CONTINUED FROM PREVIOUS PAGE

To find m_3 we again apply the balance condition, this time with the mass $(m_1 + m_2 + 0.30\text{ kg} + 0.20\text{ kg})$ at the distance 18 cm, and the mass m_3 at the distance 31 cm.

SOLUTION

1. Apply the balance condition to m_1 and m_2:
$$m_1(12\text{ cm}) = m_2(18\text{ cm})$$
$$m_1 = (1.5)m_2$$

2. Apply the balance condition to the next level up in the mobile. Solve for the sum, $m_1 + m_2$:
$$(m_1 + m_2)(6.0\text{ cm}) = (0.30\text{ kg})(24\text{ cm})$$
$$m_1 + m_2 = \frac{(0.30\text{ kg})(24\text{ cm})}{6.0\text{ cm}} = 1.2\text{ kg}$$

3. Substitute $m_1 = (1.5)m_2$ into $m_1 + m_2 = 1.2$ kg to find m_2:
$$(1.5)m_2 + m_2 = (2.5)m_2 = 1.2\text{ kg}$$
$$m_2 = 1.2\text{ kg}/2.5 = 0.48\text{ kg}$$

4. Use $m_1 = (1.5)m_2$ to find m_1:
$$m_1 = (1.5)m_2 = (1.5)0.48\text{ kg} = 0.72\text{ kg}$$

5. Apply the balance condition to the top level of the mobile:
$$(0.72\text{ kg} + 0.48\text{ kg} + 0.30\text{ kg} + 0.20\text{ kg})(18\text{ cm}) = m_3(31\text{ cm})$$

6. Solve for m_3:
$$m_3 = \frac{(1.70\text{ kg})(18\text{ cm})}{31\text{ cm}} = 0.99\text{ kg}$$

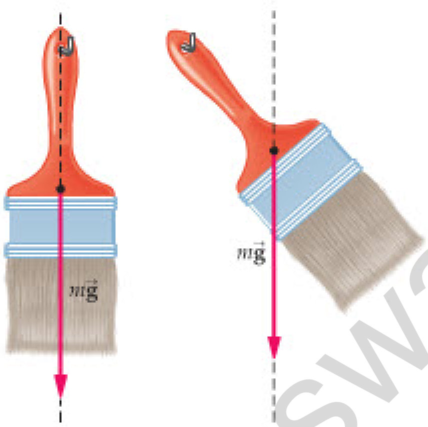
INSIGHT

With the values for m_1, m_2, and m_3 found above, the mobile balances at every level. In fact, the center of mass of the *entire* mobile is directly below the point where the uppermost string attaches to the ceiling.

PRACTICE PROBLEM

Find m_1, m_2, and m_3 if the 0.30-kg mass is replaced with a 0.40-kg mass. [**Answer:** $m_1 = 0.96$ kg, $m_2 = 0.64$ kg, $m_3 = 1.3$ kg]

Some related homework problems: Problem 43, Problem 45



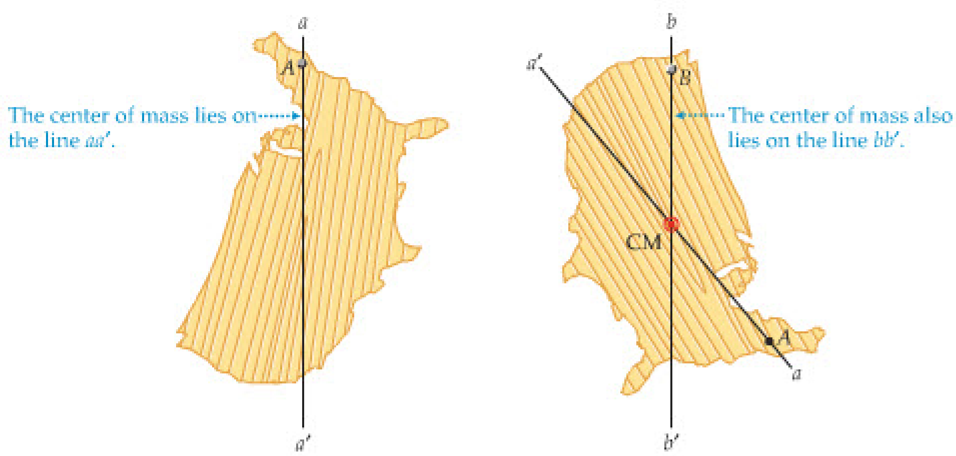
(a) Zero torque (b) Nonzero torque

▲ **FIGURE 11–8 Equilibrium of a suspended object**

(a) If an object's center of mass is directly below the suspension point, its weight creates zero torque and the object is in equilibrium. **(b)** When an object is rotated, so that the center of mass is no longer directly below the suspension point, the object's weight creates a torque. The torque tends to rotate the object to bring the center of mass under the suspension point.

In general, if you allow an arbitrarily shaped object to hang freely, its center of mass is directly below the suspension point. To see why, note that when the center of mass is directly below the suspension point, the torque due to gravity is zero, since the force of gravity extends right through the axis of rotation. This is shown in Figure 11–8 (a). If the object is rotated slightly, as in Figure 11–8 (b), the force of gravity is not in line with the axis of rotation—hence gravity produces a torque. This torque tends to rotate the object, bringing the center of mass back under the suspension point.

For example, suppose you cut a piece of wood into the shape of the continental United States, as shown in Figure 11–9, drill a small hole in it, and hang it from the



▲ **FIGURE 11–9 The geometric center of the United States**

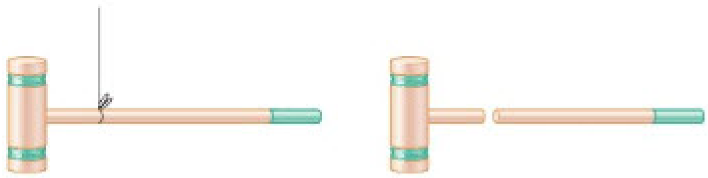
To find the center of mass of an irregularly shaped object, such as a wooden model of the continental United States, suspend it from two or more points. The center of mass lies on a vertical line extending downward from the suspension point. The intersection of these vertical lines gives the precise location of the center of mass.

point A. The result is that the center of mass lies somewhere on the line aa'. Similarly, if a second hole is drilled at point B, we find that the center of mass lies somewhere on the line bb'. The only point that is on both the line aa' *and* the line bb' is the point CM, near Smith Center, Kansas, which marks the location of the center of mass.

▲ In this scene from the movie *Mission Impossible*, Tom Cruise is attempting to download top-secret computer files without setting off the elaborate security system in the room. To accomplish this nearly impossible mission, he is suspended from the ceiling, since touching the floor would immediately give away his presence. To remain in equilibrium above the floor as he works, he must carefully adjust the position of his arms and legs to keep his center of mass directly below the suspension point.

CONCEPTUAL CHECKPOINT 11–2 COMPARE THE MASSES

A croquet mallet balances when suspended from its center of mass, as indicated in the drawing at left. If you cut the mallet in two at its center of mass, as in the drawing at right, how do the masses of the two pieces compare? **(a)** The masses are equal; **(b)** the piece with the head of the mallet has the greater mass; or **(c)** the piece with the head of the mallet has the smaller mass.



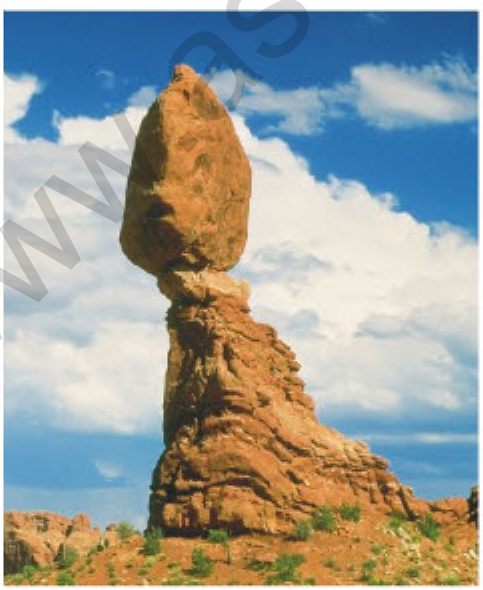
REASONING AND DISCUSSION

The mallet balances because the torques due to the two pieces are of equal magnitude. The piece with the head of the mallet extends a smaller distance from the point of suspension than does the other piece, hence its mass must be greater; that is, a large mass at a small distance creates the same torque as a small mass at a large distance.

ANSWER

(b) The piece with the head of the mallet has the greater mass.

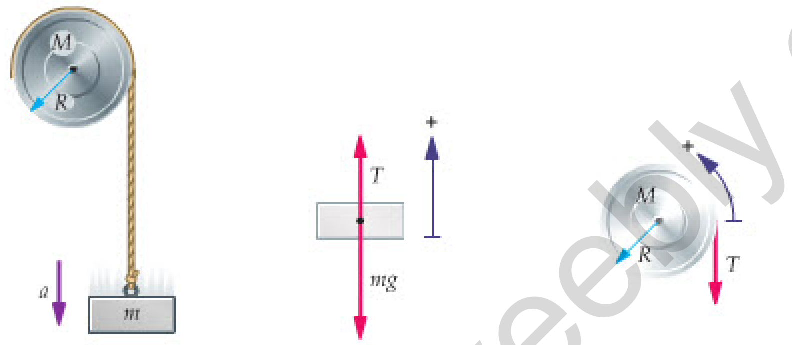
Similar considerations apply to an object that is at rest on a surface, as opposed to being suspended from a point. In such a case, the object is in equilibrium as long as its center of mass is directly above the base on which it is supported. For example, when you stand upright with normal posture your feet provide a base of support, and your center of mass is above a point roughly halfway between your feet. If you lift your right foot from the floor—without changing your posture—you will begin to lose your balance and tip over. The reason is that your center of mass is no longer above the base of support, which is now your left foot. To balance on your left foot, you must lean slightly in that direction so as to position your center of mass directly above the foot. This principle applies to everything from a performer in a high-wire act to one of the "balancing rocks" that are a familiar sight in the desert Southwest. In Problem 44 we apply this condition for stability to a stack of books on the edge of a table.



◀ (Left) Although it looks precarious, this rock in Arches National Park, Utah, has probably been balancing above the desert for many thousands of years. It will remain secure on its perch as long as its center of mass lies above its base of support. (Right) Although her knowledge may be based more on practical experience than on physics, this woman knows exactly what she must do to keep from falling. By extending one leg backward as she leans forward, she keeps her center of mass safely positioned over the foot that supports her.

11–5 Dynamic Applications of Torque

In this section we focus on applications of Newton's second law for rotation. For example, consider a disk-shaped pulley of radius R and mass M with a string wrapped around its circumference, as in **Figure 11–10 (a)**. Hanging from the string is a mass m. When the mass is released, it accelerates downward and the pulley begins to rotate. If the pulley rotates without friction, and the string unwraps without slipping, what are the acceleration of the mass and the tension in the string?



(a) Physical picture (b) Free-body diagram for mass (c) Torque producing force acting on pulley

▶ **FIGURE 11–10 A mass suspended from a pulley**

A mass m hangs from a string wrapped around the circumference of a disk-shaped pulley of radius R and mass M. When the mass is released, it accelerates downward. Positive directions of motion for the system are shown in parts (b) and (c). In part (c), the weight of the pulley acts downward at its center, and the axle exerts an upward force equal in magnitude to the weight of the pulley plus the tension in the string. Of the three forces acting on the pulley, only the tension in the string produces a torque about the axle.

At first it may seem that since the pulley rotates freely, the mass will simply fall with the acceleration of gravity. But remember, the pulley has a nonzero moment of inertia, $I > 0$, which means that it resists any change in its rotational motion. In order for the pulley to rotate, the string must pull downward on it. This means that the string also pulls upward on the mass m with a tension T. As a result, the net downward force on m is less than mg, and thus its acceleration is less than g.

To solve for the acceleration of the mass, we must apply Newton's second law to both the linear motion of the mass *and* the rotational motion of the pulley. The first step is to define a consistent choice of positive directions for the two motions. In Figure 11–10 (a) we note that when the pulley rotates counterclockwise, the mass moves upward. Thus, we choose counterclockwise to be positive for the pulley and upward to be positive for the mass.

With our positive directions established, we proceed to apply Newton's second law. Referring to the free-body diagram for the mass, shown in **Figure 11–10 (b)**, we see that

$$T - mg = ma \qquad \text{11–8}$$

Similarly, the free-body diagram for the pulley is shown in **Figure 11–10 (c)**. Note that the tension in the string, T, exerts a tangential force on the pulley at a distance R from the axis of rotation. This produces a torque of magnitude TR. Since the tension tends to cause a clockwise rotation, it follows that the torque is negative; thus, $\tau = -TR$. As a result, Newton's second law for the pulley gives

$$-TR = I\alpha \qquad \text{11–9}$$

Now, these two statements of Newton's second law are related by the fact that the string unwraps without slipping. As was discussed in Chapter 10, when a string unwraps without slipping, the angular and linear accelerations are related by

$$\alpha = \frac{a}{R}$$

Using this relation in Equation 11–9 we have

$$-TR = I\frac{a}{R}$$

or, dividing by R,

$$T = -I\frac{a}{R^2}$$

Substituting this result into Equation 11–8 yields

$$-I\frac{a}{R^2} - mg = ma$$

Finally, dividing by m and rearranging yields the acceleration, a:

$$a = -\frac{g}{\left(1 + \dfrac{I}{mR^2}\right)} \qquad \text{11–10}$$

Let's briefly check our solution for a. First, note that a is negative. This is to be expected, since the mass accelerates downward, which is the negative direction. Second, if the moment of inertia were zero, $I = 0$, or if the mass m were infinite, $m \to \infty$, the mass would fall with the acceleration of gravity, $a = -g$. When I is greater than zero and m is finite, however, the acceleration of the mass has a magnitude less than g. In fact, in the limit of an infinite moment of inertia, $I \to \infty$, the acceleration vanishes—the mass is simply unable to cause the pulley to rotate in this case.

The next Example presents another system in which Newton's laws are used to relate linear and rotational motions.

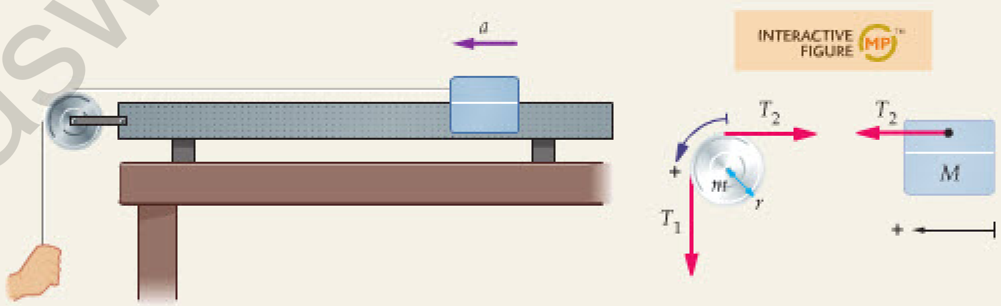
EXAMPLE 11–7 THE PULLEY MATTERS

A 0.31-kg cart on a horizontal air track is attached to a string. The string passes over a disk-shaped pulley of mass 0.080 kg and radius 0.012 m and is pulled vertically downward with a constant force of 1.1 N. Find **(a)** the tension in the string between the pulley and the cart and **(b)** the acceleration of the cart.

PICTURE THE PROBLEM

The system is shown below. We label the mass of the cart with M, the mass of the pulley with m, and the radius of the pulley with r. The applied downward force creates a tension $T_1 = 1.1$ N in the vertical portion of the string. The horizontal portion of the string, from the pulley to the cart, has a tension T_2. If the pulley had zero mass, these two tensions would be equal. In this case, however, T_2 will have a different value than T_1.

We also show the relevant forces acting on the pulley and the cart. The positive direction of rotation is counterclockwise, and the corresponding positive direction of motion for the cart is to the left.



Physical picture   Relevant forces

STRATEGY

The two unknowns, T_2 and a, can be found by applying Newton's second law to both the pulley and the cart. This gives two equations for two unknowns.

In applying Newton's second law to the pulley, note that since the pulley is a disk, it follows that $I = \frac{1}{2}mr^2$. Also, since the string is not said to slip as it rotates the pulley, we can assume that the angular and linear accelerations are related by $\alpha = a/r$.

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SOLUTION

Part (a)

1. Apply Newton's second law to the cart: $T_2 = Ma$
2. Apply Newton's second law to the pulley. Note that T_1 causes a positive torque, and T_2 causes a negative torque. In addition, use the relation $\alpha = a/r$: $\Sigma\tau = I\alpha$
$$rT_1 - rT_2 = \left(\tfrac{1}{2}mr^2\right)\left(\frac{a}{r}\right) = \tfrac{1}{2}mra$$
3. Use the cart equation, $T_2 = Ma$, to eliminate a in the pulley equation: $a = \dfrac{T_2}{M}$
$$rT_1 - rT_2 = \tfrac{1}{2}mr\left(\frac{T_2}{M}\right)$$
4. Cancel r and solve for T_2: $T_2 = \dfrac{T_1}{1 + m/2M} = \dfrac{1.1\text{ N}}{1 + 0.080\text{ kg}/[2(0.31\text{ kg})]} = 0.97\text{ N}$

Part (b)

5. Use $T_2 = Ma$ to find the acceleration: $a = \dfrac{T_2}{M} = \dfrac{0.97\text{ N}}{0.31\text{ kg}} = 3.1\text{ m/s}^2$

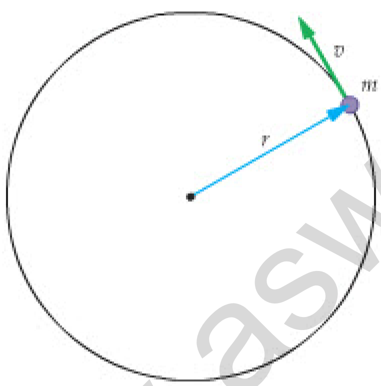
INSIGHT

Note that T_2 is less than T_1. As a result, the net torque acting on the pulley is in the counterclockwise direction, causing a rotation in that direction, as expected. If the mass of the pulley were zero ($m = 0$), the two tensions would be equal, and the acceleration of the cart would be $T_1/M = 3.5\text{ m/s}^2$.

PRACTICE PROBLEM

What applied force is necessary to give the cart an acceleration of 2.2 m/s^2? [**Answer:** $T_1 = T_2(1 + m/2M) = (Ma)(1 + m/2M) = 0.77\text{ N}$]

Some related homework problems: Problem 49, Problem 50



▲ **FIGURE 11–11 The angular momentum of circular motion**

A particle of mass m, moving in a circle of radius r with a speed v. This particle has an angular momentum of magnitude $L = rmv$.

11–6 Angular Momentum

When an object of mass m moves with a speed v in a straight line, we say that it has a linear momentum, $p = mv$. When the same object moves with an angular speed ω along the circumference of a circle of radius r, as in Figure 11–11, we say that it has an **angular momentum**, L. The magnitude of L is given by replacing m and v in the expression for p with their angular analogues I and ω (Section 10–5). Thus, we define the angular momentum as follows:

Definition of the Angular Momentum, L

$$L = I\omega \qquad 11\text{–}11$$

SI unit: $\text{kg}\cdot\text{m}^2/\text{s}$

This expression applies to any object undergoing angular motion, whether it is a point mass moving in a circle, as in Figure 11–11, or a rotating hoop, disk, or other object.

Returning for a moment to the case of a point mass m moving in a circle of radius r, recall that the moment of inertia in this case is $I = mr^2$ (Equation 10–18). In addition, the linear speed of the mass is $v = r\omega$ (Equation 10–12). Combining these results, we find

$$L = I\omega = (mr^2)(v/r) = rmv$$

Noting that mv is the linear momentum p, we find that the angular momentum of a point mass can be written in the following form:

$$L = rmv = rp \qquad 11\text{–}12$$

It is important to recall that this expression applies specifically to a point particle moving along the circumference of a circle.

More generally, a point object may be moving at an angle θ with respect to a radial line, as indicated in Figure 11–12 (a). In this case, it is only the tangential component of the momentum, $p \sin\theta = mv \sin\theta$, that contributes to the angular momentum, just as the tangential component of the force, $F \sin\theta$, is all that contributes to the torque. Thus, the magnitude of the angular momentum for a point particle is defined as:

Angular Momentum, *L*, for a Point Particle

$$L = rp \sin\theta = rmv \sin\theta \qquad 11\text{–}13$$

SI unit: $\text{kg}\cdot\text{m}^2/\text{s}$

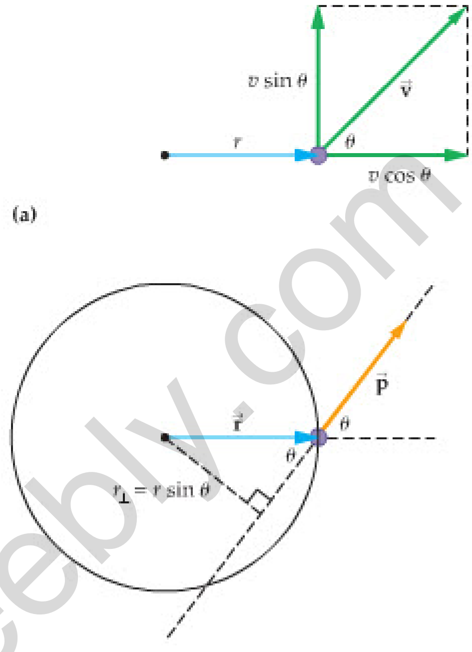


FIGURE 11–12 The angular momentum of nontangential motion

(a) When a particle moves at an angle θ with respect to the radial direction, only the tangential component of velocity, $v \sin\theta$, contributes to the angular momentum. In the case shown here, the particle's angular momentum has a magnitude given by $L = rmv \sin\theta$. **(b)** The angular momentum of an object can also be defined in terms of the moment arm, $r_\perp$. Since $r_\perp = r\sin\theta$, it follows that $L = rmv \sin\theta = r_\perp mv$. Note the similarity between this figure and Figure 11–3.

Note that if the particle moves in a circular path the angle θ is 90° and the angular momentum is $L = rmv$, in agreement with Equation 11–12. On the other hand, if the object moves radially, so that $\theta = 0$, the angular momentum is zero; $L = rmv \sin 0 = 0$.

EXERCISE 11–3

Find the angular momentum of **(a)** a 0.13-kg Frisbee (considered to be a uniform disk of radius 7.5 cm) spinning with an angular speed of 1.15 rad/s, and **(b)** a 95-kg person running with a speed of 5.1 m/s on a circular track of radius 25 m.

SOLUTION

a. Recalling that $I = \frac{1}{2}mR^2$ for a uniform disk (Table 10–1), we have

$$L = I\omega$$
$$= \left(\tfrac{1}{2}mR^2\right)\omega = \tfrac{1}{2}(0.13\ \text{kg})(0.075\ \text{m})^2(1.15\ \text{rad/s}) = 4.2 \times 10^{-4}\ \text{kg}\cdot\text{m}^2/\text{s}$$

b. Treating the person as a particle of mass m, we find

$$L = rmv = (25\ \text{m})(95\ \text{kg})(5.1\ \text{m/s}) = 12{,}000\ \text{kg}\cdot\text{m}^2/\text{s}$$

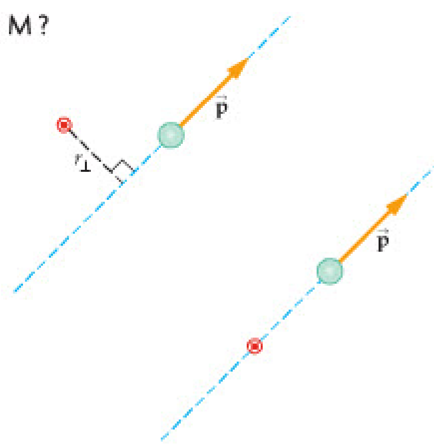
An alternative definition of the angular momentum uses the moment arm, $r_\perp$, as was done for the torque in Equation 11–3. To apply this definition, start by extending a line through the momentum vector, $\vec{\mathbf{p}}$, as in Figure 11–12 (b). Next, draw a line from the axis of rotation perpendicular to the line through $\vec{\mathbf{p}}$. The perpendicular distance from the axis of rotation to the line of $\vec{\mathbf{p}}$ is the moment arm. From the figure we see that $r_\perp = r\sin\theta$. Hence, from Equation 11–13, the angular momentum is

$$L = r_\perp p = r_\perp mv$$

If an object moves in a circle of radius r, the moment arm is $r_\perp = r$ and the angular momentum reduces to our earlier result, $L = rp$.

CONCEPTUAL CHECKPOINT 11–3 ANGULAR MOMENTUM?

Does an object moving in a straight line have nonzero angular momentum **(a)** always, **(b)** sometimes, or **(c)** never?

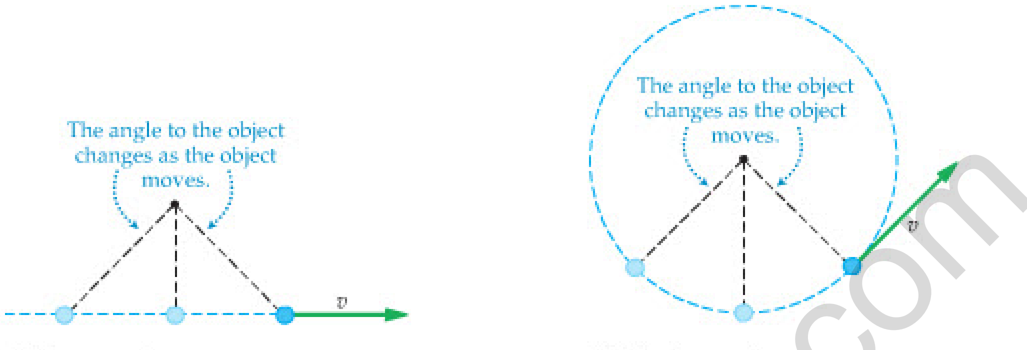


REASONING AND DISCUSSION

The answer is sometimes, because it depends on the choice of the axis of rotation. If the axis of rotation is not on the line drawn through the momentum vector, as in the left sketch at right, the moment arm is nonzero, and therefore $L = r_\perp p$ is also nonzero. If the axis of rotation is on the line of motion, as in the right sketch, the moment arm is zero; hence the linear momentum is radial and L vanishes.

