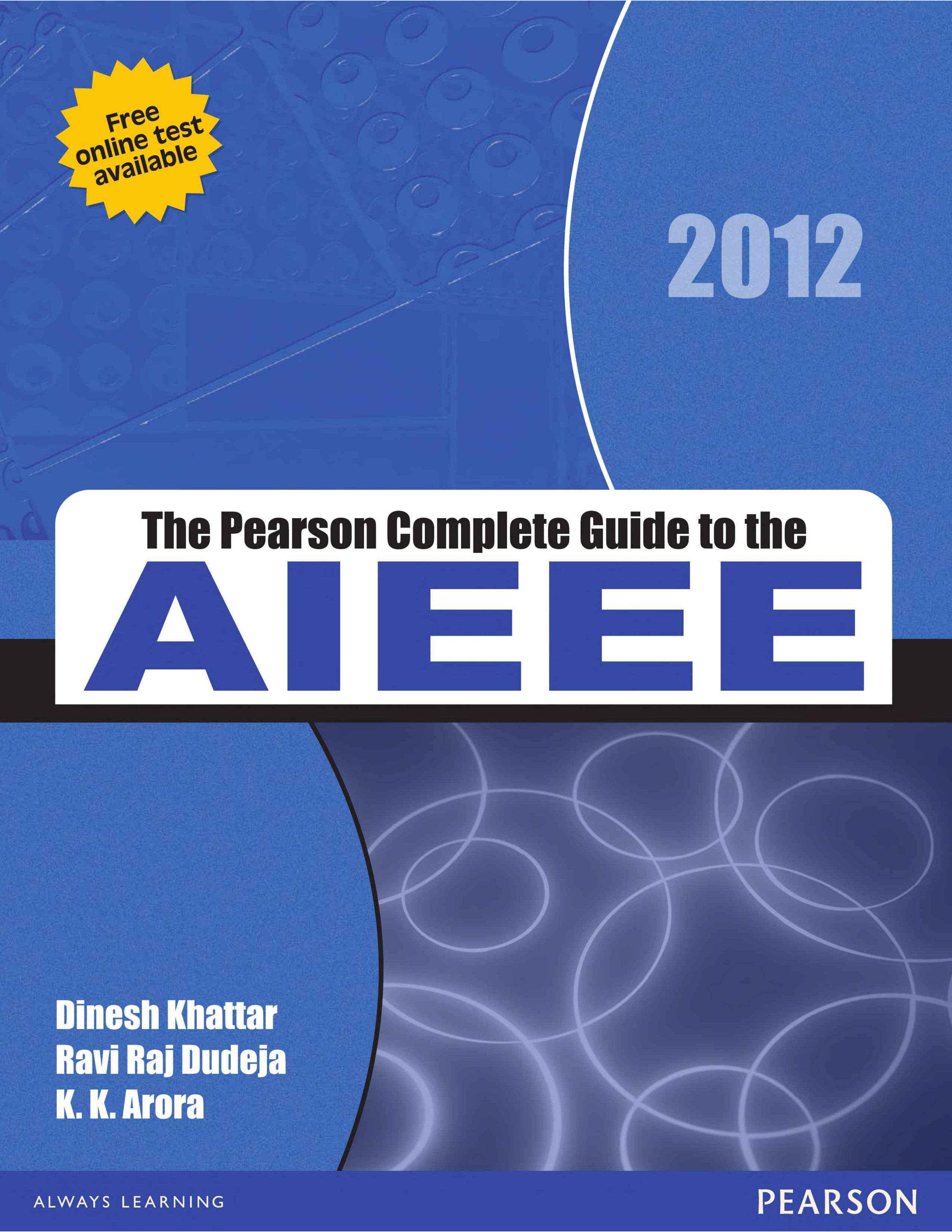

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The Pearson Complete Guide to the
AIEEE

Dinesh Khattar
Ravi Raj Dudeja
K. K. Arora

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The Pearson Complete Guide to the

AIEEE

(All India Engineering Entrance Examination)

2012

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PREFACE TO THE CURRENT EDITION

Gaining entry into India's premier engineering institutes remains as competitive as ever. We are sure that this book will help you to meet the challenge squarely. The overwhelming response to the previous editions of this book *The Pearson Complete Guide to the AIEEE*, from students and coaches in various preparatory institutes has not only boosted our confidence but has also become a shaping force for this edition.

In our effort to provide more focused study material, structured completely as per the new syllabus of AIEEE, we have withdrawn the obsolete papers from this book and added AIEEE 2011 question paper.

Suggestions and comments from the readers are always welcome.
Best of luck.

DINESH KHATTAR
RAVI RAJ DUDEJA
K. K. ARORA

PREFACE

To avoid the multiplicity of entrance examinations and solve the problem of overlap as well as to reduce the physical, mental and financial burden on students, a common entrance test for Engineering, Pharmacy and Architecture was introduced in the year 2002. Today, this is popularly known as the **All India Engineering Entrance Examination** (AIEEE). Admission to all NITs (National Institute of Technology), many universities, as well as other professional institutions such as the Army Institute of Technology, Pune and the National Institute of Foundry and Forge Technology Ranchi, is determined through the AIEEE, conducted by the Central Board of Secondary Eduction (CBSE). The Indian Institute of Information Technology at Gwalior, Jabalpur, and Ahmedabad also admit students through the AIEEE. Since competition is tough and calls for serious and systematic preparation, an effort has been made, in this book, to help students enhance their knowledge and equip them with the skills to succeed in any competitive examination.

The salient features of this book are as follows:

1. Every chapter contains a list of important definitions, formulae and problem-solving techniques. Students are expected to master all these formulae before proceeding further. Merely memorizing them is not sufficient. Students should know all the proofs of the various results found in standard text books at the senior secondary level to have the clear conceptual clarity before attempting the objective problems.
2. Plenty of short-cut methods/time-saving techniques have been discussed in every chapter.
3. At the end of every section, problems from previous years' question papers (2002–2006) have been discussed.
4. Practice exercises covering all the topics in each chapter are provided at the end for self-assessment.

We wish to thank our friends and colleagues for their help and suggestions in planning and preparing the manuscript. We would also like to thank the editorial team of Pearson Education and Mr Sachin Tyagi from Add Computers.

Suggestions and comments from our readers are always welcome.

DINESH KHATTAR
RAVI RAJ DUDEJA
K. K. ARORA

PART A PHYSICS

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Units, Dimensions and Errors

1

BRIEF REVIEW OF THE CONCEPTS

Physical quantities The quantities by means of which we describe the Laws of Physics are called physical quantities. A physical quantity may completely be specified if it has

(a) Only magnitude **(Constants or Ratio)** for example, refractive index, dielectric constant, specific gravity and so on.
(b) Magnitude and unit **(Scalars)** for example, mass, charge, current, distance and so on.
(c) Magnitude, unit and direction **(Vectors)** and follow triangle law of addition. For example displacement, torque, force etc.
(d) Magnitude and phase **(Phasors)** for example, wave, AC voltage and current, SHM and so on.
(e) **Tensors** Physical quantities which have no specified direction but have different values in different directions are called tensors. For example, moment of inertia. In anisotropic media even density, refractive index, dielectric constant, electric conductivity, stress, strain and so on become tensors. A physical quantity which has only one component is called a scalar or tensor of zero rank; if it has more than 1 component but less than or equal to four, it is called a vector or tensor of rank 1, but if it has components greater than four it is termed as tensor of higher rank.

Thus, in general a physical quantity = magnitude × unit

If the unit changes, the magnitude will also change.

Physical quantities may be divided into **fundamental** and **derived quantities.**

Fundamental quantities The quantities which do not depend upon any other quantity are called fundamental or absolute or base quantities. Initially only 3 fundamental quantities—length, mass and time were considered but with the development of science four more fundamental quantities were added. These are temperature, electric current, luminous intensity and amount of substance.

Derived quantities The quantities derived from fundamental quantities are called derived quantities like velocity, acceleration, force, momentum and so on.

Units The fixed and definite quantity taken as standard of reference with which other quantities of same kind are measured is defined as a unit.

Fundamental units The units of fundamental quantities are called fundamental units. For example units of length, mass, time and so on are called fundamental units.

Derived units units of derived quantities are called derived units. For example units of velocity, density, force and so on.

Initially three systems of units namely CGS, FPS and MKS came into existence but later on a standard international unit system was developed, more popularly known as S.I. system. In addition to seven fundamental units, there are two supplementary units, namely, angle (radian abbreviated as rad) and solid angle (stredian abbreviated str). These are also fundamental units but not considered in SI system of units. Exactly speaking S.I. system is an extension of MKS system for seven fundamental quantities tabulated below

Practical units Apart from fundamental and derived units we come across some practical units like–Light year, horse power, energy unit (1 unit = 3.6×10^6J) and so on.

Table 1.1

Physical quantity	*Unit*	*Symbol*
Mass	Kilogram	kg
Length	metre	m
Time	second	s
Temperature	Kelvin	K
Electric current	Ampere	A
Luminous intensity	Candela	cd
Amount of substance	mole	mol

While writing a unit following convention is adopted

(a) The unit named after a person starts with capital latter for example, newton is written as N (and not n).
(b) Fundamental units are written with small letters for example, metre is written as m.
(c) The symbols are not expressed in plural form for example, 50 metres will be written as 50 m.
(d) Punctuation marks such as fullstop are not used after the symbol of unit, for example, 1 litre = 10^3 cc (not 10^3c.c.)

Standards of length The most common unit is meter. Though foot is also used.

In 1960 the standard meter was defined in terms of wavelength of light, called atomic standard of length which is the distance covered in vacuum by 1650763.73 wavelengths of orange-red light of Kr-86. An accuracy 1 : 10^9 parts can be obtained with it.

In 1983 meter was defined as the length of path travelled by light in vacuum in $\frac{1}{299,792,458}$ th second.

Some other important units of length are

1 angstrom (1A°) = 10^{-10} m

1 X-ray unit (1XU) = 10^{-13} m

1 yard (yd) = 3 foot = 0.9144 m
1 ″(inch) = 2.54 cm
1 astronomical unit (1AU) = 1.49×10^{11} m
1 light year (1 ly) = 9.46×10^{15} m
1 parsec (1 pc) = 3.08×10^{16} m = $3.26\,ly$

Standards of mass Originally 1 kg mass was defined as the mass of 1 litre (10^3 cc) of water at 4°C. Now-a-days, standard kg is the mass of platinum-iridium cylinder stored in a special vault in the International Bureau of Standards in Sevres, france. The accuracy of this standard is 1 : 10^8 parts.

To measure atomic masses the unit amu (or *u*) is used.

$1u$ is the mass of $\frac{1}{12}$th mass of ${}^{12}_{6}C$ atom.

$$1\,u = \frac{1}{12} \times \left(\frac{12}{6.023 \times 10^{26}}\right) \text{kg} = 1.67 \times 10^{-27} \text{kg}.$$

In FPS system pound (lb) is the unit of mass. Sometimes **slug** is also used as the unit of mass in this system.

1 Pound mass = 0.453592737 kg
1 slug = 32.2 Pounds = 14.59 kg

Note: In astrophysics we sometimes come across chandrashekhar limit where, 1 chandrashekhar limit= 1.4 mass of sun = 2.8×10^{30} kg. Chandrashekhar showed that if the mass of an object becomes 1.4 times mass of the sun, under gravitational collapse it turns into a white dwarf.

Standards of time In 1965, atomic standard was defined. According to this standard 1s is the interval of 91192631770 vibrations of radiation corresponding to the transition between two specific hyperfine levels in ^{133}Cs (Cesium) clock which will go wrong by 1 s in 3000 years. Hydrogen maser gives a promise of producing error of 1 s only in 33,000,000 years.

It is note worthy that

(a) time can never flow back, that is, negative time does not exist.
(b) at a given instant of time a particle cannot be present in more than one position in space.

The magnitude table below describes prefixes used for multiples and submultiples of metric quantities.

Table 1.2

atto	a	10^{-18}	deca	da or D	10^{1}
femto	f	10^{-15}	hecto	h	10^{2}
pico	p	10^{-12}	kilo	k	10^{3}
nano	n	10^{-9}	mega	M	10^{6}
micro	μ	10^{-6}	Giga	G	10^{9}
milli	m	10^{-3}	terra	T	10^{12}
centi	c	10^{-2}	peta	P	10^{15}
deci	d	10^{-1}	exa	E	10^{18}

DIMENSIONS AND DIMENSIONAL FORMULAE

All physical quantities can be expressed in terms of seven fundamental units with some of their powers. These powers are called dimensions. A physical quantity expressed in terms of fundamental quantity is termed as dimensional formula of that physical quantity.

The dimensional formulae of Force, Energy, Compressibility and Capacity of a capacitor, are $[MLT^{-2}]$, $[ML^2T^{-2}]$, $[M^{-1}LT^2]$ and $[M^{-1}L^{-2}T^4A^2]$ respectively.

Significant Figures Gives the accuracy with which a physical quantity is expressed. The number of digits which are known reliably or about which we have confidence in our measurement, plus the first digit that is uncertain, are termed as significant figures. For instance, the length of the table is 122.3 cm. This has 4 significant figures and 3 is uncertain. It is worth mentioning that significant figures of a physical quantity depends upon the least count of the instrument with which it is being measured.

Rules for Determining Significant Figures

(i) All the non-zero digits are significant. For example 241.54 has five significant digits.
(ii) All zeros occurring between two non-zero digits are significant. For example, 501.002 has six significant digits.
(iii) The zeros occurring between the decimal point and the non-zero digits are not significant provided the integral part is zero. For example, 0.00243 has only 3 significant digits.
(iv) All zeros to the right of a non-zero digit in a number written without a decimal point are not significant. For example, 53200 has only three significant digits.

Exception This rule does not work when we record the values on actual measurement basis. For example, distance between two places is 1710 m. This figure has 4 significant digits.

(v) All zeros occurring to the right of non-zero digit in a number written with a decimal point are significant. For example, 2.3200 has five significant digits.

*** The number of significant figures does not vary with choice of units. For example, length of a rod is 82 cm. The number of significant figures remain 2 even if we represent it in metres or km that is, 0.82 m or 0.00082 km. This signifies that the location of decimal point does not determine the number of significant figures by itself alone.

**** In exponent form, the exponential term does not contribute to the significant figures. Thus, $8.12 \times 10^5 = 812000$ which has only 3 significant figures.

Rules for Rounding Off

(i) If the digit to be dropped is less than 5, then the preceding digit should be kept unchanged. For example, we write 3.52 as 3.5.
(ii) If the digit to be dropped is greater than 5, then the preceding digit should be increased by 1. For example, 8.76 will be written as 8.8.
(iii) If the digit to be dropped is 5 followed by digits other than zero, then the preceding digit be increased by 1. For example, 12.254 will be rounded off to first place of decimal as 12.3.

Errors in Measurements

Difference between actual and measured value of a physical quantity is called error. If a_m is measured value and a_t is actual value of a physical quantity then the error is

$$e = \Delta a = |a_m - a_t|$$

Types of Error

Errors may broadly be divided into two types: systematic errors and random errors.

(a) **Systematic error** errors arising due to the system of measurement or the errors made due to the parts involved in the system of measurement are called systematic errors. Since the system involves instrument, observer and environment, therefore, systematic error is of three types. Namely Instrumental error, Personal error or errors caused by observer and environmental error.

(b) **Random errors** (or statistical errors) Consider an example. The probability of tossing a coin is 1/2. If a coin is tossed 1000 times then the chance that we get exactly 500 times head and 500 time tail is negligible. If 1 *m*A current is passing through a wire, can we be sure that always $\left(n=\frac{10^{-3}}{1.6\times10^{-19}}=6.25\times10^{15}\right)$, constant number of electrons equal to 6.25×10^{15} are flowing per second through it. These examples illustrate how random errors creep in. Even this error cannot be removed.

Methods of Expressing Error

(a) **Absolute error** The deviation from true value of measured value or deviation of *i*th value from its mean value (of all observations).

Thus, $\Delta x_i = |x_i - x_m|$

is absolute error where x_m is mean value and x_i is the *i*th component of the observation.

(b) **Relative error** The ratio of mean absolute error to the true value of physical quantity is called relative error., that is,

$\frac{\Delta x}{x}$ or $\frac{\Delta x}{x_m}$ is called relative error.

(c) **Percentage error** = Relative error × 100

$$= \frac{\Delta x}{x} \times 100$$

Propagation or Combination of Errors

Case (i) when $x = a + b$

then maximum possible % error $= \frac{\Delta x}{x} \times 100$

$$= \frac{\Delta a + \Delta b}{a+b} \times 100$$

Case (ii) when $x = a - b$

then maximum possible % error $= \frac{\Delta x}{x} \times 100$

$$= \frac{\Delta a + \Delta b}{a-b} \times 100$$

Case (iii) when $x = a.b$

then maximum possible % error $= \frac{\Delta x}{x} \times 100$

$$= \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) \times 100$$

Case (iv) when $x = \frac{a}{b}$

then maximum possible % error $= \frac{\Delta x}{x} \times 100$

$$= \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right) \times 100$$

Case (v) when $x = \frac{a^l b^m}{y^p z^k}$

then maximum possible % error

$$= \frac{\Delta x}{x} \times 100 = \left(\frac{l\Delta a}{a} + \frac{m\Delta b}{b} + \frac{p\Delta y}{y} + \frac{k\Delta z}{z}\right) \times 100$$

SHORT-CUTS AND POINTS TO NOTE

(i) Remember all possible formulae connecting the physical quantity and see whose dimensions in the formulae are known, use that formula. For example,

$$C = \frac{\varepsilon_0 A}{d},\ C = \frac{Q}{V},\ \text{(Energy) } E = \frac{Q^2}{2C}$$

If you use $C = \frac{Q^2}{E} = (AT)^2 (ML^2T^{-2})^{-1}$

$= M^{-1}L^{-2}T^4A^2$ can be easily written.

(ii) Remember the rules to write significant numbers.

(iii) Remember that instrumental error which is equal to least count of the instrument is taken into account. Remember the rules how errors combine

(1) if $x = a + b$ then $\frac{\Delta x}{x} = \frac{\Delta a + \Delta b}{a+b}$

(2) if $x = a - b$ then $\frac{\Delta x}{x} = \frac{\Delta a + \Delta b}{a-b}$

(3) if $x = a.b$ or $\frac{a}{b}$ then $\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$

(4) if $x = \frac{a^l b^m}{y^p z^k}$

then $\frac{\Delta x}{x} = \frac{l\Delta a}{a} + \frac{m\Delta b}{b} + \frac{p\Delta y}{y} + \frac{k\Delta z}{z}$

(iv) To find an absolute error first find $\frac{\Delta x}{x}$ and then multiply it with x.

(v) Since we find maximum possible error, therefore, error due to each variable is added.

(vi) Donot consider that dimensional analysis will always work. It also has its limitations. But one must check that dimensions on left hand side and right hand side are equal.

(vii) Least count of vernier callipers

= (1*MSD* – 1*VSD*) (value of 1*MSD*), where *MSD*-Main scale division *VSD*-Vernier scale division.

If n divisions of *MSD* coincide with $(n + 1)$ divisions of *VSD*, then $1VSD = \frac{n}{n+1} MSD$

and vernier constant $= \left(1 - \frac{n}{n+1}\right)$ (value of 1*MSD*).

(viii) Least count of screwgauge or spherometer

$$= \frac{\text{Pitch}}{\text{no. of divisions on circular scale}}$$

$$\text{Pitch} = \frac{\text{number of divisions moved on linear scale}}{\text{number of rotations given}}.$$

CAUTION

(i) Not recalling a relation whose other dimensions are known.

- Assume we want to find dimensions of resistance and we write

$$R = \frac{V}{I}$$

and we donot know the dimensions of V. It is better to use

$$I^2R = P \text{ (power)}$$

Therefore, $R = \frac{P}{I^2} = [ML^2T^{-3}A^{-2}]$

is easily determined.

To find V using $V = IR$ is not a good technique, rather we can use $qV = E$ (energy) then $V = \frac{E}{q}$

$$= ML^2T^{-3}A^{-1}.$$

(ii) Not remembering significant figure rules.

- Remember the rules given in text.

(iii) Considering the fact that when $X = a - b$ or $X = \frac{a}{b}$ the error due to 'b' that is, Δb should be subtracted from Δa

- Since we find the maximum possible error, errors due to individual variables are always added and not subtracted.

(iv) Considering that only scalars and vectors are Physical quantities.

- Physical quantities are (i) Constants or Ratios (ii) Scalars (iii) vectors (iv) Phasor (v) Tensors.

(v) Considering error is made only by observer due to carelessness.

- Errors also occur due to instrument and environment. Moreover, statistical or random errors also occur.

Solved Problems

1. The terms in column A are related to column B and C. Match them correctly

A	B	C
Planck's constant	Moderator	Matter-waves
Raoult	Angstrom	Threshold frequency
Nuclear-reactor	Erg-second	E = mc^2
Einstein	Vapour pressure lowering	Heavy water
Mass defect	Photoelectric effect	6.625×10^{-27}
de-Broglie	Nuclear constant	Molecular weight

Solution

Planck's constant	Erg-second	6.625×10^{-27}
Raoult	Vapour pressure lowering	molecular weight
Nuclear reactor	Moderator	Heavy water
Einstein	Photoelectric-effect	Threshold frequency
Mass defect	Nuclear constant	E = mc^2
de-Broglie	Angstrom	Matter-waves

2. The radius of a proton is ~ 10^{-5}A° and radius of the universe is ~ 10^{26}m. Name the physical object whose size is approximately half way mark between these two on a logarithmic scale.

Solution radius of proton $r_p = 10^{-5}\text{A}^\circ = 10^{-15}\text{m}$

radius of universe $r_u = 10^{+26}\text{m}$

$$\text{Log}x = \frac{\log r_p + \log r_u}{2} = 5.5$$

or $x \cong \sqrt{10} \times 10^5 \sim 10^6\text{m}$

which is the size of moon.

3. A man on seeing a lightening starts counting seconds until he hears thunder. He then claims to have found an approximate but simple rule that if the count of second is divided by an integer, the result directly is available in km, the distance of the lightening source. What is the integer if the velocity of sound is 330 ms^{-1}?

Solution Let n be the integer, then $\frac{t(s)}{n}$ = distance in km = v(km/s) $t(s)$

or $n = \frac{1}{v \text{ kms}^{-1}} = \frac{1}{330 \times 10^{-3}} = 3$

4. Would 2 litres of benzene weight more in summer or in winter?

Solution weight $W = mg = \rho Vg$

Since V is fixed, therefore, $W \alpha \rho$. But ρ will decrease as temperature increases. Hence, it will weigh less in summer.

5. Is the time variation with position as shown in the Fig. 1.1 actually observed in nature?

Solution No, because a particle cannot be present at two positions simultaneously.

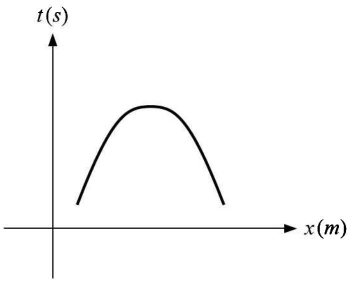


Fig. 1.1

6. A gas bubble is formed under water due to explosion and oscillates with a time period T proportional to $p^a \rho^b E^c$ where p is static pressure, ρ is density of water and E is total energy of explosion. Find the values of a, b and c.

Solution $T \propto p^a \rho^b E^c$

$[M^\circ L^\circ T] = [ML^{-1}T^{-2}]^a\,[ML^{-3}]^b\,[ML^2T^{-2}]^c$

$0 = a + b + c$

$0 = -a - 3b + 2c$

$1 = -2a - 2c$

Solving $a = \dfrac{-5}{6}$, $b = \dfrac{1}{2}$ and $c = \dfrac{1}{3}$

7. If velocity, force and time are taken to be the fundamental quantities then find the dimensional formula for (a) Mass (b) Energy

Solution (a) $[M] = [LT^{-1}]^a\,[MLT^{-2}]^b\,[T]^c$

$b = 1;\ a + b = 0$

$\therefore\ a = -1$

$-a - 2b + c = 0$

$c = +1$

Thus, $M = KV^{-1}FT$

(b) $[ML^2T^{-2}] = [LT^{-1}]^a\,[MLT^{-2}]^b\,[T]^c$

$\because\ b = 1$, and $2 = a + b$

$\therefore\ a = 1$

$-a - 2b + c = -2;$

$\therefore\ c = 1$

$\therefore\ E = KV^{+1}F^1T^1$ or $E = KVFT$

8. Find the dimensions of solar constant.

Solution Solar constant is the amount of energy received per cm² per minute

$$S = \frac{E}{TL^2} = \frac{[ML^2T^{-2}]}{[L^2T]} = [MT^{-3}]$$

9. Find the dimensions of (a) self inductance (b) resistance (c) potential.

Solution (a) $\dfrac{1}{2}Li^2 = E$

or $L = \dfrac{E}{i^2} = \dfrac{[ML^2T^{-2}]}{[A^2]} = [ML^2T^{-2}A^{-2}]$

(b) $qV = E$

or $V = \dfrac{E}{q} = \dfrac{[ML^2T^{-2}]}{[AT]} = [ML^2T^{-3}A^{-1}]$

(c) $R = \dfrac{V}{I} = \dfrac{[ML^2T^{-3}A^{-1}]}{[A]} = [ML^2T^{-3}A^{-2}]$

10. The position of a particle is given by

$x = a\sin\omega t,$

$y = a\cos 2\omega t$

find the trajectory.

Solution $y = a\cos 2\omega t = a\,(1 - 2\sin^2\omega t) = a\left(1 - \dfrac{2x^2}{a^2}\right)$ which is the equation of a parabola and hence, trajectory is a parabola.

11. In the relation $X = 3YZ^2$, X and Z represent the dimensions of capacitance and magnetic induction respectively. Find the dimensions of Y.

Solution $X = [C] = [M^{-1}L^{-2}T^2Q^2]$; $Z = [MT^{-1}Q^{-1}]$

$$Y = \frac{X}{Z^2} = \frac{M^{-1}T^2L^{-2}Q^2}{[MT^{-1}Q^{-1}]^2} = [M^{-3}T^4L^{-2}Q^4]$$

12. Match the col I with col II some of the physical quantities may have more than one choice.

	I	II
1.	Capacitance	(a) ohm × second
2.	Inductance	(b) Coul² Joule⁻¹
3.	Magnetic Induction	(c) Coulomb volt⁻¹
		(d) Newton (Ampere – m)⁻¹
		(e) Volt second (Ampere)⁻¹

Solution 1. $C = \dfrac{Q}{V} = [M^{-1}L^{-2}T^4A^2]$

(a) ohm–s $= [ML^2T^{-3}A^{-2}]\,[T] = [ML^2T^{-2}A^{-2}]$

2. $L = \dfrac{E}{i^2} = [ML^2T^{-2}A^{-2}]$

(b) coul² J⁻¹ $= [AT]^2\,[ML^2T^{-2}] = [M^{-1}L^2T^4A^2]$

3. $B = \dfrac{F}{il} = [MT^{-2}A^{-1}]$

(c) coul $V^{-1} = [AT]^2\,[ML^2T^{-2}]$

(d) N(A – m)⁻¹ $= [MT^{-2}A^{-1}]$

(e) $V - SA^{-1} = [ML^2T^{-2}A^{-2}]$

Therefore, 1 = (b) and (c); 3 = (d)

and 2 = (a) and (e)

13. Find the dimensions of $\dfrac{1}{2}\varepsilon_0 E^2$.

Solution $\dfrac{1}{2}\varepsilon_0 E^2$ represents energy density carried by a wave

$$\therefore\ \frac{1}{2}\epsilon_0 E^2 = \frac{\text{Energy}}{\text{Volume}} = \frac{ML^2T^{-2}}{L^3} = ML^{-1}T^{-2}$$

14. Name the three physical quantities having same dimensions.

Solution Work, Energy and Torque

15. The pairs of physical quantities which have same dimensions are:

(a) Reynolds number and coefficient of friction

(b) Latent heat and gravitational potential

(c) Curie and frequency of light wave

(d) Planck's constant and torque

Solution (a) both are dimensionless

(b) both have dimensions $[L^2T^{-2}]$

(c) both have dimensions $[T^{-1}]$

$\therefore$ (a), (b) and (c) are correct.

16. Compute 5.4 × 0.125

Solution $5.4 \times 0.125 = 0.675 \simeq 0.68.$

17. Two resistances $R_1 = 10.0\,\Omega \pm 0.1\,\Omega$ and $R = 5.0 \pm 0.1\,\Omega$ are connected (a) in series (b) in parallel. Calculate % error.

Solution (a) **In series:** $R_S = R_1 + R_2$

$$100 \times \frac{\Delta R_S}{R_S} = \frac{\Delta R_1 + \Delta R_2}{R_1 + R_2} \times 100$$

$$= \frac{0.1 + 0.1}{10 + 5} \times 100 = 1.33\%$$

(b) **In Parallel** $R_P = \dfrac{R_1R_2}{R_1 + R_2}$

$$\text{Percentage error} = \frac{\Delta R_P}{R_P} \times 100$$

$$= \left(\frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2} + \frac{\Delta R_1 + \Delta R_2}{R_1 + R_2}\right) \times 100$$

$$= \left(\frac{0.1}{10} + \frac{0.1}{5} + \frac{0.2}{15}\right) \times 100 = 4.33\%.$$

18. A sphere has a mass of 12.2 kg ± 0.1 kg and radius 10 cm ± 0.1 cm. Find the maximum % error in density.

Solution Density $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r^3}$

$$\frac{d\rho}{\rho} \times 100 = \left(\frac{\Delta M}{M} + \frac{3\Delta r}{r}\right) \times 100$$

$$= \left(\frac{0.1}{12.1} + 3 \times \frac{0.1}{10}\right) \times 100$$

$$= 3.83\%.$$

Problems for Practice

1. A science student takes 100 observations in an experiment. Second time he takes 500 observations in the same experiment. By doing so the possible error becomes
(a) 5 times (b) 1/5 times
(c) unchanged (d) none of these

2. If the error in the measurement of radius of a sphere is 1% then the error in the measurement of volume will be
(a) 1.1% (b) 3% (c) 5% (d) 8%

3. How many wavelengths of Kr^{86} are contained in one metre?
(a) 2347127.23 (b) 1650763.73
(c) 1553164.13 (d) 652189.63

4. Light year is the unit of
(a) speed (b) mass
(c) distance (d) time

5. Debye is the unit of
(a) density
(b) rms velocity
(c) electric dipole moment
(d) magnetic dipole moment

6. The unit of surface energy per unit area may be expressed as
(a) Nm^{-2} (b) Nm^{-1} (c) Nm (d) Nm^2

7. Density of a liquid is 13.6 gcm^{-3}. Its value in SI units is
(a) 136.0 $kg\,m^{-3}$ (b) 13600 $kg\,m^{-3}$
(c) 13.6 $kg\,m^{-3}$ (d) 1.36 $kg\,m^{-3}$

8. The ratio of the atomic radius to nuclear radius is
(a) 10^{-2} (b) 10^2 (c) 10^{-4} (d) 10^4

9. The volume of cube in m^3 is numerically equal to its surface area in m^2. The volume of the cube is
(a) 1000 m^3 (b) 512 m^3
(c) 216 m^3 (d) 64 m^3

10. Which of the following system of units is not based on the unit of mass, length and time alone?
(a) FPS (b) SI (c) CGS (d) MKS

11. If the units of force and that of length are doubled, the unit of energy will be
(a) 1/4 times (b) 1/2 times
(c) 2 times (d) 4 times

12. Estimated radius of the universe is of the order
(a) 10^{80}m (b) 10^{55}m
(c) 10^{30}m (d) 10^{25}m

13. Which of the following is the unit of latent heat?
(a) Jkg^{-1} (b) $Jkg^{-1}mol^{-1}$
(c) $J\,mol^{-1}$ (d) J

14. What are the dimensions of length in force × displacement/time?
(a) –2 (b) 0 (c) 2 (d) none of these

15. The unit of $1/l \sqrt{T/\mu}$ is the same as that of
(a) wavelength (b) wave number
(c) frequency (d) time period

16. The angular frequency is measured in rad s^{-1}. Its dimension in length are
(a) –2 (b) –1
(c) 0 (d) 2

17. If force (F) is given by $F = Pt^{-1} + \alpha t$, where t is time. The unit of P is same as that of
(a) velocity (b) displacement
(c) acceleration (d) momentum

18. The dimensional formula for strain is same as that for
(a) thrust
(b) angle of twist
(c) modulus of elasticity
(d) stress

19. The dimensions of angular frequency are the same as that of the
(a) angle (b) angle/frequency
(c) frequency (d) time period

20. Given that v is the speed, r is radius and g is acceleration due to gravity. Which of the following is dimensionless?
(a) v^2g/r (b) v^2rg (c) vr^2g (d) v^2/rg

21. If g is the acceleration due to gravity and R is the radius of earth, then the dimensional formula for gR is
(a) $M^2L^2T^{-2}$ (b) $M^\circ L^\circ T^\circ$
(c) ML^2T^{-2} (d) $M^\circ L^2T^{-2}$

22. If L and R denote the inductance and resistance, then the dimensional formula for L/R is same as that for
(a) (frequency)^2 (b) (time period)^2
(c) frequency (d) time period

23. ML^2T^{-3} are the dimensions of
(a) work (b) power
(c) momentum (d) force

24. Lux is the unit of
(a) intensity of illumination
(b) luminous efficiency
(c) luminous flux
(d) luminous intensity

25. If the units of M, L are doubled then the unit of kinetic energy will become
(a) 2 times (b) 4 times
(c) 8 times (d) 16 times

26. Which of the following is not the unit of time
(a) solar day (b) parallactic second
(c) leap year (d) lunar month

27. The correct statement about poisson's ratio is
(a) Its unit is Newton
(b) Its dimension are MLT^{-2}
(c) Its unit is Nm^{-2}
(d) It is dimensionless

28. In an experiment to determine acceleration due to gravity by simple pendulum, a student commits 1% positive error in the measurement of length and 3% negative error in the time then % error in g will be
(a) 3% (b) 4% (c) 7% (d) 10%

29. The ratio of nuclear magneton and Bohr magneton is
(a) $\frac{m_e}{m_p}$ (b) $\frac{2m_p}{m_e}$ (c) $\frac{d\rho}{\rho}$ (d) $m_e m_p$

30. One watt-hour is equivalent to
(a) 6.3×10^3 Joule
(b) 6.3×10^{-7} Joule
(c) 3.6×10^3 Joule
(d) 3.6×10^{-3} Joule

31. The mass of an electron in MeV is
(a) 51 MeV/c^2 (b) 102 MeV/c^2
(c) 1.02 MeV/c^2 (d) 0.51 MeV/c^2

32. Which of the following does not possess the same dimensions as pressure?
(a) stress (b) energy gradient
(c) bulk modulus (d) energy density

33. If C and L denote the capacitance and inductance, then the dimensional formula for CL is same as that for
(a) (frequency)^2 (b) (time period)^2
(c) frequency (d) time period

34. Given that C denotes capacitance of a capacitor and V is the potential difference across its plates. Then the dimensions of CV^2 are same as that of
(a) momentum (b) power
(c) force (d) torque

35. The product of energy and time is called action. The dimensional formula for action is same as that for
(a) power (b) angular energy
(c) force × velocity (d) impulse × distance

36. The least count of an instrument is 0.01 cm. Taking all precautions, the most possible error in the measurement can be
(a) 0.02 cm (b) 0.01 cm
(c) 0.005 cm (d) 0.001 cm

37. The length of the rod is measured with a rod calibrated in millimetres. Which of the following is the correct expression for length, keeping in view the experimental error
(a) (20.0 ± 0.5) cm (b) (20.0 ± 0.1) cm
(c) (20.00 ± 0.05) cm (d) (20.00 ± 0.01) cm

38. The time taken by a vehicle to go from one station to the another is 100 seconds. It is recorded with a stop watch having the least count of 1 second. How many significant figures are there in the value of time t = 100 s?
(a) 3 (b) 4 (c) 2 (d) 1

39. What is number of significant figures in 0.310×10^3?
(a) 6 (b) 4 (c) 3 (d) 2

40. A student performs an experiment with simple pendulum and measures time for 10 vibrations. If he measures the time for 100 vibrations, the error in the measurement of time period will be reduced by a factor of
(a) 1000 (b) 100 (c) 90 (d) 10

41. The volume of a drop is 0.175 cm^3. The volume of 20 drop is
(a) 35×10^{-1} cm^3 (b) 3.5 cm^3
(c) 0.35×10 cm^3 (d) 0.035×10^2 cm^3

42. The mass of a body is 20.00 g and its volume is 10.00 cm^3. If the measured values are expressed up to the correct significant figures, the maximum error in the value of density is
(a) 0.100 g cm^{-3} (b) 0.010 g cm^{-3}
(c) 0.001 g cm^{-3} (d) none of these

43. The least count of the metre rod is 0.1 cm. What is the permissible error in the length of the rod measured with it?
(a) ±0.1 cm (b) ±0.2 cm
(c) ±0.05 cm (d) ±0.01 cm

44. Error in the measurement of radius of a sphere is 1%. The error in the calculated value of its volume is
(a) 7% (b) 5% (c) 3% (d) 1%

45. Which of the following is the least accurately recorded observation?
(a) 50000 mm (b) 5000 mm
(c) 500 mm (d) 5 m

ANSWERS

1. (b)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (b)
11. (d)	**12.** (d)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (d)	**18.** (b)	**19.** (c)	**20.** (d)
21. (d)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (d)	**28.** (c)	**29.** (a)	**30.** (c)
31. (d)	**32.** (b)	**33.** (b)	**34.** (d)	**35.** (d)	**36.** (b)	**37.** (b)	**38.** (a)	**39.** (c)	**40.** (d)
41. (a)	**42.** (d)	**43.** (a)	**44.** (c)	**45.** (d)					

2

Motion in One Dimension

BRIEF REVIEW OF THE CONCEPTS

Frame of reference may be of two types (a) inertial (b) non-inertial. Inertial frame of reference is one which is either fixed or moves with a constant velocity in the same straight line. Non-inertial or accelerated frame of reference moves with an acceleration 'a'. Newton's laws are valid only in inertial frame. Equations of motion is to be modified by considering pseudo-vectors in accelerated frame of reference.

One dimensional motion If the particle changes its position only in one of the x, y or z directions with respect to time, then the motion is said to be one dimensional. Since the particle moves along a straight line the motion may also be termed as linear or rectilinear.

Speed The time rate of change of distance is called speed, that is, $v = \frac{dx}{dt}$

Velocity The time rate of change of displacement is called velocity, that is, $v = \frac{d\bar{x}}{dt}$

Displacement The shortest distance between initial and final position of the particle is called displacement. The length of this straight line along with the sense, initial to final, is called displacement vector.

Speed or velocity may be divided into four types.

(a) **Uniform speed** If $\frac{dx}{dt}$ is constant throughout the motion, then speed is said to be uniform.

(b) **Variable speed** If $\frac{dx}{dt}$ is not constant with respect to time but varies at different intervals of time, then the body is said to move with variable speed.

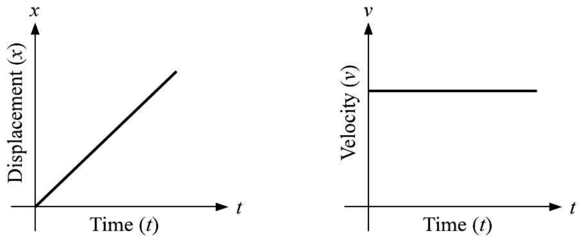
(c) **Average speed** is that uniform speed with which if the body would have moved, it would have covered the same distance as it does otherwise by moving with variable speed. Thus,

$$V_{av} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

(d) **Instantaneous speed** The speed at a particular instant of time is called instantaneous speed. For example, $\left.\frac{dx}{dt}\right|_{t=4}$ will describe speed at 4th second.

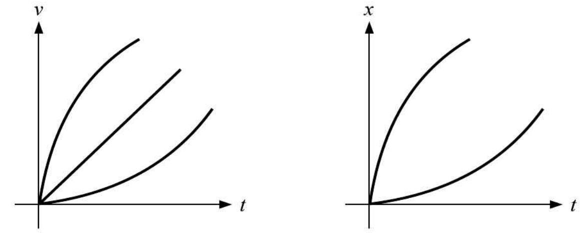
Note: If distance is changed with displacement and speed with velocity, then the same definitions may be employed for different types of velocities.

The following graphs describe various types of velocities



(a) (b)

Fig. 2.1 Uniform velocity



(a) (b)

Fig. 2.2 Variable velocity

Average speed in different circumstances

(a) **Particles covering different distances in different times** Say a particle covers s_1 distance in time t_1, s_2 in time t_2, s_3 in time t_3 and so on then average speed is

$$V_{av} = \frac{s_1 + s_2 + s_3 + \ldots}{t_1 + t_2 + t_3 + \ldots} = \frac{s_1 + s_2 + s_3 + \ldots}{\frac{s_1}{v_1} + \frac{s_2}{v_2} + \frac{s_3}{v_3} + \ldots}$$

Special case

if $s_1 = s_2 = s$

$$V_{av} = \frac{2s}{s\left(\frac{1}{v_1} + \frac{1}{v_2}\right)} = \frac{2v_1v_2}{v_1 + v_2} \quad \text{(Harmonic mean)}$$

(b) **Bodies moving with different speeds in different intervals of time** A body moves with velocity v_1 in time t_1, v_2 in time t_2, v_3 in time t_3 and so on then

$$V_{av} = \frac{v_1t_1 + v_2t_2 + v_3t_3 + \ldots}{t_1 + t_2 + t_3 + \ldots}$$

Special case

if $t_1 = t_2 = t_3 \ldots = t$ (say) then

$$V_{av} = \frac{v_1 + v_2 + v_3 + \ldots v_n}{n} \text{ (Arithmetic mean)}$$

Units of speed or velocity ms^{-1} in SI, cms^{-1} in CGS and fts^{-1} in FPS. Dimensional formula $[LT^{-1}]$.

Acceleration The time rate of change of velocity is called acceleration. Its units is ms^{-2} in SI.

$$\vec{a} = \frac{d\vec{V}}{dt}$$

(a) **Uniform acceleration** A body is said to move with uniform acceleration if acceleration remains constant throughout the motion and direction of motion does not change with time, that is,

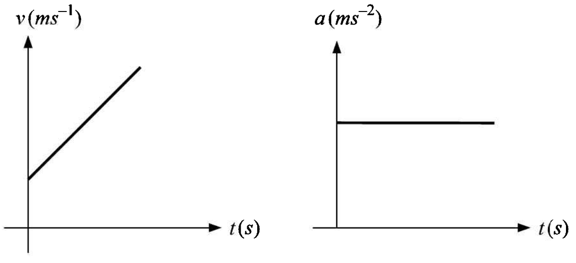
(i) $\vec{a} = \dfrac{d\vec{V}}{dt}$ = constant

(ii) Direction of motion remains unchanged.

(b) **Variable or non-uniform acceleration** If the acceleration is not constant or if the acceleration is constant but direction of motion varies or both vary, the particle is said to possess variable acceleration.

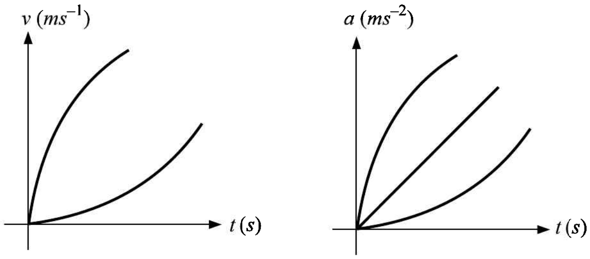
Unit of acceleration ms^{-2} in SI system, cms^{-2} in CGS system, fts^{-2} in FPS system.

Dimensional formula of acceleration $[LT^{-2}]$



(a) **(b)**

Fig. 2.3 Uniform acceleration



(a) **(b)**

Fig. 2.4 Non-uniform acceleration

Equations of motion

(a) $v = u + at$

(b) $s = ut + \frac{1}{2}at^2$

(c) $v^2 - u^2 = 2as$

(d) $s_{\text{nth}} = u + \frac{a}{2}(2n-1)$

The conditions under which these equations can be applied

(a) Motion should be one dimensional

(b) Acceleration should be uniform

(c) Frame of reference should be inertial

SHORT-CUTS AND POINTS TO NOTE

(i) Slope of $v - t$ graph is acceleration

(ii) Slope of $x - t$ graph is velocity

(iii) Average speed

$$V_{av} = \frac{\text{total distance travelled}}{\text{total time taken}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$= \frac{x_1 + x_2 + \ldots}{t_1 + t_2 + \ldots} = \frac{x_1 + x_2 + \ldots}{\frac{x_1}{v_1} + \frac{x_2}{v_2} + \ldots}$$

if a body covers equal distances with different velocities that is, $x_1 = x_2 = \ldots + x$

$$\text{or } \frac{1}{V_{av}} = \frac{1}{n}\left[\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}\right]$$

$$V_{av} = \frac{n(v_1 v_2 \ldots v_n)}{v_1 v_2 \ldots v_{n-1} + v_2 v_3 v_4 \ldots v_n + \ldots}$$

$= \dfrac{2v_1 v_2}{v_1 + v_2}$ for half distance with v_1 and other half with v_2 (that is, harmonic mean)

If a body moves with different velocities in equal interval of time then

$$V_{av} = \frac{v_1 + v_2 + \ldots + v_n}{n} \text{ (arithmetic mean).}$$

(iv) Area under $v - t$ graph is displacement.

(v) Area under $a - t$ graph is velocity at the end of time interval.

(vi) When a body leaves a moving body it acquires its velocity but not acceleration.

(vii) Instantaneous velocity $v(t_1) = \left.\dfrac{dx}{dt}\right|_{t=t_1}$

(viii) Apply $v = u + at$; $s = ut + \frac{1}{2}at^2$ and so on when

(1) motion is one-dimensional

(2) acceleration is uniform

(3) frame of reference is inertial

All the three be simultaneously true.

(ix) If acceleration is a variable always start with $\dfrac{dv}{dt} = f(t)$

velocity $v = \int dv = \int f(t)\,dt$ (if acceleration is a function of time).

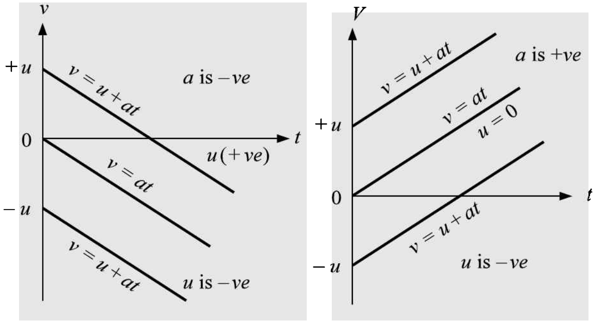
(x) If acceleration is variable and function of displacement or velocity.

For example, $a \propto -v^2$ then start

$$\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = -kv^2$$

$$\text{or } \quad \frac{vdv}{dx} = -kv^2 \quad \text{or} \quad \int \frac{dv}{v} = \int -k\,dx.$$

(xi) To draw graphs see to which mathematical function the equation resembles. For example, if acceleration is uniform then $v = u + at$ resembles $y = c + mx$ that is, equation of a straight line. Therefore, graph of velocity when acceleration is uniform is a straight line. If initial velocity $u = 0$, then it passes through origin.

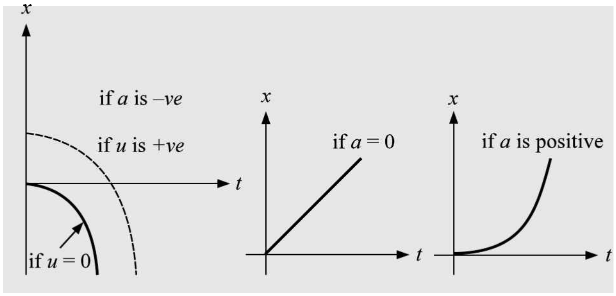


(a) **(b)**

Fig. 2.5 Graphs of straight line in various situations

(xii) If drawing graph for $x = ut + \frac{1}{2}at^2$

note that if $a = 0$ then $x = ut$ which is the equation of a straight line. Draw a straight line. If a is uniform then $x = ut + \frac{1}{2}at^2$ resembles $y = ax + bx^2$, that is, which is the general equation of a parabola. Draw a parabola.



(a) **(b)** **(c)**

Fig. 2.6

(xiii) Time cannot be negative in physics.

(xiv) If the particle starts from rest and accelerates with α, after acquiring maximum velocity then decelerates with β and finally comes to rest.

Then,

$v_{\max} = \frac{\alpha\beta t}{\alpha + \beta}$ and distance covered $s = \frac{\alpha\beta t^2}{2(\alpha + \beta)}$

Note: Remember these results as they can be directly used to solve similar questions.

CAUTION

(i) We know in uniform motion $\vec{V}$ = constant $= \vec{V}_{av}$

- However, converse is not true that is, if $\vec{V} = \vec{V}_{av}$ then motion may or may not be uniform.

(ii) Applying $v = u + at$; $s = ut + \frac{1}{2}at^2$ and so on even if acceleration is not uniform.

- Do not apply $v = u + at$; $s = ut + \frac{1}{2}at^2$ and so on when acceleration is variable. Rather use $\frac{dv}{dt} = a$ or $\frac{dx}{dt} = v$.

(iii) Applying $v = u + at$; $s = ut + \frac{1}{2}at^2$ and so on without modification even if the frame of reference is noninertial.

- For example, if the lift is moving up with an acceleration 'a' then the effective acceleration for a body falling from its ceiling and so on is $(g + a)$, that is, apply vector algebra or relative acceleration.

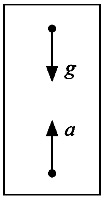


Fig. 2.7

(iv) Considering $a = 0$ if speed is constant.

- Remember if the direction changes though the speed may be constant, acceleration is not zero and acts normal to the direction of motion. Such an acceleration is called radial or normal acceleration.

(v) Not differentiating between average and instantaneous velocities.

- If a particle travels according to the equation $x = t^2 - 2t + 5$ where x is in metres and t in seconds.

  Then $v = \frac{dx}{dt}$ is instantaneous velocity while $V_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$ where t_2 and t_1 are final and initial intervals.

(vi) Not caring about inertial/noninertial frames.

- If frame of reference is inertial, only then apply $v = u + at$ and other equations of motion.

Otherwise use $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = a$ or $\frac{vdv}{dx} = a$.

Solved Problems

1. A particle moves half the distance with velocity 'u' and the other half with velocity 'v' in the same straight line. Determine average velocity.

(a) $\frac{u+v}{2}$ (b) $\frac{uv}{u+v}$

(c) $\frac{2uv}{u+v}$ (d) $u + v$

Solution (c) Let 2s be the total distance/displacement, then the time taken to move 1st half distance is $t_1 = \frac{s}{u}$ and time taken to move 2nd half distance is $t_2 = \frac{s}{v}$

$$\text{Average velocity} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

$$= \frac{2s}{\frac{s}{u}+\frac{s}{v}} = \frac{2uv}{u+v}$$

2. A particle falls from a height h. In the last 0.2 s it travels 6 m. Find the height h.

(a) 48.05 m (b) 420.05 m

(c) 32.05 m (d) none of these

Solution (a) Let the body fall from point O, then

$OB = h$

$OA = h - 6.$

Let the velocity at A be u, then

$u_1^2 - u^2 = 2g(h - 6)$

or $u_1^2 = 2g(h - 6)$ ...(i)

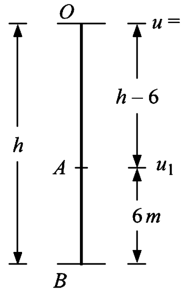


Fig. 2.8

and using $s = ut + \frac{1}{2}gt^2$

$$6 = u_1(.2) + \frac{1}{2}(10)(.2)^2 \Rightarrow u_1 = 29\,\text{ms}^{-1}$$

From (i) we get $29^2 = 20(h - 6)$ or $h = 48.05$ m

3. A particle travels 32 m in its last second of motion while it falls from a height H. Find H

(a) 58.45 m (b) 68.45 m

(c) 78.45 m (d) 89.45 m

Solution (b) $s_{\text{nth}} = u + \frac{g}{2}(2n - 1)$ or $32 = 0 + 5(2n - 1)$

$37 = 10n$ or $n = 3.7.$

using $s = ut + \frac{1}{2}gt^2$;

$$H = 0 + \frac{1}{2}(10)(3.7)^2 = \frac{136.9}{2} = 68.45\,\text{m}$$

4. A particle starts with an acceleration α from rest for some time and after achieving a maximum velocity starts retarding at a rate β and finally comes to rest. If total time taken is t then find (a) maximum velocity (b) total distance travelled.

(a) $\frac{\alpha\beta t}{\alpha+\beta}, \frac{\alpha\beta t^2}{(\alpha+\beta)}$ (b) $\frac{\alpha\beta t}{\alpha+\beta}, \frac{\alpha\beta t^2}{2(\alpha+\beta)}$

(c) $\frac{(\alpha+\beta)}{2}t, (\alpha+\beta)\frac{t^2}{2}$ (d) $(\alpha + \beta)t, (\alpha+\beta)\frac{t^2}{2}$

Solution (b) Let the particle accelerate for a time t, then maximum, velocity $v = \alpha t_1$

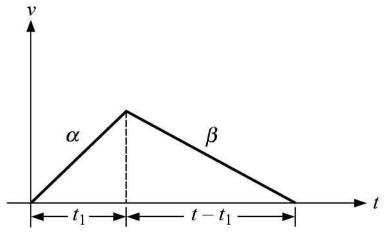


Fig. 2.9

Since it retards at a rate β and finally comes to rest therefore $0 = \alpha t_1 - \beta(t - t_1)$

or $t_1 = \frac{\beta t}{\alpha+\beta}$ $\therefore$ $v_{\max} = \frac{\alpha\beta t}{\alpha+\beta}$

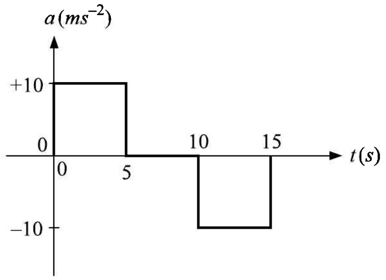
Distance travelled = area under $(v - t)$ graph

$$= \frac{1}{2}v_{\text{max.}}\,(\text{total time})$$

$$= \frac{1}{2}\frac{\alpha\beta t}{\alpha+\beta} \times t = \frac{\alpha\beta t^2}{2(\alpha+\beta)}.$$

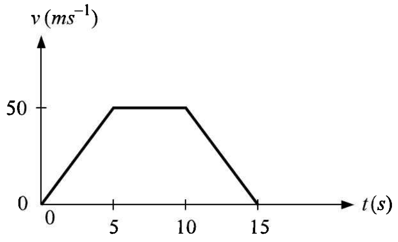
5. From the given $(a - t)$ graph, draw $(v - t)$ and $(s - t)$ curves.

Solution Velocity is the area under $(a - t)$ curve, thus at the end of 5 s, velocity $v = 50\,\text{ms}^{-1}$. Since there is no acceleration in the interval (5 – 10 s), therefore velocity remains constant, that is, $50\,\text{ms}^{-1}$. In the interval (10 – 15 s) the particle is retarded (Area = –50). Hence particle comes to rest.



(a)

To draw $(s - t)$ graph. We take help of $(v - t)$ graph. area under $(v - t)$ curve is the distance travelled.



(v – t) curve

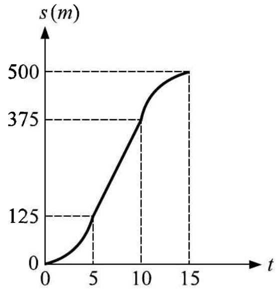
(b)

Thus

Distance moved in (0 – 5 s) is $\frac{50\times5}{2}$ = 125 m

Distance moved in (5 – 10 s) is 5 × 50 = 250 m

Distance moved in (10 – 15 s) is $\frac{50\times5}{2}$ = 125 m



(s – t) curve

(c)

Fig. 2.10

6. A particle starts from rest with an acceleration 2 ms^{-2}. If the force producing acceleration is applied for 10 s then determine the velocity of the particle after 20 s.

(a) 40 ms^{-1} (b) 10 ms^{-1}

(c) 20 ms^{-1} (d) 30 ms^{-1}

Solution (c) $\because v = u + at = 0 + 2(10) = 20$ ms^{-1}. After that force is removed, therefore, according to Newton's first law of motion, the velocity will remain unaltered that is, 20 ms^{-1}.

7. A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward and so on. Each step is 1 m long and requires 1s. Plot x – t graph of his motion and find the time when he falls in the pit 13 m away.

(a) 52 s (b) 37 s

(c) 48 s (d) none of these

Solution (b) x – t graph is shown in the given fig. From the Fig. it is clear that it takes 37 s to fall in the pit.

Alternatively: We find that in 8s the drunkard moves (5 – 3) = 2 m. Time taken to cover 8 m = (13 – 5)

$$t = \frac{8\times8}{2} = 32\text{ s}$$

Total time 32 + 5 = 37 s

Total distance (8 + 5) m = 13 m

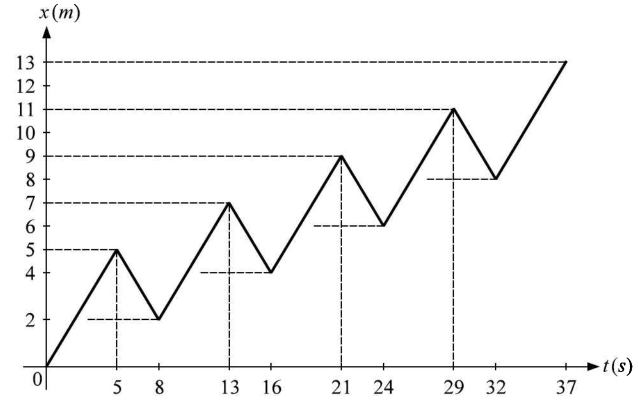


Fig. 2.11

8. If a particle moves according to the equation $x = 3t^2 + 4t$ then find (a) average velocity in 4 s and (b) instantaneous velocity at $t = 4$ s (x in cm)

Solution (a) $x(t=0) = 0$

$$x(t=4) = 3(4)^2 + 4(4) = 64$$

$$V_{av} = \frac{64}{4} = 16\text{ cm s}^{-1}$$

(b) $V = \frac{dx}{dt} = 6t + 4$

$$\left.\frac{dx}{dt}\right|_{t=4} = 28\text{ cm s}^{-1}$$

9. A particle moves according to the equation $t = ax^2 + bx$ find the retardation at any instant.

(a) $\frac{2a}{(2ax+b)^3}$ (b) $2a$

(c) $\frac{2a}{(2ax+b)^2}$ (d) none of these

Solution (a) $t = ax^2 + bx$ or $\frac{dt}{dx} = 2ax + b$

or $v = \frac{dt}{dx} = \frac{1}{2ax+b}$

$$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$$

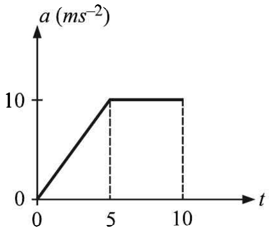
$$= \frac{-2a}{(2ax+b)^2}\left(\frac{1}{2ax+b}\right) = \frac{-2a}{(2ax+b)^3}.$$

10. Plot (v – t) and (x – t) graph for the given (a – t) graph.

Solution for (v – t) graph

$$a = 2t(0 \le t \le +5s)$$

$$v = \int_0^5 a\,dt = \int_0^5 2t\,dt$$



(a)

$$= \left.\frac{2t^2}{2}\right|_0^5 = 25\,\text{ms}^{-1}$$

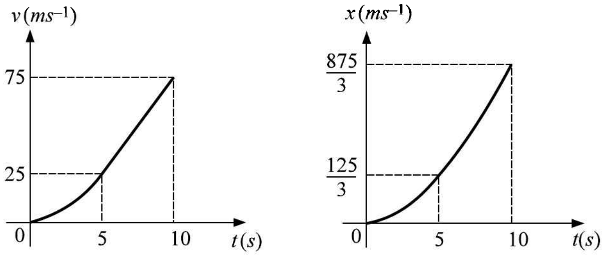
$v = u + at \quad (5 < t < 10\,\text{s})$

$= 25 + 10 \times 5$

$= 75\,\text{ms}^{-1}$ at the end of 10 s.

For $(x - t)$ graph

$$x = \int_0^5 v.dt = \left.\frac{t^3}{3}\right|_0^5 = \frac{125}{3}\,\text{m} \quad (0 < t < 5\,\text{s})$$



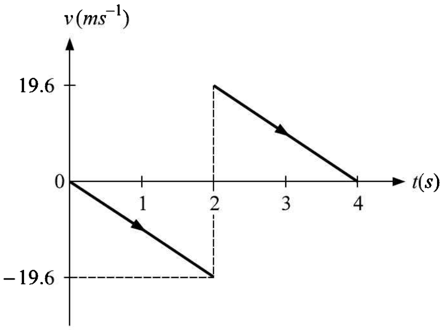
(b) (c)

Fig. 2.12

x = area under $(v - t)$ curve $(5 < t < 10\,\text{s})$

$= 250\,\text{m}.$

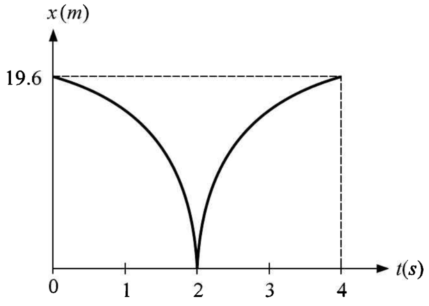
11. A ball drops from a height 19.6 m above the ground, it rebounds and rises to the same height. Plot $(v - t)$ and $(x - t)$ graphs.



(a)

Solution $x = \frac{1}{2}gt^2$

$19.6 \times 2 = 9.8\,t^2$ or $t = 2\,\text{s}.$



(b)

Fig. 2.13

12. A bus is standing at the bus stop. A man is in front of the bus at the stop. Suddenly the bus starts moving with an acceleration $2\,\text{ms}^{-2}$. The man notices the bus moving after 2 second of its motion. With what constant speed man should run so as to get in bus in two seconds when he notices the bus moving.

(a) $2\,\text{ms}^{-1}$	(b) $4\,\text{ms}^{-1}$
(c) $8\,\text{ms}^{-1}$	(d) $6\,\text{ms}^{-1}$

Solution (c) Since the bus has moved for 4 seconds, the distance moved by bus in 4 s is

$$x = \frac{1}{2}(2)\,(4)^2 = 16\,\text{m}$$

The man has to cover 16 m in 2 s so as to get into the bus

$\therefore$ Velocity of the man $v = \frac{x}{t} = \frac{16}{2} = 8\,\text{ms}^{-1}$

13. A stone drops from a height of 100 m and simultaneously a stone is thrown up with a velocity $40\,\text{ms}^{-1}$. Find the time and position when they cross each other.

(a) 2.5 s, 68.75 m	(b) 2 s, 60 m
(c) 2.2 s, 40 m	(d) none of these

Solution (a) Assume they meet when at a height x and time t from the ground. Distance travelled by the stone falling is $(100 - x)$ in the same time.

$\therefore \quad (100 - x) = \frac{1}{2}gt^2 \qquad$ ...(1)

$x = 40t - \frac{1}{2}gt^2 \qquad$ ...(2)

Substituting (1) in (2), we get

$x = 40t - (100 - x)$ or $t = 2.5\,\text{s}$

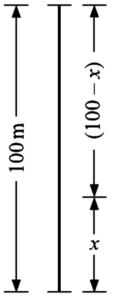


Fig. 2.14

Substituting the value of t in (1), we get $x = 68.75\,\text{m}$

14. Find the ratio of the velocity of P to the velocity of Q if two particles P and Q moving in a straight line are represented by the curves as shown in the Fig. 2.15.

(a) 3	(b) $\sqrt{3}$
(c) 2	(d) none of these

Solution (a) $\frac{v_P}{v_Q} = \frac{\tan 60}{\tan 30} = 3$

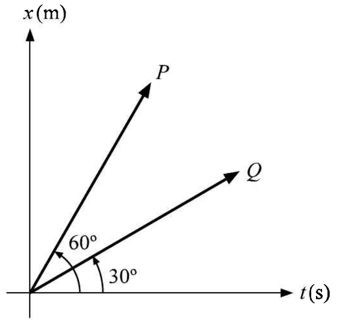


Fig. 2.15

15. A boy stretches a stone against the rubber tape of a catapult or 'gulel' through a distance of 25 cm before leaving it. The tape returns to its normal position accelerating the stone over the stretched length. The stone leaves it with a velocity $2.5\,\text{ms}^{-1}$. Assuming the acceleration to be constant while the stone was being pushed by the tape, find its magnitude.

(a) $6.5\,\text{ms}^{-2}$	(b) $12.5\,\text{ms}^{-2}$
(c) $2.5\,\text{ms}^{-2}$	(d) $10\,\text{ms}^{-2}$

Solution (b) $u = 0, s = 25\,\text{cm} = \frac{1}{4}$ m using $v^2 - u^2 = 2as$

$v = 2.5\,\text{ms}^{-1}$

$$\therefore \quad \frac{(2.5)^2}{2\times\frac{1}{4}} = a \quad \text{or} \quad a = 12.5\,\text{ms}^{-2}$$

16. When a balloon is at a height of 80 m, ascending with a velocity 10 ms^{-1} and acceleration 1.2 ms^{-2}, drops a packet. Find the time taken by the packet to reach ground. Take g = 10 ms^{-2}

(a) 6.2 s (b) 5.2 s

(c) 4.18 s (d) 3.21 s

Solution (b) Velocity of the packet is 10 ms^{-1} upwards at a height 80 m. Considering 80 m height as origin, then displacement = –80 m.

$$\therefore \quad s = ut - \frac{1}{2}gt^2$$

$$-80 = 10t - 5t^2 \quad \text{or} \quad t^2 - 2t - 16 = 0 \Rightarrow t \simeq 5.2\,\text{s}$$

17. A particle starts from rest with an acceleration 2 ms^{-1}. After acquiring maximum velocity it decelerates with 3 ms^{-2} and covers 1500 m before coming to rest. Find the time for which it has moved.

(a) 50 s (b) 45 s

(c) 35 s (d) 42.4 s

Solution (a) Apply $s = \dfrac{\alpha\beta t^2}{2(\alpha+\beta)}$, that is,

$$1500 = \frac{2\times 3t^2}{2(2+3)}$$

or $\quad t^2 = 2500$

$\quad t = 50\,\text{s}.$

18. A ball is dropped vertically from a height h above the ground. It hits the ground and bounces up vertically to a height $\dfrac{h}{2}$. Neglecting subsequent motion and air resistance draw velocity displacement (v – h) graph

Solution $v^2 = 2gh$ (while falling down)

Since it falls down, velocity vector is negative. The curve is parabolic towards the negative side.

$$v^2 = 2g\left(\frac{h}{2}\right) \text{ (while moving up)}$$

Velocity vector is positive during upward motion. The curve is a parabola.

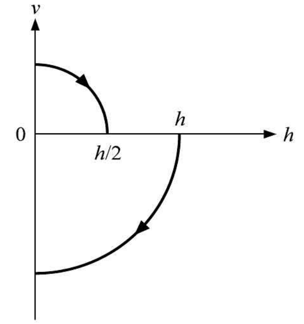


Fig. 2.16

Problems for Practice

1. A car accelerates from rest at constant rate of 2 ms^{-2} for sometime. Then, it retards, at constant rate of 4 ms^{-2} and comes to rest. If the total time for which it remains in motion be 3 seconds, what is total distance travelled?

(a) 4 m (b) 6 m

(c) 2 m (d) 3 m

2. A particle starts with initial velocity 10 ms^{-1}. It covers a distance of 20 m along a straight line in two seconds. What is acceleration of the particle?

(a) 20 ms^{-2} (b) 10 ms^{-2}

(c) 1 ms^{-2} (d) zero

3. Which of the following can be zero when the particle is in motion for some time?

(a) speed

(b) displacement

(c) distance covered

(d) none of these

4. A particle is moving at 10 ms^{-1} towards east. In one second its velocity changes to 10 ms^{-1} towards west. If the particle be uniformly accelerated, the change in velocity will be directed at

(a) 135° to east (b) 45° to east

(c) 40° to east (d) none of these

5. A car travels a distance s on a straight road in two hours and then returns to the same point in the next 3 hours. Its average velocity is

(a) $s/2 + s/3$ (b) $s/5$

(c) $2s/5$ (d) none of these

6. A car accelerates from rest at constant rate of 2 ms^{-2} for sometime. Then, it retards at a constant rate of 4 ms^{-2} and comes to rest. What is the maximum speed attained by the car, if it remains in motion for 3 second?

(a) 4 ms^{-1} (b) 6 ms^{-1}

(c) 2 ms^{-1} (d) 3 ms^{-1}

7. If the distance covered by a particle is zero, then what can we say about its displacement?

(a) It is negative

(b) It may or may not be zero

(c) It must be zero

(d) It cannot be zero

8. Fig. 2.17 shows the velocity time graph of a one dimensional motion. Which of the following characteris-tic of the particle is represented by the shaded area?

