio is $\frac{3}{4}$.

(ii) The sum of the terms of an infinite geometric progression is given by:

$$S_\infty = \frac{a}{1-r}.$$

In this case $a = 16$ and $r = \frac{3}{4}$, so:

$$S_\infty = \frac{16}{1-\frac{3}{4}} = 64.$$

A paradox

Consider the following arguments.

(i)
$$\begin{aligned} & S = 1 - 2 + 4 - 8 + 16 - 32 + 64 - \ldots \\ \Rightarrow\ & S = 1 - 2(1 - 2 + 4 - 8 + 16 - 32 + \ldots) \\ & = 1 - 2S \\ \Rightarrow\ & 3S = 1 \\ \Rightarrow\ & S = \tfrac{1}{3}. \end{aligned}$$

(ii)
$$\begin{aligned} & S = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64) + \ldots \\ \Rightarrow\ & S = 1 + 2 + 8 + 32 + \ldots \end{aligned}$$

So S diverges towards $+\infty$.

(iii)
$$\begin{aligned} & S = (1 - 2) + (4 - 8) + (16 - 32) + \ldots \\ \Rightarrow\ & S = -1 - 4 - 8 - 16 \ldots \end{aligned}$$

So S diverges towards $-\infty$.

What is the sum of the series: $\frac{1}{3}$, $+\infty$, $-\infty$, or something else?

EXERCISE 3B

1 Are the following sequences geometric?
If so, state the common ratio and calculate the seventh term.

(i) 5, 10, 20, 40, … **(ii)** 2, 4, 6, 8, …

(iii) 1, −1, 1, −1, … **(iv)** 5, 5, 5, 5, …

(v) 6, 3, 0, −3, … **(vi)** 6, 3, $1\frac{1}{2}$, $\frac{3}{4}$, …

(vii) 1, 1.1, 1.11, 1.111, …

2 A geometric sequence has first term 3 and common ratio 2.
The sequence has eight terms.

(i) Find the last term.

(ii) Find the sum of the terms in the sequence.

3 The first term of a geometric sequence of positive terms is 5 and the fifth term is 1280.

(i) Find the common ratio of the sequence.

(ii) Find the eighth term of the sequence.

4 A geometric sequence has first term $\frac{1}{9}$ and common ratio 3.

(i) Find the fifth term.

(ii) Which is the first term of the sequence which exceeds 1000?

5 **(i)** Find how many terms there are in the geometric sequence 8, 16, …, 2048.

(ii) Find the sum of the terms in this sequence.

6 **(i)** Find how many terms there are in the geometric sequence 200, 50, …, 0.195 312 5.

(ii) Find the sum of the terms in this sequence.

7 The fifth term of a geometric progression is 48 and the ninth term is 768.
All the terms are positive.

(i) Find the common ratio.

(ii) Find the first term.

(iii) Find the sum of the first ten terms.

8 The first three terms of an infinite geometric progression are 4, 2 and 1.

(i) State the common ratio of this progression.

(ii) Calculate the sum to infinity of its terms.

9 The first three terms of an infinite geometric progression are 0.7, 0.07, 0.007.

(i) Write down the common ratio for this progression.

(ii) Find, as a fraction, the sum to infinity of the terms of this progression.

(iii) Find the sum to infinity of the geometric progression 0.7 − 0.07 + 0.007 − …, and hence show that $\frac{7}{11} = 0.\dot{6}\dot{3}$.

10 The first three terms of a geometric sequence are 100, 90 and 81.

(i) Write down the common ratio of the sequence.

(ii) Which is the position of the first term in the sequence that has a value less than 1?

(iii) Find the sum to infinity of the terms of this sequence.

(iv) After how many terms is the sum of the sequence greater than 99% of the sum to infinity?

11 A geometric progression has first term 4 and its sum to infinity is 5.

(i) Find the common ratio.

(ii) Find the sum to infinity if the first term is excluded from the progression.

12 **(i)** The third term of a geometric progression is 16 and the fourth term is 12.8. Find the common ratio and the first term.

(ii) The sum of the first n terms of a geometric progression is $2^{(2n+1)} - 2$. Find the first term and the common ratio. [MEI]

13 **(i)** The first two terms of a geometric series are 3 and 4. Find the third term.

(ii) Given that x, 4, $x + 6$ are consecutive terms of a geometric series, find:

(a) the possible values of x

(b) the corresponding values of the common ratio of the geometric series.

(iii) Given that x, 4, $x + 6$ are the sixth, seventh and eighth terms of a geometric series and that the sum to infinity of the series exists, find:

(a) the first term

(b) the sum to infinity. [MEI]

14 The first four terms in an infinite geometric series are 54, 18, 6, 2.

(i) What is the common ratio r?

(ii) Write down an expression for the nth term of the series.

(iii) Find the sum of the first n terms of the series.