ANSWER

(b) An object moving in a straight line may or may not have angular momentum, depending on the location of the axis of rotation.



(a) Linear motion (b) Circular motion

FIGURE 11–13 Angular momentum in linear and circular motion

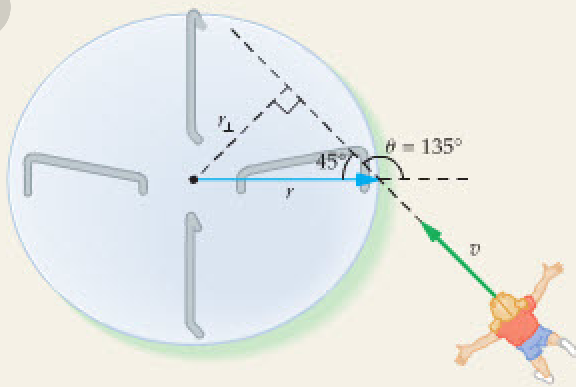
An object moving in **(a)** a straight line and **(b)** a circular path. In both cases, the angular position increases with time; hence, the angular momentum is positive.

Note that an object moving with a momentum p in a straight line that does not go through the axis of rotation has an *angular* position that changes with time. This is illustrated in **Figure 11–13 (a)**. It is for this reason that such an object is said to have an *angular* momentum.

The sign of L is determined by whether the angle to a given object is increasing or decreasing with time. For example, the object moving counterclockwise in a circular path in **Figure 11–13 (b)** has a positive angular momentum, since θ is increasing with time. Similarly, the object in Figure 11–13 (a) also has an angle θ that increases with time, hence its angular momentum is positive as well. On the other hand, if these objects were to have their direction of motion reversed, they would have angles that decrease with time and their angular momenta would be negative.

EXAMPLE 11–8 JUMP ON

Running with a speed of 4.10 m/s, a 21.2-kg child heads toward the rim of a merry-go-round. The radius of the merry-go-round is 2.00 m, and the child moves in the direction indicated. **(a)** What is the child's angular momentum with respect to the center of the merry-go-round? Use $L = rmv \sin \theta$. **(b)** What is the moment arm, $r_\perp$, in this case? **(c)** Find the angular momentum of the child with $L = r_\perp mv$.



PICTURE THE PROBLEM

Our sketch shows the child approaching the rim of the merry-go-round at an angle of 135° relative to the radial direction. Note that the line of motion of the child does not go through the axis of the merry-go-round. As a result, the child has a nonzero angular momentum with respect to that axis of rotation. We also indicate the moment arm, $r_\perp$, and the 45° angle that is opposite to it.

STRATEGY

a. The child's angular momentum can be found by applying $L = rmv \sin \theta$. In this case, we see from the sketch that $\theta = 135°$ and $r = 2.00$ m. The values of m and v are given in the problem statement.

b. and c. Our sketch shows that $r_\perp$ is the side of the right triangle opposite to the angle of 45°. It follows that $r_\perp = r \sin 45°$.

SOLUTION

Part (a)

1. Evaluate $L = rmv \sin \theta$: $L = rmv \sin \theta = (2.00 \text{ m})(21.2 \text{ kg})(4.10 \text{ m/s}) \sin 135° = 123 \text{ kg} \cdot \text{m}^2/\text{s}$

Part (b)

2. Calculate the moment arm, $r_\perp$: $r_\perp = r \sin 45° = (2.00 \text{ m}) \sin 45° = 1.41 \text{ m}$

Part (c)

3. Evaluate $L = r_\perp mv$: $L = r_\perp mv = (1.41 \text{ m})(21.2 \text{ kg})(4.10 \text{ m/s}) = 123 \text{ kg} \cdot \text{m}^2/\text{s}$

INSIGHT

When the child lands on the merry-go-round, she will transfer angular momentum to it, causing the merry-go-round to rotate about its center. This will be discussed in more detail in the next section.

Notice that we use 45° in $r_\perp = r \sin 45°$ because we calculate the length of the opposite side of the right triangle indicated in our sketch. We could have used $r_\perp = r \sin 135°$ just as well, using the same angle as in $L = rmv \sin 135°$. The results are the same in either case, since $\sin 135° = \sin 45°$.

PRACTICE PROBLEM

For what angle relative to the radial line does the child have a maximum angular momentum? What is the angular momentum in this case? [**Answer:** $\theta = 90°$, for which $L = rmv = 174\ \text{kg} \cdot \text{m}^2/\text{s}$]

Some related homework problems: Problem 56, Problem 57, Problem 58

Next, we consider the rate of change of angular momentum with time. Since the moment of inertia is a constant—as long as the mass and shape of the object remain unchanged—the change in L in a time interval Δt is

$$\frac{\Delta L}{\Delta t} = I\frac{\Delta \omega}{\Delta t}$$

Recall, however, that $\Delta\omega/\Delta t$ is the angular acceleration, α. Therefore, we have

$$\frac{\Delta L}{\Delta t} = I\alpha$$

Since $I\alpha$ is the torque, it follows that Newton's second law for rotational motion can be written as

Newton's Second Law for Rotational Motion

$$\sum \tau = I\alpha = \frac{\Delta L}{\Delta t} \qquad 11\text{–}14$$

Clearly, this is the rotational analogue of $\Sigma F_x = ma_x = \Delta p_x/\Delta t$. Just as force can be expressed as the change in *linear* momentum in a given time interval, the torque can be expressed as the change in *angular* momentum in a time interval.

EXERCISE 11–4

In a light wind, a windmill experiences a constant torque of 255 N · m. If the windmill is initially at rest, what is its angular momentum 2.00 s later?

SOLUTION

Solve Equation 11–14 for the change in angular momentum due to a single torque τ:

$$\Delta L = L_f - L_i = \left(\sum \tau\right)\Delta t = \tau\,\Delta t$$

Since the initial angular momentum of the windmill is zero, its final angular momentum is

$$L_f = \tau\,\Delta t = (255\ \text{N} \cdot \text{m})(2.00\ \text{s}) = 510\ \text{kg} \cdot \text{m}^2/\text{s}$$

11–7 Conservation of Angular Momentum

When an ice skater goes into a spin and pulls her arms inward to speed up, she probably doesn't think about angular momentum. Neither does a diver who springs into the air and folds her body to speed her rotation. Most people, in fact, are not aware that the actions of these athletes are governed by the same basic laws of physics that cause a collapsing star to spin faster as it becomes a rapidly rotating pulsar. Yet in all these cases, as we shall see, **conservation of angular momentum** is at work.

To see the origin of angular momentum conservation, consider an object with an initial angular momentum L_i acted on by a single torque τ. After a period of time, Δt, the object's angular momentum changes in accordance with Newton's second law:

$$\tau = \frac{\Delta L}{\Delta t}$$

Solving for ΔL, we find

$$\Delta L = L_f - L_i = \tau\,\Delta t$$

Thus, the final angular momentum of the object is

$$L_f = L_i + \tau\,\Delta t$$

▲ Once she has launched herself into space, this diver is essentially a projectile. However, the principle of conservation of angular momentum allows her to control the rotational part of her motion. By curling her body up into a tight "tuck," she decreases her moment of inertia, thereby increasing the speed of her spin. To slow down for an elegant entry into the water, she will extend her body, increasing her moment of inertia.

If the torque acting on the object is zero, $\tau = 0$, it follows that the initial and final angular momenta are equal—that is, the angular momentum is conserved:

$$L_f = L_i \qquad (\text{if } \tau = 0)$$

Angular momentum is also conserved in systems acted on by more than one torque, provided that the *net external torque* is zero. The reason that internal torques can be ignored is that, just as internal forces come in equal and opposite pairs that cancel, so too do internal torques. As a result, the internal torques in a system sum to zero, and the net torque acting on it is simply the net external torque. Thus, for a general system, angular momentum is conserved if $\tau_{\text{net, ext}}$ is zero:

Conservation of Angular Momentum

$$L_f = L_i \quad (\text{if } \tau_{\text{net, ext}} = 0) \qquad 11\text{–}15$$

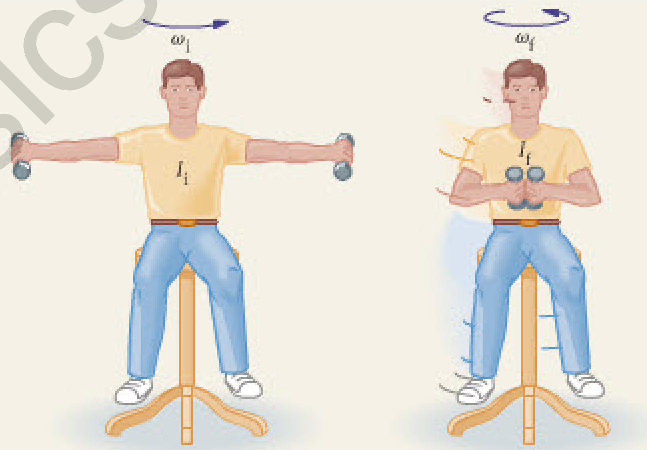
As an illustration of angular momentum conservation, we consider the case of a student rotating on a piano stool in the next Example. Notice how a change in moment of inertia results in a change in angular speed.

EXAMPLE 11–9 GOING FOR A SPIN

For a classroom demonstration, a student sits on a piano stool holding a sizable mass in each hand. Initially, the student holds his arms outstretched and spins about the axis of the stool with an angular speed of 3.72 rad/s. The moment of inertia in this case is 5.33 kg · m^2. While still spinning, the student pulls his arms in to his chest, reducing the moment of inertia to 1.60 kg · m^2. **(a)** What is the student's angular speed now? **(b)** Find the initial and final angular momenta of the student.

PICTURE THE PROBLEM

The initial and final configurations of the student are shown in our sketch. Clearly, the mass distribution in the final configuration, with the masses held closer to the axis of rotation, results in a smaller moment of inertia.



STRATEGY

Ignoring friction in the axis of the stool, since none was mentioned, we conclude that no external torques act on the system. As a result, the angular momentum is conserved. Therefore, setting the initial angular momentum, $L_i = I_i\omega_i$, equal to the final angular momentum, $L_f = I_f\omega_f$, yields the final angular speed.

SOLUTION

Part (a)

1. Apply angular momentum conservation to this system: $L_i = L_f$, $I_i\omega_i = I_f\omega_f$
2. Solve for the final angular speed, ω_f: $\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i$
3. Substitute numerical values: $\omega_f = \left(\frac{5.33 \text{ kg}\cdot\text{m}^2}{1.60 \text{ kg}\cdot\text{m}^2}\right)(3.72 \text{ rad/s}) = 12.4 \text{ rad/s}$

Part (b)

4. Use $L = I\omega$ to calculate the angular momentum. Substitute both initial and final values as a check:
$L_i = I_i\omega_i = (5.33 \text{ kg}\cdot\text{m}^2)(3.72 \text{ rad/s}) = 19.8 \text{ kg}\cdot\text{m}^2/\text{s}$
$L_f = I_f\omega_f = (1.60 \text{ kg}\cdot\text{m}^2)(12.4 \text{ rad/s}) = 19.8 \text{ kg}\cdot\text{m}^2/\text{s}$

INSIGHT

Initially the student completes one revolution roughly every two seconds. After pulling the weights in, the student's rotation rate has increased to almost two revolutions a second—quite a dizzying pace. The same physics applies to a rotating diver or a spinning ice skater.

PRACTICE PROBLEM

What moment of inertia would be required to give a final spin rate of 10.0 rad/s? [**Answer:** $I_f = (\omega_i/\omega_f)I_i = 1.99 \text{ kg}\cdot\text{m}^2$]

Some related homework problems: Problem 65, Problem 67, Problem 74

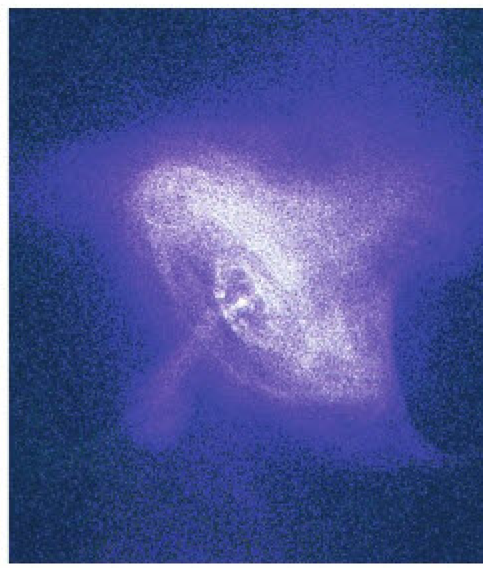
▲ This 1992 satellite photo of Hurricane Andrew (left), one of the most powerful hurricanes of recent decades, clearly suggests the rotating structure of the storm. The violence of the hurricane winds can be attributed in large part to conservation of angular momentum: as air is pushed inward toward the low pressure near the eye of the storm, its rotational velocity increases. The same principle, operating on a smaller scale, explains the tremendous destructive power of tornadoes. The tornado shown at right passed through downtown Miami on May 12, 1997.

An increasing angular speed, as experienced by the student in Example 11–9, can be observed in nature as well. For example, a hurricane draws circulating air in at ground level toward its "eye," where it then rises to an altitude of 10 miles or more. As air moves inward toward the axis of rotation, its angular speed increases, just as the masses held by the student speed up when they are pulled inward. For example, if the wind has a speed of only 3.0 mph at a distance of 300 miles from the center of the hurricane, it would speed up to 150 mph when it comes to within 6.0 miles of the center. Of course, this analysis ignores friction, which would certainly decrease the wind speed. Still, the basic principle—that a decreasing distance from the axis of rotation implies an increasing speed—applies to both the student and the hurricane. Similar behavior is observed in tornadoes and waterspouts.

Another example of conservation of angular momentum occurs in stellar explosions. On occasion a star will explode, sending a portion of its material out into space. After the explosion, the star collapses to a fraction of its original size, speeding up its rotation in the process. If the mass of the star is greater than 1.44 times the mass of the Sun, the collapse can continue until a *neutron star* is formed, with a radius of only about 10 to 20 km. Neutron stars have incredibly high densities; in fact, if you could bring a teaspoonful of neutron star material to the Earth, it would weigh about 100 million tons! In addition, neutron stars produce powerful beams of X-rays and other radiation that sweep across the sky like a gigantic lighthouse beam as the star rotates. On the Earth we see pulses of radiation from these rotating beams, one for each revolution of the star. These "pulsating stars," or *pulsars*, typically have periods ranging from about 2 ms to nearly 1 s. The Crab nebula (see Problems 9 and 106 in Chapter 10) is a famous example of such a system. The dependence of angular speed on radius for a collapsing star is considered in Active Example 11–4.

REAL-WORLD PHYSICS

Hurricanes and tornadoes



▲ Among the fastest rotating objects known in nature are pulsars: stars that have collapsed to a tiny fraction of their original size. Since all the angular momentum of a star must be conserved when it collapses, the dramatic decrease in radius is accompanied by a correspondingly great increase in rotational speed. The Crab nebula pulsar, the remains of a star whose explosion was observed on Earth nearly 1000 years ago, spins at about 30 rev/s. This X-ray photograph shows rings and jets of high-energy particles flying outward from the whirling neutron star at the center.

ACTIVE EXAMPLE 11–4 A STELLAR PERFORMANCE: FIND THE ANGULAR SPEED

A star of radius $R = 2.3 \times 10^8$ m rotates with an angular speed $\omega = 2.4 \times 10^{-6}$ rad/s. If this star collapses to a radius of 20.0 km, find its final angular speed. (Treat the star as if it were a uniform sphere, and assume that no mass is lost as the star collapses.)

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Apply conservation of angular momentum: $I_i\omega_i = I_f\omega_f$

CONTINUED ON NEXT PAGE

REAL-WORLD PHYSICS
Angular speed of a pulsar

CONTINUED FROM PREVIOUS PAGE

2. Write expressions for the initial and final moments of inertia:	$I_i = \frac{2}{5}MR_i^2$ and $I_f = \frac{2}{5}MR_f^2$
3. Solve for the final angular speed:	$\omega_f = (I_i/I_f)\omega_i = (R_i^2/R_f^2)\omega_i$
4. Substitute numerical values:	$\omega_f = 320\ \text{rad/s}$

INSIGHT
The final angular speed corresponds to a period of about 20 ms, a typical period for pulsars. Since 320 rad/s is roughly 3000 rpm, it follows that a pulsar, which has the mass of a star, rotates as fast as the engine in a racing car.

YOUR TURN
At what radius will the star's period of rotation be equal to 15 ms?
(Answers to **Your Turn** *problems are given in the back of the book.)*

Note that if the student in Example 11–9 were to stretch his arms back out again, the resulting *increase* in the moment of inertia would cause a *decrease* in his angular speed. The same effect might apply to the Earth one day. For example, a melting of the polar ice caps would lead to an increase in the Earth's moment of inertia (as we saw in Chapter 10) and thus, by angular momentum conservation, the angular speed of the Earth would decrease. This would mean that more time would be required for the Earth to complete a revolution about its axis of rotation; that is, the day would lengthen.

Since angular momentum is conserved in the systems we have studied so far, it is natural to ask whether the energy is conserved as well. We consider this question in the next Conceptual Checkpoint.

CONCEPTUAL CHECKPOINT 11–4

COMPARE KINETIC ENERGIES

A skater pulls in her arms, decreasing her moment of inertia by a factor of two, and doubling her angular speed. Is her final kinetic energy **(a)** equal to, **(b)** greater than, or **(c)** less than her initial kinetic energy?

REASONING AND DISCUSSION
Let's calculate the initial and final kinetic energies, and compare them. The initial kinetic energy is

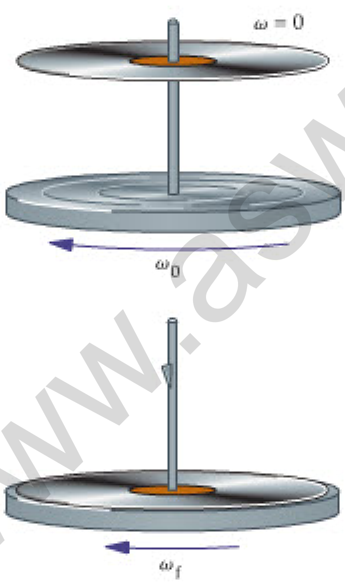
$$K_i = \tfrac{1}{2}I_i\omega_i^2$$

After pulling in her arms, the skater has half the moment of inertia and twice the angular speed. Hence, her final kinetic energy is

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}(I_i/2)(2\omega_i)^2 = 2\left(\tfrac{1}{2}I_i\omega_i^2\right) = 2K_i$$

Thus, the fact that K depends on the square of ω leads to an increase in the kinetic energy. The source of this additional energy is the work done by the muscles in the skater's arms as she pulls them in to her body.

ANSWER
(b) The skater's kinetic energy increases.



▲ FIGURE 11–14 A rotational collision
A nonrotating record dropped onto a rotating turntable is an example of a "rotational collision." Since only internal forces are involved during the collision, the final angular momentum is equal to the initial angular momentum.

Rotational Collisions

In the not-too-distant past, a person would play music by placing a record on a rotating turntable. Suppose, for example, that a turntable with a moment of inertia I_t is rotating freely with an initial angular speed ω_0. A record, with a moment of inertia I_r and initially at rest, is dropped straight down onto the rotating turntable, as in Figure 11–14. When the record lands, frictional forces between it and the turntable cause the record to speed up and the turntable to slow down, until they both have the same angular speed. Since only internal forces are involved during this process, it follows that the system's angular momentum is conserved. We can think of this event, then, as a "rotational collision."

Before the collision, the angular momentum of the system is

$$L_i = I_t\omega_0$$

After the collision, when both the record and the turntable are rotating with the angular speed ω_f, the system's angular momentum is

$$L_f = I_t\omega_f + I_r\omega_f$$

Setting $L_f = L_i$ yields the final angular speed:

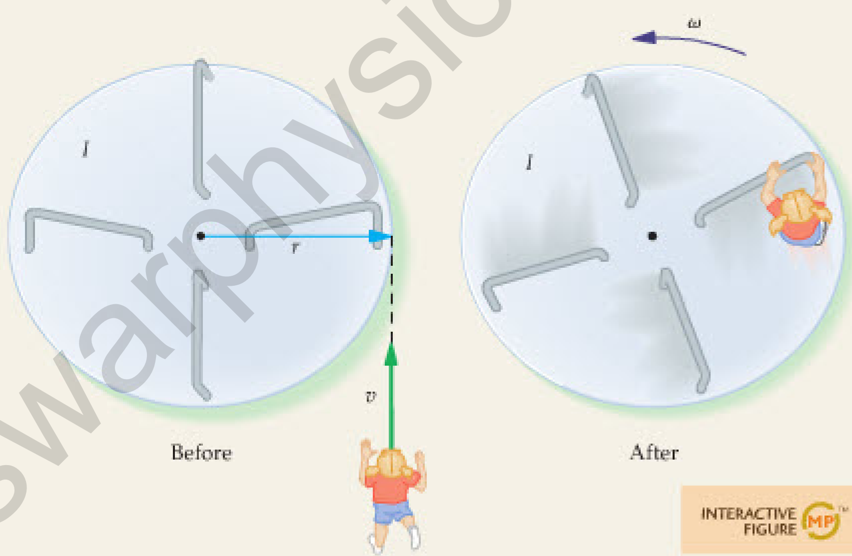
$$\omega_f = \left(\frac{I_t}{I_t + I_r}\right)\omega_0 \qquad \text{11–16}$$

Since this collision is completely inelastic, we expect the final kinetic energy to be less than the initial kinetic energy.

We conclude this section with a somewhat different example of a rotational collision. The physical principles involved are precisely the same, however.

ACTIVE EXAMPLE 11–5 CONSERVE ANGULAR MOMENTUM: FIND THE ANGULAR SPEED

A 34.0-kg child runs with a speed of 2.80 m/s tangential to the rim of a stationary merry-go-round. The merry-go-round has a moment of inertia of 512 $kg \cdot m^2$ and a radius of 2.31 m. When the child jumps onto the merry-go-round, the entire system begins to rotate. What is the angular speed of the system?



INTERACTIVE FIGURE

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Write the initial angular momentum of the child:	$L_i = rmv$
2. Write the final angular momentum of the system:	$L_f = (I + mr^2)\omega$
3. Set $L_f = L_i$ and solve for the angular speed:	$\omega = rmv/(I + mr^2)$
4. Substitute numerical values:	$\omega = 0.317$ rad/s

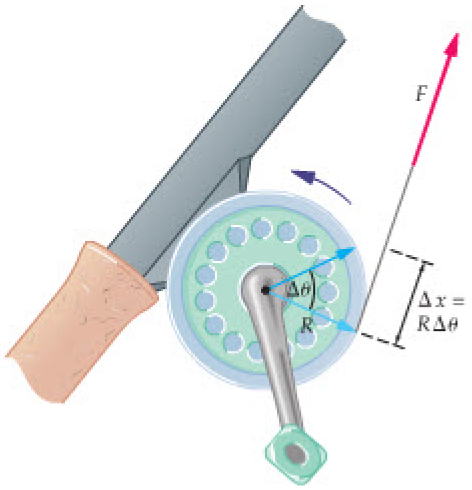
INSIGHT

If the moment of inertia of the merry-go-round had been zero, $I = 0$, the angular speed would be $\omega = v/r$. This means that the linear speed of the child, $r\omega = v$, is unchanged. If $I > 0$, however, the linear speed of the child is decreased. In this particular case, the child's linear speed after the collision is only $v = r\omega = 0.733$ m/s.

YOUR TURN

What initial speed does the child have if, after landing on the merry-go-round, it takes her 22.5 s to complete one revolution?

(Answers to **Your Turn** *problems are given in the back of the book.)*



▲ **FIGURE 11–15 Rotational work**
A force F pulling a length of line Δx from a fishing reel does the work $W = F\,\Delta x$. In terms of torque and angular displacement, the work can be expressed as $W = \tau\,\Delta\theta$.

The initial and final kinetic energies of the system in Active Example 11–5 are considered in Problem 66.

11–8 Rotational Work and Power

Just as a force acting through a distance performs work on an object, so too does a torque acting through an angular displacement. To see this, consider again the fishing line pulled from a reel. If the line is pulled with a force F for a distance Δx, as in Figure 11–15, the work done on the reel is

$$W = F\Delta x$$

Now, since the line is unwinding without slipping, it follows that the linear displacement of the line, Δx, is related to the angular displacement of the reel, $\Delta\theta$, by the following relation:

$$\Delta x = R\Delta\theta$$

In this equation, R is the radius of the reel, and $\Delta\theta$ is measured in radians. Thus, the work can be written as

$$W = F\Delta x = FR\Delta\theta$$

Finally, the torque exerted on the reel by the line is $\tau = RF$, and hence the work done on the reel is simply torque times angular displacement:

Work Done by Torque

$$W = \tau\Delta\theta \qquad 11\text{–}17$$

Note again the analogies between angular and linear quantities in $W = F\Delta x$ and $W = \tau\Delta\theta$. As usual, τ is the analogue of F, and θ is the analogue of x.

As we saw in Chapter 7, the net work done on an object is equal to the change in its kinetic energy. This is the work–energy theorem:

$$W = \Delta K = K_f - K_i \qquad 11\text{–}18$$

The work-energy theorem applies regardless of whether the work is done by a force acting through a distance or a torque acting through an angle.

Similarly, power is the amount of work done in a given time, regardless whether the work is done by a force or a torque. In the case of a torque, we have $W = \tau\Delta\theta$, and hence

Power Produced by a Torque

$$P = \frac{W}{\Delta t} = \tau\frac{\Delta\theta}{\Delta t} = \tau\omega \qquad 11\text{–}19$$

Again, the analogy is clear between $P = Fv$ for the linear case, and $P = \tau\omega$ for the rotational case.