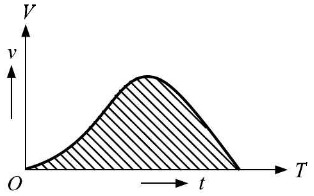


Fig. 2.17

(a) distance covered
(b) speed
(c) momentum
(d) acceleration

9. The velocity-time graph of a motion starting from rest with uniform acceleration is a straight line
(a) not passing through origin
(b) parallel to time-axis
(c) parallel to velocity-axis
(d) none of these

10. Which of the following distance time graphs represents one dimensional uniform motion?

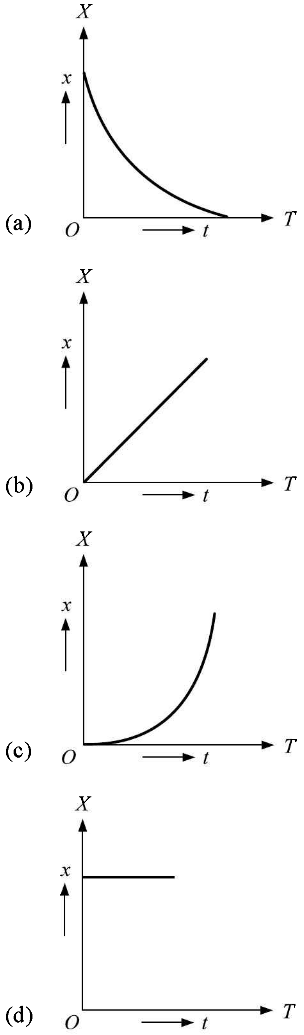


Fig. 2.18

11. A train takes 1 hour to go from one station to the other. It travels at a speed of 30 km h^{-1} for first half hour and at a speed of 50 km h^{-1} for the next half hour. The average speed of the train is

(a) 45 km h^{-1} (b) 35 km h^{-1}
(c) 40 kmh^{-1} (d) none of these

12. Which of the following graphs represents the uniform motion?

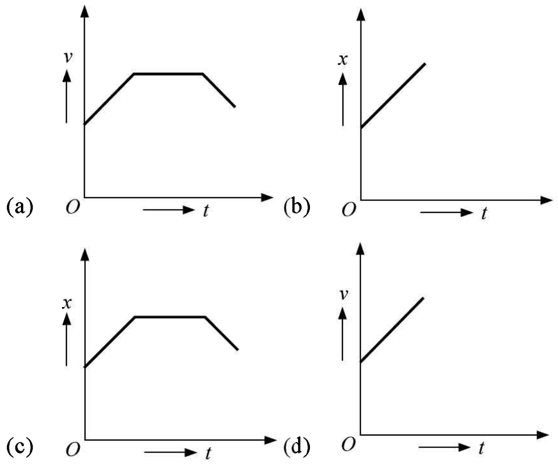


Fig. 2.19

13. A particle returns to the starting point after 10 seconds. If the rate of change of velocity during the motion is constant in magnitude, then its location after 7 seconds will be same as that after
(a) 3 s (b) 3.5 s
(c) 2 s (d) 1 s

14. The position of a particle moving on a straight line path is given by

$x = 12 + 18t + 9t^2$ metre. Its velocity after 2 seconds is

(a) 54 ms^{-1} (b) 36 ms^{-1}
(c) 84 ms^{-1} (d) 72 ms^{-1}

15. The slope of the velocity time graph for retarded motion is
(a) zero (b) positive
(c) negative (d) neutral

16. In a motion with constant acceleration after covering a distance of 100 m, the velocity is reduced to zero in 5 seconds. The distance covered by the particle in the next 5 seconds will be
(a) 500 m (b) 100 m
(c) 250 m (d) zero

17. The position of a particle moving on a straight line path is given by

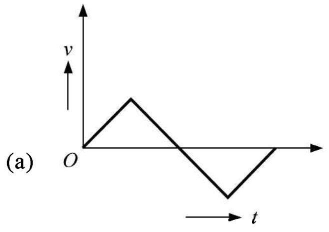
$x = 12 + 18t + 9t^2$ metre. Its acceleration at any instant is

(a) 18 ms^{-2} (b) 45 ms^{-2}
(c) 9 ms^{-2} (d) 12 ms^{-2}

18. A particle starts from rest and moves with a constant acceleration for a time t and covers a distance s_1. It continues to move with the same acceleration and covers a distance s_2 in the next time interval t. Which of the following relations is correct?

(a) $s_2 = 3s_1$ (b) $s_2 = 4s_1$
(c) $s_2 = s_1$ (d) $s_2 = 2s_1$

19. Which of the following speed time graph is not possible?



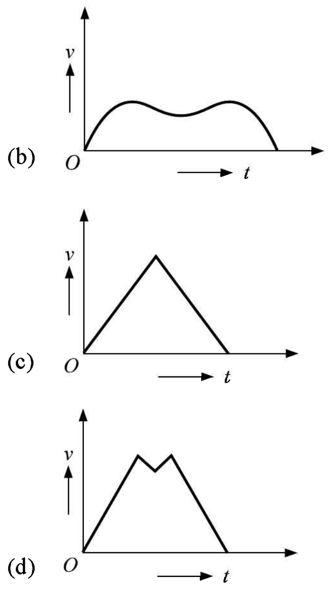


Fig. 2.20

20. A particle starts from rest and moving with constant acceleration covers a distance s_1 in the 3rd second and s_2 in the 5th second. The ratio s_1/s_2 =

(a) 9/25 (b) 25/81

(c) 3/5 (d) 5/9

21. Which of the following velocity time graphs is not possible?

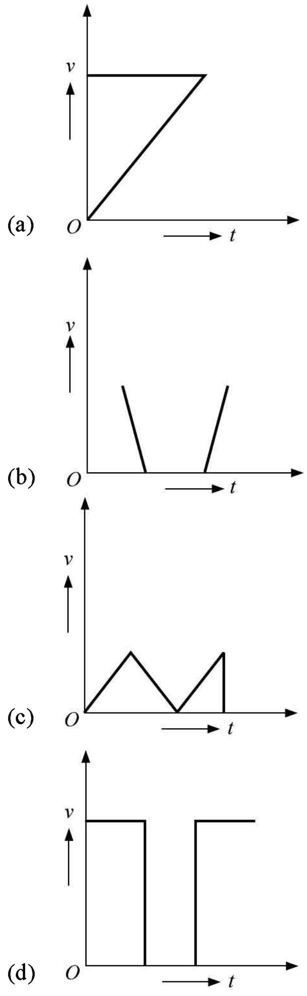


Fig. 2.21

22. A stone is thrown vertically upwards with a velocity of 30 ms^{-1}. If the acceleration due to gravity is 10 ms^{-2}, what is the distance travelled by the particle during the first second of its motion?

(a) 30 m

(b) 25 m

(c) 10 m

(d) none of these

23. Four athletes are standing at the corners *A, B, C* and *D* of a square of side *l*. They simultaneously start running, such that *A* runs towards *B*, *B* runs towards *C*, *C* runs towards *D*, *D* runs towards *A* each with velocity *v*. They will meet at *O*, after a time

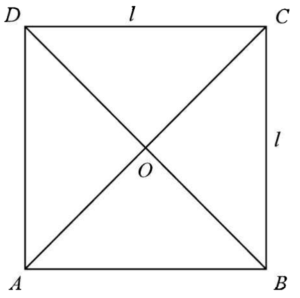


Fig. 2.22

(a) $l/\sqrt{2}\,v$ (b) $\sqrt{2/v}$

(c) l/v (d) $\sqrt{2}\,l/v$

24. A train accelerates from rest at a constant rate α for sometime and then it retards to rest at the constant rate β. If the total distance covered by the particle is *x*, then what is the maximum velocity of the train?

(a) $\left[\dfrac{2\alpha\beta}{\alpha+\beta}x\right]^{1/2}$ (b) $\left[\dfrac{2\alpha\beta}{\alpha-\beta}x\right]^{1/2}$

(c) $\left[\dfrac{\alpha+\beta}{2\alpha\beta}x\right]^{1/2}$ (d) $\left[\dfrac{\alpha-\beta}{2\alpha\beta}x\right]^{1/2}$

25. Which of the following velocity time graphs is not possible?

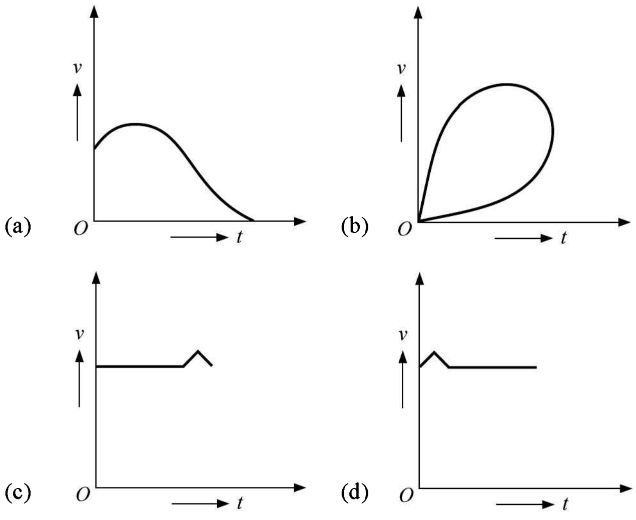


Fig. 2.23

26. A taxi leaves the station *X* for station *Y* every 10 minutes. Simultaneously, a taxi also leaves the station *Y* for station *X* every 10 minutes. The taxis move at the same constant speed and go from *X* to *Y* or vice versa in 2 hours. How many taxis coming from the other side will meet each taxi enroute from *Y* to *X*?

(a) 11 (b) 12
(c) 23 (d) 24

27. A train accelerates from rest at a constant rate α for distance x_1 and time t_1. After that it retards to rest at constant rate β for distance x_2 and time t_2. Which of the following relation is correct?

(a) $\frac{x_1}{x_2}=\frac{\alpha}{\beta}=\frac{t_2}{t_1}$ (b) $\frac{x_1}{x_2}=\frac{\beta}{\alpha}=\frac{t_2}{t_1}$

(c) $\frac{x_1}{x_2}=\frac{\alpha}{\beta}=\frac{t_1}{t_2}$ (d) $\frac{x_1}{x_2}=\frac{\beta}{\alpha}=\frac{t_1}{t_2}$

28. A rocket is fired vertically from the ground. It moves upwards with a constant acceleration $10\,\text{ms}^{-2}$ for 30 seconds. after which the fuel is consumed. After what, time from the instant of firing, the rocket will attain the maximum height? Take $g = 10\,\text{ms}^{-2}$

(a) 75 s (b) 60 s
(c) 45 s (d) 30 s

29. A stone falls from rest. The total distance covered by it in the last second of its motion is equal to the distance covered in the first three seconds. What is height from which the stone was dropped? Take $g = 10\,\text{ms}^{-2}$

(a) 200 m (b) 125 m
(c) 100 m (d) 25 m

30. A car starting from rest and moving with the uniform acceleration possess average velocities $5\,\text{ms}^{-1}$, $10\,\text{ms}^{-1}$ and $15\,\text{ms}^{-1}$ in the first, second and third seconds. What is the total distance covered by the car in these three second?

(a) 55 m (b) 30 m
(c) 15 m (d) none of these

31. First half of the distance is covered by a car with velocity v_1 and the second half is covered with velocity v_2. What is the average velocity of the car?

(a) $\frac{2v_1v_2}{v_1+v_2}$ (b) $\frac{v_1v_2}{2(v_1+v_2)}$

(c) $\frac{v_1+v_2}{v_2}$ (d) $\frac{v_1v_2}{v_1+v_2}$

32. Fig. 2.24 shows a hemispherical bowl of radius *R*. A ball bearing is released at the top of the bowl. What will be its velocity at the bottom of the bowl?

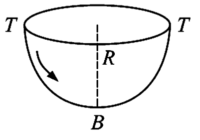


Fig. 2.24

(a) $2Rg$ (b) $\sqrt{2Rg}$
(c) Rg (d) $\sqrt{Rg}$

33. A ball dropped from a height *h* reaches the ground in time *T*. What is its height at time *T*/2?

(a) $3h/4$ (b) $h/8$
(c) $h/4$ (d) $h/2$

34. A pole is held vertically with one end on the ground. The length of the pole is 30 m. The pole is allowed to fall. Assuming that the lower end of the pole does not slip, with what velocity will the upper end strike the ground? Take $g = 10\,\text{ms}^{-2}$

(a) $30\,\text{ms}^{-1}$ (b) $20\,\text{ms}^{-1}$
(c) $10\,\text{ms}^{-1}$ (d) $5\,\text{ms}^{-1}$

35. Which of the following cannot be the speed time graph

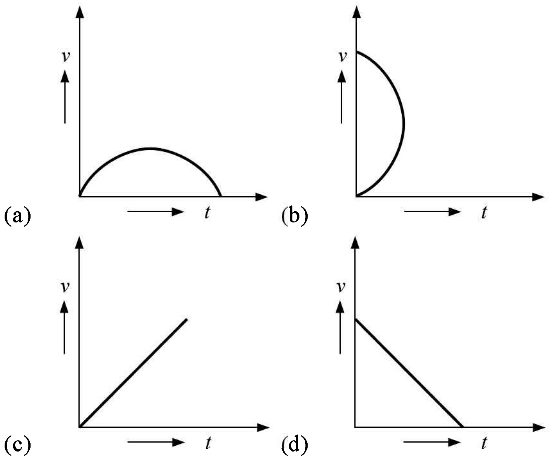


Fig. 2.25

36. A man walks in rain with a velocity of $5\,\text{kmh}^{-1}$. The rain drops strike at him at an angle of 45° with the horizontal. The downward velocity of the rain drops will be

(a) $1\,\text{kmh}^{-1}$ (b) $3\,\text{kmh}^{-1}$
(c) $4\,\text{kmh}^{-1}$ (d) $5\,\text{kmh}^{-1}$

37. A bucket is placed in the open where the rain is falling vertically. If wind begins to blow at double the velocity of the rain, how will the rate of filling of the bucket change?

(a) half (b) double
(c) quadruple (d) remains constant

38. Which of the following cannot be the distance time graph?

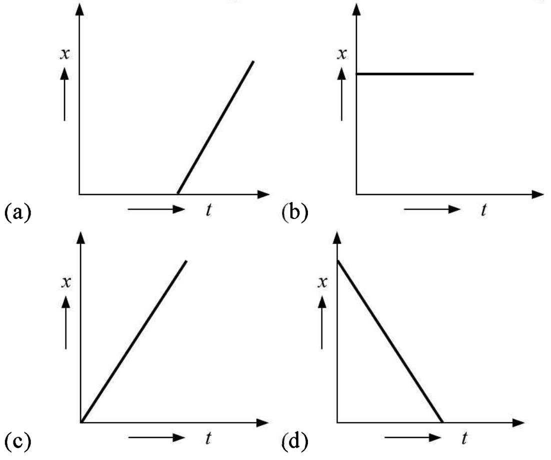


Fig. 2.26

39. Rain is falling vertically downwards with a velocity of $3\,\text{kmh}^{-1}$. A man walks in the rain with a velocity of $4\,\text{kmh}^{-1}$. The rain drops will fall on the man with a velocity of

(a) $5\,\text{kmh}^{-1}$ (b) $4\,\text{kmh}^{-1}$
(c) $3\,\text{kmh}^{-1}$ (d) $1\,\text{kmh}^{-1}$

40. A particle moves along a circular path with constant speed. What is the nature of its acceleration?

(a) its magnitude changes
(b) its direction changes
(c) it is uniform
(d) zero

41. Two cars C_1 and C_2 are moving on parallel roads in the same direction with velocity *v*. The relative velocity of C_1 with respect to C_2 is

(a) directed towards C_1
(b) directed towards C_2
(c) $2v$
(d) zero

42. The displacement of a particle is given by $\sqrt{x} = t + 1$. Which of the following statements about its velocity it true?
(a) it increases with time
(b) it decreases with time
(c) it is zero
(d) it is constant but not zero

43. Two bodies are moving in opposite directions with velocity v. The relative velocity of one with respect to the other is
(a) $\sqrt{2}\,v$ (b) $2v$
(c) v (d) zero

44. A thief is running away on a straight road in a jeep moving with a speed of 9 ms^{-1}. A policeman chases him on a motor cycle moving at a speed of 10 ms^{-1}. If the instantaneous separation of the jeep from the motorcycle is 100 m, how long will it take for the policeman to catch the thief?
(a) 100 s (b) 90 s
(c) 19 s (d) 1 s

45. A bus is moving with a velocity 10 ms^{-1} on a straight road. A scooterist wishes to overtake the bus in 100 seconds. If the bus is at a distance of 1 km from the scooterist, with what velocity should the scooterist chase the bus?
(a) 30 ms^{-1} (b) 20 ms^{-1}
(c) 40 ms^{-1} (d) 50 ms^{-1}

46. A ball is dropped from the top of the tower of height h. It covers a distance of $h/2$ in the last second of its motion. How long does the ball remain in air? Take $g = 10$ ms^{-2}
(a) 2 s (b) $(2 + \sqrt{2})$s
(c) $\sqrt{2}$ s (d) none of these

47. A stone falls from rest. The total distance covered by it in the last second of its motion is equal to the distance covered in the first three seconds of its motion. How long does the stone remain in the air. Take $g = 10$ ms$^{-2?}$
(a) 6 s (b) 7 s
(c) 4 s (d) 5 s

48. A particle moving in a straight line ultimately comes to rest. What is the angle between its initial velocity and acceleration?
(a) π (b) $\pi/2$
(c) $\pi/4$ (d) zero

49. The velocity (v) of a particle is related to the time (t) elapsed as $v = kt$, where $k = 4$ ms^{-2}. What is the distance travelled in first 2 seconds?
(a) 12 m (b) 16 m
(c) 4 m (d) 8 m

50. A simple pendulum hangs from the ceiling of a train. The string is inclined towards the rear of the train. What is the nature of motion of the train?
(a) retarted (b) at rest
(c) uniform (d) accelerated

51. A particle is moving with a velocity of 10 ms^{-1} due east. In one second its velocity changes to 10 ms^{-1} due west. If the particle is uniformly accelerated, the change in velocity will be directed as
(a) 20 ms^{-1}, east (b) 10 ms^{-1}, east
(c) 10 ms^{-1}, west (d) 20 ms^{-1}, west

52. Which of the following can be zero when the particle is in motion for some time?
(a) speed (b) velocity
(d) distance covered (d) none of these

53. The distance covered by a body moving along x-axis with initial velocity u and uniform acceleration a is given by

$$s = ut + \frac{1}{2}at^2$$

This result is a consequence of
(a) newton's III law (b) newton's II law
(c) newton's I law (d) none of these

54. Two bodies of different masses m_a and m_b are dropped from two different heights a and b. The ratio of times taken by the two to drop through these distances is
(a) $\sqrt{a} : \sqrt{b}$ (b) $a^2 : b^2$
(c) $a : b$ (c) $m_a/m_b \times b/a$

55. The relation between time t and distance x is $t = \alpha x^2 + \beta x$, where α and β are constants. The retardation is (where v is the velocity)
(a) $2\alpha\beta v^3$ (b) $2\beta^2 v^3$
(c) $2\alpha v^3$ (d) $2\beta v^3$

56. The slope of the velocity-time graph for uniform motion is
(a) zero (b) infinite
(c) positive (d) negative

57. A body is thrown vertically upwards in air. When air resistance is taken into account, the time of ascent and descent are t_1 and t_2 respectively, then which of the following is true?
(a) $t_1 > t_2$ (b) $t_1 = t_2$
(c) $t_1 < t_2$ (d) none of these

58. A stone is dropped from a height of 45 m. What will be distance travelled by it during the last one second of motion?
(a) 35 m (b) 25 m
(c) 10 m (d) 4.5 m

59. A particle falls from a height h and rebounds to $h_1 < h$, then which of the graph represents the motion correct?

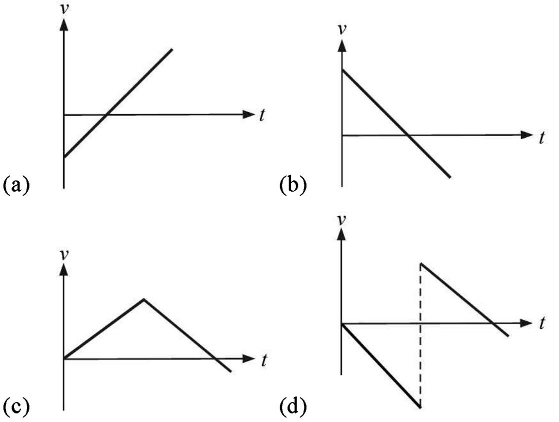


Fig. 2.27

60. The equation for the distance s of a particle along a straight line at time t is given by

$$s = a_0 + \frac{a_1 t}{2} + \frac{a_2 t^2}{3}$$

The acceleration of the particle is
(a) $a_0 + (a_2/3)$ (b) $(2/3)a_2$
(c) $a_1/2$ (d) $a_2/3$

61. A stone falls from the top of the tower in 8 seconds. How much time will it take to cover the first quarter of the distance starting from the top?

(a) 1 s
(b) 2 s
(c) 4 s
(d) none of these

62. A particle moving with constant acceleration covers a distance of 30 m in the 3rd second. It covers a distance of 50 m in the 5th second. What is the acceleration of the particle?

(a) 10 ms^{-2}
(b) 8 ms^{-2}
(c) 5 ms^{-2}
(d) 3 ms^{-2}

63. A car accelerates from rest at constant rate for the first 10 seconds and covers a distance x. It covers a distance y in the next 10 seconds at the same acceleration. Which of the following is true?

(a) $x = y$
(b) $y = 2x$
(c) $x = 3y$
(d) $y = 3x$

64. A boy projects a ball into air one after the other at an interval of one second. The next ball is thrown when the velocity of the ball thrown earlier is zero. To what height the ball will rise? Take g = 10 ms^{-2}

(a) 20 m
(b) 40 m
(c) 10 m
(d) 5 m

65. A car runs at a constant speed on a circular track of radius 100 m, taking 62.8 seconds on each lap. What is the average velocity and average speed on each lap?

(a) 0,0
(b) 10 m/s, 0
(c) 0, 10 m/s
(d) 10 m/s, 10 m/s

66. The distance travelled by a body falling freely from rest in first, second and third seconds are in the ratio

(a) 1 : 4 : 9
(b) 1 : 3 : 5
(c) 1 : 2 : 3
(d) none of these

67. The balls are dropped from the same point after an interval of 1second. If acceleration due to gravity is 10 ms^{-2}, what will be their separation 3 seconds after the release of first ball?

(a) 30 m
(b) 25 m
(c) 10 m
(d) 5 m

68. A railway train 200 m long passes over a bridge 600 m long. The time taken by the train to cross the bridge with uniform speed of 36 km/h is

(a) 20 s
(b) 60 s
(c) 80 s
(d) 800 s

69. An athlete completes one round of a circular track of radius R in 40 seconds. What will be the displacement at the end of 2 minute 20 seconds?

(a) 0
(b) $2R$
(c) $2\pi R$
(d) $7\pi R$

70. For motion on a curved path with constant acceleration, the ratio of the magnitude of the displacement to the distance covered is

(a) ≤ 1
(b) ≥ 1
(c) < 1
(d) > 1

ANSWERS

1. (b)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (b)
11. (c)	**12.** (b)	**13.** (a)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (a)	**19.** (a)	**20.** (d)
21. (a)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (b)
31. (a)	**32.** (b)	**33.** (a)	**34.** (a)	**35.** (b)	**36.** (d)	**37.** (d)	**38.** (d)	**39.** (a)	**40.** (b)
41. (d)	**42.** (a)	**43.** (b)	**44.** (a)	**45.** (b)	**46.** (b)	**47.** (d)	**48.** (a)	**49.** (d)	**50.** (d)
51. (d)	**52.** (b)	**53.** (d)	**54.** (a)	**55.** (c)	**56.** (a)	**57.** (c)	**58.** (b)	**59.** (d)	**60.** (b)
61. (c)	**62.** (a)	**63.** (d)	**64.** (d)	**65.** (c)	**66.** (b)	**67.** (b)	**68.** (c)	**69.** (b)	**70.** (c)

3

Vectors

BRIEF REVIEW OF THE CONCEPTS

Vectors are those physical quantities which have magnitude and direction and are added according to triangle law of addition. In other words, directed segments which follow triangle law of addition are called vectors.

Properties of a vector

In addition to magnitude and unit

(a) it has specified direction

(b) it obeys triangle law of addition

(c) its addition is commutative that is, $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

(d) its addition is associative that is, $\vec{A}+(\vec{B}+\vec{C}) = \left(\vec{A}+\vec{B}\right)+\vec{C}$

Types of representation of vectors Vectors may be represented in two ways

Polar form In this form $\vec{A} = (r, \theta)$ where r is magnitude and θ is the angle as shown in Fig. 3.1

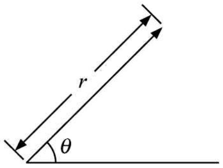


Fig. 3.1

Cartesian form In this form $\vec{A} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ where a_1, a_2 and a_3 are coefficients and $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along x, y and z directions (axes) respectively as illustrated in Fig. 3.2.

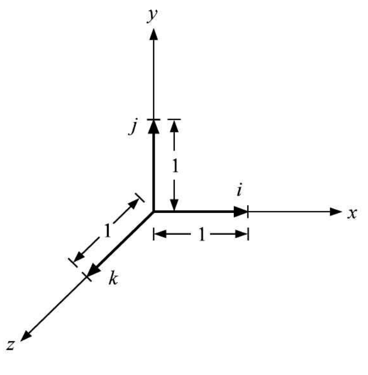


Fig. 3.2

Null vector It is a vector whose magnitude is zero and hence directions becomes indeterminate.

Unit vector Unit vector has magnitude as one and direction is specified. unit vector $\hat{a} = \dfrac{\vec{A}}{|A|}$

that is, vector divided by its magnitude represents a unit vector.

Co-planar vectors Vectors lying in the same plane are said to be co-planar.

Inertial vectors These vectors convert the noninertial frame of reference into inertial. They are also called pseudo vectors.

Addition of vectors

Vectors are added according to any of the three laws namely Triangle law, Parallelogram law and Polygon law.

Triangle law (Analytical Proof) In Fig. 3.3

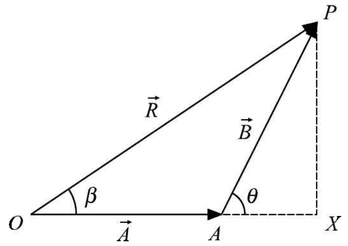


Fig. 3.3

$$
\begin{aligned}
AX &= AP\cos\theta = B\cos\theta \\
PX &= AP\sin\theta = B\sin\theta \\
OP^2 &= OX^2 + PX^2 = (OA + AX)^2 + PX^2 \\
&= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta
\end{aligned}
$$

or $$|R| = \sqrt{A^2+B^2+2\,AB\cos\theta}$$

$$\tan\beta = \frac{PX}{OA+AX} = \frac{B\sin\theta}{A+B\cos\theta}$$

$R_{\text{max}} = A + B$ (when $\theta = 0$)

that is, two vectors act in the same direction.

$R_{\text{min}} = A - B$ (when $\theta = 180°$)

that is, two vectors act in opposite direction.

This gives a clue for all other cases

The magnitude of the resultant

$$|A - |B| \le |R| \le |A| + |B|$$

Note: Minimum number of unequal co-planar vectors whose sum can be zero is three and minimum number of non-co-planar vectors whose sum can be zero is four.

Subtraction of a vector is equivalent to addition of a negative vector.

Resolution of a vector In Fig. 3.4, using triangle law

$$\vec{R} = \hat{i}R_x + \hat{j}R_y$$

obviously $R_x = R\cos\theta$ and $R_y = R\sin\theta$, therefore

$$\vec{R} = R\cos\theta\hat{i} + R\sin\theta\hat{j}$$

$$|R| = \sqrt{R_x^2 + R_y^2}$$

and $\tan\theta = \dfrac{R_y}{R_x}$ or $\theta = \tan^{-1}\left(\dfrac{R_y}{R_x}\right)$

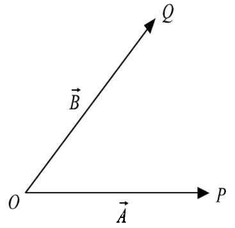


Fig. 3.4

Multiplication of Vectors

In case of vectors two types of multiplications exist.

Scalar product or dot product If the product of two vectors is scalar then the rule applied is

$\vec{A}.\vec{B} = AB\cos\theta$

$\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (Scalar product is cummutative)

$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

(Scalar product follows distributive law)

Rules

(i) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(ii) $\hat{i}.\hat{j} = \hat{j}.\hat{i} = \hat{k}.\hat{j} = \hat{j}.\hat{k} = \hat{i}.\hat{k} = \hat{k}.\hat{i} = 0$

Applications

(a) $W = \vec{F}.\vec{S}$, Power $P = \vec{F}\vec{V}$, current $I = \int \vec{J}.\vec{dS}$, magnetic flux $\phi = \vec{B}.\vec{A}$, potential energy $U = -\vec{p}.\vec{E}$ or $U = \vec{M}.\vec{B}$

(b) To find angle between two vectors

$$\theta = \cos^{-1}\frac{\vec{A}.\vec{B}}{|A||B|}$$

(c) If the scalar product of two non zero vectors is zero then vectors are perpendicular to each other.

(d) To find the component of a vector along a given direction. For instance, the component of $\vec{A}$ along $\vec{B}$ is $\dfrac{\vec{A}.\vec{B}}{|B|} = A\cos\theta$

Cross product or vector product This product is employed if the product of two vectors is a vector.

$$\vec{A}\times\vec{B} = AB\sin\theta\ \vec{n}$$

where $\vec{n}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$.

To find the direction of $\vec{n}$, Right hand screw rule is used, that is, rotate the vector $\vec{A}$ along $\vec{B}$ and the direction in which a right handed screw moves is the direction of $\vec{A}\times\vec{B}$ or $\vec{n}$.

Vector product is not commutative, that is,

$$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$$

But $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

However, $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$ but the directions are opposite.

Vector product is distributive

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

Rules

(i) $\vec{A}\times\vec{A} = 0 \quad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(ii) $\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$; $\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$

$\hat{k}\times\hat{j} = -\hat{i}\times\hat{k} = \hat{j}$

Applications

(a) Cross product is used in rotational motion for example,

Torque $\vec{\tau} = \vec{r}\times\vec{F}$,

Poynting vector $\vec{S} = \vec{E}\times\vec{H}$

Angular momentum $\vec{L} = \vec{r}\times\vec{F}$,

Linear velocity $\vec{v} = \vec{\omega}\times\vec{r}$

Magnetic force $\vec{F} = q(\vec{v}\times\vec{B})$

(b) If two non zero vectors have $\hat{i}\times\hat{j} = 0$ then they are parallel.

(c) $|\vec{A}\times\vec{B}|$ represents area of a parallelogram and $\frac{1}{2}|\vec{A}\times\vec{B}|$ represents area of a triangle $\frac{1}{2}|\vec{D}_1\times\vec{D}_2|$ = Area of a parallelogram where D_1 and D_2 are diagonals (See fig. 3.5).

(d) To find angle between two vectors

$$\theta = \sin^{-1}\frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{V}|}$$

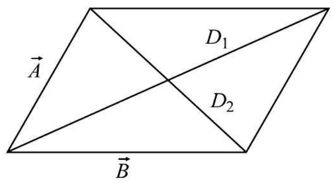


Fig. 3.5

Relative velocity Since absolute rest or absolute motion do not exist, therefore, every motion is a relative motion. Though, for convenience, we assume earth at rest and in common language measure the speed or velocity with respect to ground. But if two bodies A and B are moving with velocities V_A and V_B then relative velocity of A with respect to B may be thought of velocity of A bringing B to rest. Alternatively, vector law (triangle law) may be applied.

$$\vec{V}_{AG} = \vec{V}_{BG} + \vec{V}_{AB}$$

or $$\vec{V}_{AB} = \vec{V}_{AG} - \vec{V}_{BG}$$

or $$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$$

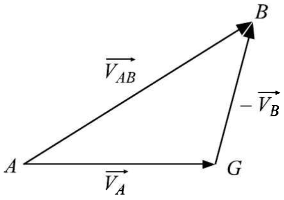


Fig. 3.6

The best way to solve the questions on relative velocity is to resolve it into x and y components as given below

$$\begin{aligned}\vec{V}_{AB} &= (V_{Ax}\hat{i} + V_{Ay}\hat{j}) - (V_{Bx}\hat{i} + V_{By}\hat{j}) \\ &= (V_{Ax} - V_{Bx})\hat{i} + (V_{Ay} - V_{By})\hat{j}\end{aligned}$$

Then $|\vec{V}_{AB}| = \sqrt{(V_{Ax} - V_{Bx})^2 + (V_{Ay} - V_{By})^2}$

and $\theta = \tan^{-1}\dfrac{V_{Ay} - V_{By}}{V_{Ax} - V_{Bx}}$ with respect to horizontal direction $(\hat{i})$

and $\phi = \tan^{-1}\dfrac{V_{Ax} - V_{Bx}}{V_{Ay} - V_{By}}$ with respect to vertical direction $(\hat{j})$.

SHORT-CUTS AND POINTS TO NOTE

(i) Laws of addition (a) Triangle law (b) Parallelogram law (c) Polygon law.

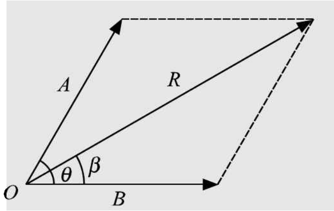


Fig. 3.7

Resultant $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$\tan\beta = \frac{A\sin\theta}{B + A\cos\theta}$$

vector subtraction is identical to addition $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. In the above case θ will be changed to $(180 - \theta)$.

(ii) If we represent the vectors in $\hat{i}$ and $\hat{j}$ form then

$$\begin{aligned}R &= A_x\hat{i} + A_y\hat{j} + B_x\hat{i} + B_y\hat{j} \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}\end{aligned}$$

$|R| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$ and

$\tan\beta = \dfrac{A_y + B_y}{A_x + B_x}$ with respect to x or $\hat{i}$ direction and

$\tan\beta' = \dfrac{A_x + B_x}{A_y + B_y}$ with respect to y or $\hat{j}$ direction.

$R_{\max} = A + B$ when $\theta = 0$;

$R_{\min} = A - B$ when $\theta = 180°$

(iii) If the system is in equilibrium then Lami's theorem may be applied.

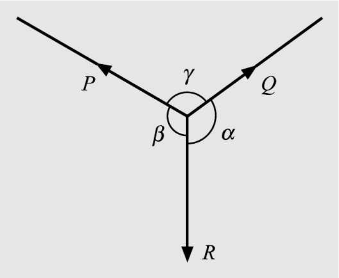


Fig. 3.8

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

(iv) A system is in equilibrium if net force $F = 0$ (linear equilibrium)

A system is in rotational equilibrium if net torque $\tau = 0$

(v) Best method to simplify questions will be to resolve the vectors into x and y components and then approach the question.

(vi) Magnitude of a vector $V = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is $|V| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

(vii) If two vectors are perpendicular their dot product is zero.

(viii) If two vectors A and B are parallel then $\vec{A} = k\vec{B}$ where k is a positive or negative real number. Moreover, $\vec{A} \times \vec{B} = 0$

(ix) $\vec{A}.(\vec{A} \times \vec{B})$ or $\vec{B}.(\vec{A} \times \vec{B}) = 0$

(x) Vector operator ∇(nabla) is used to define

$$\left\{\nabla = \frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right\}$$

(a) $\nabla\phi$, the gradient of ϕ (converts a scalar to vector. For example $-\nabla V$ is electric field strength, where V is potential)

(b) $\nabla.E$ is divergence of E

(c) $\nabla \times E$ is curl of E.

(xi) Unit vector of a vector is $\dfrac{\vec{A}}{|\vec{A}|}$. Unit vector describes direction as its magnitude is one.

(xii) Vector division is not allowed.

(xiii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

⚠ CAUTION

(i) Not resolving vectors and trying to solve with conventional method.

- Though problems can be solved using conventional methods like parallelogram law and so on but it makes the problem lengthy and time consuming.

(ii) Not remembering vector laws of addition or not understanding its full meaning.

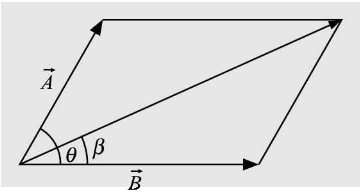


Fig. 3.9

- Parallelogram or traingle law of addition means

$$|\vec{A}+\vec{B}| = |R| = \sqrt{A^2+B^2+2AB\cos\theta}$$

$$\tan\beta = \frac{A\sin\theta}{B+A\cos\theta}$$

and $R_{max} = A + B$ if $\theta = 0$

$R_{min} = A - B$ if $\theta = 180°$

$$|\vec{A}-\vec{B}| = \sqrt{A^2+B^2+2AB\cos(180-\theta)}$$

(iii) Considering null vector has a specified direction

- Null vector has no specified direction.

(iv) Trying some other tools to prove two vectors to be perpendicular.

- Best way to prove them perpendicular is $\vec{A}.\vec{B} = 0$

  or product of the slopes $m_1.m_2 = -1$ (if vectors are two dimensional).

(v) Finding angle between vectors to prove they are parallel.

- Though in principle method is not wrong but it is time consuming.

  If $\vec{A} = k(\vec{B})$

  where k is an integer or (fraction positive or negative) then the two vectors are parallel, that is, you can judge by inspection. No need to show even $\vec{A}\times\vec{B} = 0$

(vi) Not recognizing the axes about which a vector is to be resolved.

- Recognize the axes clearly then the problem becomes small and resolve about these axes.

(vii) Considering that when a particle takes a u turn and continues to move with same velocity, average acceleration during turning is zero.

- Use the vector knowledge.

  Final velocity = $-u\hat{i}$, initial velocity = u

  change in velocity $= u_{final} - u_{initial}$

  $= -u\hat{i} - (u\hat{i}) = -2u\hat{i}$

  Since the change in velocity, $\neq 0$ therefore, average acceleration $= \frac{\Delta u}{\Delta t}$ is not zero.

Solved Problems

1. A queen is placed at the centre of carrom-board. It is hit from a position with a striker such that after rebounding from the front end, it goes to the hole in the opposite end. Determine the displacement of the queen from the point of rebounding. [The side of the carrom board is a]

(a) $\frac{a}{2}$ (b) $\sqrt{3}a$

(c) $\frac{\sqrt{10}}{3}a$ (d) $\frac{4}{3}a$

Solution (c) If $CS = DS = BD = a/3$ only then the rebounding queen goes to hole as shown in Fig. 3.15

$\therefore\ AB = \sqrt{BD^2+AD^2}$

$= \frac{\sqrt{10}\,a}{3}$

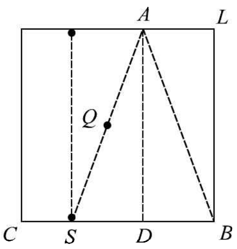


Fig. 3.10

2. A particle of mass m moving due east with a speed v collides with another particle of same mass and same speed moving due north. The two particles coalesce on collision. Find the velocity of the new particle?

(a) v (b) $2v$

(c) $v/\sqrt{2}$ (d) none of these

Solution (c) Look into Fig. 3.11

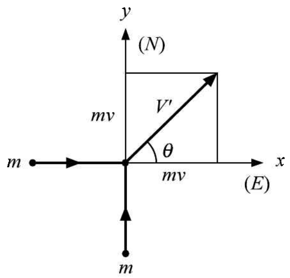


Fig. 3.11

$$mv\hat{i} + mv\hat{j} = 2m(v'_x\hat{i} + v'_y\hat{j})$$

or $\frac{1}{2}(v\hat{i} + v\hat{j}) = v'_x\hat{i} + v_y\hat{j}$

Thus $v'_x = v/2$ and $v'_y = \frac{v}{2}$

or $v' = v/\sqrt{2}$ and $\theta = \tan^{-1}(1) = 45°$

3. A particle of mass 2kg moves under a force $(4\hat{i}+6\hat{j}+8\hat{k})$. Find the acceleration to which particle is subjected to. If the particle is initially at origin and starts from rest, determine the new coordinates after 4 seconds

Solution $\hat{a} = \frac{\vec{F}}{m} = \frac{4\hat{i}+6\hat{j}+8\hat{k}}{2}$

$= (2\hat{i}+3\hat{j}+4\hat{k})\ \text{ms}^{-2}$

$s = ut + \frac{1}{2}at^2 = \frac{1}{2}(2\hat{i}+3\hat{j}+4\hat{k})(4)^2$

$= 16\hat{i}+24\hat{j}+32\hat{k}$

$\therefore$ coordinates of final position after 4 s are (16, 24, 32)

4. x and y component of vector A are 4 and 6 m respectively. The x and y component of $\vec A + \vec B$ are 10 and 9 m respectively. Calculate length of vector B and its angle with respect to x-axis.

(a) $5\sqrt{3}$, 26°36′ (b) $3\sqrt{5}$, 26°36′
(c) 5, 26°36′ (d) none of these

Solution (b) $\vec A = 4\hat i + 6\hat j$ $\vec A + \vec B = 10\hat i + 9\hat j$

$$\vec B = (\vec A + \vec B) - \vec A = 6\hat i + 3\hat j$$

length of B that is, $|\vec B| = \sqrt{6^2+3^2} = 3\sqrt{5}$ m and it makes an angle θ with x-axis, given by

$$\theta = \tan^{-1}\left(\frac{B_y}{B_x}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26°36'$$

5. A particle moves in $x - y$ plane under the action of a force $\vec F$ such that the x and y components of linear momentum $\vec P$ at any time t are $2\cos t$ and $2\sin t$. Find the angle between $\vec F$ and $\vec P$ at a given time.

(a) 60° (b) 30°
(c) 0° (d) 90°

Solution (d) $\vec P = 2\cos t\,\hat i + 2\sin t\,\hat j$ $|\vec p| = 2\sqrt{2}$

$$\vec F = \frac{d\vec p}{dt} = -2\sin t\,\hat i + 2\cos t\,\hat j \text{ As } \vec F\cdot\vec p = 0,\ \theta = 90°$$

[Since the dot product of two non zero vectors is zero, therefore, angle between $\vec F$ and $\vec P$ is 90°.]

6. A particle of mass m is moving with a constant velocity v parallel to x-axis in an $x - y$ plane as shown in Fig. 3.12. Calculate angular momentum with respect to origin at any moment.

(a) $-mvb\hat k$ (b) zero

(c) $\frac{mvb}{2}\hat k$ (d) $mvb\cos\theta\,\hat k$

Solution (a) $\vec L = \vec r \times \vec p = \begin{vmatrix} i & j & k \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix}$

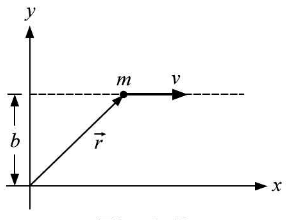


Fig. 3.12

Since the particle is moving in xy plane, therefore, $z = 0$ and $p_z = 0$

or $L = k\,(xp_y - yp_x)$ $x = vt$ (at any time t)

$$= \hat k\,(vt.0 - bmv) = -mvb\,\hat k$$

$$y = b \text{ and } p_x = mv, p_y = 0$$

7. If $\vec L$ and $\vec L'$ represent two length vectors then $|\vec L \times \vec L'|$ represents

(a) area of a triangle
(b) area of a parallelogram
(c) zero
(d) circumference of a parallelogram

Solution (b) $|L \times L'| = L\,L'\sin\theta$ which represents area of a parallelogram.

8. The sum of magnitudes of two forces is 18 N and the magnitude of their resultant is 12. If the resultant makes an angle of 90° with the smaller force, then find the magnitudes of the forces.

(a) 5 N, 13 N (b) 6 N, 12 N
(c) 10 N, 8 N (d) none of these

Solution (a) $P + Q = 18$ ...(1)

$$R = 12 = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \quad ...(2)$$

$$\tan\phi = \frac{Q\sin\theta}{P + Q\cos\theta} = \infty \quad\text{or}\quad P + Q\cos\theta = 0$$

Thus $Q(1 - \cos\theta) = 18$ ...(3)

from (1) and (2)

$(P + Q)^2 - R^2 = 2PQ(1 - \cos\theta) = 18^2 - 12^2 = 180$

or $PQ(1 - \cos\theta) = 90$ ...(4)

Dividing (4) by (3) we get, $P = 5$ and $Q = 13$ units.

Shortcut $(18 - x)^2 = x^2 + 12^2$ or $x = 5$

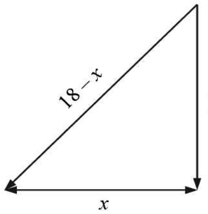


Fig. 3.13

9. In the arrangement shown in Fig. 3.14, the ends P and Q of an unstretchable string move downwards with speed u. If the pulleys are fixed then find the speed with which mass M moves up.

(a) $2u\cos\theta$ (b) $u\cos\theta$

(c) $\frac{u}{2\cos\theta}$ (d) $\frac{u}{\cos\theta}$

Solution (d) In ΔOLM $l^2 = y^2 + x^2$

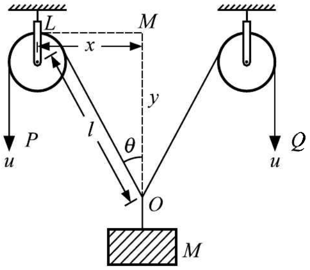


Fig. 3.14

$$2l\frac{dl}{dt} = 2y\frac{dy}{dt} + 0 \text{ (as } x \text{ does not change)}$$

$$\frac{dy}{dt} = \frac{dl/dt}{y/l} = \frac{u}{\cos\theta}$$

Note: as mass M moves up θ increases, cos θ decreases and hence vertical speed increases.

10. A man flies from Wanshigton to Manila. Find the displacement vector if the latitude and longitude of two cities are 39°N, 77°W and 15°N, 121°E.

(a) 10200 km (b) 9020 km
(c) 8020 km (d) 7200 km

Solution (a) Distance along North = $6400 \times \frac{(39-15)}{90}$

$$= 6400 \times \frac{24}{90} = \frac{5120}{3} \cong 1707\,\text{km}$$

Distance along east= $12800\ \frac{[(90-77)+121]}{180}$

$$= 12800 \times \frac{134}{180} \cong 9600\,\text{km}$$

Thus the displacement vector may be written as $9600\hat{i} + 1707\hat{j}$

or $|d| = \sqrt{(9600)^2 + (1707)^2} \cong 10200\,\text{km}$ approx.

11. The product $\vec{a}.(\vec{b}\times\vec{c})$ represents
    (a) volume of a parallelopiped
    (b) area of parallelopiped
    (c) surface area of a parallelogram
    (d) none of these

Solution (a)

12. The rain is falling vertically with a speed $4\,\text{ms}^{-1}$ and the man is moving due east with a speed $3\,\text{ms}^{-1}$. With what speed the rain appears to be coming to man and where shall he hold the umbrella to protect himself from rains?
    (a) $5\,\text{ms}^{-1}$, 37° with respect to vertical
    (b) $5\,\text{ms}^{-1}$, 37° with respect to horizontal
    (c) $5\,\text{ms}^{-1}$, 30° wrt vertical
    (d) none of these

Solution (a) $\vec{v}_{rm} = -4j - 3i$

$$|v_{rm}| = 5\,\text{ms}^{-1}$$

$$\theta = \tan^{-1}\left(+\frac{3}{4}\right) \text{ or } 37°$$

with respect to vertical

13. To a man moving due north with a speed $5\,\text{ms}^{-1}$, the rain appears to fall vertically. When the man doubles his speed, the rain appears to fall at 60°. Find the actual speed of the rain and its direction.

Solution Let $V_r = V_{rx}\hat{i} + V_{ry}\hat{j}$ and $V_m = 5\hat{i}$ (in Ist case)

$$\vec{V}_{rm} = (V_{rx} - V_m)\,\hat{i} + V_{ry}\,\hat{j}$$

Case (i) $\tan 90 = \frac{V_{ry}}{V_{rx} - 5}$

or $V_{rx} = 5\,\text{ms}^{-1}$

Case (ii) $\vec{V}_{rm} = (5\hat{i} - 10\hat{j}) + V_{ry\hat{j}}$ $\quad\because V_m = 10\hat{i}$

$$\tan 60 = \frac{V_{ry}}{5-10} \text{ or } V_{ry} = -5\sqrt{3}$$

$$-2_{i} = 5\hat{i} - 5\sqrt{3}\,\hat{j} \quad |V_r| = 10\,\text{ms}^{-1}$$

$$\angle\phi = \tan^{-1}\left(\frac{-5\sqrt{3}}{5}\right) \text{ or } \phi = 120°$$

that is, rain is falling on the ground making an angle of 120° with north or 30° south of vertical.

14. A swimmer can swim in still water at a speed $8\,\text{km h}^{-1}$. The man is on the south bank, the river is flowing at a rate $4\,\text{km h}^{-1}$ and is 1 km wide. In what direction he shall swim so as to cross the river (a) in shortest time (b) along shortest distance.

* In order to cross the river in shortest time the swimmer shall strike perpendicular to the flow of river.

** In order to cross the river along the shortest distance he shall strike making an obtuse angle with the direction of flow whose actual value can be determined as per the given question. So that the resultant velocity is perpendicular to the flow of river.

Solution (a) Towards north from south bank, that is, at right angles to the flow of river.

(b) In order to move vertically along shortest distance

$v\sin\theta = v_r$ (river velocity)

$$\theta = \sin^{-1}\left(\frac{v_r}{v}\right) = \sin^{-1}\left(\frac{4}{8}\right)$$

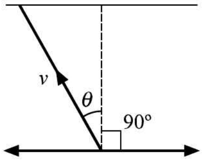


Fig. 3.15

θ = 30° or 120° with the flow of river.

15. A train is travelling due south at a rate $40\,\text{ms}^{-1}$. The rain which is falling at an angle of 30° with vertical appears falling vertically on the window pane of the train. Find the speed of the rain?
    (a) $80\,\text{ms}^{-1}$
    (b) $60\,\text{ms}^{-1}$
    (c) $40\,\text{ms}^{-1}$
    (d) none of these

Solution (a) $v_r \sin 30° = 40°$

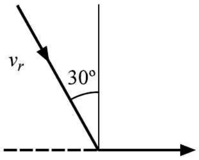


Fig. 3.16

$$v_r \times \frac{1}{2} = 40$$

$$v_r = 80\,\text{ms}^{-1}$$

16. Find the resultant of the three vectors shown in Fig. 3.17
    (a) 13.62 cm, $\tan^{-1}\frac{11}{13}$ (b) 15.62 cm, $\tan^{-1}\frac{13}{11}$
    (c) 18.62 cm, $\tan^{-1}\frac{13}{11}$ (d) none of these

Solution (a) $R = \vec{OA} + \vec{AB} + \vec{BC}$

$$R = (8\cos 37i + 8\sin 37j) + 4i + 4j$$
$$= 6.4i + 4.8j + 4i + 4j = 10.4i + 8.8j$$

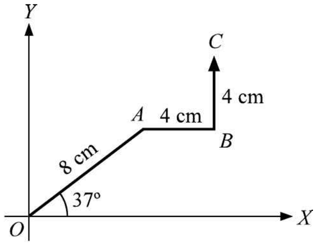


Fig. 3.17

$$|R| = \sqrt{(10.4)^2+(8.8)^2} = 13.62\,\text{cm (approx.)}$$

and $\beta = \tan^{-1}\left(\dfrac{8.8}{10.4}\right) = \tan^{-1}\left(\dfrac{11}{13}\right)$ with respect to horizontal.

17. Find the resultant of three vector $\vec{OA}, \vec{OB}, \vec{OC}$ as shown in Fig. 3.18. The radius of the circle is R.

(a) $R+\dfrac{R}{2}$ (b) $R+\dfrac{R}{\sqrt{2}}$

(c) $R\left(\sqrt{2}+1\right)$ (d) none of these

Solution (c) $|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = R$

$$X = \vec{OA}+\vec{OB}+\vec{OC}$$

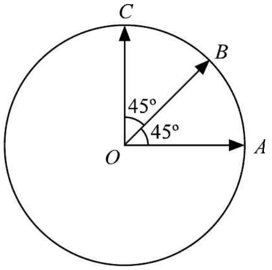


Fig. 3.18

$$\text{Resultant } X = Ri + (R\cos 45 i + R\sin 45 j) + Rj$$

$$= \left(R+\frac{R}{\sqrt{2}}\right)i+\left(R+\frac{R}{\sqrt{2}}\right)j$$

$$|X| = (\sqrt{2}R+R) \quad \text{along } \vec{OB}.$$

18. If $\vec{A}.\vec{B}=\vec{B}.\vec{C}$ then it can be concluded that

(a) $|\vec{A}| = |\vec{C}|$

(b) $|\vec{A}| \neq |\vec{C}|$

(c) $|\vec{A}|$ may or may not be equal to $|\vec{C}|$

(d) none of these

Solution (c) $\because \vec{A}.\vec{B}=\vec{B}.\vec{C}$

that is, $|\vec{A}||\vec{B}|\cos\theta_1 = |\vec{B}||\vec{C}|\cos\theta_2$

Since θ_1 and θ_2 may be different, therefore, $|\vec{A}|$ and $|\vec{C}|$ may be different.

19. Three particles A, B and C are situated at the vertices of an equilateral triangle of side l. Each of the particle starts moving with a constant velocity v such that A is always directed towards B, B towards C and C towards A. Find the time when they meet.

Solution We look into it as a problem of relative velocity and find v_{BA} in the direction of B.

$$v_{BA} = v_B - (-v_A \cos 60)$$

$$= v + \frac{v}{2} = \frac{3v}{2}$$

$$\text{time taken } t = \frac{l}{3v/2} = \frac{2l}{3v}$$

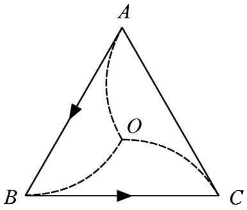


Fig. 3.19

Problems for Practice

1. To get a resultant displacement of 10 cm, two displacement vectors one of magnitude 6 m and another of 8 m should be combined
(a) at an angle 60° (b) perpendicular to each
(c) parallel (d) anti-parallel

2. When mass is rotating in a plane about a fixed point its angular momentum is directed along
(a) the axis of rotation
(b) line at an angle of 45° to the axis of rotation
(c) the radius
(d) the tangent to the orbit

3. Which of the following is a vector?
(a) force (d) mass (c) energy (d) power

4. When two vectors $\vec{A}$ and $\vec{B}$ of magnitude a and b are added, the magnitude of the resultant vector is always
(a) greater than $(a + b)$
(b) not greater than $(a + b)$
(c) equal to $(a + b)$
(d) less than $(a + b)$

5. Identity the vector quantity.
(a) heat (b) angular momentum
(c) time (d) work

6. Which of the following quantities is a scalar?
(a) magnetic moment
(b) acceleration due to gravity
(c) electric field
(d) electrostatic potential

7. Which of the following quantities is a vector?
(a) volume (b) temperature
(c) displacement (d) density

8. The rectangular components of force 5 dyne are
(a) 3 and 4 dyne (b) 2.5 and 25 dyne
(c) 1 and 2 dyne (d) 2 and 3 dyne

9. Identify the scalar quantity.
(a) work (b) impulse
(c) force (d) acceleration

10. Moment of inertia is
(a) vector (b) scalar
(c) phasor (d) tensor

11. If the magnitudes of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are 12, 5 and 13 units respectively and $\vec{A} + \vec{B} = \vec{C}$, the angle between vectors $\vec{A}$ and $\vec{B}$ is
(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 0

12. Angular displacement is

(a) a scalar (b) a vector
(c) neither (a) nor (b) (d) either (a) or (b)

13. A mosquito flies from the hole in a mosquito net top corner diametrically opposite. If the net is 3 × 2 × 2 m then the displacement of the mosquito is

(a) $\sqrt{13}$ m (b) $\sqrt{17}$ m
(c) $\sqrt{11}$ m (c) none of these

14. A man travels 1 mile due east, 5 mile due south, 2 mile due east and finally 9 miles due north. How far is he from the starting point?

(a) 3 miles (b) 5 mile
(c) 4 mile (d) between 5 and 9 mile

15. Two forces of magnitude 7N and 5N newton act on a particle at an angle θ to each other, θ can have any value. The minimum magnitude of the resultant force is

(a) 12N (b) 8N
(c) 2N (d) 5N

16. I started walking down a road to day-break facing the sun. After walking for some time, I turned to my left then I turned to the right once again. In which direction was I going then

(a) northeast (b) south
(c) east (d) northwest

17. If $\vec{A}=\vec{B}+\vec{C}$ and the magnitudes of $\vec{A},\vec{B}$ and $\vec{C}$ are 5, 4 and 3 units respectively. The angle between $\vec{A}$ and $\vec{C}$ is

(a) $\pi/2$ (b) $\sin^{-1}(3/4)$
(c) $\cos^{-1}(3/5)$ (d) $\cos^{-1}(4/5)$

18. A boat which has a speed of 5 kmhr^{-1} in still water crosses a river of width 1 km along the shortest possible path in 15 minutes. The velocity of the water in kmhr^{-1} is

(a) 4 (b) $\sqrt{41}$ (c) 1 (d) 3

19. If two waves of same frequency and same amplitude respectively on superimposition produce a resultant wave of the same amplitude, the wave differ in phase by

(a) $\pi/5$ (b) $2\pi/3$ (c) $\pi/4$ (d) zero

20. If $\vec{n}$ is a unit vector in the direction of the vector $\vec{A}$, then

(a) $\vec{n} = |\vec{A}|/\vec{A}$ (b) $\vec{n}=\vec{n}\times\vec{n}$
(c) $\vec{n} = \vec{A}/|\vec{A}|$ (d) $\vec{n} = \vec{A}\,|\vec{A}|$

21. Two forces of 4 dyne and 3 dyne act upon a body. The resultant force on the body can only be

(a) between 3 and 4 dyne
(b) between 1 and 7 dyne
(c) more than 3 dynes
(d) more than 4 dynes

22. A river is flowing from west to east at a speed of 3 m/minute. A man on the south bank of the river, capable of swimming at 10 m/minutes in still water wants to swim the river in the shortest time. He should swim in a direction

(a) 30° west of north
(b) 60° east of north
(c) 30° east of north
(d) due north

23. The resultant of two equal forces is double of either of the force. The angle between them is

(a) 0° (b) 60°
(c) 90° (d) 120°

24. An aeroplane is moving on a circular path with a speed 250 kmhr^{-1} what is the change in velocity in half revolution

(a) 0 (b) 125 kmhr^{-1}
(c) 250 kmhr^{-1} (d) 500 kmhr^{-1}

25. A body constrained to move in y direction is subject to force given by $\vec{F} = (-2\vec{i} + 15\vec{j} + 6\vec{k})$ N. What is the work done by this force, in moving the body through a distance of 10 m along y-axis

(a) 20 J (b) 150 J (c) 160 J (d) 190 J

26. I walked 4 mile turned to my left and walked 6 mile then turned to my right again and walked 4 mile. Which of the choice mentions the distance from the straight point to the place where I stopped?

(a) 10 mile (b) 14 mile
(c) 15 mile (d) 20 mile

27. A Force $\vec{F}=-2\vec{i}+15\vec{j}+6\vec{k}$ newton produces an acceleration of 1 ms^{-2} in a body the body would be

(a) $10\sqrt{2}$ kg (b) $6\sqrt{2}$ kg
(c) 20 kg (d) 200 kg

28. Maximum and minimum magnitudes of the resultant of two vectors of magnitudes P and Q are in the ratio 3 : 1. Which of the following relations is true

(a) $PQ = 1$ (b) $P = 2Q$
(c) $P = Q$ (d) none of these

29. What is the projection of $\vec{P}$ on $\hat{Q}$?

(a) $\vec{Q}.\vec{P}$ (b) $\vec{P}.\hat{Q}$
(c) $\vec{P}.\vec{Q}$ (d) $\vec{P}.\hat{Q}$

30. Rain is falling vertically 4 ms^{-1}. A man is moving due east with 3 ms^{-1}. The direction in which he shall hold the umbrella with the vertical is

(a) 53° east of vertical
(b) 37° east of vertical
(c) 53° west of vertical
(d) 37° west of vertical

31. There are N co-planar vectors each of magnitude V. Each vector is inclined to the proceeding vector at angle $2\pi/N$. What is the magnitude of their resultant

(a) zero (b) V/N (c) V (d) NV

32. Which of the following operations between the two vectors can yield a vector perpendicular to either of them?

(a) subtraction (b) division
(c) addition (d) multiplication

33. Three vectors A, B and C satisfy the relation $\vec{A}.\vec{B}=0$ and $A.C = 0$. The vector A is parallel to

(a) $\vec{B}.\vec{A}$ (b) $\vec{B}$
(c) $\vec{C}$ (d) $\vec{B}\times\vec{C}$

34. Angle between the vectors $(\hat{i}+\hat{j})$ and $(\hat{j}+\hat{k})$ is

(a) 60° (b) 90° (c) 180° (d) 0°

35. Resultant of two vectors $\vec{P}$ and $\hat{Q}$ is inclined at 45° to either of them. What is the magnitude of the resultant?

(a) $\sqrt{P^2+Q^2}$ (b) $\sqrt{P^2-Q^2}$
(c) $P+Q$ (d) $P-Q$

36. A steamer is heading due North with 20 ms^{-1}. The wind is blowing 10 ms^{-1} due west. In which direction does the flag on the mast flutters?

(a) $\tan^{-1}1/2$ west of north
(b) $\tan^{-1}1/2$ east of north
(c) $\tan^{-1}1/2$ north of east
(d) $\tan^{-1}1/2$ north of west

37. What is the angle between $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}$?

(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d) none of these

38. What is the maximum number of components into which a vector can be split?
(a) 2 (b) 3 (c) 4 (d) more than 4

39. What is the maximum number of a rectangular components into which a vector can be split in its own plane?
(a) two (b) three (c) four (d) more than 4

40. A force of 6 kg and 8 kg can be applied together to produce the effect of a single force of
(a) 20 kg (b) 15 kg (c) 11 kg (d) 1 kg

41. To a person going east in a car with a velocity of 25 kmhr^{-1} a train appears to move towards north with a velocity of 25 $\sqrt{3}$ km/h. The actual velocity
(a) 5 kmhr^{-1} (b) 25 kmhr^{-1}
(c) 50 kmhr^{-1} (d) 53 kmhr^{-1}

42. The area of a Δ formed with sides $5i + 3j - k$ and $3i + 2j - k$ is
(a) $\sqrt{6}$ (b) $\sqrt{3}$ (c) $\sqrt{3/2}$ (d) $\sqrt{5/2}$

43. At what angle should be the two forces $2p$ and $\sqrt{2}\,p$ act so that the resultant force is $P\sqrt{10}$
(a) 45° (b) 60° (c) 90° (d) 120°

44. Two cars are moving. A along east with 10 ms^{-1}. At any instant it is 1500m away from the crossing. B at the same instant is 1800 m away from the crossing and is moving towards the crossing with 15 ms^{-1}. When do they come closest?
(a) 109.3 s (b) 129.2 s
(c) 119.3 s (d) 99.3 s

45. What is the angle between $\vec{P}$ and the resultant of $(\vec{P}+\vec{Q})$ and $(\vec{P}-\vec{Q})$?
(a) $\dfrac{\tan^{-1}\left|(\vec{P}-\vec{Q})\right|}{\left|P+Q\right|}$ (b) $\tan^{-1}(Q/P)$
(c) $\tan^{-1}(P/Q)$ (d) zero

46. The length of second's hand in a watch is 1 cm. The change in velocity of its tip in 15 seconds is
(a) zero (b) $(\pi/15\sqrt{2})$ cms^{-1}
(c) $(\pi/15)$cms^{-1} (d) $(\pi/15\sqrt{2})$ cms^{-1}

47. Rain falling vertically downwards with a velocity of 3 kmh^{-1}. A person moves on a straight road with a velocity of 4 kmh^{-1}. Then the apparent velocity of the rain with respect to the person is
(a) 1 kmh^{-1} (b) 5 kmh^{-1}
(c) 4 kmh^{-1} (d) 3 kmh^{-1}

48. A large number of particles are moving towards each other with velocity v having directions of motion randomly distributed. What is the average relative velocity between any two particles averaged over all the pairs?
(a) $4v/\pi$ (b) $4\pi v$ (c) v (d) $\pi v/4$

49. The magnitudes of the X and Y components $\vec{P}$ are 7 and 6. The magnitudes of the X and Y components of $\vec{P}+\vec{Q}$ are 11 and 9 respectively. What is the magnitude of Q?
(a) 9 (b) 8 (c) 6 (d) 5

50. A swimmer can swim in still water with speed v and the river flowing with velocity $v/2$. To cross the river in shortest time, he should swim making angle θ with the upstream. What is the ratio of the time taken to swim across in the shortest time to that in swimming across over shortest distance?
(a) sin θ (b) cos θ (c) tan θ (d) cot θ

51. A vector of magnitude a is rotated through angle θ. What is the magnitude of the change in the vector?
(a) $2a\sin\theta$ (b) $2a\cos\theta$
(c) $2a\sin(\theta/2)$ (d) $2a\cos(\theta/2)$

52. Consider a vector $\vec{F} = 4\hat{i} - 3\hat{j}$. Another vector that is perpendicular to $\vec{F}$ is
(a) $4i + 3j$ (b) $7k$
(c) $3i - 4j$ (d) $6i$

53. A helicopter is flying south with a speed of 50 km h^{-1}. A train is moving with the same speed towards east. The relative velocity of the helicopter as seen by the passengers in the train will be 50 $\sqrt{2}$ km h^{-1} towards.
(a) northwest (b) southwest
(c) northeast (d) southeast

54. A man is walking due east at the rate of 4 kmhr^{-1} and the rain is falling at an angle of 30° east of vertical with a velocity of 6 kmhr^{-1}. The velocity of the rain relative to the man will be
(a) 5 kmhr^{-1} (b) 7.118 kmhr^{-1}
(c) 8.718 kmhr^{-1} (d) 10 kmhr^{-1}

55. A truck travelling due north at 20 ms^{-1} turns west and travels at the same speed. Then the change in velocity is
(a) 40 ms^{-1} northwest
(b) 20 $\sqrt{2}$ ms^{-1} northwest
(c) 20 $\sqrt{2}$ ms^{-1} southwest
(d) 40 ms^{-1} southwest

56. Given that P is a point on a wheel rolling on a horizontal ground. The radius of the wheel is R. Initially if the point P is in contact with the ground the wheel rolls through half the revolution. What is the displacement of the point P?
(a) $R\sqrt{\pi^2+1}$ (b) $R\sqrt{\pi^2+4}$
(c) πR (d) $2\pi R$

57. A vector $\vec{F}_1$ is along x axis. If $\vec{F}_1 \cdot \vec{F}_2$ is zero then $\vec{F}_2$ could be
(a) $(j + k)$ (b) $-(i + j)$
(c) $4(i + k)$ (d) $-4i$

58. A parallelogram is formed with $\vec{a}$ and $\vec{b}$ as the sides. Let $\vec{d}_1$ and $\vec{d}_2$ be the diagonals of the parallelogram then $a^2 + b^2 =$
(a) $(d_1^2 + d_2^2)/2$ (b) $(d_1^2 - d_2^2)/2$
(d) $d_1^2 + d_2^2$ (d) $d_1^2 - d_2^2$

59. A particle starts from centre O towards OA then moves along AB and stops at B. If $R = 100$ m the displacement of the particle is
(a) 100 m (b) $100\sqrt{2}$ m
(c) $\dfrac{100}{\sqrt{2}}$ m (d) none of these

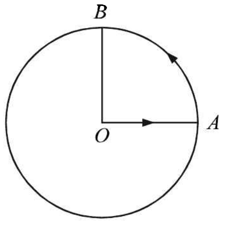


Fig. 3.20

60. If $|\vec{A}| = |\vec{B}|$, then what is the angle between $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$
(a) 90° (b) 60°
(c) 30° (d) 0°

ANSWERS

1. (b)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (d)
11. (b)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (b)	**20.** (c)
21. (b)	**22.** (d)	**23.** (a)	**24.** (d)	**25.** (b)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (d)	**30.** (b)
31. (a)	**32.** (d)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (b)	**37.** (d)	**38.** (d)	**39.** (a)	**40.** (c)
41. (c)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (d)	**46.** (b)	**47.** (b)	**48.** (d)	**49.** (d)	**50.** (a)
51. (c)	**52.** (b), (c)	**53.** (a)	**54.** (c)	**55.** (c)	**56.** (b)	**57.** (a)	**58.** (a)	**59.** (a)	**60.** (a)

4

Motion in Two Dimensions

BRIEF REVIEW OF THE CONCEPTS

The best way to solve a two dimensional problem is to convert it into two, one dimensional motions. Say along$x$- and y-directions. Thus resolve the velocity v to its x and y components say

$$v = v_x\hat{i} + v_y\hat{j}\,.$$

Treat the motion in x-directions and y-directions as one dimensional motion. The acceleration can also be resolved and taken separately along x-directions and y-directions.

Projectile A freely falling body having constant horizontal velocity may be termed as a projectile. In general, we can say that if velocity along one-direction, that is, x-direction remains constant and there exists along y-direction an accelerated motion, then a body undergoing such a motion is called a projectile. It always follows a parabolic path as illustrated in Fig. 4.1. Consider a projectile fired with a velocity v making an angle θ with the horizontal from the point O (considered origin) at t = 0. Resolving v along x- and y-directions

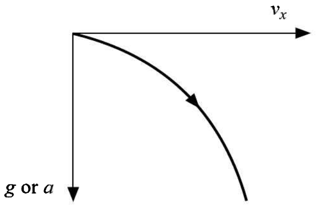


Fig. 4.1

$$v_x = v\cos\theta$$

and $v_y = v\sin\theta$ (as shown in Fig. 4.2).