(iv) Find the sum to infinity.

(v) How many terms are needed for the sum to be greater than 80.999?

15 A tank is filled with 20 litres of water. Half the water is removed and replaced with anti-freeze and thoroughly mixed. Half this mixture is then removed and replaced with anti-freeze. The process continues.

(i) Find the first five terms in the sequence of amounts of water in the tank at each stage.

(ii) Find the first five terms in the sequence of amounts of anti-freeze in the tank at each stage.

(iii) Is either of these sequences geometric? Explain.

16 A pendulum is set swinging. Its first oscillation is through an angle of 30°, and each succeeding oscillation is through 95% of the angle of the one before it.

(i) After how many swings is the angle through which it swings less than 1°?

(ii) What is the total angle it has swung through at the end of its tenth oscillation?

17 A ball is thrown vertically upwards from the ground. It rises to a height of 10 m and then falls and bounces. After each bounce it rises vertically to $\frac{2}{3}$ of the height from which it fell.

(i) Find the height to which the ball bounces after the nth impact with the ground.

(ii) Find the total distance travelled by the ball from the first throw to the tenth impact with the ground.

18 The first three terms of an arithmetic sequence, a, $a + d$ and $a + 2d$, are the same as the first three terms, a, ar, ar^2, of a geometric sequence ($a \neq 0$).

Show that this is only possible if $r = 1$ and $d = 0$.

19 The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression

(ii) the sum to infinity of the progression.

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 June 2008]

20 A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression in each of the following cases:

(i) the progression is arithmetic

(ii) the progression is geometric with a positive common ratio.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q3 November 2009]

21 **(i)** Find the sum to infinity of the geometric progression with first three terms 0.5, 0.5^3 and 0.5^5.

(ii) The first two terms in an arithmetic progression are 5 and 9. The last term in the progression is the only term which is greater than 200. Find the sum of all the terms in the progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 June 2009]

22 The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term.

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$.

(iii) Find the common ratio of the geometric progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 November 2007]

INVESTIGATIONS

Snowflakes

Draw an equilateral triangle with sides 9 cm long.
Trisect each side and construct equilateral triangles on the middle section of each side as shown in diagram **(b)**.

Repeat the procedure for each of the small triangles as shown in **(c)** and **(d)** so that you have the first four stages in an infinite sequence.

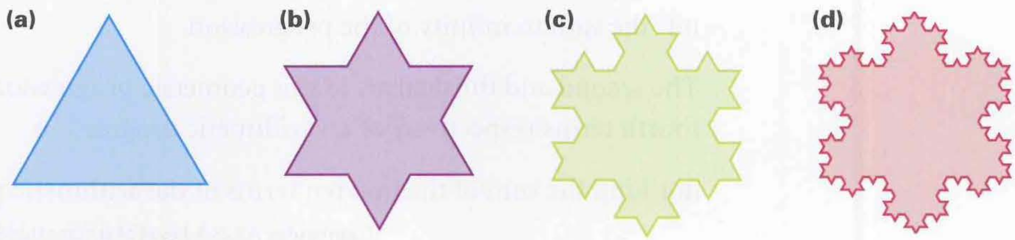


Figure 3.4

Calculate the length of the perimeter of the figure for each of the first six steps, starting with the original equilateral triangle.

What happens to the length of the perimeter as the number of steps increases?

Does the area of the figure increase without limit?

Achilles and the tortoise

Achilles (it is said) once had a race with a tortoise. The tortoise started 100 m ahead of Achilles and moved at $\frac{1}{10}$ ms^{-1} compared to Achilles' speed of 10 ms^{-1}.

Achilles ran to where the tortoise started only to see that it had moved 1 m further on. So he ran on to that spot but again the tortoise had moved further on, this time by 0.01 m. This happened again and again: whenever Achilles got to the spot where the tortoise was, it had moved on. Did Achilles ever manage to catch the tortoise?

Binomial expansions

A special type of series is produced when a binomial (i.e. two-part) expression like $(x + 1)$ is raised to a power. The resulting expression is often called a *binomial expansion*.

The simplest binomial expansion is $(x + 1)$ itself. This and other powers of $(x + 1)$ are given below.

$$
\begin{aligned}
(x+1)^1 &= 1x + 1 \\
(x+1)^2 &= 1x^2 + 2x + 1 \\
(x+1)^3 &= 1x^3 + 3x^2 + 3x + 1 \\
(x+1)^4 &= 1x^4 + 4x^3 + 6x^2 + 4x + 1 \\
(x+1)^5 &= 1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1
\end{aligned}
$$

Expressions like these, consisting of integer powers of x and constants are called *polynomials*.

If you look at the coefficients on the right-hand side above you will see that they form a pattern.

$$
\begin{array}{ccccccccccc}
 & & & & & (1) & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

These numbers are called *binomial* coefficients.