EXERCISE 11–5

It takes a good deal of effort to make homemade ice cream. **(a)** If the torque required to turn the handle on an ice cream maker is 5.7 N·m, how much work is expended on each complete revolution of the handle? **(b)** How much power is required to turn the handle if each revolution is completed in 1.5 s?

SOLUTION

a. Applying Equation 11–17 yields

$$W = \tau\Delta\theta = (5.7\ \text{N}\cdot\text{m})(2\pi\ \text{rad}) = 36\ \text{J}$$

b. Power is the work per time; that is,

$$P = W/\Delta t = (36\ \text{J})/(1.5\ \text{s}) = 24\ \text{W}$$

Equivalently, the angular speed of the handle is $\omega = (2\pi)/T = (2\pi)/(1.5\ \text{s}) = 4.2\ \text{rad/s}$, and therefore Equation 11–19 yields $P = \tau\omega = (5.7\ \text{N}\cdot\text{m})(4.2\ \text{rad/s}) = 24\ \text{W}$.

*11–9 The Vector Nature of Rotational Motion

We have mentioned many times that the angular velocity is a vector, and that we must be careful to use the proper sign for ω. But if the angular velocity is a vector, what is its direction?

To address this question, consider the rotating wheel shown in **Figure 11–16**. Each point on the rim of this wheel has a velocity vector pointing in a different direction in the plane of rotation. Since different parts of the wheel move in different directions, how can we assign a single direction to the angular velocity vector, $\vec{\omega}$? The answer is that there is only one direction that remains fixed as the wheel rotates; the direction of the axis of rotation. By definition, then, the angular velocity vector, $\vec{\omega}$, is taken to point along the axis of rotation.

Given that $\vec{\omega}$ points along the axis of rotation, we must still decide whether it points to the left or to the right in Figure 11–16. The convention we use for assigning the direction of $\vec{\omega}$ is referred to as the right-hand rule:

Right-Hand Rule for the Angular Velocity, $\vec{\omega}$
Curl the fingers of the right hand in the direction of rotation.
The thumb now points in the direction of the angular velocity, $\vec{\omega}$.

The right-hand rule for $\vec{\omega}$ is illustrated in Figure 11–16.

The same convention for direction applies to the angular momentum vector. First, recall that the angular momentum has a magnitude given by $L = I\omega$. Hence, we choose the direction of $\vec{L}$ to be the same as the direction of $\vec{\omega}$. That is

$$\vec{L} = I\vec{\omega} \qquad 11\text{–}20$$

The angular momentum vector is also illustrated in Figure 11–16.

Similarly, torque is a vector, and it too is defined to point along the axis of rotation. The right-hand rule for torque is similar to that for angular velocity:

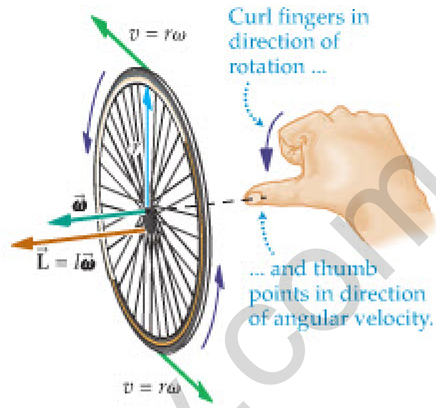
Right-Hand Rule for Torque, $\vec{\tau}$
Curl the fingers of the right hand in the direction of rotation that this torque would cause if it acted alone.
The thumb now points in the direction of the torque vector, $\vec{\tau}$.

Examples of torque vectors are given in **Figure 11–17**.

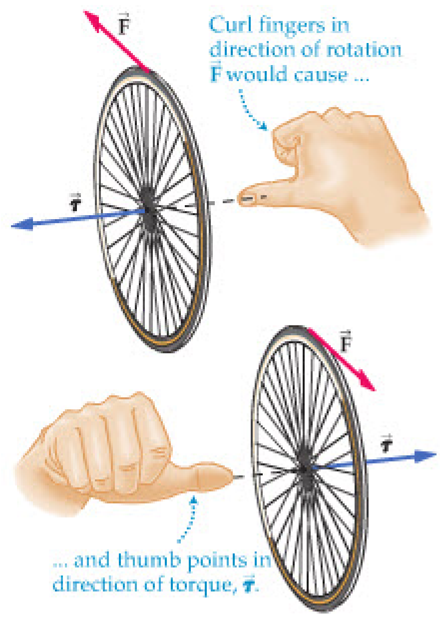
As an example of torque and angular momentum vectors, consider the spinning bicycle wheel shown in **Figure 11–18**. The angular momentum vector for the wheel points to the left, along the axis of rotation. If a person pushes on the rim of the wheel in the direction indicated, the resulting torque is also to the left, as shown in the figure. If this torque lasts for a time Δt, the angular momentum changes by the amount

$$\Delta\vec{L} = \vec{\tau}\,\Delta t$$

Adding $\Delta\vec{L}$ to the original angular momentum $\vec{L}_i$ yields the final angular momentum, $\vec{L}_f$, shown in Figure 11–18. Since $\vec{L}_f$ is in the same direction as $\vec{L}_i$, but with a greater magnitude, it follows that the wheel is spinning in the same direction as



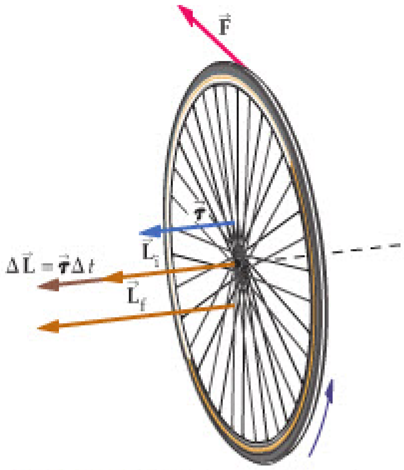
▲ **FIGURE 11–16 The right-hand rule for angular velocity**
The angular velocity, $\vec{\omega}$, of a rotating wheel points along the axis of rotation. Its direction is given by the right-hand rule.



▲ **FIGURE 11–17 The right-hand rule for torque**
Examples of torque vectors obtained using the right-hand rule.

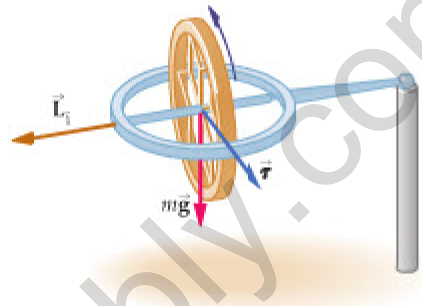
◀ Children have always been fascinated by tops—but not only children. The physicists in the photo at right, Wolfgang Pauli and Niels Bohr, seem as delighted by a spinning top as any child. Their contributions to modern physics, discussed in Chapter 30, helped to show that subatomic particles, the ultimate constituents of matter, have a property (now referred to as "spin") that is in some ways analogous to the rotation of a top or a gyroscope.

▲ The 1.5-inch fused quartz sphere shown here is no ordinary ball. In fact, it is the most perfect sphere ever manufactured. If the Earth were this smooth, the change in elevation from the deepest ocean trench to the highest mountain peak would be only 16 feet. Such precision is required because this sphere is designed to serve as the rotor for an extremely sensitive gyroscope. The device, a million times more sensitive than those used in the best inertial navigation systems, orbits the Earth as part of an experiment to test predictions of Einstein's theory of general relativity.



▲ **FIGURE 11–18 Torque and angular momentum vectors**

A tangential push on the spinning wheel in the direction shown causes a torque to the left. As a result, the angular momentum increases. Hence, the wheel spins faster, as expected.



▲ **FIGURE 11–19 The torque exerted on a gyroscope**

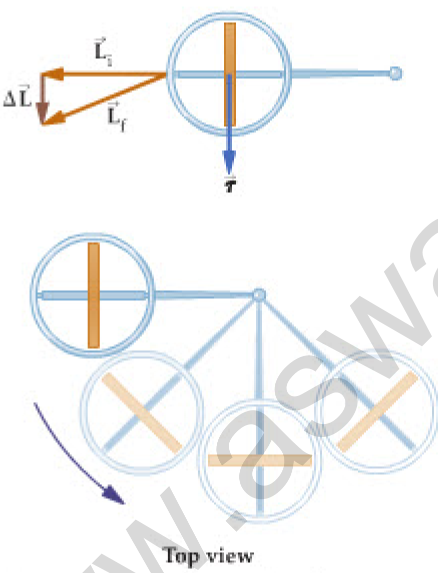
A spinning gyroscope has an initial angular momentum to the left. The torque due to gravity is out of the page.

before, only faster. This is to be expected, considering the direction of the person's push on the wheel.

On the other hand, if a person pushes on the wheel in the opposite direction, the torque vector points to the right. As a result, $\Delta\vec{L}$ points to the right as well. When we add $\Delta\vec{L}$ and $\vec{L}_i$ to obtain the final angular momentum, $\vec{L}_f$, we find that it has the same direction as $\vec{L}_i$, but a smaller magnitude. Hence, we conclude that the wheel spins more slowly, as one would expect.

Finally, a case of considerable interest is when the torque and angular momentum vectors are at right angles to one another. The classic example of such a system is the **gyroscope**. To begin, consider a gyroscope whose axis of rotation is horizontal, as in **Figure 11–19**. If the gyroscope were to be released with no spin it would simply fall, rotating counterclockwise downward about its point of support. Curling the fingers of the right hand in the counterclockwise direction, we see that the thumb, and hence the torque due to gravity, points out of the page.

Next, imagine the gyroscope to be spinning rapidly—as would normally be the case—with its angular momentum pointing to the left in Figure 11–19. If the gyroscope is released now, it doesn't fall as before, even though the torque is the same. To see what happens instead, consider the change in angular momentum, $\Delta\vec{L}$, caused by the torque, $\vec{\tau}$, acting for a small interval of time. As shown in **Figure 11–20**, the small change, $\Delta\vec{L}$, is at right angles to $\vec{L}_i$; hence the final angular momentum, $\vec{L}_f$, is essentially the same length as $\vec{L}_i$, but pointing in a direction slightly out of the page. With each small interval of time, the angular momentum vector continues to change in direction so that, viewed from above as in Figure 11–20, the gyroscope as a whole rotates in a counterclockwise sense around its support point. This type of motion, where the axis of rotation changes direction with time, is referred to as **precession**.



▲ **FIGURE 11–20 Precession of a gyroscope**

The gyroscope as viewed from above. In a time Δt the angular momentum changes by the amount $\Delta\vec{L} = \vec{\tau}\Delta t$. This causes the angular momentum vector, and hence the gyroscope as a whole, to rotate in a counterclockwise direction.

REAL-WORLD PHYSICS
Precession of the Earth

Because of its spinning motion about its rotational axis, the Earth may be considered as one rather large gyroscope. Gravitational forces exerted on the Earth by the Sun and the Moon subject it to external torques that cause its rotational axis to precess. At the moment, the rotational axis of the Earth points toward Polaris, the "North Star," which remains almost fixed in position in time-lapse photographs while the other stars move in circular paths about it. In a few hundred years, however, Polaris will also move in a circular path in the sky because the Earth's axis of rotation will point in a different direction. After 26,000 years the Earth will complete one full cycle of precession, and Polaris will again be the pole star.

On a smaller scale, gyroscopes are used in the navigational systems of a variety of vehicles. In such applications, the rapidly spinning wheel of a gyroscope is mounted on nearly frictionless bearings so that it is practically free from external torques. If no torque acts on the gyroscope, its angular momentum vector remains unchanged both in magnitude and—here is the important point—in direction. With the axis of its gyroscope always pointing in the same, known direction, it is possible for a vehicle to maintain a desired direction of motion relative to the gyroscope's reference direction. On the Hubble Space Telescope, for example, six gyroscopes are used for pointing and stability, though it can operate with only three working gyroscopes if necessary.

REAL-WORLD PHYSICS
Gyroscopes in navigation and space

THE BIG PICTURE PUTTING PHYSICS IN CONTEXT

LOOKING BACK

The concept of force (Chapters 5 and 6) is extended to torque, its rotational equivalent, in Section 11–1. We also apply Newton's laws to rotation in Section 11–6, just as for linear motion in Chapters 5 and 6.

The connection between rotational and linear quantities (Chapter 10) is used in Section 11–2 to relate torque to angular acceleration. In addition, we extend linear momentum (Chapter 9) to angular momentum in Sections 11–6 and 11–7.

Work and kinetic energy (Chapter 7) are applied to rotational systems in Section 11–8.

LOOKING AHEAD

Angular momentum and the conservation of angular momentum play important roles in the study of gravity. See, in particular, the discussion of Kepler's third law in Section 12–3.

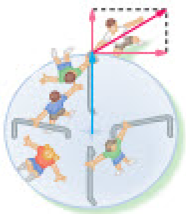
Torque arises in the discussion of magnetic fields and the forces they exert. See Section 22–5 in particular. The torques due to magnetic fields are also the key element in the operation of electric motors, as we see in Section 23–6.

Angular momentum is quantized (given discrete values) in the Bohr model of the hydrogen atom in Section 31–4.

CHAPTER SUMMARY

11–1 TORQUE

A force applied so as to cause an angular acceleration is said to exert a torque, τ.



Tangential Force
A force is tangential if it is tangent to a circle centered on the axis of rotation.

Torque Due to a Tangential Force
A tangential force F applied at a distance r from the axis of rotation produces a torque

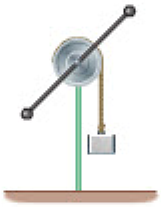
$$\tau = rF \qquad 11\text{–}1$$

Torque for a General Force
A force exerted at an angle θ with respect to the radial direction, and applied at a distance r from the axis of rotation, produces the torque

$$\tau = rF\sin\theta \qquad 11\text{–}2$$

11–2 TORQUE AND ANGULAR ACCELERATION

A single torque applied to an object gives it an angular acceleration.



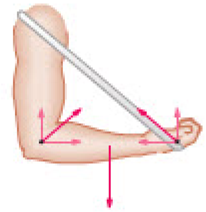
Newton's Second Law for Rotation
The connection between torque and angular acceleration is

$$\sum\tau = I\alpha \qquad 11\text{–}4$$

In this expression, I is the moment of inertia about the axis of rotation and α is the angular acceleration about this axis.

Rotational/Translational Analogies
Torque is analogous to force, the moment of inertia is analogous to mass, and the angular acceleration is analogous to linear acceleration. Therefore, the rotational analogue of $F = ma$ is $\tau = I\alpha$.

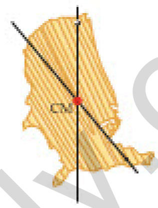
11–3 ZERO TORQUE AND STATIC EQUILIBRIUM



The conditions for an object to be in static equilibrium are that the total force and the total torque acting on the object must be zero:

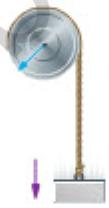
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum \tau = 0$$

11–4 CENTER OF MASS AND BALANCE



An object balances when it is supported at its center of mass.

11–5 DYNAMIC APPLICATIONS OF TORQUE

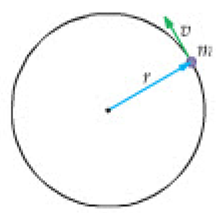


Newton's second law can be applied to rotational systems in a way that is completely analogous to its application to linear systems.

Systems Involving Both Rotational and Linear Elements

In a system with both rotational and linear motions—such as a string passing over a pulley and attached to a mass—Newton's second law must be applied separately to the rotational and linear motions of the system. Connections between the two motions, such as $\alpha = a/r$, can be used to solve for all the accelerations in the system.

11–6 ANGULAR MOMENTUM



A moving object has angular momentum as long as its direction of motion does not extend through the axis of rotation.

Angular Momentum and Angular Speed

Angular momentum can be expressed in terms of angular speed and the moment of inertia as follows:

$$L = I\omega \tag{11–11}$$

This is the rotational analogue of $p = mv$.

Tangential Motion

An object of mass m moving tangentially with a speed v at a distance r from the axis of rotation has an angular momentum, L, given by

$$L = rmv \tag{11–12}$$

General Motion

If an object of mass m is a distance r from the axis of rotation and moves with a speed v at an angle θ with respect to the radial direction, its angular momentum is

$$L = rmv \sin\theta \tag{11–13}$$

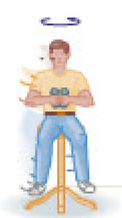
Newton's Second Law

Newton's second law can be expressed in terms of the rate of change of the angular momentum:

$$\sum \tau = I\alpha = \frac{\Delta L}{\Delta t} \tag{11–14}$$

This is the rotational analogue of $\Sigma F = \Delta p/\Delta t$.

11–7 CONSERVATION OF ANGULAR MOMENTUM



If the net external torque acting on a system is zero, its angular momentum is conserved:

$$L_f = L_i$$

Rotational Collisions

Systems in which two rotational objects come into contact can be thought of in terms of a "rotational collision." In such a case, the total angular momentum of the system is conserved.

11–8 ROTATIONAL WORK AND POWER

A torque acting through an angle does work, just as does a force acting through a distance.

Work Done by a Torque

A torque τ acting through an angle $\Delta\theta$ does a work W given by

$$W = \tau\Delta\theta \quad \text{11–17}$$

Work–Energy Theorem

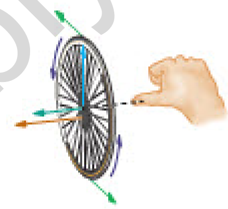
The work–energy theorem is

$$W = \Delta K = K_f - K_i \quad \text{11–18}$$

This theorem applies whether the work is done by a force or by a torque. In the linear case the kinetic energy is $\frac{1}{2}mv^2$; in the rotational case, the kinetic energy is $K = \frac{1}{2}I\omega^2$ (Equation 10–17).

*11–9 THE VECTOR NATURE OF ROTATIONAL MOTION

Rotational quantities have directions that point along the axis of rotation. The precise direction is given by the right-hand rule.



Right-Hand Rule

If the fingers of the *right hand* are curled in the direction of rotation, the thumb points in the direction of the rotational quantity in question. This rule applies to the angular velocity vector, $\vec{\omega}$, the angular acceleration vector, $\vec{\alpha}$, the angular momentum vector, $\vec{L}$, and the torque vector, $\vec{\tau}$.

PROBLEM-SOLVING SUMMARY

Type of Problem	Relevant Physical Concepts	Related Examples
Find the torque exerted on a system.	The torque exerted by a tangential force a distance r from the axis of rotation is $\tau = rF$. If the force is at an angle θ to the radial direction, the torque is $\tau = rF\sin\theta$.	Example 11–1
Determine the angular acceleration of a system.	First, calculate the torque exerted on the system. Next, find the angular acceleration using Newton's second law as applied to rotation, namely, $\tau = I\alpha$.	Examples 11–2, 11–3
Find the forces required for static equilibrium.	Static equilibrium requires that both the net force and the net torque acting on a system be zero.	Examples 11–4, 11–5, 11–6 Active Examples 11–1, 11–2, 11–3
Find the final angular momentum of a system.	A torque changes the angular momentum L of a system with time as follows: $\tau = \Delta L/\Delta t$. If no net torque acts on a system, its angular momentum is conserved.	Examples 11–8, 11–9 Active Examples 11–4, 11–5

CONCEPTUAL QUESTIONS

For instructor-assigned homework, go to www.masteringphysics.com

(Answers to odd-numbered Conceptual Questions can be found in the back of the book.)

1. Two forces produce the same torque. Does it follow that they have the same magnitude? Explain.
2. A car pitches down in front when the brakes are applied sharply. Explain this observation in terms of torques.
3. A tightrope walker uses a long pole to aid in balancing. Why?
4. When a motorcycle accelerates rapidly from a stop it sometimes "pops a wheelie"; that is, its front wheel may lift off the ground. Explain this behavior in terms of torques.
5. Give an example of a system in which the net torque is zero but the net force is nonzero.
6. Give an example of a system in which the net force is zero but the net torque is nonzero.
7. Is the normal force exerted by the ground the same for all four tires on your car? Explain.
8. Give two everyday examples of objects that are not in static equilibrium.
9. Give two everyday examples of objects that are in static equilibrium.
10. Can an object have zero translational acceleration and, at the same time, have nonzero angular acceleration? If your answer is no, explain why not. If your answer is yes, give a specific example.
11. Stars form when a large rotating cloud of gas collapses. What happens to the angular speed of the gas cloud as it collapses?
12. What purpose does the tail rotor on a helicopter serve?

13. Is it possible to change the angular momentum of an object without changing its linear momentum? If your answer is no, explain why not. If your answer is yes, give a specific example.

14. Suppose a diver springs into the air with no initial angular velocity. Can the diver begin to rotate by folding into a tucked position? Explain.

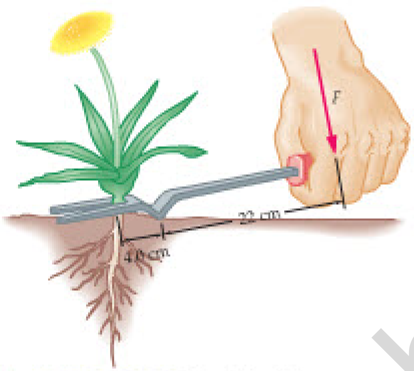
PROBLEMS AND CONCEPTUAL EXERCISES

Note: Answers to odd-numbered Problems and Conceptual Exercises can be found in the back of the book. **IP** *denotes an integrated problem, with both conceptual and numerical parts;* **BIO** *identifies problems of biological or medical interest;* **CE** *indicates a conceptual exercise.* **Predict/Explain** *problems ask for two responses:* **(a)** *your prediction of a physical outcome, and* **(b)** *the best explanation among three provided. On all problems, red bullets (•, ••, •••) are used to indicate the level of difficulty.*

SECTION 11–1 TORQUE

1. • To tighten a spark plug, it is recommended that a torque of 15 N · m be applied. If a mechanic tightens the spark plug with a wrench that is 25 cm long, what is the minimum force necessary to create the desired torque?

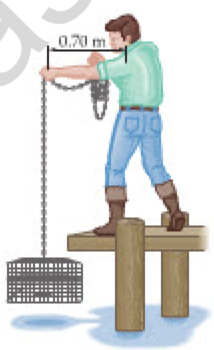
2. • **Pulling a Weed** The gardening tool shown in Figure 11–21 is used to pull weeds. If a 1.23-N · m torque is required to pull a given weed, what force did the weed exert on the tool?



▲ **FIGURE 11–21** Problem 2

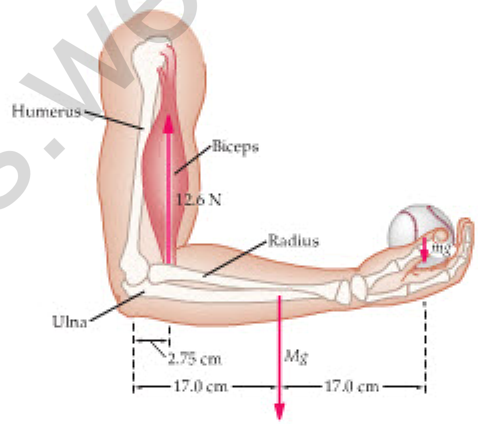
3. • A 1.61-kg bowling trophy is held at arm's length, a distance of 0.605 m from the shoulder joint. What torque does the trophy exert about the shoulder if the arm is **(a)** horizontal, or **(b)** at an angle of 22.5° below the horizontal?

4. • A person slowly lowers a 3.6-kg crab trap over the side of a dock, as shown in Figure 11–22. What torque does the trap exert about the person's shoulder?



▲ **FIGURE 11–22** Problem 4

5. •• **IP BIO Force to Hold a Baseball** A person holds a 1.42-N baseball in his hand, a distance of 34.0 cm from the elbow joint, as shown in Figure 11–23. The biceps, attached at a distance of 2.75 cm from the elbow, exerts an upward force of 12.6 N on the forearm. Consider the forearm and hand to be a uniform rod with a mass of 1.20 kg. **(a)** Calculate the net torque acting on the forearm and hand. Use the elbow joint as the axis of rotation. **(b)** If the net torque obtained in part (a) is nonzero, in which direction will the forearm and hand rotate? **(c)** Would the torque exerted on the forearm by the biceps increase or decrease if the biceps were attached farther from the elbow joint?



▲ **FIGURE 11–23** Problems 5 and 19

6. •• At the local playground, a 16-kg child sits on the end of a horizontal teeter-totter, 1.5 m from the pivot point. On the other side of the pivot an adult pushes straight down on the teeter-totter with a force of 95 N. In which direction does the teeter-totter rotate if the adult applies the force at a distance of **(a)** 3.0 m, **(b)** 2.5 m, or **(c)** 2.0 m from the pivot?

SECTION 11–2 TORQUE AND ANGULAR ACCELERATION

7. • **CE Predict/Explain** Consider the pulley–block systems shown in Conceptual Checkpoint 11–1. **(a)** Is the tension in the string on the left-hand rotating system greater than, less than, or equal to the weight of the mass attached to that string? **(b)** Choose the *best explanation* from among the following:
   I. The mass is in free fall once it is released.
   II. The string rotates the pulley in addition to supporting the mass.
   III. The mass accelerates downward.

8. • **CE Predict/Explain** Consider the pulley–block systems shown in Conceptual Checkpoint 11–1. **(a)** Is the tension in the string on the left-hand rotating system greater than, less than, or equal to the tension in the string on the right-hand rotating system? **(b)** Choose the *best explanation* from among the following:
   I. The mass in the right-hand system has the greater downward acceleration.
   II. The masses are equal.

III. The mass in the left-hand system has the greater downward acceleration.

9. • **CE** Suppose a torque rotates your body about one of three different axes of rotation: case A, an axis through your spine; case B, an axis through your hips; and case C, an axis through your ankles. Rank these three axes of rotation in increasing order of the angular acceleration produced by the torque. Indicate ties where appropriate.