Maximum height attained

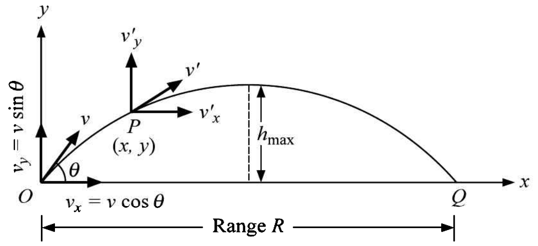


Fig. 4.2

$$h_{\max} = \frac{v^2\sin^2\theta}{2g} \quad \text{as} \quad v'_y = 0$$

$$\text{time of flight } T = 2t = \frac{2v\sin\theta}{g}$$

Horizontal range (*R*)

$$R = v_x \times \text{time of flight} = v\cos\theta\left(\frac{2v\sin\theta}{g}\right) = \frac{v^2\sin 2\theta}{g}$$

$R_{\max}$ (Maximum range) $R_{\max}$ will be obtained when $\sin 2\theta = 1$

That is, $2\theta = 90°$ or $\theta = 45°$ and $R_{\max} = \dfrac{v^2}{g}$

projectile will possess same range when projected with same velocity making complementary angles, that is, θ or (90 – θ).

$$\text{Trajectory } y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} = x\tan\theta\left(1 - \frac{x}{R}\right)$$

Comparing it with $y = ax + bx^2$

We say that it is a general equation of parabola. Hence the path followed by the projectile is a Parabola.

Instantaneous velocity (*v*') Let v' be the velocity at any instant t, then

$$v' = v_x i + v'_y j = v\cos\theta\, i + (v\sin\theta - gt)j$$

$$|v'| = \sqrt{v^2 + g^2t^2 - 2vgt\sin\theta}$$

$$\alpha = \tan^{-1}\left(\frac{v\sin\theta - gt}{v\cos\theta}\right)$$

Range and time of flight on an inclined plane Consider an inclined plane of inclination α. Let a projectile be fired making an angle θ along horizontal. Call axis along the inclined plane to be x-axis. Thus velocity makes an angle (θ – α) along x-axis. (See Fig. 4.3) The time of flight

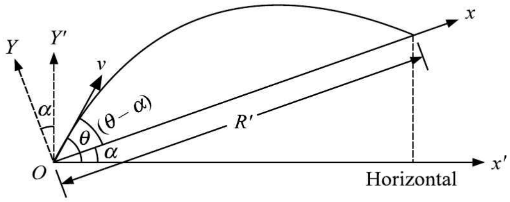


Fig. 4.3

$$T' = \frac{2v\sin(\theta - \alpha)}{g\cos\alpha}$$

$$R' = \frac{v^2}{g\cos^2\alpha}\,[\sin(2\theta - \alpha) - \sin\alpha]$$

Range R' on inclined plane will be maximum when $2\theta - \alpha = \pi/2$

or $\theta - \alpha = \pi/2 - \theta$

That is, when the direction of projection bisects the angle that the inclined plane makes with OY' (vertical direction).

Note: From the above discussion we conclude that the motion in a plane is two dimensional motion. Projectile is the example of motion in a vertical plane under certain conditions mentioned earlier.

Motion in horizontal and vertical circles are other examples of two dimensional motion.

Motion in space may be termed as three dimensional motion.

SHORT-CUTS AND POINTS TO NOTE

(i) The best method to solve a problem in two-dimension motion is to convert it into two one-dimension motion problems, whenever direct formulae are inapplicable.

(ii) For oblique projection [Fig (4.4)]

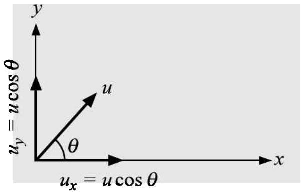


Fig.4.4

(1) Time of flight $T = \dfrac{2u\sin\theta}{g}$

(2) Height attained $H = \dfrac{u^2\sin^2\theta}{2g}$

(3) Range $R = \dfrac{u^2\sin 2\theta}{g}$

(4) Maximum Range $R_{\max} = \dfrac{u^2}{g}$

when $\sin 2\theta = 1$ or $\theta = 45°$

(5) Trajectory $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$ (Parabolic)

or $y = x\tan\theta\left(1 - \dfrac{x}{R}\right)$.

(6) Instantaneous velocity

$$v = \sqrt{u_x^2 + v_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

and direction $\beta = \tan^{-1}\dfrac{v_y}{u_x} = \tan^{-1}\dfrac{u\sin\theta - gt}{u\cos\theta}$

Note: Range is equal if projected at complementary angles, that is, θ and $(90 - \theta)$ with same velocity $\boldsymbol{u}$.

(7) Radius of curvature $R_C = \dfrac{v^2}{g}$ where v is tangential velocity at that point where radius of curvature is desired.

In general $R_C = \dfrac{v^2}{a_r}$ where a_r is radial or normal acceleration.

(iii) Average acceleration

$$a_{av} = \frac{v_f - v_i}{t} = \frac{(v_{fx}\hat{i} + v_{fy}\hat{j}) - (v_{ix}\hat{i} + v_{iy}\hat{j})}{t}$$

and direction $\beta = \tan^{-1}(v_{fy} - v_{iy})/(v_{fx} - v_{ix})$

where v_f is the final velocity and v_i is the initial velocity with respect to x-direction v_{fx} and v_{fy} are x and y components of final velocity ; v_{ix} and v_{iy} are x and y components of initial velocity.

(iv) Try to solve the problem by applying vector laws if the physical quantities are vectors.

(v) Relative velocity Note that relative velocity is actually a vector law.

In Fig. 4.5 A represents point A, G represents ground and B as point B.

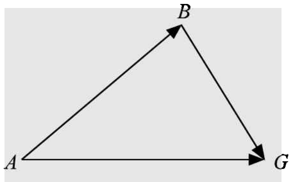


Fig. 4.5

Then from Δ-Law $v_{AG} = v_{AB} + v_{BG}$

or $v_{AB} = v_{AG} - v_{BG} = v_A - v_B$

where v_{AG} is velocity of A (with respect to Ground) normally written as v_A. Similarly v_{BG} may be written as v_B

$$v_{AB} = (v_{Ax}\hat{i} + v_{Ay}\hat{j}) - (v_{bx}\hat{i} + v_{By}\hat{j})$$

$$= (v_{Ax} - v_{Bx})\hat{i} - (v_{Ay} - v_{By})\hat{j}$$

Thus $|v_{AB}| = \sqrt{(v_{Ax} - v_{Bx})^2 + (v_{Ay} - v_{By})^2}$

and direction $\tan\beta = \dfrac{\text{Coefficient of } j}{\text{Coefficient of } i} = \dfrac{v_{Ay} - v_{By}}{v_{Ax} - v_{Bx}}$

with respect to i-direction or x-direction as shown in Fig. 4.6.

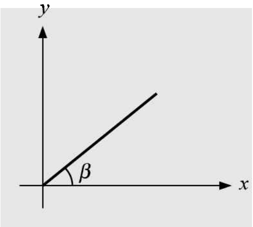


Fig. 4.6

or direction $\tan\beta' = \dfrac{\text{Coefficient of } i}{\text{Coefficient of } j}$

$$= \frac{v_{Ax} - v_{Bx}}{v_{Ay} - v_{By}}$$

with respect to j or y direction or illustrated in Fig. 4.7

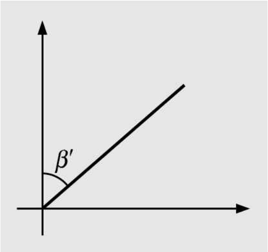


Fig. 4.7

(vi) Whenever solving problems for inclined plane, consider axis along the inclined plane as x-axis and perpendicular to it as y-axis. Consider Fig. 4.8.

Note: that $u_x = u\cos(\theta - \alpha)$

and $a_x = -g\sin\alpha$

$u_y = u\sin(\theta - \alpha)$

$a_y = -g\cos\alpha$

since acceleration is present along both directions, therefore, while finding x or y use

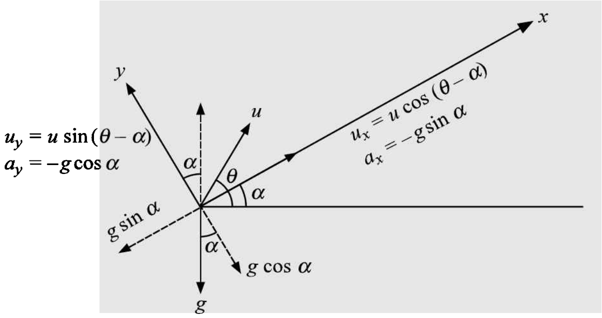


Fig. 4.8

$$x = u_x t + \frac{1}{2}a_x t^2$$

$$y = u_y t + \frac{1}{2}a_y t^2 \text{and so on.}$$

Also keep in mind the directions

If projected upwards take a_x and a_y negative.

If particle is projected down the plane as illustrated in Fig. 4.9 Along the plane $a_x = g\sin\alpha$ (+ve) and perpendicular to the plane $a_y = -g\cos\alpha$ (–ve).

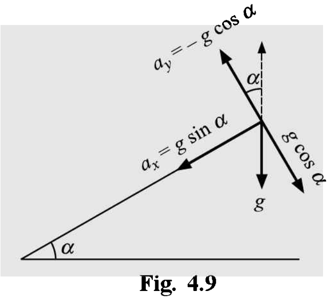


Fig. 4.9

(vii) To cross the river along shortest path the swimmer shall strike at an obtuse angle to the flow of river so that resultant velocity v is along the normal as illustrated in Fig. 4.10.

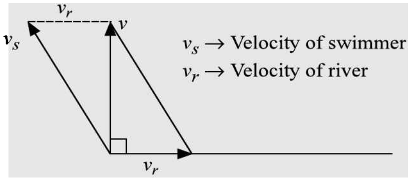


Fig. 4.10

obviously $v^2 = v_s^2 - v_r^2$ (Δ law)

or $v = \sqrt{v_s^2 - v_r^2}$

If the width of the river is l then the time taken to cross the river is $t = \dfrac{l}{v} = \dfrac{l}{\sqrt{v_s^2 - v_r^2}}$

(viii) To cross the river in shortest time strike the river at right angle to its flow and shortest time is

$t_{\text{min}} = \dfrac{l}{v_s}$ where l is width of the river and v_s is the velocity of the swimmer.

(ix) In oblique projected motion average velocity between points of equal height is equal to horizontal velocity.

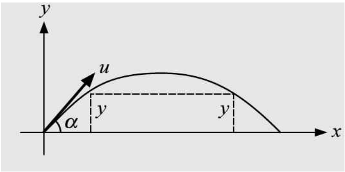


Fig. 4.11

(x) Maximum range on an inclined plane

$$R_{\text{(inc), max}} = \frac{u^2}{g[1+\sin\alpha]} \text{ (projected up the incline)}$$

(xi) Angle of projection for maximum range

$$\theta - \alpha = \frac{\pi}{2} - \theta \quad \text{or} \quad \theta = \frac{\pi}{4} + \frac{\alpha}{2} = 45° + \frac{\alpha}{2}$$

(xii) Radius of curvature $R = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$

(xiii) Equation of trajectory is the relation between x and y. Try to establish this relation by eliminating 't'

(xiv) Maximum height along an inclined plane

$$h'_{\text{max}} = \frac{u^2\sin^2(\theta-\alpha)}{2g\cos\alpha}$$

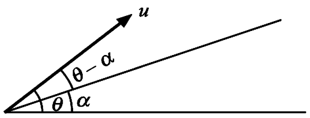


Fig. 4.12

(xv) Total acceleration $a_{\text{net}} = \sqrt{a_r^2 + a_t^2}$

and $\tan\beta = \dfrac{a_t}{a_r}$

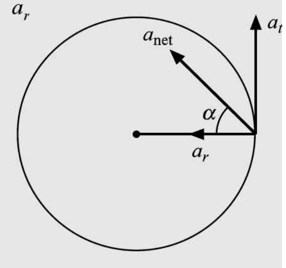


Fig. 4.13

(xvi) If the resultant of two motions is to be determined use

$$v_R = v_1 + v_2 = (v_{1x} + v_{2x})\hat{i} + (v_{1y} + v_{2y})\hat{j}$$

or $|v_R| = \sqrt{(v_{1x}+v_{2x})^2 + (v_{1y}+v_{2y})^2}$

and $\beta = \tan^{-1} \dfrac{v_{1y}+v_{2y}}{v_{1x}+v_{2x}}$

with respect to x-direction.

⚠ CAUTION

(i) Applying direct equations of projectile when starting or terminating points are not at the same vertical height or vertical displacement between starting and terminating point is nonzero.

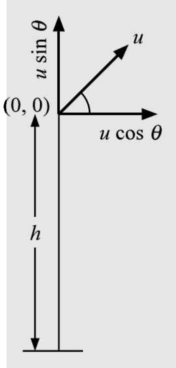


Fig. 4.14

- Apply two one-dimension motion approach with time of flight as combining factor.
- Whenever the particle starts from a height, consider that point to be origin and find time of flight with vertical displacement as $-h$, that is, use

$$-h = u\sin\theta\, t - \frac{1}{2}gt^2$$

(ii) If common trigonometric relations are not correctly remembered

- Remember common trigonometric relations like

$\sin 2\theta = 2\sin\theta\cos\theta$; $\sin(180 - \theta) = \sin\theta$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

(iii) Considering vertical distance given in the problem as maximum height.

- In most of the cases vertical distance given is not maximum height.

(iv) Confusion between vectors/scalars/tensors.

- Certain scalars also have directions but they are not added according to vector laws.

(v) Only applying trajectory equation whenever y (vertical displacement) or x(horizontal displacement) is to be determined.

- Remember all the formulae

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta};$$

$$y = x\tan\theta\left(1 - \frac{x}{R}\right);$$

$y = u_y t - \frac{1}{2}gt^2$ and $x = u_x t$ and so on.

and see which suits best.

(vi) When a particle starts from a height h, to find the maximum range applying $\theta = 45°$ (angle of projection).

- Angle will be less than 45° and may be determined using

$$\frac{d}{d\theta}(x) = 0 \text{ and } x = u_x\,(t_{\text{gravitational field}})$$

Solved Problems

1. The displacement of a particle in a plane at any instant is given by

$$s = at^2 i + bt^2 j.$$

Determine the velocity of the particle at any instant.

(a) $2t(a + b)$ (b) $2t\sqrt{a^2 + b^2}$

(c) $\dfrac{2at}{a+b}$ (d) $\dfrac{2bt}{a+b}$

Solution (b) $v_x = \left.\dfrac{\partial s}{\partial t}\right|_{y = \text{const}} = 2at$ $v_y = \left.\dfrac{\partial s}{\partial t}\right|_{x = \text{const}} = 2bt$

or $v = 2ati + 2btj$

$$|v| = \sqrt{v_x^2 + v_y^2} = 2t\sqrt{a^2 + b^2}$$

$\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$ with respects horizontal.

2. A particle travels according to the equation $y = x - \dfrac{x^2}{2}$. Find the maximum height it achieves.

(a) 1m (b) 2m

(c) 1/2m (d) 1/4m

Solution (c) Compare it with the equation of trajectory of a projectile

$$y = x\tan\theta - \frac{g\,x^2}{2\cos^2\theta\, v^2}$$

$\tan\theta = 1$ or $\theta = 45°$, $\dfrac{2\times g}{2v^2} = \dfrac{1}{2}$ or $\dfrac{g}{v^2} = \dfrac{1}{2}$

$h_{\max} = \dfrac{v^2\sin^2\theta}{2g} = \dfrac{1}{2}$ Thus, $h_{\max} = \dfrac{1}{2}$.

Alternatively $\dfrac{dy}{dx} = 1 - \dfrac{2x}{2} = 0$

or $x = 1$, $y = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ m.

3. An athlete runs with a speed 12 ms^{-1}. Determine the longest jump he can undertake.
(a) 14.4 (b) 12m
(c) 12.1m (d) 16.2m

Solution (a) $R_{\max} = \dfrac{v^2}{g} = \dfrac{12^2}{10} = 14.4$ m

4. A staircase has 5 steps each 10 cm high and 10 cm wide. A ball is placed at the top of the staircase. What minimum velocity (horizontal) be given to the ball so that it directly hits the lowest plane?
(a) 2ms^{-1} (b) 1ms^{-1}
(c) $\sqrt{2}$ ms^{-1} (d) none of these

Solution (c) If it is to hit the lowest plane and velocity be minimum then one step should be less. Time taken to travel a vertical distance of 40 cm is

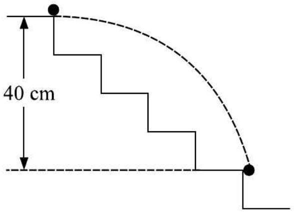


Fig. 4.15

$t = \sqrt{\dfrac{2h}{g}} = \sqrt{\dfrac{2\times 0.4}{10}}$

$= 0.2\sqrt{2}$ s.

Horizontal velocity $v = \dfrac{x}{t} = \dfrac{0.4}{0.2\sqrt{2}} = \sqrt{2}$ ms^{-1}.

5. There is a 11.2 ft ditch with the approach roads at an angle of 15° with the horizontal. With what minimum speed should a motorbike (5 ft long) be moving on the road to cross the ditch safely?

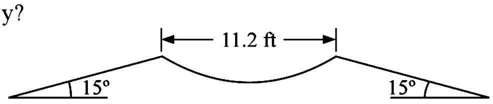


Fig. 4.16

(a) 32fts^{-1} (b) 16fts^{-1}
(c) 24fts^{-1} (d) none ot these

Solution (a) $R = 11.2 + 5\cos 15 = 16$ ft

$R = \dfrac{u^2\sin 2\theta}{g}$ $\therefore u^2 = \dfrac{R.g}{\sin 2\theta} = \dfrac{16\times 32}{1/2}$

or $u = 32.0$ fts^{-1}.

6. A boy standing on a long rail road car throws a ball up (straight) with a velocity 9.8 ms^{-1}. The car is moving on the horizontal road with an acceleration 1 ms^{-2}. How far behind the boy the ball will fall?
(a) 2m (b) 1m
(c) 1/2m (d) 3 m

Solution (a) $v = u + at$

$0 = 9.8 - 9.8t$ or $t = 1$ s

The total time the ball takes to return is 2 s (1 s going up 1 s coming down)

The distance moved by the rail road car in 2 s is

$$s = \frac{1}{2}at^2 = \frac{1}{2} \times 1\,(2)^2 = 2\text{ m}$$

Thus, the ball will fall 2 m behind the boy.

7. The benches of a gallery in a cricket stadium are 1 m wide and 1 m high. A batsman strikes the ball at a level 1 m above the ground and hits a mammoth sixer. The ball starts at 35 ms^{-1}, at an angle of 53° with the horizontal. The benches are perpendicular to the plane of motion and the first bench is 110 m away from the batsman. On which bench will the ball hit?
(a) 5 (b) 4
(c) 7 (d) 6

Solution (d) $y = x\tan\theta - \dfrac{g\,x^2}{2u^2\cos^2\theta}$

$$(n - 1) = (110 + n)\frac{4}{3} - \frac{4.9(110+n)^2}{35\times 35(.6)^2}$$

or $x^2 + 190n - 1190 = 0$

or $n = \dfrac{-190+\sqrt{(190)^2+4\times(1190)}}{2}$

$= \dfrac{-190+202}{2} = 6.$

8. A rifle with a muzzle velocity 1500fts^{-1} shoots a bullet at a small target 150 ft away. How high above the target must the gun be aimed so that the bullet hits the target?
(a) 2.02 inch (b) 1.72 inch
(c) 1.82 inch (d) 1.92 inch

Solution (d) Time taken to travel 150 ft $= \dfrac{150}{1500} = \dfrac{1}{10}$ s

$$s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32\ \frac{1}{100} = \frac{16}{100}\text{ ft}$$

$= \dfrac{16\times 12}{100}$ inch $= 1.92$ inch.

9. A shell is fired with a muzzle velocity 20 ms^{-1} at an angle of 60° with the horizontal. At the top of the trajectory the shell explodes into two fragments P and Q of equal masses. One of them say P comes to rest immediately, how far from the gun will the fragment Q fall?
(a) $30\sqrt{3}$ (b) $20\sqrt{3}$
(c) $10\sqrt{3}$ m (d) $40\sqrt{3}$ m

Solution (a) Applying conservation of linear momentum at the highest point

$$m\,20\cos 60 = \frac{m}{2}(0) + \frac{m}{2}v_Q$$

$$v_Q = 20\text{ ms}^{-1}$$

Time taken to reach the ground $= \dfrac{u\sin\theta}{g} = \dfrac{20\times\sqrt{3}}{10\times 2}$
$= \sqrt{3}$ s

Total horizontal distance $x = x_1 + x_2 = R/2 + 20\sqrt{3}$
$= 10\sqrt{3} + 20\sqrt{3} = 30\sqrt{3}$ m.

10. An astronaut on a strange planet finds he can jump a maximum horizontal distance of 30 m. If his initial speed is 9 ms^{-1} then find (a) acceleration due to gravity at that planet (b) the maximum height to which he can jump.

(a) 9.8 ms^{-2}, 30m (b) 4.9 ms^{-2}, 30m

(c) 2.7 ms^{-2}, 30m (d) 2.7 ms^{-2}, 15m

Solution (d) $x = u^2/g \Rightarrow g = u^2/x = \frac{81}{30} = 2.7\,\text{ms}^{-2}$

$h = u^2/2g = 15\,\text{m}.$

11. In a detective story a body is found 5 m from the base of a building and 25 m beneath the open window. Would you guess the death to be accidental or not?

Solution $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 25}{10}} = \sqrt{5}$ s

The horizontal velocity of the body = $5/\sqrt{5} = 2.23\,\text{ms}^{-1}$, which is 23% of the world class sprint speed 10 ms^{-1}. Such a speed cannot be achieved by a standing person unless a push is given from behind. Hence the death is not accidental.

12. A particle of mass m is projected with a velocity v making an angle θ with the horizontal, find (i) linear momentum (ii) angular momentum at the highest point.

(a) 0, 0

(b) 0, $mv^3\sin^2\theta$

(c) $mv\cos\theta, \dfrac{mv^3\sin^2\theta\cos\theta}{2g}$

(d) $mv\cos\theta, \dfrac{mv^3\cos^2\theta\sin\theta}{2g}$

Solution (i) $p = mv\cos\theta$

(ii) $l = p$ (perpendicular distance)

$$= mv\cos\theta \times \frac{v^2\sin^2\theta}{2g}$$

$$= \frac{mv^3\sin^2\theta\cos\theta}{2g}.$$

13. Two projectiles each of mass m are projected with same velocity v making an angle α and β from the same point in opposite directions. Find the change in their momentum at any instant.

(a) $2mv\sin(\alpha + \beta)$ (b) $2mv\sin\dfrac{\alpha+\beta}{2}$

(c) $2mv\cos(\alpha + \beta)$ (d) $2mv\cos\dfrac{(\alpha+\beta)}{2}$

Solution (d) $\Delta p_x = m[v\cos\alpha - (-v\cos\beta)]$

$\Delta p_y = m[v\sin\alpha - gt - (v\sin\beta - gt)]$

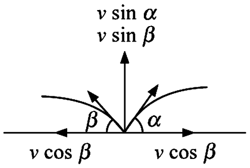


Fig. 4.17

$$\Delta p = \sqrt{\Delta p_x^2 + \Delta p_y^2}$$

$$= m\sqrt{(v\cos\alpha + v\cos\beta)^2 + (v\sin\alpha + v\sin\beta)^2}$$

$$= mv\sqrt{2 + 2[\cos\alpha\cos\beta - \sin\alpha\sin\beta]}$$

$$= mv\sqrt{2[1+\cos(\alpha+\beta)]} = 2mv\cos\frac{\alpha+\beta}{2}$$

14. An aircraft is flying at a height of 2800 m above the ground. The angle subtended by it in 10 s is 30°. Find the speed of the aircraft.

(a) 150 ms^{-1} (b) 100 ms^{-1}

(c) 140 ms^{-1} (d) 125 ms^{-1}

Solution (a) $AC = OC\tan 15$

$= 2800 \times 0.2679$

$= 750.12$

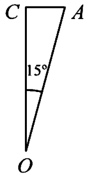


Fig. 4.18

$$v = \frac{750.12}{5} = 150.02\,\text{ms}^{-1}$$

15. A train is moving due east with a velocity 10 ms^{-1}. It takes 10 s to move along north with a speed 10 ms^{-1}. Find the average acceleration of the train.

(a) $10\sqrt{2}$ northeast (b) $10\sqrt{2}$ southeast

(c) $10\sqrt{2}$ northwest (d) $10\sqrt{2}$ southwest

Solution (c) $dv = 10j - 10i$ $a = \frac{dv}{dt} = \frac{10\sqrt{2}}{10} = \sqrt{2}$ ms^{-1}

$= 10\sqrt{2}$ ms^{-1}

$\alpha = \tan^{-1}(-1)$ northwest.

Problems for Practice

1. A projectile is projected with a linear momentum $\vec{p}$ making angle θ with the horizontal. The change in momentum of the projectile on return to the ground will be

(a) $2p\sin\theta$ (b) $2p\tan\theta$

(c) $2p$ (d) $2p\cos\theta$

2. Which of the following is the largest when the height attained by the projectile is the greatest?

(a) angle of projectile with the vertical

(b) range

(c) time of flight

(d) none

3. A projectile is thrown in to space so as to have the maximum possible horizontal range equal to 400 m. Taking the point of projection as the origin, the co-ordinate of the point where the velocity of the projectile is minimum are

(a) (400, 200) (b) (200, 200)
(c) (400, 100) (d) (200, 100)

4. A ball is thrown horizontally form the top of a tower. What happens to the horizontal component of its velocity?
(a) first increases and then decreases
(b) increases
(d) decreases
(d) remains the same

5. A bullet is fired horizontally with a velocity of $2000\,\text{ms}^{-1}$. If acceleration due to gravity is $10\,\text{ms}^{-2}$, in first second it will fall through a height of
(a) 200 m (b) 20 m
(c) 10 m (d) 5 m

6. A projectile is thrown at an angle of 40° with the horizontal and its range is R_1. Another projectile is thrown at an angle 40° with the vertical and its range is R_2. What is the relation between R_1 and R_2?
(a) $R_2 = 2R_1$ (b) $R_1 = 4R_2/5$
(c) $R_1 = 2R_2$ (d) $R_1 = R_2$

7. In case of a projectile, what is the angle between the instantaneous velocity and acceleration at the highest point?
(a) 180° (b) 90° (c) 45° (d) 0°

8. A projectile can have the same range R for two angles of projection. It t_1 and t_2 be the time of flight in the two cases then what is the product of the two times of flight?
(a) $t_1t_2 \propto 1/R$ (b) $t_1t_2 \propto 1/R^2$
(c) $t_1t_2 \propto R^2$ (d) $t_1t_2 \propto R$

9. A ball is thrown at an angle θ with the horizontal. Its horizontal range is equal to its maximum height. This is possible when $\tan\alpha$
(a) 4 (b) 2 (c) 1 (d) 0.5

10. A projectile is fired with a velocity of $10\,\text{ms}^{-1}$ at an angle of 60° with the horizontal. Its velocity at the highest point is
(a) $10\,\text{ms}^{-1}$ (b) $8.66\,\text{ms}^{-1}$
(c) $5\,\text{ms}^{-1}$ (d) zero

11. A heavy and lighter body are dropped from the top of a tower. Which will reach the ground first?
(a) heavier one
(b) lighter one
(c) both will reach simultaneously
(d) cannot be said

12. The range of a projectile when fired at 75° with the horizontal is 0.5 km. What will be its range when fired at 45°?
(a) 2 km (b) 1.5 km (c) 1 km (d) 0.5 km

13. A ball is thrown horizontally and another is just dropped from the top of tower. Which will reach the ground first?
(a) first ball
(b) second ball
(c) both will reach at same time
(d) depends upon the masses of the balls

14. A particle moves in a plane with uniform acceleration having direction different than that of the instantaneous velocity. What is the nature of the trajectory?
(a) circle (b) ellipse
(c) straight line (d) parabola

15. A ball thrown by one player reaches the other in 2 seconds. The maximum height attained by the ball above the point of projection will be about
(a) 10 m (b) 7.5 m (c) 5 m (d) 2.5 m

16. Four projectiles are fired with the same velocities at angle 25°, 40°, 55° and 70° with the horizontal. The range of projectile will be largest for the one projected at angle
(a) 70° (b) 40° (c) 55° (d) 25°

17. Two bullets are fired horizontally with different velocities from the same height. Which will reach the ground first?
(a) slower one
(b) faster one
(c) both will reach simultaneously
(d) cannot be said

18. A ball is projected upwards from the top of a tower with a velocity $50\,\text{ms}^{-1}$ making an angle 30° with the horizontal. The height of the tower is 70 m. After how many seconds from the instant of throwing will the ball reach the ground?
(a) 9 s (b) 7 s (c) 5 s (d) 2 s

19. Two projectiles are fired at different angles with the same magnitude of velocity such that they have the same range. At what angles they might have been projected?
(a) 35° and 75° (b) 25° and 65°
(c) 10° and 50° (d) none of the above

20. What should be the angle of projection, so that the horizontal range is equal to the maximum height?
(a) $\tan^{-1}4$ (b) $\tan^{-1}3$
(c) $\tan^{-1}2$ (d) $\tan^{-1}1$

21. A passenger travelling in a train moving with constant velocity drops a stone from the window. To the passenger the path of projectile will appear to be
(a) arc of a circle (b) elliptical
(c) straight line (d) parabola

22. The projectile goes farthest away from the earth, when the angle of projection is
(a) 180° (b) 90° (c) 45° (d) 0°

23. Two stones are projected with the same (magnitude) velocity but making different angles with the horizontal. Their ranges are equal. If the angle of projection of one is $\pi/3$, and its maximum height is y_1 then the maximum height of the other will be
(a) $y_1/2$ (b) $y_1/3$ (c) $3y_1$ (d) $2y_1$

24. A projectile has the maximum range 500 m. If the projectile is thrown up an inclined plane of 30° with the same (magnitude) velocity, the distance covered by it along the inclined plane will be
(a) 990 m (b) 670.5 m
(c) 333.3 m (d) 250 m

25. A projectile is projected with kinetic energy *KE*. If it has the maximum possible horizontal range, then its kinetic energy at the highest point will be
(a) *KE* (b) 0.75*KE*
(c) 0.5*KE* (d) 0.25*KE*

26. An object is projected with a kinetic energy k. Its range is r. It will have the minimum kinetic energy, after covering a horizontal distance equal to
(a) r (b) $0.75r$ (c) $0.5r$ (d) $0.25r$

27. A block of mass 200g is compressed against a spring of spring constant $2000\ \text{Nm}^{-1}$ by 5 cm. Where will the block land from point O?
(a) 2.5 m (b) 3.2 m (c) 4.4 m (d) 5.0 m

28. An object is projected upwards. Its acceleration at the highest point is

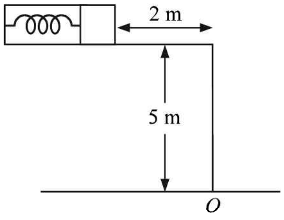


Fig. 4.19

(a) upwards
(b) downwards
(c) cannot be predicted
(d) zero

29. A particle travels according to the equaiton $x = at^3$, $y = bt^3$. The equation of the trajectory is
(a) $y = ax^2/b$ (b) $y = bx^2/a$
(c) $y = bx/a$ (d) $y = bx^3/a$

30. A particle travels according to the equation $x = a\,\mathrm{cost}\,\omega t$ and $y = b\cos(\omega t + \pi/4)$. The trajectory is
(a) rectangular parabola
(b) rectangular hyperbola
(c) oblique ellipse
(d) cycloid

31. A projectile is fired at an angle θ with the horizontal. Find the condition under which it lands perpendicular on an inclined plane inclination α as shown in Fig. 4.20.

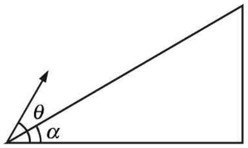


Fig. 4.20

(a) $\sin\alpha = \cos(\theta - \alpha)$
(b) $\cos\alpha = \sin(\theta - \alpha)$
(c) $2\tan\alpha = \cot(\theta - \alpha)$
(d) none of these

32. If a man wants to hit a target, he should aim his rifle in a direction
(a) lower than the target
(b) of the target
(c) higher than the target
(d) none of these

33. Two bullets are fired at angle θ and $(90 - \theta)$ to the horizontal with the same speed. The ratio of their times of flight is
(a) $1 : \tan\theta$ (b) $\tan^2\theta : 1$
(c) $1 : 1$ (d) $\tan\theta : 1$

34. Two particles are projected upwards with the same initial velocity v_0 in two different angles of projection such that their horizontal ranges are the same. The ratio of the heights of their highest points will be
(a) $1/\cos\theta$ (b) $\sin^2\theta$
(c) $\tan^2\theta$ (d) $\sin\theta$

35. A steam boat goes across a lake and comes back (i) on a quiet day when the water is still (ii) on a rough day when there is a uniform current so as to help the journey onwards and to impede the journey back wards. If the speed of the launch, on both days, was same, the time required for the complete journey on the rough day, as compared to that on the quiet day will be
(a) less (b) more
(c) same (d) cannot be predicted

36. The velocity of light emitted by a source S, observed by an observer O, who is at rest with respect to S is c, If the observer moves towards S with velocity V, then the velocity of light as observed will be
(a) $[1 - (c^2/v^2)]^{1/2}$ (b) c
(c) $c + v$ (d) $c - v$

37. The x and y co-ordinates of a particle at any time t are given by

$$x = 7t + 4t^2 \text{ and } y = 5t$$

where x and y are in m and t in s. The acceleration of the particle at 5 s is
(a) $40\,\mathrm{ms^{-2}}$ (b) $20\,\mathrm{ms^{-2}}$
(c) $8\,\mathrm{ms^{-2}}$ (d) zero

38. A particle is projected at angle of 45° with a velocity of $9.8\,\mathrm{ms^{-1}}$. The horizontal range will be
(a) $9.8/\sqrt{2}$ m (b) $9.8\sqrt{2}$ m
(c) 9.8 m (d) 4.9 m

39. The range of a projectile, when launched at an angle of 15° with the horizontal is 1.5 km. What is the range of the projectile when launched at an angle of 45° to the horizontal
(a) 0.75 km (b) 1.5 km
(c) 3 km (d) 6 km

40. A particle moves according to the equation $y = x - \dfrac{x^2}{2}$ then the maximum height reached is
(a) 1m (b) 1/3 m (c) 1/2 m (d) 1/4 m

41. Two projectiles, one fired from the surface of the earth with speed $5\,\mathrm{ms^{-1}}$ and the other fired from the surface of a planet with initial speed $3\,\mathrm{ms^{-1}}$ trace identical trajectories. Neglecting friction, the value of acceleration due to gravity on the planet is
(a) $8.5\,\mathrm{ms^{-2}}$ (b) $5.9\,\mathrm{ms^{-2}}$
(c) $3.5\,\mathrm{ms^{-2}}$ (d) $16.3\,\mathrm{ms^{-2}}$

42. A food packet is released from a helicopter flying at a height of 1 km with a velocity 80 $\mathrm{ms^{-1}}$ (when on a flood relief mission), the distance at which the packet falls from the point of release is nearly
(a) 1500 m (b) $800\sqrt{2}$ m
(c) 1000 m (d) none of these

43. A rocket is projected vertically upwards with resultant acceleration of $10\,\mathrm{ms^{-2}}$. The fuel in the rocket is exhausted in one minute, but the rocket continues to move up. After the fuel has exhausted, how long the rocket will take to reach the maximum height
(a) 60 s (b) 30 s (c) 20 s (d) 10 s

44. Two blocks are attached to the two ends of a string passing over a smooth pulley as shown in the Fig. 4.21. The acceleration of the block will be in ($\mathrm{ms^{-2}}$)

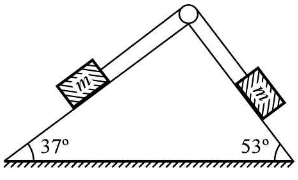


Fig. 4.21

(a) 0.33 (b) 0.66 (c) 1 (d) 1.32

45. A particle is projected upwards. The time corresponding to height h while ascending and while descending are t_1 and t_2 respectively. The velocity of projection will be

(a) $g(t_1 + t_2)$ (b) $\frac{g(t_1+t_2)}{2}$

(c) gt_1 (d) gt_2

46. A frictionless wire is fixed between A and B inside a sphere of radius R. A small ball slips along the wire. The time taken by the ball to slip from A to B will be

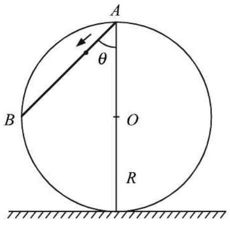


Fig. 4.22

(a) $2\sqrt{R/g}$

(b) $gR/\sqrt{g\cos\theta}$

(c) $\frac{2\sqrt{gR}}{g\cos\theta}$

(d) $\frac{2\sqrt{gR\cos\theta}}{g}$

47. The time of flight of a projectile is 10 seconds and its range is 500 m. The maximum height reached by it will be ($g = 10\,ms^2$)

(a) 125 m (b) 82 m (c) 50 m (d) 25 m

48. A toy gun consists of a spring and a rubber dart of mass 10 g. When compressed by 4 cm and released, it projects the dart to a height of 2 m. If compressed by 6 cm, the height achieved will be

(a) 6 m (b) 3 m (c) 4.5 m (d) 4 m

49. A motorcylist is moving 18 kmh^{-1} due east. He turns to north and continues to move with the same speed. The time taken is 5 seconds. The acceleration (average) during turning is

(a) $\sqrt{2}\ ms^{-2}$ northeast

(b) $\sqrt{2}\ ms^{-2}$ southeast

(c) 0

(d) $\sqrt{2}\ ms^{-2}$ northwest

50. The velocity of a swimmer with respect to the direction of river flow is u. In what direction should the swimmer swim in order to reach a point just in front of the starting point. (v is speed of river)

(a) $\sin^{-1}\left(\frac{u}{\sqrt{u^2+v^2}}\right)$ (b) $\sin^{-1}\left(\frac{v}{\sqrt{u^2+v^2}}\right)$

(c) $\sin^{-1}\left(\frac{v}{u}\right)$ (d) $\sin^{-1}\left(\frac{u}{v}\right)$

51. The coordinates of a moving particle at any instant of time t are $x = at$ and $y = bt^2$. The trajectory of the particle is

(a) $y = \frac{b^2x}{a^2}$ (b) $y = \frac{b^2x^2}{a^2}$

(c) $y^2 = \frac{b^2x}{a^2}$ (d) $y = \frac{bx^2}{a^2}$

52. The angle of projection, for which the horizontal range and the maximum height of a projectile are same, will be

(a) 180° (b) 90° (c) 67° (d) 76°

53. A particle of a mass m is thrown with velocity u at an angle of 45° with the horizontal. The angular momentum of the particle about the point of projection when it is at the maximum height h will be

(a) $m\sqrt{2gh^3}$ (b) $m\sqrt{gh^3}$

(c) $2\sqrt{2gh}$ (d) zero

54. The horizontal range of a projectile is 4 times its maximum height. Its angle of projection will be

(a) 90° (b) 60° (c) 45° (d) 30°

55. Two bodies P and Q, attached to an inextensible string passing over two smooth stationary pulleys A and B, move vertically downwards with a uniform speed v. The weight W will move upwards with a velocity

(a) $2v\cos\theta$ (b) $\frac{2v}{\cos\theta}$

(c) $v\cos\theta$ (d) $\frac{v}{\cos\theta}$

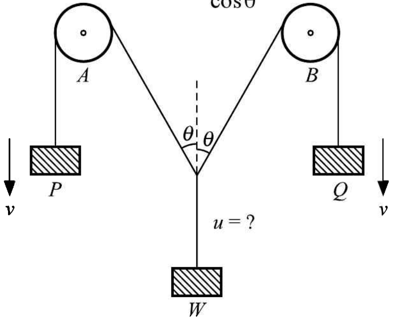


Fig. 4.23

56. A particle is projected with a velocity u making an angle θ with the horizontal such that the trajectory just grazes the vertices of the triangle then

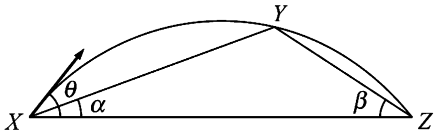


Fig. 4.24

(a) $\tan\theta = \cot\alpha + \cot\beta$
(b) $\cot\theta = \cot\alpha + \cot\beta$
(c) $\sin\theta = \sin\alpha + \sin\beta$
(d) $\tan\theta = \tan\alpha + \tan\beta$

57. The velocity at the maximum height of a projectile is half of its initial velocity u. Its range on the horizontal plane is

(a) $\frac{2u^2}{3g}$ (b) $\frac{u^2}{3g}$ (c) $\frac{3u^2}{g}$ (d) $\frac{\sqrt{3}u^2}{2g}$

58. Two parallel high buildings are x distance apart. A boy throws a ball horizontally with 10 ms^{-1} which falls 10 m below in the opposite building. Then the x is

(a) 20 m (b) 10 m

(c) $10\sqrt{2}$ m (d) $10\sqrt{3}$ m

59. A projectile is fired with a velocity v making an angle θ with the horizontal, from a distance 23 m from a 10 m high wall, such that it just clears the wall and falls 36 m away from the wall on the other side of the wall. Find v and θ.

(a) $49.1\,\text{ms}^{-1}, \tan^{-1}\dfrac{295}{414}$

(b) $59.1\,\text{ms}^{-1}, \tan^{-1}\dfrac{295}{414}$

(c) $39.1\ \text{ms}^{-1}, \tan^{-1}\dfrac{414}{295}$

(d) $29.1\ \text{ms}^{-1}, \tan^{-1}\dfrac{414}{295}$

60. A ball is projected upwards. Its velocity at the highest point is

(a) downwards (b) upwards

(c) zero (d) horizontal

61. The time elapsed is plotted along the x-axis and the acceleration is plotted along the y-axis. The area between the graph and the x-axis gives.

(a) difference in acceleration

(b) average velocity

(c) distance covered

(d) difference in velocities

62. A particle is projected with velocity u making an angle θ with the vertical. The average velocity between the points of half the maximum height is

(a) $u/2$ (b) $u\cos\theta$

(c) $u\tan\theta$ (d) $u\sin\theta$

63. The maximum height attained by a projectile is increased by 5%. Keeping the angle of projection constant, what is the percentage increase in the horizontal range?

(a) 20% (b) 15% (c) 10% (d) 5%

64. A projectile is fired with speed u making angle θ with the horizontal. Its potential energy at the highest point is

(a) $1/2\,mu^2\sin^2\theta$ (b) $1/2\ mu^2\sin^2 2\theta$

(c) $1/2\ mu^2\cos^2\theta$ (d) $1/2\ mu^2$

ANSWERS

1. (a)	**2.** (c)	**3.** (d)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (c)
11. (c)	**12.** (c)	**13.** (c)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (a)
21. (c)	**22.** (b)	**23.** (b)	**24.** (c)	**25.** (c)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (c)	**32.** (c)	**33.** (d)	**34.** (c)	**35.** (b)	**36.** (b)	**37.** (c)	**38.** (c)	**39.** (c)	**40.** (c)
41. (c)	**42.** (b)	**43.** (a)	**44.** (c)	**45.** (b)	**46.** (a)	**47.** (a)	**48.** (c)	**49.** (d)	**50.** (c)
51. (d)	**52.** (d)	**53.** (a)	**54.** (c)	**55.** (d)	**56.** (d)	**57.** (d)	**58.** (c)	**59.** (a)	**60.** (d)
61. (d)	**62.** (d)	**63.** (d)	**64.** (a)						

5

Laws of Motion

BRIEF REVIEW OF THE CONCEPTS

Force Force is a pull or push which generates or tends to generate motion in a body at rest; stops or tends to stop a body in motion; increases and decreases the magnitude of velocity of the moving body or changes the direction of motion of the moving body; changes or tends to change the shape of the body.

Newton's first law of motion A body at rest remains in its state of rest and a body in uniform motion remains in its state of uniform motion unless it is compelled by some **external force** to change its state.

This inherent property with which the body cannot change its state of rest or of uniform motion unless acted upon by an external force, is called **inertia**. Hence Newton's first Law of motion may also be called Law of inertia.

Mass In newtonion mechanics mass is considered to be a measure of inertia of a body and is considered independent of its velocity. It is a scalar quantity and measured in kg (SI System)

Momentum (p) The total quantity of motion contained in a body is called momentum. It is a vector quantity and is measured in kg ms^{-1}.

$$\vec{p} = m\vec{v}\ .$$

If two bodies of different masses have same momentum, the lighter has more velocity (or more KE).

Newton's second law of motion The time rate of change of momentum is directly proportional to external force applied on it and the change in momentum occurs in the direction of force.

$$\vec{F}\ \alpha\ \frac{d\vec{p}}{dt} \quad \text{or} \quad \vec{F} = \frac{d\vec{p}}{dt}$$

where k is a constant of proportionality. In cgs or in SI system k = 1. Thus the equation of motion becomes

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{md\vec{v}}{dt} = m\,\vec{a}$$

In scalar form $F = ma$ is called equation of motion.

Force is a vector quantity. Its unit is Newton (N). kilogram weight (kg wt) or kilogram force (kgf) is gravitational unit of force.

$$1\,\text{kg}\,wt = 1\,\text{kgf} = 9.8\,\text{N}$$

Accelerated motion may be produced (a) due to change in its speed (b) due to change in its direction.

Impulse Product of force and time for which it acts is called impulse.

$F = dp/dt$ or $F.dt = dp$ (change in momentum)

or $F_{av}.t = \Delta p$ and it is called impulse momentum theorem.

Basic forces in nature There are four basic forces in nature. All other forces are manifestation of these forces. The basic forces are Gravitational force, Electromagnetic force, Strong force and Weak nuclear force.

Gravitational force : Electromagnetic force : Weak force : Strong force : : 1 : 10^{38} : 10^{26} : 10^{40}

In situations like rocket v may be constant (escape velocity) while m changes.

Therefore $\vec{F} = -\text{grad}\ u = -\frac{d(u)}{dr}$ where u is P.E (potential energy).

Newton's third law of motion To every action there is an equal and opposite reaction.

$$F_{AB} = -F_{BA}$$

that is, forces in nature occur in pair. Single isolated force is not possible. Action and reaction act on different bodies

**Note that Newton's third law may not be valid in certain cases in Electrostatics and in springs.

Law of conservation of linear momentum If no external force acts, then the total momentum of the system is conserved

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{or} \quad \vec{p} = \text{constant}$$

Types of equilibrium

Translatory equilibrium When several forces act on a body such that the resultant force is zero that is, $\Sigma F = 0$, the body is said to be in translatory equilibrium. It may be mentioned if $\Sigma F = 0$.

Then $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$

$F = 0$ implies $a = 0$ or $\frac{dv}{dt} = 0$, that is, either $v = 0$ or v is constant.

It means the body is in the state of rest (static equilibrium) or in uniform motion (dynamic equilibrium).

If the force is conservative $F = 0 = \frac{du}{dr}$, the PE (Potential Energy) is maximum, minimum or constant.

Stable equilibrium If on slight displacement from equilibrium position, body has the tendency to regain its original position, it is said to be in stable equilibrium. In case of stable equilibrium, PE is minimum $\left(\frac{d^2u}{dr^2} = +\text{ve}\right)$, so the centre of gravity is lowest.

Unstable equilibrium If on slight displacement from equilibrium position the body moves in the direction of displacement, the equilibrium is known to be unstable. PE in this case is maximum $\left(\frac{d^2u}{dr^2} = -\text{ve}\right)$, so centre of gravity is highest.

Neutral equilibrium If the body on slight displacement has no tendency to come back or to move in the direction of displacement, the equilibrium is known to be neutral and centre of gravity remains at constant height.

Figure illustrates the points of stable, unstable and neutral equilibriums on a PE versus displacement curve.

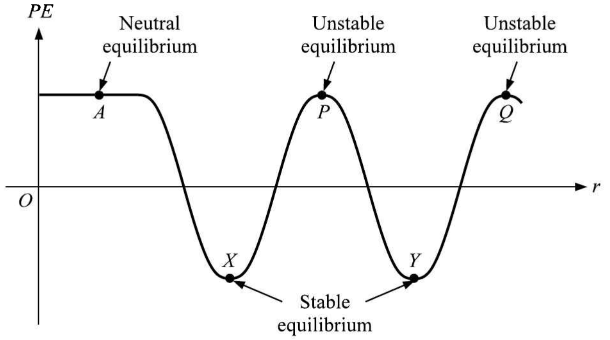


Fig. 5.1

Strings String is considered to be massless unless stated and hence tension remains constant throughout the string.

String is assumed to be inextensible (perfectly elastic) unless stated. That is why the magnitude of acceleration of any number of masses connected to it is always same.

If the pulley is massless and smooth, string is also massless hence, tension on two sides is equal (Fig. 5.2).

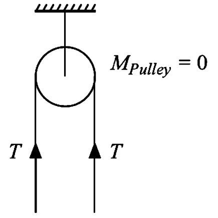


Fig. 5.2

If string changes tension changes (Fig. 5.3).

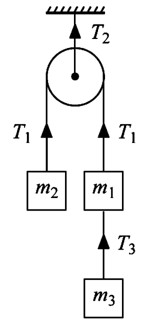


Fig. 5.3

If forces are equal and opposite on a massless string tension is equal to either of the force.

The maximum tension which a string can bear is called breaking strength.

Springs Springs are assumed to be massless unless stated. Restoring force is same every where

$$F = -kx$$

springs can be stretched or compressed stretch or compression is taken positive.

Restoring force is linear as is clear from $F = -kx$. where k is called spring constant

$k \propto 1/l$ (k also depends upon radius, length and material used)

In series $\frac{1}{k_{\text{effective}}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$

In parallel $k_{\text{effective}} = k_1 + k_2 + \ldots$

If masses m_1 and m_2 connected by a spring as shown in Fig. 5.4 is oscillating or both masses move take reduced mass μ. $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.

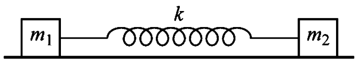


Fig. 5.4

If the spring has mass m_s, then $\frac{m_s}{3}$ is used to produce extension.

If the string has mass, tension at each point is different and is calculated as

$\lambda = m/l$ is mass per unit length.

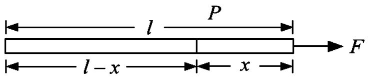


Fig. 5.5

We have to find tension at P. Mass of length $(l - x)$ of the string $= \lambda(l - x) = \frac{m(l-x)}{l}$

Tension at $P = \frac{F}{m}\left[\frac{m(l-x)}{l}\right] = \frac{F(l-x)}{l}$

Pseudo forces The imaginary force/s added while dealing with non-inertial or accelerated frame of reference problem, so that Newton's laws may become valid are called pseudo forces or inertial forces. If the frame of reference is moving with an acceleration a_0 then on a particle of mass m its value is ma_0. In the force equation a force $-ma_0$ is to be added.

Friction If we try to slide a body over a surface the motion is resisted by a bonding between the body and the surface. This resistance is represented by a single force and is called friction. The force of friction is parallel to the surface and opposite to the direction of intended motion.

If a body is at rest and it is not being pulled, force of friction is zero.

If a pulling force is applied and the body does not move, friction still acts and is called static friction. The maximum value of static friction is called limiting friction, that is, static friction is self adjusting with an upper limit called limiting friction.

Limiting friction $F_{fL} = \mu_s N$ where N is normal reaction

$\mu_s = \tan\theta$ where θ is angle of limiting friction.

$$\mu_s > \mu_k > \mu_R$$

where μ_s stands for coefficient of static friction, μ_k stands for coefficient of kinetic friction and μ_R stands for coefficient of rolling friction.

Friction is indepen-dent of the surface area in contact but however depends upon the nature of material of the surfaces in contact (that is, force of adhesion orcohesion), their rough-ness or smoothness, inclination. Normally, with increase in smooth-ness, friction decreases. However, if the surfaces are made too smooth by polishing, the bonding force of cohesion or adhesion increases resulting in cold welding.

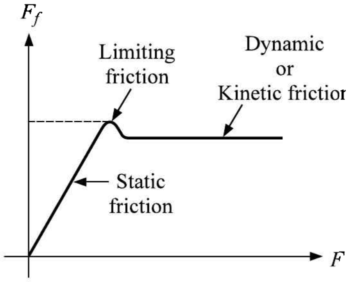


Fig. 5.6

Friction may not oppose the motion always. In certain cases it causes motion. In cycling, the rear wheel moves by the force communicated to it by pedalling while front wheel moves by itself. Thus while pedalling the force of friction acts on it in forward direction. Front wheel moves by itself experiencing friction in backward direction. If pedalling is stopped both the wheels move by themselves and so experience a force in backward direction.

In practice $0 < \mu < 1$, but $\mu > 1$ is observed. Coefficient of friction between Cu-Cu surfaces is 1.6, between glass and glass is 1.

Friction is a non conservative force.

If force is applied and still the body is at rest then the force of friction is equal to force applied.

SHORT-CUTS AND POINTS TO NOTE

(i) Tension is a reaction force produced in a string.

(ii) In a massless string (if not passing over a pulley) tension is equal at all points.

(iii) If pulley is massless, smooth and string is massless then tension on two sides is equal.

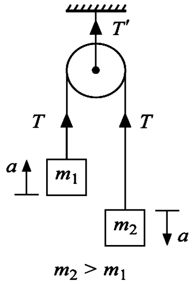


Fig. 5.7

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2}$$

$$T = \frac{2\,m_1 m_2 g}{(m_1 + m_2)}\ ;\ T' = 2T$$

(iv) If the string changes, tension will change. Assume pulley is smooth and massless string is also massless.

In Fig. 5.8 $T = \dfrac{2(m_1 + m_3)m_2 g}{(m_1 + m_2 + m_3)}$;

$$a = \frac{[(m_1 + m_3) - m_2]\,g}{m_1 + m_2 + m_3}$$

$T' = m_3(g - a)$; $T'' = 2T$

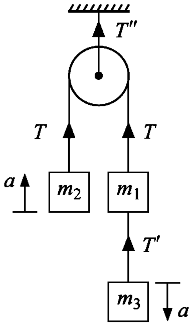


Fig. 5.8

(v) If the pulley system shown in Fig. 5.7 is moved up with an acceleration $a' \le g$ then

$$a = \frac{(m_2 - m_1)(g + a')}{m_1 + m_2} \text{ and } T = \frac{2m_1 m_2(g + a)}{m_1 + m_2}$$

(vi) If the pulley system shown in Fig. 5.8 is moved up with an acceleration $a' \le g$ then

$$a = \frac{[(m_1 + m_3) - m_2](g + a')}{m_1 + m_2 + m_3};$$

$$T = \frac{2(m_1 + m_3)m_2(g + a')}{m_1 + m_2 + m_3}$$

$$T' = m_3(g + a' - a)$$

(vii) If $F > 2T$ in Fig. 5.9 then a' with which the pulley system moves upwards is

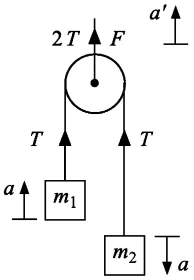


Fig. 5.9

$$a' = \frac{F - 2T}{m_1 + m_2}$$

$$a = \frac{(m_2 - m_1)(g + a')}{m_1 + m_2}$$

$$T = \frac{m_1 m_2 (g + a')}{m_1 + m_2}$$

If $F \le 2T$ then conditions mentioned in Point (iii) will be applicable that is,

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2} \text{ and } T = \frac{2m_1 m_2 g}{m_1 + m_2}$$

(viii) The springs act in parallel if the displacement in all the springs is equal. For example, in the Fig. 5.10 shows springs are in $||^{l}$ (Parallel) as displacement is equal.

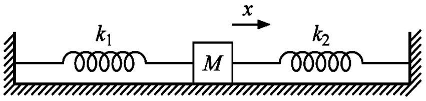


Fig. 5.10

(ix) The springs are in series if the displacement in the two springs are unequal as illustrated in Fig. 5.11 The displacement in spring of spring constant k_1 is x_1 and the displacement in spring of spring constant k_2 is x_2. Total displacement

$$x = x_1 + x_2$$

(x) If spring is cut then spring constant $k \propto \dfrac{1}{l}$

(xi) In case of Fig. 5.12, if the block comes down by y the spring stretches by $y/2$

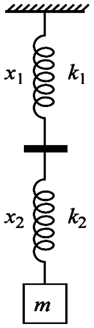


Fig. 5.11

and $F' = 2T = 2F$

or $F = \frac{F'}{2} = \frac{k(y/2)}{2} = \frac{ky}{4}$

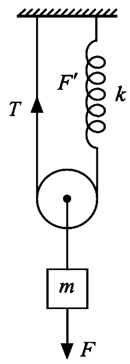


Fig. 5.12

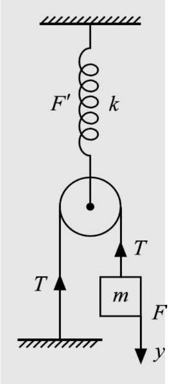


Fig. 5.13

In Fig 5.12

$$T = F'$$
$$F = 2F'$$

If mass m shifts down by 'y' the spring will stretch by $2y$

Hence $F = 2(k2y) = 4k(y)$.

(xii) In case of Fig. 5.14, if F is the force applied on the pulley and m is the mass then $F = 2T$

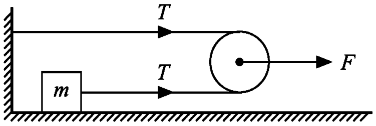


Fig. 5.14

$$T = ma_{\text{block}} \therefore a_{\text{block}} = \frac{F}{2m}$$

If the block moves a distance x, pulley moves $\frac{x}{2}$, therefore,

acceleration of the pulley $= a_{\text{pulley}} = a_{\text{block}}/2$

(xiii) Since force is a vector, apply vector laws whenever there are more than one forces.

(xiv) Draw free body diagram before you solve the problems.

(xv) If Force is applied on the body and body does not move, then, friction = force applied and not μN where N is normal reaction.

(xvi) Friction $= \mu_s N$ when body just begins to move and $\mu_k N$ when it is in motion.

(xvii) Barring few exceptions $\mu_s < 1$ and hence $\mu_k < 1$.

(xviii) In conservative forces workdone depends upon the initial and final position. It is independent of the path followed. Net workdone in a closed loop equals zero.

(xix) Frictional force is non conservative

(xx) If there is no friction along the incline then $a = g \sin\theta$

(xxi) If there is friction then $a = g \sin\theta - \mu g \cos\theta$ while moving down and $a = g \sin\theta - \mu g \cos\theta$ while moving up.

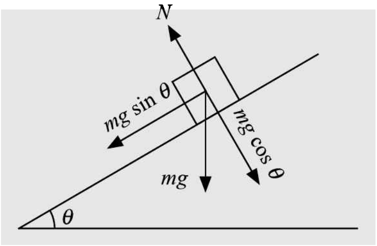


Fig. 5.15

(xxii) Decceleration is $a = \mu g$ on horizontal plane due to friction.

(xxiii) If a lift moves up with an acceleration 'a' then effective acceleration is $(g + a)$ and if it moves down with an acceleration 'a' then effective acceleration is $(g - a)$.

⚠ CAUTION

(i) Applying Newton's laws without caring about inertial/noninertial frames.

- ■ In noninertial frame of reference first apply pseudo vectors to make inertial frame of reference and then apply Newton's laws.

(ii) Considering action and reaction act on different bodies always.

- ■ In case of elastic bodies and springs, the action and reaction act on same body, that is, in restoring force and deforming forces action and reaction are on same bodies. These are called internal forces.

(iii) Considering Newton's third law is always valid.

- ■ In electrostatics in certain cases Newton's third law is not valid.

(iv) Considering that force of friction is always opposite to the motion

- ■ If force of friction or frictional torque causes the motion then it is in the same direction of motion.

(v) Considering that if friction is present, for a massless pulley then tensions on two sides are unequal.

- ■ If pulley is massless, tension on two sides of the pulley in a string passing over is equal.

(vi) Considering in cases as shown in Fig. 5.16 the acceleration of pulley and blocks to be equal.

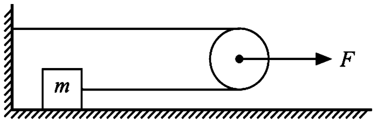


Fig. 5.16

- ■ Acceleration of pulley is half the acceleration of block

(vii) Considering force constant of a spring does not change when cut

- ■ In such cases force constant $k \propto \frac{1}{l}$

(viii) Considering friction is always μmg or μN

- If body does not move on application of force then friction is equal to force applied. If body comes in motion limiting force is $\mu_s N$ and during proper motion it is $\mu_k N$

(ix) Considering in equilibrium body essentially is at rest.

- In only static equilibrium body is at rest but in dynamic equilibrium body is in uniform motion in a straight line.

(x) If a rope is pulled by equal forces on two sides then considering that tension in it is zero.

- Tension is equal to either of the force applied.

(xi) Considering that impulse always provides acceleration.

- Impulse may provide only velocity if it is sharp.

(xii) Considering rough surfaces have more coefficient of friction.

- Coefficient of friction between glass and glass is 1.0 and between copper and copper is 1.6.

Solved Problems

1. A block of mass 2 kg is resting on a frictionless plane. It is struck by a jet releasing water at 1 kgs^{-1} and at a speed 5 ms^{-1}. The initial acceleration of the block is

(a) 1 ms^{-2} (b) 2.5 ms^{-2}
(c) 5 ms^{-2} (d) zero

Solution (b) $F = \frac{vdm}{dt} = 5 \times 1 = 5\,\text{N}$

$a = F/m = 5/2 = 2.5\,\text{ms}^{-2}$

2. Calculate the volume of a balloon filled with hydrogen gas, which will be sufficient to lift a load of 25 kg in air. Given that densities of air and hydrogen are $12.9 \times 10^{-4}\,\text{gcm}^{-3}$ and $9 \times 10^{-5}\,\text{gcm}^{-3}$

(a) 25 m^3 (b) 24 m^3
(c) 20.83 m^3 (d) none of these

Solution (c) Thrust ≥ Weight of hydrogen + Load

$$V\rho_{\text{air}}g \geq V\rho_H g + mg \quad \text{or} \quad m = V(\rho_{\text{air}} - \rho_H)$$

or $$V = \frac{m}{\rho_{\text{air}} - \rho_H} = \frac{25\times10^3}{12.9\times10^{-4} - 9\times10^{-5}}$$

$$= 2.083 \times 10^7\text{cc or } 20.83\,\text{m}^3$$

3. Water from a hose pipe of area of cross-section A is emerging out with a velocity v and strikes a block of mass m. Find the acceleration produced in the block. [density of water is ρ]

(a) $\frac{\rho Av}{m}$ (b) $\frac{\rho Av^2}{m}$

(c) ρAv (d) $\frac{dv}{dt}$

Solution (b) Let ρAv be mass of water coming out, that is, $\frac{dm}{dt} = \rho Av$ then $F = \frac{vdm}{dt} = \rho Av^2$ and $a = \frac{\rho Av^2}{m}$.

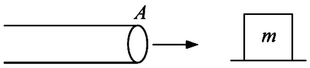


Fig. 5.17

4. Which of the following graph depicts spring constant k versus length l of the spring correctly?

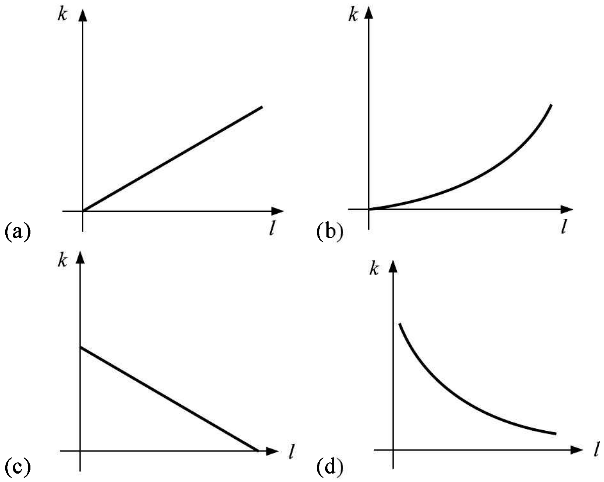


Fig. 5.18

Solution (d) Since $k \propto 1/l$.

5. Two masses m_1 and m_2 ($m_2 > m_1$) are connected at two ends of a light string (inextensible) passing over a smooth pulley. Find the tension in the strings and acceleration of the pulleys. See Fig. 5.19

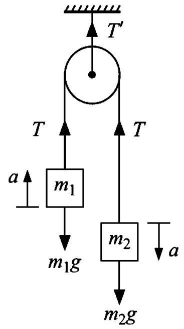


Fig. 5.19

(a) $\frac{2m_1m_2g}{m_1+m_2}, \frac{(m_2-m_1)g}{m_1+m_2}$

(b) $\frac{m_1m_2g}{m_1+m_2}, \frac{m_2-m_1g}{2(m_2+m_1)}$

(c) $(m_1 + m_2)g, \frac{g}{2}$

Solution (a) Because $m_2 > m_1$, therefore, m_2 moves downward and m_1 upwards with an acceleration a.

$$m_2g - T = m_2a \quad \text{...(i)}$$

$$T - m_1g = m_1a \quad \text{...(ii)}$$

Adding (i) and (ii) $a = \frac{(m_2 - m_1)g}{m_2 + m_1}$

and $T = m_1g + m_1\left(\frac{m_2 - m_1}{m_2 + m_1}\right)g = \frac{2m_1m_2}{(m_1 + m_2)}g$

6. Three blocks A, B and C tied with inextensible strings are placed on a smooth horizontal table. If a force F is applied as shown in Fig 5.20 then find tensions T_1 and T_2.

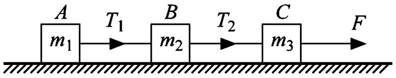


Fig. 5.20

(a) $\frac{m_1F}{m_1 + m_2 + m_3}, \frac{(m_1 + m_2)F}{m_1 + m_2 + m_3}$

(b) $\frac{m_1F}{m_1 + m_2 + m_3}, \frac{m_2F}{m_1 + m_2 + m_3}$

(c) $\frac{m_1F}{m_1 + m_2 + m_3}, \frac{(m_2 + m_3)F}{m_1 + m_2 + m_3}$

(d) none

Solution (a) $a = \frac{F}{m_1 + m_2 + m_3}$

$$T_1 = m_1a = \frac{m_1F}{m_1 + m_2 + m_3}$$

$$T_2 = (m_1 + m_2)\,a = \frac{(m_1 + m_2)F}{m_1 + m_2 + m_3}$$

7. A 2 kg plate is kept suspended in air by allowing 10 marbles hitting per second with a speed v from the downward direction. If the mass of each marble is 20 g then determine v.

(a) $20\,\text{ms}^{-1}$ (b) $100\,\text{ms}^{-1}$
(c) $40\,\text{ms}^{-1}$ (d) $50\,\text{ms}^{-1}$

Solution (d) Look at Fig. 5.21 change in momentum of each marble

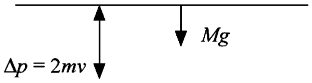


Fig. 5.21

$$\Delta p = 2mv = 4 \times 10^{-2}v$$

$$F = Mg = \frac{\Delta p}{\Delta t} \Rightarrow \frac{4\times10^{-2}v}{\frac{1}{10}} = 2 \times 10 \quad v = 50\,\text{ms}^{-1}$$

8. A ball of mass 100 g falls from a height 100 m and rebounds to 25 m. Find the average force acting on the floor if time of collision is 1/10 s.

(a) $2.1\sqrt{10}$ N (b) $21\sqrt{10}$ N
(c) 10/2.1 (d) 21 N

Solution (b) Impulse $= \Delta p = m[v_2 - (-v_1)]$

$$= 0.1\,[\sqrt{2gh_2} + \sqrt{2gh_1}]$$

$$= 0.1\left[\sqrt{9.8\times50} + \sqrt{9.8\times200}\right]$$

$$= 0.1\left[7\sqrt{10} + 14\sqrt{10}\right]$$

$$= 0.1 \times 21\sqrt{10} = 2.1\sqrt{10}\ \text{Ns}$$

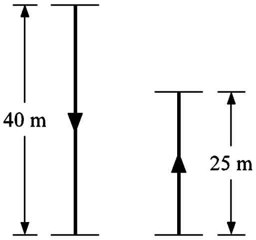


Fig. 5.22

Average force $F_{av} = \frac{\Delta p}{\Delta t} = \frac{2.1\sqrt{10}}{\frac{1}{10}} = 21\sqrt{10}$ N

9. A balloon with mass M is descending down with an acceleration $a\,(a < g)$. What mass m be detatched from it, so that it starts moving up with an acceleration a.

(a) $2Ma/g$ (b) $2Mg/a$
(c) $2Ma/(g + a)$ (d) $Ma/(g + a)$

Solution (c) **Case (i)** If force acting downward is F then $F = Mg - Ma$. **Case (ii)** When mass m is removed the weight acting downwards becomes $(M - m)g$ and force equation is

$$F - (M - m)g = (M - m)a$$

$$Mg - Ma - (M - m)g = (M - m)a$$

$$mg = (2M - m)a$$

$$m(g + a) = 2Ma \quad \text{or} \quad m = \frac{2Ma}{g + a}$$

10. A pendulum is suspended from the roof and makes an angle of 3° with the vertical. Find the acceleration of the train.

(a) $5\,\text{ms}^{-1}$ (b) $5.23\,\text{ms}^{-1}$
(c) $2.53\,\text{ms}^{-1}$ (d) $0.523\,\text{ms}^{-1}$

Solution (d) In Fig. 5.23 $ma = T\sin4°$

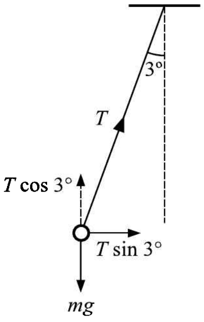


Fig. 5.23

$$mg = T\cos3°$$

$$\frac{a}{g} = \tan3° \quad a = g\tan3°$$

($\because$ θ is small $\tan\theta = \theta$ in radians)

$$= g \times \frac{3}{180} \times \pi = 10 \times \frac{3\times3.14}{180}$$

$$= 3.14/6 = 0.523\,\text{ms}^{-2}$$

11. A body of 2 kg has an initial speed $5\,\text{ms}^{-1}$. A force acts on it for 4 seconds in the direction of motion. The force time graph is shown in Fig 5.24. Find the impulse of the force and the final speed of the body.