This is called *Pascal's triangle*, or the *Chinese triangle*. Each number is obtained by adding the two above it, for example

$$4 + 6$$

gives 10

This pattern of coefficients is very useful. It enables you to write down the expansions of other binomial expressions. For example,

$$
\begin{aligned}
(x+y) &= 1x + 1y \\
(x+y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3
\end{aligned}
$$

Notice how in each term the sum of the powers of x and y is the same as the power of $(x + y)$.

This is a binomial expression.

These numbers are called binomial coefficients.

EXAMPLE 3.11 Write out the binomial expansion of $(x + 2)^4$.

SOLUTION

The binomial coefficients for power 4 are 1 4 6 4 1.

In each term, the sum of the powers of x and 2 must equal 4.

So the expansion is

$$1 \times x^4 + 4 \times x^3 \times 2 + 6 \times x^2 \times 2^2 + 4 \times x \times 2^3 + 1 \times 2^4$$

i.e. $x^4 + 8x^3 + 24x^2 + 32x + 16.$

EXAMPLE 3.12

Write out the binomial expansion of $(2a-3b)^5$.

SOLUTION

The binomial coefficients for power 5 are 1 5 10 10 5 1.

The expression $(2a-3b)$ is treated as $(2a+(-3b))$.

So the expansion is

$$1\times(2a)^5+5\times(2a)^4\times(-3b)+10\times(2a)^3\times(-3b)^2+10\times(2a)^2\times(-3b)^3 +5\times(2a)\times(-3b)^4+1\times(-3b)^5$$

i.e. $32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5$.

Historical note

Blaise Pascal has been described as the greatest might-have-been in the history of mathematics. Born in France in 1623, he was making discoveries in geometry by the age of 16 and had developed the first computing machine before he was 20.

Pascal suffered from poor health and religious anxiety, so that for periods of his life he gave up mathematics in favour of religious contemplation. The second of these periods was brought on when he was riding in his carriage: his runaway horses dashed over the parapet of a bridge, and he was only saved by the miraculous breaking of the traces. He took this to be a sign of God's disapproval of his mathematical work. A few years later a toothache subsided when he was thinking about geometry and this, he decided, was God's way of telling him to return to mathematics.

Pascal's triangle (and the binomial theorem) had actually been discovered by Chinese mathematicians several centuries earlier, and can be found in the works of Yang Hui (around 1270 A.D.) and Chu Shi-kie (in 1303 A.D.). Pascal is remembered for his application of the triangle to elementary probability, and for his study of the relationships between binomial coefficients.

Pascal died at the early age of 39.

Tables of binomial coefficients

Values of binomial coefficients can be found in books of tables. It is helpful to use these when the power becomes large, since writing out Pascal's triangle becomes progressively longer and more tedious, row by row.

EXAMPLE 3.13

Write out the full expansion of $(x+y)^{10}$.

SOLUTION

The binomial coefficients for the power 10 can be found from tables to be

1 10 45 120 210 252 210 120 45 10 1

and so the expansion is

$$x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}.$$

There are 10 + 1 = 11 terms.

As the numbers are symmetrical about the middle number, tables do not always give the complete row of numbers.

The formula for a binomial coefficient

There will be times when you need to find binomial coefficients that are outside the range of your tables. The tables may, for example, list the binomial coefficients for powers up to 20. What happens if you need to find the coefficient of x^{17} in the expansion of $(x+2)^{25}$? Clearly you need a formula that gives binomial coefficients.

The first thing you need is a notation for identifying binomial coefficients. It is usual to denote the power of the binomial expression by n, and the position in the row of binomial coefficients by r, where r can take any value from 0 to n. So for row 5 of Pascal's triangle

$n = 5$:	1	5	10	10	5	1
	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$

The general binomial coefficient corresponding to values of n and r is written as $\binom{n}{r}$. An alternative notation is ${}^nC_r$, which is said as 'N C R'.

Thus $\binom{5}{3} = {}^5C_3 = 10.$

The next step is to find a formula for the general binomial coefficient $\binom{n}{r}$. However, to do this you must be familiar with the term *factorial*.

The quantity '8 factorial', written 8!, is

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320.$$

Similarly, $12! = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479\,001\,600$,

and $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$, where n is a positive integer.

Note that 0! is defined to be 1. You will see the need for this when you use the formula for $\binom{n}{r}$.

ACTIVITY 3.1 The table shows an alternative way of laying out Pascal's triangle.

		Column (r)								
		0	**1**	**2**	**3**	**4**	**5**	**6**	**…**	r
Row (n)	**1**	1	1							
	2	1	2	1						
	3	1	3	3	1					
	4	1	4	6	4	1				
	5	1	5	10	10	5	1			
	6	1	6	15	20	15	6	1		
	…	…	…	…	…	…	…	…	…	
	…	…	…	…	…	…	…	…	…	…
	n	1	n	?	?	?	?	?	?	?