10. • A torque of 0.97 N · m is applied to a bicycle wheel of radius 35 cm and mass 0.75 kg. Treating the wheel as a hoop, find its angular acceleration.

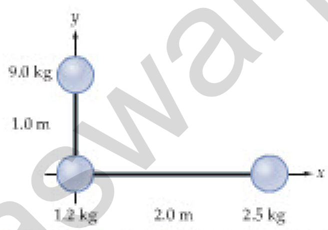
11. • When a ceiling fan rotating with an angular speed of 2.75 rad/s is turned off, a frictional torque of 0.120 N · m slows it to a stop in 22.5 s. What is the moment of inertia of the fan?

12. • When the play button is pressed, a CD accelerates uniformly from rest to 450 rev/min in 3.0 revolutions. If the CD has a radius of 6.0 cm and a mass of 17 g, what is the torque exerted on it?

13. •• A person holds a ladder horizontally at its center. Treating the ladder as a uniform rod of length 3.15 m and mass 8.42 kg, find the torque the person must exert on the ladder to give it an angular acceleration of 0.302 rad/s^2.

14. •• **IP** A wheel on a game show is given an initial angular speed of 1.22 rad/s. It comes to rest after rotating through 0.75 of a turn. **(a)** Find the average torque exerted on the wheel given that it is a disk of radius 0.71 m and mass 6.4 kg. **(b)** If the mass of the wheel is doubled and its radius is halved, will the angle through which it rotates before coming to rest increase, decrease, or stay the same? Explain. (Assume that the average torque exerted on the wheel is unchanged.)

15. •• The L-shaped object in Figure 11–24 consists of three masses connected by light rods. What torque must be applied to this object to give it an angular acceleration of 1.20 rad/s^2 if it is rotated about **(a)** the x axis, **(b)** the y axis, or **(c)** the z axis (which is through the origin and perpendicular to the page)?

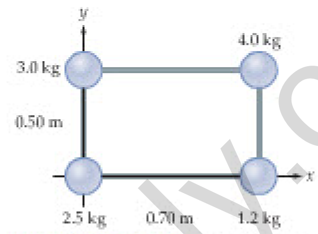


▲ **FIGURE 11–24** Problems 15, 16, and 82

16. •• **CE** The L-shaped object described in Problem 15 can be rotated in one of the following three ways: case A, about the x axis; case B, about the y axis; and case C, about the z axis (which passes through the origin perpendicular to the plane of the figure). If the same torque τ is applied in each of these cases, rank them in increasing order of the resulting angular acceleration. Indicate ties where appropriate.

17. •• **CE** A motorcycle accelerates from rest, and both the front and rear tires roll without slipping. **(a)** Is the force exerted by the ground on the rear tire in the forward or in the backward direction? Explain. **(b)** Is the force exerted by the ground on the front tire in the forward or in the backward direction? Explain. **(c)** If the moment of inertia of the front tire is increased, will the motorcycle's acceleration increase, decrease, or stay the same? Explain.

18. •• **IP** A torque of 13 N · m is applied to the rectangular object shown in Figure 11–25. The torque can act about the x axis, the y axis, or the z axis, which passes through the origin and points out of the page. **(a)** In which case does the object experience the greatest angular acceleration? The least angular acceleration? Explain. Find the angular acceleration when the torque acts about **(b)** the x axis, **(c)** the y axis, and **(d)** the z axis.



▲ **FIGURE 11–25** Problems 18 and 83

19. •• A fish takes the bait and pulls on the line with a force of 2.2 N. The fishing reel, which rotates without friction, is a cylinder of radius 0.055 m and mass 0.99 kg. **(a)** What is the angular acceleration of the fishing reel? **(b)** How much line does the fish pull from the reel in 0.25 s?

20. •• Repeat the previous problem, only now assume the reel has a friction clutch that exerts a restraining torque of 0.047 N · m.

SECTION 11–3 ZERO TORQUE AND STATIC EQUILIBRIUM

21. • **CE Predict/Explain** Suppose the person in Active Example 11–3 climbs higher on the ladder. **(a)** As a result, is the ladder more likely, less likely, or equally likely to slip? **(b)** Choose the *best explanation* from among the following:
   I. The forces are the same regardless of the person's position.
   II. The magnitude of f_2 must increase as the person moves upward.
   III. When the person is higher, the ladder presses down harder on the floor.

22. • A string that passes over a pulley has a 0.321-kg mass attached to one end and a 0.635-kg mass attached to the other end. The pulley, which is a disk of radius 9.40 cm, has friction in its axle. What is the magnitude of the frictional torque that must be exerted by the axle if the system is to be in static equilibrium?

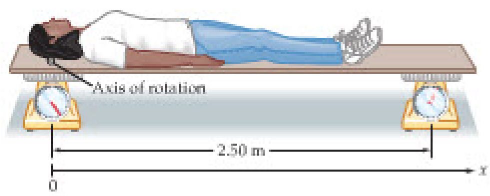
23. • To loosen the lid on a jar of jam 8.9 cm in diameter, a torque of 8.5 N · m must be applied to the circumference of the lid. If a jar wrench whose handle extends 15 cm from the center of the jar is attached to the lid, what is the minimum force required to open the jar?

24. • Consider the system in Active Example 11–1, this time with the axis of rotation at the location of the child. Write out both the condition for zero net force and the condition for zero net torque. Solve for the two forces.

25. •• **IP BIO** Referring to the person holding a baseball in Problem 5, suppose the biceps exert just enough upward force to keep the system in static equilibrium. **(a)** Is the force exerted by the biceps more than, less than, or equal to the combined weight of the forearm, hand, and baseball? Explain. **(b)** Determine the force exerted by the biceps.

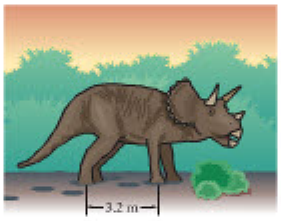
26. •• **IP BIO A Person's Center of Mass** To determine the location of her center of mass, a physics student lies on a lightweight plank supported by two scales 2.50 m apart, as

indicated in Figure 11–26. If the left scale reads 290 N, and the right scale reads 122 N, find **(a)** the student's mass and **(b)** the distance from the student's head to her center of mass.



▲ **FIGURE 11–26** Problem 26

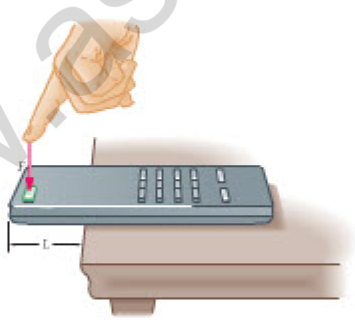
27. •• **Triceratops** A set of fossilized triceratops footprints discovered in Texas show that the front and rear feet were 3.2 m apart, as shown in Figure 11–27. The rear footprints were observed to be twice as deep as the front footprints. Assuming that the rear feet pressed down on the ground with twice the force exerted by the front feet, find the horizontal distance from the rear feet to the triceratops's center of mass.



▲ **FIGURE 11–27** Problem 27

28. •• **IP** A schoolyard teeter-totter with a total length of 5.2 m and a mass of 38 kg is pivoted at its center. A 19-kg child sits on one end of the teeter-totter. **(a)** Where should a parent push vertically downward with a force of 210 N in order to hold the teeter-totter level? **(b)** Where should the parent push with a force of 310 N? **(c)** How would your answers to parts (a) and (b) change if the mass of the teeter-totter were doubled? Explain.

29. •• A 0.122-kg remote control 23.0 cm long rests on a table, as shown in Figure 11–28, with a length L overhanging its edge. To operate the power button on this remote requires a force of 0.365 N. How far can the remote control extend beyond the edge of the table and still not tip over when you press the power button? Assume the mass of the remote is distributed uniformly, and that the power button is 1.41 cm from the overhanging end of the remote.



▲ **FIGURE 11–28** Problem 29

30. •• **IP** A 0.16-kg meterstick is held perpendicular to a vertical wall by a 2.5-m string going from the wall to the far end of the stick. **(a)** Find the tension in the string. **(b)** If a shorter string is used, will its tension be greater than, less than, or the same as that found in part (a)? **(c)** Find the tension in a 2.0-m string.

31. •• Repeat Example 11–4, this time with a uniform diving board that weighs 225 N.

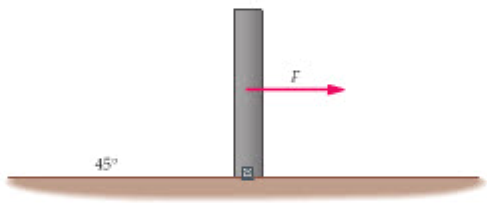
32. •• Babe Ruth steps to the plate and casually points to left center field to indicate the location of his next home run. The mighty Babe holds his bat across his shoulder, with one hand holding the small end of the bat. The bat is horizontal, and the distance from the small end of the bat to the shoulder is 22.5 cm. If the bat has a mass of 1.10 kg and has a center of mass that is 67.0 cm from the small end of the bat, find the magnitude and direction of the force exerted by **(a)** the hand and **(b)** the shoulder.

33. •• A uniform metal rod, with a mass of 3.7 kg and a length of 1.2 m, is attached to a wall by a hinge at its base. A horizontal wire bolted to the wall 0.51 m above the base of the rod holds the rod at an angle of 25° above the horizontal. The wire is attached to the top of the rod. **(a)** Find the tension in the wire. Find **(b)** the horizontal and **(c)** the vertical components of the force exerted on the rod by the hinge.

34. •• **IP** In the previous problem, suppose the wire is shortened, so that the rod now makes an angle of 35° with the horizontal. The wire is horizontal, as before. **(a)** Do you expect the tension in the wire to increase, decrease, or stay the same as a result of its new length? Explain. **(b)** Calculate the tension in the wire.

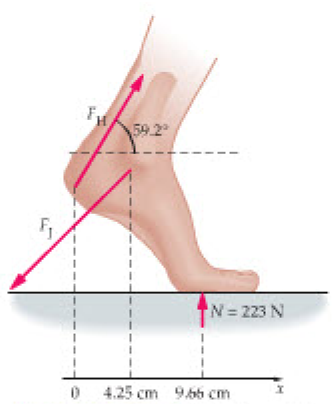
35. •• Repeat Active Example 11–3, this time with a uniform 7.2-kg ladder that is 4.0 m long.

36. •• A rigid, vertical rod of negligible mass is connected to the floor by a bolt through its lower end, as shown in Figure 11–29. The rod also has a wire connected between its top end and the floor. If a horizontal force F is applied at the midpoint of the rod, find **(a)** the tension in the wire, and **(b)** the horizontal and **(c)** the vertical components of force exerted by the bolt on the rod.



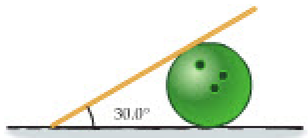
▲ **FIGURE 11–29** Problems 36, 111, and 112

37. ••• **BIO Forces in the Foot** Figure 11–30 shows the forces acting on a sprinter's foot just before she takes off at the start of the race. Find the magnitude of the force exerted on the heel by the Achilles tendon, F_H, and the magnitude of the force exerted on the foot at the ankle joint, F_J.



▲ **FIGURE 11–30** Problem 37

38. ••• A stick with a mass of 0.214 kg and a length of 0.436 m rests in contact with a bowling ball and a rough floor, as shown in Figure 11–31. The bowling ball has a diameter of 21.6 cm, and the angle the stick makes with the horizontal is 30.0°. You may assume there is no friction between the stick and the bowling ball, though friction with the floor must be taken into account. **(a)** Find the magnitude of the force exerted on the stick by the bowling ball. **(b)** Find the horizontal component of the force exerted on the stick by the floor. **(c)** Repeat part (b) for the vertical component of the force.



▲ **FIGURE 11–31** Problem 38

39. ••• **IP** A uniform crate with a mass of 16.2 kg rests on a floor with a coefficient of static friction equal to 0.571. The crate is a uniform cube with sides 1.21 m in length. **(a)** What horizontal force applied to the top of the crate will initiate tipping? **(b)** If the horizontal force is applied halfway to the top of the crate, it will begin to slip before it tips. Explain.

40. ••• In the previous problem, **(a)** what is the minimum height where the force *F* can be applied so that the crate begins to tip before sliding? **(b)** What is the magnitude of the force in this case?

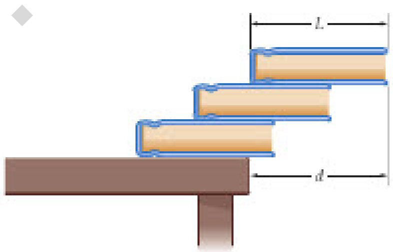
SECTION 11–4 CENTER OF MASS AND BALANCE

41. • A hand-held shopping basket 62.0 cm long has a 1.81-kg carton of milk at one end, and a 0.722-kg box of cereal at the other end. Where should a 1.80-kg container of orange juice be placed so that the basket balances at its center?

42. • If the cat in Active Example 11–2 has a mass of 2.8 kg, how close to the right end of the two-by-four can it walk before the board begins to tip?

43. •• **IP** A 0.34-kg meterstick balances at its center. If a necklace is suspended from one end of the stick, the balance point moves 9.5 cm toward that end. **(a)** Is the mass of the necklace more than, less than, or the same as that of the meterstick? Explain. **(b)** Find the mass of the necklace.

44. •• **Maximum Overhang** Three identical, uniform books of length *L* are stacked one on top the other. Find the maximum overhang distance *d* in Figure 11–32 such that the books do not fall over.



▲ **FIGURE 11–32** Problems 44 and 107

45. •• A baseball bat balances 71.1 cm from one end. If a 0.560-kg glove is attached to that end, the balance point moves 24.7 cm toward the glove. Find the mass of the bat.

SECTION 11–5 DYNAMIC APPLICATIONS OF TORQUE

46. •• A 2.85-kg bucket is attached to a disk-shaped pulley of radius 0.121 m and mass 0.742 kg. If the bucket is allowed to fall, **(a)** what is its linear acceleration? **(b)** What is the angular acceleration of the pulley? **(c)** How far does the bucket drop in 1.50 s?

47. •• **IP** In the previous problem, **(a)** is the tension in the rope greater than, less than, or equal to the weight of the bucket? Explain. **(b)** Calculate the tension in the rope.

48. •• A child exerts a tangential 42.2-N force on the rim of a disk-shaped merry-go-round with a radius of 2.40 m. If the merry-go-round starts at rest and acquires an angular speed of 0.0860 rev/s in 3.50 s, what is its mass?

49. •• **IP** You pull downward with a force of 28 N on a rope that passes over a disk-shaped pulley of mass 1.2 kg and radius 0.075 m. The other end of the rope is attached to a 0.67-kg mass. **(a)** Is the tension in the rope the same on both sides of the pulley? If not, which side has the largest tension? **(b)** Find the tension in the rope on both sides of the pulley.

50. •• Referring to the previous problem, find the linear acceleration of the 0.67-kg mass.

51. ••• A uniform meterstick of mass *M* has an empty paint can of mass *m* hanging from one end. The meterstick and the can balance at a point 20.0 cm from the end of the stick where the can is attached. When the balanced stick–can system is suspended from a scale, the reading on the scale is 2.54 N. Find the mass of **(a)** the meterstick and **(b)** the paint can.

52. ••• **Atwood's Machine** An Atwood's machine consists of two masses, m_1 and m_2, connected by a string that passes over a pulley. If the pulley is a disk of radius *R* and mass *M*, find the acceleration of the masses.

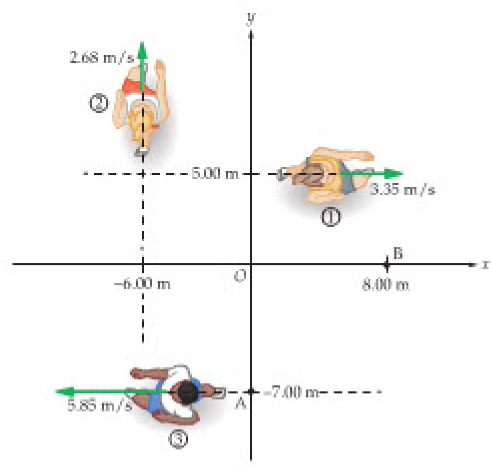
SECTION 11–6 ANGULAR MOMENTUM

53. • Calculate the angular momentum of the Earth about its own axis, due to its daily rotation. Assume that the Earth is a uniform sphere.

54. • A 0.015-kg record with a radius of 15 cm rotates with an angular speed of $33\frac{1}{3}$ rpm. Find the angular momentum of the record.

55. • In the previous problem, a 1.1-g fly lands on the rim of the record. What is the fly's angular momentum?

56. • Jogger 1 in Figure 11–33 has a mass of 65.3 kg and runs in a straight line with a speed of 3.35 m/s. **(a)** What is the magnitude



▲ **FIGURE 11–33** Problems 56, 57, and 58 of the jogger's linear momentum? **(b)** What is the magnitude of the jogger's angular momentum with respect to the origin, *O*?

57. • Repeat the previous problem for the case of jogger 2, whose speed is 2.68 m/s and whose mass is 58.2 kg.

58. •• **IP** Suppose jogger 3 in Figure 11–33 has a mass of 62.2 kg and a speed of 5.85 m/s. **(a)** Is the magnitude of the jogger's angular momentum greater with respect to point A or point B? Explain. **(b)** Is the magnitude of the jogger's angular momentum with respect to point B greater than, less than, or the same as it is with respect to the origin, *O*? Explain. **(c)** Calculate the magnitude of the jogger's angular momentum with respect to points A, B, and *O*.

59. •• A torque of 0.12 N · m is applied to an egg beater. **(a)** If the egg beater starts at rest, what is its angular momentum after 0.65 s? **(b)** If the moment of inertia of the egg beater is 2.5×10^{-3} kg · m^2, what is its angular speed after 0.65 s?

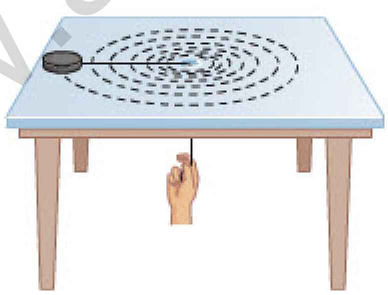
60. •• A windmill has an initial angular momentum of 8500 kg · m^2/s. The wind picks up, and 5.86 s later the windmill's angular momentum is 9700 kg · m^2/s. What was the torque acting on the windmill, assuming it was constant during this time?

61. •• Two gerbils run in place with a linear speed of 0.55 m/s on an exercise wheel that is shaped like a hoop. Find the angular momentum of the system if each gerbil has a mass of 0.22 kg and the exercise wheel has a radius of 9.5 cm and a mass of 5.0 g.

SECTION 11–7 CONSERVATION OF ANGULAR MOMENTUM

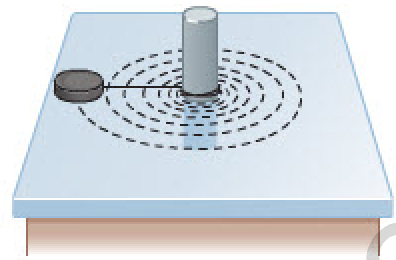
62. • **CE Predict/Explain** A student rotates on a frictionless piano stool with his arms outstretched, a heavy weight in each hand. Suddenly he lets go of the weights, and they fall to the floor. As a result, does the student's angular speed increase, decrease, or stay the same? **(b)** Choose the *best explanation* from among the following:
   I. The loss of angular momentum when the weights are dropped causes the student to rotate more slowly.
   II. The student's moment of inertia is decreased by dropping the weights.
   III. Dropping the weights exerts no torque on the student, but the floor exerts a torque on the weights when they land.

63. • **CE** A puck on a horizontal, frictionless surface is attached to a string that passes through a hole in the surface, as shown in Figure 11–34. As the puck rotates about the hole, the string is pulled downward, bringing the puck closer to the hole. During this process, do the puck's **(a)** linear speed, **(b)** angular speed, and **(c)** angular momentum increase, decrease, or stay the same?



▲ **FIGURE 11–34** Problems 63 and 93

64. • **CE** A puck on a horizontal, frictionless surface is attached to a string that wraps around a pole of finite radius, as shown in Figure 11–35. **(a)** As the puck moves along the spiral path, does its speed increase, decrease, or stay the same? Explain. **(b)** Does its angular momentum increase, decrease, or stay the same? Explain.



▲ **FIGURE 11–35** Problem 64

65. • As an ice skater begins a spin, his angular speed is 3.17 rad/s. After pulling in his arms, his angular speed increases to 5.46 rad/s. Find the ratio of the skater's final moment of inertia to his initial moment of inertia.

66. • Calculate both the initial and the final kinetic energies of the system described in Active Example 11–5.

67. • A diver tucks her body in midflight, decreasing her moment of inertia by a factor of two. By what factor does her angular speed change?

68. •• **IP** In the previous problem, **(a)** does the diver's kinetic energy increase, decrease, or stay the same? **(b)** Calculate the ratio of the final kinetic energy to the initial kinetic energy for the diver.

69. •• A disk-shaped merry-go-round of radius 2.63 m and mass 155 kg rotates freely with an angular speed of 0.641 rev/s. A 59.4-kg person running tangential to the rim of the merry-go-round at 3.41 m/s jumps onto its rim and holds on. Before jumping on the merry-go-round, the person was moving in the same direction as the merry-go-round's rim. What is the final angular speed of the merry-go-round?

70. •• **IP** In the previous problem, **(a)** does the kinetic energy of the system increase, decrease, or stay the same when the person jumps on the merry-go-round? **(b)** Calculate the initial and final kinetic energies for this system.

71. •• A student sits at rest on a piano stool that can rotate without friction. The moment of inertia of the student–stool system is 4.1 kg · m^2. A second student tosses a 1.5-kg mass with a speed of 2.7 m/s to the student on the stool, who catches it at a distance of 0.40 m from the axis of rotation. What is the resulting angular speed of the student and the stool?

72. •• **IP** Referring to the previous problem, **(a)** does the kinetic energy of the mass–student–stool system increase, decrease, or stay the same as the mass is caught? **(b)** Calculate the initial and final kinetic energies of the system.

73. •• **IP** A turntable with a moment of inertia of 5.4×10^{-3} kg · m^2 rotates freely with an angular speed of $33\frac{1}{3}$ rpm. Riding on the rim of the turntable, 15 cm from the center, is a cute, 32-g mouse. **(a)** If the mouse walks to the center of the turntable, will the turntable rotate faster, slower, or at the same rate? Explain. **(b)** Calculate the angular speed of the turntable when the mouse reaches the center.

74. •• A student on a piano stool rotates freely with an angular speed of 2.95 rev/s. The student holds a 1.25-kg mass in each outstretched arm, 0.759 m from the axis of rotation. The combined moment of inertia of the student and the stool, ignoring the two masses, is 5.43 kg · m^2, a value that remains constant. **(a)** As the student pulls his arms inward, his angular speed increases to 3.54 rev/s. How far are the masses from the axis of rotation at this time, considering the masses to be points? **(b)** Calculate the initial and final kinetic energies of the system.

75. ••• **Walking on a Merry-Go-Round** A child of mass m stands at rest near the rim of a stationary merry-go-round of radius R and moment of inertia I. The child now begins to walk around the circumference of the merry-go-round with a tangential speed v with respect to the merry-go-round's surface. **(a)** What is the child's speed with respect to the ground? Check your result in the limits **(b)** $I \rightarrow 0$ and **(c)** $I \rightarrow \infty$.

SECTION 11–8 ROTATIONAL WORK AND POWER

76. • **CE Predict/Explain** Two spheres of equal mass and radius are rolling across the floor with the same speed. Sphere 1 is a uniform solid; sphere 2 is hollow. Is the work required to stop sphere 1 greater than, less than, or equal to the work required to stop sphere 2? **(b)** Choose the *best explanation* from among the following:
   I. Sphere 2 has the greater moment of inertia and hence the greater rotational kinetic energy.
   II. The spheres have equal mass and speed; therefore, they have the same kinetic energy.
   III. The hollow sphere has less kinetic energy.

77. • How much work must be done to accelerate a baton from rest to an angular speed of 7.4 rad/s about its center? Consider the baton to be a uniform rod of length 0.53 m and mass 0.44 kg.

78. • Turning a doorknob through 0.25 of a revolution requires 0.14 J of work. What is the torque required to turn the doorknob?

79. • A person exerts a tangential force of 36.1 N on the rim of a disk-shaped merry-go-round of radius 2.74 m and mass 167 kg. If the merry-go-round starts at rest, what is its angular speed after the person has rotated it through an angle of 32.5°?

80. • To prepare homemade ice cream, a crank must be turned with a torque of 3.95 N · m. How much work is required for each complete turn of the crank?

81. • **Power of a Dental Drill** A popular make of dental drill can operate at a speed of 42,500 rpm while producing a torque of 3.68 oz · in. What is the power output of this drill? Give your answer in watts.

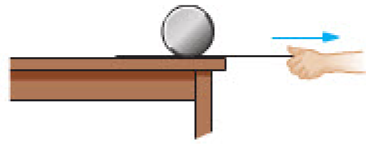
82. •• The L-shaped object in Figure 11–24 consists of three masses connected by light rods. Find the work that must be done on this object to accelerate it from rest to an angular speed of 2.35 rad/s about **(a)** the x axis, **(b)** the y axis, and **(c)** the z axis (which is through the origin and perpendicular to the page).

83. •• The rectangular object in Figure 11–25 consists of four masses connected by light rods. What power must be applied to this object to accelerate it from rest to an angular speed of 2.5 rad/s in 6.4 s about **(a)** the x axis, **(b)** the y axis, and **(c)** the z axis (which is through the origin and perpendicular to the page)?

84. •• **IP** A circular saw blade accelerates from rest to an angular speed of 3620 rpm in 6.30 revolutions. **(a)** Find the torque exerted on the saw blade, assuming it is a disk of radius 15.2 cm and mass 0.755 kg. **(b)** Is the angular speed of the saw blade after 3.15 revolutions greater than, less than, or equal to 1810 rpm? Explain. **(c)** Find the angular speed of the blade after 3.15 revolutions.