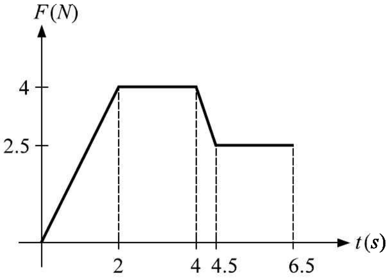


Fig. 5.24

(a) $9.25\,\text{ms}^{-1}$ (b) $5\,\text{ms}^{-1}$

(c) $14.25\,\text{ms}^{-1}$ (d) $4.25\,\text{ms}^{-1}$

Solution (c) Impulse = area under $(F - t)$ graph = 18.51 Ns

$$\Delta v = \frac{\text{Impulse}}{m} = \frac{18.51}{2} = 9.26\,\text{ms}^{-1}$$

$$v_f = v_i + \Delta v = 5 + 9.25 = 14.26\,\text{ms}^{-1}$$

12. A uniform rope of length L, resting on a frictionless horizontal surface is pulled at one end by a force F. Find the tension in the rope at a distance l from the end where force F is applied [Fig 5.25].

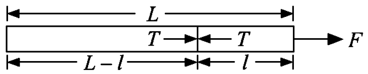


Fig. 5.25

(a) $\frac{Fl}{L}$ (b) $\frac{F(L-l)}{L}$

(c) $\frac{Fl}{L+l}$ (d) $\frac{Fl}{L-l}$

Solution (b) Let $\frac{M}{L}$ be mass per unit length. Then mass of length $(L - l]$ is $M' = M\frac{(L-l)}{L}$.

$$T = M' \times \frac{F}{M} = \frac{M\,(L-l)}{L} \times \frac{F}{M} = \frac{(L-l)}{L}F$$

13. A uniform chain of length l and mass m lies on a smooth horizontal table with its length perpendicular to the edge of the table and small part overhanging. The chain starts sliding down from rest due to the weight of hanging part. Find the acceleration and velocity of the chain when length of the hanging portion is x.

(a) $\frac{gx}{l}, \sqrt{\frac{gx^2}{l}}$ (b) $\frac{gx}{l}, \sqrt{gx}$

(c) $\frac{gx}{l}, \sqrt{gl}$ (d) $\frac{gx}{l}, \sqrt{g(l-x)}$

Solution (a) When the length x of the chain overhangs, the force is proportional to its length.

According to Newton's law $\frac{m}{l}xg = m\frac{d^2x}{dt^2}$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{x}{l}g \quad \text{or} \quad a = \frac{x}{l}g$$

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{gx}{l}$$

$$\frac{v^2}{2} = \frac{g}{l}\frac{x^2}{2} \quad \text{or} \quad v = \sqrt{\frac{gx^2}{l}}$$

14. A body of mass m is suspended by two strings making angles α and β with the horizontal. Find the tension in the strings. See Fig. 5.26

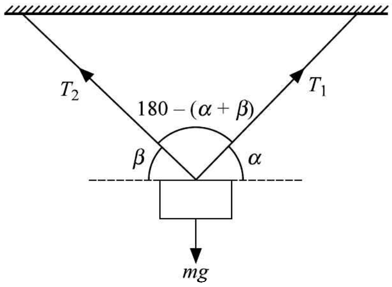


Fig. 5.26

(a) $T_1 = \frac{mg\cos\beta}{\sin(\alpha+\beta)} = T_2$

(b) $T_1 = T_2 = \frac{mg\sin\beta}{\sin(\alpha+\beta)}$

(c) $T_1 = \frac{mg\cos\beta}{\sin(\alpha+\beta)}, T_2 = \frac{mg\cos\alpha}{\sin\alpha+\beta}$

(d) none of these

Solution Applying Lami's theorem

$$\frac{T_1}{\sin(90+\beta)} = \frac{T_2}{\sin(90+\alpha)} = \frac{mg}{\sin(180-(\alpha+\beta)]}$$

or $T_1 = \frac{mg\cos\beta}{\sin(\alpha+\beta)}$ $T_2 = \frac{mg\cos\alpha}{\sin(\alpha+\beta)}$

15. The springs shown in Fig. 5.27 are unstretched. If the block is displaced by a small distance x and then released find the initial acceleration.

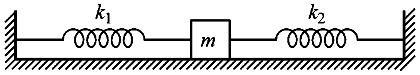


Fig. 5.27

(a) $(k_1 - k_2)\frac{x}{m}$ (b) $(k_1 + k_2)\frac{x}{m}$

(c) $(k_2 - k_1)\frac{x}{m}$ (d) none of these

Solution (b) $F = (k_1 + k_2)x = ma$ $a = (k_1 + k_2)\frac{x}{m}$

16. A small block B is placed on block A of mass 5 kg and length 20 cm. If initially the block is placed at the right end of block A. A constant horizontal force of 10 N is applied on the block A. All the surfaces are assumed frictionless. Find the time in which B separates from A.

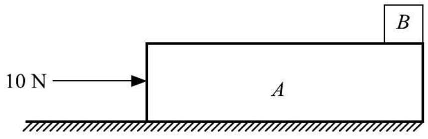


Fig. 5.28

(a) 0.2 s
(b) 0.32 s
(c) 0.39 s
(d) 0.45 s

Solution (d) $a = \frac{10}{2} = 2\,\text{ms}^{-2}$

$$s = ut + \frac{1}{2}at^2$$

$$0.2 = 0 + \frac{1}{2} \times 2\,t^2 \quad \text{or} \quad t^2 = 0.2$$

$$t = 0.45\,\text{s}$$

17. A block A is able to slide on the frictionless incline of angle θ and length l, kept inside an elevator going up with uniform velocity v. Find the time taken by the block to slide down the length of the incline if released from rest. What would be the time taken if the elevator is accelerated at a ms^{-2}.

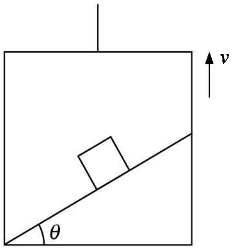


Fig. 5.29

(a) $\sqrt{\frac{2l}{g\sin\theta}}, \sqrt{\frac{2l}{(g+a)\sin\theta}}$

(b) $\sqrt{\frac{2l}{g\sin\theta}}, \sqrt{\frac{2l}{(g-a)\sin\theta}}$

(c) $\sqrt{\frac{2l}{g\sin\theta}}, \sqrt{\frac{2l}{g\sin\theta}}$

(d) none of these

Solution (a) **Case (i)** $\frac{1}{2} g \sin\theta \; t^2 = l$ or $t = \sqrt{\frac{2l}{g\sin\theta}}$

Case (ii) $\frac{1}{2}(g + a) \sin\theta \; t^2 = l$ or $t = \sqrt{\frac{2l}{(g+a)\sin\theta}}$

18. A mass m is hung with a light inextensible string as shown in Fig 5.30, find T_1.

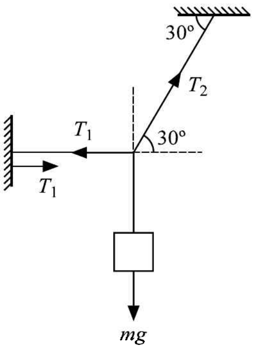


Fig. 5.30

(a) mg
(b) $mg/2$
(c) $mg/\sqrt{3}$
(d) $\frac{2}{\sqrt{3}}mg$

Solution (c) $T_2 \sin 30 = T_1$

$$T_2 \cos 30 = mg$$

$$\tan 30° = \frac{T_1}{mg}$$

or $T_1 = mg \tan 30° = mg/\sqrt{3}$

19. Three equal weights of mass m each are hanging on a string passing over a fixed pulley as shown in Fig. 5.31. Find the tension in the string connecting B and C.

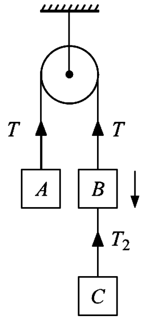


Fig. 5.31

(a) mg
(b) $mg/3$
(c) $mg/2$
(d) $2mg/3$

Solution (d) Acceleration $a = \frac{(2m-m)\,g}{3m} = \frac{g}{3}$

$$T_2 = m(g - a) = m(g - g/3) = 2mg/3$$

20. If applying a force F the mass m is displaced vertically down by y from equilibrium position. Find the force constant k of the spring and displacement in the spring for case shown in Fig. 5.32.

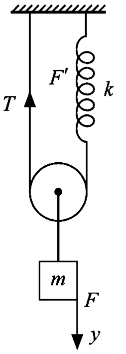


Fig. 5.32

Solution Obviously $T = F'$

$$2F' = F$$

If the mass m shifts by y, the spring will stretch by $2y$ (as string is inextensible)

$\therefore$ $F' = k(2y)$ or $F = 2.F' = 4ky$.

or $k = F/4y$.

21. A chain of length l and mass M, uniformly distributed throughout the length is placed over a table partially overhanging. Find the maximum length of the chain which can be overhung without sliding down. Assuming coefficient of friction to be μ.

(a) $\mu l/(\mu + 1)$ (b) $\mu l/(\mu - 1)$

(c) μl (d) $(\mu - l)l$

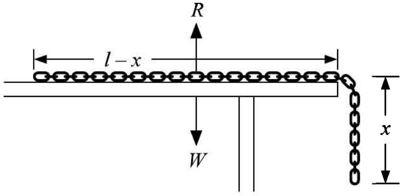


Fig. 5.33

Solution (a) Let x be the part over hanging

weight of length $x = kxg \quad k = \dfrac{m}{l}$

weight of length $(l - x) = k(l - x)\,g$

then $k\,\mu\,(l - x)g = kxg$

or $(\mu + 1)\,x = \mu l$

$$x = \mu l(\mu + 1)$$

22. If the coefficient of friction between an insect and bowl is μ and the radius of the bowl is r, find the maximum height to which the insect can crawl in the bowl.

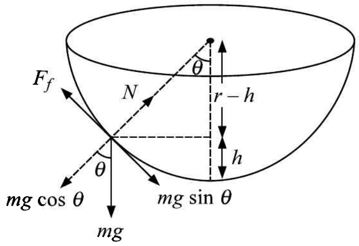


Fig. 5.34

(a) r/μ (b) $r(\mu - 1)$

(c) $r/\sqrt{1+\mu^2}$ (d) $r\left[1-1/\sqrt{1+\mu^2}\right]$

Solution (d) It is clear from the figure

$N = mg\cos\theta$

force of friction $F_f = \mu N = mg\sin\theta \Rightarrow \mu = \tan\theta$

$$\sqrt{\frac{r^2-(r-h)^2}{r-h}} = \tan\theta = \mu$$

$$r^2 - (r - h)^2 = \mu^2(r - h)^2$$

$$h = \left[r-\frac{r}{\sqrt{1+\mu^2}}\right] = r\left[1-\frac{1}{\sqrt{1+\mu^2}}\right]$$

23. A body of mass m rests on a horizontal floor with which it has a coefficient of static friction μ. What minimum force F be applied to make the body move. Also find the direction of force.

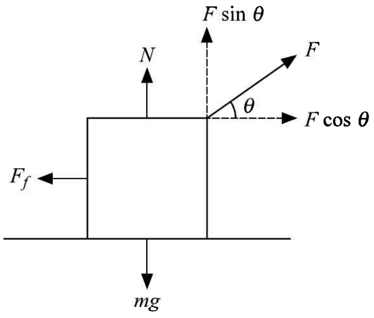


Fig. 5.35

(a) $\dfrac{\mu mg}{\sqrt{1+\mu^2}}$ (b) $\mu\,mg$

(c) $\mu\,mg\cos\theta$ (d) $\dfrac{\mu mg}{1+\mu^2}$

Solution (a) Let force F be applied making an angle θ as shown in Fig. 5.36

$N = mg - F\sin\theta$

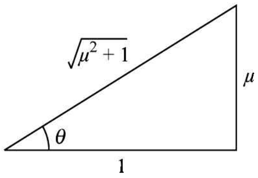


Fig. 5.36

$F_f = \mu N = \mu(mg - F\sin\theta)$

For the body to move

$F\cos\theta \geq \mu(mg - F\sin\theta)$

$F(\mu\sin\theta + \cos\theta) \geq \mu mg$

or $$F \geq \frac{\mu mg}{\mu\dfrac{(\mu)}{\sqrt{\mu^2+1}}+\dfrac{1}{\sqrt{\mu^2+1}}} = \frac{\mu mg}{\sqrt{\mu^2+1}}.$$

The direction is shown in the above figure.

24. The friction coefficient between the board and floor shown in Fig. 5.37 is μ. Find the minimum force that the man can exert on the rope so that the board does not move. Given mass of man M, mass of board m.

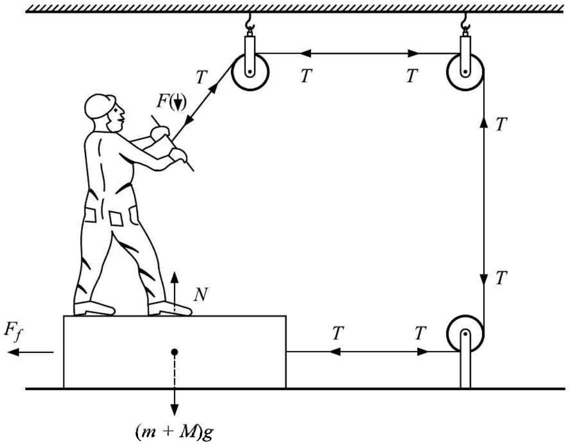


Fig. 5.37

(a) $\mu(m + M)g$ (b) $\dfrac{\mu(m+M)g}{\mu+1}$

(c) $\dfrac{\mu(m+M)g}{\mu-1}$ (d) none of these

Solution (b) $N + T = (m + M)g$

$N = (m + M)g - T$ The system will not move till

$T > F_f$

that is, $T \leq \mu[(m + M)g - T]$ or $T \leq \dfrac{\mu(m+M)g}{\mu+1}$

or $$F_{\max} = \frac{\mu(m+M)g}{\mu+1}$$

Problems for Practice

1. A toy train consists of three identical compartment A, B and C. It is being pulled by a constant force F along C. The ratio of the tensions in the string connecting AB and BC is
(a) 2 : 1 (b) 1 : 3 (c) 1 : 1 (d) 1 : 2

2. A block of mass M is pulled along a smooth horizontal surface with a rope of mass m. The acceleration of the block will be
(a) $F/(M + m)$ (b) $F/(M - m)$
(c) F/M (d) F/m

3. A body of weight 50N is dragged on a horizontal surface with a force of 28.2N. The frictional force acting on the body and the normal reactional force will be

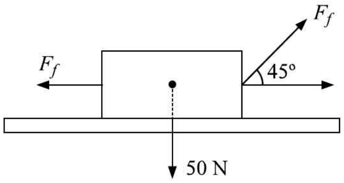


Fig. 5.38

(a) 2N, 3N (b) 5N, 6N
(c) 10N, 15N (d) 20N, 30N

4. Two blocks of mass 4kg and 6kg are placed in contact with each other on a frictionless horizontal surface. If we apply a push of 5N on the heavier mass, the force on the lighter mass will be
(a) 2N (b) 4N (c) 5N (d) None of these

5. A jar containing water is placed in a train. The train accelerates from left to right. Which of the following shows the water level in a jar correctly?

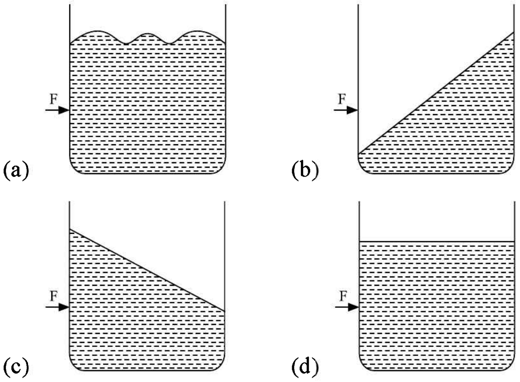


Fig. 5.39

6. A block of mass m is placed on a smooth inclined plane of inclination θ with the horizontal. The force exerted by the plane on the block has a magnitude
(a) $mg\tan\theta$ (b) $mg\cos\theta$
(c) $mg/\cos\theta$ (d) mg

7. The work done in dragging a block of mass 5kg on an inclined plane of height 2m is 150Joule. The work done against the frictional force will be
(a) 200Joule (b) 150Joule
(c) 100Joule (d) 50Joule

8. A monkey of mass 20kg is holding a vertical rope. The rope can break when a mass of 25kg is suspended from it. What is the maximum acceleration with which the monkey can climb up along the rope?
(a) 7ms^{-2} (b) 10ms^{-2}
(c) 5ms^{-2} (d) 2.5ms^{-2}

9. A force of 5Newton acts on a body of weight 9.8Newton. What is the acceleration produced in ms^{-2}?
(a) 0.51 (b) 1.46 (c) 49.00 (d) 5.00

10. A body of mass m is released from the top of a rough inclined plane of length l. If the frictional force is f then the velocity of the body of the bottom in ms^{-1} will be
(a) $\sqrt{\frac{2}{m}(mgh - fl)}$ (b) $2gh - f/l$
(c) $\sqrt{\frac{2}{m}}gh$ (d) zero

11. Three bodies, each of mass 2kg., are connected by means of a string passing over a frictionless pulley as shown in the Fig. 5.40. The tension in the string connecting Y to Z will be

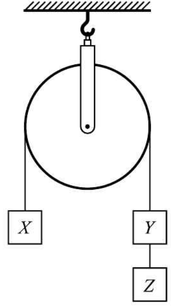


Fig. 5.40

(a) 1.30N (b) 3.3N
(c) 13N (d) 33N

12. A block of mass 2kg is lying on a floor. The coefficient of static friction is 0.54. What will be the value of frictional force if the applied force is 2.8N and g = 10ms^{-2}?
(a) zero (b) 2N (c) 2.8N (d) 10.8N

13. A cube weighing 10N is lying on a rough inclined plane of slope 3 in 5. The coefficient of friction between the plane and the cube is 0.6. The force necessary to move the cube up the plane will be
(a) 6.4N (b) 10.8N (c) 21.6N (d) 108N

14. A block of metal is lying on the floor of a bus. The maximum acceleration which can be given to the bus so that the block may remain at rest, will be
(a) μg^2 (b) $\mu^2 g$ (c) μg (d) μ/g

15. A body of weight w is lying at rest on a rough horizontal surface. If the angle of friction is θ, then the minimum force required to move the body along the surface will be
(a) $w\cos\theta$ (b) $w\tan\theta$
(c) $w\sin\theta$ (d) $w\cot\theta$

16. A block of mass 0.5kg. rests against a wall exerting a horizontal force of 15N on the wall. If the coefficient of friction between the wall and the block is 0.5 then the frictional force acting on the block will be
(a) 49.9N (b) 9.8N
(c) 4.90N (d) 0.49N

17. A block of mass 2kg is lying on a floor. The coefficient of kinetic friction is 0.4. If a force of 2.5N is applied on the

block as shown in Fig. 5.41 then the magnitude of frictional force will be

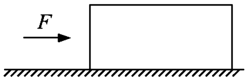


Fig. 5.41

(a) 10 N (b) 7.5 N (c) 5 N (d) 2.5 N

18. A rope of length l is pulled with a constant force f. T is the tension in the rope at a point distant x from the end where the force is applied. Then T is

(a) $f(l - x)/l$ (b) $fl/(l - x)$

(c) $\frac{(f - x)}{l - x}$ (d) $\frac{fl}{x}$

19. Two masses m_1 and m_2 are attached to a string which pass over a frictionless fixed pulley. Given that $m_1 = 10$ kg and $m_2 = 6$ kg and $g = 10\,\text{ms}^{-2}$. What is the acceleration of the masses?

(a) $2.5\,\text{ms}^{-2}$ (b) $5\,\text{ms}^{-2}$

(c) $20\,\text{ms}^{-2}$ (d) $40\,\text{ms}^{-2}$

20. A book is lying on the table. What is the angle between the action of the book on the table and the reaction of the table on the book?

(a) 180° (b) 90° (c) 45° (d) 0°

21. A parachutist of weight w strikes the ground with his legs fixed and comes to rest with an upward acceleration of magnitude $3g$. Force exerted on him by ground during landing is

(a) $4w$ (b) $3w$ (c) $2w$ (d) w

22. The force that prevents the relative motion between the layers of a liquid is called

(a) static friction
(b) sliding friction
(c) contact friction
(d) none of these

23. Gravels are dropped on a conveyer belt at the rate of $0.5\,\text{kgs}^{-1}$. The extra force required in newtons to keep the belt moving at 2ms^{-1} is

(a) 0.5 (b) 1 (c) 2 (d) 4

24. Starting from rest, a body slides down a 45° inclined plane in twice the time it takes to slide down the same distance in the absence of friction. The coefficient of friction between the body and the inclined plane is

(a) 0.25 (b) 0.33 (c) 0.75 (d) 0.80

25. When we walk on ice, we should take small steps to avoid slipping. This is because smaller steps ensure

(a) larger friction
(b) smaller friction
(c) larger normal force
(d) smaller normal force

26. A chain of length L and mass m is allowed to fall on a table such that the part falling on the table comes to rest instantaneously. The force acting on the table when l part of it has lied on the table is

(a) $\frac{3mlg}{L}$ (b) $\frac{2mlg}{L}$

(c) $\frac{mlg}{L}$ (d) $\frac{3mlg}{2L}$

27. Two balls of mass 1 kg and 2 kg respectively are connected to the two ends of the spring. The two balls are pressed together and placed on a smooth table. When released, the lighter ball moves with an acceleration of $2\,\text{ms}^{-2}$. The acceleration of the heavier ball will be

(a) $0.5\,\text{ms}^{-2}$ (b) $1\,\text{ms}^{-2}$

(c) $2\,\text{ms}^{-2}$ (d) $4\,\text{ms}^{-2}$

28. A fireman wants to slide down a rope. The breaking load for the rope is 3/4th of the weight of the man. With what minimum acceleration should the fireman slide down? Acceleration due to gravity is g.

(a) zero (b) $g/4$ (c) $3g/4$ (d) $g/2$

29. A rain drop of mass 0.1 g is falling with uniform speed of $10\,\text{cm s}^{-1}$. What is the net weight of the drop?

(a) 10^{-2}N (b) 10^{-3}N

(c) 2×10^{-3}N (d) zero

30. Two masses m_1 and m_2 are attached to the ends of a string which passes over a pulley attached to the top of an inclined plane. The angle of inclination of the plane in θ. Take $g = 10\,\text{ms}^{-2}$

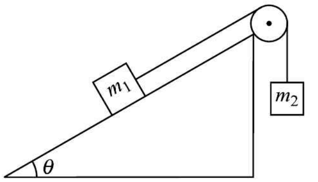


Fig. 5.42

If $m_1 = 10$ kg, $m_2 = 5$ kg, $\theta = 30°$, what is the acceleration of mass m_2?

(a) zero (b) $(2/3)\,\text{ms}^{-2}$

(c) $5\,\text{ms}^{-2}$ (d) $10\,\text{ms}^{-2}$

31. A heavy uniform bar is being carried by two men on their shoulders. The weight of the bar is w. If one man lets it fall from the end carried by him, what will be the weight experienced by the other?

(a) none of these (b) $w/4$

(c) $w/2$ (d) w

32. The coefficient of friction of an inclined plane is $1/\sqrt{3}$. If it is inclined at angle 30° with the horizontal, what will be the downward acceleration of the block placed on the inclined plane?

(a) 0 (b) $\sqrt{2}\,\text{ms}^{-2}$

(c) $\sqrt{3}\,\text{ms}^{-2}$ (d) $3\,\text{ms}^{-2}$

33. A body is projected upwards with a kinetic energy of 100 J. Taking the friction of air into account, when it returns to earth, its kinetic energy will be

(a) more than 100 J (b) less than 100 J

(c) 100 J (d) none of these

34. Which of the following is a self adjusted force?

(a) sliding friction (b) static friction

(c) limiting friction (d) dynamic friction

35. A body is placed over an inclined plane of angle $\pi - \theta$. The angle between normal reaction and the weight of the body is

(a) equal to the angle of friction
(b) more than θ
(c) less than θ
(d) θ

36. The frictional force due to air on a body of mass 0.25 kg falling with an acceleration of $9.2\,\text{ms}^{-1}$ will be

(a) 0.15 N (b) 1.5 N

(c) 15 N (d) zero

37. If a rough surface is polished beyond a certain limit then the magnitude of frictional force will

(a) nothing can be said
(b) some time increases and some time decreases
(c) Increase
(d) decrease

38. The force F to be applied on the triangular block of mass M is ... so that the block of mass m placed on it appears stationary

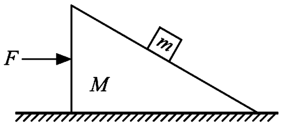


Fig. 5.43

(a) $mg\tan\theta$ (b) $(M + m)g \tan\theta$
(c) $(M + m)g \cos\theta$ (d) $(M + m)g \sin\theta$

39. A car is moving on a straight horizontal road with a speed of $72\,\text{kmh}^{-1}$. If the coefficient of static friction between the tyre of the car and the road is 0.5, then the minimum distance, within which the car can be stopped will be
(a) 72 m (b) 40 m (c) 30 m (d) 20 m

40. When we kick a stone, we get hurt. Due to which one of the following properties does it happens?
(a) velocity (b) momentum
(c) inertia (d) reaction

41. A cricket player catches a ball of mass 100 g and moving with a velocity of $25\,\text{ms}^{-1}$. If the ball is caught in 0.1 s, the force of the blow exerted on the hand of the player is
(a) 4N (b) 40N (c) 25N (d) 250N

42. A body is moving with a velocity $1\,\text{ms}^{-1}$ and a force F is needed to stop it within a distance x. If the speed of the body is $3\,\text{ms}^{-1}$, the force needed to stop it within the same distance (x) will be
(a) $9F$ (b) $6F$ (c) $3F$ (d) $1.5F$

43. A bullet of mass m moving with a velocity v is fired into a large wooden block of mass M. If the bullet remains embedded in the wooden block, the velocity of the system will be
(a) $\dfrac{M}{M-m}v$ (b) $\dfrac{M}{M+m}v$
(c) $\dfrac{m}{M+m}v$ (d) $\dfrac{M}{M-m}v$

44. A particle of mass 2 kg is moving along a circular path of radius 1 m. If its angular speed is $2\pi\,\text{rad s}^{-1}$, then the centripetal force on it is
(a) $8\pi^2$N (b) $4\pi^4$N (c) 8π N (d) 4π N

45. The velocity time graph of a lift moving downwards is a straight line inclined to the time axis at 45°. If mass of the lift is m kg. What is the effective weight (in newton) of the lift? Take $g = 10\,\text{ms}^{-2}$
(a) m (b) $9m$ (c) $10m$ (d) none of these

46. A man weighing 100 kgf carries a load of 10 kgf on his head. He jumps from a tower with the load on his head. What will be the weight of the load as experienced by the man? Take $g = 10\,\text{ms}^{-2}$
(a) 0
(b) 10 kgf
(c) slightly more than 10 kgf
(d) 110 kgf

47. A mass is allowed to move from rest down an inclined plane of inclination 60°. It reaches the bottom in time t. If it moves down an inclined plane of inclination 30° but of same height, the time taken to reach the bottom will be
(a) $t/2$ (b) $\sqrt{3}\,t$ (c) $t/\sqrt{3}$ (d) t

48. Water is flowing through a right angled bend with a velocity v. The force along the support is ... if A is area of cross-section and ρ is density of water.

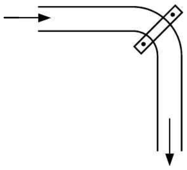


Fig. 5.44

(a) $\sqrt{2}\,A\rho v^2$ (b) $A\rho v^2$
(c) $2A\rho v^2$ (d) $\left(\sqrt{2}+1\right)A\rho v^2$

49. A sparrow flying in the air sits on a stretched telegraph wire. If weight of the sparrow is W, which of the following is true about the additional tension T produced in the wire?
(a) $T = 0$ (b) $T > W$
(c) $T = W$ (d) $T < W$

50. A rocket is ejecting 50 g of gases per second at a speed of $500\,\text{ms}^{-1}$. What is the accelerating force on the rocket?
(a) 125N (b) 25N (c) 6N (d) 1N

51. A body starts climbing up an inclined plane having inclination 30° and stops at a distance of 17.3 m. If the angle of inclination be 60°, how long will it be able to climb up starting with the same speed?
(a) 17.3 m (b) 12.9 m
(c) 10.0 m (d) 8.6 m

52. A bob is suspended with the help of a thread whose breaking load is twice the weight of the bob. Taking $g = 10\,\text{ms}^{-2}$, what is the minimum time in which the bob can be raised by 10 m?
(a) $\sqrt{2}$ s (b) $2\sqrt{2}$ s
(c) $1/\sqrt{2}$ s (d) 1 s

53. A box of mass 10 kg is placed on a rough inclined plane of inclination 30°. The box slides down with acceleration $2.5\,\text{ms}^{-2}$. What is the coefficient of friction between the box and the inclined plane?
(a) $\dfrac{1}{3\sqrt{2}}$ (b) $\dfrac{1}{2\sqrt{3}}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$

54. A gun of mass M fires a bullet of mass m with maximum speed v. Given that $m < M$. The kinetic energy of the gun will be
(a) $1/2Mv^2$ (b) $1/2mv^2$
(c) $< 1/2mv^2$ (d) more than $1/2mv^2$

55. A balloon contains 10 g of air. The air begins to escape from a small hole in the balloon at the steady rate of $4\,\text{cms}^{-1}$ and it completely shrinks in 2 seconds. What is the average force on the balloon?
(a) 20 dyne (b) 15 dyne
(c) 10 dyne (d) 5 dyne

56. A truck is moving with an acceleration 'a'. A block of mass m placed against its vertical wall fails to slip down. The coefficient of friction is μ. The minimum value of 'a' is
(a) $\dfrac{g}{\mu}$ (b) μg (c) $\dfrac{\mu g}{2}$ (d) $(\mu + 1)g$

57. A body of mass m is acted upon by a force F and the acceleration produced is a. If three forces each equal to F and inclined to each other at 120° act on the same body, the acceleration produced will be
(a) $a/\sqrt{3}$ (b) $\sqrt{2}\,a$ (c) $3a$ (d) zero

58. A ball of mass 100 g falls vertically down from a height 80 m and rebounds to 60 m. The impulse imparted to the ground is

(a) 0.536 kg ms^{-1} (b) 3.46 kg ms^{-1}
(c) 7.46 N-s (d) 4 N-s

59. A rocket of mass 120 kg is fired in the gravity free space. It ejects gases which velocity 600 ms^{-1} at the rate of 1 kgs^{-1}, what will be the initial acceleration of the rocket

(a) 15 ms^{-2} (b) 10 ms^{-2}
(c) 5 ms^{-2} (d) 1 ms^{-2}

60. A body of mass 16 kg accelerated down a smooth inclined plane with a force 8 N. The force required to make it move upward with the same acceleration

(a) 16N (b) 12N (c) 8N (d) 4N

61. A bullet of mass 5g is fired with impulse equal to 120 Ns. If the mass of the gun is 12 kg, what is the recoil velocity of the gun?

(a) 20 ms^{-1} (b) 15 ms^{-1}
(c) 10 ms^{-1} (d) 5 ms^{-1}

62. A ball of mass m collides elasticity against a wall with velocity v and rebounds with the same velocity in the opposite direction. What is the impulse on the wall?

(a) $1/2mv$ (b) $2mv$
(c) $(3/2)mv$ (d) mv

63. If 4 kg block is held after 2s of start then how high 2 kg block will rise before coming to rest momentarily?

(a) $\frac{g}{9}$ m

(b) $\frac{2g}{9}$ m

(c) $\frac{g}{3}$ m

(d) $\frac{2g}{3}$ m

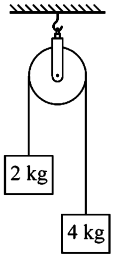


Fig. 5.45

64. A body falling with a speed 2 ms^{-1} strikes the floor and rebounds with a speed of 1 ms^{-1}. The loss of energy is

(a) 75% (b) 50% (c) 25% (d) 12.5%

65. The linear momentum of a particle is given by $p = a + bt^2$, where t is time and a and b are constants. The force acting on the body varies directly as

(a) t (b) t^2 (c) t^3 (d) t^0

66. The mechanical advantage of a system of pulleys is 3. How much force need to be applied to rise a load of 60 kg through 50 cm?

(a) 30 kgf (b) 90 kgf
(c) 20 kgf (d) 120 kgf

67. A solid ball of density half that of water falls from a height of 20 m and then enters water. To what depth will it go in the water? Take $g = 10$ ms^{-2}

(a) 5 m (b) 40 m (c) 20 m (d) 10 m

68. A balloon of mass M is under a drag force F and upthrust T. It is moving down with a uniform velocity v. What amount of mass m be removed that it starts rising up with same velocity v?

(a) $M - T/g$ (b) $2T/g$
(c) T/g (d) $2(M - T/g)$

69. Consider the situation shown in Fig. 5.46. All the surfaces are smooth. The tension in the string connected to $2m$ is

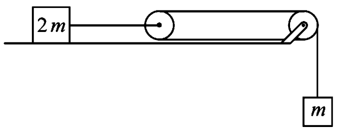


Fig. 5.46

(a) $\frac{mg}{3}$ (b) $\frac{4mg}{3}$ (c) $\frac{2mg}{3}$ (d) mg

70. The mass of an elevator is 4000 kg. The tension in the cable is 56 KN. Starting from rest how much distance will it rise is 2 s?

(a) 16 m (b) 8 m (c) 4 m (d) 2 m

71. A rocket of mass 6000 kg is set for vertical firing. If the exhaust speed be 1 kms^{-1} how much gas must be ejected to give the rocket an upward acceleration of 20 ms^{-2}. Take $g = 10$ ms^{-2}?

(a) 180 kgs^{-1} (b) 120 kgs^{-1}
(c) 90 kgs^{-1} (d) 45 kgs^{-1}

72. The potential energy of a conservative system is given by $PE = ay^2 - by$, where y represents the position of the particle and a as well as b are constants. What is the force acting on the system?

(a) $b - 2ay$ (b) $2ay - b$
(c) $-by$ (d) $-ay$

73. Consider the situation shown in Fig. 5.47. The acceleration of block of mass m is

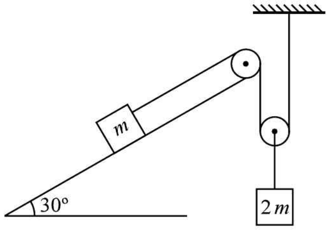


Fig. 5.47

(a) $g/3$ up the plane (b) $g/3$ down the plane
(c) $g/2$ up the plane (d) $g/2$ down the plane

74. An electron starting from rest has a velocity that increases linearly with time, that is, $v = kt$ were$k = 2$ ms^{-2}. The distance covered in the first 3 seconds will be

(a) 36 m (b) 27 m
(c) 16 m (d) 9 m

75. A monkey B shown in Fig. 5.48 is holding on to the tail of monkey A who is climbing up a rope. The masses of the monkeys A and B are 5 kg and 2 kg respectively. If A can tolerate a tension of 30 N in its tail what force should it apply on rope to carry the monkey B with it?

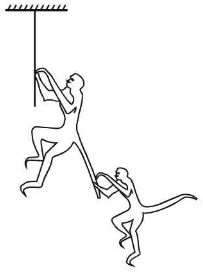


Fig. 5.48

(a) 70 N (b) 105 N
(c) between 70 and 105 N (d) cannot be said

ANSWERS

1. (d)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (d)	**10.** (a)
11. (c)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (a)
21. (a)	**22.** (d)	**23.** (b)	**24.** (c)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (a)
31. (a)	**32.** (a)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (a)	**37.** (c)	**38.** (b)	**39.** (b)	**40.** (d)
41. (c)	**42.** (a)	**43.** (c)	**44.** (a)	**45.** (b)	**46.** (a)	**47.** (b)	**48.** (a)	**49.** (b)	**50.** (b)
51. (c)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (a)	**56.** (a)	**57.** (d)	**58.** (c)	**59.** (c)	**60.** (a)
61. (c)	**62.** (b)	**63.** (b)	**64.** (a)	**65.** (a)	**66.** (c)	**67.** (c)	**68.** (d)	**69.** (c)	**70.** (b)
71. (a)	**72.** (a)	**73.** (a)	**74.** (d)	**75.** (c)					

6 Circular Motion

BRIEF REVIEW OF THE CONCEPTS

Circular motion Circular motion may be divided into two types (a) motion in a horizontal circle (b) motion in a vertical circle. The basic difference being that in case of vertical circle acceleration due to gravity plays a role at every point and changes the speed at every point. We will deal with them separately.

Motion in a horizontal circle Here, acceleration acts continuously even when speed is uniform, therefore, simple equation of motion as used in translation motion cannot be applied. We define new variables and equations to describe motion.

(a) **Angular displacement (θ)** Change in angular position (initial to final) is called angular displacement.

(b) **Angular velocity (ω)** Time rate of change of angular displacement is called angular velocity.

Thus $\omega = \dfrac{d\theta}{dt}$

(c) **Angular acceleration (α)** Time rate of change of angular velocity is called angular acceleration.

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

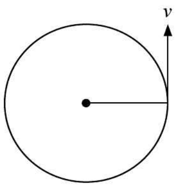


Fig. 6.1

If the angular acceleration is constant then

$$\omega = \omega_0 + \alpha t$$

$$\omega_{aV} = \frac{2\pi}{T}$$

where T is time period to complete circle.

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

where ω is final angular velocity, ω_0 is initial angular velocity, α is angular acceleration, θ is angular displacement and t is time. If $\alpha = 0$, particle moves in uniform circular motion, in such a case v = constant.

Relation between v and ω; a and α

$v = r\omega$ where r is radius of the circular path.

$a_t = r\alpha$ where a_t is tangential acceleration as explained in Fig 6.2

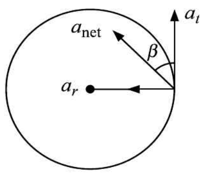


Fig. 6.2

Centripetal or radial acceleration $a_r = \dfrac{v^2}{r} = r\omega^2$

net acceleration $a_{\text{net}} = \sqrt{a_t^2 + a_r^2}$ $\quad \tan\beta = \dfrac{a_r}{a_t}$

Centripetal force $F = \dfrac{mv^2}{r} = mr\omega^2$. It is a pseudoforce. It acts towards the centre.

Centrifugal force The inertial reaction required to take into account the acceleration of frame of reference is called centrifugal force. It is equal to $-mr\omega^2$

Motion in a vertical circle When a body moves in a vertical circle then at the highest point

$\dfrac{mv^2}{r} \geq mg \quad v_{\min} = \sqrt{rg}$ at highest point

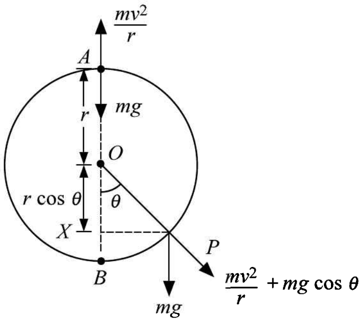


Fig. 6.3

$v_{\min}$ at any point P can be determined using the fact that the body has come down by a vertical distance

$$AX = AO + OX = r + r\cos\theta \quad \text{(From Fig. 6.3)}$$

Thus
$$v_{\min P}^2 = v_{\min A}^2 + 2gh$$
$$= rg + 2g(r + r\cos\theta)$$
$$= 3rg + 2rg\cos\theta$$
$$v_{\min P} = \sqrt{3rg + 2rg\cos\theta}$$

$v_{\min}$ at lowest point or bottom is obtained by using $\theta = 0$ that is, $v_{\min\,\text{bottom}} = \sqrt{5rg}$

Tension, if the string is used or normal reaction at any point P is obtained as

$$T \text{ or } N = \frac{mv^2}{r} + mg\cos\theta$$

where v is velocity at that point.

SHORT-CUTS AND POINTS TO NOTE

(i) Circular motion is uniform if $a_t = \alpha = 0$, that is, angular velocity remains constant and radial acceleration $a_r = \frac{v^2}{r} = r\omega^2$ remains constant.

(ii) When α or a_t is present ω (angular velocity) varies with time

$$v = r\omega \text{ and } a_t = r\alpha;\ \omega = \frac{d\theta}{dt} \text{ and } \alpha = \frac{d^2\theta}{dt^2}$$

(iii) Since a_t and a_r are at right angles to one another

$\therefore\ \alpha_{\text{net}} = \sqrt{a_r^2 + a_t^2}$

and $\beta = \tan^{-1}\frac{a_r}{a_t}$.

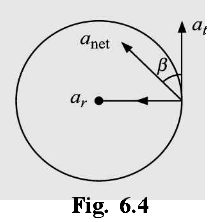


Fig. 6.4

(iv) If $a_t = 0$ or $\alpha = 0$, no work is done in circular motion. that is, in uniform circular motion no work is done as force and displacement are perpendicular. If $a_t \neq 0$ or $\alpha \neq 0$ then work is done.

(v) If the particle moves from A to B in uniform circular motion with a velocity v then change in velocity

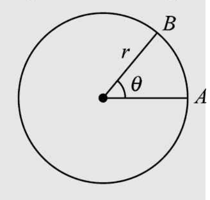


Fig. 6.5

$\Delta v = 2v\sin(\theta/2)$ and average acceleration

$$a_{aV} = \frac{2v\sin(\theta/2)}{\frac{r\theta}{v}} = \frac{2v^2\sin(\theta/2)}{r\theta(\text{radian})}.$$

(vi) Centripetal force $F = \frac{mv^2}{r} = mr\omega^2$.

(vii) While dealing with vertical circle apply conservation of energy or gain in KE = Loss in PE or vice versa.

(viii) In vertical circle $T_{\text{Lowest point}} - T_{\text{highest point}} = 6mg$.

(ix) In vertical circle if velocity at the top is given then velocity at the bottom is

$$v_{\text{bottom}}^2 = v_{\text{top}}^2 + 2g(2r).$$

and if velocity at the bottom is given then velocity at the top is given by

$$v_{\text{top}}^2 = v_{\text{bottom}}^2 - 2g(2r).$$

(x) Minimum velocity to be given at the bottom to complete the vertical loop is $\sqrt{5rg}$ where r is radius of the loop.

Minimum velocity at the top = $\sqrt{rg}$

Minimum velocity at any point in the vertical circle is = $\sqrt{3rg + 2rg\cos\theta}$.

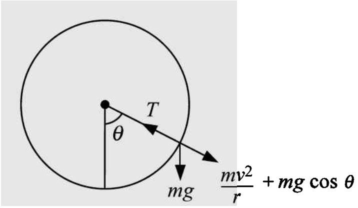


Fig. 6.6

(xi) Tension at any point $T = \frac{mv^2}{r} + mg\cos\theta$

If the particle is not tied with a string then tension may be read as action or normal reaction depending on the case.

(xii) In rotational motion radius of each point is different, therefore, different particles move in different radii as illustrated in Fig. 6.7. In circular motion radius in fixed. Each particle is assumed to follow same track of radius r.

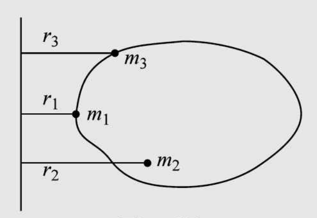


Fig. 6.7

(xiii) Radius of a projectile at any instant will be

$$r = \frac{v^2}{a} \text{ because } a = \frac{v^2}{r}.$$

(xiv) If a road is banked then angle of banking

$$\tan\theta = \frac{v^2}{rg}$$

A cyclist will bend at the same angle while taking a turn.

(xv) Maximum and minimum velocities when friction is present

$$v_{\max} = \left[\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}\right]^{1/2}$$

and $$v_{\min} = \sqrt{\frac{rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}.$$

CAUTION

(i) Considering no acceleration acts in uniform circular motion as v is constant.

- Acceleration in uniform circular motion is v^2/r called radial or centripetal acceleration or normal acceleration.

(ii) Applying equations like $v = u + at$ and so on, in horizontal circular motion.

- $v = u + at$ and so on cannot be applied.
  Apply $\omega = \omega_0 + \alpha t$, $\theta = \omega_0 t + 1/2\alpha t^2$
  and $\omega^2 - \omega_0^2 = 2\alpha\theta$.

(iii) Considering that work is done in circular motion as force acts on the particle.

- In uniform circular motion no work will be done as force and displacement are perpendicular. However, work will be done in non uniform circular motion.

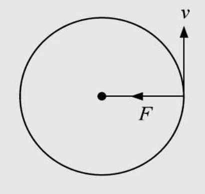


Fig. 6.8

(iv) Considering minimum velocity to be given to complete vertical circle is $\sqrt{rg}$.

- Minimum velocity to be given to a particle to complete vertical circle is $\sqrt{5rg}$.

(v) Considering change in velocity $\Delta v = 0$ in uniform circular motion.

- Change in speed is zero as speed is constant but since acceleration is present or direction is continuously changing, therefore, velocity is changing even in uniform circular motion.

(vi) Considering velocity will always be given by equation $v = \sqrt{3rg + 2rg\cos\theta}$ at any point.

- This is the minimum velocity which is required, if velocity is other than minimum then apply the following equation.
  If v_{top} is given then $v_{\text{any point}}$ is
  $v^2_{\text{any point}} = v^2_{\text{top}} + 2g(h)$
  h is the distance of descend

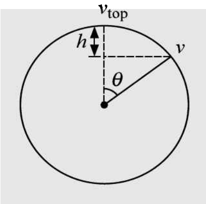


Fig. 6.9

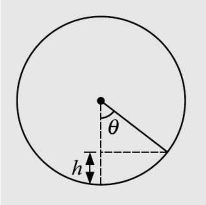


Fig. 6.10

or $h = r - r\cos\theta = v^2_{\text{top}} + 2g\,(r - r\cos\theta)$

If v_{bottom} is given then $v_{\text{any point}}$ is
$v^2_{\text{any point}} = v^2_{\text{bottom}} - 2gh = v^2_{\text{bottom}} - 2g(r - r\cos\theta)$.

(vii) If a car accelerates on a circular road with acceleration a, then considering net acceleration is a

- Since $\frac{v^2}{r}$ also acts, therefore, net acceleration is $\sqrt{\left(\frac{v^2}{r}\right)^2 + a^2} = \sqrt{(r\omega^2)^2 + a^2}$.

(viii) Considering tension is $\frac{mv^2}{r}$ in vertical circle as well as in the horizontal circle.

- Tension in vertical circle at any point is $\frac{mv^2}{r} + mg\cos\theta$ where v is velocity at that point.

Solved Problems

1. A particle starts from rest with an acceleration $2\,\text{rads}^{-2}$ in a circle of radius 2 m. Find its linear speed after 6 s.

(a) $12\,\text{ms}^{-1}$	(b) $24\,\text{ms}^{-1}$
(c) $4\,\text{ms}^{-1}$	(d) none of these

Solution (b) $\omega = \omega_0 + \alpha t = 0 + 2(6) = 12\,\text{rads}^{-1}$
$v = r\omega = 24\,\text{ms}^{-1}$

2. Two particles move in concentric circles of radius r_1 and r_2 such that they maintian a straight line with the centre. The ratio of their angular velocities is

(a) 1	(b) r_1/r_2
(c) r_2/r_1	(d) none of these

Solution (a) Since the particles move in circles of radius r_1 and r_2 such that the line joining them with the centre is maintained, therefore, their angular displacements are euqal in equal time,

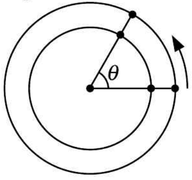


Fig. 6.11

Hence $\frac{\omega_1}{\omega_2} = 1$.

3. Two wheels having radii in the ratio 1 : 3 are connected by a common belt. If the smaller wheel is accelerated from rest at a rate $1.5\,\text{rads}^{-2}$ for 10 s. Find the velocity of bigger wheel.

(a) $5\,\text{rads}^{-1}$	(b) $15\,\text{rads}^{-1}$
(c) $45\,\text{rads}^{-1}$	(d) none of these

Solution (a) $\omega = \omega_0 + \omega t = 0 + 1.5 \times 10 = 15\,\text{rads}^{-1}$

Since the wheels are connected with a common belt, their linear velocity is constant, that is, $v_1 = v_2$

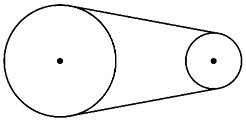


Fig. 6.12

$r_1\omega_1 = r_2\omega_2$

or $\omega = \omega_1 \frac{r_1}{r_2} = 15 \times \frac{1}{3} = 5\,\text{rads}^{-1}$.

4. A car is moving on a circular road of diameter 50 m with a speed 5 ms^{-1}. It is suddenly accelerated at a rate 1 ms^{-2}. If its mass is 500 kg. Find the net force acting on the car.

(a) 500 N (b) 1000 N

(c) $500\sqrt{2}$ N (d) $500/\sqrt{2}$ N

Solution (c) $a_{\text{net}} = \sqrt{\left(\frac{v^2}{r}\right)^2 + a_t^2} = \sqrt{\left(\frac{25}{25}\right)^2 + 1^2}$

$= \sqrt{2}$ ms^{-2}

$F = m\, a_{\text{net}} = 500\sqrt{2}$ N.

5. A particle of mass 1 kg is placed 40 cm away from the centre of a disc. The coefficient of friction between the two surfaces is 0.25. At what maximum speed in rpm the disc can rotate so that the particle is not displaced from its position?

(a) 2.5 (b) 12.5

(c) 2.0 (d) 2.3

Solution (d) Let ω be maximum speed. Then

$mr\omega^2 = \mu mg$

$\omega = \sqrt{\frac{\mu g}{r}} = \sqrt{\frac{.25 \times 10}{.4}} = \frac{5}{2} = 2.5 \text{ rads}^{-1}$

$= \frac{2.5 \times 60}{2\pi} = \frac{150}{6.28} = 23.8 \text{ rpm}$

6. A rod of length l is pivoted at one of the ends and is made to rotate in a horizontal plane as shown in Fig. 6.13 with a constant angular speed. A ball of mass m suspended by a string of length l from the other end of the rod. If the angle made by the string with the vertical is θ then determine angular speed of rotation.

(a) $\sqrt{\frac{g}{l(1+\sin\theta)}}$ (b) $\sqrt{\frac{g}{l\cos\theta}}$

(c) $\sqrt{\frac{g\tan\theta}{l(1+\sin\theta)}}$ (d) $\sqrt{\frac{g\cot\theta}{l(1+\cot\theta)}}$

Solution (c) The radius of the horizontal circle becomes

$r = l(1 + \sin\theta)$

If T is tension in the string

$T\cos\theta = mg$

$T\sin\theta = mr\omega^2$

$\tan\theta = \frac{r\omega^2}{g}$ $\omega = \sqrt{\frac{g\tan\theta}{l(1+\sin\theta)}}$

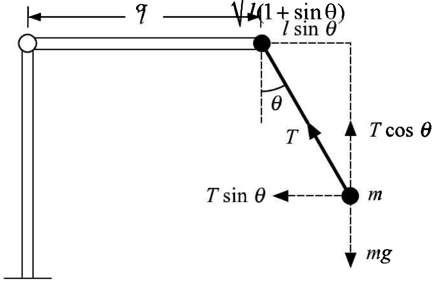


Fig. 6.13

7. A particle of mass m is suspended from the ceiling through a string of length l. The particle moves in a horizontal circle of radius r as shown in Fig. 6.14. Find the speed of the particle and tension in the string.

(a) $\sqrt{rg\tan\theta}, \frac{mgl}{\sqrt{(2r^2)}}$ (b) $\sqrt{rg\sin\theta}, \frac{mgl}{\sqrt{l^2+r^2}}$

(c) $\sqrt{rg\tan\theta}, \frac{mgl}{\sqrt{l^2-r^2}}$ (d) none of these

Solution (c) $T\cos\theta = mg$

$T\sin\theta = \frac{mv^2}{r}$ $\tan\theta = \frac{v^2}{rg}$

or $v = \sqrt{rg\tan\theta}$

$T = \frac{mg}{\cos\theta} = \frac{mgl}{\sqrt{l^2-r^2}}$

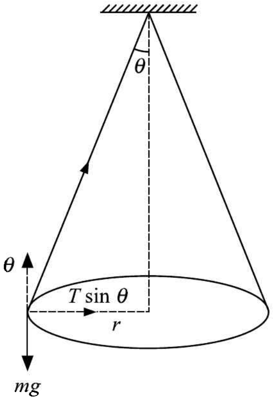


Fig. 6.14

8. One end of a spring of spring constant k and natural length l is fixed and the other end is connected to a particle of mass m lying on a frictionless horizontal table. The spring remains horizontal. If the mass is made to rotate at an angular velocity ω rads^{-1}. Find the elongation in the spring.

(a) $\frac{ml\omega^2}{k}$ (b) $\frac{ml\omega^2}{k - m\omega^2}$

(c) $\frac{ml\omega^2}{k + m\omega^2}$ (d) none of these

Solution (b) Let Δl be the elongation in the spring then

$k\Delta l = mr\omega^2 = m(l + \Delta l)\,\omega^2$

$\Delta l(k - m\omega^2) = ml\omega^2$ or $\Delta l = \frac{ml\omega^2}{k - m\omega^2}$

9. A table with smooth horizontal surface is rotating at a speed ω about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the centre. Find the speed of the particle with respect to the table as its distance from the centre becomes l.

Solution $a = \frac{F}{m} = \frac{m\omega^2 x}{m}$ or $\frac{dv}{dt} = \omega^2 x$

$\frac{dv}{dx} \cdot \frac{dx}{dt} = \omega^2 x$

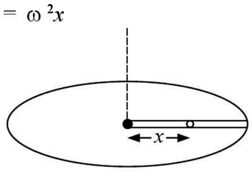


Fig. 6.15

$v\,dv = \omega^2 x\,dx$

$\int_0^v v\,dv = \int_a^L \omega^2 x\,dx$

$v = \omega\sqrt{l^2 - a^2}$.

10. A car moves on a horizontal track of radius r, the speed increasing constantly at rate $\frac{dv}{dt} = a$. The coefficient of friction between road and tyre is μ. Find the speed at which car will skid.

(a) $[(\mu^2g^2 + a^2)r^2]^{1/4}$ (b) $\sqrt{\mu g r}$

(c) $[(\mu^2g^2 - a^2)r^2]^{1/4}$ (d) $\sqrt{ar}$

Solution (c) net acceleration is $\sqrt{a^2 + \left(\frac{v^2}{r}\right)^2} \geq \mu g$

or $\left(\frac{v^2}{r}\right)^2 = \mu^2g^2 - a^2$ or $v = [(\mu^2g^2 - a^2)r^2]^{1/4}$.

11. A particle is moving in a horizontal circle at a height h from the vertex in a conical funnel with uniform speed. Find its speed.

(a) $\sqrt{hg}$

(b) $\sqrt{hg\cos\theta}$

(c) $\sqrt{hg\tan\theta}$

(d) connot be determined.

Solution (a) Let m be the mass of the particle and N, normal reaction. Then resolving N, we have

$$N\cos\theta = mg$$

$$N\sin\theta = \frac{mv^2}{r}$$

$$\tan\theta = \frac{v^2}{rg} \quad \tan\theta = \frac{h}{r}$$

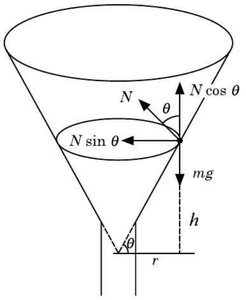


Fig. 6.16

$$\therefore \quad \frac{h}{r} = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{hg}.$$

12. A nut-bolt system has 20 threads per cm. Find the time taken to travel the nut 4 cm if it moves in the bolt at a rate 4 rotations per second.

(a) 80 s (b) 40 s

(c) 30 s (d) 20 s

Solution (d) Total no. of threads in 4 cm = 80

In one rotation 1 thread is moved

∴ To move 80 threads 80 rotations are required

Speed is 4 rotation/s. Hence 20 s are required.

13. A spot light is fixed 4 m from the vertical wall and is rotating at a rate 1 rads^{-1}. The spot moves horizontally on the wall. Find the speed of the spot on the wall when spot light makes an angle of 45° with the wall.

(a) 4 ms^{-1} (b) 6 ms^{-1}

(c) 8ms^{-1} (d) none of these

Solution (c) s is the spot light shown in Fig. 6.17 and x is the distance moved by spot in time t.

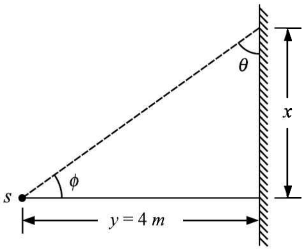


Fig. 6.17

From the Fig. 6.17 $\frac{x}{y} = \tan\phi$

$$x = y\tan\phi$$

$$v = \frac{dx}{dt} = y\sec^2\phi\,\frac{d\phi}{dt} \qquad \frac{d\phi}{dt} = \omega = 1$$

$$v = 4(2)\,1 \text{ (when } \theta = 45°, \phi = 45°)$$

$$= 8\,\text{ms}^{-1}.$$

14. An inclined plane ends into a loop of radius r. Find the minimum height from where a particle shall slip so as to loop the loop.

(a) $\frac{5r}{2}$ (b) $\frac{5r}{3}$

(c) $\frac{5r}{4}$ (d) $\frac{10}{3}r$

Solution (a) Assume the particle slips from a height h then

$$mgh = \frac{1}{2}mv^2$$

or $v^2 = 2gh$.

At the lowest point

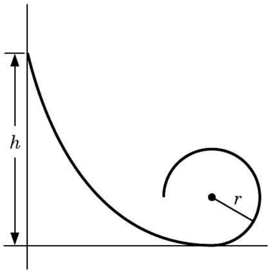


Fig. 6.18

$$v_{\min} = \sqrt{5rg}$$

$$5rg = 2gh \quad \text{or} \quad h = 5r/2.$$

15. A particle slips from a height $4r$ from an inclined plane which ends into a loop of radius r. Find the normal reaction at the highest point and at the lowest point.

(a) $3mg$, $9mg$ (b) $3mg$, $6mg$

(c) 0, $6mg$ (d) $4mg$, $8mg$

Solution (a) (i) For the highest point $h' = 2r$

using $mgh = \frac{1}{2}mv^2$

or $\frac{mv^2}{r} = 4mg$

$$N = \frac{mv^2}{r} + mg\cos\theta$$

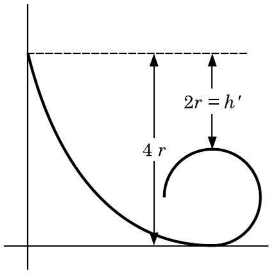


Fig. 6.19

At the highest point $\theta = 180° = 4mg - mg = 3mg$.

(ii) At the lowest point $h' = 4r$

using $mgh = \frac{1}{2}mv^2$ we get $\frac{mv^2}{r} = 8mg$

$$N = \frac{mv^2}{r} + mg\cos\theta.$$

At the lowest point $\theta = 0$

$$N = 8mg + mg = 9mg.$$

16. A particle has velocity $\sqrt{2rg}$ at the highest point in a vertical circle of radius r. Find the ratio of the normal reactions at the highest to the lowest points.

(a) 1/7 (b) 1/3
(c) 1/5 (d) 2/9

Solution (a) (i) At the highest point

$$N_{HP} = \frac{mv^2}{r} + mg\cos\theta \qquad (\because \theta = 180°)$$

$$= 2mg - mg = mg.$$

(ii) At the lowest point

$$v^2 = u^2 + 2gh = 2rg + 2g(2r) = 6rg$$

$$\therefore \quad N_{LP} = \frac{mv^2}{r} + mg\cos\theta = 6mg + mg = 7\text{mg}$$

$$\frac{N_{HP}}{N_{LP}} = \frac{1}{7}.$$

17. A marble of mass m and radius b is placed in a hemispherical bowl of radius r. Find the minimum velocity to be given to the marble so that it reaches the highest point.

(a) $\sqrt{2g(r-b)}$ (b) $\sqrt{2gr}$
(c) $\sqrt{2g(r+b)}$ (d) $\sqrt{g(r-b)}$

Solution (a) $\frac{mv^2}{2} = mgh$

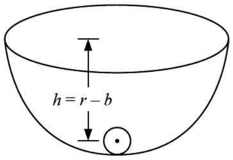


Fig. 6.20

$$= mg(R - b)$$

$$v = \sqrt{2g(r-b)}$$

18. A particle is placed at the top of a sphere of radius r. It is given a little jerk so that it just starts slipping down. Find the point from where it leaves the sphere.

(a) $r/2$ (b) $r/3$ (c) $r/4$ (d) r

Solution (b) Assume the particle leaves from the point where it has travelled down a distance h

$$h = r - r\cos\theta.$$

velocity at point P is given by

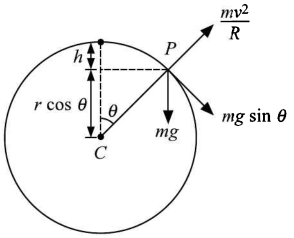


Fig. 6.21

$$\frac{mv^2}{2} = mgh = mg(r - r\cos\theta)$$

or $\frac{mv^2}{r} = 2mg(1 - \cos\theta)$

If the particle leaves at P then $mg\cos\theta = \frac{mv^2}{r}$

or $mg\cos\theta = 2mg(1 - \cos\theta)$

or $3\cos\theta = 2$ or $\cos\theta = \frac{2}{3}$

$$h = r - r\cos\theta = r - \frac{2r}{3} = r/3.$$

19. A pendulum of length l has a bob of mass m. It is released from rest when it makes an angle θ with the vertical. Find the tension when it makes an angle $\theta/2$ with the vertical.

(a) $mg\cos\theta/2$
(b) $3mg\cos\theta/2$
(c) $3mg\cos\theta/2 - 2mg\cos\theta$
(d) $mg\cos\theta/2 - mg\cos\theta$

Solution (c) The bob travels down a distance AB

$$AB = l\cos\theta/2 - l\cos\theta$$

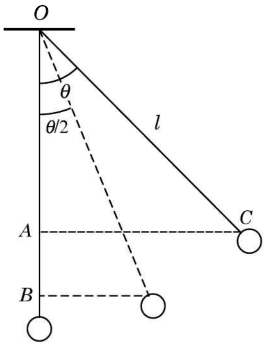


Fig. 6.22

$$\frac{mv^2}{2} = mgl(\cos\theta/2 - \cos\theta)$$

or $\frac{mv^2}{l} = 2mg(\cos\theta/2 - \cos\theta)$

$$T = \frac{mv^2}{l} + mg\cos\theta/2$$

$$= 3mg\cos\theta/2 - 2mg\cos\theta$$

Problems for Practice

1. A particle is moving along a circular path with uniform speed. Through what angle does its angular velocity change when it completes half of the circular path?
(a) 360° (b) 180° (c) 45° (d) 0°

2. A fly wheel rotates at a constant speed of 3000 rpm. The angle described by the shaft in radian in one second is
(a) 3000π (b) 100π
(c) 50π (d) 2π

3. A scooter is going round a circular track with a speed of $30\,ms^{-1}$. The radius of the circular track is 50 m. The angular velocity of the scooter is
(a) $1000\,rad\,s^{-1}$ (b) $0.6\,rad\,s^{-1}$
(c) $2.5\,rad\,s^{-1}$ (d) $0.4\,rad\,s^{-1}$

4. A particle is moving along a circular path of radius 5 m and with uniform speed $5\,ms^{-1}$. What will be the average acceleration when the particle completes half revolution?
(a) $10\pi\,ms^{-2}$ (b) $10/\pi\,ms^{-2}$
(c) $10\,ms^{-2}$ (d) zero

5. What determines the nature of the path followed by the particle?
(a) acceleration (b) speed
(c) velocity (d) none of these

6. A particle moves along a circular path of radius r with uniform speed v. The angle described by the particle in one second is given by
(a) vr^{-2} (b) $v^{-2}r$
(c) vr^{-1} (d) $v^{-1}r$

7. To enable a particle describe a circular path what should be the angle between its velocity and acceleration?
(a) 180° (b) 90°
(c) 45° (d) 0°

8. A particle is describing the circular path of radius 10 m every 2 s. The average angular speed of the particle, during 4 s is
(a) $20\pi\,rad\,s^{-1}$ (b) $4\pi\,rad\,s^{-1}$
(c) $2\pi\,rad\,s^{-1}$ (d) none of these

9. The angular speed of the minute hand of the clock in degrees per second is
(a) 1.0 (b) 0.1
(c) 0.001 (d) none of these

10. The angular speed of a motor increases from 600 to 1200 rpm in 10 s. What is the angular acceleration of the motor?
(a) $60\,rad\,s^{-2}$ (b) $2\pi\,rad\,s^{-2}$
(c) $60\pi\,rad\,s^{-2}$ (d) $600\,rad\,s^{-2}$

11. A wheel is subjected to uniform angular acceleration about its axis. Initially its angular velocity is zero. In the first two seconds it rotates through angle θ_1. In next two second it roates through angle θ_2. What is the ratio θ_2/θ_1?
(a) 4 (b) 3 (c) 2 (d) 1

12. Two bullets are fired at angle θ and $(90 - \theta)$ to the horizontal with the same speed v. The ratio of their radii at the highest points is
(a) $\cos^2\theta$ (b) $\tan^2\theta : 1$
(c) $1 : 1$ (d) $\cot^2\theta : 1$

13. A stone is just released from the window of a train moving along a horizontal straight track. The stone will hit the ground folowing
(a) hyperbolic path (b) circular path
(c) a straight line path (d) parabolic path

14. A metal ring of mass m and radius r is placed on a smooth horizontal table and is set rotating about its axis so that each part of the ring moves with a speed v. The tension in the ring is
(a) $\dfrac{mv^2}{2\pi r}$ (b) $\dfrac{mv^2}{r}$ (c) $\dfrac{mv^2}{\pi r^2}$ (d) $\dfrac{mv^2}{2r}$

15. A particle moves in a circular path of radius r. In half the period of revolution its displacement and distance covered are
(a) $2r$, πr (b) r, πr
(c) $2r$, $2\pi r$ (d) $r\sqrt{2}$, πr

16. The angular acceleration of a particle moving along a circular path with uniform speed is
(a) zero
(b) variable
(c) uniform but non-zero
(d) incomplete information

17. A hollow cylinder of radius r rotates about its axis with a frequency f such that a person on the wall does not fall. The minimum value of coefficient of friction is

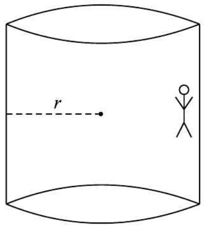


Fig. 6.23

(a) $\dfrac{4\pi^2 f^2 r}{g}$ (b) $\dfrac{f^2 r}{g}$
(c) $\dfrac{\pi^2 f^2 r}{g}$ (d) $\dfrac{g}{4\pi^2 r f^2}$

18. A bucket tied at the end of a 1.6 m long string is whirled in a vertical circle with a constant speed. What should be the minimum speed so that the water from the bucket does not spill out during rotation? ($g = 10\,ms^{2}$)
(a) $9\,ms^{-1}$ (b) $6.25\,ms^{-1}$
(c) $16\,ms^{-1}$ (d) none of these

19. A stone of mass m tied to a string of length l is rotated in a circle with the other end of the string as the centre. The speed of the stone is v. If the string breaks, the stone will move
(a) along a tangent
(b) towards the centre
(c) away from the centre
(d) will stop

20. An automobile is turning around a circular road of radius r. The coefficient of friction between the tyres and the road is μ. The velocity of the vehicle should not be more than

(a) $\sqrt{\mu g/r}$ (b) $\sqrt{\mu rg}$

(c) μrg (d) $\mu g/r$

21. The roadway bridge over a canal is in the form of a circular arc of radius 18 m. What is the greatest speed with which a motorcycle can cross the bridge without leaving the ground?

(a) $18.98\,\text{ms}^{-1}$ (b) $18/9.8\,\text{ms}^{-1}$

(c) $\sqrt{9.8}\ \text{ms}^{-1}$ (d) $\sqrt{18\times 9.8}\ \text{ms}^{-1}$

22. An electric fan has blades of length 30 cm as measured from the axis of rotation. If the fan is rotating at 1200 rpm, the acceleration of a point on the tip of the blade is about

(a) $2370\,\text{ms}^{-2}$ (b) $5055\,\text{ms}^{-2}$

(c) $1000\,\text{ms}^{-2}$ (d) $4740\,\text{ms}^{-2}$

23. A bottle of soda water is held by the neck and swing briskly in a horizontal circle. Near which position of the bottle do the bubbles collect

(a) in the middle of the bottle
(b) near the bottom
(c) bubbles uniformly distributed
(d) near the neck

24. Two cars c_1 and c_2 are going round concentric circles of radii r_1 and r_2. They complete the circular paths in the same time then $\dfrac{\text{speed of } c_1}{\text{speed of } c_2}$

(a) r_2/r_1 (b) r_1/r_2 (c) 1 (d) none of these

25. A fan has 3 blades. The edges of two blades are 1 m apart. A coin of mass 2g is placed on one of the blade edge. If the fan rotates with 600 rpm, the force experienced by the coin is nearly

(a) 3.1N (b) 4.6N (c) 5.2N (d) 2.8N

26. A stone of mass 1 kg is tied to the end of a string 1 m long. It is whirled in a vertical circle. If the velocity of stone at the top be $4\,\text{ms}^{-1}$. What is the tension in the string at the lowest point? Take $g = 10\,\text{ms}^{-2}$

(a) 6N (b) 66N (c) 16N (d) 76N

27. A body of mass 1 kg is rotating in a vertical circle of radius 1 m. What will be the difference in its kinetic energy at the top and bottom of the circle? Take $g = 10\,\text{ms}^{-2}$

(a) 50 J (b) 30 J (c) 20 J (d) 10 J

28. A thin uniform rod of length l is hinged at the lower end to the level floor and stands vertically. If the rod is allowed to fall, its upper end will strike the floor with a velocity

(a) $\sqrt{3gl}$ (b) $\sqrt{5gl}$ (c) $\sqrt{gl}$ (d) $\sqrt{2gl}$

29. A small block slides down from the top of a hemisphere of radius r as shown in the Fig. 6.24. It is assumed that there is no friction between the block and the hemisphere. At what height h the block will lose contact with the surface of the sphere?