Show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, by following the procedure below.

The numbers in column 0 are all 1.

To find each number in column 1 you multiply the 1 in column 0 by the row number, n.

(i) Find, in terms of n, what you must multiply each number in column 1 by to find the corresponding number in column 2.

(ii) Repeat the process to find the relationship between each number in column 2 and the corresponding one in column 3.

(iii) Show that repeating the process leads to

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\times\ldots\times r} \text{ for } r \geqslant 1.$$

(iv) Show that this can also be written as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and that it is also true for $r = 0$.

EXAMPLE 3.14 Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to calculate these.

(i) $\binom{5}{0}$ **(ii)** $\binom{5}{1}$ **(iii)** $\binom{5}{2}$

(iv) $\binom{5}{3}$ **(v)** $\binom{5}{4}$ **(vi)** $\binom{5}{5}$

SOLUTION

(i) $\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{120}{1 \times 120} = 1$

(ii) $\binom{5}{1} = \frac{5!}{1!4!} = \frac{120}{1 \times 24} = 5$

(iii) $\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{2 \times 6} = 10$

(iv) $\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \times 2} = 10$

(v) $\binom{5}{4} = \frac{5!}{4!1!} = \frac{120}{24 \times 1} = 5$

(vi) $\binom{5}{5} = \frac{5!}{5!0!} = \frac{120}{120 \times 1} = 1$

Note

You can see that these numbers, 1, 5, 10, 10, 5, 1, are row 5 of Pascal's triangle.

Most scientific calculators have factorial buttons, e.g. [x!]. Many also have [ⁿCᵣ] buttons. Find out how best to use your calculator to find binomial coefficients, as well as practising non-calculator methods.

EXAMPLE 3.15 Find the coefficient of x^{17} in the expansion of $(x+2)^{25}$.

SOLUTION

$$(x+2)^{25} = \binom{25}{0}x^{25} + \binom{25}{1}x^{24}2^1 + \binom{25}{2}x^{23}2^2 + \ldots + \binom{25}{8}x^{17}2^8 + \ldots \binom{25}{25}2^{25}$$

So the required term is $\binom{25}{8} \times 2^8 \times x^{17}$

$$\binom{25}{8} = \frac{25!}{8!17!} = \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18 \times \cancel{17!}}{8! \times \cancel{17!}}$$

$$= 1\,081\,575.$$

So the coefficient of x^{17} is $1\,081\,575 \times 2^8 = 276\,883\,200$.

Note

Notice how 17! was cancelled in working out $\binom{25}{8}$. Factorials become large numbers very quickly and you should keep a look-out for such opportunities to simplify calculations.

The expansion of $(1 + x)^n$

When deriving the result for $\begin{pmatrix} n \\ r \end{pmatrix}$ you found the binomial coefficients in the form

$$1 \qquad n \qquad \frac{n(n-1)}{2!} \quad \frac{n(n-1)(n-2)}{3!} \quad \frac{n(n-1)(n-2)(n-3)}{4!} \quad \ldots$$

This form is commonly used in the expansion of expressions of the type $(1+x)^n$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{1\times 2} + \frac{n(n-1)(n-2)x^3}{1\times 2\times 3} + \frac{n(n-1)(n-2)(n-3)x^4}{1\times 2\times 3\times 4} + \ldots$$

$$+ \frac{n(n-1)}{1\times 2}x^{n-2} + nx^{n-1} + 1x^n$$

EXAMPLE 3.16

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1+x)^9$.

SOLUTION

The power of x is the same as the largest number underneath.

$$(1+x)^9 = 1 + 9x + \frac{9\times 8}{1\times 2}x^2 + \frac{9\times 8\times 7}{1\times 2\times 3}x^3 + \ldots$$

Two numbers on top, two underneath.

Three numbers on top, three underneath.

$$= 1 + 9x + 36x^2 + 84x^3 + \ldots$$

The expression $1 + 9x + 36x^2 + 84x^3$ ... is said to be in *ascending* powers of x, because the powers of x are increasing from one term to the next.

An expression like $x^9 + 9x^8 + 36x^7 + 84x^6$ ... is in *descending* powers of x, because the powers of x are decreasing from one term to the next.

EXAMPLE 3.17

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1-3x)^7$. Simplify the terms.

SOLUTION

Think of $(1-3x)^7$ as $(1+(-3x))^7$. Keep the brackets while you write out the terms.

$$(1+(-3x))^7 = 1 + 7(-3x) + \frac{7\times 6}{1\times 2}(-3x)^2 + \frac{7\times 6\times 5}{1\times 2\times 3}(-3x)^3 + \ldots$$

$$= 1 - 21x + 189x^2 - 945x^3 + \ldots$$

Note how the signs alternate.