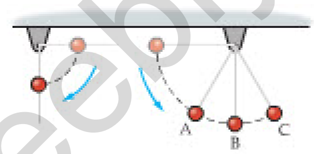
GENERAL PROBLEMS

85. • **CE** A uniform disk stands upright on its edge, and rests on a sheet of paper placed on a tabletop. If the paper is pulled horizontally to the right, as in Figure 11–36, **(a)** does the disk rotate clockwise or counterclockwise about its center? Explain. **(b)** Does the center of the disk move to the right, move to the left, or stay in the same location? Explain.



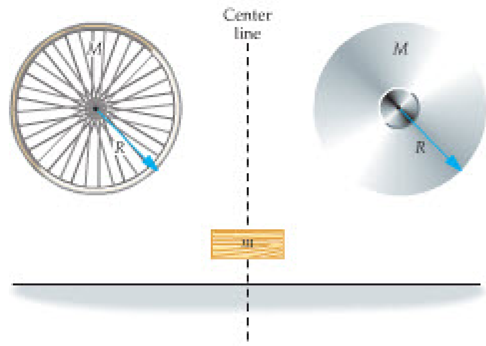
▲ **FIGURE 11–36** Problem 85

86. • **CE** Consider the two rotating systems shown in Figure 11–37, each consisting of a mass m attached to a rod of negligible mass pivoted at one end. On the left, the mass is attached at the midpoint of the rod; to the right, it is attached to the free end of the rod. The rods are released from rest in the horizontal position at the same time. When the rod to the left reaches the vertical position, is the rod to the right not yet vertical (location A), vertical (location B), or past vertical (location C)? Explain.



▲ **FIGURE 11–37** Problem 86

87. • **CE Predict/Explain** A disk and a hoop (bicycle wheel) of equal radius and mass each have a string wrapped around their circumferences. Hanging from the strings, halfway between the disk and the hoop, is a block of mass m, as shown in Figure 11–38. The disk and the hoop are free to rotate about their centers. When the block is allowed to fall, does it stay on the center line, move toward the right, or move toward the left? **(b)** Choose the *best explanation* from among the following:
   I. The disk is harder to rotate, and hence its angular acceleration is less than that of the wheel.
   II. The wheel has the greater moment of inertia and unwinds more slowly than the disk.
   III. The system is symmetric, with equal mass and radius on either side.



▲ **FIGURE 11–38** Problem 87

88. • **CE** A beetle sits at the rim of a turntable that is at rest but is free to rotate about a vertical axis. Suppose the beetle now begins to walk around the perimeter of the turntable. Does the beetle move forward, backward, or does it remain in the same location relative to the ground? Answer for two different cases, **(a)** the turntable is much more massive than the beetle and **(b)** the turntable is massless.

89. • **CE** A beetle sits near the rim of a turntable that is rotating without friction about a vertical axis. The beetle now begins to walk toward the center of the turntable. As a result, does the angular speed of the turntable increase, decrease, or stay the same? Explain.

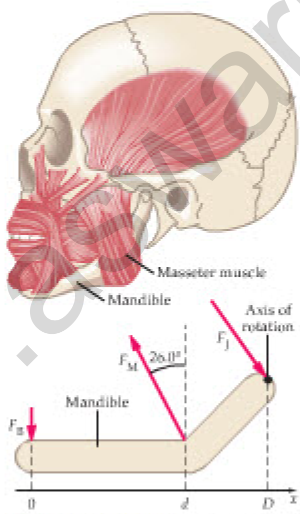
90\. • **CE** Suppose the Earth were to magically expand, doubling its radius while keeping its mass the same. Would the length of the day increase, decrease, or stay the same? Explain.

91\. • After getting a drink of water, a hamster jumps onto an exercise wheel for a run. A few seconds later the hamster is running in place with a speed of 1.3 m/s. Find the work done by the hamster to get the exercise wheel moving, assuming it is a hoop of radius 0.13 m and mass 6.5 g.

92\. •• A 47.0-kg uniform rod 4.25 m long is attached to a wall with a hinge at one end. The rod is held in a horizontal position by a wire attached to its other end. The wire makes an angle of 30.0° with the horizontal, and is bolted to the wall directly above the hinge. If the wire can withstand a maximum tension of 1450 N before breaking, how far from the wall can a 68.0-kg person sit without breaking the wire?

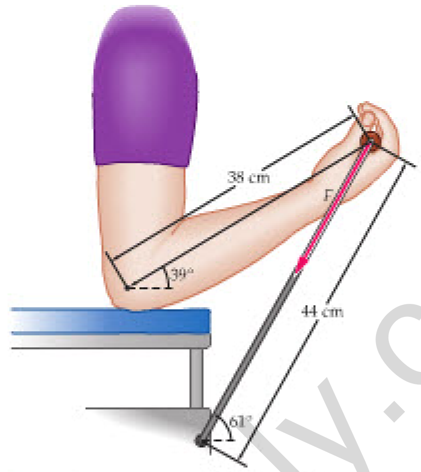
93\. •• **IP** A puck attached to a string moves in a circular path on a frictionless surface, as shown in Figure 11–34. Initially, the speed of the puck is v and the radius of the circle is r. If the string passes through a hole in the surface, and is pulled downward until the radius of the circular path is $r/2$, **(a)** does the speed of the puck increase, decrease, or stay the same? **(b)** Calculate the final speed of the puck.

94\. •• **BIO** **The Masseter Muscle** The masseter muscle, the principal muscle for chewing, is one of the strongest muscles for its size in the human body. It originates on the lower edge of the zygomatic arch (cheekbone) and inserts in the angle of the mandible. Referring to the lower diagram in **Figure 11–39**, where $d = 7.60$ cm and $D = 10.85$ cm, **(a)** find the torque produced about the axis of rotation by the masseter muscle. The force exerted by the masseter muscle is $F_M = 455$ N. **(b)** Find the biting force, F_B, exerted on the mandible by the upper teeth. Find **(c)** the horizontal and **(d)** the vertical component of the force F_J exerted on the mandible at the joint where it attaches to the skull. Assume that the mandible is in static equilibrium, and that upward is the positive vertical direction.



▲ **FIGURE 11–39** Problem 94

95\. •• **Exercising the Biceps** You are designing exercise equipment to operate as shown in **Figure 11–40**, where a person pulls upward on an elastic cord. The cord behaves like an ideal spring and has an unstretched length of 31 cm. If you would like the torque about the elbow joint to be 81 N · m in the position shown, what force constant, k, is required for the cord?



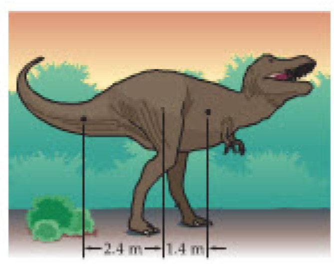
▲ **FIGURE 11–40** Problem 95

96\. •• **Horsepower of a Car** Auto mechanics use the following formula to calculate the horsepower (HP) of a car engine:

$$\text{HP} = \text{Torque} \cdot \text{RPM}/C$$

In this expression, Torque is the torque produced by the engine in ft · lb, RPM is the angular speed of the engine in revolutions per minute, and C is a dimensionless constant. **(a)** Find the numerical value of C. **(b)** The Shelby Series 1 engine is advertised to generate 320 hp at 6500 rpm. What is the corresponding torque produced by this engine? Give your answer in ft · lb.

97\. •• **Balancing a *T. rex*** Paleontologists believe that *Tyrannosaurus rex* stood and walked with its spine almost horizontal, as indicated in **Figure 11–41**, and that its tail was held off the ground to balance its upper torso about the hip joint. Given that the total mass of *T. rex* was 5400 kg, and that the placement of the center of mass of the tail and the upper torso was as shown in Figure 11–41, find the mass of the tail required for balance.

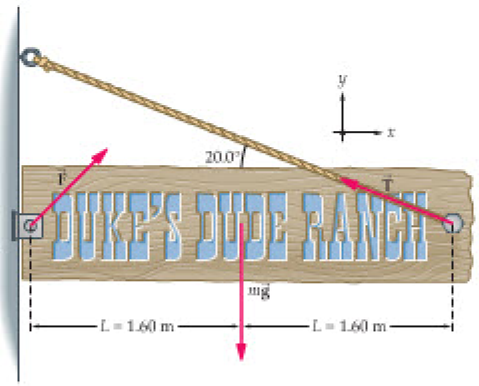


▲ **FIGURE 11–41** Problem 97

98\. •• **IP** You hold a uniform, 28-g pen horizontal with your thumb pushing down on one end and your index finger pushing upward 3.5 cm from your thumb. The pen is 14 cm long. **(a)** Which of these two forces is greater in magnitude? **(b)** Find the two forces.

99\. •• In Active Example 11–3, suppose the ladder is uniform, 4.0 m long, and weighs 60.0 N. Find the forces exerted on the ladder when the person is **(a)** halfway up the ladder and **(b)** three-fourths of the way up the ladder.

100\. •• When you arrive at Duke's Dude Ranch, you are greeted by the large wooden sign shown in **Figure 11–42**. The left end of the sign is held in place by a bolt, the right end is tied to a rope that makes an angle of 20.0° with the horizontal. If the sign is uniform, 3.20 m long, and has a mass of 16.0 kg, what are **(a)** the tension in the rope, and **(b)** the horizontal and vertical components of the force, $\vec{F}$, exerted by the bolt?

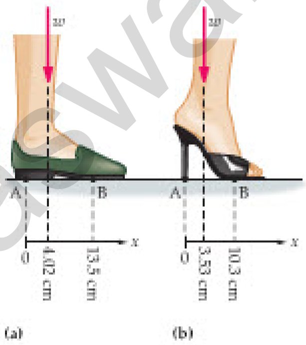


▲ **FIGURE 11–42** Problem 100

101. •• A 67.0-kg person stands on a lightweight diving board supported by two pillars, one at the end of the board, the other 1.10 m away. The pillar at the end of the board exerts a downward force of 828 N. **(a)** How far from that pillar is the person standing? **(b)** Find the force exerted by the second pillar.

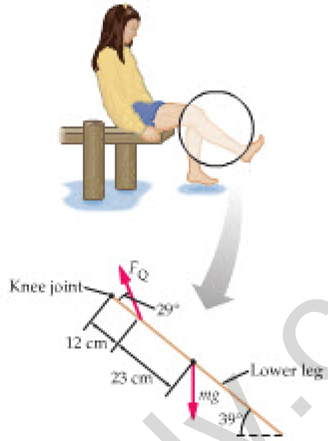
102. •• In Example 11–4, find $\vec{F}_1$ and $\vec{F}_2$ as a function of the distance, x, of the swimmer from the left end of the diving board. Assume that the diving board is uniform and has a mass of 85.0 kg.

103. •• **Flats Versus Heels** A woman might wear a pair of flat shoes to work during the day, as in Figure 11–43 (a), but a pair of high heels, Figure 11–43 (b), when going out for the evening. Assume that each foot supports half her weight, $w = W/2 = 279$ N, and that the forces exerted by the floor on her feet occur at the points A and B in both figures. Find the forces F_A (point A) and F_B (point B) for **(a)** flat shoes and **(b)** high heels. **(c)** How have the high heels changed the weight distribution between the woman's heels and toes?



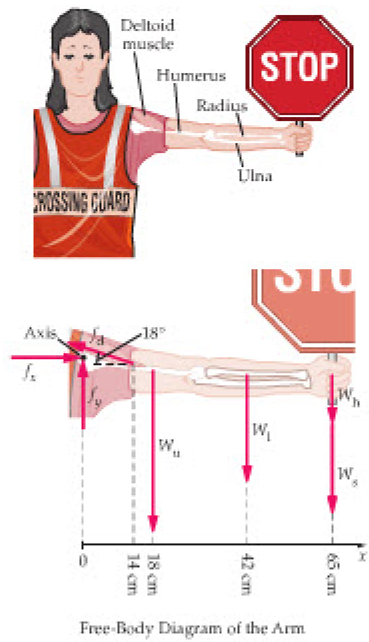
▲ **FIGURE 11–43** Problem 103

104. •• **BIO** A young girl sits at the edge of a dock by the bay, dipping her feet in the water. At the instant shown in Figure 11–44, she holds her lower leg stationary with her quadriceps muscle at an angle of 39° with respect to the horizontal. Use the information given in the figure, plus the fact that her lower leg has a mass of 3.4 kg, to determine the magnitude of the force, F_Q, exerted on the lower leg by the quadriceps.



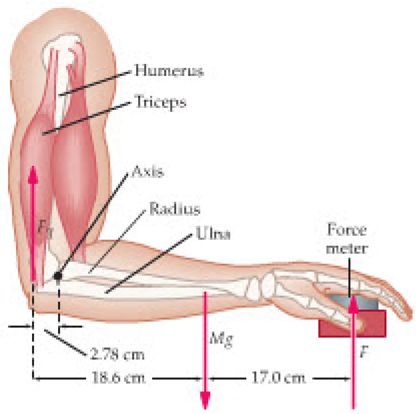
▲ **FIGURE 11–44** Problem 104

105. •• **BIO Deltoid Muscle** A crossing guard holds a STOP sign at arm's length, as shown in Figure 11–45. Her arm is horizontal, and we assume that the deltoid muscle is the only muscle supporting her arm. The weight of her upper arm is $W_u = 18$ N, the weight of her lower arm is $W_l = 11$ N, the weight of her hand is $W_h = 4.0$ N, and the weight of the sign is $W_s = 8.9$ N. The location where each of these forces acts on the arm is indicated in the figure. A force of magnitude f_d is exerted on the humerus by the deltoid, and the shoulder joint exerts a force on the humerus with horizontal and vertical components given by f_x, and f_y, respectively. **(a)** Is the magnitude of f_d greater than, less than, or equal to the magnitude of f_x? Explain. Find **(b)** f_d, **(c)** f_x, and **(d)** f_y. (The weights in Figure 11–45 are drawn to scale; the unknown forces are to be determined. If a force is found to be negative, its direction is opposite to that shown.)



▲ **FIGURE 11–45** Problem 105

106. •• **BIO Triceps** To determine the force a person's triceps muscle can exert, a doctor uses the procedure shown in Figure 11–46, where the patient pushes down with the palm of his hand on a force meter. Given that the weight of the lower arm is $Mg = 15.6$ N, and that the force meter reads $F = 89.0$ N, what is the force F_T exerted vertically upward by the triceps?

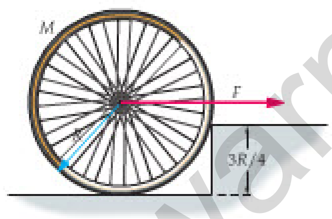


▲ **FIGURE 11–46** Problem 106

107. •• **IP** Suppose a fourth book, the same as the other three, is added to the stack of books shown in Figure 11–32. **(a)** What is the maximum overhang distance, d, in this case? **(b)** If the mass of each book is increased by the same amount, does your answer to part (a) increase, decrease, or stay the same? Explain.

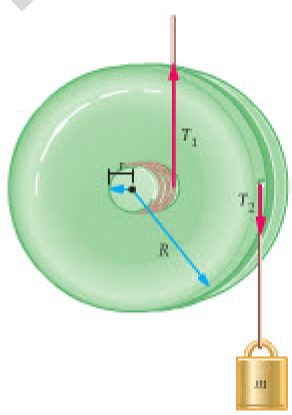
108. •• **IP** Suppose partial melting of the polar ice caps increases the moment of inertia of the Earth from $0.331\ M_E R_E^2$ to $0.332\ M_E R_E^2$. **(a)** Would the length of a day (the time required for the Earth to complete one revolution about its axis) increase or decrease? Explain. **(b)** Calculate the change in the length of a day. Give your answer in seconds.

109. ••• A bicycle wheel of radius R and mass M is at rest against a step of height $3R/4$, as illustrated in **Figure 11–47**. Find the minimum horizontal force F that must be applied to the axle to make the wheel start to rise up over the step.



▲ **FIGURE 11–47** Problem 109

110. ••• A 0.101-kg yo-yo has an outer radius R that is 5.60 times greater than the radius r of its axle. The yo-yo is in equilibrium if a mass m is suspended from its outer edge, as shown in **Figure 11–48**. Find the tension in the two strings, T_1 and T_2, and the mass m.



▲ **FIGURE 11–48** Problem 110

111. ••• In Problem 36, assume that the rod has a mass of M and that its bottom end simply rests on the floor, held in place by static friction. If the coefficient of static friction is μ_s, find the maximum force F that can be applied to the rod at its midpoint before it slips.

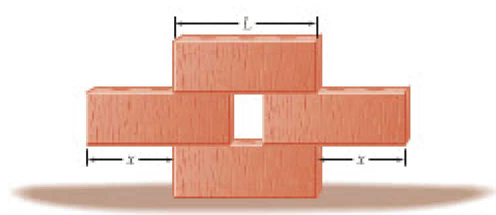
112. ••• In the previous problem, suppose the rod has a mass of 2.3 kg and the coefficient of static friction is 1/7. **(a)** Find the greatest force F that can be applied at the midpoint of the rod without causing it to slip. **(b)** Show that if F is applied 1/8 of the way down from the top of the rod, it will never slip at all, no matter how large the force F.

113. ••• A cylinder of mass m and radius r has a string wrapped around its circumference. The upper end of the string is held fixed, and the cylinder is allowed to fall. Show that its linear acceleration is $(2/3)g$.

114. ••• Repeat the previous problem, replacing the cylinder with a solid sphere. Show that its linear acceleration is $(5/7)g$.

115. ••• A mass M is attached to a rope that passes over a disk-shaped pulley of mass m and radius r. The mass hangs to the left side of the pulley. On the right side of the pulley, the rope is pulled downward with a force F. Find **(a)** the acceleration of the mass, **(b)** the tension in the rope on the left side of the pulley, and **(c)** the tension in the rope on the right side of the pulley. **(d)** Check your results in the limits $m \to 0$ and $m \to \infty$.

116. ••• **Bricks in Equilibrium** Consider a system of four uniform bricks of length L stacked as shown in **Figure 11–49**. What is the maximum distance, x, that the middle bricks can be displaced outward before they begin to tip?



▲ **FIGURE 11–49** Problem 116

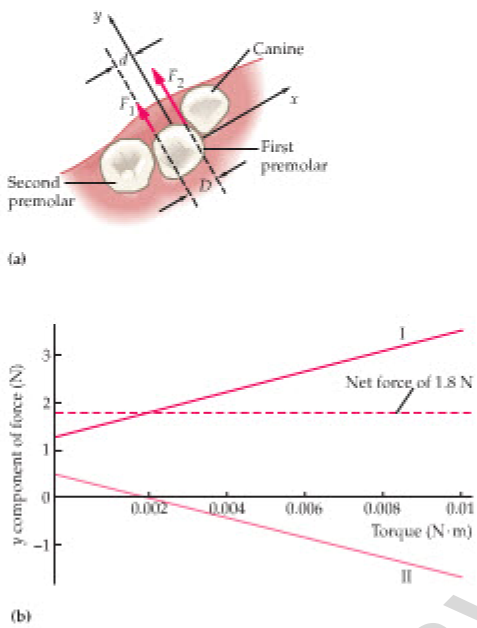
PASSAGE PROBLEMS

BIO Correcting Torsiversion

Torsiversion is a medical condition in which a tooth is rotated away from its normal position about the long axis of the root. Studies show that about 2 percent of the population suffer from this condition to some degree. For those who do, the improper alignment of the tooth can lead to tooth-to-tooth collisions during eating, as well as other problems. Typical patients display a rotation ranging from 20° to 60°, with an average around 30°.

An example is shown in **Figure 11–50 (a)**, where the first premolar is not only displaced slightly from its proper location in the negative y direction, but also rotated clockwise from its normal orientation. To correct this condition, an orthodontist might use an archwire and a bracket to apply both a force and a torque to the tooth. In the simplest case, two forces are applied to the tooth in different locations, as indicated by F_1 and F_2 in Figure 11–50 (a). These two forces, if chosen properly, can reposition the tooth by exerting a net force in the positive y direction, and also reorient it by applying a torque in the counterclockwise direction.

In a typical case, it may be desired to have a net force in the positive y direction of 1.8 N. In addition, the distances in Figure 11–50 (a) can be taken to be $d = 3.2$ mm and $D = 4.5$ mm. Given these conditions, a range of torques is possible for various values of the y components of the forces, F_{1y} and F_{2y}. For example, Figure 11–50 (b) shows the values of F_{1y} and F_{2y} necessary to produce a given torque, where the torque is measured about the center of the tooth (which is also the origin of the coordinate system). Notice that the two forces always add to 1.8 N in the positive y direction, though one of the forces changes sign as the torque is increased.



▲ **FIGURE 11–50** Problems 117, 118, 119, and 120

117. • The two, solid straight lines in Figure 11–50 (b) represent the two forces applied to the tooth. Which line corresponds to which force?

A. I = F_{1y}, II = F_{2y} **B.** I = F_{2y}, II = F_{1y}

118. • What is the value of the torque that corresponds to one of the forces being equal to zero?

A. 0.0023 N · m **B.** 0.0058 N · m

C. 0.0081 N · m **D.** 0.017 N · m

119. •• Find the values of F_{1y} and F_{2y} required to give zero net torque.

A. $F_{1y} = -1.2$ N, $F_{2y} = 3.0$ N **B.** $F_{1y} = 1.1$ N, $F_{2y} = 0.75$ N

C. $F_{1y} = -0.73$ N, $F_{2y} = 2.5$ N **D.** $F_{1y} = 0.52$ N, $F_{2y} = 1.3$ N

120. •• Find the values of F_{1y} and F_{2y} required to give a net torque of 0.0099 N · m. This is a torque that would be effective at rotating the tooth.

A. $F_{1y} = -1.7$ N, $F_{2y} = 3.5$ N **B.** $F_{1y} = -3.8$ N, $F_{2y} = 5.6$ N

C. $F_{1y} = -0.23$ N, $F_{2y} = 2.0$ N **D.** $F_{1y} = 4.0$ N, $F_{2y} = -2.2$ N

INTERACTIVE PROBLEMS

121. •• **Referring to Example 11–7** Suppose the mass of the pulley is doubled, to 0.160 kg, and that everything else in the system remains the same. **(a)** Do you expect the value of T_2 to increase, decrease, or stay the same? Explain. **(b)** Calculate the value of T_2 for this case.

122. •• **Referring to Example 11–7** Suppose the mass of the cart is doubled, to 0.62 kg, and that everything else in the system remains the same. **(a)** Do you expect the value of T_2 to increase, decrease, or stay the same? Explain. **(b)** Calculate the value of T_2 for this case.

123. •• **Referring to Active Example 11–5** Suppose the child runs with a different initial speed, but that everything else in the system remains the same. What initial speed does the child have if the angular speed of the system after the collision is 0.425 rad/s?

124. •• **Referring to Active Example 11–5** Suppose everything in the system is as described in Active Example 11–5 except that the child approaches the merry-go-round in a direction that is not tangential. Find the angle θ between the direction of motion and the outward radial direction (as in Example 11–8) that is required if the final angular speed of the system is to be 0.272 rad/s.

PHYSICS IN

PERSPECTIVE

Momentum: A Conserved Quantity

When objects interact, momentum may be conserved while mechanical energy is dissipated. Why? These pages explore momentum conservation and point out key differences between momentum and mechanical energy.

1 How do linear and angular momentum relate?

The equations of linear and angular momentum are analogous, and all the principles presented on these pages apply to angular as well as linear momentum.

	Definition	Newton's 2nd law
Linear momentum:	$\vec{p} = m\vec{v}$	$\Sigma\vec{F} = m\vec{a} = \dfrac{\Delta\vec{p}}{\Delta t}$
Angular momentum:	$\vec{L} = I\vec{\omega}$	$\Sigma\vec{\tau} = I\vec{\alpha} = \dfrac{\Delta\vec{L}}{\Delta t}$

Analogous quantities

Linear	Angular
Acceleration $\vec{a}$	Angular acceleration $\vec{\alpha}$
Force $\vec{F}$	Torque $\vec{\tau}$
Velocity $\vec{v}$	Angular velocity $\vec{\omega}$
Mass m	Moment of inertia I

2 Why is momentum conserved?

Momentum conservation follows from Newton's laws.

Recall that the general form of Newton's second law relates force to momentum:

An object's change in momentum … … equals the net force acting on the object …

$$\Sigma\vec{F} = \Delta\vec{p}/\Delta t \quad \text{or} \quad \Delta\vec{p} = (\Sigma\vec{F})\Delta t$$

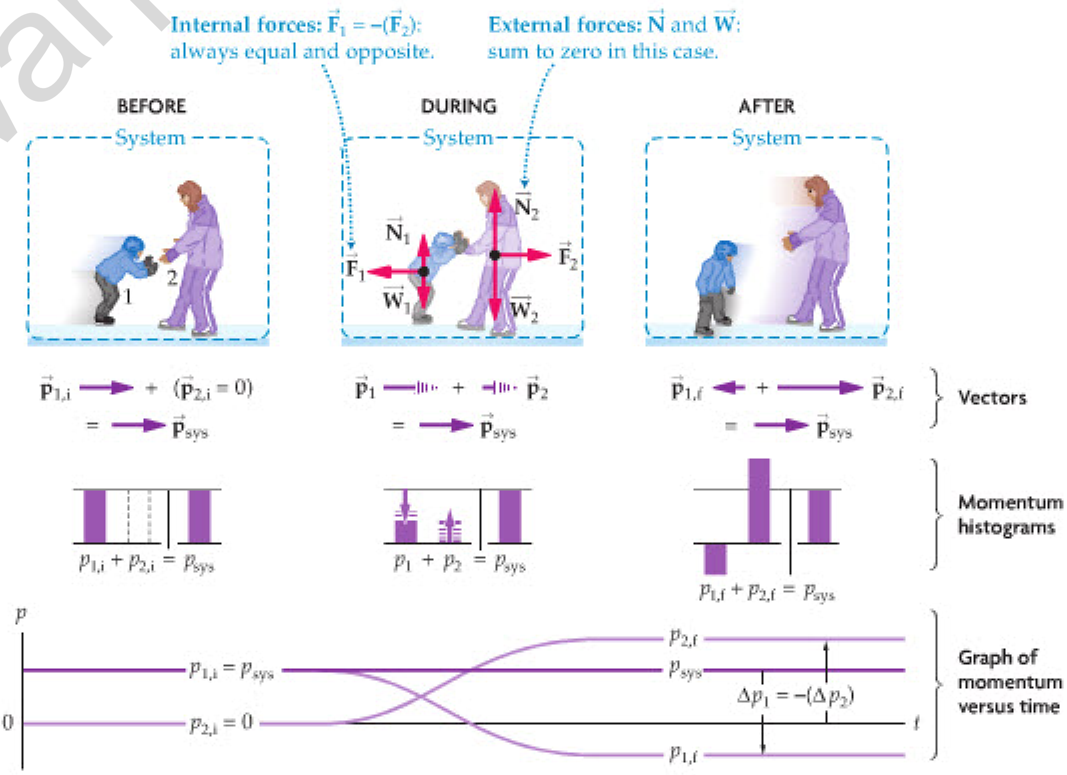
… multiplied by the time over which the force acts.