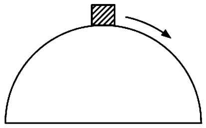


Fig. 6.24

(a) $r/3$ (b) $r/4$ (c) $2r/3$ (d) $r/2$

30. A car is taking a turn on a level road. It may be thrown outwards because of the

(a) reaction of the ground
(b) frictional force
(c) weight
(d) lack of centripetal force

31. The velocity of a particle at highest point of the vertical circle is $\left[\dfrac{g(\sin\theta-\mu\cos\theta)}{r\sin\theta(\cos\theta+\mu\sin\theta)}\right]^{1/2}$. The tension at the lowest point if mass of the particle is m is

(a) $2mg$ (b) $4mg$ (c) $6mg$ (d) $8mg$

32. An inclined plane ends into a vertical loop of radius r as shown in Fig. 6.25. A particle is released from a height $5r$ as shown in Fig. 6.25. The reaction at point C is

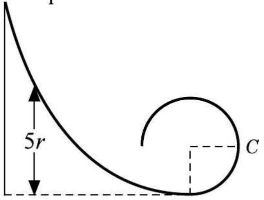


Fig. 6.25

(a) $8mg$ (b) $7mg$ (c) $1mg$ (d) $3mg$

33. A second's pnedulum is suspended in a car that is travelling with a constant speed of $10\,\text{ms}^{-1}$ round a circle of radius 10 m. If the pendulum undergoes small oscillations, the time period will be

(a) 2 s
(b) 4 s
(c) less then 2 s
(d) greater than 2s but less than 4s

34. A hemispherical bowl of radius r is rotated about its axis of symmetry which is kept vertical. A small block is kept at a position where the radius makes an angle θ with the vertical. The block rotates with the bowl without any slipping. The friction coefficient between the block and the bowl is μ. The maximum speed for which the block will not slip

(a) $\left[\dfrac{g(\sin\theta-\mu\cos\theta)}{r\sin\theta(\cos\theta+\mu\sin\theta)}\right]^{1/2}$

(b) $\left[\dfrac{g(\sin\theta+\mu\cos\theta)}{r\sin\theta(\cos\theta+\mu\sin\theta)}\right]^{1/2}$

(c) $\left[\dfrac{g(\sin\theta+\mu\cos\theta)}{r\sin\theta(\cos\theta+\mu\sin\theta)}\right]^{1/2}$

(d) none of these

35. A car is travelling with linear velocity v on a circular road of radius r. If it is increasing its speed at the rate of 'a' ms^{-2} then the resultant acceleration will be

(a) $\sqrt{\dfrac{v^4}{r^2}+a^2}$ (b) $\sqrt{\dfrac{v^4}{r^2}-a^2}$

(c) $\sqrt{\dfrac{v^2}{r^2}+a^2}$ (d) $\sqrt{\dfrac{v^2}{r^2}-a^2}$

ANSWERS

1. (d) **2.** (b) **3.** (b) **4.** (b) **5.** (a) **6.** (c) **7.** (b) **8.** (d) **9.** (b) **10.** (b)
11. (b) **12.** (d) **13.** (d) **14.** (a) **15.** (a) **16.** (a) **17.** (d) **18.** (a) **19.** (a) **20.** (b)
21. (d) **22.** (d) **23.** (b) **24.** (b) **25.** (b) **26.** (b) **27.** (c) **28.** (a) **29.** (c) **30.** (d)
31. (d) **32.** (a) **33.** (c) **34.** (c) **35.** (a)

Work, Power and Energy

7

BRIEF REVIEW OF THE CONCEPTS

Work and Power In our daily life 'work' implies an activity, resulting in muscular or mental exertion. However, in Physics the term 'work' is used when a particle is displaced by the action of a force. Thus, work is a scalar quantity. Its SI unit is Joule, CGS unit erg.

$$1\text{J} = 10^7 \text{ erg}$$

$$dW = \vec{F}.\vec{ds}$$

$$W = F\cdot s = Fs\cos\theta \quad \text{if } F \text{ is constant}$$

$$W = \int \vec{F}.\vec{ds} \qquad \text{if } F \text{ is variable}$$

$$W = \int mdv\cdot\frac{ds}{dt} = \int m\vec{v}.d\vec{v} = \Delta\text{KE}$$

work done on particle by the resultant force acting on it is always equal to change in KE.

Thus for positive work $\text{KE}_{\text{final}} > \text{KE}_{\text{initial}}$

This theorem is valid for all types of forces (external or internal, conservative or non conservative) and is called work energy theorem.

$W = \Delta$PE(change in potential energy) (for conservative forces)

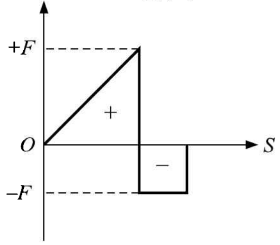


Fig. 7.1

$$W = \frac{1}{2}kx^2$$

where k is spring constant and x is displacement. It could be extension or compression in case of spring.

$$W = \frac{1}{2} \text{ stress} \times \text{strain} \times \text{volume (in elastic bodies.)}$$

Since work is independent of time, the time rate of doing work is defined as Power.

$$P = \frac{dW}{dt} \quad P_{\text{average}} = \frac{W}{t}$$

If force is constant

$$P = \frac{dW}{dt} = \frac{d}{dt}(F\cdot s) = F\cdot\frac{ds}{dt} = \vec{F}\cdot\vec{v}$$

Power is a scalar quantity. Its SI unit is Watt or J/s. Practical unit is HP. 1 HP = 746 W = 550 ft-lb/s.

$$W = \int P\cdot dt \text{ or area under } P - t \text{ curve.}$$

Practical unit of work or energy is kWh (kilowatt-hour).

$1\,\text{kWh} = 3.6 \times 10^6\text{J}$

Kinetic Energy can never be negative.

Potential energy is the energy stored in a body or system by virtue of its position or configuration. In other words, it is a function whose negative gradient is force. Potential energy can be negative or positive. Potential energy is defined only for conservative forces. It does not exist for non-conservative forces.

(a) Elastic potential energy

Potential Energy $= \frac{1}{2}ky^2$ is always positive

(b) Electric potential energy

Potential Energy $= \frac{q_1q_2}{4\pi\epsilon_0 r}$ may be positive or negative

(c) Gravitational Potential Energy

Potential Energy $= \frac{q_1q_2}{4\pi\epsilon_0 r}$ may be positive or negative

$= mgh$ (if height is not very large)

Mechanical energy E = KE + PE which is conserved if internal forces are conservative and no work is done by external forces. If some of the internal forces are non conservative, mechanical energy of the system is not conserved.

Mass-energy E = mc^2 relates mass with energy or vice versa.

Nuclear energy results due to nuclear force between neutrons and protons.

Quantization of energy Planck has shown that radiations emitted by a black body are quantised. Quantum nature of energy is confirmed in atomic and subatomic world. Even light energy is quantised. Thus $E = hf$.

SHORT-CUTS AND POINTS TO NOTE

(i) Work done $W = \vec{F}.\vec{s} = Fs\cos\theta$ (force is constant)

$W = \int F.ds$ (if force is variable)

$W = \Delta PE$ (change in potential energy)

$W = \Delta KE$ (work energy theorem)

$W = 1/2kx^2$ (in a spring)

$W =$ 1/2 stress × strain × volume (in elastic bodies)

$W = \int p\, dV$ where P is pressure and V is volume

(ii) Power $P = \int F.dv$

$P = \frac{dW}{dt}$

$P = .\vec{F}.\vec{v}$

work $W = \int P.dt$

(iii) Potential energy exists only for conservative forces. It does not exist for nonconservative forces. All the central forces are conservative.

(a) Elastic potential energy PE $= 1/2ky^2$ is always positive

(b) Electric potential energy PE $= \frac{q_1q_2}{4\pi\epsilon_0 r}$ may be positive or negative

(c) Gravitational potential energy

PE $= \frac{-Gm_1m_2}{r}$ may be positive or negative

$= mgh$ (if height is not very large)

(iv) If a body is in static or dynamic equilibrium then work done is zero.

(v) If a force is always perpendicular to velocity work done is zero.

(vi) Total mechanical energy of a system remain constant if the internal forces are conservative and external forces do no work.

Total mechanical energy KE + PE is not conserved if nonconservative forces like friction act.

(vii) KE $= \frac{p^2}{2m}$ where p is linear momentum.

(viii) Area under Power-time graph gives work done.

(ix) F $= \frac{-du}{dr}$ for a conservative force and

$\Delta u = -\int F.dr$.

(x) If $\frac{du}{dr} = 0$ body is in equilibrium.

(xi) When a chain hangs by l/n part, l being total length work done to pull it on the table is $mg\ l/2n^2$

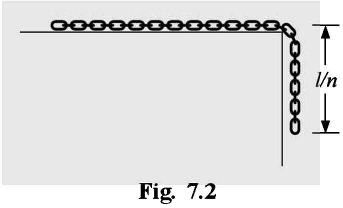


Fig. 7.2

(xii) To find maximum displacement produced in a spring utilize the potential energy of the spring

that is, $\frac{1}{2}kx^2$ = workdone

⚠ CAUTION

(i) Applying $W = \vec{F}.\vec{s}$ even when force is variable.

- ■ When force is variable apply $W = \int F.ds$ or $W = \Delta PE$ or $W = \Delta KE$.

(ii) To find workdone even when the force is perpendicular to the velocity.

- ■ No work is done when force is perpendicular to the velocity.

  For example, no work is done when particle moves in uniform circular motion.

(iii) Considering that when a moving body strikes a spring as shown in Fig. 7.3 it imparts its complete energy to the spring.

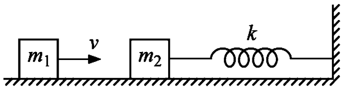


Fig. 7.3

- ■ In spring system first momentum should be conserved.

(iv) To find maximum extension in the spring in the case shown in Fig. 7.4. Applying tension

$T = kx = mg.$

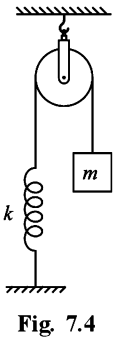


Fig. 7.4

- ■ Apply $mgx = \frac{1}{2}kx^2$ or $x = \frac{2mg}{k}$.

(v) Considering work done by an external force F on spring is always $1/2kx^2$

- Work done by external force is Fx in case the block attached to the spring moves very fast, that is, block accelerates.
  However if the block moves slowly and steadily work done = $\frac{1}{2}kx^2$.

(vi) Conserving energy when nonconservative force is involved.

- If nonconservative forces act on the body and do work then energy cannot be conserved.

(vii) Considering that for a rolling body even KE is $1/2mv^2$

- For a rolling body KE = $1/2mv^2 + 1/2I\omega^2$.

(viii) Assuming that the gravitational potential energy is always mgh.

- If h is small then gravitational potential energy is mgh otherwise gravitational potential energy = $\frac{-GMm}{R+h}$.

(ix) In a system of mutual forces considering KE due to one particle only.

- Take total ΔKE as the sum of $\Delta \cdot (KE_1 + KE_2)$ of both the particles or more particles = ΔPE..

Solved Problems

1. A force $F = 5 + 2x$ acts on a body in x-direction where x is in metres and F in newton. Find the work done in displacing the body from $x = 0$ to $x = 2$ m.

(a) 14 J (b) 18 J

(c) 16 J (d) none of these

Solution (a) $dW = F.dx$

and $$W = \int_{x_1}^{x_2} F.dx = \int_0^2 (5+2x)\,dx$$

$$= \left|5x + x^2\right|_0^2 = 14\text{ J}.$$

2. A block of mass m is pulled along a horizontal surface by applying a force at an angle θ with the horizontal. The friction coefficient between the block and the surface is μ. If the block travels at a uniform velocity then calculate the work done during its displacement d.

(a) $\frac{\mu mgd}{\cos\theta}$ (b) $\frac{\mu mgd}{\sin\theta}$

(c) $\frac{\mu mgd}{\cos\theta + \mu\cos\theta}$ (d) $\frac{\mu mgd}{\cos\theta - \mu\sin\theta}$

Solution (c) For uniform motion

$$F\cos\theta = \mu N$$

$$N = mg - F\sin\theta$$

Thus $F\cos\theta = \mu(mg - F\sin\theta)$

or $F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$

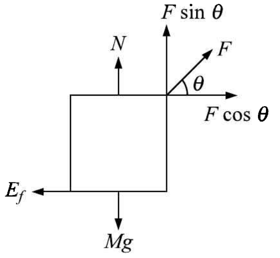


Fig. 7.5

$$W = F \cdot d = \frac{\mu mg.d}{\cos\theta + \mu\sin\theta}.$$

3. A particle of mass 2 kg travels according to equation $3t^3 + 2t$. Find the workdone in 3 seconds of its motion.

(a) 6889 J (b) 6885 J

(c) 1782 J (d) none of these

Solution (b) $W = \Delta KE = \frac{1}{2}m\left[v_f^2 - v_i^2\right]$ $v = \frac{dx}{dt} = 9t^2 + 2$

$$= \frac{1}{2} \times 2[83^2 - 2^2]$$

$v(3) = 83\text{ ms}^{-1}$ and $v(0) = 2\text{ ms}^{-1}$

$$= [6889 - 4] = 6885\text{ J}.$$

4. A uniform chain of length l and mass m overhangs from a smooth table so that 2/3rd part of it is on the table. Find the velocity of the chain when it completely slips off the table.

(a) $\sqrt{2gl}$ (b) $2/3\sqrt{2gl}$

(c) $\sqrt{2gl/3}$ (d) none of these

Solution (b) PE when $\frac{l}{3}$ of chain overhangs

$$PE_i = \int_0^{l/3} -\frac{m}{l}gx\,dx = -\frac{1}{18}mgh$$

PE when the complete chain slips

$$PE_f = \int_0^{l} -\frac{m}{l}gx\,dx = -\frac{1}{2}mgl$$

Loss of $PE = PE_i - PE_f = \frac{4}{9}mgl$

Loss in PE = gain in KE

$$\frac{4}{9}mgl = \frac{1}{2}mv^2 \text{ or } v = \frac{2}{3}\sqrt{2gl}.$$

5. A block of mass m is pushed against a spring of spring constant k fixed to one end of the wall. The block can slide on a frictionless table as shown in Fig. 7.6. The natural length of the spring is l and it is compressed to half its natural length when the block is released. Find the velocity of the block as a function of its distance x from the wall.

(a) $\sqrt{\frac{k}{m}\left[\frac{l^2}{4} - (l-x)^2\right]}$ (b) $\sqrt{\frac{k}{m}(l-x)^2}$

(c) $\sqrt{\frac{k}{m}\left(\frac{l^2}{2}-(l-x)^2\right)}$ (d) none of these

Solution (a) Assume at any instant the block is at a place x, then applying, the law of conservation of energy

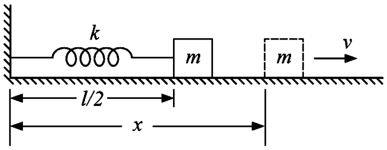


Fig. 7.6

$$\frac{1}{2}k\left(\frac{l}{2}\right)^2=\frac{1}{2}k(l-x)^2+\frac{1}{2}mv^2$$

or $$v=\sqrt{\frac{k}{m}\left[\frac{l^2}{4}-(l-x)^2\right]}.$$

when the spring acquires the length $x = l$, then the block moves with a constant velocity

$$v=\sqrt{\frac{k}{m}}\frac{l}{2}.$$

6. A pump motor is used to deliver water at a certain rate from a given pipe. To obtain n-time water from the same pipe in the same time by what amount (a) the force and (b) power of the motor be increased.

(a) n^2F, nP (b) nF, nP
(c) nF, n^2P (d) n^2F, n^3P

Solution (d) $\frac{dm}{dt}=Av\rho$

where A is area of cross-section of pipe v is velocity with which water is pumped out and ρ is density of the liquid.

To get n times water

$$\frac{dm'}{dt}=n\frac{dm}{dt}$$

$Av'\rho = nAv\rho$ or $v' = nv$

(a) as $F=v\frac{dm}{dt}$ $\therefore$ $\frac{F'}{F}=\frac{v'\frac{dm'}{dt}}{v\frac{dm}{dt}}=n^2$ or $F'=n^2F$

(b) as $P = F\cdot v$ $\therefore$ $\frac{P'}{P}=\frac{(n^2F).nv}{F.v}=n^3$ or $P'=n^3P.$

7. The momentum of a body is increased by 20%. Find the percentage increase in kinetic energy.

(a) 40% (b) 22%
(c) 41% (d) 44%

Solution (d) initial KE $=\frac{P^2}{2m}$

new momentum $=P+\frac{P}{5}=\frac{6P}{5}$

Final KE $=\frac{(6/5P)^2}{2m}=\frac{36}{25}\frac{P^2}{2m}$

Increase in KE $=\frac{36}{25}\frac{P^2}{2m}-\frac{P^2}{2m}=\frac{11}{25}\left(\frac{P^2}{2m}\right)$

% Increase in KE $=\frac{\frac{11}{25}\left(\frac{P^2}{2m}\right)}{P^2/2m}\times 100 = 44\%.$

8. The earth moves round the sun once a year. How much work would have been done on the earth to bring it to rest relative to the sun (ignore the rotation of earth about its own axis) Given mass of the earth 6×10^{24}kg and distance between sun and earth 1.5×10^{11}m.

(a) 27×10^{24}J (b) 26×10^{28}J
(c) 27×10^{32}J (d) none of these

Solution (c) $\omega=\frac{2\pi}{T}=\frac{2\pi}{3.15\times10^7}=1.99\times10^{-7}\text{rads}^{-1}$

$v = r\omega = 1.5 \times 10^{11} \times 1.99 \times 10^{-7}$
$\simeq 3 \times 10^4\text{ms}^{-1}$

Work done to stop the sun is

$W=\frac{1}{2}mv^2=\frac{1}{2}\times 6\times 10^{24}\times(3\times10^4)^2=27\times10^{32}$J.

It is done opposite to the direction of motion of the earth.

9. A particle of mass m is moving in a horizontal circle of radius r, under a centripetal froce k/r^2 where k is constant. Find the total energy of the particle.

(a) $-k/r$ (b) k/r
(c) $k/(2r)$ (d) $-k/(2r)$

Solution (d) $\frac{mv^2}{r}=\frac{k}{r^2}$ $KE=\frac{1}{2}mv^2=\frac{k}{2r}$

or $PE=\int_\infty^r -F.dr=\frac{-k}{r}$

Total energy E = PE + KE $=\frac{-k}{r}+\frac{k}{2r}=-\frac{k}{2r}$ Negative sign shows bound system.

10. A spring gun having a spring of spring constant k is placed at a height h. A ball of mass m is placed in its barrel and compressed by a distance x. Where shall we place a box on the ground so that the ball lands in the box?

(a) $\frac{kh}{mg}x$ (b) $\sqrt{\frac{2kh}{mg}}x$
(c) $\sqrt{\frac{kh}{2mg}}x$ (d) $\frac{kh}{2mg}$

Solution (b) $\frac{1}{2}mv^2=\frac{1}{2}kx^2$ $v=\sqrt{\frac{k}{m}}x$

Time taken to fall $t=\sqrt{\frac{2h}{g}}$

so the horizontal distance travelled is

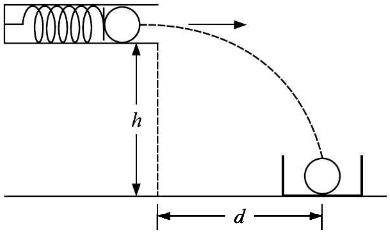


Fig. 7.7

$$d = v.t = \sqrt{\frac{k}{m}} \times \sqrt{\frac{2h}{g}}\, x = \sqrt{\frac{2kh}{mg}}\, x.$$

11. In a water fall the water falls from a height 200 m. Calculate the temperature of water risen when it touches the ground.

(a) 0.24°C (b) 0.48°C

(c) 0.61°C (d) 0.58°C

Solution (b) $mgh = mc\,\Delta\theta$

or $$\Delta\theta = \frac{gh}{c} = \frac{10 \times 200}{4200} = 0.48°\text{C}.$$

12. A man weighing 60 kg climbs up a staircase with 20 kg weight on his head. The staircase has 20 steps each being 15 cm high. If he takes 10 s to climb, find the power.

[Take g = 9.8]

(a) 325.2 W (b) 235.2 W

(c) 123.5 W (d) 202.5 W

Solution (b) $P = \frac{(M+m)\,gh}{t} = \frac{80 \times 9.8 \times 3}{10} = 235.2\text{W}.$

13. Atomic power station at Tarapur has a generating capacity of 200 MW. The energy generated in a day is

(a) 1.725×10^{10}J (b) 172.8×10^{9}J

(c) 1728×10^{10}J (d) none of these

Solution (c) $200 \times 10^6 \times 24 \times 60 \times 60 = 1728 \times 10^{10}$J.

14. Three identical masses are placed at the vertices of an equilateral triangle of side l. Find the workdone to displace the masses to the vertices of an equilateral triangle of side $2l$.

(a) $\frac{3Gm^2}{l}$ (b) $\frac{3Gm^2}{2l}$

(c) $\frac{3Gm^2}{4l}$ (d) $\frac{26Gm^2}{3l}$

Solution (b) Work done to displace the particle on one side = change in potential energy

$$W_1 = Gm^2\left[\frac{-1}{2l} + \frac{1}{l}\right] = \frac{Gm^2}{2l}$$

Total work done on three sides $W = 3W_1 = \frac{3Gm^2}{2l}$.

15. Two blocks are connected by a string as shwon in Fig. 7.8. Thus after moving a distance l is

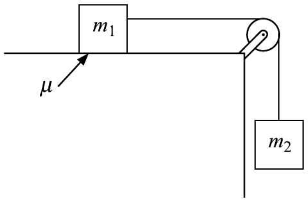


Fig. 7.8

Solution Loss in potential energy = Gain in KE + Work done against friction

$$m_2gl = \frac{1}{2}(m_1 + m_2)v^2 + \mu m_1 gl$$

or $$v = \sqrt{\frac{2(m_2 - \mu m_1)gl}{m_1 + m_2}}.$$

16. If the kinetic energy of a particle increases by 44%. Find the increase in momentum.

(a) 44% (b) 20%

(c) 12% (d) 32%

Solution (b) In $\text{KE}_f = \text{KE}_i + 0.44\text{KE}_i$

$= 1/2mv^2(1 + .44) = 1/2m(1.2v)^2$

Increase in velocity = $0.2v$ or 20%

∴ momentum increases by 20%.

17. A bullet loses $\frac{1}{20}$th of its velocity when it passes through a plank. Find the maximum number of planks through which it can pass.

(a) 19 (b) 20

(c) 11 (d) 10

Solution (d) velocity left $= v - \frac{v}{20} = \frac{19v}{20}$

KE lost = Work done in passing through a plank

$$\frac{1}{2}m\left[v^2 - \left(\frac{19v}{20}\right)^2\right] = F \cdot t$$

Let t be the thickness of a plank and F be the resistive force

$$\frac{mv^2}{2}\left[\frac{39}{400}\right] = F \cdot t$$

To loose $$\frac{mv^2}{2} = nFt = \frac{400}{39}\,Ft$$

$$n = \frac{400}{39} = 10.25$$

that is, it will pass through 10 planks.

18. A block of mass m falls through a height h on a spring of spring constant k. The maximum displacement in the spring is

(a) $\frac{2mgh}{k}$ (b) $\frac{mg}{k}$

(c) $\sqrt{\frac{2mgh}{k}}$ (d) none of these

Solution (d) If the spring is displaced by x then

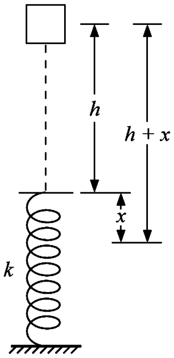


Fig. 7.9

$$mg(h + x) = \frac{1}{2}kx^2$$

or $$x = \frac{1}{2}\left[\frac{2mg}{k} \pm \sqrt{\left(\frac{2mg}{k}\right)^2 + \frac{8mgh}{k}}\right].$$

19. The heart of a man pumps 4l blood per minute at 120 mm of Hg. The power of the pump (heart) is

(a) 1.28 W (b) 1.18 W
(c) 1.08 W (d) 0.98 W

Solution (c) Pressure $p = h\rho g$

$= 120 \times 10^{-3} \times 13.6 \times 10^{3} \times 9.8$

Volume of blood $= 4l = 4 \times 10^{-3}\text{m}^3$

Power $P = \frac{p\Delta V}{t} = \frac{120\times13.6\times9.8\times4\times10^{-3}}{60} = 1.08\,\text{W}$.

20. When a spring is stretched by 2 cm its potential energy is u. When it is stretched by 10 cm then the potential energy will be

(a) $\frac{4}{5}u$ (b) $5u$
(c) $25u$ (d) $10u$

Solution (c) $u = \frac{k(2)^2}{2}$ $u' = \frac{k(10)^2}{2}$ $\therefore u' = 25u$.

21. A car weighing 500 kg working against a resistance of 500 N accelerates from rest to 20 ms^{-1} in 10 s (Take $g = 10\,\text{ms}^{-2}$). The work done by the engine will be

(a) 1.05×10^5 J (b) 1.0×10^5 J
(c) 1.5 J (d) 1.5×10^5

Solution (d) $F = ma + F_f = 500 \times 2 + 500 = 1.5 \times 10^3$

$v = u + at$ $20 = 0 + a \times 10$

$a = 2\,\text{ms}^{-2}$

$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 2\,(10)^2 = 10^2$

$W = F\cdot s = 1.5 \times 10^5\text{J}$.

22. A wire suspended vertically from one end is stretched by attaching a weight of 200 N to the lower end. The weight stretches the wire by 1 mm. The elastic energy stored in the wire is

(a) 10 J (b) 20 J
(c) 0.1 J (d) 0.2 J

Solution (c) Elastic energy stored $= \frac{1}{2} \times$ stress $\times$ strain $\times$ volume

$= \frac{1}{2}\frac{F\times\Delta l}{2} \times (A \times l) = \frac{F\times\Delta l}{2}$

$= \frac{200\times10^{-3}}{2} = 0.1$ J.

23. A car moving with a speed of 50 kmh^{-1} can be stopped by brakes after at least 6 m. If the same car is moving with 100 kmh^{-1}, the minimum stopping distance is

(a) 6 m (b) 12 m
(c) 18 m (d) 24 m

Solution (d) $s = \frac{v^2}{2a}$ or $\frac{s_1}{s_2} = \frac{2v_1^2}{2v_2^2}$

as breaking force and hence deceleration produced is constant

$s_2 = 6 \times 4 = 24$ m.

24. The spring constant of a spring balance is $5 \times 10^3\,\text{Nm}^{-1}$. It is initially stretched by 5 cm from the unstretched position. Then the work done to stretch it further by another 5 cm is

(a) 18.75 J (b) 25.0 J
(c) 6.25 J (d) 12.5 J

Solution (a) $\Delta W = \frac{1}{2}\,k(x_2^2 - x_1^2)$

$= \frac{5\times10^3}{2}\,[(.1)^2 - (.05)^2]$

$= 18.75$ J.

Problems for Practice

1. A body is thrown vertically upwards from the surface of earth in such a way that it reaches up to a height equal to 10 R_e. The velocity imparted to the body will be

(a) zero (b) 0.106 kms^{-1}
(c) 1.06 kms^{-1} (d) 10.6 kms^{-1}

2. Two bodies have same kinetic energy. They are stopped by applying same retarding force. Which body will stop first?

(a) lighter one (b) heavier one
(c) both (d) cannot be predicted

3. In hydrogen atom the radius of the orbit of electron changes from r_1 to r_2 and angular frequency changes from ω_1 to ω_2. The ratio of ω_1 to ω_2 will be

(a) r_1/r_2 (b) $(r_1/r_2)^2$
(c) $(r_2/r_1)^2$ (d) $(r_2/r_1)^3$

4. Water from a stream is falling on the blades of a turbine at the rate of 100 kgs^{-1}. If the height of the stream is 100 m then the power delivered to the turbine is

(a) 1 kW (b) 100 W (c) 10 kW (d) 100 kW

5. A ball falls from a height of 5 m and strikes the roof of a lift. If at the time of collision, lift is moving in the upward direction with a velocity of 1 ms^{-1}, then the velocity with which the ball rebounds after collision will be

(a) 13 ms^{-1} upwards
(b) 12 ms^{-1} downwards
(c) 11 ms^{-1} downwards
(d) 12 ms^{-1} upwards

6. The kinetic energy of a man is half the kinetic energy of a boy of half of his mass. If the man increases his speed by 1 ms^{-1} then his kinetic energy becomes equal to that of the boy. The ratio of the velocity of the boy and that of the man is

(a) 3/4 (b) 4/3 (c) 1/2 (d) 2

7. A block of mass m moving with velocity v strikes another identical mass connected to a spring as shown in Fig. 7.10. The maximum displacement produced in the spring (Assume surfaces to be smooth) is

(a) $\sqrt{\frac{mv^2}{2k}}$ (b) $\sqrt{\frac{mv^2}{k}}$

(c) $\sqrt{\frac{mv^2}{3k}}$ (d) none of these

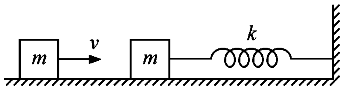


Fig. 7.10

8. A person of mass 60 kg carries a 15 kg body on the top of building 10 m high in 5 minutes. He puts a power in carrying the body
(a) 10 W (b) 5 W (c) 30 W (d) 15 W

9. A body of mass 2 kg is lying on a plane indined at an angle of 30° with the horizontal, the coeficient of static friction between the plane and the body is 0.7. The frictional force acting on the body will be
(a) $0.7 \times 9.8 \times \sqrt{3}$ N (b) 0.7×9.8 N
(c) 9.8 N (d) $9.8 \times \sqrt{3}$ N

10. A truck moves up an incline rising 1 : 10 with a velocity $10\,\text{ms}^{-1}$. The coefficient of friction is 0.2 mass of the truck is 20 tonne and the efficiency of the engine is 60%. Find the power of the engine.
(a) 6×10^5 W (b) 4×10^6 W
(c) 7.5×10^6 W (d) 10^6 W

11. A 50 gm bullet moving with a velocity of $10\,\text{ms}^{-1}$ gets embeded into a 950 g stationery body. The loss in kinetic energy of the system will be
(a) 100% (b) 95%
(c) 50% (d) 5%

12. The velocity of a rocket jet is $600\,\text{ms}^{-1}$. If 300 kg gas is coming out of the rocket per second then the force acting on the rocket will be
(a) 18×10^{-5} N (b) 8×10^5 N
(c) 18×10^5 N (d) 1.8×10^5 N

13. A block of mass 16 kg is moving on a frictioless horizontal surface with velocity $4\,\text{ms}^{-1}$ and comes to rest after pressing a spring. If the force constant of the string is $100\,\text{Nm}^{-1}$ then compression in the spring will be
(a) 0.6 m (b) 1.6 m (c) 3.2 m (d) 6.1 m

14. A 10 kg satellite completes one revolution around the earth at a height of 100 km in 108 minutes. The work done by the gravitational force of earth will be
(a) 0 J (b) $\frac{100 \times 10}{108}$ J
(c) $\frac{108 \times 10}{100}$ J (d) $108 \times 100 \times 10$ J

15. A 1 kg ball falls from a height of 25 cm and rebounds up to a height of 9 cm. The coefficient of restitution is
(a) 0.6 (b) 0.32 (c) 0.40 (d) 0.56

16. A crane lifts 300 kg weight from earth's surface up to a height of 2 m in 3 seconds. The average power generated by it will be
(a) 0 W (b) 1960 W
(c) 2205 W (d) 4410 W

17. A 55 gm tennis ball strikes the ground and rebounds back. If its velocities are $5\,\text{ms}^{-1}$ and $4\,\text{ms}^{-1}$ before and after the collision respectively then the impulse imparted will be
(a) –495 Ns (b) –0.495 Ns
(c) 0.495 Ns (d) 495 Ns

18. A car is mvoing with a speed of $40\,\text{kmh}^{-1}$. If the car engine generates 7 kilowatt power the resistance in the path of motion of the car will be
(a) zero (b) 280 N (c) 360 N (d) 630 N

19. A ball of mass 1 kg is released from the tower of pisa. The kinetic energy generated in it after 10 m will be
(a) 0.98 J (b) 9.8 J (c) 10 J (d) 98 J

20. The retarding force required to reduce velocity of a 3 kg body from $0.75\,\text{ms}^{-1}$ to $0.25\,\text{ms}^{-1}$ in 0.02 seconds will be
(a) 100 N (b) 75 N (c) 50 N (d) 25 N

21. A force of 10 N displaces a body by 6 m in 3 seconds. The power of the agency applying the forces is
(a) 180 W (b) 5 W
(c) 1.8 W (d) none of these

22. An engine pumps out 40 kg of water through a height of 10 m in 1 s. Given that the efficiency of the engine is 60%. What is the power of the engine? (Take $g = 10\,\text{ms}^{-2}$)
(a) 3.3 kJ (b) 6.33 kJ (c) 800 kJ (d) none of these

23. How much work is done in raising a stone of mass 5 kg and relative density 3 lying at the bed of lake through a height of 5 metre? (Take $g = 10\,\text{ms}^{-2}$)
(a) 75 J (b) $\frac{980}{3}$ J (c) $\frac{250}{3}$ J (d) $\frac{500}{3}$ J

24. A person holds a bucket of weight 60 N. He walks 7 m along the horizontal and then climbs up a vertical distance of 5 m. The work done by the man is
(a) 720 J (b) 420 J (c) 300 J (d) none of these

25. A train of weight 10^7 N is running on a level track with uniform speed of $36\,\text{kmh}^{-1}$. The frictional force is 0.5 kgf per quintal. What is the power of engine? $g = 10\,\text{ms}^{-2}$
(a) 500 kW (b) 50 kW
(c) 5 kW (d) 0.5 kW

26. A particle moves up a rough incline of inclination 37° along the plane by 2 m and returns back to the starting point. $\mu = 0.4$ The net work done by the body is ... (mass of the body = 2 kg)
(a) zero (b) 25.6 J (c) 73.6 J (d) none of these

27. A uniform chain of length l and mass m is lying on a smooth table. One fourth of its length is hanging vertically down, over the edge of the table. How much work need to be done to pull the hanging part back to the table.
(a) $mgl/8$ (b) $mgl/32$ (c) mgl (d) $mgl/2$

28. The unit of power SI (watt) is equivalent to
(a) $\text{kg}\,\text{m}^2\text{s}^{-3}$ (b) $\text{kg}\,\text{ms}^{-2}$
(c) $\text{kg}\,\text{m}^2\text{s}^{-2}$ (d) none of these

29. A block is moved from rest through a distance of 4 m along a straight line path. The mass of the block is 5 kg and the force acting on it is 20 N. If the kinetic energy acquired by the block be 40 J, at what angle to the path the force is acting?
(a) 60° (b) 45° (c) 30° (d) none of these

30. A heavy weight is suspended from the spring. A person raises the weight till the spring becomes slack. The work done by him is W. The energy stored in the stretched spring was E. What will be the gain in gravitational potential energy?
(a) $W + E$ (b) $W - E$
(c) W (d) E

31. A body starts from rest and acquires a velocity v in time T. The work done on the body in time t will be proportional to
(a) $\frac{v^2}{T^2}t$ (b) $\frac{v^2}{T^2}t^2$ (c) $\frac{v}{T}t$ (d) $\frac{v^2}{T}t^2$

32. A man drags a box with uniform speed across a 10 m rough floor. The coefficient of friction of the floor is 0.5. If the man pulls the box with a force of $\sqrt{3}$ kN at an angle 30° with the horizontal, what is the work done in draging the box?

(a) $\frac{15}{\sqrt{3}}$ kJ (b) 5 k J (c) 30 k J (d) 15 k J

33. A body moving at $2\,\text{ms}^{-1}$ can be stopped over a distance x. If the kinetic energy of the body is doubled, how long will it go before coming to the rest, if the retarding force remains unchanged?
(a) $8x$ (b) $4x$ (c) $2x$ (d) x

34. The linear momentum of a particle increases by 50%. The increase in kinetic energy should be
(a) 225% (b) 100% (c) 50% (d) 125%

35. A car is acclerated from rest to a speed of $10\,\text{ms}^{-1}$. Let the energy spent be E. If we accelerate the car from $10\,\text{ms}^{-1}$ to $20\,\text{ms}^{-1}$, the energy spent will be
(a) $4E$ (b) $3E$ (c) $2E$ (d) E

36. The kinetic energy of a body decreases by 19%. What is the percentage decrease in momentum?
(a) 5% (b) 10% (c) 15% (d) 20%

37. A block of mass m slides down the surface of a bowl of radius r from its rim to the bottom. What will be the kinetic energy of the block at the bottom?
(a) $mgr/2$ (b) $mgr/4$
(c) $2mgr$ (d) mgr

38. A body falls freely under gravity. Its velocity is v when it has lost potential energy equal to PE. What is the mass of the body?
(a) $2\text{PE}/v^2$ (b) PE/v^2
(c) PE^2/v^2 (d) $2\text{PE}^2/v^2$

39. The momentum of particle is numerically equal to its kinetic energy. What is the velocity of the particle?
(a) $9\,\text{ms}^{-1}$ (b) $3\,\text{ms}^{-1}$
(c) $2\,\text{ms}^{-1}$ (d) $1\,\text{ms}^{-1}$

40. A bullet is fired on a wooden block. Its velocity reduces by 50% on peneterating 30 cm. How long will it peneterate further before coming to rest?
(a) 5 cm (b) 10 cm
(c) 15 cm (d) 30 cm

41. A body of mass 2 kg is projected vertically upwards with a speed of $3\,\text{ms}^{-1}$. The maximum value of the gravitational potential energy by the body is
(a) 2.25 J (b) 4.5 J (c) 9 J (d) 18 J

42. The magnitude of the momentum of a particle is increased by 100%. If the initial kinetic energy be KE, then what will be the kinetic energy now?
(a) $2KE$ (b) $3KE$ (c) $4KE$ (d) $8KE$

43. A body is acted upon by a force which varies inversely as the distance (x) covered. The work done on the body is proportional to
(a) x^2 (b) x (c) $\sqrt{x}$ (d) none of these

44. The displacement x and time t far a particle are related to each other as $t = \sqrt{x} + 3$. What is the work done in first six seconds of its motion if mass is 2 kg.
(a) 2 J (b) 4 J (c) 6 J (d) zero

45. A vehicle is moving on a rough horizontal road with velocity v. The stopping distance will be directly proportioanl to
(a) v^3 (b) v^2 (c) v (d) $\sqrt{v}$

46. A constant force F is applied on a body. The power (P) generated is related to the time elapsed (t) as
(a) $P \propto t^{3/2}$ (b) $P \propto \sqrt{t}$
(c) $P \propto t$ (d) $P \propto t^2$

47. A gun of mass M fires a bullet of mass m. What will be the ratio of the recoil energy of the gun to that of the bullet?
(a) $\frac{M}{M+m}$ (b) $\frac{M}{m}$
(c) $\frac{m}{M}$ (d) $\frac{m}{m+M}$

48. A particle of mass M is moving in a horizontal circle of radius r under the centripetal force equal to k/r^2, where k is a constant. What is the total energy of the particle?
(a) $-k/r$ (b) $-k/2r$ (c) $k/2r$ (d) k/r

49. A simple pendulum of length 1 m has a bob of 200 g. It is displaced through 60° and then released. What will be its kinetic energy when it passes through the mean position?
(a) 2.0 J (b) 1.5 J (c) 1.0 J (d) 0.5 J

50. A shell of mass $4m$ at rest explodes into three fragments. Two fragments each of mass m move with speed v in mutually perpendicular directions. What is the amount of energy released in the process?
(a) mv^2 (b) $1/2\,mv^2$
(c) $3\,mv^2$ (d) $\frac{3}{2}\,mv^2$

51. The kinetic energy of a body varies directly as the time (t) elapsed. The force acting varies directly as
(a) $1/\sqrt{t}$ (b) $\sqrt{t}$ (c) $1/t$ (d) t

52. An elastic string of unstretched length l and force cosntant k is stretched by a small amount x. It is further stretched by another small length y. What is the work done in second stretching?
(a) $1/2\ k\ (y^2 - x^2)$ (b) $1/2\ ky\ (2x + y)$
(c) $1/2\ ky^2$ (d) $\frac{1}{2}\ k(x^2 + y^2)$

53. An automobile engine of mass m accelerates and a constant power P is applied by the engine. The instantaneous speed of the engine will be
(a) $[Pt/2m]^{1/2}$ (b) $[Pt/4m]^{1/2}$
(c) $[Pt/m]^{1/2}$ (d) $[2Pt/m]^{1/2}$

54. Sea water strikes a sea wall with a velocity of 20 ms^{-1}. The force acting per m^2 on the wall will be
(a) 412×10^5 N (b) -412×10^5 N
(c) 4.12×10^5 N (d) 4.00×10^5 N

55. A spherical ball of radius r and specific gravity 0.5 is floating in water and is in equilibrium such that half of it is inside water. The ball is pressed so that it is just submerged. The work done will be (d is density of water).
(a) $n\pi r^4 dg$ (b) $\pi dg/r^4$
(c) $5/12\pi r^4 dg$ (d) $12/5\pi r^4 dg$

56. Which of the following quantities is conserved if $\Sigma\tau = 0$
(a) only $\vec{P}$ (b) only $\vec{L}$
(c) both $\vec{L}$ and $\vec{P}$ (d) both $\vec{J}$ and KE

57. A bomb of mass 9 kg explodes into two pieces of 3 kg and 6 kg. The velocity of 3 kg piece is $16\,\text{ms}^{-1}$. The kinetic energy of 6 kg piece is
(a) 192 J (b) 687 J (c) 768 J (d) 786 J

58. Two electrons are brought closer together. The potential eergy of the system will be
(a) zero (b) less (c) more (d) infinity

59. A ball is dropped from a hegith of 10 m. If 40% of its energy is lost on collision with the earth then after collision the ball will rebound to a height of

(a) 4 m (b) 6 m (c) 8 m (d) 10 m

60. A ball, with initial kinetic energy *KE*, is projected at an angle of 45° from the horizontal. Its kinetic energy at the highest point of motion will be

(a) $KE/\sqrt{2}$ (b) KE
(c) $KE/2$ (d) zero

61. A body of mass m is accelerated to velocity v in time t'. The work done by the force as a function of time t will be

(a) $\frac{m}{2}\frac{v^2t^2}{t'^2}$ (b) $\frac{mvt^2}{2t'}$
(c) $\frac{m}{2}\frac{v}{t'}t^2$ (d) $\frac{1}{2}\left(\frac{mv}{t'}\right)^2 t^2$

62. A ship of mass 3×10^7 kg is initially at rest. It is being pulled by a force of 5×10^4 N through a distance 3 m. If the air resistance is negligible, then the speed of the ship will be

(a) 0.1 ms^{-1} (b) 1.5 ms^{-1}
(c) 5 ms^{-1} (d) 60 ms^{-1}

63. A particle moves under the influence of a force $F = cx$ from $x = 0$ to $x = x_1$. The work done in this process will be

(a) cx_1^3 (b) $\frac{cx_1^2}{2}$ (c) cx_1^2 (d) zero

64. Two masses of 1 gm and 4 gm are moving with the same kinetic nergy. The ratio of their linear momenta will be

(a) 1 : 16 (b) 1 : 2
(c) $\sqrt{2}$: 1 (d) 4 : 1

65. An electric motor produces a tension of 4500 N in a load lifting cable and rolls it at the rate of 2 ms^{-1}. The power of the motor is

(a) 9×10^3 HP (b) 225 KW
(c) 15 KW (d) 9 KW

66. In the tug of war team *A* is slowly being defeated by team *B*. Net work is being done by

(a) team *B* (b) team *A*
(c) both (d) none of these

67. A car starts from rest with an acceleration α on a circular road of radius r. The work done by the car in t seconds is ... if mass of the car is m

(a) $\frac{1}{2}m\alpha^2t^2r^2$ (b) $\frac{1}{2}m\alpha^2r^2t$
(c) $m\alpha^2r^2t^2$ (d) none of these

68. A spring of force constant k is cut into two halves and they are connected as shown in Fig. 7.11. The block is displaced by x and released. The velocity of the block at $x/2$ is

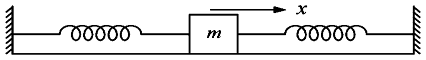


Fig. 7.11

(a) $\sqrt{\frac{3kx^2}{4m}}$ (b) $\sqrt{\frac{7kx^2}{2m}}$
(c) $\sqrt{\frac{7kx^2}{4m}}$ (d) none of thesee

69. A motor car of 100 HP is moving with a constant velocity of 72 kmh^{-1}. The forward force exerted by the engine on the car is

(a) 2.73×10^1 N (b) 2.73×10^2 N
(c) 2.73×10^3 N (d) 2.73×10^9 N

70. A gun of mass 20 kg has bullet of mass 0.1 kg in it. The gun is free to recoil. 804 Joules of recoil energy are released on firing the gun. The speed of bullet in ms^{-1} is

(a) $\sqrt{804 \times 2010}$ (b) $\sqrt{804 \times 4 \times 10^3}$
(c) $\sqrt{\frac{804}{2010}}$ (d) $\sqrt{\frac{2010}{804}}$

71. A pump pulls 1000 kg water per minute from a 15 m deep well and provides 4 ms^{-1} velocity to it. The power of pump is (Take $g = 10$ ms^{-2})

(a) 0.6 *W* (b) 0.6 *KW*
(c) 2.6 *KW* (d) 2.6 *W*

72. A 2 kg mass lying on a table is displaced in the horizontal direction through 50 cm. The work done by the normal reaction will be

(a) 0 (b) 10 Joule
(c) 100 erg (d) 100 Joule

73. A particle of mass 2 kg moves according to the equation $x = t^3 - 2t$. The work done in first 2 seconds is

(a) 100 J (b) 144 J (c) 96 J (d) 104 J

74. The maximum and minimum distance of a planet from sun are 14×10^9 km and 7×10^7 km respectively. If the maximum speed of the planet is 8×10^2 kms^{-1} then its minimum speed will be

(a) zero (b) 0.4 kms^{-1}
(c) 2 kms^{-1} (d) 4 kms^{-1}

75. The work done by a compound pendulum in one complete oscillation is

(a) $2\pi\sqrt{\frac{l}{g}}$ J (b) $\pi\sqrt{\frac{l}{g}}$ J
(c) 10 J (d) 0

76. A radioactive nucleus intially at rest decays by emitting an electron and a neutrino in mutually perpendicular directions. The momentum of the recoil nucleus will be

(a) $\sqrt{p_e^2 + p_\nu^2}$ (b) $p_e p_\nu$
(c) $\frac{p_e}{p_\nu}$ (d) $\frac{p_\nu}{p_e}$

77. A spring of spring constant k is given a sharp blow by a force F so that the spring shows a compression x. Then work done on the spring is

(a) $\frac{1}{2}kx^2$ (b) Fx
(c) $\frac{kx^2}{4}$ (d) none of these

ANSWERS

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (d)
11. (b)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (c)	**18.** (d)	**19.** (d)	**20.** (b)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (a)
31. (b)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (b)	**36.** (b)	**37.** (d)	**38.** (a)	**39.** (c)	**40.** (b)
41. (c)	**42.** (c)	**43.** (d)	**44.** (d)	**45.** (b)	**46.** (c)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (d)
51. (a)	**52.** (b)	**53.** (d)	**54.** (d)	**55.** (c)	**56.** (b)	**57.** (a)	**58.** (c)	**59.** (b)	**60.** (c)
61. (a)	**62.** (a)	**63.** (b)	**64.** (b)	**65.** (d)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (c)	**70.** (b)
71. (c)	**72.** (a)	**73.** (c)	**74.** (d)	**75.** (d)	**76.** (a)	**77.** (b)			

Conservation of Momentum

8

BRIEF REVIEW OF THE CONCEPTS

Momentum Linear momentum of a particle is a physical quantity with state of motion and is defined as the product of mass and velocity. (not speed)

that is, $\vec{p} = m\vec{v}$

Though the word linear is not commonly employed but it is sometimes used to distinguish it from angular momentum.

Units of linear momentum are kgms^{-1} or N–s.

Linear momentum depends upon the frame of reference for example, the linear momentum of a body at rest in a moving train, is zero relative to a person sitting in the train while it is not zero for a person standing on the ground.

Two bodies of same mass and moving with same speed will have different momentum unless their directions of motion are same.

For a body or a particle $\vec{p} = m\vec{v}$, so the linear momentum will change if mass or velocity or both change.

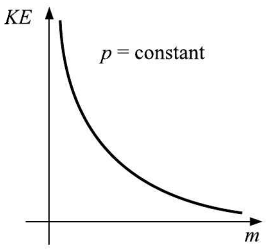
(a) If m = constant then $p \propto v$, that is, if different bodies have same mass, the momentum will be largest for the fastest moving body.

(b) If v = constant then $p \propto m$, that is, If different bodies are moving with same velocity then momentum is largest for the heaviest body.

(c) If p = constant then $v \propto 1/m$, that is, if different bodies have same momentum the lightest body will have greatest velocity.

$$\frac{p^2}{2m} = \text{(KE)} \qquad \therefore\ p = \sqrt{2(KE)m}$$

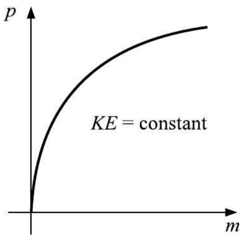
(d) If p = constant kinetic enercy $\propto 1/m$, that is, if different bodies have same momentum, kinetic enercy will be maximum for lightest body.



(a)

(e) If kinetic enercy = constant then $p \propto \sqrt{m}$

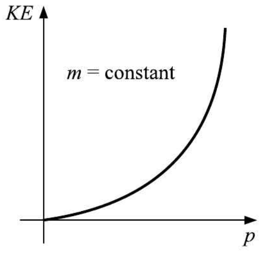
that is, different bodies have same kinetic enercy, then heaviest body will possess maximum momentum.



(b)

(f) If m = constant then $p \propto \sqrt{(KE)}$

that is,, if different bodies have same mass, the body having largest KE will have maximum momentum.



(c)

Fig. 8.1

Since $\vec{F} = \dfrac{d\vec{p}}{dt}$, the slope of $p - t$ curve will yield force and the area under $F - t$ curve will give the change in momentum.

$p = \dfrac{h}{\lambda}$ for a particle wave and $p = \dfrac{E}{c}$ for photons.

Law of conservation of linear momentum In an isolated system (if no external force acts) of constant mass (closed) the total momentum of the system remains constant (with time). According to this law Σp_i = constant.

This equation clarifies that linear momentum of various particles may change but their sum remains unchanged.

Law of conservation of momentum is independent of frame of reference though momentum depends on the frame of reference.

Conservation of linear momentum is equivalent to Newton's third law of motion

$$\vec{p}_1 + \vec{p}_2 = \text{constant}$$

Differentiating with respect to time

$$\frac{d\vec{p}_1}{dt}+\frac{d\vec{p}_2}{dt} = 0 \quad \text{or} \quad F_1 + F_1 = 0 \quad \text{or} \quad F_2 = -F_1$$

that is, for every action there is an equal and opposite reaction.

Law of conservation of momentum is universal, that is, it applies to both microscopic and macroscopic particles. It holds good even in atomic and nuclear physics where classical physics fails.

In collisions, we consider the situation before and after collision. The term before collision and after collision refers to the conditions when the interaction force between the particles become effectively zero. The duration of collision is negligibly small as compared to the time for which event is observed. During collision internal forces act between the colliding particles.

If the motion of the colliding particles before and after impact remains in the same line, the collision is said to be 'direct' or 'head on' or 'one dimensional'.

If the two particles after collision do not maintain the same line of motion, the collision is called oblique. If in oblique collision before and after the collision same plane is maintained the collision is called two dimensional otherwise three dimensional.

In collision the effect of external forces such as friction and gravity are not taken into account as the duration of collision is small. Average impulsive force responsible for collision is much larger than external forces acting on the system.

If the charge on interacting particles do not change, collision is said to be scattering and if they change reaction is said to have taken place. However, during a collision charge remains conserved.

The impulsive force acting during collision is internal and hence the total momentum of the system is conserved.

If in a collision *KE* before collision and *KE* after collision are equal then collision is said to be elastic. Most of the collisions between atoms, nuclei, protons, electrons and so on are elastic.

If in a collision two particles stick together or move with the same common velocity after collision then collision is called perfectly inelastic, for example, a bullet embedded in a wooden block after collision.

Most of the collisions in our world (macroscopic) are imperfect or semielastic. We define for such collisions coefficient of restitution.

$$e = -\frac{\text{velocity of separation}}{\text{velocity of approach}}$$

or
$$e = \frac{|\text{relative velocity of recession}|}{|\text{relative velocity of approach}|}$$

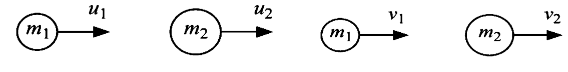
For perfectly elastic collisions $e = 1$; for perfectly inelastic collisions $e = 0$; and for all other collisions $0 < e < 1$

A problem related to imperfect collisions can be solved by using law of conservation of momentum and the definition of e, the coefficient of restitution, that is,

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\vec{v}_2 - \vec{v}_1 = e(u_1 - u_2)$$

One dimensional collision (Elastic) Consider two bodies moving initially along the line joining their centres as shown in Fig. 8.2. Assuming the initial direction of motion to be positive and applying law of conservation of momentum, we have



before collision **after collision**

Fig. 8.2

Special cases

(i) If $m_1 = m_2$, that is, colliding particles have equal mass, then

$$v_1 = u_2 \quad \text{and} \quad v_2 = u_1$$

(ii) If target particle say m_2 is at rest, that is, $u_2 = 0$ and $m_1 = m_2$

then $v_1 = 0$, and $v_2 = u_1$

(iii) If target is massive, that is, $m_2 >> m_1$ and is at rest, then

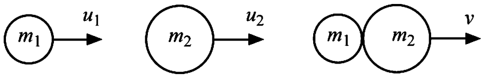
$$v_1 = -u_1 \text{ and } v_2 = u_1$$

(iv) If projectile is massive, that is, $m_1 >> m_2$ then

$$v_1 = u_1 \quad \text{and} \quad v_2 = 2u_1 - u_2$$

or $v_2 = 2u_1$ if target is at rest.

Perfectly inelastic collision In case of inelastic collision, after collision two bodies move with a common velocity v. Applying law of conservation of momentum



before collision **after collision**

Fig. 8.3

or
$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

$$\text{Decrease in KE} = \text{KE}_{\text{initial}} - \text{KE}_{\text{final}}$$

$$= \frac{m_1m_2}{2(m_1 + m_2)}(u_1 - u_2)^2.$$

Oblique collisions In oblique collisions, assuming the collision to be two dimensional, momentum in x and y directions be separately conserved. But if the collision is elastic kinetic energy should remain conserved. If two particles have same mass in oblique elastic collision then the two particles after collision move at right angles to each other.

Motion of two masses connected by a spring Consider two blocks of masses m_1 and m_2 respectively, connected by a massless spring, resting on a frictionless surface as shown in Fig. 8.4. If the spring is stretched or compressed and then released from rest

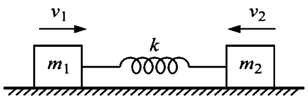


Fig. 8.4

Then $F_{\text{ext}} = 0$

$$\vec{p}_{\text{system}} = \vec{p}_1 + \vec{p}_2 = \text{constant}$$

Because both the blocks were at rest

$$\vec{p}_1 + \vec{p}_2 = 0 \quad \text{or} \quad \vec{p}_2 = -\vec{p}_1$$

At any instant the momentum is equal in magnitude but opposite in direction. Though the value of momentum at different places is different.

as $p = mv$, therefore, $\vec{p}_2 = -\vec{p}_1$ hence $\vec{v}_2 = -\frac{m_1\vec{v}_1}{m_2}$ that is, the two blocks always move in opposite directions with lighter block moving faster.

as $KE = \frac{p^2}{2m}$ and $|\vec{p}_1| = |\vec{p}_2|$ $\frac{\text{KE}_1}{\text{KE}_2} = \frac{m_2}{m_1}$, that is, kinetic energy of two blocks is not equal, the lighter block has more kinetic energy.

Kinetic energy of blocks is not constant. Initially it is zero but when the blocks are in motion it is nonzero. However, during motion kinetic energy is converted to elastic potential energy of spring or vice versa such that KE + PE = ME = constant.

Centre of mass Centre of mass of a body is a point where the whole mass of the body is supposed to be concentrated for dealing with its translatory motion. For a discrete system of particles centre of mass is defined as

$$\vec{R}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i},$$

If the system has homogeneous distribution of mass, then treating the mass element *dm* at positive $\vec{r}$ as a point mass and replacing summation by integration

$$\vec{R}_{\text{cm}} = \frac{1}{M}\int \vec{r}dm$$

so that $x_{\text{cm}} = \frac{1}{M}\int xdm$; $y_{\text{cm}} = \frac{1}{M}\int ydm$; $z_{\text{cm}} = \int zdm$ ·

There may or may not be any mass present physically at the centre of mass and it may lie with in or outside the body.

For symmetrical bodies having homogeneous distribution of mass, it coincides with centre of symmetry or geometrical centre.

If the centre of masses of the parts of a system are known, combined centre of mass may be obtained by treating the mass of the parts concentrated at their respective centre of masses.

If centre of mass forms the origin of a coordinate system then sum of moments of the masses of the system about its centre of mass is zero.

Centre of mass and centre of gravity in uniform gravitational field coincide but may be different for varying gravitational fields. For example, centre of mass and centre of gravity of a mountain may not be the same.

Motion of centre of mass For a system of particles centre of mass is given by

$$\vec{R}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots}{m_1 + m_2 + \ldots} \qquad \ldots(1)$$

$$\text{so} \quad \frac{d}{dt}(\vec{R}_{\text{cm}}) = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots}{m_1 + m_2 + \ldots} \qquad \ldots(2)$$

$$\vec{v}_{\text{cm}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots}{m_1 + m_2 + \ldots} \qquad \ldots(3)$$

The velocity of particles in centre of mass frame will be given by

$$\vec{v}_i{}' = \vec{v}_i - \vec{v}_{\text{cm}} \qquad \ldots(4)$$

as $m_1 + m_2 + \ldots = M$, mass of the system.

Therefore, equation (3) may be re-written as

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots = \vec{p}_1 + \vec{p}_2 + \ldots$$

$$= \vec{p}_i = \vec{p} \qquad \ldots(5)$$

that is, linear momentum of a system of particles is equal to the product of mass of the system with velocity of centre of mass. Furthermore, if $\vec{v}_{\text{cm}} = 0$ and $\vec{p} = 0$, that is, in the frame of reference of centre of mass the momentum of a system is zero. That is why centre of mass frame is usually referred to as zero momentum frame.

If we differentiate equation (5) with respect to time, we get

$$\frac{Md\vec{v}_{\text{cm}}}{dt} = \frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} \text{ or } M\vec{a}_{\text{cm}} = \vec{F}_{\text{ext}} \qquad \ldots(6)$$

If $F_{\text{ext}} = 0$ $\vec{a}_{\text{cm}} = 0$, that is, $\vec{v}_{\text{cm}}$ = constant

that is, if no external force is acting on the system, the velocity of its centre of mass remains constant that is. In other words velocity of centre of mass remains unaffected by internal forces.

SHORT-CUTS AND POINTS TO NOTE

(i) Linear momentum is conserved whenever external force is zero or in a two body system net external force is zero.

(ii) Elastic collision occurs only in atomic or subatomic particles. In real world, collisions are inelastic. Only collision between two ping-pong balls may be considered elastic. During elastic collision kinetic energy and linear momentum are conserved.

(iii) Impulse ($F.dt = \Delta p$). Since during collision contact time is very small, therefore, we consider that no external force or impulse has been imparted to the body.

(iv) $p = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$ is relativistic momentum.

(v) $p = \dfrac{h}{\lambda}$ (de-Broglie relation). p (photon) $= \dfrac{E}{c}$.

(vi) In one dimensional elastic collision, coefficient of restitution $e = 1$

$$v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2}u_2$$

$$v_2 = \frac{2m_1u_1}{m_1 + m_2} + \frac{(m_2 - m_1)u_2}{m_1 + m_2}$$

Fig. 8.5

If $m_1 = m_2$ then $v_2 = u_1$ and $v_1 = u_2$; if $m_2 \to \infty$ and $v_2 = 0$ then $v_1 = -u_1$.

(vii) If collision is oblique and elastic then equal masses fly off at right angles to one another.

(viii) Collision is one dimensional if the centre of mass of the colliding particles before and after collision are in the same straight line. Collision shown in Fig. 8.6 is not one-dimensional as centre of mass is not in the same straight line.

Fig. 8.6

(ix) In partially inelastic collision

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ and $e = -\dfrac{(v_2 - v_1)}{(u_2 - u_1)}$

coefficient of restitution $0 < e < 1$

(x) In perfectly inelastic collision, the colliding particles join together after collision and move with a common velocity

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

Loss in KE $= \dfrac{1}{2}\dfrac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2$.

(xi) Radioactivity is also considered to be a collision process, therefore, momentum is conserved before and after emission.

(xii) In oblique elastic collision conserve momentum in x and y direction separately. Since kinetic energy is scalar, therefore do not take components of kinetic energy. However kinetic energy is conserved just before and just after collision.

(xiii) In rocket propulsion acceleration

$$a = \frac{V_g\left(\frac{dm}{dt}\right)}{M} - g \quad \text{where } M = M_0 - \frac{dm}{dt}\cdot t$$

Force (upward) or upward thrust (net)

$$Ma = V_g\frac{dm}{dt} - Mg$$

only upthrust $= v_g\frac{dm}{dt}$ ($v_g \rightarrow$ velocity of gases, $\frac{dm}{dt}$ is rate of burning of fuel)

Velocity at any instant

$$v = v_0 + v_g \log_e \frac{M_0}{M_0 - \frac{dm}{dt}.t} - gt$$

where v_0 is velocity at $t = 0$.

(xiv) If a radioactive nucleus decays by γ-emission and energy of γ-ray is E, then momentum of γ-ray $= \frac{E}{c}$ where c is speed of light and velocity of recoil of the nucleus is $v_{\text{nucleus}} = \frac{E}{c \times m_{\text{nucleus}}}$

and kinetic energy of recoiling nucleus

$$= \frac{p^2}{2M_{\text{nucleus}}} = \frac{E^2}{2c^2M_{\text{nucleus}}}$$

(xv) If a radioactive nucleus of mass number A decays by α-emission then conserving momentum

Momentum before emission = Momentum after emission

$$0 = v_1(A - 4) + 4(v_2)$$

where v_1 and v_2 are velocities of recoiling nucleus and α-particle respectively.

(xvi) If two particles strike a rod and after collision stick with the rod as shown in Fig. 8.7 In such cases conserve linear momentum as well as angular momentum. Find the new centre of mass about which the rod will rotate to conserve angular momentum.

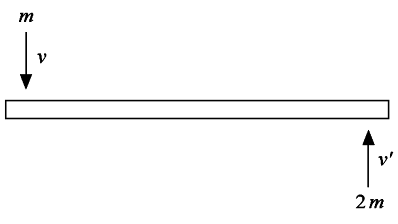


Fig. 8.7

(xvii) If a particle strikes a wall and gets reflected then average force exerted by it on the wall is

$$F_{aV} = \frac{\text{change in momentum}}{\text{time of contact}}.$$

(xviii) In oblique collisions coefficient of restitution normally acts in one direction, that is, along common normal. For instance, if the particles were moving under gravity then coefficient of restitution will be applied only in y-direction in which gravity acts.

$$e = -\frac{v_{2y} - v_{1y}}{u_{2y} - u_{1y}}$$

(xix) Area under $F - t$ graph is impulse or change in momentum

(xx) Since momentum and energy are treated invariant quantities of motion, therefore, action is regarded as impulse or change in momentum and not force. Moreover, it is assumed in an isolated system that there may be exchange of energy and of momentum between different parts but total of each quantity is conserved.

(xxi) If the mass of a spaceship moving around a planet varies then do not apply conservation of momentum to find new speed as the orbital velocity is independent of mass of the spaceship.

(xxii) Product of mass and speed is called pomentum. Pomentum conservation can be applied if speed is to be found out.

⚠ CAUTION

(i) Conservation of momentum can be applied only in collisions.

- ■ Conservation of linear momentum is applied whenever external force is zero.

(ii) Considering that linear momentum is not conserved when motion is rotational.

- ■ If a body collides with a disc as shown in Fig. 8.8 as no external force acts. Linear momentum is conserved.

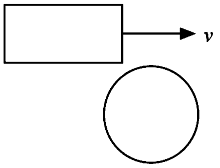


Fig. 8.8

(iii) Considering that conservation of linear momentum cannot be applied if there is external force.

- ■ In general the law is true. But if a two body system is considered then these forces become internal and conservation of momentum can be applied. The forces should be mutual.

Fig. 8.9

For example, m_1 and m_2 move towards each other due to gravitational attraction. If we consider single body system then force is external and momentum is not conserved. If however, we consider two body system then force becomes internal and momentum conservation can be applied.

(iv) Considering that linear momentum is always conserved if collision occurs.

- ■ Consider the situation that a hinged rod strikes a body. Here linear momentum is not conserved as there is an external force acting on the body, however angular momentum is conserved.

(v) Not using proper sign while conserving momentum

- Must apply proper sign. Consider that a projectile at the highest point splits into two parts of equal mass and one retraces the path. To find the velocity of the other half

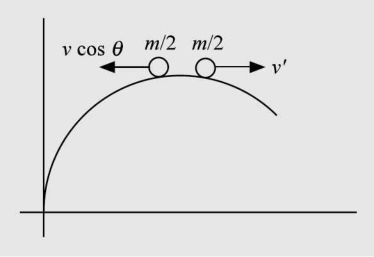


Fig. 8.10

$$mv\cos\theta = -\frac{mv\cos\theta}{2} + \frac{mv'}{2}$$

that is, momentum before break up = momentum after break up. Note the application of proper sign.

(vi) While finding net acceleration of the rocket, not taking into account acceleration due to gravity

- Use $a_{\text{net}} = \dfrac{v_g \dfrac{dm}{dt}}{M} - g$

  where $M = M_0 - \dfrac{t\,dm}{dt}$

  and $v = v_0 + v_g \log_e \dfrac{M_0}{M} - gt$ where v_0 is velocity at $t = 0$.

(vii) Considering that coefficient of restitution be applied in both the directions in oblique collision.

- Coefficient of restitution is applied in the direction of common normal. For example, in case of motion under gravity, apply coefficient of restitution in y direction.

(viii) Considering that when particles are moving in same direction and collide, collision is one dimensional.

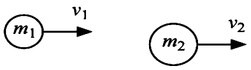


Fig. 8.11

- If the centre of mass of the particles move in same straight line then collision is one dimensional. For example, in. Fig. 8.11 though the particles are moving in the same straight line but the centre of masses are not in the same straight line. Therefore, it is a two dimension or oblique collision.

(ix) Considering that when masses of the colliding particles are equal then the velocities after collision is exchanged.

- Their velocities exchange after collision only if the collision is elastic and one dimensional

(x) Taking components of kinetic energy also along x and y directions in elastic two dimension collision.

- Kinetic energy is a scalar quantity and hence its components cannot be taken.

(xi) Considering only linear momentum is conserved during collision and not angular momentum.

- If a body rotates as a result of collision then both linear and angular momentum are conserved.

(xii) Conserving momentum in a space ship moving around the earth/or a planet when its mass is varied.

- Since the orbital velocity is independent of mass of spaceship, therefore, do not apply conservation of momentum even if the mass is varied.

Solved Problems

1. A liquid of density ρ flows along a horizontal pipe of uniform cross-section A with a velocity v through a right angled bend as shown in Fig. 8.12. What force has to be exerted at the bend to hold the pipe in equilibrium?

(a) $\rho A v^2$ (b) $\sqrt{2}\rho A v^2$

(c) $\rho A v^2/\sqrt{2}$ (d) $\rho A v^2/\sqrt{3}$

Solution (b) $\Delta p = p_f - p_i$ (change in momentum as Δm of liquid passes through the bend) = $\sqrt{2}\,\Delta m v$

$$F = \frac{\Delta p}{\Delta t} = \sqrt{2}.\frac{\Delta m v}{\Delta t} = \sqrt{2}\, v\rho A \frac{\Delta l}{\Delta t} = \sqrt{2}\,\rho A v^2$$

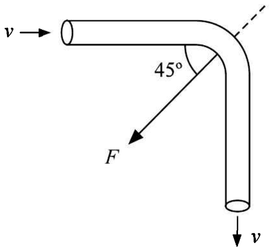


Fig. 8.12

∴ force required to hold it in position is $\sqrt{2}\,A\rho v^2$ in the direction shown otherwise it will bend in opposite direction.

2. A radioactive nucleus of mass number A initially at rest emits an α-particle having kinetic energy K. Find the KE of recoiling nucleus.

(a) K (b) $K/A - 4$

(c) $\dfrac{K4}{A-4}$ (d) $K/4$

Solution (c) $p_\alpha + p_{A-4} = 0$

$p_\alpha^2 = (p_{A-4})^2$

$K_\alpha m_\alpha = K_{A-4} \cdot m_{A-4}$ or $K_{A-4} = K\dfrac{4}{A-4}$.

3. A particle of mass $4m$ explodes at rest to three pieces each of separate masses m, m and $2m$. The equal mass fragments move at right angles to each other with speeds $4\,\text{ms}^{-1}$ and $6\,\text{ms}^{-1}$ respectively. Find the velocity of the heavier fragment.