For an individual object, momentum is conserved (does not change) when the net force acting on the object is zero (that is, $\Delta\vec{p} = 0$ when $\Sigma\vec{F} = 0$).

For a system of objects, momentum conservation follows from Newton's third law:

- The momentum of a system of objects is the vector sum of the momenta of the individual objects.
- The forces between objects in the system (**internal forces**) cannot change the system's momentum because, by Newton's third law, the objects exert *equal but opposite forces* on each other, which cause *equal and opposite momentum changes.*
- **Thus, only external forces can change the momentum of a system.**

In the following interaction, the two skaters undergo equal and opposite momentum changes, whereas the system's momentum $\vec{p}_{sys}$ is conserved.



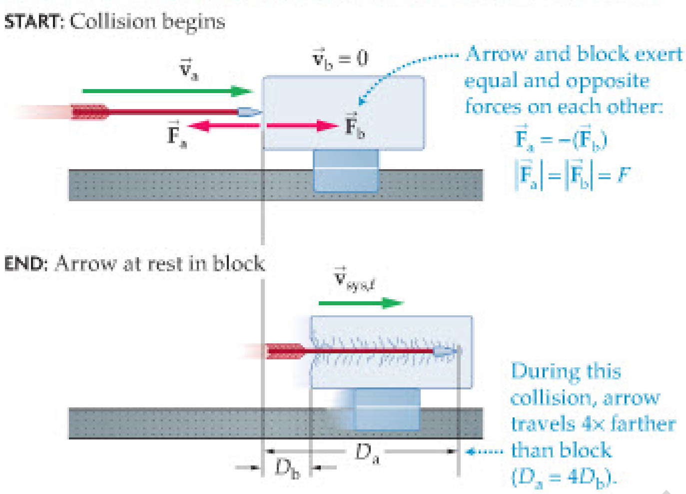
3 How can momentum be conserved when mechanical energy is not?

Force times time versus force times distance: Momentum change is due to *a force acting over a time* Δt, whereas changes in mechanical energy result from *a force acting over a distance D* (i.e., from work):

$$\Delta\vec{p} = \vec{F}(\Delta t) \qquad \Delta E = W = F(D)$$

How do these relationships apply to the inelastic collision shown below?

Arrow shot into styrofoam block attached to air-track cart



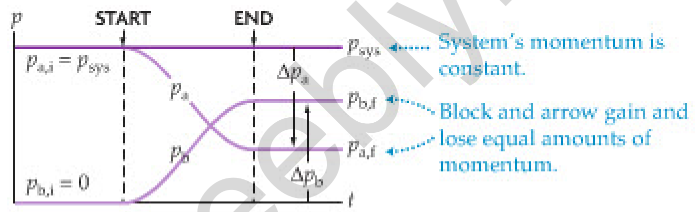
Momentum: The collision lasts the same time Δt for the arrow and block, so their mometum changes are equal and opposite:

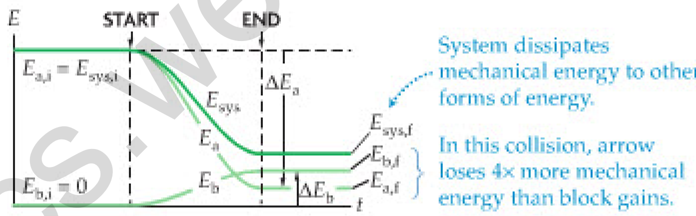
$$\Delta\vec{p}_a = \vec{F}_a\Delta t = -(\vec{F}_b)\Delta t = -\Delta\vec{p}_b$$

Mechanical energy: The objects exert the same force magnitude F on each other, but the arrow travels farther during the collision because it penetrates the block: $D_a > D_b$. Thus, the arrow loses more mechanical energy than the block gains:

$$\Delta E_a = F_a(D_a) = -40 \text{ J} \qquad \Delta E_b = F_b(D_b) = +10 \text{ J}$$

Conclusion: The collision dissipates mechanical energy while conserving momentum, as the following graphs show:



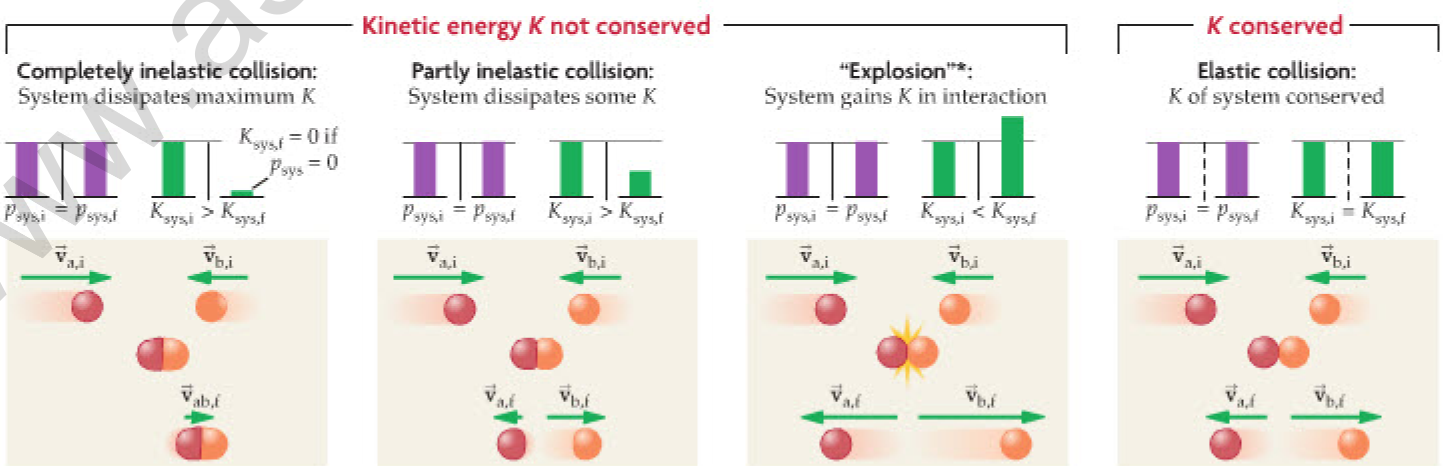


4 How does momentum conservation help us solve problems?

- You can use momentum conservation to analyze any interaction between objects for which the net external force acting on the system during the collision is zero (or is negligible compared to the internal forces).
- If the net external force is not negligible, you cannot use momentum conservation! This applies to the players at right, who push on the ground while colliding.
- For elastic collisions, you must use conservation of mechanical energy as well as conservation of momentum. (Section 9.6 solves these simultaneous equations for special cases.)

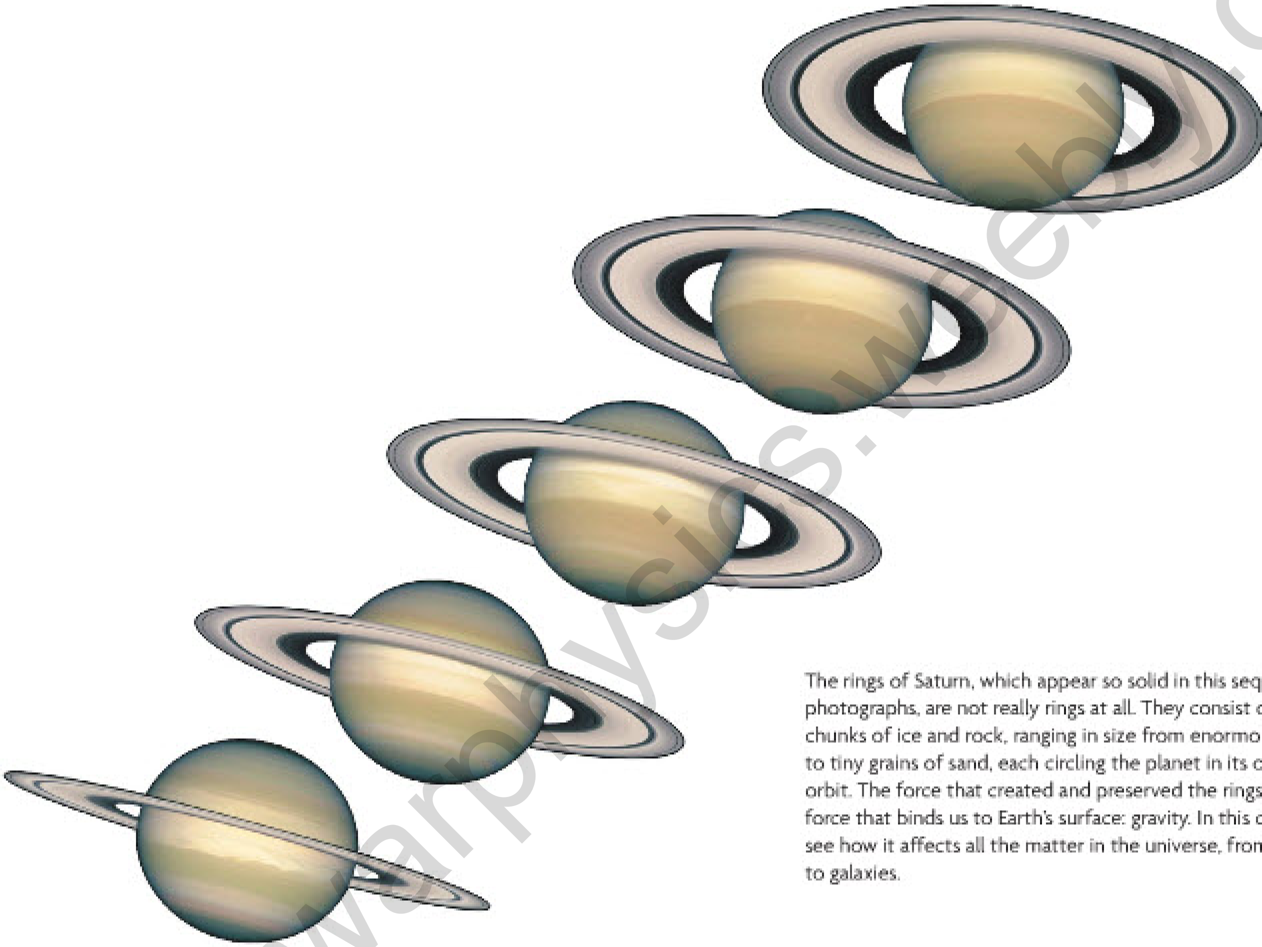
The external reaction forces of the earth on these players' feet cannot be ignored.

Collisions in which momentum is conserved can be categorized as follows, according to how kinetic energy changes or is conserved.



*Physicists use the term "explosion" to mean any interaction that adds kinetic energy to the system. Thus, the collision in Step 2 on the facing page is an explosion.

12 Gravity



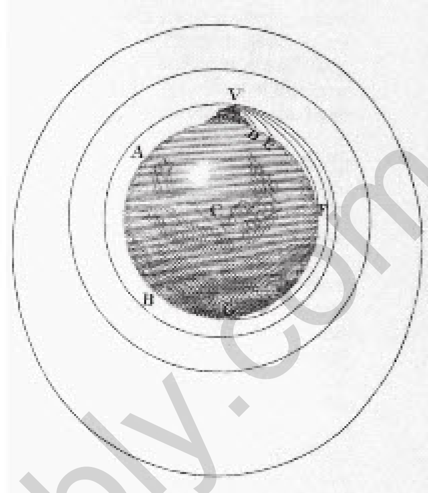
The rings of Saturn, which appear so solid in this sequence of photographs, are not really rings at all. They consist of countless chunks of ice and rock, ranging in size from enormous boulders to tiny grains of sand, each circling the planet in its own individual orbit. The force that created and preserved the rings is the same force that binds us to Earth's surface: gravity. In this chapter we'll see how it affects all the matter in the universe, from dust motes to galaxies.

The study of gravity has always been a central theme in physics, from Galileo's early experiments on free fall in the seventeenth century, to Einstein's general theory of relativity in the early years of the twentieth century, and Stephen Hawking's work on black holes in recent years. Perhaps the grandest milestone in this endeavor, however, was the discovery by Newton of the **universal law of gravitation**. With just one simple equation to describe the force of gravity, Newton was able to determine the orbits of planets, moons, and comets, and to explain such earthly phenomena as the tides and the fall of an apple.

Before Newton's work, it was generally thought that the heavens were quite separate from the Earth, and that they obeyed their own "heavenly" laws. Newton showed, on the contrary, that the same law of gravity that operates on the surface of the Earth applies to the Moon and to other astronomical objects. As a result of Newton's efforts, physics expanded its realm of applicability to natural phenomena throughout the universe.

So successful was Newton's law of gravitation that Edmond Halley (1656–1742) was able to use it to predict the return of the comet that today bears his name. Though he did not live to see its return in 1758, the fact that the comet did reappear when predicted was an event unprecedented in human history. Roughly a hundred years later, Newton's theory of gravity scored an even more impressive success. Astronomers observing the planet Uranus noticed small deviations in its orbit, which they thought might be due to the gravitational tug of a previously unknown planet. Using Newton's law to calculate the predicted position of the new planet—now called Neptune—it was found on the very first night of observations, September 23, 1846. The fact that Neptune was precisely where the law of gravitation said it should be still stands as one of the most astounding triumphs in the history of science.

Today, Newton's law of gravitation is used to determine the orbits that take spacecraft from the Earth to various destinations within our solar system and beyond. Appropriately enough, spacecraft were even sent to view Halley's comet at close range in 1986. In addition, the law allows us to calculate with pinpoint accuracy the time of solar eclipses and other astronomical events in the distant past and remote future. This incredibly powerful and precise law of nature is the subject of this chapter.



▲ In this illustration from his great work, the *Principia*, published in 1687, Newton presents a "thought experiment" to show the connection between free fall and orbital motion. Imagine throwing a projectile horizontally from the top of a mountain. The greater the initial speed of the projectile, the farther it travels in free fall before striking the ground. In the absence of air resistance, a great enough initial speed could result in the projectile circling the Earth and returning to its starting point. Thus, an object orbiting the Earth is actually in free fall—it simply has a large horizontal speed.

12–1 Newton's Law of Universal Gravitation

It's ironic, but the first fundamental force of nature to be recognized as such, **gravity**, is also the weakest of the fundamental forces. Still, it is the force most apparent to us in our everyday lives, and is the force responsible for the motion of the Moon, the Earth, and the planets. Yet the connection between falling objects on Earth and planets moving in their orbits was not known before Newton.

The flash of insight that came to Newton—whether it was due to seeing an apple fall to the ground or not—is simply this: The force causing an apple to accelerate downward is the same force causing the Moon to move in a circular path around the Earth. To put it another way, Newton was the first to realize that the Moon is *constantly falling* toward the Earth, though without ever getting closer to it, and that it falls for the same reason that an apple falls. This is illustrated in a classic drawing due to Newton, shown to the right.

To be specific, in the case of the apple the motion is linear as it accelerates downward toward the center of the Earth. In the case of the Moon the motion is circular with constant speed. As discussed in Section 6–5, an object in uniform circular motion accelerates toward the center of the circle. It follows, therefore, that the Moon *also* accelerates toward the center of the Earth. In fact, the force responsible for the Moon's centripetal acceleration is the Earth's gravitational attraction, the same force responsible for the fall of the apple.

To describe the force of gravity, Newton proposed the following simple law:

Newton's Law of Universal Gravitation

The force of gravity between any two point objects of mass m_1 and m_2 is attractive and of magnitude

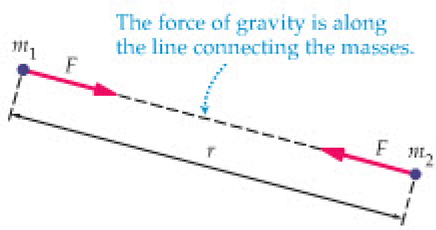
$$F = G\frac{m_1 m_2}{r^2} \qquad 12\text{–}1$$

In this expression, r is the distance between the masses, and G is a constant referred to as the **universal gravitation constant**. Its value is

$$G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \qquad 12\text{–}2$$

The force is directed along the line connecting the masses, as indicated in **Figure 12–1**.

Note that each mass experiences a force of the same magnitude, $F = Gm_1m_2/r^2$, but acting in opposite directions. That is, the force of gravity between two objects forms an action-reaction pair.



▲ **FIGURE 12–1 Gravitational force between point masses**

Two point masses, m_1 and m_2, separated by a distance r exert equal and opposite attractive forces on one another. The magnitude of the forces, F, is given by Equation 12–1.

According to Newton's law, all objects in the universe attract all other objects in the universe by way of the gravitational interaction. It is in this sense that the force law is termed "universal." Thus, the net gravitational force acting on you is due not only to the planet on which you stand, which is certainly responsible for the majority of the net force, but also to people nearby, planets, and even stars in far-off galaxies. In short, everything in the universe "feels" everything else, thanks to gravity.

The fact that G is such a small number means that the force of gravity between objects of human proportions is imperceptibly small. This is shown in the following Exercise.

EXERCISE 12–1

A man takes his dog for a walk on a deserted beach. Treating people and dogs as point objects for the moment, find the force of gravity between the 105-kg man and his 11.2-kg dog when they are separated by a distance of **(a)** 1.00 m and **(b)** 10.0 m.

SOLUTION

a. Substituting numerical values into Equation 12–1 yields

$$F = G\frac{m_1 m_2}{r^2} = (6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})\frac{(105\ \mathrm{kg})(11.2\ \mathrm{kg})}{(1.00\ \mathrm{m})^2} = 7.84 \times 10^{-8}\ \mathrm{N}$$

b. Repeating the calculation for $r = 10.0$ m gives

$$F = G\frac{m_1 m_2}{r^2} = (6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})\frac{(105\ \mathrm{kg})(11.2\ \mathrm{kg})}{(1.00\ \mathrm{m})^2} = 7.84 \times 10^{-10}\ \mathrm{N}$$

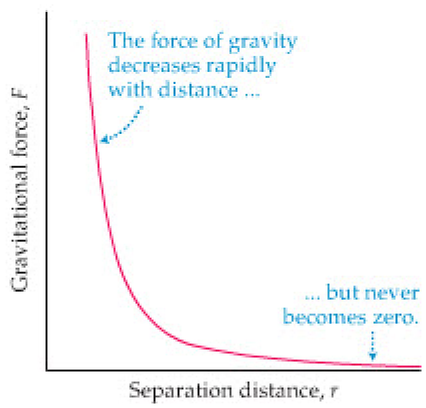


FIGURE 12–2 Dependence of the gravitational force on separation distance, *r*
The $1/r^2$ dependence of the gravitational force means that it decreases rapidly with distance. Still, it never completely vanishes. For this reason, we say that gravity is a force of infinite range; that is, every mass in the universe experiences a nonzero force from every other mass in the universe, no matter how far away.

The forces found in Exercise 12–1 are imperceptibly small. In comparison, the force exerted by the Earth on the man is 1030 N and the force exerted on the dog is 110 N—these forces are several orders of magnitude greater than the force between the man and the dog. In general, gravitational forces are significant only when large masses, such as the Earth or the Moon, are involved.

Exercise 12–1 also illustrates how rapidly the force of gravity decreases with distance. In particular, since F varies as $1/r^2$, it is said to have an **inverse square dependence** on distance. Thus, for example, an increase in distance by a factor of 10 results in a decrease in the force by a factor of $10^2 = 100$. A plot of the force of gravity versus distance is given in Figure 12–2. Note that even though the force diminishes rapidly with distance, it never completely vanishes; thus, we say that gravity is a force of infinite range.

Note also that the force of gravity between two masses depends on the product of the masses, m_1 times m_2. With this type of dependence, it follows that if either mass is doubled, the force of gravity is doubled as well. This would not be the case, for example, if the force of gravity depended on the *sum* of the masses, $m_1 + m_2$.

Finally, if a given mass is acted on by gravitational interactions with a number of other masses, the net force acting on it is simply the vector sum of each of the forces individually. This property of gravity is referred to as **superposition**. As an example, superposition implies that the net gravitational force exerted on you at this moment is the vector sum of the force exerted by the Earth, plus the force exerted by the Moon, plus the force exerted by the Sun, and so on. The following Example illustrates superposition.

PROBLEM-SOLVING NOTE
Net Gravitational Force

To find the net gravitational force acting on an object, you should (i) resolve each of the forces acting on the object into components and (ii) add the forces component by component.

EXAMPLE 12–1 HOW MUCH FORCE IS WITH YOU?

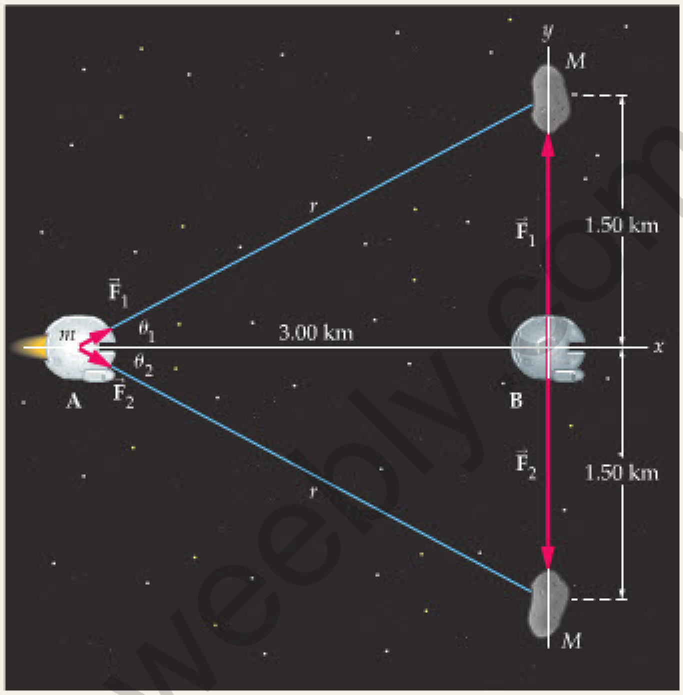
As part of a daring rescue attempt, the *Millennium Eagle* passes between a pair of twin asteroids, as shown. If the mass of the spaceship is 2.50×10^7 kg and the mass of each asteroid is 3.50×10^{11} kg, find the net gravitational force exerted on the *Millennium Eagle* **(a)** when it is at location A and **(b)** when it is at location B. Treat the spaceship and the asteroids as if they were point objects.

PICTURE THE PROBLEM

Our sketch shows the spaceship as it follows a path between the twin asteroids. The relevant distances and masses are indicated, as are the two points of interest, A and B. Note that at location A the force $\vec{F}_1$ points above the x axis at the angle θ_1 (to be determined); the force $\vec{F}_2$ points below the x axis at the angle $\theta_2 = -\theta_1$, as can be seen by symmetry. At location B, the two forces act in opposite directions.

STRATEGY

To find the net gravitational force exerted on the spaceship, we first determine the magnitude of the force exerted on it by each asteroid. This is done by using Equation 12–1 and the distances given in our sketch. Next, we resolve these forces into x and y components. Finally, we sum the force components to find the net force.



SOLUTION

Part (a)

1. Use the Pythagorean theorem to find the distance r from point A to each asteroid. Also, refer to the sketch to find the angle between $\vec{F}_1$ and the x axis. The angle between $\vec{F}_2$ and the x axis has the same magnitude but the opposite sign:

$$r = \sqrt{(3.00 \times 10^3\ \text{m})^2 + (1.50 \times 10^3\ \text{m})^2} = 3350\ \text{m}$$

$$\theta_1 = \tan^{-1}\left(\frac{1.50 \times 10^3\ \text{m}}{3.00 \times 10^3\ \text{m}}\right) = \tan^{-1}(0.500) = 26.6°$$

$$\theta_2 = -\theta_1 = -26.6°$$

2. Use r and Equation 12–1 to calculate the magnitude of the forces $\vec{F}_1$ and $\vec{F}_2$ at point A:

$$F_1 = F_2 = G\frac{mM}{r^2}$$

$$= (6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\frac{(2.50 \times 10^7\ \text{kg})(3.50 \times 10^{11}\ \text{kg})}{(3350\ \text{m})^2}$$

$$= 52.0\ \text{N}$$

3. Use the values of θ_1 and θ_2 found in Step 1 to calculate the x and y components of $\vec{F}_1$ and $\vec{F}_2$:

$$F_{1,x} = F_1 \cos\theta_1 = (52.0\ \text{N}) \cos 26.6° = 46.5\ \text{N}$$

$$F_{1,y} = F_1 \sin\theta_1 = (52.0\ \text{N}) \sin 26.6° = 23.3\ \text{N}$$

$$F_{2,x} = F_2 \cos\theta_2 = (52.0\ \text{N}) \cos(-26.6°) = 46.5\ \text{N}$$

$$F_{2,y} = F_2 \sin\theta_2 = (52.0\ \text{N}) \sin(-26.6°) = -23.3\ \text{N}$$

4. Add the components of $\vec{F}_1$ and $\vec{F}_2$ to find the components of the net force, $\vec{F}$:

$$F_x = F_{1,x} + F_{2,x} = 93.0\ \text{N}$$

$$F_y = F_{1,y} + F_{2,y} = 0$$

Part (b)

5. Use Equation 12–1 to find the magnitude of the forces exerted on the spaceship by the asteroids at location B:

$$F_1 = F_2 = G\frac{mM}{r^2}$$

$$= (6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\frac{(2.50 \times 10^7\ \text{kg})(3.50 \times 10^{11}\ \text{kg})}{(1.50 \times 10^3\ \text{m})^2}$$

$$= 259\ \text{N}$$

6. Use the fact that $\vec{F}_1$ and $\vec{F}_2$ have equal magnitudes and point in opposite directions to determine the net force, $\vec{F}$, acting on the spaceship:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = 0$$

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INSIGHT

We find that the net force at location A is in the positive x direction, as one would expect by symmetry. At location B, where the force exerted by each asteroid is about 5 times greater than it is at location A, the *net* force is zero since the attractive forces exerted by the two asteroids are equal and opposite, and thus cancel. Note that the forces in our sketch have been drawn in correct proportion.

Rocket scientists often use the gravitational force between astronomical objects and spacecraft to accelerate the spacecraft and send them off to distant parts of the solar system. In fact, this gravitational attraction makes possible the "slingshot" effect illustrated in Figure 9–31.

PRACTICE PROBLEM

Find the net gravitational force acting on the spaceship when it is at the location $x = 5.00 \times 10^3$ m, $y = 0$. **[Answer:** 41.0 N in the negative x direction]

Some related homework problems: Problem 9, Problem 11, Problem 12

12–2 Gravitational Attraction of Spherical Bodies

Newton's law of gravity applies to point objects. How, then, do we calculate the force of gravity for an object of finite size? In general, the approach is to divide the finite object into a collection of small mass elements, then use superposition and the methods of calculus to determine the net gravitational force. For an arbitrary shape, this calculation can be quite difficult. For objects with a uniform spherical shape, however, the final result is remarkably simple, as was shown by Newton.

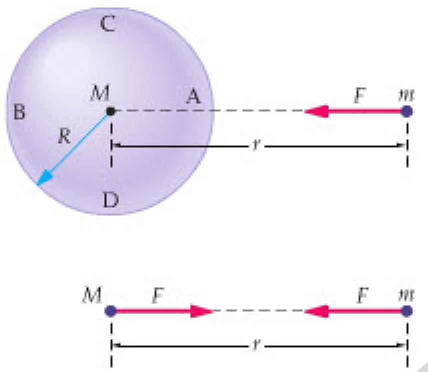


FIGURE 12–3 Gravitational force between a point mass and a sphere
The force is the same as if all the mass of the sphere were concentrated at its center.

Uniform Sphere

Consider a uniform sphere of radius R and mass M, as in Figure 12–3. A point object of mass m is brought near the sphere, though still outside it at a distance r from its center. The object experiences a relatively strong attraction from mass near the point A, and a weaker attraction from mass near point B. In both cases the force is along the line connecting the mass m and the center of the sphere; that is, along the x axis. In addition, mass at the points C and D exert a net force that is also along the x axis—just as in the case of the twin asteroids in Example 12–1. Thus, the symmetry of the sphere guarantees that the net force it exerts on m is directed toward the sphere's center. The magnitude of the force exerted by the sphere must be calculated with the methods of calculus—which Newton invented and then applied to this problem. As a result of his calculations, Newton was able to show that **the net force exerted by the sphere on the mass m is the same as if all the mass of the sphere were concentrated at its center.** That is, the force between the mass m and the sphere of mass M has a magnitude that is simply

$$F = G\frac{mM}{r^2} \qquad \text{12–3}$$

Let's apply this result to the case of a mass m on the surface of the Earth. If the mass of the Earth is M_E, and its radius is R_E, it follows that the force exerted on m by the Earth is

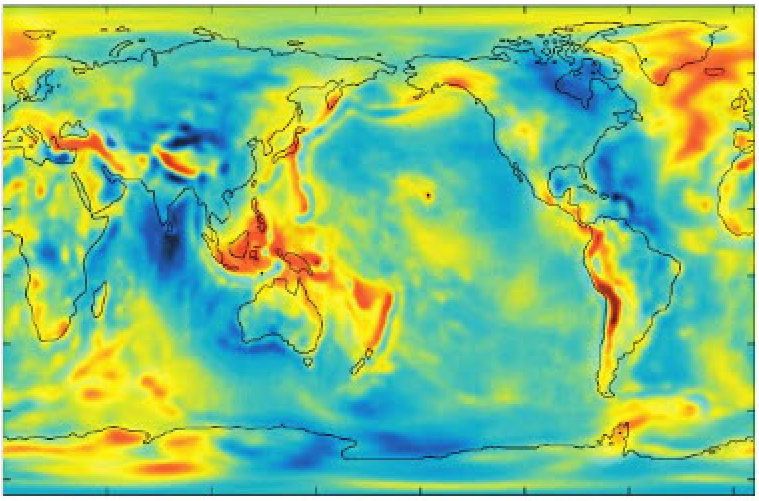
$$F = G\frac{mM_E}{{R_E}^2} = m\left(\frac{GM_E}{{R_E}^2}\right)$$

We also know, however, that the gravitational force experienced by a mass m on the Earth's surface is simply $F = mg$, where g is the acceleration due to gravity. Therefore, we see that

$$m\left(\frac{GM_E}{{R_E}^2}\right) = mg$$

or

$$g = \frac{GM_E}{{R_E}^2} = \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{(6.37 \times 10^6\ \mathrm{m})^2} = 9.81\ \mathrm{m/s^2} \qquad \text{12–4}$$



◀ This global model of the Earth's gravitational strength was constructed from a combination of surface gravity measurements and satellite tracking data. It shows how the acceleration of gravity varies from the value at an idealized "sea level" that takes into account the Earth's nonspherical shape. (The Earth is somewhat flattened at the poles—its radius is greatest at the equator.) Gravity is strongest in the red areas and weakest in the dark blue areas.

This result can be extended to objects above the Earth's surface, and hence farther from the center of the Earth, as we show in the next Example.

EXAMPLE 12–2 THE DEPENDENCE OF GRAVITY ON ALTITUDE

REAL-WORLD PHYSICS

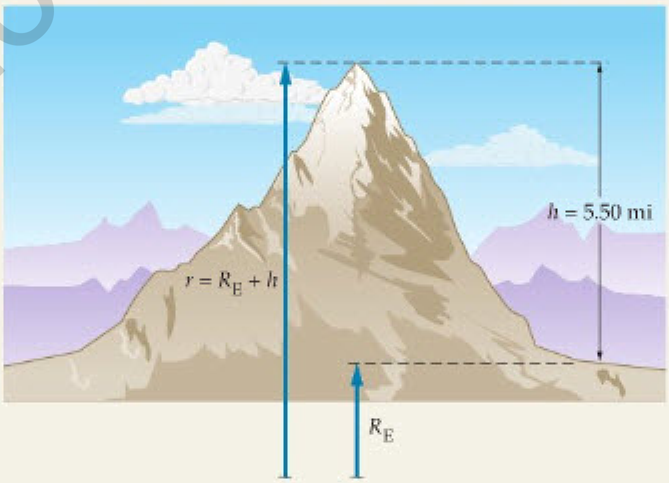
If you climb to the top of Mt. Everest, you will be about 5.50 mi above sea level. What is the acceleration due to gravity at this altitude?

PICTURE THE PROBLEM

At the top of the mountain, your distance from the center of the Earth is $r = R_E + h$, where $h = 5.50$ mi is the altitude.

STRATEGY

First, use $F = GmM_E/r^2$ to find the force due to gravity on the mountaintop. Then, set $F = mg_h$ to find the acceleration g_h at the height h.



SOLUTION

1. Calculate the force F due to gravity at a height h above the Earth's surface:

$$F = G\frac{mM_E}{(R_E + h)^2}$$

2. Set F equal to mg_h and solve for g_h:

$$F = G\frac{mM_E}{(R_E + h)^2} = mg_h$$

$$g_h = G\frac{M_E}{(R_E + h)^2}$$

3. Factor out R_E^2 from the denominator, and use the fact that $GM_E/R_E^2 = g$:

$$g_h = \left(\frac{GM_E}{R_E^2}\right)\frac{1}{\left(1 + \frac{h}{R_E}\right)^2} = \frac{g}{\left(1 + \frac{h}{R_E}\right)^2}$$

4. Substitute numerical values, with $h = 5.50$ mi $= (5.50 \text{ mi})(1609 \text{ m/mi}) = 8850$ m, and $R_E = 6.37 \times 10^6$ m:

$$g_h = \frac{g}{\left(1 + \frac{h}{R_E}\right)^2} = \frac{9.81 \text{ m/s}^2}{\left(1 + \frac{8850 \text{ m}}{6.37 \times 10^6 \text{ m}}\right)^2} = 9.78 \text{ m/s}^2$$

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INSIGHT

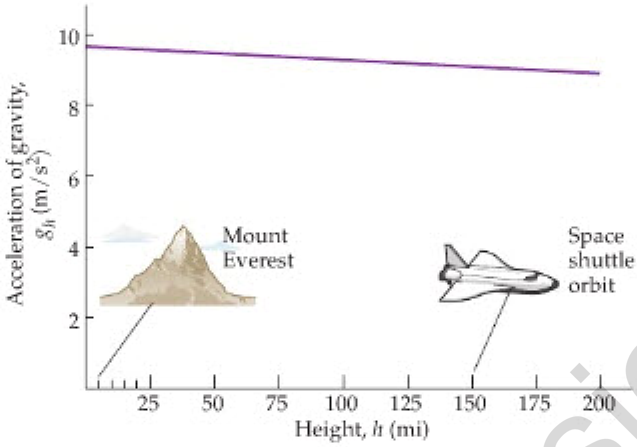
As expected, the acceleration due to gravity is less as one moves farther from the center of the Earth. Thus, if you were to climb to the top of Mt. Everest, you would lose weight—not only because of the physical exertion required for the climb, but also because of the reduced gravity. In particular, a person with a mass of 60 kg (about 130 lb) would lose about half a pound of weight just by standing on the summit of the mountain.

A plot of g_h as a function of h is shown in Figure 12–4 (a). The plot indicates the altitude of Mt. Everest and the orbit of the space shuttle. Figure 12–4 (b) shows g_h out to the orbit of communications and weather satellites, which orbit at an altitude of roughly 22,300 mi.

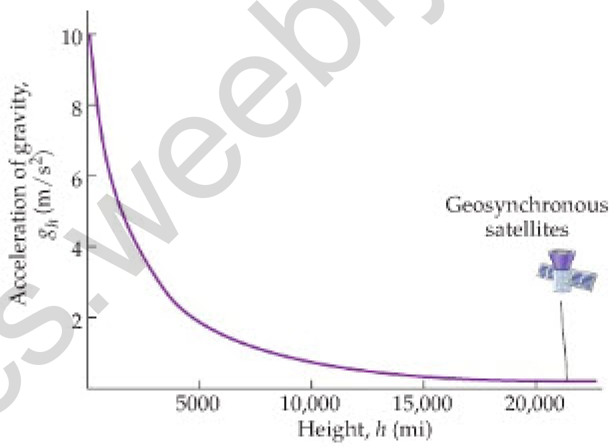
PRACTICE PROBLEM

Find the acceleration due to gravity at the altitude of the space shuttle's orbit, 250 km above the Earth's surface. [**Answer:** $g_h = 9.08\ \mathrm{m/s^2}$, a reduction of only 7.44% compared to the acceleration of gravity on the surface of the Earth.]

Some related homework problems: Problem 15, Problem 17



(a) Acceleration of gravity near the Earth's surface

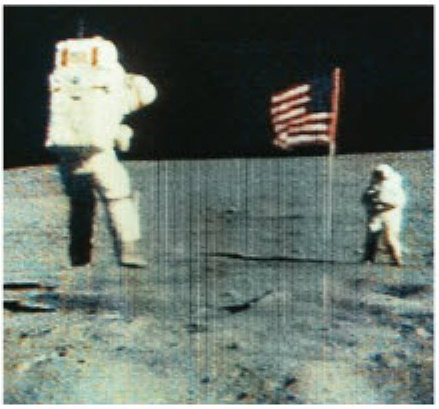


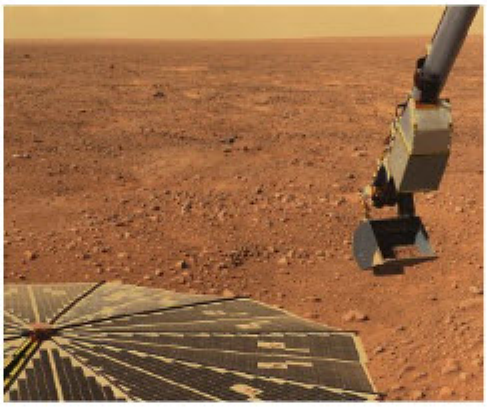
(b) Acceleration of gravity far from the Earth

FIGURE 12–4 ▲ **The acceleration due to gravity at a height h above the Earth's surface**

(a) In this plot, the peak of Mt. Everest is at about $h = 5.50$ mi, and the space shuttle orbit is at roughly $h = 150$ mi. **(b)** This shows the decrease in the acceleration of gravity from the surface of the Earth to an altitude of about 25,000 mi. The orbit of geosynchronous satellites—ones that orbit above a fixed point on the Earth—is at roughly $h = 22{,}300$ mi.

Equation 12–4 can be used to calculate the acceleration due to gravity on other objects in the solar system besides the Earth. For example, to calculate the acceleration due to gravity on the Moon, g_m, we simply use the mass and radius of the Moon in Equation 12–4. Once g_m is known, the weight of an object of mass m on the Moon is found by using $W_m = mg_m$.





▶ (Left) The weak lunar gravity permits astronauts, even encumbered by their massive space suits, to bound over the Moon's surface. The low gravitational pull, only about one-sixth that of Earth, is a consequence not only of the Moon's smaller size, but also of its lower average density. (Right) The force of gravity on the surface of Mars is only about 38% of its strength on Earth. This was an important factor in designing NASA's Phoenix Mars Lander, shown here lifting a scoop of dirt on its 16th Martian day after landing in May 2008.

EXERCISE 12–2

a. Find the acceleration due to gravity on the surface of the Moon.

b. The lunar rover had a mass of 225 kg. What was its weight on the Earth and on the Moon? (*Note:* The mass of the Moon is $M_m = 7.35 \times 10^{22}$ kg and its radius is $R_m = 1.74 \times 10^6$ m.)

SOLUTION

a. For the Moon, the acceleration due to gravity is

$$g_m = \frac{GM_m}{R_m^{\,2}} = \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(7.35 \times 10^{22}\ \mathrm{kg})}{(1.74 \times 10^{6}\ \mathrm{m})^2} = 1.62\ \mathrm{m/s^2}$$

This is about one-sixth the acceleration due to gravity on the Earth.

b. On the Earth, the rover's weight was

$$W = mg = (225\ \mathrm{kg})(9.81\ \mathrm{m/s^2}) = 2210\ \mathrm{N}$$

On the Moon, its weight was

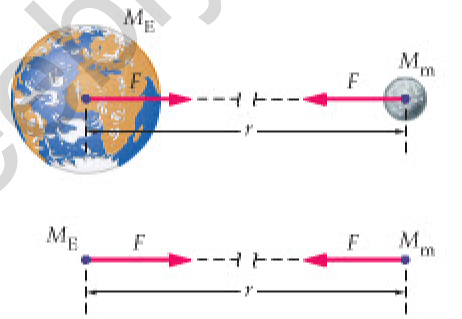
$$W_m = mg_m = (225\ \mathrm{kg})(1.62\ \mathrm{m/s^2}) = 365\ \mathrm{N}$$

As expected, this is roughly one-sixth its Earth weight.

The replacement of a sphere with a point mass at its center can be applied to many physical systems. For example, the force of gravity between two spheres of finite size is the same as if *both* were replaced by point masses. Thus, the gravitational force between the Earth, with mass M_E, and the Moon, with mass M_m, is

$$F = G\frac{M_E M_m}{r^2}$$

The distance r in this expression is the center-to-center distance between the Earth and the Moon, as shown in Figure 12–5. It follows, then, that in many calculations involving the solar system, moons and planets can be treated as point objects.



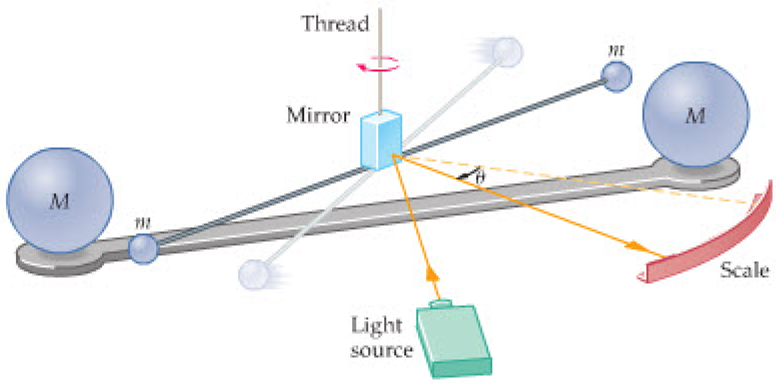
▲ FIGURE 12–5 Gravitational force between the Earth and the Moon

The force is the same as if both the Earth and the Moon were point masses. (The sizes of the Earth and Moon are in correct proportion in this figure, but the separation between the two should be much greater than that shown here. In reality, it is about 30 times the diameter of the Earth, and so would be about 2 ft on this scale.)

Weighing the Earth

The British physicist Henry Cavendish performed an experiment in 1798 that is often referred to as "weighing the Earth." What he did, in fact, was measure the value of the universal gravitation constant, G, that appears in Newton's law of gravity. As we have pointed out before, G is a very small number; hence a sensitive experiment is needed for its measurement. It is because of this experimental difficulty that G was not measured until more than 100 years after Newton published the law of gravitation.

In the Cavendish experiment, illustrated in Figure 12–6, two masses m are suspended from a thin thread. Near each suspended mass is a large stationary mass M, as shown. Each suspended mass is attracted by the force of gravity toward the large mass near it; hence the rod holding the suspended masses tends to rotate and twist the thread. The angle through which the thread twists can be measured by bouncing a beam of light from a mirror attached to the thread. If the force required to twist the thread through a given angle is known (from previous experiments), a measurement of the twist angle gives the magnitude of the force of gravity. Finally, knowing the masses m and M, and the distance between their centers, r, we can use Equation 12–1 to solve for G. Cavendish found $6.754 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, in good agreement with the currently accepted value given in Equation 12–2.



◀ FIGURE 12–6 The Cavendish experiment

The gravitational attraction between the masses m and M causes the rod and the suspending thread to twist. Measurement of the twist angle allows for a direct measurement of the gravitational force.

To see why Cavendish is said to have weighed the Earth, recall that the force of gravity on the surface of the Earth, mg, can be written as follows:

$$mg = G\frac{mM_E}{{R_E}^2}$$

Canceling m and solving for M_E yields

$$M_E = \frac{g{R_E}^2}{G} \qquad 12\text{–}5$$

Before the Cavendish experiment, the quantities g and R_E were known from direct measurement, but G had yet to be determined. When Cavendish measured G, he didn't actually "weigh" the Earth, of course. Instead, he calculated its mass, M_E.

EXERCISE 12–3

Use $M_E = g{R_E}^2/G$ to calculate the mass of the Earth.

SOLUTION

Substituting numerical values, we find

$$M_E = \frac{g{R_E}^2}{G} = \frac{(9.81\ \text{m/s}^2)(6.37 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2} = 5.97 \times 10^{24}\ \text{kg}$$

As soon as Cavendish determined the mass of the Earth, geologists were able to use the result to calculate its average density; that is, its average mass per volume. Assuming a spherical Earth of radius R_E, its total volume is

$$V_E = \tfrac{4}{3}\pi {R_E}^3 = \tfrac{4}{3}\pi(6.37 \times 10^6\ \text{m})^3 = 1.08 \times 10^{21}\ \text{m}^3$$

Dividing this into the total mass yields the average density, ρ:

$$\rho = \frac{M_E}{V_E} = \frac{5.97 \times 10^{24}\ \text{kg}}{1.08 \times 10^{21}\ \text{m}^3} = 5530\ \text{kg/m}^3 = 5.53\ \text{g/cm}^3$$

This is an interesting result because typical rocks found near the surface of the Earth, such as granite, have a density of only about $3.00\ \text{g/cm}^3$. We conclude, then, that the interior of the Earth must have a greater density than its surface. In fact, by analyzing the propagation of seismic waves around the world, we now know that the Earth has a rather complex interior structure, including a solid inner core with a density of about $15.0\ \text{g/cm}^3$ (see Section 10–5).

REAL-WORLD PHYSICS
The internal structure of the Earth and the Moon

A similar calculation for the Moon yields an average density of about $3.33\ \text{g/cm}^3$, essentially the same as the density of the lunar rocks brought back during the Apollo program. Hence, it is likely that the Moon does not have an internal structure similar to that of the Earth.

Since G is a universal constant—with the same value everywhere in the universe—it can be used to calculate the mass of other bodies in the solar system as well. This is illustrated in the following Example.

EXAMPLE 12–3 MARS ATTRACTS!

After landing on Mars, an astronaut performs a simple experiment by dropping a rock. A quick calculation using the drop height and the time of fall yields a value of $3.73\ \text{m/s}^2$ for the rock's acceleration. **(a)** Find the mass of Mars, given that its radius is $R_M = 3.39 \times 10^6$ m. **(b)** What is the acceleration of gravity due to Mars at a distance $2R_M$ from the center of the planet?

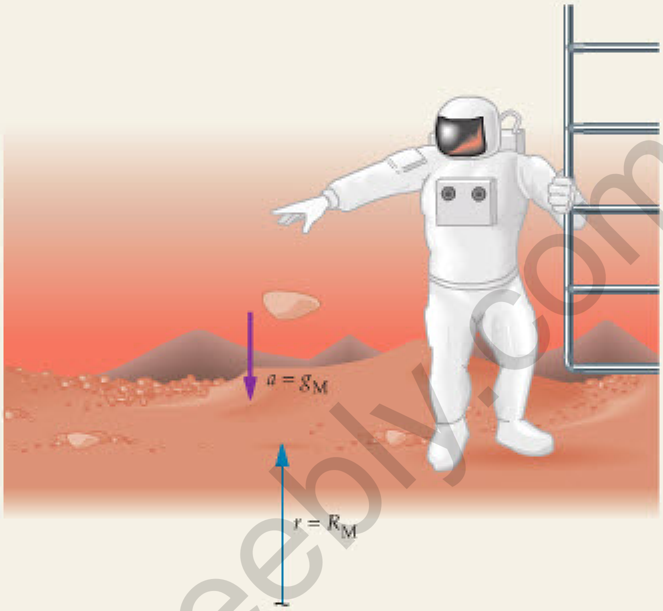
PICTURE THE PROBLEM

Our sketch shows an astronaut dropping a rock to the ground on the surface of Mars. If the acceleration of the rock is measured, we find $g_M = 3.73\ \text{m/s}^2$, where the subscript M refers to Mars. In addition, we indicate the radius of Mars in our sketch, where $R_M = 3.39 \times 10^6$ m.

STRATEGY

a. Since the acceleration of gravity is g_M on the surface of Mars, it follows that the force of gravity on an object of mass m is $F = mg_M$. This force is also given by Newton's law of gravity—that is, $F = GmM_M/R_M{}^2$. Setting these expressions for the force equal to one another yields the mass of Mars, M_M.

b. Set $F = ma$ equal to $F = GmM_M/(2R_M)^2$ and solve for the acceleration, a.



SOLUTION

Part (a)

1. Set mg_M equal to $GmM_M/R_M{}^2$:

$$mg_M = G\frac{mM_M}{R_M{}^2}$$

2. Cancel m and solve for the mass of Mars:

$$M_M = \frac{g_M R_M{}^2}{G}$$

3. Substitute numerical values:

$$M_M = \frac{(3.73\ \text{m/s}^2)(3.39 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2} = 6.43 \times 10^{23}\ \text{kg}$$

Part (b)

4. Apply Newton's law of gravity with $r = 2R_M$. Use the fact that $g_M = GM_M/R_M{}^2$ from Step 1 to simplify the calculation:

$$ma = G\frac{mM_M}{(2R_M)^2} \quad \text{or}$$

$$a = G\frac{M_M}{(2R_M)^2} = \tfrac{1}{4}\left(G\frac{M_M}{R_M{}^2}\right) = \tfrac{1}{4}(g_M) = \tfrac{1}{4}(3.73\ \text{m/s}^2) = 0.933\ \text{m/s}^2$$

INSIGHT

The important point here is that the universal gravitation constant, G, applies as well on Mars as on Earth, or any other object. Therefore, knowledge of the size and acceleration of gravity of an astronomical body is sufficient to determine its mass.

PRACTICE PROBLEM

If the radius of Mars were reduced to 3.00×10^6 m, with its mass remaining the same, would the acceleration of gravity on Mars increase, decrease, or stay the same? Check your answer by calculating the acceleration of gravity for this case. [**Answer:** The acceleration of gravity increases to $4.77\ \text{m/s}^2$.]