(a) $\sqrt{13}\,\text{ms}^{-1}$ (b) $10\,\text{ms}^{-1}$

(c) $5\,\text{ms}^{-1}$ (d) $\sqrt{17}\,\text{ms}^{-1}$

Solution (a) Look into Fig 8.13(a) conserving momentum about y-axis.

$$p_{y1} = p_{y2} \Rightarrow 6\text{m} \sin\alpha = 4\text{m} \cos\alpha$$

$\tan\alpha = 2/3$

Conserving linear momentum about x-axis

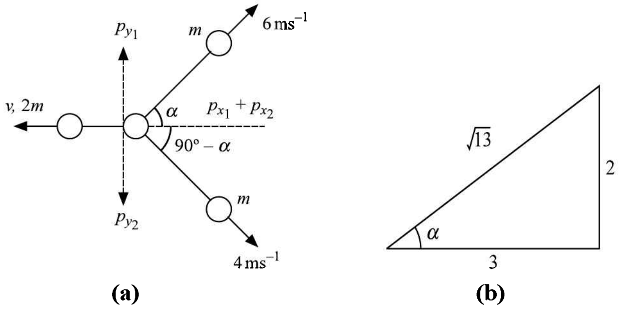


Fig. 8.13

$2mv = 6m \cos\alpha + 4m \sin\alpha$

$v = 3 \times \dfrac{3}{\sqrt{13}} + 2\times\dfrac{2}{\sqrt{13}} = \sqrt{13}\ \text{ms}^{-1}$.

4. A block of mass 2 kg is moving on a frictionaless horizontal track with a velocity $1\,\text{ms}^{-1}$ towards another block of equal mass at rest. The spring of force constant $100\,\text{Nm}^{-1}$ is attached at one end. Find the maximum compression in the spring.

(a) 5 cm (b) 7.5 cm

(c) 10 cm (d) none

Solution (c) On conserving the momentum

$2 \times 1 = (2 + 2)v$ or common speed $v = 1/2\,\text{ms}^{-1}$

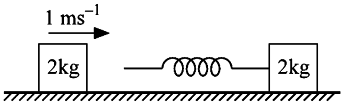


Fig. 8.14

KE before collision = $1/2 \times 2 \times (1)^2 = 1\,\text{J}$

KE after collision = $\dfrac{1}{2}\times 2\times\left(\dfrac{1}{2}\right)^2 + \dfrac{1}{2}\times 2\left(\dfrac{1}{2}\right)^2 = \dfrac{1}{2}\,\text{J}$

The difference of KE stored in spring = $1/2\,\text{J}$

$\dfrac{1}{2} = \dfrac{1}{2}kx^2$ $x^2 = 10^{-2}$ or $x = 0.1\,\text{m} = 10\,\text{cm}$.

5. Two blocks of masses m_1 and m_2 are connected by a spring of spring constant k as shown in Fig. 8.15. The block of mass m_2 is given a sudden impulse so that it acquires a velocity v_0 towards right. Find (a) the velocity of the centre of mass (b) maximum elongation that the spring will suffer.

(a) $\dfrac{m_2 v_0}{m_1 + m_2}, v_0\sqrt{\dfrac{m_1 m_2}{(m_1 + m_2)k}}$

(b) $\dfrac{m_1 v_0}{m_1 + m_2}, v_0\sqrt{\dfrac{m_1 m_2}{2(m_1 + m_2)k}}$

(c) $\dfrac{(m_1 - m_2)v_0}{m_1 + m_2}, \sqrt{\dfrac{m_1 m_2}{m_1 + m_2}}v_0$

(d) none of these

Solution (a) Conserving the momentum

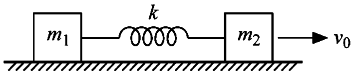


Fig. 8.15

$$m_2 v_0 = (m_1 + m_2)v \quad \text{or} \quad v = \frac{m_2 v_0}{m_1 + m_2}$$

Loss in KE= $\dfrac{1}{2}m_2 v_0^2 - \dfrac{1}{2}[m_1 + m_2]\left[\dfrac{m_2 v_0}{m_1 + m_2}\right]^2$

$= \dfrac{m_1 m_2}{2(m_1 + m_2)}v_0^2$

(a) Velocity of centre of mass

$$\frac{m_1 \times 0 + m_2 v_0}{m_1 + m_2} = \frac{m_2 v_0}{m_1 + m_2}$$

(b) To find compression in the spring we apply
Energy stored by spring = kinetic energy lost

$$\frac{1}{2}kx^2 = \frac{m_1 m_2}{2(m_1 + m_2)}v_0^2 \quad \text{or} \quad x = v_0\sqrt{\frac{m_1 m_2}{(m_1 + m_2)k}}.$$

6. A ball of mass m moving with a speed v makes a head on collision with an identical ball at rest. The kinetic energy after collision of the balls is three fourth the original kinetic energy. Find coefficient of restitution e.

(a) 1/2 (b) 1/3

(c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

Solution (c) Applying conservation of momentum

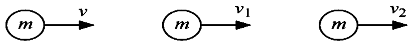


Fig. 8.16

$$mv = mv_1 + mv_2$$

or $v = v_1 + v_2$ ...(1)

$v_2 - v_1 = ev$ ...(2)

Squaring (1) $v^2 = v_1^2 + v_2^2 + 2v_1 v_2$

Given $v_1^2 + v_2^2 = \dfrac{3}{4}v^2$

($\because$ KE is $\dfrac{3}{4}$ the original after collision)

$\therefore$ $2v_1 v_2 = 1/4v^2$

Squaring (2), we have

$$v_1^2 + v_2^2 - 2v_1 v_2 = e^2 v^2$$

$$\frac{3}{4}v^2 - \frac{1}{4}v^2 = e^2v^2 \text{ or } \frac{1}{2} = e^2 \text{ or } e = \frac{1}{\sqrt{2}}.$$

7. In a γ-decay process, the internal energy of a nucleus of mass *m* decreases, a γ-ray of energy *E* is emitted and the nucleus recoils. Find decrease in internal energy.

(a) $E^2/2mc^2$ (b) E^2/mc^2
(c) $E + E^2/(mc^2)$ (d) $E + E2/(mc^2)$

Solution (d) Conserving the momentum

$0 = mv + E/c$ or $mv = -E/c$

$v = -E/(mc)$

Loss of KE of nucleus $= E + mv^2/2$

$= E + 1/2m[E/(mc)]^2$

$= E + E^2/(2mc^2)$

8. A ball falls on an inclined plane of inclination θ from a height *h* above the point of impact and makes a perfectly elastic collision. Where will it hit the inclined plane again?

(a) $8h/\sin\theta$ (b) $8h\sin\theta$
(c) $4h\sin\theta$ (d) $4h/\sin\theta$

Solution (b) $v = \sqrt{2gh}$ ...(1)

Applying conservation of momentum the ball will recoil with velocity *v* making an angle θ with the perpendicular to the plane.

Along horizontal direction

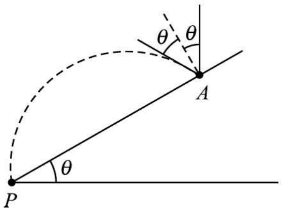


Fig. 8.17

$$AP = v\sin\theta\, t + \frac{g}{2}\sin\theta\, t^2 \quad ...(2)$$

Along vertical direction displacement is zero

$$0 = v\cos\theta t - \frac{1}{2}g\cos\theta t^2 \text{ or } t = \frac{2v}{g} \quad ...(3)$$

Substituting the value of *t* from (3) in eq. (2)

$$AP = v\sin\theta\left(\frac{2v}{g}\right) + \frac{g}{2}\sin\theta\left(\frac{2v}{g}\right)^2$$

$$= \frac{4v^2}{g}\sin\theta = 8h\sin\theta \text{ along the incline.}$$

9. Two masses m_1 and m_2 are connected by a spring of spring constant *k* and are placed on a smooth horizontal surface. Initially the spring is stretched through a distance x_0 when the system is released from rest. Find the distance moved by two blocks before they again come to rest.

Solution Conserving the momentum

$$m_1\vec{v}_1 = -m_2\vec{v}_2 \text{ or } m_1\vec{x}_1 = -m_2\vec{x}_2$$

or $$\frac{x_1}{x_2} = \frac{m_2}{m_1} \text{ or } \frac{x_1}{x_1+x_2} = \frac{m_2}{m_1+m_2}$$

or $$\frac{x_1}{2x_0} = \frac{m_2}{m_1+m_2} \text{ or } x_1 = \frac{2m_2x_0}{m_1+m_2}$$

Similarly $x_2 = \dfrac{2m_1x_0}{m_1+m_2}$.

10. A block of mass 0.2 kg when suspended through a spring produces an extension of 1.0 cm in equilibrium. A particle of mass 0.12 kg is dropped on the block from a height of 45 cm. The particle sticks to the block after the impact. Find the maximum extension.

(a) 5 cm (b) 6 cm
(c) 1 cm (d) none of these

Solution (b) Spring constant $k = \dfrac{0.2\times10}{10^{-2}} = 200\,\text{Nm}^{-1}$ velocity of the falling block $v_1 = \sqrt{2g(h+x)}$ where *x* is the extension in the spring $h = 45 + 1 = 46$ cm

Conserving the momentum

$m_1v_1 = (m_1 + m)v$

or $$v = \frac{0.12}{0.32}\sqrt{9.2+20x} = \frac{3}{8}\sqrt{9.2+20x}$$

Since the collision is inelastic, the energy gained by the block is utilised to extend the spring further.

$$\frac{1}{2}(m_1 + M)v^2 = \frac{1}{2}kx^2$$

$$0.32\left[\frac{9}{64}(9.2+20x)\right] = 200x^2$$

Simplifying, we get $x = 5$ cm. Since the spring has 1 cm extension already, the net extension is 6 cm.

11. A 60 kg man skating with a speed of 10 ms^{-1} collides with a 40 kg skater at rest and they cling to each other. Find the loss in kinetic energy.

(a) 1200 J (b) 600 J
(c) 3000 J (d) 2000 J

Solution (a) Loss in KE $= \dfrac{m_1m_2}{2(m_1+m_2)}(v_1 - v_2)^2$

$$= \frac{60\times40}{2(100)}(10-0)^2 = 1200\,\text{J}.$$

12. A small block of super dense material has mass 3×10^{24} kg. It is situated at a height *h* ($<< R_e$) from where it falls on the earth's surface. Calculate its speed when its height has reduced to *h*/2. The mass of the earth is 6×10^{24} kg.

(a) $\sqrt{2gh/3}$ (b) $\sqrt{gh/2}$
(c) $\sqrt{gh}$ (d) $\sqrt{3gh/2}$

Solution (a) Change in Potential energy

$$\Delta\text{PE} = m_1g\left(h - \frac{h}{2}\right) = m_1g\frac{h}{2}$$

$$\text{KE} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1v_1 = -m_2v_2 \text{ or } v_2 = -\frac{m_1}{m_2}v_1$$

Using conservation principle

ΔPE = gain in KE

$$m_1g\frac{h}{2} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_2}v_1\right)^2$$

or $$gh = v_1^2 + \frac{m_1}{m_2}v_1^2 \quad (\because m_2 = 2m_1)$$

$$= v_1^2 + \frac{1}{2}v_1^2 = \frac{3}{2}v_1^2 \text{ or } v_1 = \sqrt{\frac{2gh}{3}}.$$

13. Find the centre of mass of a uniform disc of radius a from which a circular section of radius b is removed and the centre of hole is at a distance c from the centre of the disc.

(a) $\dfrac{-b^2c}{(a^2-b^2)}$ (b) $\dfrac{-c^2b}{(a^2-c^2)}$

(c) $\dfrac{-b^2c}{(b^2-c^2)}$ (d) $\dfrac{-c^2b}{(b^2-c^2)}$

Solution (a) Let σ be the surface density.

$$\vec{x} = \frac{\sigma a^2\pi.0 - \sigma\pi b^2 c}{\sigma\pi(a^2-b)^2}$$

$$= \frac{-b^2c}{(a^2-b^2)}$$

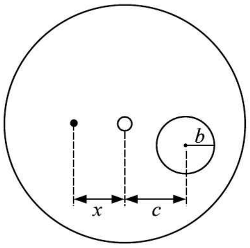


Fig. 8.18

Thus $\vec{x}$ shifts left of the centre O of the disc on the line joining the two centres by

$$\vec{x} = \frac{-b^2c}{(a^2-b^2)}$$

14. A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface. A uniform cylinder of radius r and mass m is released from rest at the top point A as shown in Fig. 8.19. The cylinder slips in the semicircular frictionless track. (a) How far the block moved when the cylinder reaches the bottom (point B) of the track? (b) what is the speed of the block when cylinder reaches the bottom of the track?

Solution Initially system is at rest ∴ $\vec{v}_{\text{cm}} = 0$

If v_1 and v_2 are velocities of cylinder and block respectively then

$$\frac{mv_1 + Mv_2}{m+M} = 0 \quad \text{or} \quad mv_1 + Mv_2 = 0$$

$$m\frac{d\vec{r}_1}{dt} + \frac{Md\vec{r}_2}{dt} = 0$$

or $$m\Delta r_1 + M\Delta r_2 = 0$$

When cylinder reaches bottom it has been displaced by $(R - r)$ horizontally to the right relative to the block and if the displacement of block relative to ground is d_2 then displacement of cylinder with respect to ground is

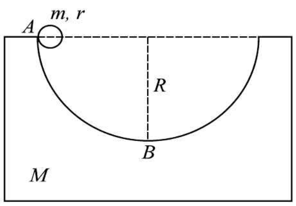


Fig. 8.19

$$d_1 = (R - r) - d_2$$

or $$md_1 = Md_2$$

$$m[(R - r) - d_2] = Md_2$$

or $$d_2 = \frac{m(R-r)}{M+m} \text{ to the left}$$

Applying conservation of mechanical energy between points A and B

$$mg(R - r) = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$$

or $$mg(R - r) = \frac{1}{2}m\left(\frac{M}{m}v_2\right)^2 + \frac{1}{2}Mv_2^2$$

$$2m^2g(R - r) = M(M + m)v_2^2$$

or $$v_2 = m\sqrt{\frac{2g(R-r)}{M(M+m)}} \text{ to the left.}$$

15. A nucleus of mass M emits a γ-ray photon of frequency f_0. Show that the loss of internal energy by the nucleus is $hf_0\left[1 + \dfrac{hf_0}{2Mc^2}\right]$.

Solution $\vec{p}_{\text{photon}} + \vec{p}_{\text{nucleus}} = 0$

(∵ initially the nucleus is at rest)

∴ nucleus and photon will move in opposite directions.

Kinetic energy acquired by nucleus

$$\text{KE} = \frac{p^2_{\text{nucleus}}}{2M} = \frac{\left(\dfrac{hf_0}{c}\right)^2}{2M}$$

Total internal energy lost by nucleus = ΔE = energy of photon + KE

$$= hf_0 + \frac{(hf_0)^2}{2Mc^2} = hf_0\left[1 + \frac{hf_0}{2Mc^2}\right].$$

16. Two persons each of mass m are standing at the two extremes of a rail-road car of mass M resting on a smooth track. The person on the left jumps with a horizontal velocity u with respect to the state of the car before the jump. Thereafter the other person jumps to the right with the same speed u with respect to state of the car before jump. Find the velocity of the car after both have jumped down.

(a) $\dfrac{2mu}{m+M}$ (b) zero

(c) $\dfrac{m^2u}{M(M+m)}$ (d) $\dfrac{M^2u}{m(M+m)}$

Solution When the first person jumps to left with a speed u, applying law of conservation of momentum

$$mu = -(M + m)v \quad \text{or} \quad v = \frac{-mu}{M+m}$$

that is, the rail-road car will move towards right.

Now the second man jumps to right with a speed u, applying law of conservation of momentum

$$mv + mu = (m + M)v + Mv'$$

or $$m\left(\frac{mu}{M+m}\right) + mu = mu + Mv'$$

or $$v' = \frac{m^2u}{M(m+M)}$$

17. A block of mass m rests on a wedge of Mass M which in turn rests on a horizontal table. All the surfaces are smooth and the system is initially at rest. Find the velocity of the wedge when block after sliding down just touches the ground.

Solution The horizontal and vertical components of the velocity of block with respect to the wedge at the instant when it just touches the ground are

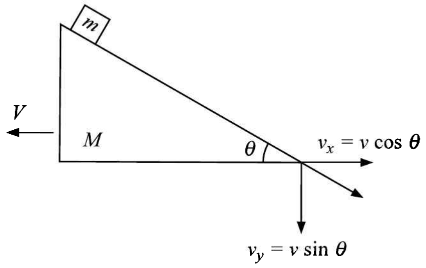


Fig. 8.20

$$\left.\begin{array}{l} v_x = v\cos\theta - V \\ v_y = v\sin\theta \end{array}\right\} \text{(1) where } V \text{ is velocity of wedge relative to table}$$

As no external force acts along the horizontal direction, linear momentum is conserved.

$$MV = mv_x = m(v\cos\theta - V) \qquad \text{...(2)}$$

Using conservation of energy

$$mgh = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 = \frac{1}{2}MV^2 + \frac{1}{2}m(v_x^2 + v_y^2)$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}m[(v\cos\theta - V)^2 + v^2\sin^2\theta]$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}mV^2 + \frac{1}{2}mv^2 - \frac{1}{2}m(2Vv\cos\theta) \text{ ...(3)}$$

Solving (2) and (3) to eliminate v

$$V = \left[\frac{2m^2 g\cos^2\theta\, h}{(M+m)(M+m\sin^2\theta)}\right]^{1/2}$$

18. A bullet of mass 10 g strikes a ballistic pendulum of mass 2.0 kg. The centre of mass of the pendulum rises a vertical distance of 12 cm. Assuming the bullet remains embedded in the pendulum, calculate its initial velocity.

(a) 210 ms^{-1} (b) 310 ms^{-1}
(c) 110 ms^{-1} (d) 410 ms^{-1}

Solution (b) Conserving the momentum we get

$$0.01v = (2.01)v' \qquad \text{...(1)}$$

conserving the energy

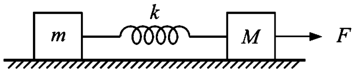
$$(2.01)\,gh = \frac{1}{2}(2.01)v'^2$$

or $v' = \sqrt{2gh}$ ...(2)

From equations (1) and (2)

$$v = \frac{2.01}{.01}\sqrt{2gh} = 201\sqrt{2\times9.8\times0.12} = 310\,\text{ms}^{-1}.$$

19. A block of mass m is connected to another block of mass M by a massless spring as shown Fig. 8.22(a). The spring constant of the spring is k. Surface below the blocks is smooth and horizontal. Initially the blocks are at rest and the spring is unstretched. A constant Force F is applied to the block of mass M as shown in the Fig. 8.21 (a) below. Find the maximum extension in the spring.



(a)

(a) $\dfrac{2mF}{(m+M)k}$ (b) $\dfrac{2MF}{(m+M)k}$

(c) $\dfrac{mF}{2(m+M)k}$ (d) $\dfrac{MF}{2(m+M)k}$

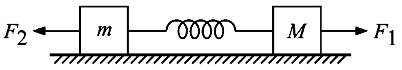
Solution (a) $a = \dfrac{F}{m+M}$

$$0.01v = (2.01)v'$$

Net external force on m is $F_2 = ma = \dfrac{F}{m+M}$

net external force on M is F_1

$$F_1 = F - Ma = F - \frac{M\,F}{M+m} = \frac{mF}{m+M}$$



(b)

Fig. 8.21

work done

$$W = F_1x_1 + F_2x_2 = \frac{mF}{m+M}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2$$

or $(x_1 + x_2) = \dfrac{2mF}{k(m+M)}$.

20. Three particles A, B, C of equal masses move with speed v along the medians of an equilateral triangle as shown Fig. 8.22. They collide at centroid G of the triangle. After collision A comes to rest, B retraces its path with speed v. What is the velocity of C?

Solution Since $p_A + p_B + p_C = 0$ (before collision). After collision $p_A = 0$

$\therefore$ $p_C = -p_B$ that is, C moves along GY with velocity v after collision.

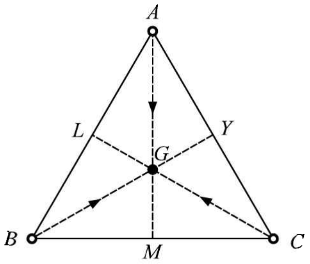


Fig. 8.22

Problems for Practice

1. A light and a heavy body have equal kinetic energy. Which one has a greater momentum?

(a) the heavy body (b) the light body
(c) both (d) cannot be said

2. The linear momentum of a body is increased by 50%, its kinetic energy will increase by

(a) 150% (b) 125%
(c) 100% (d) 50%

3. Two unequal masses are tied together with a compressed spring. When the cord is burnt with a match stick releasing the spring, the two masses fly apart with equal

(a) momentum (b) acceleration
(c) kinetic energy (d) speed

4. A particle at rest suddenly disintegrates into two particles of equal masses which start moving. The two fragments will

(a) move opposite with unequal speeds
(b) move in any direction with any speed
(c) move in same direction with equal speeds
(d) move opposite with equal speed

5. Two masses of 1 gm and 4 gm are moving with kinetic energy in the ratio 4 : 1. What is the ratio of their linear mementum?

(a) 6 : 1 (b) 4 : 1 (c) 1 : 2 (d) 1 : 1

6. On a stationary sail boat air is blown from a fan attached to the boat. The boat

(a) moves in opposite direction in which air is blown
(b) does not move
(c) moves in same direction in which air blows
(d) spins around

7. The spacecraft of mass M moving with a velocity v in free space explodes and breaks up into two pieces. If after explosion a piece of mass m comes to rest, the other piece of spacecraft will have a velocity

(a) $\frac{Mv}{(M-m)}\sqrt{a^2+b^2}$ (b) $\frac{Mv}{(M+m)}$
(c) $mV/(M-m)$ (d) $\frac{mv}{(M+m)}$

8. A shell explodes and many pieces fly off in different directions. The following is conserved

(a) kinetic energy (b) momentum
(c) both (d) none

9. A light and a heavy body have equal momenta. Which one has greater kinetic energy?

(a) the heavy body (b) the light body
(c) both (d) none

10. A bird resting on the floor of an air tight box which is being carried by a boy starts flying. The boy feels that the box is now

(a) lighter
(b) heavier
(c) shows no change in weight
(d) lighter in the beginning and then heavier

11. Two bodies of mass m_A and m_B have equal kinetic energy. The ratio of their momenta is

(a) $\sqrt{m_A} : \sqrt{m_B}$ (b) $m_A^2 : m_B^2$
(c) $m_A : m_B$ (d) $m_B : m_A$

12. A bullet weighing 50 gm leaves the gun with a velocity of 30 ms^{-1}. If the recoil speed imparted to the gun is 1 ms^{-1}, the mass of the gun

(a) 1.5 kg (b) 15 kg (c) 20 kg (d) 30 kg

13. If the kinetic energy of a body becomes four times of its initial value, then the new momentum will be

(a) double (b) 3 times
(c) 4 times (d) unchanged

14. A particle of mass m is moving in a horizontal circle of radius r with uniform speed v. When it moves from one point to a diametrically opposite point its

(a) KE changes by mv^2
(b) KE changes by $(1/4)mv^2$
(c) momentum does not change
(d) momentum changes by $2mv$

15. Kinetic energy of a body of mass m and momentum p is given by

(a) p^2m (b) $m^2/2p$ (c) mp (d) $p^2/2m$

16. When the velocity of a body is doubled

(a) kinetic energy is doubled
(b) acceleration is doubled
(c) momentum is doubled
(d) potential energy is doubled

17. A surface is hit elastically and normally by n balls per unit time. All the balls have the same mass m and move with the same velocity u. The force on the surface is

(a) $1/2\,mnu^2$ (b) mnu^2
(c) $2mnu^2$ (d) $2mnu$

18. A bomb of mass 9 kg explodes into two pieces of mass 3 kg and 6 kg. The velocity of mass 3 kg is 16 ms^{-1}. The kinetic energy of mass 6 kg in joule is

(a) 768 (b) 96 (c) 384 (d) 192

19. A body of mass m at rest explodes into three pieces, two of which of mass $m/4$ each are thrown off in perpendicular directions with velocity of 3 ms^{-1} and 4 ms^{-1} respectively. The third piece will be thrown off with a velocity of

(a) 3 ms^{-1} (b) 2.5 ms^{-1}
(c) 2 ms^{-1} (d) 1.5 ms^{-1}

20. A body of mass 1 kg initially at rest explodes and breaks into three fregments of masses in the ratio 1 : 1 : 3. The two pieces of equal mass fly off perpendicular to each other with a speed of 15 ms^{-1} each. The speed of the heavier fragment is

(a) 45 ms^{-1} (b) 15 ms^{-1}
(c) 5 ms^{-1} (d) $5\sqrt{2}$ ms^{-1}

21. A block of mass $2m$ moving with constant velocity $3\vec{v}$ collides with another block of mass m which is at rest and stick to it. The velocity of the compound block after the collision is

(a) $\vec{v}$ (b) $3\vec{v}/2$ (c) $2\vec{v}$ (d) $3\vec{v}$

22. When two bodies collide elastically then the quantity conserved is

(a) kinetic energy (b) momentum
(c) both (d) none

23. Two balls at the same temperature collide. What is conserved?

(a) momentum (b) kinetic energy
(c) velocity (d) temperature

24. Two balls each of mass 0.25 kg are moving towards each other in a straight line, one at 3 ms^{-1} and the other at 1 ms^{-1} collide. The balls stick together after the collision. The magnitude of the final velocity of the combined mass is

(a) 1/2 ms^{-1} (b) 1 ms^{-1}
(c) 2 ms^{-1} (d) 4 ms^{-1}

25. A bullet weighing 10 gm and moving at 300 ms^{-1} strikes a 5 kg block of ice and drops dead. The ice block is sitting on frictionless level surface. The speed of the block, after the collision is

(a) 60 cm s^{-1} (b) 60 ms^{-1}
(c) 6 cm s^{-1} (d) 6 ms^{-1}

26. A body of mass m moving with a constant velocity v hits another body of the same mass moving with same velocity v but in the opposite direction, and sticks to it. The velocity of the compound body after collision is

(a) zero (b) v
(c) $2v$ (d) $v/2$

27. A bullet of mass 'a' and velocity 'b' is fired into a large block of wood of mass 'c'. The final velocity of the system is

(a) $\frac{ab}{a+c}$ (b) $\frac{b}{a}(a+c)$

(c) $\frac{cb}{(a+b)}$ (d) $\frac{b}{c}(a+b)$

28. A mass m_1 moves with a great velocity. If it strikes another mass m_2 at rest in a head on collision it comes back along its path with a low speed after collision. Then

(a) $m_1 > m_2$

(b) $m_1 = m_2$

(c) $m_1 < m_2$

(d) there is no relation between m_1 and m_2.

29. Two solid rubber balls A and B having masses 200 g and 400 g respectively are moving in opposite directions with velocity of A equal to 0.3 ms^{-1}. After collision the two balls come to rest then the velocity of B is

(a) $-0.15\,ms^{-1}$ (b) $0.15\,ms^{-1}$

(c) $1.5\,ms^{-1}$ (d) none of these

30. A body of mass m moving with a constant velocity v hits another body of the same mass at rest and sticks to it. The velocity o the compound body after collision is

(a) zero (b) $v/2$ (c) v (d) $2v$

31. A body of mass 2 kg moving with a velocity of 3 ms^{-1} collides head on with a body of mass 1 kg moving with a velocity of 4 ms^{-1} in opposite direction. After collision the two bodies stick together and move with a common velocity

(a) $(2/3)\,ms^{-1}$ (b) $3/4\,ms^{-1}$

(c) $1/4\,ms^{-1}$ (d) $1/4\,ms^{-1}$

32. A steel ball moving with a velocity $\vec{v}$ collides with an identical ball originally at rest. The velocity of the first ball after the collision is

(a) $(-1/2)\vec{v}$ (b) $-\vec{v}$

(c) $\vec{v}$ (d) zero

33. Two perfectly elastic particles A and B of equal masses travelling along the line joining them with velocity 15 ms^{-1} and 10 ms^{-1} respectively collide. Their velocities after the elastic collision will be (in ms^{-1}) respectively

(a) 20 and 5 (b) 10 and 15

(c) 5 and 20 (d) 0 and 25

34. A neutron travelling with a velocity v and kinetic energy KE collides elastically head on with the nucleus of an atom of mass number A at rest. The fraction of total energy retained by the neutron is

(a) $(A/A + 1)$ (b) $(A/A + 1)^2$

(c) $(A - 1)/(A + 1)$ (d) $(A - 1)/(A + 1)^2$

35. A massive ball moving with speed v collides with a tiny ball having a mass very much smaller than the mass of the first ball. The collision is elastic, then immediately after the impact, the second ball will move with a speed approximately equal to

(a) ∞ (b) $v/2$ (c) v (d) $2v$

36. A sphere of mass m moving with a constant velocity v hits another stationary sphere of the same mass. If e is the coefficient of restitution, then the ratio of the velocities of the two spheres after collision will be

(a) $(e + 1)/(e - 1)$ (b) $(e - 1)/(e + 1)$

(c) $(1 + e)/(1 - e)$ (d) $(e + 1)/(1 - e)$

37. A system consists of mass M and m. The centre of mass of the system is

(a) nearer to m

(b) at the position of large mass

(c) nearer to M

(d) in the middle

38. A man weighing 80 kg is standing on a trolley weighing 320 kg. The trolley is resting on frictionless horizontal rails. If the man starts walking on the trolley along the rails at a speed 1 ms^{-1} then after 4 second his displacement relative to the ground will be

(a) 3.0 m (b) 3.2 m (c) 4.8 m (d) 5 m

ANSWERS

1. (a)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (c)
11. (a)	**12.** (a)	**13.** (a)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (d)
21. (c)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (b)
31. (a)	**32.** (d)	**33.** (b)	**34.** (d)	**35.** (d)	**36.** (c)	**37.** (c)	**38.** (b)		

9 Gravitation

BRIEF REVIEW OF THE CONCEPTS

Newton's law of gravitation Newton in 1665 formulated $F \propto m_1 m_2$

$F \propto \frac{1}{r^2}$

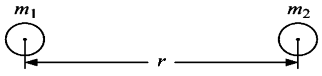


Fig. 9.1

$F = \frac{Gm_1m_2}{r^2}$ where $G = 6.67 \times 10^{-11}$. Its unit is Nm^2kg^{-2} and is called universal gravitational constant. The value of G was first experimentally determined by Cavendish in 1736. The value of G measured for small distances is about 1% less than the value of G measured for large distances.

Gravitational field intensity Gravitational force per unit mass is called gravitational field intensity. Gravitational field intensity of earth is 'g'

$$E_g = \frac{F}{m} = \frac{GM}{r^2}$$

Gravitational potential (V_g) The amount of work done to bring a unit mass from infinity to that point under the influence of gravitational field of a given mass M, without changing the velocity.

$$V_g = -\frac{GM}{r}$$

Gravitational potential energy The amount of work done to bring a mass m from infinity to that point under the influence of gravitational field of a given mass M without changing the velocity.

$$u_g = -\frac{GMm}{r}$$

Note that $W = \Delta U_g$ and $U_g = mV_g$.

Variation of 'g' due to height $g' = g\left(1 - \frac{2h}{R}\right)$ if $h << R$

$g' = \frac{g}{(1 + h/R)^2}$ if h is comparable to R

Variation of *g* due to depth

$g' = g(1 - x/R)$ where x is the depth.

$= 0$ at the centre of the earth

Variation of *g* with rotation of earth/latitude

$$g' = g\left(1 - \frac{R\omega^2}{g}\cos^2\lambda\right)$$

that is, g is maximum at the poles and minimum at the equator

Orbital velocity $v_0 = \sqrt{GM/r}$

where v_o is speed with which a planet or a satellite moves in its orbit and r is the radius of the orbit.

Escape velocity $v_e = \sqrt{2GM/r}$

Escape velocity is the minimum velocity required to escape from the surface of the earth/planet from its gravitational field. Note $v_e = \sqrt{2}\, v_0$.

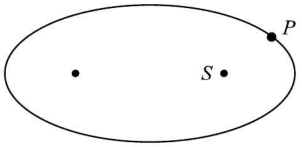
Time period $T = \frac{2\pi r}{v_0}$ or $T^2 = \frac{4\pi^2 r^3}{GM}$

Kinetic Energy KE $= \frac{1}{2} m v_0^2 = \frac{GMm}{2a}$, PE $= -\frac{GMm}{a}$

Net energy E = KE + PE $= -\frac{GMm}{2a}$

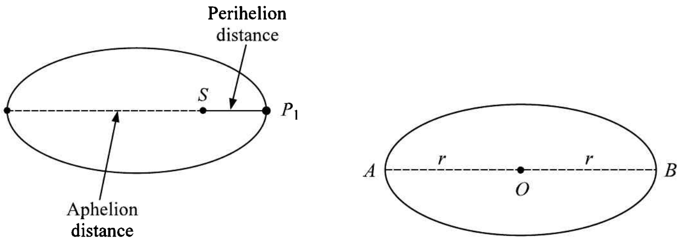
Kepler's laws

First law The planets revolve around the sun in the elliptical orbits with sun at one of the focus.



(a)

Second law The radius vector sweeps out equal area in equal interval of time. This law may be derived from law of conservation of angular momentum.



(b) (c)

Fig. 9.2

Thus $\frac{v_1}{v_2} = \frac{r_2}{r_1}$ or $\frac{v_{\text{perihelion}}}{v_{\text{aphelion}}} = \frac{r_{\text{aphelion}}}{r_{\text{perihelion}}}$

that is, when the planet is closer to the sun it moves fast.

Third law The square of the time period of a planet is proportional to the cube of a semimajor axis

$$T^2 \alpha\, r^3$$

If eccentricity of the orbit is e then $\dfrac{r_{\text{aphelion}}}{r_{\text{prihelion}}} = \dfrac{1+e}{1-e}$

and $r_{\text{perihelion}} + r_{\text{aphelion}} = 2r$; r being semimajor axis

Weightlessness in a satellite

$\dfrac{GMm}{r^2} - N = m\left(\dfrac{GM}{r^2}\right)$ or $N = 0$ where N is contact force by the surface

that is, the surface does not exert any force on the body and hence its apparent weight is zero.

SHORT-CUTS AND POINTS TO NOTE

(i) Gravitational force is only attractive force and $F = \dfrac{Gm_1m_2}{r^2}$. The force is conservative. If $r \le 10^{-8}$m then intermolecular forces dominate.

(ii) Gravitational field $E_g = \dfrac{GM}{r^2}$ is force per unit mass.

(iii) Aryabhatt in 5th century AD first described that the earth revolves around the sun in a circular orbit and the moon moves around the earth.

(iv) The moon takes 27.3 days to revolve around the earth and the radius of its orbit is 3.85×10^5km.

(v) Cavendish measured the value of G experimentally for the first time.

(vi) Gravitational potential due to a point mass $= \dfrac{-GM}{r}$

(vii) Gravitational potential due to a ring at any point on its axis, assuming mass of the ring is uniformly distributed is $V = \dfrac{-GM}{\sqrt{R^2 + x^2}}$.

(viii) Gravitational potential due to a shell

$$V_{\text{in}} = V_{\text{sur}} = \frac{-GM}{R};\ V_{\text{out}} = \frac{-GM}{x}.$$

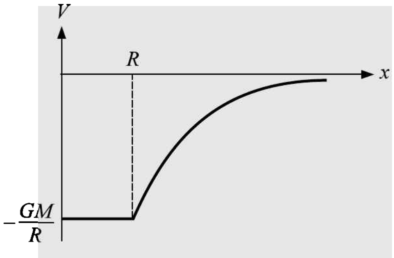


Fig. 9.3

(ix) Gravitational potential due to a solid sphere of radius R

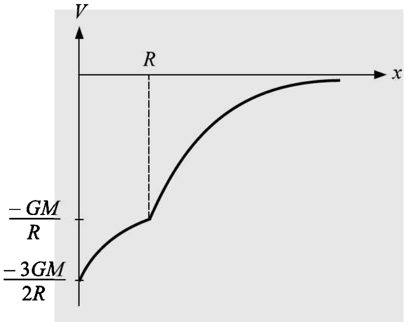


Fig. 9.4

$$V_{\text{in}} = \frac{-GM}{2R^3}(3R^2 - x^2) \text{ for } x < R$$

$$V_{\text{sur}} = -\frac{GM}{R}, \text{ for } x = R;\ V_{\text{out}} = \frac{-GM}{x} \text{ for } x > R$$

(x) Gravitational Potential energy $U_g = -\dfrac{GMm}{r}$; $u_g = mV$.

Negative sign shows force is attractive.

(xi) Work done = ΔPE; $F = -\dfrac{du}{dr}$. At equilibrium $\dfrac{du}{dr} = 0$

(xii) Gravitational field intensity of the earth $= g = \dfrac{GM}{R^2}$ and is valid upto 2 – 3 kms above the earth. With height or depth g varies.

(xiii) Gravitational field intensity due to a ring at any point of the axis

$$E_g = \frac{GMx}{(x^2 + R^2)^{3/2}}$$

The field is directed towards the centre. At the centre of the ring E_g is minimum (= 0) and E_g is maximum at $x = \dfrac{R}{\sqrt{2}}$.

(xiv) Gravitational field intensity due to a disc

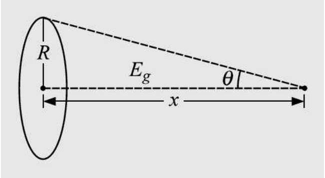


Fig. 9.5

$$E = \frac{2GM}{R^2}\left[1 - \frac{x}{\sqrt{x^2 + R^2}}\right] = \frac{2GM}{R^2}\,[1 - \cos\theta\,]$$

$$E_{\text{centre}} = \frac{2GM}{R^2}$$

(xv) Gravitational field intensity due to a shell

$$E_{\text{in}} = 0,\ E_{\text{sur}} = \frac{GM}{R^2},\ E_{\text{out}} = \frac{GM}{x^2}$$

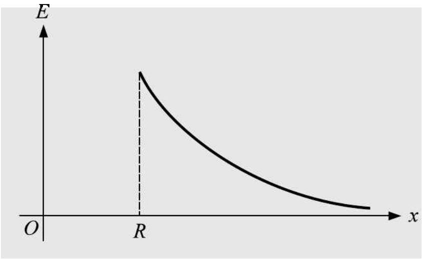


Fig. 9.6

(xvi) Gravitational field intensity due to a solid sphere

$$E_{\text{in}} = \frac{GMx}{R^2} \text{ for } x < R$$

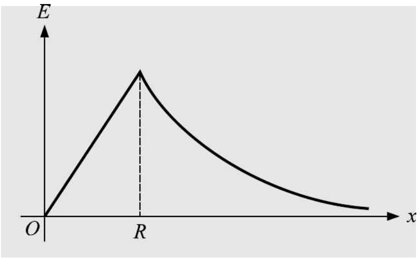


Fig. 9.7

$$E_{\text{sur}} = \frac{GM}{R^2},\ E_{\text{out}} = \frac{GM}{x^2} \text{ for } x > R$$

(xvii) Variation of g with height

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \quad \text{if } h > \frac{R}{10}$$

$$g' = g\left(1-\frac{2h}{R}\right) \quad \text{if } h < \frac{R}{10}$$

Note: g never becomes zero with height, that is, $g \to 0$ if $h \to \infty$

(xviii) Variation of g with depth $g' = g\left(1-\frac{x}{R}\right)$

At the centre of the earth $g' = 0$, that is, body will be weightless.

(xix) Variation of g with latitude

$$g' = g\left(1-\frac{R\omega^2\cos^2\lambda}{g}\right)$$

where λ is angle of latitude.

At the poles $g' = g$ (max) $\quad \because \lambda = 90°$

At the equator $g' = g\left(1-\frac{R\omega^2}{g}\right) \quad \because \lambda = 0°$.

(xx) Kepler's laws First law The planets move around the sun in elliptical orbits with the sun at a focus.

Second law Areal velocity is constant. The law is based on conservation of angular momentum.

$$\frac{r_1}{r_2} = \frac{v_2}{v_1}$$

Perihelion distance is the shortest distance between the sun and the planet.

Aphelion distance is the largest distance between the sun and the planet

$$\frac{v_{\text{aphelion}}}{v_{\text{perihelion}}} = \frac{r_{\text{perihelion}}}{r_{\text{aphelion}}}$$

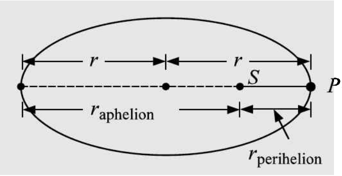


Fig. 9.8

if e is the eccentricity of the orbit then

$$\frac{1+e}{1-e} = \frac{r_{\text{aphelion}}}{r_{\text{perihelion}}}$$

$$r_{\text{aphelion}} + r_{\text{perihelion}} = 2r.$$

- If $e < 1$ and total energy (KE + PE) > 0 the path of the satellite is hyperbolic and it escapes.
- If $e < 1$ and total energy is negative it moves in elliptical path.
- If $e = 0$ and total energy is negative it moves in circular path.
- If $e = 0$ and total energy is zero it will take parabolic path.

Third law $T^2 \propto r^3$(r = semimajor axis)

(xxi) Orbital velocity $v_o = \sqrt{\frac{GM}{r}}$ where r is radius of the orbit.

For a satellite $r = R_e + h$.

If $v > v_o$ it acquires elliptical path ($v_o < v < v_e$).

(xxii) Escape velocity $v_e = \sqrt{\frac{2GM}{r}}$

Note that $v_e = \sqrt{2}\, v_o$.

If $v > v_e$ the satellite takes hyperbolic path and leaves the gravitational field of the earth.

(xxiii) Time period of revolution $T = \frac{2\pi r}{v_o} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$

or $T^2 = \frac{4\pi^2 r^3}{GM}$ It also proves Kepler's third law

$$T^2 \propto r^3$$

(xxiv) Total energy possessed by a body revolving around the earth/planet or the sun is

$$E_{\text{net}} = \text{KE} + \text{PE} = \frac{GMm}{2r} - \frac{GMm}{r} = \frac{-GMm}{2r}$$

Note that PE = twice the KE with a negative sign or PE = –2KE

Net energy = –KE

(xxv) The path of the projectiles thrown to lower heights is parabolic and thrown to greater heights is elliptical.

CAUTION

(i) Not remembering that gravitational field intensity depends upon shape/geometry and distance

- Inside the shell it is zero; at the centre of the ring it is zero.

  If a part is cut the gravitational field will vary not only due to the fact that mass has varied but also because the shape has varied.

(ii) Assuming gravitational potential depends only on distance

- Gravitational potential also depends upon the shape and geometry. Gravitational field inside the shell is zero but gravitational potential is nonzero and at the surface it is constant

(iii) Assuming work done = $\vec{F}\cdot\vec{d}$

- Work done $= \int F.dx$

$$= PE_{\text{final}} - PE_{\text{initial}} = \Delta PE$$

Since the force is variable, work $\neq F \cdot d$.

(iv) Assuming g varies with height as $g' = g\left(1-\frac{2h}{R}\right)$

- $g' = g\left(1-\frac{2h}{R}\right)$ is valid only if the distance (height) $\leq \frac{R}{10}$ otherwise use

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}.$$

(v) Assuming that when the sun is closer only then there will be summer.

- Though in principle it appears true but in case of earth the solar radiations are incident oblique. During winter, sun is closer to the earth, therefore winter is of short duration.

(vi) Assuming $g = 0$ at the equator or g to be constant at the surface of the earth.

- g is maximum at the poles and minimum at the equator but this variation is quite small and occurs due to the rotation of the earth at latitudes.

(vii) Assuming that only one geostationary satellite is sufficient for global communication.

- One satellite can cover at the most 42% area. Therefore at least 3 satellites would be required. At the most 180 geostanouary Satellites can be launched at an angular separation of 2°.

(viii) Assuming that while finding the orbtal velocity

$v_o = \sqrt{\frac{GM}{r}}$ for a satellite, M is mass of the satellite

- M is the mass of the planet, that is, escape velocity and the orbital velocity are independent of the mass of the body being escaped or put into the orbit.

(ix) Considering that gravitational potential energy is the gravitational binding energy.

- Gravitational binding energy is sum of the kinetic energy and potential energy

$$\text{Binding energy} = \text{KE} + \text{PE} = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$= \frac{-GMm}{2r}$$

Note that in a conservative system PE = –2KE and binding energy is equal to –KE.

(x) Considering that Kepler's laws can be applied only to planets.

- Kepler's laws may be applied to natural and artificial satellites as well.

(xi) Considering that there may be shielding effect in gravitational force.

- Since gravitational force does not depend upon medium so no medium can shield it or block it.

(xii) Considering centripetal or centrifugal force cannot be applied on the earth.

- The particles on the poles or on the axis of rotation do not have such forces because they rotate in a circle of zero radius.

(xiii) Assuming gravitational field inside a shell is zero always.

- Gravitational field inside the shell is zero because of its own mass. It exists due to other masses placed nearby, inside or outside.

(xiv) Considering that escape velocity depends upon direction.

- Theoretically it does not depend upon direction. However, practically it has little dependence.

(xv) Considering that gravitational potential energy is always negative.

- If the surface of the earth is selected as a reference and potential energy is taken as zero at the surface of the earth then net potential energy at a point outside the earth will be taken +ve. That is, positive or negative potential energy depends upon the reference point.

Solved Problems

1. A spherical shell is cut into pieces A and B. P is a point along a chord on piece A and Q is a point along a chord on piece B. If E_P and E_Q are gravitational field intensities then

(a) $E_P + E_Q = 0$ (b) $E_P = E_Q > 0$

(c) $E_P > E_Q$ (d) $E_P < E_Q$

Solution (a) Since gravitational field inside the shell $= 0$ $\therefore$ $E_P + E_Q = 0$ or $E_P = -E_Q$.

2. Three equal masses each equal to m placed at the vertices of an equilateral triangle of side l the force exerted on a particle of mass m placed at the centroid of the triangle is

(a) $\frac{4Gm^2}{2l^2}$ (b) $\frac{3Gm^2}{2l^2}$

(c) $\frac{Gm^2}{l^2}$ (d) none of these

Solution (d) $\Sigma F = 0$ since the forces F_1, F_2 and F_3 form a triangle such that all the forces are in order.

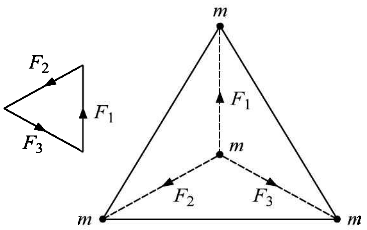


Fig. 9.9

3. The gravitational intensity due to a shell of mass m and radius R varies as

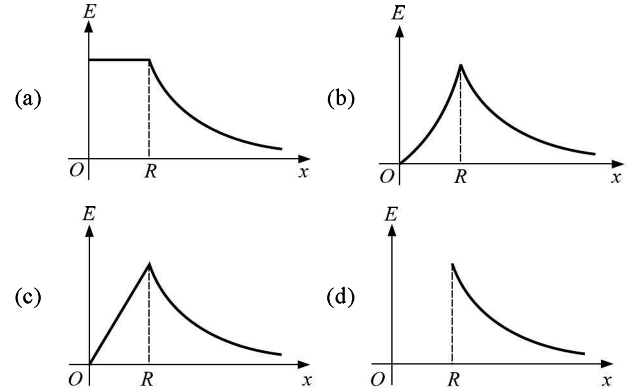


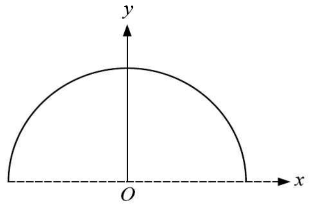
Fig. 9.10

Solution (d) Because gravitational field intensity inside a shell is zero.

4. A wire of length l and mass m is bent in the form of a semicircle. The gravitational field intensity at the centre of semicircle is

(a) $\frac{Gm}{\pi l}$ along x-axis (b) $\frac{Gm}{\pi l}$ along y axis

(c) $\frac{2\pi Gm}{l^2}$ along y-axis (d) $\frac{2\pi Gm}{l^2}$ along x-axis

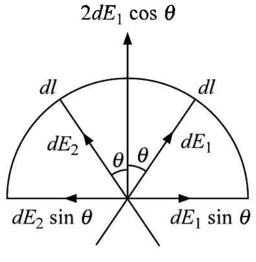


(a)

Solution (c) Consider two small elements each of length $dl = rd\theta$ symmetrically.

$$\pi r = l$$

Resolve the elemental field intensities $\because |dE_1| = |dE_2|$



(b)
Fig. 9.11

$$dE = 2dE_1\cos\theta = \frac{G2\lambda rd\theta}{r^2}\cos\theta$$

or $$dE = \frac{2\lambda G}{r}\int_0^{90^\circ}\cos\theta\, d\theta = \frac{2\lambda G}{r} = \frac{2\pi Gm}{l^2}.$$

5. The Planet X moves in an elliptical orbit around the sun. The eccentricity of the orbit is e. The ratio of $v_{\min}$ to $v_{\max}$ is

(a) $\frac{1+e}{1-e}$ (b) $\frac{1-e}{1+e}$

(c) $\frac{e}{1-e}$ (d) $\frac{e}{1+e}$

Solution (b) $\frac{v_{\max}}{v_{\min}} = \frac{r_{\min}}{r_{\max}} = \frac{1-e}{1+e}$.

6. If the total energy of an earth satellite is positive, it means that
   (a) the satellite is bound to the earth
   (b) the satellite may no longer be bound to the earth's field
   (c) the satellite moves away from the orbit making a parabolic path
   (d) the satellite escapes in a hyperbolic path

Solution (d) E_{net} = KE + PE ≥ 0 then KE ≥ PE and hence the satellite escapes in a hyperbolic path.

7. A satellite is orbiting just above the surface of a planet of average density D with period T. If G is the universal gravitational constant, the quantity $3\pi/G$ is equal to

(a) T^2D (b) $2\pi T^2D$

(c) $3\pi D^2T$ (d) D^2T

Solution (a) $T = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi\sqrt{\frac{R^3}{G\frac{4}{3}\pi R^3 D}}$

or $T^2 = \frac{4\pi^2R^3}{G\frac{4}{3}\pi R^3D} = \frac{3\pi}{DG}$ or $T^2D = 3\pi/G$.

8. In the Fig. 9.14 shown equal mass m are hung in a balance. The error in the weighing if $h_1 > h_2$ and R is the radius of the earth is

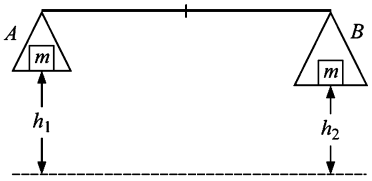


Fig. 9.12

(a) $\frac{2mg}{R}(h_1 - h_2)$ (b) $\frac{2mg}{R}(h_2 - h_1)$

(c) $mg\left(1-\frac{2h_1}{R}\right)$ (d) $2mg\left(\frac{h_2+h_1}{h_1h_2}\right)$

Solution (a) $W_1 = mg\left(1-\frac{2h_1}{R}\right)$; $W_2 = mg\left(1-\frac{2h_2}{R}\right)$

$$\Delta W = W_2 - W_1 = mg\left(\frac{2h_1}{R}-\frac{2h_2}{R}\right) = \frac{2mg}{R}(h_1 - h_2).$$

9. A body is projected with 15 kms^{-1}. What is its velocity in the outer space far off the earth (away from its gravitational field)?

(a) 3.8 kms^{-1} (b) 7.6 kms^{-1}

(c) 10 kms^{-1} (d) none of these

Solution (c) $\frac{1}{2}mv^2 = \frac{1}{2}mu_i^2 - \frac{1}{2}mv_e^2$

or $v^2 = u_i^2 - v_e^2 = (15)^2 - (11.2)^2$

$v \simeq 10$ kms^{-1}

10. To overcome the effect of weightlessness in an artificial satellite

(a) the satellite is rotated around its axis with compartment of astronauts at the centre of the satellite

(b) the satellite is shaped like a wheel.

(c) the satellite is rotated around and around till weightlessness disappears.

(d) the compartment of astronaut is kept on the periphery of rotating wheel like satellite

Solution (d) The centrifugal force acting provides artificial weight.

11. By what percentage the velocity of moon be increased so that it runs away from the gravitational field of the earth?

(a) 1% (b) 21.2%

(c) 41.4% (d) 81.1%

Solution (c) $v_e = \sqrt{2}\,V_0 = 1.414v_0$

$\Delta v = 0.414v_0$

that is, velocity be increased by 41.4%.

12. A mass *m* is at a distance *x* from one end of a uniform rod of length *l* and mass *M*. The gravitational force on the mass due to the rod is

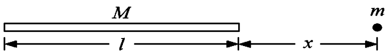


Fig. 9.13

(a) $\frac{GMm}{(x+l)^2}$ (b) $\frac{GMm}{x(l+x)}$

(c) $\frac{GMm}{x^2}$ (d) $\frac{GmM}{\left(\frac{l}{2}+x\right)^2}$

Solution (b) Consider a small element *dy* at distance *y* from the mass *m*. Then force due to this element

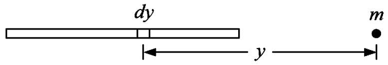


Fig. 9.14

$$dF = \frac{GM\,dy\,m}{l\,y^2}$$

$$F = \int_x^{x+l}\frac{GM\,dy\,m}{l\,y^2} = \frac{GMm}{l}\left[\frac{1}{x}-\frac{1}{x+l}\right] = \frac{GMm}{x(x+l)}.$$

13. A cavity of radius *R*/2 is created in a sphere of radius *R* and mass *M* at the periphery and this mass is placed a distance 3*R* from the centre of the sphere. The force between the two is

(a) $\frac{41GM^2}{3600R^2}$ (b) $\frac{16GM^2}{225R^2}$

(c) $\frac{2GM^2}{22R^2}$ (d) none of these

Solution (a) Let $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ be the density.

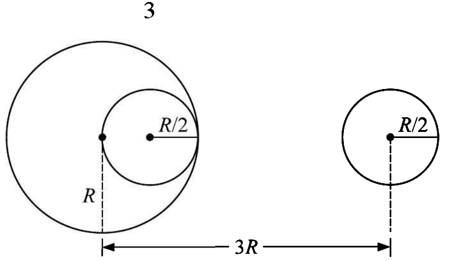


Fig. 9.15

Mass of cavity

$$M' = \rho\frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{M}{8}$$

$$F = \frac{M}{8}E = \frac{M}{8}\left[\frac{GM}{(3R)^2} - \frac{GM/8}{(5/2\,R)^2}\right] = \frac{41GM^2}{3600R^2}.$$

14. A geostationary satellite orbits around the earth in a circular orbit of radius 36000 km. Then the period of a spy satellite orbiting a few hundred kilometres above the earth's surface ($R_{\text{earth}} = 6400$ km) will approximately be

(a) 1/2*h* (b) *h*

(c) 2*h* (d) 4*h*

Solution (c) $T = \left(\frac{6400}{36000}\right)^{3/2} \times 24 \simeq 2h.$

15. A simple pendulum has a time period T_1 when on earth's surface and T_2 when taken to a height *R* above the earth's surface. *R* is the radius of the earth. The value of T_2/T_1 is

(a) 1 (b) $\sqrt{2}$

(c) 4 (d) 2

Solution (d) $T_1 = 2\pi\sqrt{\frac{l}{g}}$

$$T_2 = 2\pi\sqrt{\frac{l}{\frac{g}{\left(1+\frac{R}{R}\right)^2}}} = 2\pi\sqrt{\frac{4l}{g}} \quad \text{or} \quad \frac{T_2}{T_1} = 2.$$

16. An artificial satellite moving in a circular orbit around the earth has a total energy E_0 (KE + PE). Its PE is

(a) $-E_0$ (b) $1.5E_0$

(c) $2E_0$ (d) E_0

Solution (c)

17. A satellite is launched into a circular orbit of radius *R* around the earth. A second satellite is launched into an orbit of radius (1.01) *R*. The period of the second satellite is larger than the first one by approximately

(a) 0.7% (b) 1%

(c) 1.5% (d) 3%

Solution (c) $\frac{T_1^2}{T_2^2} = \frac{R^3}{(1.0R)^3}$

or $\frac{T_2}{T_1} = (1.01)^{3/2} = \left(1+.01\times\frac{3}{2}\right)$

% increase $.01 \times \frac{3}{2} \times 100 = 3/2\%$.

18. A solid sphere of uniform density and radius u units is located with its centre at the origin O of coordinates. Two spheres of equal radii 1 unit each with their centres at $(-2, 0, 0)$ and $(2, 0, 0)$ respectively are taken out of the solid leaving behind spherical cavities as shown in Fig. 9.16. Then

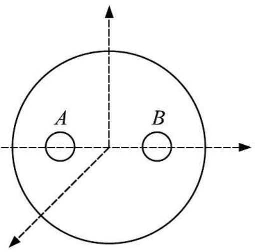


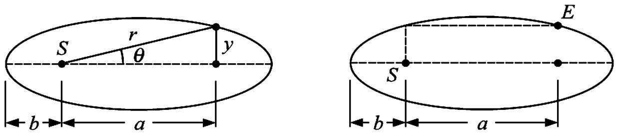
Fig. 9.16

(a) gravitational force due to this object at the origin is zero
(b) gravitational force at the point $B(2, 0, 0)$ is zero
(c) the gravitational potential is the same at all points of the circle $z^2 + y^2 = 36$
(d) the gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$

Solution (a), (c) and (d). The mass are symmetrically distributed along the circle $y^2 + z^2 = 36$ and $y^2 + z^2 = 4$.

19. The largest and the shortest distance of the earth from the sun are a and b respectively. The distance of the earth from the sun when it is at a point where its distance from the major axis is same as that of perpendicular distance of sun form the major axis

Solution Let θ be the angle between r and a



(a) **(b)**

Fig. 9.17

The $\frac{y}{r} = 1 - e\cos\theta$ where e is the eccentricity.

At $r = a$, $\theta = 0$

$$y/a = 1 - e$$

At $r = b$, $\theta = 180°$

$\frac{y}{b} = 1 + e$ Thus $\frac{y}{a} + \frac{y}{b} = 2$ or $y = \frac{2ab}{a+b}$.

20. Assume the acceleration due to gravity at the earth's surface is $10\,\text{ms}^{-2}$ and at the surface of mars it is $4\,\text{ms}^{-2}$. Then the weight of a person (60 kg) moving with a constant velocity on a spaceship is best given by the curve as a function of time (Assume no other object in the space)

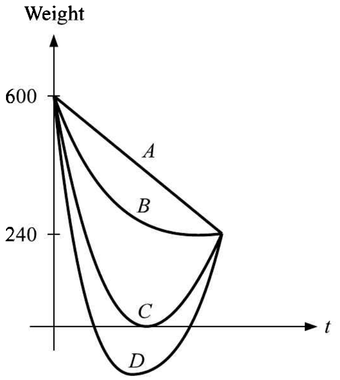


Fig. 9.18

Solution (c) In between somewhere the weight will become zero where the gravitational pull of earth and mars will be equal.

21. The Kinetic energy required to project a body of mass m from the earth's surface to infinity is

(a) $\frac{mgR}{4}$ (b) $\frac{mgR}{2}$
(c) mgR (d) $2mgR$

Solution (c) KE $\frac{1}{2}mv^2 \geq \frac{GMm}{R} = m\left(\frac{GM}{R^2}\right)R = mgR$

22. Which of the following quantities remain constant in a planetary motion (consider elliptical orbits) as seen from the sun.
(a) speed (b) angular speed
(c) kinetic energy (d) angular momentum

Solution (d) Angular momentum is conserved as there is no external torque (system is in equilibrium).

23. If V and E are gravitational potential and gravitation field intensity at a point then which of the following in not possible?
(a) $V = 0, E = 0$ (b) $E \neq 0, V = 0$
(c) $V \neq 0, E = 0$ (d) $V \neq 0, E \neq 0$

Solution (b).

24. The true weight of an object in a geostationary satellite whose weight at N-pole is 10N is
(a) 0.23 N (b) 10 N
(c) 2.3 N (d) zero

Solution (a) $mg' = \frac{mg}{\left(1+\frac{h}{R}\right)^2} = \frac{10}{(1+5.66)^2} = 0.23\text{N}$.

25. At what rate the earth should rotate so that the weight of a body at the equator of the earth becomes zero?
(a) 17 times the present speed
(b) 8 times the present speed
(c) 170 times the present speed
(d) 80 times the present speed

Solution (a) $mg' = mg\left(1-\frac{R\omega^2}{g}\right)$ at the equator

for $mg' = 0$

$$\omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{9.8}{6.4\times10^6}} = 1.237 \times 10^{-3}\,\text{rads}^{-1}$$

about 17 times the present speed of the earth.

26. The gravitational potential at a point is $V = 20(x + y)$, the gravitational field at a point (x, y) is

(a) $-10x^2\hat{i} - 10y^2\hat{j}$ (b) $-20(\hat{i}+\hat{j})$

(c) $10(x^2\hat{i}+10y^2\hat{j})$ (d) $20(i+\hat{j})$

Solution (b) $E = -\frac{\partial v}{dx}\hat{i} - \frac{\partial v}{\partial y}\hat{j} = -20\hat{i} - 20\hat{j}$

27. Three particles of mass *m* are placed at the vertices of an equilateral triangle of side *l*. The work done to bring the masses at the vertices of an equilateral triangle of side 2*l* is

(a) $\frac{3m^2G}{l}$ (b) $\frac{3m^2G}{2l}$

(c) $\frac{-3m^2G}{2l}$ (d) $\frac{3m^2G}{l}$

Solution (b) $W_{\text{one side}} = -\frac{Gm^2}{2l} - \left(\frac{-Gm^2}{l}\right) = \frac{Gm^2}{2l}$

for three sides $W_{\text{Total}} = 3W_{\text{one side}} = \frac{3Gm^2}{2l}$.

28. Two satellites are moving at a height of *R* and 5*R*. The ratio of the velocities of the two is

(a) $\sqrt{5} : 1$ (b) $\sqrt{3} : 1$

(c) $\sqrt{5} : \sqrt{2}$ (d) 1 : 1

Solution (b) $\frac{v_1}{v_2} = \frac{\sqrt{GM/2R}}{\sqrt{GM/6R}} = \frac{\sqrt{3}}{1}$.

29. Two spheres (identical) of mass *m* and radius *R* are separated by 3*R* from their centres. The force between them is proportional to

(a) R^{-2} (b) R^2

(c) R^{-4} (d) R^4

Solution (d) $F = \frac{GM_1M_2}{(3R)^2}$

$$= \frac{G\rho\left(\frac{4}{3}\pi R^3\right)\rho\left(\frac{4}{3}\pi R^3\right)}{(3R)^2} \propto R^4.$$

30. The height from the south-pole where acceleration due to gravity is equal to that at the equator is

(a) 160 km (b) 110 km

(c) 11 km (d) 16 km

Solution (c) $g' = g\left(1 - \frac{2h}{R}\right) = g\left(1 - \frac{R\omega^2}{g}\right)$

$$\therefore \quad \frac{2h}{R} = \frac{R\omega^2}{g}$$

or $h = \frac{R^2\omega^2}{2g} = \frac{(6.4\times10^6)^2 \times \left(\frac{2\pi}{3600\times24}\right)^2}{2\times10}$ = 11 km.

31. The radius of the orbit of a planet is 2 times that of the earth, the time period of the planet is

(a) 2.8 years (b) 4.2 years

(c) 5.6 years (d) 7.0 years

Solution (a) $\frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^{3/2} = \frac{T_P}{T} = \left(\frac{2}{1}\right)^{3/2}$ = 2.8 years.

32. The radius of the orbit of a Satellite is *R*. Its kinetic energy is proportional to

(a) $1/\sqrt{R}$ (b) $1/R$

(c) R (d) $1/R^{3/2}$

Solution (b) KE = $\frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{R}}\right)^2$ that is, KE ∝ 1/*R*.

33. The escape velocity for a body projected vertically upwards from the surface of earth is 11 kms^{-1}. If the body is projected at an angle 45° with the vertical, the escape velocity will be

(a) 22 kms^{-1} (b) 11 kms^{-1}

(c) $11\sqrt{2}$ kms^{-1} (d) 11 kms^{-1}

Solution (b) Escape velocity theoretically is independent of direction.

34. The time period of a satellite is 5 h. If the separation between the earth and the satellite is increased 4 times the previous value, the new time period will be

(a) 80 h (b) 40 h

(c) 20 h (d) 10 h

Solution (b) $\frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^{3/2}$ or $T_1 = 5\left(\frac{4}{1}\right)^{3/2}$ = 40h.

35. Two spherical bodies of mass *M* and 5*M* and radii *R* and 2*R* are released in free space with initial separation 12*R* between them. The smaller body moves a distance-----just before collision if they move towards each other due to gravitational force

(a) 4.5*R* (b) 7.5*R*

(c) 1.5*R* (d) 2.5*R*

Solution (b) Actual separation = 12*R* – 3*R* = 9*R*

$$\frac{a_1}{a_2} = \frac{M}{5M} = \frac{1}{5} = \frac{S_1}{S_2}$$

$\therefore \quad S_2 = 5S_1$ or $6S_1 = 9$

$S_2 = \frac{9\times5}{6}$ = 7.5 m.

36. A spring balance is attached to the ceiling of a lift. A man hangs his bag on the spring balance and reads 49 N when the lift is stationary. When the lift moves downwards with 5 ms^{-2} the reading on the balance is

(a) 74 N (b) 15 N

(c) 49 N (d) 24 N

Solution (d) $W' = M(g-a) = \frac{49}{9.8}(9.8-5)$ = 24 N

37. If W_1, W_2, W_3 represent the work done in moving a particle from *A* to *B* along 3 different paths 1, 2 and 3 respectively as shown in Fig. 9.19 in the gravitational field of a point mass *m*. The correct relation is

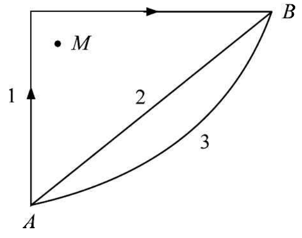


Fig. 9.19

(a) $W_1 > W_2 > W_3$

(b) $W_1 = W_2 = W_3$
(c) $W_1 > W_2 < W_3$
(d) $W_2 > W_1 > W_3$

Solution (b) Since gravitational force is conservative

38. Two spheres of masses *m* and *M* are situated in air and the gravitational force between them is *F*. The space around the masses is now filled with a liquid of specific gravity 3. The gravitational force will now be
(a) $F/9$
(b) $3F$
(c) F
(d) $F/3$

Solution (c) gravitational force is independent of the medium.

39. The radius of earth is about 6400 km. The mass of the earth is 10 times the mass of mars. An object weighs 200 N on the earth surface, then its weight on the surface of the mars is
(a) 80 N (b) 40 N
(c) 8 N (d) 20 N

Solution (d) $F = \dfrac{GMm}{R^2} = W$ as M is $\dfrac{M}{10}$

$$\therefore\ F' = W' = \frac{M}{10}$$

Assuming R to be same = 20 N.

40. The escape velocity for a body of mass 1 kg from the earth's surface is 11.2 kms^{-1}. The escape velocity for a body of mass 100 kg would be
(a) 11.2×10^2 kms^{-1}
(b) 11.2 kms^{-1}
(c) 112.0 kms^{-1}
(d) 11.2×10^{-2} kms^{-1}

Solution (b) escape velocity is independent of mass of the body.