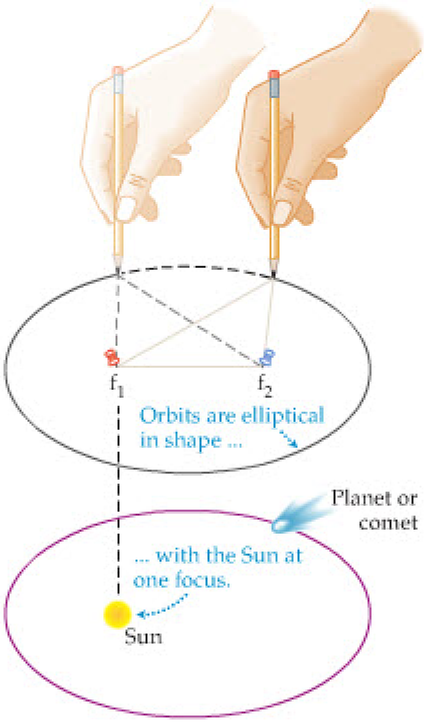
Some related homework problems: Problem 20, Problem 21

12–3 Kepler's Laws of Orbital Motion

If you go outside each clear night and observe the position of Mars with respect to the stars, you will find that its apparent motion across the sky is rather complex. Instead of moving on a simple curved path, it occasionally reverses direction (this is known as *retrograde motion*). A few months later it reverses direction yet again and resumes its original direction of motion. Other planets exhibit similar odd behavior.

The Danish astronomer Tycho Brahe (1546–1601) followed the paths of the planets, and Mars in particular, for many years, even though the telescope had not yet been invented. He used, instead, an elaborate sighting device to plot the precise position of the planets. Brahe was joined in his work by Johannes Kepler (1571–1630) in 1600, and after Brahe's death, Kepler inherited his astronomical observations.

Kepler made good use of Brahe's life work, extracting from his carefully collected data the three laws of orbital motion we know today as Kepler's laws. These laws make it clear that the Sun and the planets do not orbit the Earth, as Ptolemy—the ancient Greek astronomer—claimed, but rather that the Earth, along with the other planets, orbit the Sun, as proposed by Copernicus (1473–1543).



▲ **FIGURE 12–7 Drawing an ellipse**

To draw an ellipse, put two tacks in a piece of cardboard. The tacks define the "foci" of the ellipse. Now connect a length of string to the two tacks, and use a pencil and the string to sketch out a smooth closed curve, as shown. This closed curve is an ellipse. In a planetary orbit a planet follows an elliptical path, with the Sun at one focus. Nothing is at the other focus.

Why the planets obey Kepler's laws no one knew—not even Kepler—until Newton considered the problem decades after Kepler's death. Newton was able to show that each of Kepler's laws follows as a direct consequence of the universal law of gravitation. In the remainder of this section we consider Kepler's three laws one at a time, and point out the connection between them and the law of gravitation.

Kepler's First Law

Kepler tried long and hard to find a circular orbit around the Sun that would match Brahe's observations of Mars. After all, up to that time everyone from Ptolemy to Copernicus believed that celestial objects moved in circular paths of one sort or another. Though the orbit of Mars was exasperatingly close to being circular, the small differences between a circular path and the experimental observations just could not be ignored. Eventually, after a great deal of hard work and disappointment over the loss of circular orbits, Kepler discovered that Mars followed an orbit that was elliptical rather than circular. The same applied to the other planets. This observation became Kepler's first law:

> Planets follow elliptical orbits, with the Sun at one focus of the ellipse.

This is a fine example of the scientific method in action. Though Kepler expected and wanted to find circular orbits, he would not allow himself to ignore the data. If Brahe's observations had not been so accurate, Kepler probably would have chalked up the small differences between the data and a circular orbit to error. As it was, he had to discard a treasured—but incorrect—theory, and move on to an unexpected, but ultimately correct, view of nature.

Kepler's first law is illustrated in **Figure 12–7**, along with a definition of an ellipse in terms of its two foci. In the case where the two foci merge, as in **Figure 12–8**, the ellipse reduces to a circle. Thus, a circular orbit *is* allowed by Kepler's first law, but only as a special case.

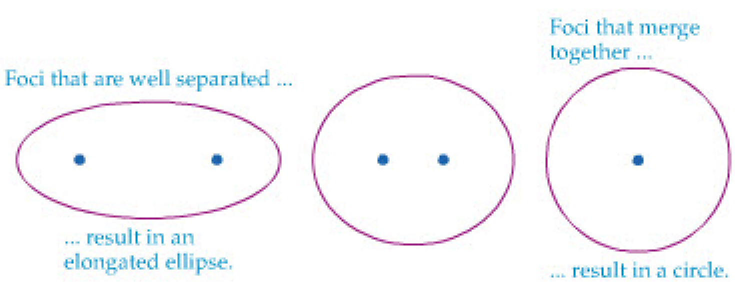
Newton was able to show that, because the force of gravity decreases with distance as $1/r^2$, closed orbits must have the form of ellipses or circles, as stated in Kepler's first law. He also showed that orbits that are not closed—say the orbit of a comet that passes by the Sun once and then leaves the solar system—are either parabolic or hyperbolic.

Kepler's Second Law

When Kepler plotted the position of a planet on its elliptical orbit, indicating at each position the time the planet was there, he made an interesting observation. First, draw a line from the Sun to a planet at a given time. Then a certain time later—perhaps a month—draw a line again from the Sun to the new position of the planet. The result is that the planet has "swept out" a wedge-shaped area, as indicated in **Figure 12–9 (a)**. If this procedure is repeated when the planet is on a different part of its orbit, another wedge-shaped area is generated. Kepler's observation was that the areas of these two wedges are equal:

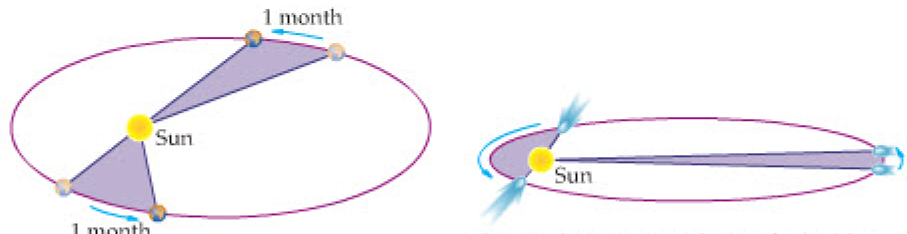
> As a planet moves in its orbit, it sweeps out an equal amount of area in an equal amount of time.

Kepler's second law follows from the fact that the force of gravity on a planet is directly toward the Sun. As a result, gravity exerts zero torque about the Sun,



▶ **FIGURE 12–8 The circle as a special case of the ellipse**

As the two foci of an ellipse approach one another, the ellipse becomes more circular. In the limit that the foci merge, the ellipse becomes a circle.



(a) Equal areas in equal times

(b) Equal areas in equal times for highly elliptical orbit

FIGURE 12–9 Kepler's second law

(a) The second law states that a planet sweeps out equal areas in equal times. **(b)** In a highly elliptical orbit, the long, thin area is equal to the broad, fan-shaped area.

which means that the angular momentum of a planet in its orbit must be conserved. As Newton showed, conservation of angular momentum is equivalent to the equal-area law stated by Kepler.

CONCEPTUAL CHECKPOINT 12–1 COMPARE SPEEDS

The Earth's orbit is slightly elliptical. In fact, the Earth is closer to the Sun during the northern hemisphere winter than it is during the summer. Is the speed of the Earth during winter **(a)** greater than, **(b)** less than, or **(c)** the same as its speed during summer?

REASONING AND DISCUSSION

According to Kepler's second law, the area swept out by the Earth per month is the same in winter as it is in summer. In winter, however, the radius from the Sun to the Earth is less than it is in summer. Therefore, if this smaller radius is to sweep out the same area, the Earth must move more rapidly.

ANSWER

(a) The speed of the Earth is greater during the winter.

Though we have stated the first two laws in terms of planets, they apply equally well to any object orbiting the Sun. For example, a comet might follow a highly elliptical orbit, as in **Figure 12–9 (b)**. When it is near the Sun, it moves very quickly, for the reason discussed in Conceptual Checkpoint 12–1, sweeping out a broad wedge-shaped area in a month's time. Later in its orbit, the comet is far from the Sun and moving slowly. In this case, the area it sweeps out in a month is a long, thin wedge. Still, the two wedges have equal areas.

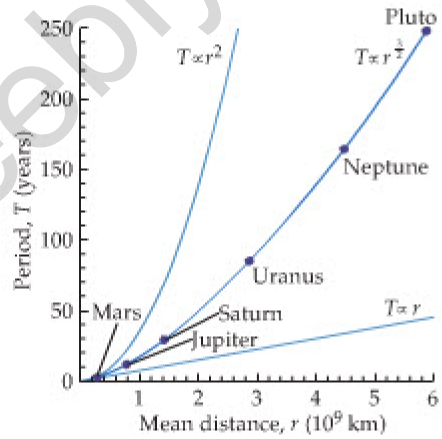


FIGURE 12–10 Kepler's third law and some near misses

These plots represent three possible mathematical relationships between period of revolution, T (in years), and mean distance from the Sun, r (in kilometers). The lower curve shows $T = (\text{constant})r$; the upper curve is $T = (\text{constant})r^2$. The middle curve, which fits the data, is $T = (\text{constant})r^{3/2}$. This is Kepler's third law.

Kepler's Third Law

Finally, Kepler studied the relation between the mean distance of a planet from the Sun, r, and its period—that is, the time, T, it takes for the planet to complete one orbit. **Figure 12–10** shows a plot of period versus distance for the planets of the solar system. Kepler tried to "fit" these results to a simple dependence between T and r. If he tried a linear fit—that is, T proportional to r (the bottom curve in Figure 12–10)—he found that the period did not increase rapidly enough with distance. On the other hand, if he tried T proportional to r^2 (the top curve in Figure 12–10), the period increased too rapidly. Splitting the difference, and trying T proportional to $r^{3/2}$, yields a good fit (the middle curve in Figure 12–10). This is Kepler's third law:

> The period, T, of a planet increases as its mean distance from the Sun, r, raised to the 3/2 power. That is,
>
> $$T = (\text{constant})r^{3/2} \qquad 12\text{–}6$$

It is straightforward to derive this result for the special case of a circular orbit. Consider, then, a planet orbiting the Sun at a distance r, as in **Figure 12–11**. Since the planet moves in a circular path, a centripetal force must act on it, as we saw in Section 6–5. In addition, this force must be directed toward the center of the circle; that is, toward the Sun. It is as if you were to swing a ball on the end of a string in a circle above your head, as in Figure 6–12 (p. 169). In order for the ball to move in a circular path, you have to exert a force on the ball toward the center of the circular path. This force is exerted through the string. In the case of a planet orbiting the Sun, the centripetal force is provided by the force of gravity between the Sun and the planet.

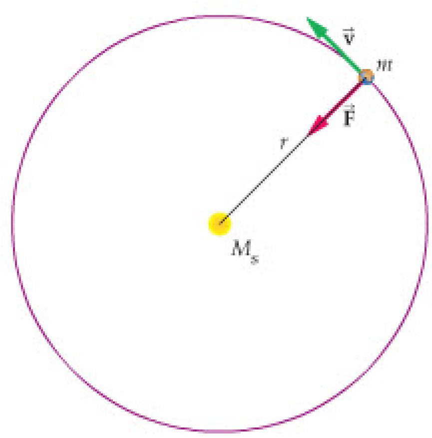


FIGURE 12–11 Centripetal force on a planet in orbit

As a planet revolves about the Sun in a circular orbit of radius r, the force of gravity between it and the Sun, $F = GmM_s/r^2$, provides the required centripetal force.

If the planet has a mass m, and the Sun has a mass M_s, the force of gravity between them is

$$F = G\frac{mM_s}{r^2}$$

Now, this force creates the centripetal acceleration of the planet, a_{cp}, which, according to Equation 6–15, is

$$a_{cp} = \frac{v^2}{r}$$

Thus, the centripetal force necessary for the planet to orbit is ma_{cp}:

$$F = ma_{cp} = m\frac{v^2}{r}$$

Since the speed of the planet, v, is the circumference of the orbit, $2\pi r$, divided by the time to complete an orbit, T, we have

$$F = m\frac{v^2}{r} = m\frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 rm}{T^2}$$

Setting the centripetal force equal to the force of gravity yields

$$\frac{4\pi^2 rm}{T^2} = G\frac{mM_s}{r^2}$$

Eliminating m and rearranging, we find

$$T^2 = \frac{4\pi^2}{GM_s}r^3$$

or

$$T = \left(\frac{2\pi}{\sqrt{GM_s}}\right)r^{3/2} = (\text{constant})r^{3/2} \qquad \text{12–7}$$

As predicted by Kepler, T is proportional to $r^{3/2}$.

Deriving Kepler's third law by using Newton's law of gravitation has allowed us to calculate the constant that multiplies $r^{3/2}$. Note that the constant depends on the mass of the Sun; that is, *T depends on the mass being orbited*. It does not depend on the mass of the planet orbiting the Sun, however, as long as the planet's mass is much less than the mass of the Sun. As a result, Equation 12–7 applies equally to all the planets.

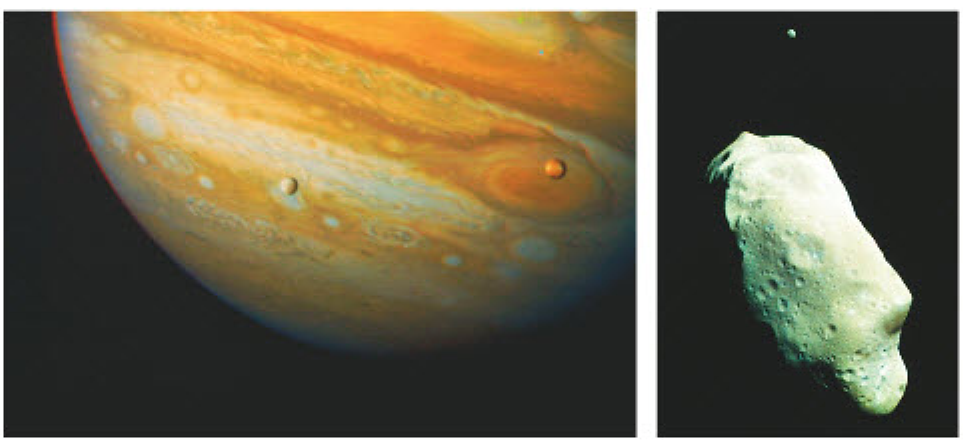
This result can also be applied to the case of a moon or a satellite (an artificial moon) orbiting a planet. To do so, we simply note that it is the planet that is being orbited, not the Sun. Hence, to apply Equation 12–7, we just replace the mass of the Sun, M_s, with the mass of the appropriate planet.

As an example, let's calculate the mass of Jupiter. One of the four moons of Jupiter discovered by Galileo is Io, which completes one orbit every 42 h 27 min =

PROBLEM-SOLVING NOTE

The Mass in Kepler's Third Law

When applying Kepler's third law, recall that the mass in Equation 12–7, M_s, refers to the mass of the object being orbited. Thus, the third law can be applied to satellites of any object, as long as M_s is replaced by the orbited mass.



▶ Kepler's laws of orbital motion apply to planetary satellites as well as planets. Jupiter, the largest planet in the solar system, has at least 16 moons, all of which travel in elliptical orbits that obey Kepler's laws. (The moons in the photo at left, passing in front of Jupiter, are Io and Europa, two of the four largest Jovian satellites discovered by Galileo in 1609.) Even some asteroids have been found to have their own satellites. The large cratered object in the photo at right is 243 Ida, an asteroid some 56 km long; its miniature companion at the top of the photo is Dactyl, about 1.5 km in diameter. Like all gravitationally bound bodies, Ida and Dactyl orbit their common center of mass.

1.53×10^5 s. Given that the average distance from the center of Jupiter to Io is 4.22×10^8 m, we can find the mass of Jupiter as follows:

$$T = \left(\frac{2\pi}{\sqrt{GM_J}}\right)r^{3/2}$$

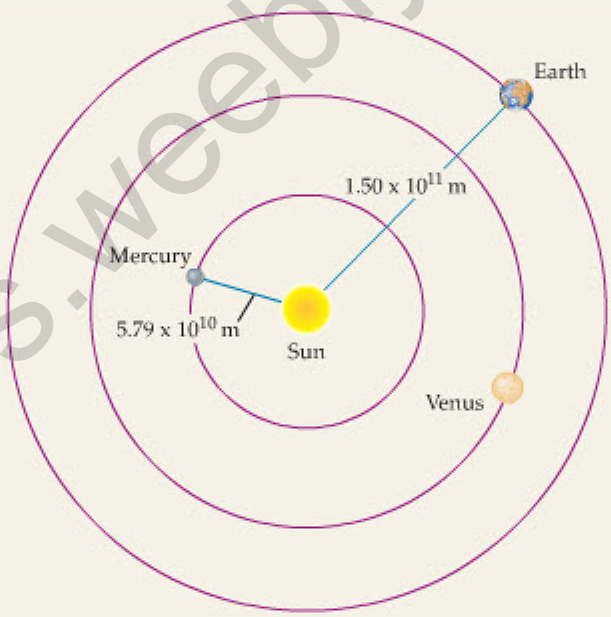
$$M_J = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(4.22 \times 10^8 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(1.53 \times 10^5 \text{ s})^2} = 1.90 \times 10^{27} \text{ kg}$$

EXAMPLE 12–4 THE SUN AND MERCURY

The Earth revolves around the Sun once a year at an average distance of 1.50×10^{11} m. **(a)** Use this information to calculate the mass of the Sun. **(b)** Find the period of revolution for the planet Mercury, whose average distance from the Sun is 5.79×10^{10} m.

PICTURE THE PROBLEM

Our sketch shows the orbits of Mercury, Venus, and the Earth in correct proportion. In addition, each of these orbits is slightly elliptical, though the deviation from circularity is too small for the eye to see. Finally, we indicate that the orbital radius for Mercury is 5.79×10^{10} m and the orbital radius for Earth is 1.50×10^{11} m.



STRATEGY

a. To find the mass of the Sun, we solve Equation 12–7 for M_s. Note that the period $T = 1$ yr must be converted to seconds before we evaluate the formula.

b. The period of Mercury is found by substituting $r = 5.79 \times 10^{10}$ m in Equation 12–7.

SOLUTION

Part (a)

1. Solve Equation 12–7 for the mass of the Sun:

$$T = \left(\frac{2\pi}{\sqrt{GM_s}}\right)r^{3/2}$$

$$M_s = \frac{4\pi^2 r^3}{GT^2}$$

2. Calculate the period of the Earth in seconds:

$$T = 1\text{ y}\left(\frac{365.24 \text{ days}}{1 \text{ y}}\right)\left(\frac{24 \text{ hr}}{1 \text{ day}}\right)\left(\frac{3600 \text{ s}}{1 \text{ hr}}\right) = 3.16 \times 10^7 \text{ s}$$

3. Substitute numerical values in the expression for the mass of the Sun obtained in Step 1:

$$M_s = \frac{4\pi^2 r^3}{GT^2}$$

$$= \frac{4\pi^2(1.50 \times 10^{11} \text{ m})^3}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(3.16 \times 10^7 \text{ s})^2}$$

$$= 2.00 \times 10^{30} \text{ kg}$$

Part (b)

4. Substitute $r = 5.79 \times 10^{10}$ m into Equation 12–7. In addition, use the mass of the Sun obtained in part (a):

$$T = \left(\frac{2\pi}{\sqrt{GM_s}}\right)r^{3/2}$$

$$= \left(\frac{2\pi}{\sqrt{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(2.00 \times 10^{30} \text{ kg})}}\right) \times (5.79 \times 10^{10} \text{ m})^{3/2}$$

$$= 7.58 \times 10^6 \text{ s} = 0.240 \text{ y} = 87.7 \text{ days}$$

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INSIGHT

In part (a), notice that the mass of the Sun is almost a million times more than the mass of the Earth, as determined in Exercise 12–3. In fact, the Sun accounts for 99.9% of all the mass in the solar system.

In part (b) we see that Mercury, with its smaller orbital radius, has a shorter year than the Earth.

PRACTICE PROBLEM

Venus orbits the Sun with a period of 1.94×10^7 s. What is its average distance from the Sun? **[Answer:** $r = 1.08 \times 10^{11}$ m**]**

Some related homework problems: Problem 28, Problem 32

▲ Many weather and communications satellites are placed in geosynchronous orbits that allow them to remain "stationary" in the sky—that is, fixed over one point on the Earth's equator. Because the Earth rotates, the period of such a satellite must exactly match that of the Earth. The altitude needed for such an orbit is about 36,000 km (see Active Example 12–1). Other satellites, such as those used in the Global Positioning System (GPS), the Hubble Space Telescope, and the American space shuttles, operate at much lower altitudes—typically just a few hundred miles. The photo at left shows the communications satellite Intelsat VI just prior to its capture by astronauts of the space shuttle *Endeavour*. A launch failure had left the satellite stranded in low orbit. The astronauts snared the satellite (right) and fitted it with a new engine that boosted it to its geosynchronous orbit, where it is still in operation today.

REAL-WORLD PHYSICS
Geosynchronous satellites

A *geosynchronous satellite* is one that orbits above the equator with a period equal to one day. From the Earth, such a satellite appears to be in the same location in the sky at all times, making it particularly useful for applications such as communications and weather forecasting. From Kepler's third law, we know that a satellite has a period of one day only if its orbital radius has a particular value. We determine this value in the following Active Example.

ACTIVE EXAMPLE 12–1 FIND THE ALTITUDE OF A GEOSYNCHRONOUS SATELLITE

Find the altitude above the Earth's surface where a satellite orbits with a period of one day ($R_E = 6.37 \times 10^6$ m, $M_E = 5.97 \times 10^{24}$ kg, $T = 1$ day $= 8.64 \times 10^4$ s).

SOLUTION *(Test your understanding by performing the calculations indicated in each step.)*

1. Rewrite Equation 12–7, using the mass of the Earth in place of the mass of the Sun: $T = \left(2\pi/\sqrt{GM_E}\right)r^{3/2}$
2. Solve for the radius, r: $r = (T/2\pi)^{2/3}(GM_E)^{1/3}$
3. Substitute numerical values: $r = 4.22 \times 10^7$ m
4. Subtract the radius of the Earth to find the altitude: $r - R_E = 3.58 \times 10^7$ m

INSIGHT
Thus, *all* geosynchronous satellites orbit 3.58×10^7 m $\approx$ 22,300 mi above our heads.

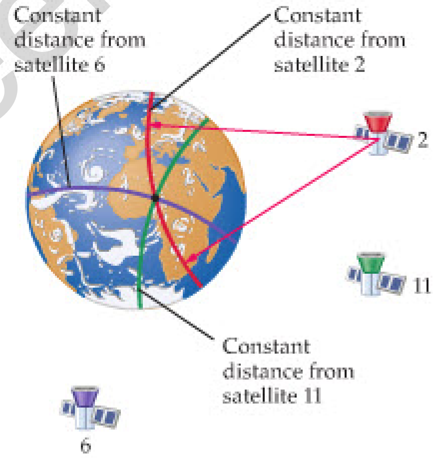
YOUR TURN
Find the altitude above the surface of the Moon where a "lunasynchronous" satellite would orbit. [*Note*: The length of a lunar day is one month (27.332 days), which is why we see only one side of the Moon.]

(Answers to **Your Turn** *problems are given in the back of the book.)*

Not all spacecraft are placed in geosynchronous orbits, however. The U.S. space shuttle, for example, orbits at an altitude of about 150 mi. At that altitude, it takes less than an hour and a half to complete one orbit. The International Space Station, operational although still under construction, orbits at a similar altitude.

REAL-WORLD PHYSICS
The Global Positioning System (GPS)

The 24 satellites of the Global Positioning System (GPS) are also in relatively low orbits. These satellites, which have an average altitude of 12,550 mi and orbit the Earth every 12 hours, are used to provide a precise determination of an observer's position anywhere on Earth. The operating principle of the GPS is illustrated in Figure 12–12. Imagine, for example, that satellite 2 emits a radio signal at a particular time (all GPS satellites carry atomic clocks on board). This signal travels away from the satellite with the speed of light (see Chapter 25) and is detected a short time later by an observer's GPS receiver. Multiplying the time delay by the speed of light gives the distance of the receiver from satellite 2. Thus, in our example, the observer must lie somewhere on the red circle in Figure 12–12. Similar time delay measurements for signals from satellite 11 show that the observer is also somewhere on the green circle; hence the observer is either at the point shown in Figure 12–12, or at the second intersection of the red and green circles on the other side of the planet. Measurements from satellite 6 can resolve the ambiguity and place the observer at the point shown in the figure. Measurements from additional satellites can even determine the observer's altitude. GPS receivers, which are used by hikers, boaters, and others who need to know their precise location, typically use signals from as many as 12 satellites. As currently operated, the GPS gives positions with a typical accuracy of 2 m to 10 m.



▲ **FIGURE 12–12 The Global Positioning System**
A system of 24 satellites in orbit about the Earth makes it possible to determine a person's location with great accuracy. Measuring the distance of a person from satellite 2 places the person somewhere on the red circle. Similar measurements using satellite 11 place the person's position somewhere on the green circle, and further measurements can pinpoint the person's location.

Orbital Maneuvers

We now show how Kepler's laws can give insight into maneuvering a satellite in orbit. Suppose, for example, that you are piloting a spacecraft in a circular orbit, and you would like to move to a lower circular orbit. As you might expect, you should begin by using your rockets to decrease your speed—that is, fire the rockets that point in the forward direction so that their thrust (Section 9–8) is opposite to your direction of motion. The result of firing the decelerating rockets at a given point A in your original orbit is shown in Figure 12–13 (a). Note that your new orbit is not a circle, as desired, but rather an ellipse. To produce a circular orbit you can simply fire the decelerating rockets once again at point B, on the opposite side of the Earth from point A. The net result of these two firings is that you now move in a circular orbit of smaller radius.

REAL-WORLD PHYSICS
Maneuvering spacecraft

Similarly, to move to a larger orbit, you must fire your accelerating rockets twice. The first firing puts you into an elliptical orbit that moves farther from the Earth, as Figure 12–13 (b) shows. After the second firing you are again in a circular orbit. This simplest type of orbital transfer, requiring just two rocket burns, is referred to as a *Hohmann transfer*. The Hohmann transfer is the basic maneuver used to send spacecraft such as the Mars lander from Earth's orbit about the Sun to the orbit of Mars.