Problems for Practice

1. The ratio of gravitational mass and inertial mass is
(a) 2 : 1 (b) 1 : 1 (c) 1 : 2 (d) 3 : 1

2. The escape velocity from moon is
(a) 7.2 kms^{-1} (b) 42 kms^{-1}
(c) 2.41 kms^{-1} (d) 11.2 kms^{-1}

3. To determine time, an astronaut in an earth satellite should use
(a) either a spring watch or a pendulum clock
(b) a spring watch
(c) neither a spring watch nor a pendulum clock
(d) a pendulum clock

4. Four particles each of mass *m*, are placed at the vertices of square and are moving along a circle of radius *r* under the influence of mutual gravitational attraction. The speed of each particle will be

(a) $\sqrt{\dfrac{Gm}{r}\left(2\sqrt{2}+1\right)}$ (b) $\sqrt{\dfrac{Gm}{r}}$

(c) $\sqrt{\dfrac{Gm}{r}\left(2\sqrt{2}+1\right)}$ (d) $\sqrt{\dfrac{2\sqrt{2}\,GM}{r}}$

5. A bomb blasts on the moon. Its sound will be heard on earth after
(a) sound will never be heard
(b) 138 minutes
(c) 10 minutes
(d) 3.7 minutes

6. A tunnel is dug along one of the diameters of the earth. The force on a particle of mass *m* distant *x* from the centre in this tunnel will be

(a) $\dfrac{GM_e m}{R^2 x}$ (b) $\dfrac{GM_e m R^3}{x}$

(c) $\dfrac{GM_e m x}{R^2}$ (d) $\dfrac{GM_e m x}{R^3}$

7. A satellite has to revolve round the earth in a circular orbit of radius 8×10^3 km. The velocity of projection of satellite in this orbit will be
(a) 16 kms^{-1} (b) 7.08 kms^{-1}
(c) 3 kms^{-1} (d) 8 kms^{-1}

8. The length of the day from today when the sun is directly overhead till tomorrow again when the sun is directly overhead can be determined by the
(a) rotation of the earth about its on axis
(b) revolution of the earth around the sun
(c) inclination of axis of rotation of the earth from the plane of revolution
(d) rotation of earth about its own axis as well as its revolution around the sun

9. The orbital velocity of Jupiter is
(a) less than the orbital velocity of earth
(b) more than the orbital velocity of earth
(c) zero
(d) equal to the orbital velocity of earth

10. Two metallic spheres each of mass *m* are suspended by two strings each of length *l*. The distance between the upper ends of strings is *l*. The angle which the string made with the vertical due to mutual attraction of the spheres is

(a) $\tan^{-1}\dfrac{Gm}{gl^2}$ (b) $\tan^{-1}\dfrac{Gm}{2gl^2}$

(c) $\tan^{-1}\dfrac{2Gm}{gl^2}$ (d) $\tan^{-1}\dfrac{2Gm}{gl}$

11. The value of acceleration due to gravity at height *h* from earth surface will become half its value on the surface if (R = radius of earth)
(a) $h = (\sqrt{2}+1)R$ (b) $h = 2R$
(c) $h = (\sqrt{2}-1)R$ (d) $h = R$

12. The velocity of a satellite in a parking orbit is

(a) $3.1\,\text{kms}^{-1}$ (b) zero

(c) $2.35\,\text{kms}^{-1}$ (d) $8\,\text{kms}^{-1}$

13. A satellite of mass *m* is revolving at height *h* from earth's surface. Its orbital velocity will be

(a) $\sqrt{\dfrac{gR_e^2}{R_e+h}}$ (b) $\sqrt{gR_e}$

(c) $\sqrt{\dfrac{gR_e}{R_e+h}}$ (d) $\dfrac{gR_e}{R_e+h}$

14. The number of communication satellites necessary for intercontinental telecast will be

(a) 6 (b) 4 (c) 5 (d) 3

15. A force of 75 Newton acts on a body of mass 2.5 kg at a centre point. The intensity of gravitation field at this point will be

(a) $40\,\text{Nkg}^{-1}$ (b) $13\,\text{Nkg}^{-1}$

(c) $20\,\text{Nkg}^{-1}$ (d) $30\,\text{Nkg}^{-1}$

16. The correct graph between the gravitational potential (v_g) due to hollow sphere and distance from its centre will be

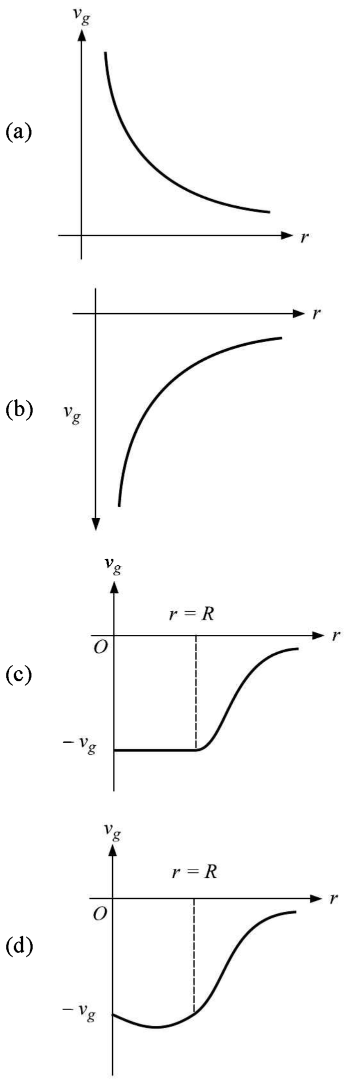


Fig. 9.20

17. The motion of a planet around then sun in an elliptical orbit is shown in the following Fig. 9.21. Sun is situated on the focus. The shaded areas are equal. If the planet takes time t_1 and t_2 in moving from *A* to *B* and from *C* to *D* respectively then

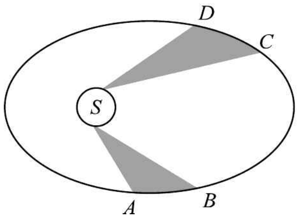


Fig. 9.21

(a) incomplete information

(b) $t_1 = t_2$

(c) $t_1 > t_2$

(d) $t_1 < t_2$

18. Three particles of equal mass *m* are situated at the vertices of an equilateral triangle of side *l*. What should be the velocity of each particle so that they move on a circular path without changing *l*?

(a) $\sqrt{\dfrac{GM}{2l}}$ (b) $\sqrt{\dfrac{2GM}{l}}$

(c) $\sqrt{\dfrac{GM}{3l}}$ (d) $\sqrt{\dfrac{GM}{l}}$

19. The escape velocity from a planet of mass *m* and radius *r* is

(a) $\sqrt{\dfrac{Gm}{r}}$ (b) $\sqrt{Gmr}$

(c) $\sqrt{\dfrac{2Gm}{r}}$ (d) $\sqrt{2Gmr}$

20. The weight of a body at the centre of the earth will be

(a) zero (b) MgR_e

(c) $\dfrac{MgR_e}{2}$ (d) infinity

21. A satellite is revolving close to the surface of a planet. If its time period is *T* and mean density of the planet is *D* then the value of the $T\sqrt{D}$ will be equal to

(a) $\sqrt{3\pi/G}$ (b) $\sqrt{g}$

(c) V_g (d) $T/\sqrt{D}$

22. The decrease in the value of *g* at height *h* from the earth's surface is

(a) $\dfrac{R}{2hg}$ (b) $\dfrac{2h}{R}g$ (c) $\dfrac{2h}{R}$ (d) $\dfrac{h}{R}g$

23. A planet of mass *M* is revolving round the sun in an elliptical orbit. If its angular momentum is *J* then the area swept per second by the line joining planet to sun will be

(a) $\dfrac{JM}{2}$ (b) $\dfrac{J}{M}$ (c) $\dfrac{J}{2M}$ (d) JM

24. A stone is dropped freely into a tunnel along the diameter of the earth, when it reaches the centre of earth then it possesses only

(a) weight (b) mass

(c) kinetic energy (d) potential energy

25. The escape velocity on moon as compared to that on earth will be

(a) equal

(b) more

(c) less

(d) sometimes less and sometimes more

26. The escape velocity from a spherical satellite is v_e. The escape velocity from another satellite of double radius and half the mean density will be

(a) $\sqrt{2}\,v_e$ (b) $v_e/2$ (c) $2v_e$ (d) $4v_e$

27. A person jumps from the fifth storey of a building with load of mass m on the head. The weight experienced by him before reaching the earth will be

(a) mg kg.wt (b) $m(g + a)$
(c) zero (d) mg

28. A body of mass 5 kg is hanging from a spring balance inside a lift. If the lift is moving downwards with an acceleration 10 ms^{-2} then the reading of the spring balance will be

(a) zero (b) 20 kg (c) 5 kg (d) 10 kg

29. Newton's law of gravitation is valid

(a) only for charged bodies
(b) all bodies
(c) only heavenly bodies
(d) only for small bodies

30. If the orbital speed of moon is increased by 41.4% then the moon will

(a) fall on earth
(b) attract all bodies on earth between it
(c) have time period equal to 27 days
(d) leave its orbit and escape out.

31. The magnitudes of gravitational field at distance t_1 and t_2 from the centre of a uniform sphere of mass m and radius r are F_1 and F_2 respectively then

(a) $\frac{F_1}{F_2} = \frac{t_1}{t_2}$ If $t_1 < r$ and $t_2 < r$

(b) $\frac{F_1}{F_2} = \frac{t_1^2}{t_2^2}$ If $t_1 > r$ and $t_2 > r$

(c) $\frac{F_1}{F_2} = \frac{t_1}{t_2}$ $t_1 > r$ and $t_2 > r$

(d) $\frac{F_1}{F_2} = \frac{t_1^2}{t_2^2}$ If $t_1 < r$ and $t_2 < r$

32. The semi major axis of the orbits of mercury and mars in the astronomical units are 0.387 and 1.524 respectively. If the time period of mercury is 0.241 year, then the time period of mars will be

(a) 2.9 years (b) 0.19 years
(c) 1.9 years (d) 6.9 years

33. Where does a body weigh maximum?

(a) in artificial satellite
(b) at poles
(c) at equator
(d) on moon

34. The orbital velocity of a satellite revolving close to earth's surface is

(a) 3.1 kms^{-1} (b) 11.2 kms^{-1}
(c) 8 kms^{-1} (d) 2.4 kms^{-1}

35. The time taken by the radio waves in coming from communication satellite to earth and back is

(a) 1 s (b) 1/2 s (c) 1/4 s (d) 1/8 s

36. A clock P is based on pendulum motion and a watch S is based on spring motion. Both are made isochronous on earth. On a planet with density equal to that of earth but radius half that of earth

(a) S will run faster than P
(b) both will run with same period as that on earth
(c) both will run equal with period different as that on earth
(d) P will run faster than S

37. How far must a body be on the line joining the earth and the sun, in order that the gravitational pull on it due to the sun is counterbalanced by that due to the earth? (Given orbital radius of earth is 10^8 km and $M_s = 3.24 \times 10^5 M_e$)

(a) 64×10^2 km (b) 1.75×10^2 km
(c) 1.75×10^9 km (d) 6400 km

38. An artificial satellite remains in an orbit around the earth because

(a) earth's attraction provides the necessary centripetal force to the satellite
(b) earth does not exert gravitational force on satellite as it is far away from earth
(c) the engine provided in the satellite constantly exerts a force in the upward direction to balance the weight of satellite
(d) attraction due to earth is balanced by attraction due to other planets.

39. On moving from poles to equator the weight of a body

(a) constantly increases
(b) constantly decreases
(c) remain constant
(d) sometimes decreases and some times increases

40. If a tunnel is dug along the diameter of the earth and a body is dropped into it, then the time taken by it to cross the tunnel once is

(a) $2\pi\sqrt{R_e/g}$ (b) $\pi\sqrt{R_e/g}$
(c) $2\pi\sqrt{g/R_e}$ (d) $\pi\sqrt{g/R_e}$

41. Two spheres of same radius and same material are placed in contact with each other. The gravitational force between them is

(a) $F \propto R^2$ (b) $F \propto R^6$
(c) $F \propto R^4$ (d) $F \propto 1/R^2$

42. A satellite is launched in a circular orbit of radius R and other satellite is launched in circular orbit of radius $1.01R$. The time period of second satellite is different from that of the first satellite

(a) 1% decrease (b) 1% increase
(c) 1.5% increase (d) 1.5% decrease

43. An artificial satellite is revolving close to the earth. Its orbital velocity mainly depends upon

(a) the radius of earth
(b) the orbital radius
(c) the mass of earth (d) the mass of satellite.

44. The work done in carrying a body of mass m from earth's surface to infinite height will be

(a) $mgR_e/2$ (b) zero
(c) infinity (d) mgR_e

45. The gravitational potential at a point above the earth's surface is -5.12×10^7 Jkg^{-1} and acceleration due to gravity is 6.4 ms^{-2}. The distance of this point from centre of earth is

(a) 36×10^3 km (b) 4×10^3 km
(c) 8×10^3 km (d) 4×10^3 km

46. A boy is playing on a swing in sitting position. The time period of oscillation of the swing is T. If the boy stands up, the time period of the swing will be

(a) more than T
(b) T
(c) less than T
(d) cannot be predicted

47. A clock S works on the oscillations of a spring. Another clock P works on the oscillations of a pendulum. Both the clocks keep correct time on the earth. What will happen if they are taken to the moon?
(a) none of them will keep correct time
(b) both of them will keep correct time
(c) only clock S will keep correct time
(d) only clock P will keep correct time

48. The gravitational mass of a body on the earth is M. The inertial mass of the same body on the moon will be
(a) zero (b) $6M$ (c) M (d) $M/6$

49. Assuming the earth as a sphere of uniform density, the acceleration due to gravity half way towards the centre of the earth will be
(a) $0.25g$ (b) $0.50g$ (c) $0.75g$ (d) $0.125g$

50. Gravitational attraction of the earth on a stone of mass M is F_s and that on hydrogen balloon of mass M located at the same place is F_H. Which of the following is true?
(a) $\vec{F}_s = -\vec{F}_H$ (b) $F_s < F_H$
(c) $\vec{F}_s = \vec{F}_H$ (d) $F_s > F_H$

ANSWERS

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (a)
11. (c)	**12.** (a)	**13.** (a)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (a)
21. (a)	**22.** (b)	**23.** (c)	**24.** (b), (c)	**25.** (c)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (d)
31. (a)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (b)
41. (c)	**42.** (c)	**43.** (a)	**44.** (d)	**45.** (c)	**46.** (c)	**47.** (c)	**48.** (c)	**49.** (b)	**50.** (c)

10

Rotational Motion

BRIEF REVIEW OF THE CONCEPTS

Moment of inertia (MOI) Moment of inertia plays the same role in rotational motion as mass in linear motion. Moment of inertia $I = \Sigma m_i r_i^2$.

$I = \int r^2 dm$ for continuous distribution of mass

$I = M k^2$

where M is the mass of the body and k is radius of gyration MOI is a tensor quantity.

Moment of inertia of bodies about an axis passing through their centre of mass (COM) and perpendicular to the plane of the body are given as

MOI of a ring $= mr^2$

MOI of a disc(solid) $= \dfrac{mr^2}{2}$

MOI of a disc(annular) $= \dfrac{m}{2}(r_1^2 + r_2^2)$

MOI of a cylinder(solid) $= \dfrac{mr^2}{2}$

MOI of a hollow cylinder $= mr^2$ (if shell type)

$= \dfrac{m}{2}(r_1^2 + r_2^2)$

MOI of a sphere (shell) $= \dfrac{2}{3}mr^2$

MOI of a solid sphere $= \dfrac{2}{5}mr^2$

MOI of a hollow sphere $= \dfrac{2}{5}m(r_1^2 + r_2^2)$

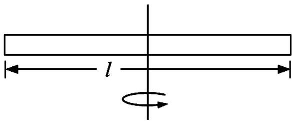


Fig. 10.1

MOI of a rod (cylindrical) $= \dfrac{ml^2}{12}$

MOI of a rectangular rod $= \dfrac{m(l^2 + b^2)}{12}$

MOI of a rectangular lamina $= \dfrac{m(l^2 + b^2)}{12}$

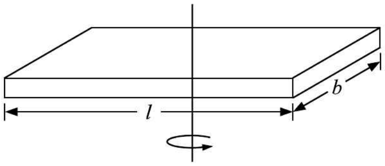


Fig. 10.2

MOI of a ring about one of the diameter is $\dfrac{mr^2}{2}$

MOI of a cylindrical rod about the edge is $\dfrac{ml^2}{3}$

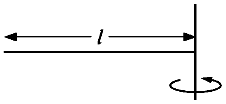


Fig. 10.3

Parallel axis theorem If MOI about an axis passing through COM of a body is known, then MOI of the body about an axis passing parallel to this axis at a distance x from it is given by

$$I = I_{COM} + mx^2$$

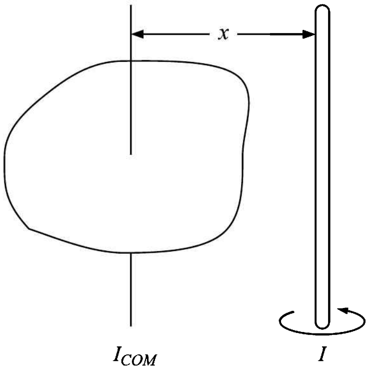


Fig. 10.4

where I_{COM} is MOI about an axis passing through their COM.

Perpendicular axis theorem This theorem is applicable only to the plane bodies. If X- and Y- be the axes chosen in the plane of the body and Z-axis be perpendicular to this plane, three being mutually perpendicular then

$$I_Z = I_X + I_Y$$

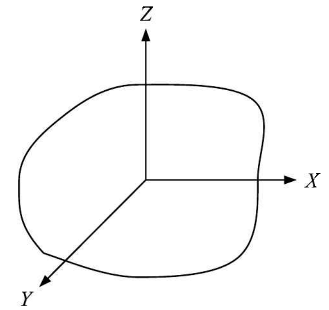


Fig. 10.5

where I_X and I_Y are MOI about X- and Y- axes respectively. Thus

Angular velocity (At any instant t is) $\omega = \dfrac{d\theta}{dt}$

Angular acceleration $\alpha = \dfrac{d\omega}{dt}$

linear velocity $\vec{v} = \vec{r}\omega$

tangential acceleration $\vec{a} = \vec{r}\alpha$

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + 1/2\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Torque (τ) $\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{\tau}| = I\alpha$

$|\vec{\tau}| = F$ (Perpendicular distance)

$\vec{\tau}_{\text{ext.}} = \dfrac{dL}{dt}$ where L is angular momentum

Torque is moment of the force about a point. The dimensions of the torque are same as that of energy but it is not energy. Its unit is (N–m). The dimensional formula is $[ML^2T^{-2}]$.

If the line of action of a force passes through its COM then it does not contribute to torque. In general, if line of action of a force passes through axis of rotation then no torque will be formed.

Angular momentum (L) $\vec{L} = \vec{r} \times \vec{p}$ where p is linear momentum

$|\vec{L}| = p \times$ (perpendicular distance)

$|\vec{L}| = I\omega$

If external torque is zero then angular momentum is conserved. Angular momentum is the moment of the linear momentum.

Angular impulse $J = \int_{t_1}^{t_2} \tau \, dt = L_2 - L_1$

Rotational kinetic energy $= \dfrac{1}{2} I\omega^2$

If a body only rotates it possesses only rotational kinetic energy. However if a body rolls on a surface then it possesses both linear kinetic energy and rotational kinetic energy.

Total KE $= \dfrac{1}{2}mv^2 + \dfrac{1}{2}I\omega^2$

work done $W = \int \vec{\tau} \cdot d\vec{\theta}$

Rotational Power $P_{\text{rot.}} = \vec{\tau}.\vec{\omega}$

MOI of a Parallelopiped

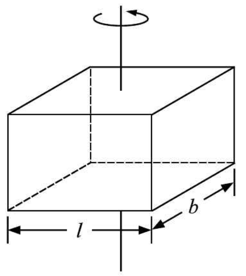


Fig. 10.6

$$I = m\frac{(l^2 + b^2)}{12}$$

MOI of an elliptical disc

$$I = \frac{m}{4}(a^2 + b^2)$$

MOI of a cone (Right circular cone)

$$I = \frac{3}{10}mr^2$$

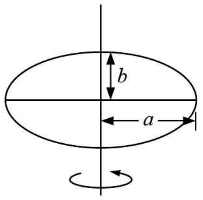


Fig. 10.7

MOI of a triangular prism or equilateral triangle $= \dfrac{ma^2}{6}$

MOI about the base of a triangular lamina is $\dfrac{mb^2}{6}$, about hypotenuse is $\dfrac{mp^2}{6}$ and about perpendicular is

$$I_h = \frac{mb^2p^2}{6(p^2 + b^2)}$$

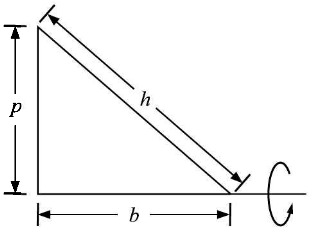


Fig. 10.8

Acceleration of a body rolling down an inclined plane

$$a = \frac{g\sin\theta}{1 + \frac{k^2}{r^2}} = \frac{g\sin\theta}{1 + \frac{I^2}{mr^2}}$$

Time taken to reach the ground

$$t = \sqrt{\frac{2l\left(1 + \frac{k^2}{r^2}\right)}{g\sin\theta}} \quad \text{(Fig. 10.9)}$$

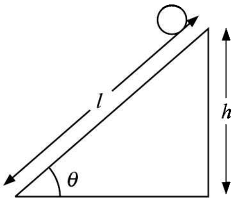


Fig. 10.9

Velocity on reaching the ground $v = \sqrt{\dfrac{2gh}{1+\dfrac{k^2}{r^2}}}$ (Fig. 10.9)

For a system to be in equilibrium (rotational) $\Sigma\tau = 0$

For a system to be in equilibrium (linear) $\Sigma F = 0$

For equilibrium (rotational + linear) $\Sigma F = 0$, $\Sigma\tau = 0$

Combined rotation and translation

$$\vec{a}_{\text{COM}} = \frac{\vec{F}_{\text{ext}}}{m} \text{ and } \alpha = \frac{\tau^{\text{ext}}_{\text{COM}}}{I_{\text{COM}}}$$

These equations together with initial conditions completely define the motion where $\tau^{\text{ext}}_{\text{COM}}$ is external torque about COM.

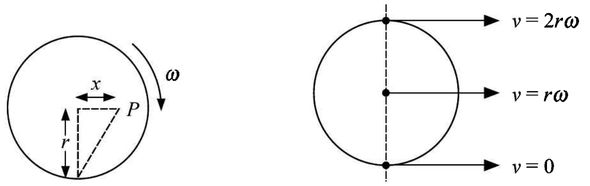
pure rolling means COM moves in a straight line and the wheel rotates in its plane about its centre with $v = r\omega$.

In pure rolling, velocity of the contact point is zero. Velocity of COM

$$v_{\text{COM}} = r\omega,\ v_{\text{top}} = 2r\omega = 2v_{\text{COM}} \text{ (Fig. 10.10(b))}$$

wheel moves a distance $2\pi r$ in one complete rotation.

In Fig. 10.10(a), in pure rolling, $V_p = \omega\sqrt{r^2 + x^2}$



(a) (b)

Fig. 10.10

If the wheel moves a distance > $2\pi r$ in one complete rotation then

$$v_{\text{COM}} > r\omega$$

and the motion is termed as rolling with forward slipping. The extreme case is pure translation where brakes are very tightly applied.

$$\omega = 0 \quad v = v_{\text{COM}}$$

Another type of rolling with slipping is when the wheel covers a distance < $2\pi r$ in one complete rotation. $v_{\text{COM}} < r\omega$ and the motion is termed as rolling with backward slipping.

Angular momentum of a body in combined rotation and translation

$$L = L_{\text{COM}} + m\vec{r}_0 \times \vec{v}_0$$

$M(\vec{r}_0 \times \vec{v}_0)$ is assumed to be the angular momentum of the body as if the mass is assumed concentrated at the COM translating with v_0.

Table 10.1 Equivalence between rotational and linear motion

Linear motion	*Rotational motion*
Displacement x	Angular displacement θ
Linear velocity $v = \dfrac{dx}{dt}$	Angular velocity $\omega = \dfrac{d\theta}{dt}$
Linear momentum $p = mv$	Angular momentum $L = I\omega$
Mass m	Moment of Inertia I
Acceleration $a = \dfrac{dv}{dt}$	Angular acceleration $\alpha = \dfrac{d\omega}{dt}$
Force $F = ma$	Torque $\tau = I\alpha$
Impulse $I = F\Delta t = \Delta p$	Rotational impulse $J = \int \tau.dt = \Delta L$
Work $W = \int \vec{F}.\vec{ds}$	Work $= \int \vec{\tau}.\vec{d\theta}$
Power $P = \vec{F}.\vec{v}$	Rotational Power $P = \vec{\tau}.\vec{\omega}$

Apply equations

$$\omega = \omega_0 + \alpha t,\quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2 \text{ and } \omega^2 - \omega_0^2 = 2\alpha\theta$$

In an accelerating wheel force of friction acts in a direction in which the wheel is moving so that frictional torque acts in a direction opposite to the applied torque.

If the wheel is rolling with forward slipping then force of friction acts in the direction opposite to the motion of the wheel until pure rolling begins.

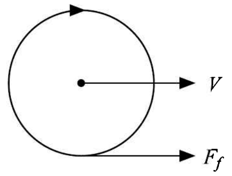


Fig. 10.11

Coriolis force If a body moves along the diameter of a turntable (rotating with an angular velocity ω) with a velocity v, then an additional force called coriolis force acts and coriolis force $F_c = 2mv\omega$.

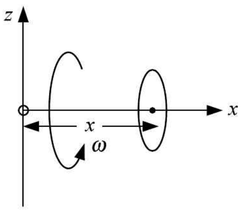


Fig. 10.12

Precession angular speed. If a fly wheel of MOI I and angular momentum L is spring with ω then precession angular speed $\Omega = \dfrac{L_z}{L_z} = \dfrac{mgr}{I\omega}$

SHORT-CUTS AND POINTS TO NOTE

(i) MOI of a ring about a diameter (*XY*) is $\frac{mr^2}{2}$

(ii) MOI of a ring about a tangent parallel to a diameter is $\frac{3mr^2}{2}$ (about *TT'*).

(iii) MOI of a disc parallel to one of the diameters is $\frac{mr^2}{4}$.

(iv) MOI of a disc about a tangent parallel to a diameter is $\frac{5}{4}mr^2$.

(v) MOI of a ring about the tangent parallel to *Z*-axis is $2mr^2$.

(vi) MOI of a disc about the tangent parallel to *Z*-axis is $\frac{3}{2}mr^2$.

(vii) MOI is a tensor the value of which changes with the direction. However they are added like scalars.

(viii) MOI of hollow bodies is more than solid bodies.

(ix) If bodies roll down an inclined plane, assuming their radii to be equal then solid sphere scales down first (the one with least MOI) and the ring reaches last (the one with maximum MOI) provided they start from the same initial point.

(x) Acceleration of bodies rolling down an inclined plane is

$$a = \frac{g\sin\theta}{1+\frac{I}{mr^2}} = \frac{g\sin\theta}{1+\frac{k^2}{r^2}}$$

(xi) A body rolling forward has friction in the forward direction provided it is being accelerated or a torque to move it acts on it.

(xii) A body rolling with forward slipping has friction in the direction opposite to the motion until pure rolling begins. In such cases friction slows down linear velocity but aids angular velocity to increase until $v = r\omega$.

(xiii) While deciding about which axis MOI is maximum, think of $I = \Sigma m_i r_1^2$ also. (before making your opinion).

(xiv) Rotational kinetic energy is $\frac{1}{2}I\omega^2$ and total kinetic energy in rotational motion is $\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$.

(xv) *x*-component of the torque is

$$\hat{i}(yF_z - F_y z) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & y & z \\ 0 & F_y & F_z \end{vmatrix}$$

Similarly *y* component is $-\hat{j}\,(xF_z - F_x z)$

and *z* component is $\vec{k}\,(xF_y - F_x y)$.

(xvi) If mass of the pulley is M_p, string is massless and pulley is smooth then

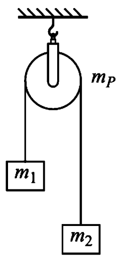


Fig. 10.13

$$a = \frac{(m_2 - m_1)g}{\left(m_1 + m_2 + \frac{m_p}{2}\right)}$$

$$= \frac{(m_2 - m_1)g}{\left(m_1 + m_2 + \frac{I}{r^2}\right)}$$

if *I* is MOI of the pulley.

⚠ CAUTION

(i) Adding MOI like vectors while finding MOI of a composite body

- MOI is a tensor, it is added like scalar.

(ii) Considering acceleration of a body rolling down an incline as $a = g\sin\theta$

- Note that acceleration is $a = \frac{g\sin\theta}{1+\frac{I}{mr^2}}$

(iii) Considering equations of motion, that is, $v = u + at$ and so on can be applied

- If motion is pure rotational apply

  $\omega = \omega_0 + \alpha t$, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

  $\omega^2 - \omega_0^2 = 2\alpha\theta$.

  If motion is combination of rotation and translation then apply both the equation sets. For conducting rotational motion apply $\omega = \omega_0 + \alpha t$ and so on and while dealing with linear motion apply $v = u + at$ and so on. To combine use $a = r\alpha$ and $v = r\omega$ when rolling.

(iv) Considering frictional force stops rotation/rolling.

- Rotational motion is stopped by frictional torque. If the body is rolling with forward slipping then friction acts in a direction opposite to the motion or when torque applied is removed then it acts in opposite direction.

(v) Considering solid bodies rotate more

- Hollow bodies have large MOI. They rotate more. Therefore all wheels are made either hollow or mass is concentrated at the rim.

(vi) Considering that a sphere can roll on a smooth inclined plane

- Minimum amount of coefficient of friction required is $\frac{2}{7}\tan\theta$, θ being angle of inclination.

(vii) When the pulley is smooth but has mass, string is massless, considering mass of the pulley is redundant

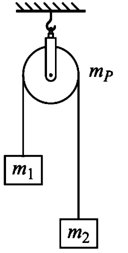


Fig. 10.14

- Acceleration $a = \dfrac{(m_2 - m_1)g}{\left(m_1 + m_2 + \dfrac{m_P}{2}\right)}$

where m_p is mass of the pulley

$$= \frac{(m_2 - m_1)g}{\left(m_1 + m_2 + \dfrac{I}{r^2}\right)}$$

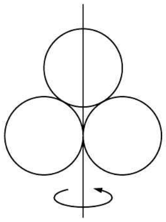
where I is MOI of the pulley.

(viii) Considering $v = r\omega$ during rolling in all cases

- $v = r\omega$ is valid in pure rolling. $v = v_{\text{COM}} = r\omega$. $v_{\text{contact point}} = 0$ and $v_{\text{top}} = 2r\omega$ during pure rolling. If rolling with forward slipping is the case then $v > r\omega$. If the case is rolling with backward slipping $v < r\omega$.

Solved Problems

1. Three rings each of mass m and radius r are so placed that they touch each other. Find the MOI about the axis as shown in the Fig. 10.15(a)



(a)

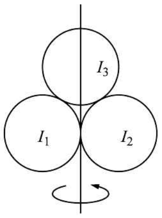
(a) $5mr^2$ (b) $\dfrac{5}{2}mr^2$

(c) $7mr^2$ (d) $\dfrac{7}{2}mr^2$

Solution (d) $I = I_1 + I_2 + I_3$

$$I_1 = I_2 = \frac{3}{2}mr^2$$

$$I_3 = \frac{mr^2}{2}$$



(b)
Fig. 10.15

$$\therefore \quad I = I_1 + I_2 + I_3 = \frac{7}{2}mr^2.$$

2. A wheel has mass of the rim 1 kg, having 50 spokes each of mass 5 g. The radius of the wheel is 40 cm. Find the MOI.

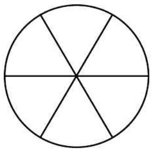


Fig. 10.16

(a) 0.273 kgm²
(b) 1.73 kgm²
(c) 0.173 kgm²
(d) 2.73 kgm²

Solution (c) $I = mr^2 + 50\dfrac{ml^2}{3}$

$$= 1 \times (0.4)^2 + \frac{50(5\times10^{-3})(.4)^2}{3}$$

$$= 0.16(1.083) = 0.173\,\text{kgm}^2.$$

3. The radius of gyration of a disc about a tangent parallel to one of the diameters is

(a) $\dfrac{r}{\sqrt{2}}$ (b) $\dfrac{\sqrt{5}}{2}r$

(c) $\dfrac{2}{\sqrt{5}}r$ (d) $\sqrt{\dfrac{5}{2}}\,r$

Solution (b) $I = \dfrac{5}{4}mr^2$ about the tangent

$$\frac{5}{4}mr^2 = mk^2 \quad \text{or} \quad k = \frac{\sqrt{5}}{2}r.$$

4. About which axis MOI in the given triangular lamina is maximum

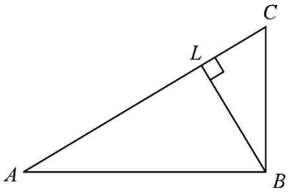


Fig. 10.17

(a) AB (b) BC

(c) AC (d) BL

Solution (b) $\because$ MOI is $\Sigma m_i r_i^2$. About BC masses are spread far away than any other axis.

5. The radius of gyration for a right circular cone is

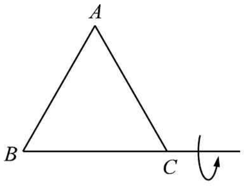
(a) $\sqrt{\frac{3}{5}}\,r$ (b) $\sqrt{\frac{5}{3}}\,r$

(c) $\sqrt{\frac{3}{10}}\,r$ (d) $\sqrt{\frac{10}{3}}\,r$

Solution (c) MOI of cone is $\frac{3}{10}mr^2 = mk^2$

$$\therefore \quad k = \sqrt{\frac{3}{10}}\,r$$

6. Three identical rods each of mass m and length l form an equilateral triangle. MOI about one of the sides is



(a)

(a) $ml^2/2$

(b) ml^2

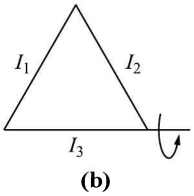
(c) $3ml^2/4$

(d) $2ml^2/4$

Solution (a) $I = I_1 + I_2 + I_3$

$$= \frac{m(l\sin 60°)^2}{3} + \frac{m(l\sin 60°)^2}{3} + 0$$

$$= ml^2/2$$



(b)

Fig. 10.18

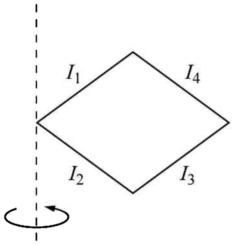
7. The MOI of a system of four rods each of length l and mass m about the axis shown is

(a) $\frac{2}{3}ml^2$ (b) $2ml^2$

(c) $3ml^2$ (d) $\frac{8}{3}ml^2$

Solution (d) Consider a square lamina, then MOI about COM

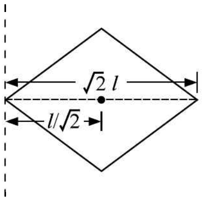
$$= 4m\left(\frac{l^2}{12} + \frac{l^2}{12}\right) = \frac{2ml^2}{3}$$



(a)

Apply perpendicular axis theorem

$$= \frac{2ml^2}{3} + 4m\left(\frac{l}{\sqrt{2}}\right)^2 = \frac{8}{3}ml^2.$$



(b)

Fig. 10.19

8. The surface density of a circular disc of radius a depends on the distance as $\rho(r) = A + Br$. The MOI about the line perpendicular to the plane of the disc is

(a) $\pi a^4\left(\frac{A}{2} + \frac{2a}{5}B\right)$ (b) $\pi a^4\left(\frac{A}{2} + \frac{2B}{5}\right)$

(c) $2\pi a^3\left(\frac{A}{2} + \frac{Ba}{5}\right)$ (d) none of these

Solution (a) $dm = 2\pi r dr\,(\rho)$

$= (A + Br)\,(2\pi\, rdr)$

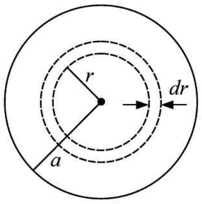


Fig. 10.20

$$I = \int_0^a dmr^2$$

$$= \frac{\pi A a^4}{2} + \frac{2\pi B a^5}{5}$$

9. One quarter of the disc of mass m is removed. The new MOI is (Radius is r)

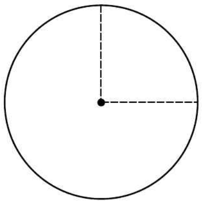


Fig. 10.21

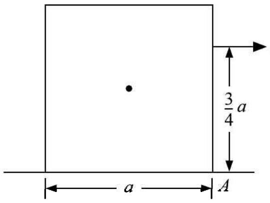
(a) $\frac{3}{2}mr^2$ (b) $\frac{mr^2}{2}$

(c) $\frac{3}{8}mr^2$ (d) none of these

Solution (c) New mass $m' = \frac{3}{4}m$

$$I = \frac{1}{2}m'r^2 = \frac{3}{8}mr^2.$$

10. A cube of side a, mass m is to be tilted at point A by applying a Force F as shown in Fig. 10.22(a) below. The minimum force required is

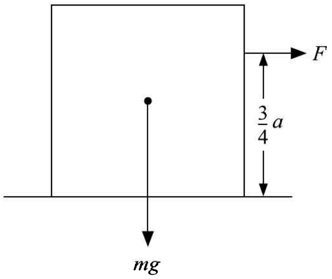


(a)

(a) mg (b) $\frac{2}{3}mg$

(c) $\frac{3}{2}mg$ (d) $\frac{3}{4}mg$

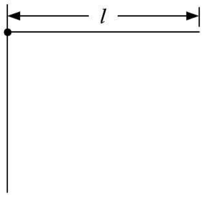
Solution See Fig. 10.22(b).



(b)
Fig. 10.22

$$mg \times \frac{a}{2} = \frac{F3a}{4} \Rightarrow F = \frac{mgl}{2} =$$

11. A rod of length l is hinged at one end and kept horizontal. It is allowed to fall. The velocity of the other end of the rod is



(a)

(a) $\sqrt{lg}$

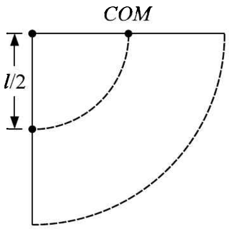
(b) $\sqrt{2lg}$

(c) $\sqrt{3lg}$

(d) none of these
COM comes down by $l/2$

Solution (c) $\frac{mgl}{2} = \frac{1}{2}I\omega^2$

$$mg\ l/2 = \frac{1}{2}\frac{ml^2\omega^2}{3} \quad \text{or} \quad v = l\omega = \sqrt{3gl}$$



(b)
Fig. 10.23

12. A light metre rod is pivoted at the centre as shown in Fig. 10.24. Masses 3 kg and 5 kg are hung from the ends. The initial angular acceleration is

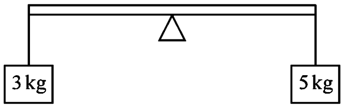


Fig. 10.24

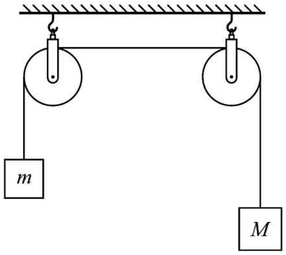
(a) $2\,\text{rad}\,\text{s}^{-2}$ (b) $5\,\text{rad}\,\text{s}^{-2}$

(c) $8\,\text{rad}\,\text{s}^{-2}$ (d) none of these

Solution (b) $I = (0.5)^2\,(3 + 5) = 2.0$

$2\alpha = (5 - 3)g\,(0.5)$ or $\alpha = 5\,\text{rad}\,\text{s}^{-2}$

13. Each pulley in Fig. 10.25(a) below has radius r and MOI I. The acceleration of the blocks is



(a)

(a) $\frac{(M-m)g}{\left(M+m+\frac{2I}{r^2}\right)}$ (b) $\frac{(M-m)g}{\left(M+m-\frac{2I}{r^2}\right)}$

(c) $\frac{(M-m)g}{\left(M+m+\frac{I}{r^2}\right)}$ (d) $\frac{(M-m)g}{\left(M+m-\frac{I}{r^2}\right)}$

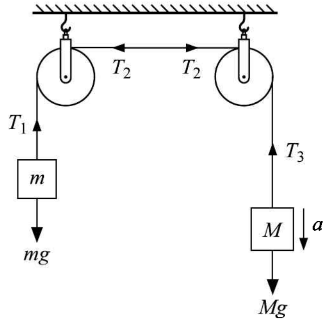
Solution (a)

$T_1 - mg = ma$ ...(1)

$r(T_2 - T_1) = I\alpha$ ...(2)

$Mg - T_3 = Ma$ ...(3)

$r(T_3 - T_2) = I\alpha$ ...(4)



(b)
Fig. 10.25

and $a = R\alpha$ from (2) and (4) we get $T_3 - T_1 = \frac{2Ia}{r^2}$

from (1) and (3) $(M-m)g = (M+m)a + T_3 - T_1$

or $(M-m)\quad g = (M+m)\,a + \frac{2Ia}{r^2}$

or $$a = \frac{(M-m)g}{\left(M+m+\frac{2I}{r^2}\right)}$$

14. A ladder of length l and mass m is placed against a smooth vertical wall, but the ground is not smooth. coefficient of friction between the ground and the ladder is μ. The angle θ at which ladder will stay in equilibrium is

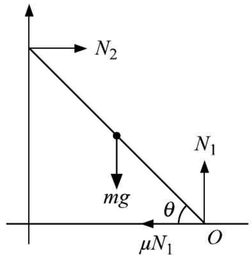


Fig. 10.26

(a) $\theta = \tan^{-1}(\mu)$
(b) $\theta = \tan^{-1}(2\mu)$
(c) $\theta = \tan^{-1}\left(\frac{\mu}{2}\right)$
(d) none of these

Solution (d) $mg = N_1$

$\mu N_1 = N_2$ or $N_2 = \mu mg$

Taking moment about O

$$\mu mgl \sin\theta = mg\frac{1}{2}\cos\theta$$

$$\tan\theta = \frac{1}{2\mu} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{1}{2\mu}\right).$$

15. The door of an almirah is 6 ft high, 1.5 ft wide and weighs 6 kg. The door is supported by two hinges situated at a distance of 1 ft from the ends. Assuming forces exerted on the hinges are equal, the magnitude of the force is
(a) 15N (b) 10N
(c) 28N (d) 43N

Solution (d) $\tan\theta = \frac{2}{3/4}$

$$= \frac{8}{3} \quad \cos\theta = \frac{3}{\sqrt{73}}$$

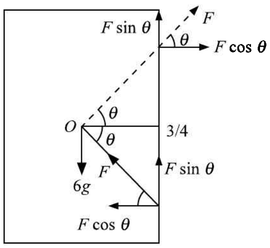


Fig. 10.27

Taking torque about O

$$6g\left(\frac{3}{4}\right) = 4F\cos\theta$$

$$6g = 4F\left(\frac{3}{\sqrt{73}}\right)$$

or $$F = 5\sqrt{73} = 43\text{N}$$

16. A gramophone record has radius r and mass M. It is rotating with a speed ω. Two drops of wax each of mass m fall on it at a distance $r/2$ on opposite sides. The new velocity is
(a) $\frac{\omega}{2}$ (b) $\frac{M\omega}{(M+m)}$
(c) $\frac{m\omega}{M+m}$ (d) $\frac{(M-m)\omega}{(M+m)}$

Solution (b) Conserve angular momentum

$$\frac{Mr^2}{2}\omega = \left[\frac{Mr^2}{2} + 2m\left(\frac{r}{2}\right)^2\right]\omega' \quad \text{or} \quad \omega' = \frac{M\omega}{(M+m)}.$$

17. If the earth shrinks half its radius without changing its mass then the length of the day will be
(a) 6h (b) 12h
(c) 8h (d) 24h

Solution (a) Conserve angular momentum

$$\frac{2}{5}mr^2\left(\frac{2\pi}{T}\right) = \frac{2}{5}m\left(\frac{B}{2}\right)^2\left(\frac{2\pi}{T}\right) \quad T' = 6h.$$

18. A diver having MOI 6 kgm^2 about an axis through centre of mass rotates at 2 rads^{-1}. If he folds his hands and feet to decrease the MOI to 5 kgm^2, the new angular velocity is
(a) 2 rads^{-1} (b) 2.4 rads^{-1}
(c) 1.67 rads^{-1} (d) 2.2 rads^{-1}

Solution (b) Using $I_1\omega_1 = I_2\omega_2$

$$\frac{6\times 2}{5} = \omega'$$

19. A sphere of radius r rolls down an inclined plane of inclination θ without slipping. The minimum value of coefficient of friction should be

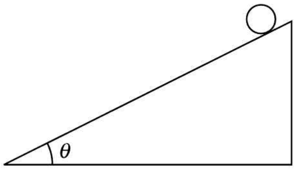


Fig. 10.28

(a) $2\sin\theta/7$
(b) $2\tan\theta/7$
(c) $2\cos\theta/7$
(d) $2\cot\theta/7$

Solution (b) Let F_f be force of friction

$$F_f r = \frac{2}{5}mr^2(\alpha) \quad \text{or} \quad F_f = \frac{2}{5}ma$$

$$F_f = \mu mg\cos\theta; \quad a = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\sin\theta$$

Thus $\mu mg\cos\theta = \frac{2}{7}mg\sin\theta$ or $\mu = \frac{2}{7}\tan\theta$.

20. A point P is fixed on a wheel rolling on a smooth horizontal plane. The point P describes
(a) straight line (b) cycloid
(c) hyperbola (d) parabola

Solution (b) Point P though covers a linear distance $2\pi R$ but moves in a cycloid as illustrated in Fig. 10.29.

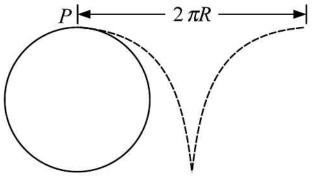


Fig. 10.29

21. A cylinder of mass m is suspended through two strings wrapped around it. The tension in the string as it falls through h is

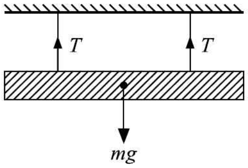


Fig. 10.30

(a) $\frac{mg}{3}$ (b) $\frac{mg}{6}$
(c) $\frac{2mg}{3}$ (d) none of these

Solution (b) $a = \frac{g\sin\theta}{1+1/2} = \frac{2}{3}g\sin\theta = \frac{2}{3}g$ $\theta = 90°$

$$2T = mg - ma = mg - \frac{2}{3}mg = \frac{mg}{3} \text{ or } T = \frac{mg}{6}.$$

22. A dancer on a turn table if suddenly folds her hands then the speed of the turn table
(a) decreases
(b) remain unchanged
(c) increases
(d) none of these

Solution (c) As MOI decreases, therefore, speed increases to conserve angular momentum.

23. Fig. 10.31 shows I_X, I_Y, I_1 and I_2 MOI of a square lamina about the axis shown. The MOI about an axis perpen-dicular to the plane and passing through COM is

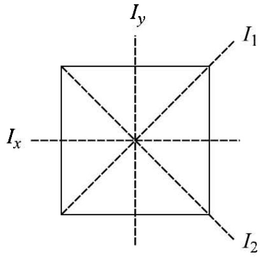


Fig. 10.31

(a) $I_1 + I_X$ (b) $I_2 + I_Y$
(c) $\frac{I_1 + I_Y}{2}$ (d) $I_1 + I_2$

Solution (a), (b), (d) $I_Z = I_X + I_Y = 2I_X$ or $I_X = I_Y = \frac{I_Z}{2}$

Similarly $I_1 = I_2 = \frac{I_Z}{2}$

Hence $I_1 = I_2 = I_X = I_Y = \frac{I_Z}{2}$

$\therefore$ $I_Z = I_1 + I_X = I_2 + I_Y = I_1 + I_2$ are correct.

24. A uniform rod is kept vertical on a smooth horizontal surface at a point A. If it is slightly rotated and released to fall on the horizontal surface, the lower end will remain
(a) at A
(b) at a distance less than $l/2$ from A
(c) at a distance $l/2$ from A
(d) at a distance $> l/2$ from A

Solution (c) So that centre of mass remain at A.

25. A sphere cannot roll on
(a) smooth horizontal surface
(b) rough horizontal surface
(c) rough inclined plane
(d) smooth inclined plane.

Solution (d) Minimum coefficient of friction required is $\frac{2}{7}$ $\tan\theta$.

26. A smooth inclined plane is fixed in a car accelerating at $a = g\tan\theta$. If the sphere is set pure rolling on the incline then

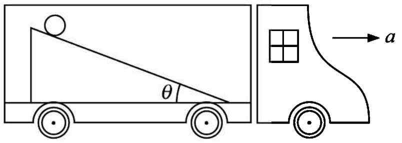


Fig. 10.32

(a) it will continue rolling
(b) it will slip down
(c) its linear velocity will increase
(d) its linear velocity will decrease.

Solution (a) It will continue rolling because incline and sphere are at rest with respect to car.

27. A sphere starts rolling down an incline plane of inclination θ. The speed of its centre when it has covered a distance l
(a) $\sqrt{2gl\sin\theta}$ (b) $\sqrt{\frac{5}{7}gl\sin\theta}$
(c) $\sqrt{\frac{10}{7}gl\sin\theta}$ (d) $\sqrt{\frac{15}{7}gl\sin\theta}$

Solution (c) $v = \sqrt{2as} = \sqrt{2\times\frac{5}{7}\text{lg}\sin\theta}$

$= \sqrt{\frac{10}{7}gl\sin\theta}$

$\because$ $a = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\sin\theta.$

28. A small solid sphere of radius r rolls down an incline without slipping which ends into a vertical loop of radius R. Find the height above the base so that it just loops the loop

(a) $\frac{5}{2}R$ (b) $\frac{5}{2}(R-r)$

(c) $\frac{25}{10}(R-r)$ (d) $\frac{27}{10}R-\frac{17r}{10}$

Solution (d) The minimum velocity at P, top of the loop should be $v = \sqrt{g(R-r)} = r\omega$

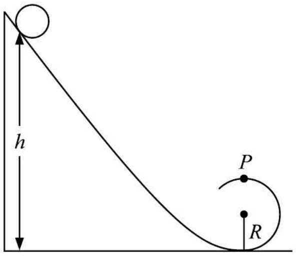


Fig. 10.33

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(2R-r)$$

$$= \frac{1}{2}mg(R-r) + \frac{1}{2}\left(\frac{2}{5}\right)mr^2\omega^2 + mg(2R-r)$$

$$= \frac{7}{10}mg(R-r) + mg(2R-r) = \frac{mg}{10}(27R-17r)$$

or $h = \frac{1}{10}(27R-17r)$.

29. A sphere is rolling down an inclined plane without slipping. The ratio of rotational kinetic energy to total kinetic energy is

(a) 5/7
(b) 2/5
(c) 2/7
(d) none of these

Solution (c) $\frac{KE_{\text{rot}}}{KE_{\text{tot}}} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2+\frac{1}{2}I\omega^2} = \frac{\frac{2}{5}mv^2}{\frac{7}{5}mv^2} = \frac{2}{7}$

30. A particle is moving along a straight line parallel to X-axis with constant velocity. Its angular velocity about the origin is

(a) remains unchanged
(b) increases
(c) decreases
(d) Insufficient data to reply

Solution (a) $L = r \times p = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ vt & y_0 & 0 \\ v & 0 & 0 \end{vmatrix}$

$L = -\hat{k}\, mvy_0$

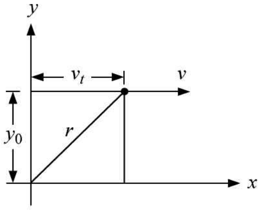


Fig. 10.34

Problems for Practice

1. Torque per unit moment of inertia is equivalent to
(a) angular velocity (b) angular acceleration
(c) radius of gyration (d) inertia

2. A circular disc starts slipping without rolling down an inclined plane then its velocity will be
(a) gh (b) $2gh$ (c) $\sqrt{gh}$ (d) $\sqrt{2gh}$

3. A spherical shell first rolls and then slips down an inclined plane. The ratio of its acceleration in two cases will be
(a) 5/3 (b) 3/5 (c) 15/13 (d) 13/15

4. The moment of inertia of a ring about its geometrical axis is I, then its moment of inertia about its diameter will be
(a) $2I$ (b) $I/2$ (c) I (d) $I/4$

5. A car is moving with a speed of 72 kmh^{-1}. The radius of its wheel is 50 cm. If its wheels come to rest after 20 rotations as a result of application of brakes, then the angular retardation produced in the car will be
(a) 23.5 rads^{-2} (b) 0.25 rads^{-2}
(c) 6.35 rads^{-2} (d) zero

6. The unit of moment of inertia is
(a) Joule/sec
(b) Joule-second/radian
(c) Joule-second^2/radian^2
(d) Joule/radian

7. A solid cylinder of mass 0.1 kg and radius 0.025 metre is rolling on a horizontal smooth table with uniform velocity of 0.1 ms^{-1}. Its total energy will be
(a) 7.5×10^{-2} Joule (b) 7.5×10^{-3} Joule
(c) 7.5×10^{-4} Joule (d) 0.07×10^{-4} Joule

8. A ring is rolling on an inclined plane. The ratio of the linear and rotational kinetic energies will be
(a) 2 : 1 (b) 1 : 2 (c) 1 : 1 (d) 4 : 1

9. The angular momentum and the moment of the inertia are respectively
(a) vector and tensor quantities
(b) scalar and vector quantities
(c) scalar and scalar quantities
(d) vector and vector quantities

10. In an arrangement four particles, each of mass 2 gm are situated at the coordinates point (3,2,0), (1,–1,0), (0,0,0) and (–1,1,0). The moment of inertia of this arrangement about the Z-axis will be
(a) 8 units (b) 16 units
(c) 43 units (d) 34 units

11. The kinetic energy of rotation of a particle is 18 Joule. If the angular momentum vector coincides with the axis of rotation and the moment of inertia of the particle about this axis is 0.01 Kgm^2, then its angular momentum will be

(a) 0.06 J-sec (b) 0.6 J-sec

(c) 0.006 J-sec (d) zero

12. The relation between the linear velocity and angular velocity is

(a) $\vec{\omega} = \vec{r} \times \vec{v}$ (b) $\vec{v} = \vec{r} \times \vec{\omega}$

(c) $\vec{v} = \vec{\omega} \times \vec{r}$ (d) $\vec{\omega} = \vec{v} \times \vec{r}$

13. A thin rod of length l and mass m is suspended from one of its ends. It is set into oscillations about a horizontal axis. If the angular speed is ω while passing through its mean position, how high will its centre of mass rise from the lowest position?

(a) $\omega^2 l^2/(6g)$ (b) $\omega^2 l^2/(2g)$

(c) $\omega^2 l^2/(3g)$ (d) $\omega^2 l^2/g$

14. Two rotating bodies have same angular momentum but their moments of inertia are I_1 and I_2 respectively ($I_1 > I_2$), which body will have higher kinetic energy of rotation

(a) first

(b) second

(c) both will have same kinetic energy

(d) not possible to predict

15. The angular velocity of the hour hand of a watch is

(a) 5 rad.h^{-1} (b) 10 rad.h^{-1}

(c) 1 rad.h^{-1} (d) 0.52 rad.h^{-1}

16. Which of the following quantities is zero about the centre of mass of a body?

(a) mass (b) moment of mass

(c) acceleration (d) angular acceleration

17. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 seconds. The value of torque is

(a) $4A_0$ (b) $12A_0$ (c) A_0 (d) $3A_0/4$

18. The ratio of kinetic energies of two spheres rolling with equal centre of mass velocities is 2 : 1. If their radii are in the ratio 2 : 1, then the ratio of their masses will be

(a) 1 : 1 (b) 1 : 8 (c) 1 : 7 (d) $2\sqrt{2}$: 1

19. The value of angular momentum of the earth rotating about its own axis is

(a) 7×10^{33} kgm^{2}s^{-1} (b) 7×10^{32} kgm^{-2}s^{-1}

(c) 0.7×10^{30} kgm^{2}s^{-1} (d) zero

20. A long thread is wrapped round a reel. If one end of thread is held in hand and the reel is allowed to fall under gravity, then the acceleration of the reel will be

(a) g (b) $2/3g$ (c) $3/2g$ (d) zero

21. A chain couples and rotates two wheels in a bicycle. The radii of bigger and smaller wheels are 0.5 m and 0.1 m respectively. The bigger wheel rotates at the rate of 200 rotations per minute, then the rate of rotation of smaller wheel will be

(a) 1000 rpm (b) $\dfrac{50}{3}$ rpm

(c) 200 rpm (d) 40 rpm

22. Minimum time period in a compound pendulum is obtained when

(a) $l = \pm k/2$ (b) $l = \pm k$

(c) $l = \pm k/\sqrt{2}$ (d) $l = 0$

23. A girl sits near the edge of rotating circular platform. If the girl moves from circumference towards the centre of the platform, then the angular velocity of the platform will

(a) decrease (b) increase

(c) remain same (d) becomes zero

24. The moment of inertia of a circular disc of mass 300 g and radius 5 cm about a tangential axis normal to the plane of the disc will be

(a) 750 g.cm^2 (b) 7500 g.cm^2

(c) 75 g.cm^2 (d) zero

25. A particle is revolving in a circle of radius r. Its displacement after completing half the revolution will be

(a) πr (b) $2r$ (c) $2\pi r$ (d) $r/2$

26. The work done in rotating a body from angle θ_1 to angle θ_2 will be

(a) $\tau/(\theta_1 - \theta_2)$ (b) $\tau(\theta_2 - \theta_1)$

(c) zero (d) $(\theta_1 - \theta_2)/\tau$

27. The velocity of centre of mass of a disc rolling on an inclined plane changes from v to $2v$ at any instant of time. If m is the mass of disc, then increase in its kinetic energy will be

(a) $\dfrac{9}{4}mv^2$ (b) $mv^2/2$ (c) mv^2 (d) $3mv^2$

28. The relation between angular momentum and angular velocity is

(a) $\vec{L} = \vec{r} \times \vec{\omega}$ (b) $\vec{L} = \vec{\omega} \times \vec{r}$

(c) $L = I/\vec{\omega}$ (d) $L = I\vec{\omega}$

29. The moment of inertia of a fly-wheel is 4 kg-m^2. A torque of 10 Newton-metre is applied on it. The angular acceleration produced will be

(a) 25 rad s^{-2} (b) 0.25 rad s^{-2}

(c) 2.5 rad s^{-2} (d) zero

30. A ring of mass 10 kg and diameter 0.4 metre is rotating about its geometrical axis at 1200 rotations per minute. Its moment of inertia and angular momentum will be respectively

(a) 0.4 kg.m^2 and 50.28 J-s

(b) 50.28 kg.m^2 and 0.4 J-s

(c) 0.4 J-s and 50.28 kg.m^2

(d) 0.4 kg.m^2 and zero

31. The angular momentum of particle is

(a) parallel to its linear momentum

(b) perpendicular to its linear momentum

(c) inclined to its linear momentum

(d) a scalar quantity

32. It is believed that many dense stars rotate at the rate of 1 rotation per second. If the radius of the star is 20 km., then the acceleration of the body situated at the equator of the star will be

(a) zero (b) 4×10^8 ms^{-2}

(c) 8×10^5 ms^{-2} (d) 20×10^3 ms^{-2}

33. Two metallic discs have same mass and same thickness but different densities. The moment of inertia about the geometrical axis will be more of the disc

(a) with lower density

(b) with higher density

(c) moment of inertia of both the discs will be same

(d) nothing can be said

34. A gramophone disc is rotating at 78 rotations per minute. Due to power cut, it comes to rest after 30 seconds. The angular retardation of the disc will be

(a) 0.27 rads^{-2} (b) 0.127 rads^{-2}

(c) 12.7 rads^{-2} (d) zero

35. A rigid body is rotating about a vertical axis at n rotations per minute. If the axis slowly becomes horizontal in t seconds as the body keeps on rotating at n rotations per minute then the torque acting on the body will be – – – if the moment of inertia of the body about axis of rotation is I

(a) zero (b) $\dfrac{2\pi nI}{60t}$

(c) $\dfrac{2\sqrt{2}\pi nI}{60t}$ (d) $\dfrac{4\pi nI}{60t}$

36. Four masses are arranged on a massless rod as shown in the Fig. 10.35. The moment of inertia of the system about yy' axis is nearly.

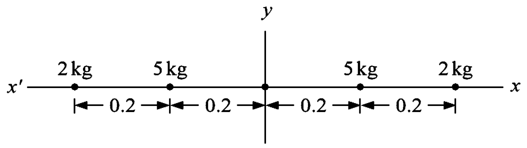


Fig. 10.35

(a) 2 kg.m^2 (b) 0.1 kg.m^2

(c) 1.04 kg.m^2 (d) zero

37. A heavy disc is rotating with uniform angular velocity ω about its own axis. A piece of wax sticks to it. The angular velocity of the disc will

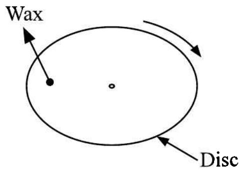


Fig. 10.36

(a) increase

(b) decrease

(c) becomes zero

(d) remain unchanged

38. A cylinder of mass M and radius r is mounted on a frictionless axle over a well. A rope of negligible mass is wrapped around the cylinder and a bucket of mass m is suspended from the rope. The linear acceleration of the bucket will be

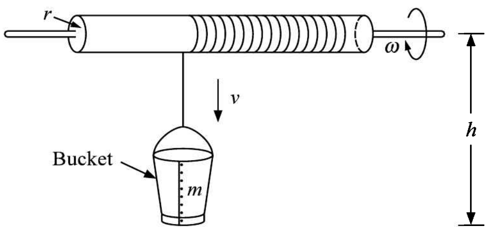


Fig. 10.37

(a) $\dfrac{mg}{M+2m}$ (b) $\dfrac{2mg}{M+2m}$

(c) $\dfrac{Mg}{2M+m}$ (d) $\dfrac{2mg}{M+2m}$

39. A disc oscillates about a tangential axis lying in its own plane. The equivalent length of a simple pendulum will be

(a) $3/2r$ (b) $\dfrac{5}{4}r$ (c) $2r$ (d) $\dfrac{5}{2}r$

40. The minimum time period of a compound pendulum is

(a) $T = 2\pi\sqrt{\dfrac{2k}{g}}$ (b) $T = 2\pi\sqrt{\dfrac{k}{g}}$

(c) $T = 2\pi\sqrt{\dfrac{l}{g}}$ (d) $T = \pi\sqrt{\dfrac{2k}{g}}$

41. A wheel is rolling on a plane road. The linear velocity of its centre of mass is v. Then velocities of the point A and B on the circumference of the wheel relative to road will be

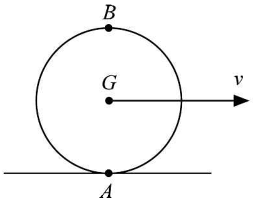


Fig. 10.38

(a) $v_A = v_B = 0$ (b) $v_A = v, v_B = 0$

(c) $v_A = 0, v_B = v$ (d) $v_A = 0, v_B = 2v$

42. A body of mass m is moving with a constant velocity parallel to x-axis. Its angular momentum about the origin will

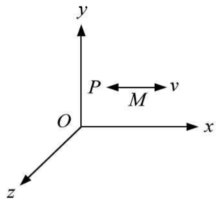


Fig. 10.39

(a) go on decreasing

(b) go on increasing

(c) remain constant

(d) becomes zero

43. If a body is lying in the $y - z$ plane, then according to theorem of perpendicular axis the correct expression will be

(a) $I_z = I_x + I_y$ (b) $I_y = I_x + I_z$

(c) $I_x = I_y + I_z$ (d) $I_y = I_z + md^2$

44. The analogues in rotatory motion of the quantities m, $\vec{v}, \vec{p}, \vec{F}$ and $\vec{a}$ in linear motion are

(a) I, $\vec{\omega}, \vec{L}\vec{\tau}$ and $\vec{a}$ (b) I, $\vec{\omega}, \vec{K}, \vec{\tau}$ and $\vec{a}$

(c) I, $\vec{\omega}, \vec{L}, \vec{E}$ and $\vec{a}$ (d) I, $\vec{\omega}, L, \vec{\tau}$ and k

45. A thin uniform rod PQ of mass m and length l is free to rotate about a hinge at P in the floor. Initially the rod is vertical. If it is released from this position then its angular velocity while striking the floor will be

(a) $3g/2$ (b) $\sqrt{g/3l}$

(c) $\sqrt{3g/l}$ (d) $\sqrt{gl}$

46. In the following Fig. 10.40 about which axis the moment inertia of the triangular lamina will be maximum

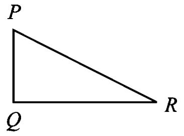


Fig. 10.40

(a) PQ (b) QR

(c) RP (d) $PQ = QR = RP$

47. A circular disc of radius r is made to oscillate in a vertical plane about a pivot situated at a distance of $r/2$ from its centre. Its angular frequency will be

(a) $\sqrt{4g/(3r)}$ (b) $\sqrt{2g/(3r)}$

(c) $\sqrt{3g/(2r)}$ (d) $\sqrt{3r/(4g)}$

48. The correct relation connecting the quantities rotational power P_r, $\vec{\omega}$ and $\vec{\tau}$ is

(a) $P_r = \omega \times \vec{\tau}$ (b) $P_r = \vec{\tau} \times \vec{\omega}$

(c) $P_r = \vec{\tau} \cdot \vec{\omega}$ (d) $P_r = \vec{\omega} \times \vec{\alpha}$

49. A thin wire of length l and mass m is bent in the form of a semicircle. Its moment of inertia about an axis joining its free ends will be

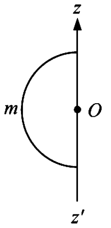


Fig. 10.41

(a) zero (b) ml^2

(c) $\frac{ml^2}{\pi^2}$ (d) $\frac{ml^2}{2\pi^2}$

50. About which axis in the following Fig. 10.42 the moment of inertia of the rectangular lamina is the maximum?

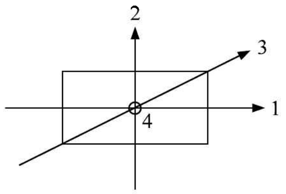


Fig. 10.42

(a) 2 (b) 3

(c) 4 (d) 1

51. A solid sphere is rolling down a plane inclined at an angle 30° from horizontal. Its acceleration will be

(a) $\frac{5}{14}g$ (b) $\frac{14}{5}g$ (c) $\frac{2}{5}g$ (d) $\frac{5}{2}g$

52. The angular speed of the earth around the sun is

(a) $\frac{2\pi}{365 \times 24 \times 60 \times 60}$ rads^{-1}

(b) $\frac{365 \times 24 \times 60 \times 60}{2\pi}$ rads^{-1}

(c) $\frac{2\pi}{24 \times 60}$ rad/sec.

(d) $\frac{2\pi}{60}$ rads^{-1}

53. Moon is revolving round the earth as well as it is rotating about its own axis. The ratio of its angular momenta in two cases will be (orbital radius of moon is = 3.82×10^8 m and radius of moon = 1.74×10^6 m)

(a) $\frac{1.22 \times 10^5}{4}$ (b) $\frac{1.22 \times 10^5}{3}$

(c) $\frac{1.22 \times 10^5}{2}$ (d) $\frac{1.22 \times 10^5}{1}$

54. A block of mass 12 kg is attached to a string wrapped around a wheel of radius 10 cm. The acceleration of the block moving down an inclined plane is measured as 2 ms^{-2}. The tension in the string is [see Fig. 10.43]

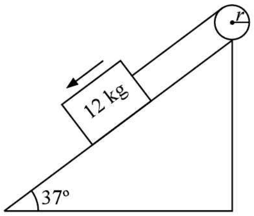


Fig. 10.43

(a) 24.5 N

(b) 68.7 N

(c) 23.4 N

(d) 46.8 N

ANSWERS

1. (b)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (d)
11. (b)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (b)	**27.** (a)	**28.** (d)	**29.** (c)	**30.** (a)
31. (b)	**32.** (d)	**33.** (a)	**34.** (a)	**35.** (c)	**36.** (c)	**37.** (b)	**38.** (d)	**39.** (b)	**40.** (a)
41. (d)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (c)	**46.** (a)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (c)
51. (a)	**52.** (a)	**53.** (b)	**54.** (d)						

Interatomic Forces and Elasticity

11

BRIEF REVIEW OF THE CONCEPTS

Interatomic and Intermolecular forces The force between two atoms can be typically represented by the potential energy curve shown in Fig. 11.1. The zero of potential energy is taken when the atoms are widely separated ($r \to \infty$). The minimum potential energy indicates equilibrium position ($r = r_0$). Note that inter atomic force is a result of attractive and repulsive forces. When $r < r_0$ repulsive force dominates. For $r > r_0$ attractive force dominates. At $r = r_0$ the attractive and repulsive forces are balanced. Therefore, at $r = r_0$, $\frac{du}{dr} = 0$

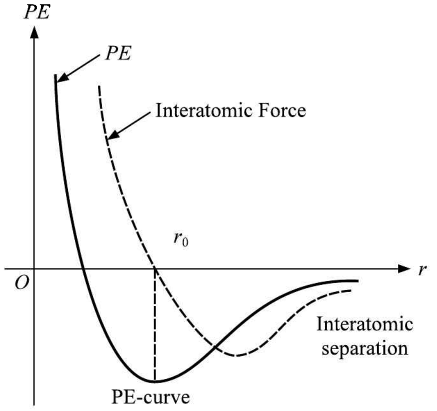


Fig. 11.1

Attractive forces $\propto \frac{1}{r^7}$ and repulsive forces are proportional to $\frac{1}{r^{13}}$

that is, $u = \frac{A}{r^{12}} - \frac{B}{r^6}$, $F = \frac{-du}{dr}$

Repulsive force contribution is $+\frac{12A}{r^{13}}$ and attractive force contribution is $= -\frac{6B}{r^7}$

Force between identical molecules is called cohesive force and force between dissimilar molecules is called adhesive force.

The range of molecular force is of the order of 10^{-10}m. A sphere around a molecule having a radius equal to the range of molecular force is called sphere of molecular action of that molecule.

Bonds The electromagnetic interaction between the electrons and nuclei result in bonds. Ionic or electrovalent bonding, covalent bonding, and metallic bonding are some important examples.

In solids the intermolecular forces are so strong that the moclecules or ions stay fixed at their equilibrium positions. If these equilibrium positions have a very regular three dimensional arrangement, we call such solids as **crystals.** The position occupied by the molecules or the ions are called **lattice points.** Crystalline solids may be divided into four categories depending upon the nature of the bonding between the basic units.

Elasticity A body is said to be elastic if on releasing deforming force it regains its original shape. If it retains its new size and shape on removal of deforming force, it is said to be plastic.

Fig. 11.2 and 11.3 show stress strain diagram for a brittle material and for a ductile material respectively. Yield point is that point from where permanent disturbance begins and the material shows elastic hysteresis.

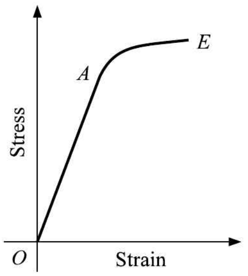


Fig. 11.2 Stress strain diagram for a brittle material

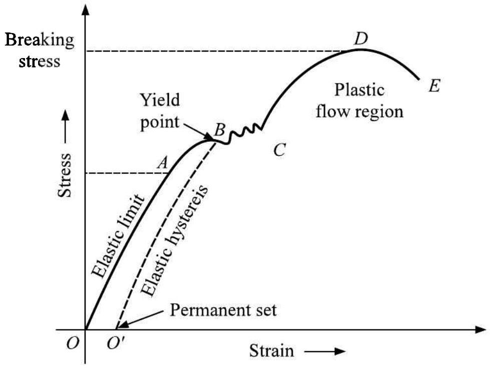


Fig.11.3 Stress strain diagram for a ductile material

More is, the *CD* part more ductile is the material.

$$\text{Working stress} = \frac{\text{Breaking stress}}{\text{Safety factor}}$$

Young's modulus *Y* It is the ratio of longitudinal stress to the longitudinal strain.

$$Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = \frac{F/A}{\Delta l/l} = \frac{Fl}{A\Delta l} = \frac{mgl}{A\Delta l}$$

Bulk modulus $B = \frac{\text{Volumetic stress}}{\text{Volumetric strain}} = \frac{P}{\frac{\Delta V}{V}} = \frac{PV}{\Delta V}$

$B_{\text{isothermal}} = P$ and $B_{\text{adiabatic}} = \gamma P$ where $\gamma = \frac{C_P}{C_V}$

Compressibility $C = \frac{1}{B}$ (reciprocal of Bulk modulus)

Shear modulus (η) It is the ratio of tangential stress to tangential strain.

$$\eta = \frac{F/A}{\frac{CC'}{CA}} = \frac{F}{A\tan\theta}$$

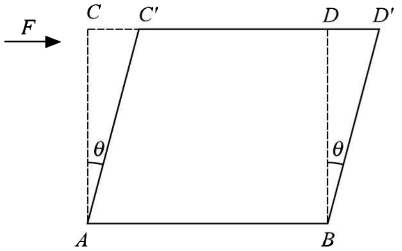


Fig. 11.4

Poisson's ratio It is the ratio of lateral strain to longitudinal strain

$$\sigma = \frac{-\Delta r/r}{\Delta l/l}$$

For most of the materials σ lies between 0.18 to 0.25. Though theoretically it may lie between –1 to +0.5. So far no substance has been found having negative Poisson's ratio. Negative value of Poisson's ratio would mean that when the wire is subjected to stress, its radius will increase.

Relation between Y, B, η and σ

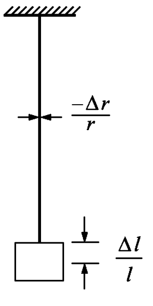


Fig. 11.5

$$B = \frac{Y}{3(1-2\sigma)}; \quad \frac{Y}{\eta} = 2(1+\sigma)$$

$$Y = \frac{9\eta B}{\eta + 3B}; \quad \frac{\eta}{3B} = \frac{1-2\sigma}{2(1+\sigma)}$$

$$\sigma = \frac{3B-2\eta}{2(\eta+3B)}$$

Torsional Couple G = where C = Torsional rigidity

Torsional Couple $G = \frac{\pi\eta r^4}{2l}$ and $\frac{T}{\phi} = \eta$ where T is tangential stress

$$\phi = \frac{x\theta}{l}$$

Elastic potential energy $u = \frac{AY}{2l}(\Delta l)^2$ (in a stretched wire)

$u = \frac{1}{2} \times$ stress $\times$ strain $\times$ volume $= \frac{1}{2}C\theta^2$

Elastic potential energy per unit volume = 1/2 stress × strain

Note: An impurity with higher elasticity increases the elasticity of material if added and an impurity of less elasticity decreases the net elasticity of the material.

Note: Elasticity of the material decreases on heating. Elasticity of **INVAR** steel remains unaffected with temperature.

Resilence is the term used to define the amount of work done in straining a body with in elastic limit. It is measure of the capacity to with stand a blow or a mechanical shock without acquiring permanent set. For a uniform cross-section rod resilence = $P^2/2Y$ where P is force per unit area. For a beam resilience = 1/2ω × (deflection at the load) and ω is load.

Proof resilence defines the maximum strain energy which can be stored without permanent strain. Stiffness of beam is the ratio of maximum deflection to its span.

SHORT-CUTS AND POINTS TO NOTE

(i) Note that from the potential energy curve at the lowest point for small variation it shows harmonic nature. Therefore, if we compress a rigid body or elongate it, it will return to mean position or acquire its original shape. that is, a body remains perfectly elastic when the compression or elongation is small.

(ii) Repulsion is strong when the interatomic distance < r_0 and attraction is strong when $r > r_0$. These forces in one word may be called restoring force.

(iii) Potential energy $u = \frac{A}{r^{12}} - \frac{B}{r^6}$

or $F = \frac{-\partial u}{\partial r} = \frac{12A}{r^{13}} - \frac{6B}{r^7}$

and $r_0 = \left(\frac{2A}{B}\right)^{1/6}$ obtained by putting $F = 0$

Interatomic force constant

$$K = \frac{\text{Interatomic force}}{\text{Change in interactomic distance}}$$

(iv) A molecule having a radius equal to the range of molecular force is called sphere of molecular action of that molecule.

(v) Solids may be molecular, ionic, covalent and metallic if they are crystalline. Amorphous solids are glassy or supersaturated supercooled fluid with strong intermolecular forces. Ionic solids have quite high melting points and are poor conductors in solid form and good conductors in molten or aqueous solution.

Polar molecules have dipole-dipole bond and the bond between nonpolar molecules is van der Waal's bond.

(vi) Brittle substances cannot bear more strain while ductile substances can bear large strain. Hooke's Law → stress ∝ strain within elastic limits.

(vii) Working stress = $\dfrac{\text{Breaking stress}}{\text{Safety factor}}$. It is an important parameter while designing bridges/building/gas cylinders and so on.

(viii) Young's modulus $Y = \dfrac{Fl}{A\Delta l} = \dfrac{Mgl}{A\Delta l}$.

(ix) Bulk modulus $B = \dfrac{PV}{\Delta V}$, $B_{\text{isothermal}} = P$ and

$$B_{\text{adiabatic}} = \gamma P \text{ where } \gamma = C_p/C_V$$

(x) Compressibility $C = 1/B$ (reciprocal of Bulk odulus).

(xi) Shear modulus η is the ratio of tangential stress to tangential strain.

$$\eta = F/(A \tan\theta).$$

(xii) Poisson's ratio $\sigma = \dfrac{-\Delta r / r}{\Delta l / l}$.

(xiii) $\dfrac{Y}{\eta} = 2(1 + \sigma)$, $B = \dfrac{Y}{3(1-2\sigma)}$; $Y = \dfrac{9\eta B}{\eta + 3B}$

$\dfrac{\eta}{3B} = \dfrac{1-2\sigma}{2(1+\sigma)}$, $\sigma = \dfrac{3B - 2\eta}{2(\eta + 3B)}$

Torsional couple $G = \dfrac{\pi\eta r^4}{2l}$

$\eta = \dfrac{T}{\phi}$ where $\phi = \dfrac{x\theta}{l}$ $G = C\theta$, where T is tangential stress

(xiv) Elastic energy $u = \dfrac{AY\Delta l^2}{2l} = \dfrac{1}{2}$ × stress × strain × volume.

Elastic energy per unit volume = $\dfrac{1}{2}$ × stress × strain.

(xv) Elastic fatigue When the deforming force applied on the body is changed rapidly, it loses the property of elasticity temporarily. This is called elastic fatigue.

(xvi) Elastic relaxation The property by virtue of which body does not regain its original shape immediately after removal of the deforming force. It takes some time to regain its original shape.

(xvii) Bending moment Torque necessary to bend a beam is defined as the bending moment, $G = \dfrac{YI_G}{R}$ where I_G = geometric MOI of bent beam and R = Radius of curvature.

Flexural rigidity $YI_a = GR$

For a rod of breadth b and thickness d $I_G = \dfrac{bd^3}{12}$

For a rod of circular cross-section $I_G = \dfrac{\pi r^4}{4}$

$I_G = \dfrac{\pi(r_2^4 - r_1^4)}{4}$ if the rod is hollow.

(xviii) Cantilever The beam clamped at one end and loaded at another end is called a cantilever.

$$\delta = \frac{Wl^3}{3YI_G}$$

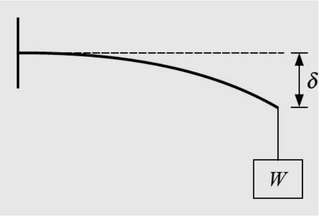


Fig. 11.6

$I_G = \dfrac{bd^3}{12}$ for a rectangular rod of width b and thickness d.

$I_G = \dfrac{\pi r^4}{4}$ for a circular cross-sectional rod.

$$\therefore \quad \delta = \frac{4Wl^3}{3\pi r^4 Y}$$

(xix) Depression produced in a beam supported at two ends and loaded in the middle as shown in Fig. 11.7

$$\delta = \frac{Wl^3}{48 I_G Y}$$

For a rectangular cross-section $\delta = \dfrac{Wl^3}{4bd^3 Y}$

for a circular cross-section $\delta = \dfrac{Wl^3}{12\pi r^4 Y}$.

(xx) Elastic relaxation time The time delay in regaining the original shape after removal of deforming forces. Elastic relaxation time for Gold, Silver and phosphor bronze is negligible. For quartz it is minimum, therefore, quartz fibre is nearly perfectly elastic. Putty is perfectly plastic body (Putty-gypsum).

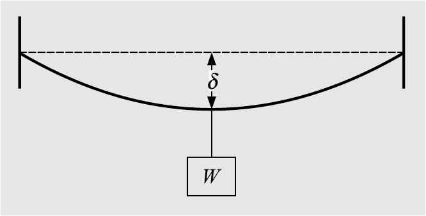


Fig. 11.7

(xxi) Young's modulus is practically equal to stress which will double the length of a wire.

(xxii) Remember stress is a tensor as its value changes when direction changes. Pressure is a scalar as it acts normally.

⚠ CAUTION

(i) Considering molecular forces as only attractive.

- If $r > r_0$, net force is attractive. If $r < r_0$, the net force is repulsive. It forms the basis of elastic force which is restoring in nature.

(ii) Considering all type of bonds are equally strong.

- Strong bonds like ionic and covalent make the material hard and give it high melting point.

(iii) Considering that ionic solids are conductors of electricity.

- In solid form they are poor conductors. They become good conductors in molten form or in aqueous solution.

(iv) Considering bulk modulus and compressibility as same quantities.

- Compressibility is reciprocal of bulk modulus that is,

$$\text{Compressibility} = \frac{1}{\text{bulk modulus}}.$$

(v) Considering that shear strain can be applied only in cubiodal solids.

- It can be applied to any shape of the body.

  During torsional oscillations shear strain is applied to the wire/string supporting the disc/rod and so on or shear may produce torsion.

(vi) Not remembering formulae between Y, σ, B and η

- Note $B = \frac{Y}{3(1-2\sigma)}$; $\frac{Y}{\eta} = 2(1+\sigma)$,

$$Y = \frac{9\eta B}{\eta + 3B}$$

$$\sigma = \frac{3B-2\eta}{2(\eta+3B)} \quad \text{Torsional couple } C = \frac{\pi\eta r^4}{2l}.$$

If Tangential stress is T then $\frac{T}{\phi} = \eta$ where ϕ is shear angle.

$$\phi = \frac{x\theta}{l} \text{ where } \theta \text{ is angle of twist.}$$

(vii) Confusing with elastic potential energy and elastic potential energy per unit volume.

- Elastic potential energy per unit volume

$$\text{Potential energy/vol} = \frac{1}{2}\text{stress} \times \text{strain}$$

Elastic potential energy

$$= \frac{1}{2}\text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{AY(\Delta l)^2}{2l}.$$

(viii) Confusing Young's modulus with breaking stress.

- The units of all modulus of elasticity are Nm^{-2} or Pascal (Pa), that is, of stress or pressure but all modulus of elasticity are determined as the ratio of stress to strain. Breaking stress is simply a stress. When a stress greater than breaking stress is applied, the rigid body under investigation breaks.

(ix) Considering breaking stress and working stress are equal

- Working stress $= \frac{\text{breaking stress}}{\text{safety limit}}$.

Solved Problems

1. A student plots a graph from his readings on the determination of Young's modulus of a metal wire but forgets to label. The quantities on X and Y axes may be respectively

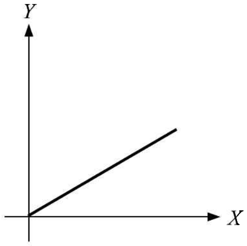


Fig. 11.8

(a) weight hung and length increased
(b) stress applied and length increased
(c) stress applied and strain developed
(d) length increased and weight hung

Solution Since stress ∝ strain, therefore, all of these can be used.

2. When a metal wire is stretched by a load, the fractional change in the volume $\Delta V/V$ is proportional to

(a) $\frac{\Delta l}{l}$ (b) $\left(\frac{\Delta l}{l}\right)^2$
(c) $\sqrt{\frac{\Delta l}{l}}$ (d) none of these

Solution (a) since $V = \pi r^2 l$ $\quad\therefore\quad \frac{dV}{V} \propto \frac{dl}{l}$.

3. A uniform heavy rod of weight W, cross-sectional area A and length l is hanging from a fixed support. Young's modulus of the material of the rod is Y. Neglect the lateral contraction. The elongation of the rod is

(a) $\frac{Wl}{AY}$ (b) $\frac{Wl}{2AY}$
(c) $\frac{Wl}{4AY}$ (d) $\frac{Wl}{3AY}$

Solution (b) Consider a small length dx of the rod at a distance x from the fixed end. The part below this small element has length $(l - x)$. The tension of the rod at the element equals the weight of the rod below

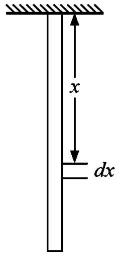


Fig. 11.9

$F = (l - x)\frac{W}{l}$;

$\Delta l = \frac{\text{orignal length} \times \text{stress}}{Y}$

$= \frac{F.dx}{AY} = \frac{(l-x)Wdx}{lAY}$

Total elongation $= \int_0^l \frac{(l-x)Wdx}{lAY}$

$= \frac{W}{lAY}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{Wl}{2AY}$.

4. The length of a metal wire is l_1 when the tension in it is T_1 and is l_2 when the tension is T_2. The natural length of the wire is

(a) $\frac{l_1 + l_2}{2}$ (b) $\sqrt{l_1 l_2}$

(c) $\frac{l_1T_2 - l_2T_1}{T_2 - T_1}$ (d) $\frac{l_1T_2 + l_2T_1}{T_2 + T_1}$

Solution (c) $\frac{\Delta l_1}{\Delta l_2} = \frac{T_1}{T_2}$ or $\Delta l_1 T_2 = \Delta l_2 T_1$

Put $l_1 = l + \Delta l_1$ and $l_2 = l + \Delta l_2$

$l_1T_2 - l_2T_1 = (l + \Delta l_1)T_2 - (l + \Delta l_2)T_1$

or $l = \frac{l_1T_2 - l_2T_1}{T_2 - T_1}$.

5. A rope 1 cm in diameter breaks if the tension in it exceeds 500 N. The maximum tension that may be given to a similar rope of diameter 2 cm is

(a) 500 N (b) 250 N

(c) 1000 N (d) 2000 N

Solution (d) Breaking force = breaking stress × Area = breaking stress × πr^2.

6. A wire can sustain a weight of 20 kg before breaking. The wire is cut into two halves, each part can sustain a force of

(a) 10 kg (b) 20 kg

(c) 40 kg (d) 80 kg

Solution (b) Breaking force ∝ area of cross-section. As area of cross-section remains unchanged, therefore, breaking force remains the same.

7. A cylinder manufacturer for LPG along with checking against leakage has to test for

(a) Young's modulus (b) Bulk modulus

(c) Shear modulus (d) Working stress

Solution (d)

8. What is the density of lead under a pressure of 2×10^4 Ncm^{-2} (density of lead = 11.4 gcm^{-3} and bulk modulus = 8×10^9 Pa)

(a) 11.4 g cm^{-3} (b) 11.115 g cm^{-3}

(c) 11.685 g cm^{-3} (d) none of these

Solution (c) $B = \frac{P}{\frac{\Delta V}{V}} = \frac{P}{\frac{d\rho}{\rho}}$

or $d\rho = \frac{P\rho}{B} = \frac{11.4 \times 2 \times 10^4}{8 \times 10^9 \times 10^{-4}} = 0.285$

$\rho = 11.4 + 0.285 = 11.685$ gcm^{-3}.

9. Assuming that shear stress at the base of the mountain is equal to the force/area due to its weight. Calculate the maximum possible height of a mountain on the earth if breaking shear stress for a typical rock is 30×10^7 Nm^{-2} and the density is 3×10^3 kgm^{-3}

(a) 11 kg (b) 10 kg

(c) 12 kg (d) 13 kg

Solution (b) $W = Ah\rho g$ $p = W/A = h\rho g$

$h\rho g <$ breaking stress $h < \frac{30 \times 10^7}{3 \times 10^3 \times 10} = 10$ km

$h_{max} = 10$ km.

10. The force F is needed to punch a 1.46 cm diameter hole in a steel plate 11.27 cm thick. The ultimate shear strength of steel is 3.45×10^6 Nm^{-2}. The value of F is

(a) 2×10^3 N (b) 2×10^4 N

(c) 2×10^5 N (d) 200 N

Solution (c) $F/A >$ ultimate shear stress

$F > A \times$ (ultimate shear stress)

$= 3.45 \times 10^6 (2 \times 3.14 \times .73 \times 10^{-2} \times 11.27 \times 10^{-2}) = 200$ KN.

11. A uniform pressure p is exerted on all sides of a solid cube at temperature t°C. By what amount should the temperature of the cube be raised in order to bring the volume back to the volume it had before the pressure was applied.

(a) $P/(\gamma B)$ (b) $2p(\gamma B)$

(c) $3p(\gamma B)$ (d) none of these

Solution (a) $\Delta V = \frac{Vp}{B} = V\gamma\Delta\theta$

{$\therefore \Delta V = V\gamma\Delta\theta$ volumetric expansion}

or $\Delta\theta = p(\gamma B)$.

12. A solid sphere of radius r made of a material of bulk modulus B is surrounded by a liquid in a cylindrical container. A massless piston of area A floats on the surface of a liquid. The fractional change in radius of the sphere when a mass m is placed on the piston to compress the liquid

(a) $\frac{mg}{AB}$ (b) $\frac{mg}{3AB}$

(c) $\frac{3mg}{AB}$ (d) none of these

Solution (b) For a spherical body

$V = \frac{4}{3}\pi r^3$ $\frac{\Delta r}{r} = \frac{\Delta V}{3V}$ $\left(\Delta P = \frac{Mg}{A}\right)$

$B = \frac{V\Delta P}{\Delta V}$; $\frac{\Delta V}{V} = \frac{mg}{AB}$ or $\frac{\Delta R}{R} = \frac{mg}{3AB}$.

13. Consider the Fig. 11.10 shown below. A wire of radius 0.2 mm is used. The extension produced is 1 cm. The Young's modulus is nearly

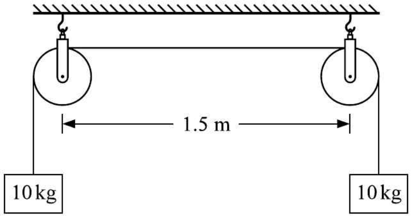


Fig. 11.10

(a) 2.4×10^{11} Pa (b) 1.2×10^{11} Pa

(c) 2.0×10^{11} Pa (d) none of these

Solution (b) $Y = \frac{mgl}{\pi r^2 \Delta l} = \frac{10\times10\times1.5}{3.14\times4\times10^{-8}\times1\times10^{-2}}$

$= 1.2 \times 10^{11}$ Pa.

14. The greatest length of steel wire that can hang vertically without breaking is ---. Breaking stress of steel is $7.8 \times 10^8\text{Nm}^{-2}$. Density of steel is $7.8\,\text{gcm}^{-3}$.

(a) 10 km (b) 1 km

(c) 10^4 cm (d) 10^2 km

Solution (a) Breaking force F = breaking stress × area

$\rho \pi r^2 lg = 7.8 \times 10^8 (\pi r^2)$

or $l = \frac{7.8\times10^8}{7.8\times10^3\times10} = 10^4\text{m} = 10\text{km}.$

15. Two wires of equal length and of same material are connected as shown in Fig. 11.11. What minimum weight be hanged in the hanger so that one of the wire breaks. Breaking stress = 10^{10}Pa.

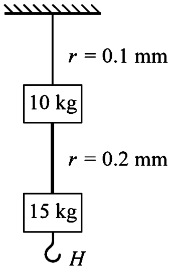


Fig. 11.11

(a) 6.4 kg (b) 10.4 kg

(c) 1.4 kg (d) 16.4 kg

Solution (a) $F = (\pi r^2)$ (breaking stress)

$= 3.14\,(10^{-4})^2\,(10^{10}) = 314\text{N}$

As thinner wire will break, therefore, we take the radius of thinner wire.

Force already applied = 250 N

Additional Force to be applied

= 314 − 250 = 64 N or 6.4 kg.

16. A bar has cross-sectional area A. It is subjected to equal and opposite tensile forces F as shown. Consider a plane through the bar making an angle θ with a plane at right angles to the bar. Find shear stress and tensile stress.

(a) $\frac{F\sin2\theta}{2A}, \frac{F\cos\theta}{A}$ (b) $\frac{F\sin2\theta}{A}, \frac{F\cos^2\theta}{A}$

(c) $\frac{F\sin2\theta}{A}, \frac{F\cos^2\theta}{2A}$ (d) $\frac{F\sin2\theta}{2A}, \frac{F\cos^2\theta}{A}$

Solution (d) shear stress $= \frac{F\sin\theta}{\frac{A}{\cos\theta}} = \frac{F\sin\theta\cos\theta}{A}$

$= \frac{F\sin2\theta}{2A}$

tensile stress $= \frac{F\cos\theta}{A/\cos\theta} = \frac{F\cos^2\theta}{A}$ (normal force)

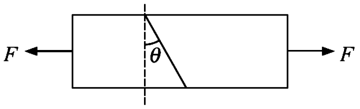


Fig. 11.12

17. Find the increase in pressure required to decrease the volume of a water sample by 0.01%. Bulk modulus of water = $2.1 \times 10^9\text{Nm}^{-2}$.

(a) $2.1 \times 10^5\text{Nm}^{-2}$ (b) $1.2 \times 10^5\text{Nm}^{-2}$

(c) $1.2 \times 10^6\text{Nm}^{-2}$ (d) $2.1 \times 10^6\text{Nm}^{-2}$

Solution (a) $B = \frac{P}{\frac{\Delta V}{V}}$

or $P = B\frac{\Delta V}{V} = 2.1 \times 10^9\left(\frac{0.01}{100}\right) = 2.1 \times 10^5\text{Pa}.$

19. A copper wire of negligible mass, length 1 m and cross-sectional area 10^{-6}m^2 is kept on a smooth horizontal table with one end fixed. A ball of mass 1 kg is attached to the other end. If the wire and ball are rotating with an angular velocity of $100\,\text{rads}^{-1}$, the wire breaks down. Find the breaking stress.

Solution $F = mr\omega^2$; Breaking stress $= \frac{mr\omega^2}{\text{area}}$

Breaking stress $= \frac{1\times1\times(100)^2}{10^{-6}} = 10^{10}\text{Pa}.$

20. A wooden wheel of diameter D is fitted with a metallic sleeve of diameter d and cross-sectional area A, Diameter d is slightly less than D. If Young's modulus of steel is Y, the force used to expand the sleeve is given by

(a) $\frac{YAd}{D-d}$ (b) $YA\frac{d}{D}$

(c) $\frac{YA(D-d)}{d}$ (d) $YA\frac{D}{d}$

Solution (c) $Y = \frac{Fl}{A\Delta l} = \frac{F(2\pi d)}{A\,4\pi\left(\frac{D-d}{2}\right)} = \frac{Fd}{A(D-d)}$

or $F = \frac{YA(D-d)}{d}.$

Problems for Practice

1. The thermal conductivity of silver and copper is high because they have the following bonding.

(a) covalent (b) metallic

(c) ionic (d) van der waal's

2. The volume of an fcc crystal having N atoms is (a = lattice parametre).

(a) $Na^3/2$ (b) $2Na^3$

(b) $Na^3/4$ (d) Na^3

3. The distance between two neutral atoms in equilibrium is
(a) $(2A/B)^{1/6}$ (b) $(2A/B)^{-1/2}$
(c) $\sqrt{2A/B}$ (d) $2A/B$

4. The another name for a strong electric interaction is
(a) van der waal (b) covalent
(c) metallic (d) ionic

5. The ratio of kinetic energy to potential energy for solids is
(a) $KE/U < 1$ (b) $KE > U$
(c) $KE = U$ (d) $KE/U > 1$

6. The lattice of Na and Al are bcc and fcc respectively. Presuming them to be closely packed, their packing factors are respectively
(a) 0.52 and 0.52 (b) 0.68 and 0.34
(c) 0.68 and 0.74 (d) 0.34 and 0.34

7. In order to separate out two molecules from each other, the potential energy between them must be
(a) minimum (b) positive
(c) negative (d) zero

8. On reducing the distance between atoms to half, the Van der waal‘s attraction force due to dipole-dipole interaction will become
(a) 32 times less
(b) 128 times more
(c) 32 times more
(d) 128 times less

9. LiF_2 is
(a) a metallic crystal
(b) a covalent crystal
(c) a molecular crystal
(d) an ionic crystal

10. The points of maximum and minimum attraction in the curve between potential energy (u) and distance (r) of a diatomic molecules are respectively

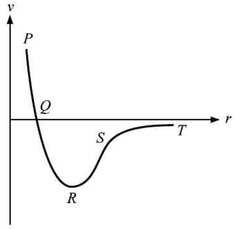


Fig. 11.13

(a) S and R (b) T and S
(c) R and S (d) S and T

11. In polar molecules the positive and negative charge centres
(a) coincide
(b) do not coincide
(c) sometimes coincide and some times do not
(d) nothing can be predicted

12. The interaction abundantly found in Cu, Ag and Au is
(a) electron-electron (b) attraction
(c) repulsion (d) ion-ion

13. Two lead balls, when pressed against each other, stick together because of
(a) surface tension (b) viscosity
(c) friction (d) intermolecular forces

14. The volume of a crystal in bcc structure having N atoms is – (a = lattice parameter)
(a) $0.5Na^3$ (b) $0.25Na^3$
(c) $0.33Na^3$ (d) Na^3

15. Two electrons each of two atoms P and Q form a compound PQ. This is an example of
(a) polar covalent bond
(b) non-polar covalent band
(c) polar molecules
(d) non-polar molecules

16. The electrical conductivity of materials bound by ionic binding is
(a) zero (b) very low
(c) very high (d) infinity

17. The number of atoms per unit cell for a simple cubic lattice is
(a) 1 (b) 2 (c) 4 (d) 6

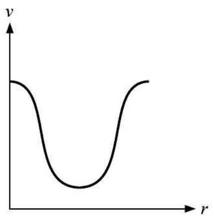
18. The distance between two atoms in contact is known as
(a) atomic radius (b) molecular sphere
(c) lattice parameter (d) packing factor

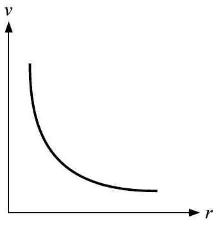
19. The intermolecular force is related to distance as
(a) $F \propto r^7$ (b) $F \propto r^2$
(c) $F \propto 1/r^7$ (d) $F \propto 1/r^2$

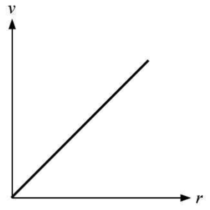
20. In diamond crystal lattice, the angle enclosed by the bonds between different carbon atoms is
(a) 45° (b) 90°
(c) 109°28’ (d) 60°

21. The relation connecting the energy u and distance r between dipole and induced dipole is
(a) $u \propto r^2$(b) $u \propto r^6$
(c) $u \propto r$ (d) $u \propto r^{-6}$

22. The correct curve for a stable diatomic molecule is

(a)


(b)


(c)


(d) None of these

Fig. 11.14

23. The atomic radius for a body centered cubic lattice is
(a) $\frac{a}{2\sqrt{2}}$ (b) $\frac{a}{2}$ (c) a (d) $\frac{\sqrt{3}}{4}a$

24. The molecular range for materials is
(a) 10^{-12}m (b) 10^{-9}m
(c) 10^{-10}m (d) 10^{-11}m

25. The packing factor for a simple cubic lattice is
(a) $\pi/6$ (b) $\frac{\sqrt{3}\pi}{8}$ (c) π (d) $\frac{\pi\sqrt{2}}{6}$

26. The melting point of a material with low binding energy is
(a) negative (b) high
(c) low (d) infinity

27. Water is filled in a balloon. To what depth of a lake should it be carried in order to decrease its volume by 0.01%? The bulk modulus of elasticity for water is $K = 2 \times 10^9 \text{Nm}^{-2}$
(a) 204 meter (b) 2.04 meter
(c) 2.04 km (d) 20.4 meter

28. The force required to break a wire of radius R is F. The force necessary to break another wire of radius $2r$ will be
(a) $F/4$ (b) $2F$ (c) $F/2$ (d) $4F$

29. The ratio of change in dimension at right angles to applied force to the initial dimension is defined as
(a) η (shear modulus)
(b) β (lateral strain)
(c) σ (posson's ratio)
(d) Y (Young's modulus)

30. The end of a wire of length 0.5 m and radius 10^{-3}m is twisted through 0.80 radian. The shearing strain at the surface wire will be
(a) 1.6×10^3 (b) 1.6×10^2
(c) 1.6×10^{-3} (d) 1.6×10^6

31. If the tensile force is suddenly removed from a wire then its temperature will
(a) increase (b) decrease
(c) become zero (d) remain constant

32. If a wire is stretched by applying force at one of its ends then the elastic potential energy density in term's of young's modulus Y and linear strain α will be
(a) $Y\alpha/2$ (b) $2\alpha^2 Y$ (c) $Y\alpha^2/2$ (d) $\alpha^2 Y/2$

33. Out of the following whose elasticity is independent of temperature?
(a) invar steel (b) copper
(c) glass (d) steel

34. A spherical ball is compressed by 0.01% when a pressure of 100 atmosphere is applied on it. Its bulk modulus of elasticity in dyne cm^{-2} will be approximately
(a) 10^{24} (b) 10^{14} (c) 10^{12} (d) 10^6

35. The limit upto which the stress is directly proportional to strain in called
(a) elastic relaxation (b) breaking limit
(c) elastic limit (d) elastic fatigue

36. The work done in twisting a steel wire of length 25 cm and radius 2 mm through 45° will be ($\eta = 8 \times 10^{10} \text{Nm}^{-2}$)
(a) 15.47J (b) 18.79 J (c) 2.48 J (d) 3.1 J

37. For the Hooke's law to be held good, the intermolecular distance, as compared to the equilibrium distance, must be
(a) much more
(b) very much less
(c) approximately same
(d) zero

38. The Young's modulus for steel is $Y = 2 \times 10^{11} \text{Nm}^{-2}$. If the interatomic distance is 3.2 Å, then the interatomic force constant in NÅ^{-1} will be
(a) 3.2×10^{-9} (b) 6.4×10^9
(c) 3.2×10^9 (d) 6.4×10^{-9}

39. How many times is the adiabatic modulus of elasticity of a gas as compared to its isothermal modulus of elasticity?
(a) γ times (b) three times
(c) two times (d) $\frac{1}{\gamma}$ times

40. Steel is more elastic than rubber because for a given load the strain produced in steel as compared to that produced in rubber is
(a) less (b) more
(c) equal (d) nothing can be said

41. The limits of σ are
(a) from 0 to –1 (b) from –1 to 1/2
(c) from –1/2 to 1/2 (d) from 1 to 1

42. The young's modulus of elasticity of a wire is Y and stress is F. The elastic potential energy per unit volume in wire will be
(a) $F^2/2Y$ (b) $Y^2/2F$ (c) F^2/Y (d) $2Y/F^2$

43. A Force of 600 kg-wt can break a wire. The force necessary to break another wire of half the cross-sectional area will be
(a) 900 kg-wt (b) 1200 kg-wt
(c) 300 kg-wt (d) 150 kg-wt

44. A cable that can support a load W is cut into two equal parts. The maximum load that can be supported by either part is
(a) W (b) $2W$ (c) $W/4$ (d) $W/2$

45. A wire of length L, area of cross-section A is stretched by a Force F. The change in length of the wire is l. Which of the following statement is wrong?
(a) $l \propto A$ (b) $l \propto L$ (c) $l \propto F$ (d) none of these

46. According to the Hooke's law the force required to change the length of a wire by 'l' is proportional to
(a) l (b) l^2 (c) l^{-2} (d) l^{-1}

47. A wire is stretched to double its length, the strain is
(a) 0.5 (b) 1 (c) 2 (d) zero

48. A long string is stretched by 0.12 cm. The energy stored per unit volume is 0.25 Jm^{-3}. If it is stretched by 1.0 cm, the potential energy per unit volume stored in it will be
(a) 6.25 Jm^{-3} (b) 1.25 Jm^{-3}
(c) 0.05 Jm^{-3} (d) 0.01 Jm^{-3}

49. A metallic rod breaks when strain produced is 0.2%. The young's modulus of the material of the rod is $7 \times 10^9 \text{Nm}^{-2}$. What should be its area of cross-section to support a load of 10^4N?
(a) $7.1 \times 10^{-2}\text{m}^2$ (b) $7.1 \times 10^{-8}\text{m}^2$
(c) $7.1 \times 10^{-4}\text{m}^2$ (d) $7.1 \times 10^{-6}\text{m}^2$

50. There is no change in the volume of a wire due to change in its length on stretching. The poisson ratio of the material of the wire is
(a) –0.25 (b) –0.50 (c) 0.25 (d) 0.50

51. The fractional change in volume per unit increase in pressure is called
(a) bulk modulus
(b) volume coefficient
(c) compressibility
(d) pressure coefficient

52. Given that Y represents young's modulus and σ represents the strain. Which of the following gives the energy stored per unit volume of a strained wire?
(a) $1/2\, Y^2\sigma^2$ (b) $\frac{1}{2} Y\sigma$
(c) $\frac{1}{2} Y^2\sigma$ (d) $\frac{1}{2} Y\sigma^2$

53. The correct graph verifying Hooke's law is

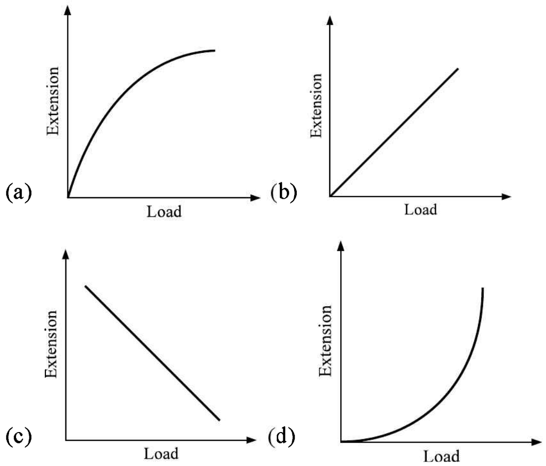


Fig. 11.15

54. The breaking stress of material is 10^{10}Pa. The wires of radius 0.1 mm and 0.2 mm as shown in Fig. 11.16. are made of this material. How much load be added in the hanger (assume) mass (es) so that one of the wire breaks? (Take $g = 10\text{ms}^{-2}$)

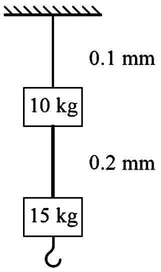


Fig. 11.16

(a) 61.4 kg (b) 614 kg
(c) 6.14 kg (d) .614 kg

55. The length of a wire is l_1 when tension is T_1 and l_2 when tension is T_2. The natural length of the wire is

(a) $\frac{l_1 + l_2}{2}$ (b) $\sqrt{l_1 l_2}$
(c) $\frac{l_1 T_2 - l_2 T_1}{T_2 - T_1}$ (d) $\frac{l_1 T_2 + l_2 T_1}{T_1 + T_2}$

56. In the load-extension graph for a wire, the elastic limit lies between the points

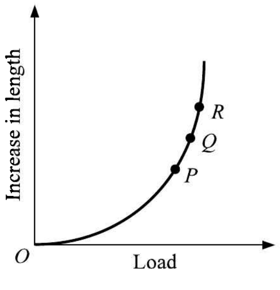


Fig. 11.17

(a) P and R (b) O and P
(c) Q and P (d) Q and R

57. While making a bridge iron rods are to be used. The thickness of rods is determined by

(a) breaking stress (b) lateral stress
(c) working stress (d) young's modulus

58. Two cylinders A and B are made of the same material. The lengths and radii of the two cylinders are in the ratio of 1 : 2. Both are twisted by the same external torque. The ratio of the angle of twist of A and B is

(a) 8 : 1 (b) 1 : 8 (c) 1 : 4 (d) 1 : 2

59. Two rods of different materials having coefficient of linear expansion α_1, α_2 and young's modulli Y_1 , Y_2 respectively are fixed between two rigid massive walls. The rods are heated such that they undergo the same increase in temperature. There is no bending of rods. If $\alpha_1 : \alpha_2 = 2 : 3$, the thermal stresses developed in the two rods are equal, provided $y_1 : y_2$ is equal to

(a) 3 : 2 (b) 4 : 9 (c) 1 : 1 (d) 2 : 3

60. A wire of length l and cross sectional area A is made of a material of young's modulus Y. If the wire is stretched by the amount x, the work done is

(a) $YAx/2l$ (b) YAx^2l
(c) YAx^2/l (d) $YAx^2/2l$

61. The young's modulus of a perfectly rigid body is

(a) one (b) zero
(c) infinity (d) none of these

62. The breaking stress of a wire depends on

(a) radius of the wire
(b) shape of the cross section
(c) material of the wire
(d) length of the wire

63. One litre of a gas is maintained at pressure 72 cm of Hg. It is compressed isothermally so that its volume becomes 900 cm^3. The value of stress and strain will be respectively

(a) 106.24Nm^{-2} and 0.1
(b) 0.106Nm^{-2} and 0.1
(c) 10662.4Nm^{-2} and 0.1
(d) 1.06Nm^{-2} and 0.1

64. Find the strain developed in the wire as shown in the Fig. 11.18 when longitudinal force F is applied. Young's modulus is Y.

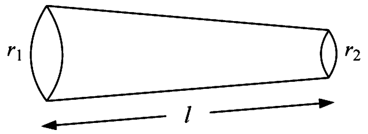


Fig. 11.18

(a) $\frac{F}{\pi(r^2 - r_2^2)\gamma}$ (b) $\frac{F}{\pi\left(\frac{r_1 + r_2}{2}\right)^2 \gamma}$

(c) $\frac{F}{\pi\left(\frac{r_1^2 + r_2^2}{2}\right)\gamma}$ (d) $\frac{F}{\pi r_1 r_2 Y}$

65. A wire is stretched through 1 mm by certain load. The extension produced in the wire of same material with double the length and radius with the same load will be

(a) 0.5 mm (b) 1 mm
(c) 3 mm (d) 4 mm

ANSWERS

1. (b) **2.** (c) **3.** (a) **4.** (d) **5.** (a) **6.** (c) **7.** (d) **8.** (b) **9.** (d) **10.** (d)
11. (b) **12.** (d) **13.** (d) **14.** (a) **15.** (a) **16.** (b) **17.** (a) **18.** (c) **19.** (c) **20.** (c)
21. (d) **22.** (d) **23.** (d) **24.** (b) **25.** (a) **26.** (c) **27.** (d) **28.** (d) **29.** (b) **30.** (c)
31. (a) **32.** (d) **33.** (a) **34.** (c) **35.** (c) **36.** (c) **37.** (c) **38.** (d) **39.** (a) **40.** (a)
41. (b) **42.** (a) **43.** (c) **44.** (a) **45.** (c) **46.** (a) **47.** (b) **48.** (a) **49.** (c) **50.** (d)
51. (c) **52.** (d) **53.** (b) **54.** (c) **55.** (c) **56.** (b) **57.** (c) **58.** (a) **59.** (a) **60.** (d)
61. (c) **62.** (c) **63.** (c) **64.** (d) **65.** (a)

Hydrodynamics and Properties of Fluids

12

BRIEF REVIEW OF THE CONCEPTS

Fluid is something that can flow, therefore **liquids** and **gases** come in this category. A **perfect** liquid cannot be compressed and shearing stress cannot be maintained.

Streamlined, steady state flow By **steady state** or **stationary flow** we mean that at any place in a fluid, the velocity never changes.

Streamline is a curve tangent to which at any point gives the direction of fluid velocity at that point.

Equation of continuity Volume entering per second = volume leaving per second

or $$A_1v_1 = A_2v_2$$

where A_1 and A_2 are area of cross-sections and v_1 and v_2 are velocities at the two ends/points of a pipe.

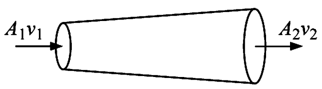


Fig. 12.1

Bernoulli's theorem It states that the total energy is constant.

KE + PE + Pressure head Energy = constant

$$1/2mv^2 + mgh + P\Delta V = \text{constant}$$

or $$1/2\rho v^2 + \rho gh + P = \text{constant}$$

In a horizontal pipe $\frac{1}{2}\rho v^2 + P = \text{constant}$

Torricelli's theorem According to this theorem velocity of efflux $v_{\text{efflux}} = \sqrt{2g(H - h)}$

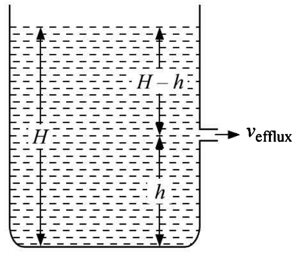


Fig. 12.2

Dynamic lift or Magnus effect When a ball is spinning in a fluid as illustrated in Fig. 12.3. The resultant velocity below the ball decreases and the ball experiences an upward thrust. Such a phenomenon is called dynamic lift or Magnus effect.

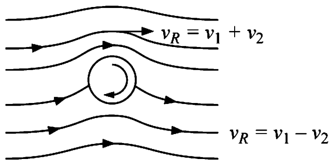


Fig. 12.3

Surface tension The property of the liquid with which its surface behaves as a stretched membrane and can support small objects placed on its surface.

Surface tension = surface energy

Surface energy = work done due to surface tension/area

Dimensional formula (MT^{-2}) (force/length)

Excess pressure in a drop/bubble $\Delta P = \frac{2T}{r}$

where T is surface tension and r is radius.

If a bubble has two free surfaces, like soap bubble, then the excess pressure is $\Delta P = \frac{4T}{r}$.

Angle of contact is the angle between the tangent to the liquid vapour interface and the tangent to liquid solid interface. In other words, it is the angle between the tangent to the meniscus at the point of contact and wall of the container.

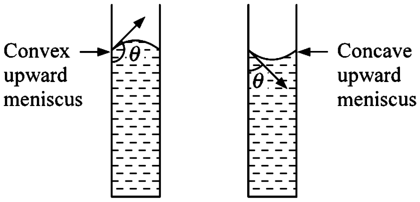


Fig. 12.4

Liquids like water, alcohol, ether, carbon tetrachloride xylene, glycerine and acetic acid have angle of contact zero or nearly zero with glass. Meniscus may be concave upward or convex upward.

Table 12.1

Concave upward meniscus	*Convex upward meniscus*
1. Angle of contact is acute.	Angle of contact should be obtuse.
2. Adhesive force between liquid and the walls of container is greater than the cohesive force between liquid molecules.	Cohesive force between liquid molecules is greater than the adhesive force between liquid molecules and walls of the container.
3. The liquid wets the walls of the container.	The liquid does not wet the walls of the container.

Ascent of liquid in a capillary tube

$h = \dfrac{2T\cos\theta}{r\rho g}$ if meniscus is not taken into account.

$h = \dfrac{2T\cos\theta}{r\rho g} - \dfrac{r}{3}$ if meniscus is also taken into account.

$h = \dfrac{2T}{r\rho g}$ if $\theta = 0$.

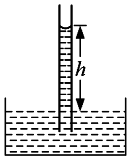


Fig. 12.5

Note that if angle of contact θ is acute, the liquid will rise in the capillary tube.

It will dip in the cases where angle of contact θ is obtuse, for example, in Hg as illustrated in Fig. 12.6.

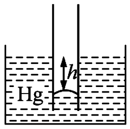


Fig. 12.6

If the liquid rises in a capillary tube and the tube is of insufficient height then the excess liquid will stay as a drop above the surface as illustrated in the Fig. 12.7. No overflow will occur. If overflow takes place, it means the liquid will rise again. Thus, a perpetual motion will begin and such a motion is disallowed.

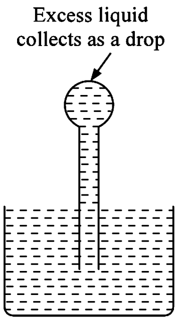


Fig. 12.7

Energy (E) required to split a big drop into n small drops each of radius r

$$E = 4\pi r^2\, n^{2/3}\, T(n^{1/3} - 1)$$

Radius of a big drop $R = n^{1/3}r$

Same amount of energy will be released when a large drop is formed when n small drops each of radius r will coalesce.

Viscosity The property of a fluid to oppose relative motion between its layers is called viscosity. This property can be observed when the flow is steady or the liquid moves with a constant velocity. The flow may be called laminar. The opposition or resistance is due to intermolecular forces (cohesive force). It is clear from the Fig. 12.8 that velocity of layers decrease perpendicular to the flow

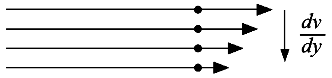


Fig. 12.8

$\therefore$ Shearing stress $\dfrac{F}{A} \propto$ velocity gradient (rate of change of strain)

or $$F = -\eta A \frac{dv}{dy}$$

where η is coefficient of viscosity, A is area of cross-section and $\dfrac{dv}{dy}$ is velocity gradient.

Since the coefficient of viscosity is the ratio of shearing stress to the rate of change of strain, it may be regarded as **transient** or **fugitive rigidity**. Maxwell regarded viscosity as the limiting case of elastic solid, η is coefficient of viscosity. Its unit is Poise (CGS), Poiseuille (SI) 1 Poise = 0.1 Poiseuille.

Critical velocity The velocity at which the **steady, laminar** or **streamline** flow changes to **turbulent** or **eddy** flow is called **critical** velocity.

Stoke's formula $F = 6\pi\eta r v$

This formula is valid if Reynolds number for the system is < 1.

Terminal velocity when a drop or a spherical body falls under gravity then

$v_{\text{Terminal}} = \dfrac{2r^2(\rho - \sigma)g}{9\eta}$ where r is radius of drop/body

and $\rho \to$ density of drop/body, $\sigma \to$ density of viscous fluid, $\eta \to$ coefficient of viscosity

Reynold's number $R = \dfrac{\text{Inertial force/area}}{\text{Viscous drag/area}}$

If D is diameter of the tube then

$$R = \frac{\rho v D}{\eta} = \frac{\text{Inertial force}}{\text{Viscous drag}} = \frac{\frac{1}{2}\rho v^2}{\frac{\eta v}{r}}$$

For water If $v < 20$ cm/s or $R < 2000$ flow is laminar at 20°C.

If $v > 30$ cm/s or $R > 3000$ flow is turbulent at 20°C.

Kinematic viscosity is the ratio of $\dfrac{\eta}{\rho}$. Its unit is **stokes.**

Poiseuille's equation Amount of liquid flowing per second through a tube of radius r is given by

$$\frac{dv}{dt} = \frac{\pi \text{P} r^4}{8\eta l}$$

where $\dfrac{P}{l}$ is pressure gradient r is radius of the tube $P = \rho(h_1 - h_2)$ g. Scheme for Posiseuille's law.

Fig. 12.9 shows how the rate of flow varies with pressure head.

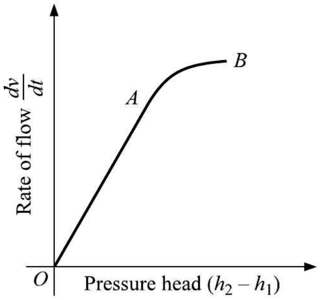


Fig. 12.9

Variation of viscosity with temperature in liquids

$$\eta\rho^{-1/2} = Ae^{C\rho/T}$$

where A and C are constant ρ is density and T is temperature in Kelvin.

that is, viscosity of the liquids in general decreases with rise in temperature.

$\eta = a\eta_0 T^{+1/2}$ that is, coefficient of viscosity increases with temperature where λ is the mean free path and c is rms velocity of the gas moecules.

Variation of pressure due to height $P_2 = P_1 + \rho gh$ where P_1 is pressure at point A and P_2 is pressure at point B.

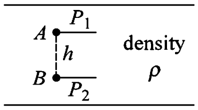


Fig. 12.10

Pascal's law If the pressure at a point is changed, the change is transmitted to the entire liquid without being diminished in magnitude.

Hydraulic lift Principles (a) Pascals law (b) Liquids are incompressible

If A_1 and A_2 are the area of cross-sections in the U-tube on two sides as shown in the Fig. 12.11 and F is the force applied on the side of cross-section A_1, then weight lifted is

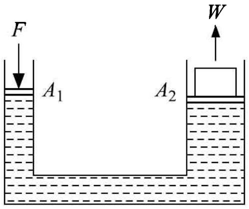


Fig. 12.11

$$W = F\frac{A_2}{A_1}$$

Barometer measures atmospheric pressure. Fortin's barometer is most commonly used to measure atmospheric pressure. Manometers are used to measure partial pressure.

Archimedes' principle When a body is weighed in a fluid it loses its weight. The weight lost is equal to the weight of the amount of liquid displaced. This is in accordance with Newton's laws of motion. The buoyant force or upthrust is responsible to decrease in the weight

$$\text{weight in the liquid} = V(\rho - \sigma)g$$

where V is volume displaced, ρ is density of the body and σ is density of fluid

$$\text{weight lost} = V\sigma g$$

Specific gravity or relative density = $\frac{\text{density of body}}{\text{density of water}}$. In CGS system, density of a body = specific gravity (without unit)

If m_1 and m_2 are masses of densities ρ_1 and ρ_2 then the combined density is

$$\rho = \frac{m_1 + m_2}{\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}} = \frac{\Sigma m_i}{\Sigma \frac{m_i}{\rho_i}} \text{ if } m_1 = m_2 \text{ then } \rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

If V_1 is volume of density ρ_1 and V_2 is volume of density ρ_2 then the combined density

$$\rho = \frac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2} = \frac{\Sigma \rho_i V_i}{\Sigma V_i}$$

Variation of density with rise in temperature

$$\rho = \rho_0(1 - \gamma\Delta\theta)$$

where $\Delta\theta$ is change in temperature and γ is volumetric coefficient of expansion.

Variation of density with volume/Bulk modulus

$$\rho = \rho_0 \frac{V_0}{V} = \rho_0\left(1 + \frac{\Delta P}{B}\right)$$

Floatation A body will float if the weight of the body $mg \le$ upthrust or buoyant force

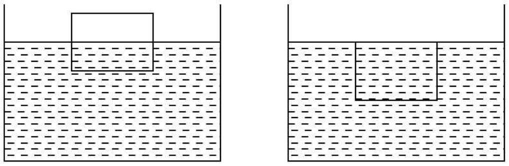


Fig. 12.12

(a) body floating **(b) body just floating**

that is, $\rho_{\text{body}} \le \rho_{\text{liquid}}$

If $\rho_{\text{body}} < \rho_{\text{liquid}}$ it floats

If $\rho_{\text{body}} = \rho_{\text{liquid}}$ it just floats

SHORT-CUTS AND POINTS TO NOTE

(i) In a steady flow of incompressible and nonviscous fluid, volume entering per second = volume leaving per second, that is, $A_1v_1 = A_2v_2$ or Av = constant.

If you note carefully the water falling down from a tap becomes narrow in cross-section as its velocity increases on falling down under gravity as illusrated in Fig. 12.13.

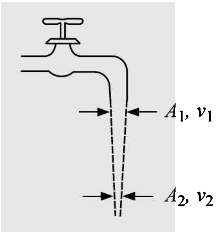


Fig. 12.13

Fig. 12.14 show graph v versus A (area of cross-section).

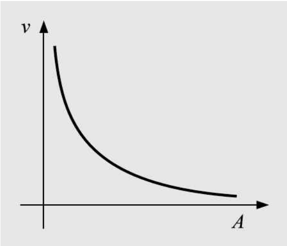


Fig. 12.14

(ii) Bernoulli's Principle is based on conservation of energy. Total energy which is sum of kinetic energy, potential energy and Pressure head energy is constant, that is,

$$\frac{1}{2}\rho v^2 + \rho gh + P = \text{constant}$$

or $$\frac{1}{2}\rho v_1^2 + \rho gh_1 + P_1 = \frac{1}{2}\rho v_2^2 + \rho gh_2 + P_2$$

In a horizontal pipe $\frac{1}{2}\rho v^2 + P = \text{constant}$.

that is, liquid flows due to variation of pressure difference in a horizontal pipe.

During a hurricane or wind tornado when the high speed wind blows over a straw or tin roof, it creates a low pressure P_{out} and high pressure P_{in} according to Bernoulli's principle and the roof is lifted up and then blown off by the wind.

Two boats moving in same direction come closer as water/wind passing in between is quite fast and creates a low pressure region.

A spinning ball shows a dynamic lift or magnus effect as resultant velocity below the spinning ball decreases and resultant velocity above the ball increases. Hence a high pressure is created below the ball and it is lifted up.

(iii) $v_{\text{efflux}} = \sqrt{2(H-h)g}$ in open tank Fig. 12.23. The liquid coming out will cover maximum horizontal distance if $H - h = h$ or $h = \frac{H}{2}$.

(iv) In venturimeter volume flowing per second $\frac{V}{t} = aA\sqrt{\frac{2gH}{A^2 - a^2}}$

(v) Surface tension is the resultant molecular force at the surface per unit length.

surface tension = surface energy

surface energy = work done due to surface tension per unit area

Surface tension only depends upon the nature of liquid and is independent of the area of surface or length of line considered.

Surface tension is a scalar. It has a unique direction and need not be specified.

Surface tension decreases with rise in temperature. It becomes zero at critical temperature (where interface between liquid and vapour disappears).

Surface tension decreases with impurities.

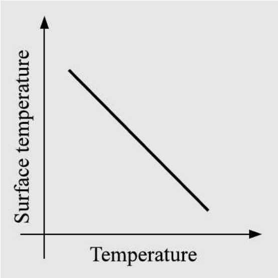


Fig. 12.15

In case of soluble impurities surface tension may increase or decrease depending upon the nature of impurity. Highly soluble impurities like salt increase surface tension while sparingly soluble impurities like soap decreases the surface tension.

Balloon is based on elasticity and soap bubble or air bubble is based on surface tension.

Excess pressure in a drop/bubble having single surface $\Delta P = \frac{2T}{r}$.

Excess pressure in a bubble having two surfaces like soap bubble $\Delta P = \frac{4T}{r}$

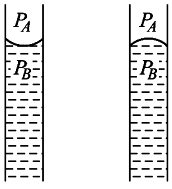
Pressure just above concave meniscus [Fig. 12.16(a)]

$$P_A = P_B + \frac{2T}{r}.$$

that is, it is larger by $\frac{2T}{r}$; $P_A = P_0$ (atmospheric pressure)

or $P_B = P_0 - 2T/r$.

Pressure just below convex meniscus [Fig. 12.26(b)]



(a) (b)

Fig. 12.16

$$P_B = P_A + \frac{2T}{r}.$$

that is, it is larger by $2T/r$ or $P_B = P_0 + \frac{2T}{r}$.

(vi) Height to which a liquid rises

$$h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3}$$

$h = \frac{2T}{r\rho g}$ if θ = 0° (for example, for water, alcohol, ether, CCl_4) from fig. 12.17 $h \propto \frac{1}{r}$

If angle of contact θ is acute, the meniscus is concave upwards and such a liquid wets the walls of the container.

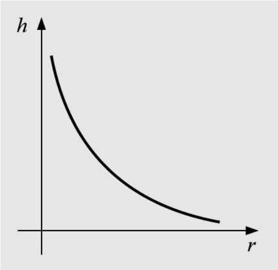


Fig. 12.17

If angle of contact is obtuse such a liquid does not wet the walls of the container. It makes a convex upward meniscus and instead of rising in a capillary, it actually dips as is clear from the relation because $\cos\theta$ will be negative.

hr = constant (more is the radius the less it will rise)

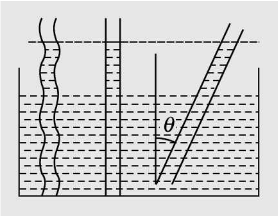


Fig. 12.18

However in equilibrium, if the radius of meniscus is same, liquid will rise to equal height irrespective of shape and angle of tilt. Oil in a wick of the lamp, action of soaking by towel, blotting paper, root hairs of plants are based on capillary action.

(vii) In a capillary tube of insufficient height the excess liquid will collect as a drop but it will not overflow.

(viii) When a big drop of radius R splits to n drops each of radius r or n drops each of radius r coalesce to form a big drop, the energy required or released is

$$\Delta E = T4\pi r^2 n^{2/3}[n^{1/3} - 1] = 4\pi R^2 T[n^{1/3} - 1].$$

(ix) Viscosity is fluid friction or internal friction due to intermolecular forces. SI unit of coefficient of viscosity is Poiseuille (Pl) 1 Pl = 10 poise. Poise is CGS unit

Shearing stress $\frac{F}{A}$ ∝ rate of change of strain (velocity gradient). Dimension formula of coefficient of viscosity is $[ML^{-1}T^{-1}]$

$$\frac{F}{A} = -\eta A\frac{dv}{dy}$$

viscosity may be termed as fugitive rigidity.

(x) Terminal velocity

$$v_T = \frac{2r^2(\rho-\sigma)g}{9\eta}$$

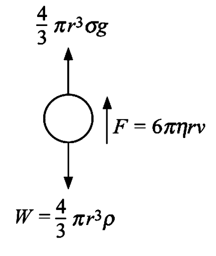


Fig. 12.19

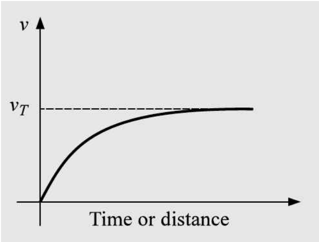


Fig. 12.20

Note: $v_T \propto r^2$. More the radius, more is the terminal velocity. Fig. 12.20 shows that terminal velocity remain constant. When a drop falls or a spherical body falls initially velocity increases until it enters viscous fluid and then becomes constant. More the density of fluid in which it is falling, less will be the terminal velocity.

(xi) Critical velocity is the maximum velocity upto which flow is steady.

Reynold's number $R = \frac{\rho vD}{\eta} = \frac{\text{internal force}}{\text{viscous drag}}$.

R is dimensionless.

For water if $R < 2000$ or $v < 20\,\text{cms}^{-1}$, flow is steady.

and if $R > 3000$ or $v > 30\,\text{cms}^{-1}$ flow is turbulant

Kinematic viscosity has unit stokes and it is the ratio of coefficient of viscosity to the density of the liquid, that is, η/ρ is kinematic viscosity.

(xii) Poiseuille's equation Amount of liquid flowing per second through a tube of radius r is given by

$$\frac{dV}{dt} = \frac{\pi Pr^2}{8\eta l}$$

When the liquid from a vessel is poured into another vessel it falls in such a way that the upper layer has more velocity than lower layers.

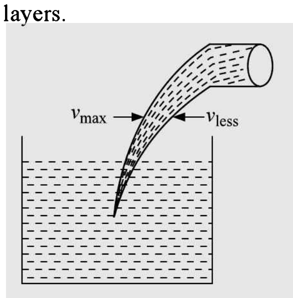


Fig. 12.21

Compare the Poiseuille's equation with ohm's law

$$I = \frac{V}{R}$$

$\frac{dV}{dt}$ = flow current (i_{flow}) P acts as potential difference or $V_{\text{flow}} = P$.

flow resistance $R_{\text{flow}} = \frac{8\eta l}{\pi r^4}$.

Viscosity of liquids fall with rise in temperature.

$$\eta\rho^{-1/2} = A\,e^{c\rho/T}$$

T → temperature, A and c are constant.

Viscosity of gases increases with rise in temperature due to increased collisions.

$$\eta = a\eta_0 T^{1/2}$$

Using a viscometer, coefficients of viscositites of two liquids can be compared as $\frac{\eta_1}{\eta_2} = \frac{\rho_1 t_1}{\rho_2 t_2}$.

where t_1 and t_2 are the time taken to vacate a given volume for two liquids of densities ρ_1 and ρ_2 respectively.

(xiii) Variation of pressure due to height $P_2 = P_1 + \rho gh$

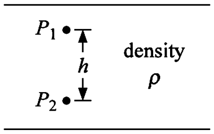


Fig. 12.22

(xiv) Pascal's law If we leave gravitational pressure $P = \rho gh$ then pressure exerted by the liquid is constant. It may also be stated that if the pressure at a point is changed, the change is transmitted to the entire liquid without being diminished in magnitude.

On this principle is based a) hydraulic lift b) hydraulic press and c) hydraulic brakes.

In hydraulic lift, weight lifted $W = \dfrac{FA_2}{A_1}$

where F is force applied A_2 and A_1 are area of cross-sections at weight and Force ends respectively.

(xv) Archimedes principle Weight lost by a body when weighed in a fluid is equal to $V \rho g$, where V is the volume of the fluid displaced, ρ is density of the fluid.

$$\text{Specific gravity} = \frac{\text{density of a body}}{\text{density of water at 4°C}}$$

$$= \frac{\text{weight of a body in air}}{\text{weight of body in air} - \text{weight of body in water}}$$

$$\frac{\rho_{\text{liquid}}}{\rho_{\text{water}}} = \frac{\text{loss in weight in a liquid}}{\text{loss in weight in water}}.$$

(xvi) The volume of cavity

Outer volume of the body $V = \dfrac{W_{\text{air}} - W_{\text{water}}}{\rho g}$ $\rho \rightarrow$ density of water

volume of material of the body

$$V_0 = \frac{\text{mass of the body in air}}{\text{density of the body}}$$

Volume of the cavity
= outer volume of body
– volume of material of the body
= $(V - V_0)$.

(xvii) Impurity determination

Let the total mass of the body be m, densities of materials are ρ_1 and ρ_2(impurity). Let x be the amount of impurity.

Then $\dfrac{x}{\rho_2} + \dfrac{m - x}{\rho_1} = V$ (total volume of the body).

Note that in the CGS system weight lost in water in grams is equal to volume of the body in cc.

(xviii) A body will float if $\rho_{\text{body}} \le \rho_{\text{liquid}}$.

(xix) Small fog particles act like a rigid body due to excess pressure in it $\left(P \propto \dfrac{1}{r}\right)$.

(xx) If two bubbles merge in isothermal conditions then $r = \sqrt{r_1^2 + r_2^2}$.

when two bubbles coaelsce $r = \dfrac{r_1 r_2}{r_2 - r_1}$

(xxi) Cohesisive force $F_C \propto \dfrac{1}{r^8}$

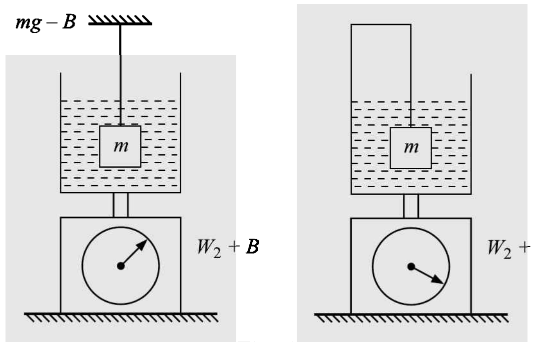


Fig. 12.23

Consider the situation shown in Fig. 12.23 and note that how the reading of the spring balance is affected.

CAUTION

(i) Considering when two bubbles coalesce the pressure inside increases

- Pressure inside will decrease or excess pressure will decrease as $P \propto \dfrac{1}{r}$.

∴ use $P_{\text{net}} = P_1 - P_2$

or $\dfrac{2T}{R} = \dfrac{2T}{r_1} - \dfrac{2T}{r_2}$ or $\dfrac{1}{R} = \dfrac{1}{r_1} - \dfrac{1}{r_2}$.

If two bubbles merge isothermally then

$$P_1V_1 + P_2V_2 = PV$$

$$\frac{2T}{r_1}\left(\frac{4}{3}\pi r_1^3\right) + \frac{2T}{r_2}\left(\frac{4}{3}\pi r_2^3\right) = \frac{2T}{r}\left(\frac{4}{3}\pi r^3\right)$$

or $r^2 = r_1^2 + r_2^2$.

(ii) Considering that on opening valve, bigger bubble will become smaller and smaller bubble will grow in size.

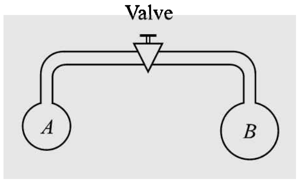


Fig. 12.24

- Since $P = \dfrac{2T}{r}$, therefore, pressure inside the smaller bubble is more than in the bigger bubble. Air flows from higher pressure to lower pressure, therefore, smaller bubble will become more small and bigger one will grow in size.

(iii) Considering that if a capillary tube is of insufficient height, the excess liquid will overflow or the liquid will not rise more than the height of the tube.

- Excess liquid will collect in the form of a drop above the surface of the capillary. But it will not over flow as perpetual motion is not possible.

(iv) Considering Archimedes principle different from Newton's laws of motion.

- Archimedes principle follows Newton's laws of motion and loss in weight = buoyant force or upthrust is basically Newtons's third law.

(v) Considering that melting of ice in a liquid will raise the level of the liquid.

- ■ The level does not change as on melting the volume of ice decreases.

(vi) Considering that if a body sinks in one liquid, it will sink in other liquid also.

- ■ Floatation depends upon density. If effective density of the body is less than or equal to the density of liquid it will float.

(vii) Considering that density of a body will remain unaltered.

- ■ If the body has a cavity its density will decrease

(viii) Considering a water proof material fills the pores

- ■ It changes angle of contact from acute to obtuse and hence it will not wet the surface.

(ix) Considering ρgh as the normal pressure exerted by the liquid

- ■ The normal force per unit area (not only on bottom) but on the side walls is pressure of the liquid. It always acts perpendicular to the surface in contact.

(x) Considering that flowing liquid offers no resistance

- ■ Resistance offered by flowing liquid through a tube is

$$\frac{P}{\frac{dV}{dt}} = \frac{8\eta l}{\pi r^4}.$$

If tubes are added in series then $R = R_1 + R_2$.

In series $\frac{dV}{dt}$ remains unchanged $= \frac{8\eta}{\pi}\left[\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right]$

If tubes are added in parallel then

$$r = \frac{r_1 r_2}{r_1 + r_2} = \frac{8\eta l_1 l_2}{\pi r_1^4 r_2^4}\left[\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right]^{-1}$$

In parallel $\frac{dV}{dt}$ of the two tubes will add up.

(xi) Considering that liquid possess only kinetic energy and potential energy.

- ■ A moving liquid possesses kinetic energy, potential energy and pressure head energy.

(xii) Considering that velocity of efflux or range depends upon density of the liquid.

- ■ Velocity of efflux and hence range does depend upon the density of the liquid. Rather

$$v_{\text{efflux}} = \sqrt{2g(H-h)}.$$

is independent of density of the liquid.

Solved Problems

1. Drops of liquid of density d are floating half immersed in a liquid of density ρ. If the surface tension of liquid is T then the radius of the drop is

(a) $\sqrt{\frac{3T}{g(3d-\rho)}}$ (b) $\sqrt{\frac{6T}{g(2d-\rho)}}$

(c) $\sqrt{\frac{3T}{g(2d-\rho)}}$ (d) $\sqrt{\frac{3T}{g(4d-3\rho)}}$

Solution (c) weight of the drop = forces due to surface tension + buoyant force

$$\frac{4}{3}\pi r^3 dg = 2\pi rT + \frac{1}{2}\times\frac{4}{3}\pi r^3\rho g \quad \text{or } r = \sqrt{\frac{3T}{g(2d-\rho)}}.$$

2. A block of mass m just floats in a liquid. It is pushed down and released then

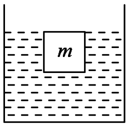


Fig. 12.25

(a) it will oscillate
(b) it will rise to original position and stay there
(c) it will sink
(d) it will rise to another position and stay there

Solution (c) The additional push gives an unbalanced force and according to Newton's first law it will continue to move down and sink.

3. A spring balance has a beaker almost full of water placed on it. A suspended body of mass m is lowered into the beaker without touching the walls and the bottom of the beaker then the reading of the spring balance

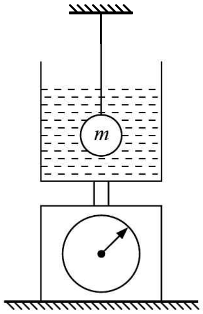


Fig. 12.26

(a) increases by an amount equal to upthrust
(b) decreases by an amount equal to upthrust
(c) remains unchanged
(d) increases equal to the weight of the body

Solution (a) Weight will increase equal to the upthrust as action and reaction are equal and opposite. Body exerts a force equal to upthrust on the beaker.

4. If W is the work done to form a bubble of volume V in a liquid then the work done to form a bubble of double the volume is

(a) $4^{1/3}W$ (b) $2^{1/3}W$

(c) $3^{1/2}W$ (d) $3^{1/3}W$

Solution (a) New radius $R = 2^{1/3}r$

$$\frac{W_2}{W_1} = \frac{4\pi R^2 T}{4\pi r^2 T} = 2^{2/3} \quad \text{or} \quad W_2 = W\,2^{2/3} = 4^{1/3}W.$$

5. The pressure of water in a pipe when the tap is open and closed are respectively $3 \times 10^5\,\text{Nm}^{-2}$ and 3.5×10^5. With the open tap the velocity of water flowing is

(a) $10\,\text{ms}^{-1}$ (b) $5\,\text{ms}^{-1}$

(c) $20\,\text{ms}^{-1}$ (d) $15\,\text{ms}^{-1}$

Solution (a) $P_{\text{closed}} = P_{\text{open}} + \frac{1}{2}\rho v^2$

or
$$v = \sqrt{\frac{2(P_{\text{closed}} - P_{\text{open}})}{\rho}}$$
$$= \sqrt{\frac{2 \times (.5 \times 10^5)}{10^3}} = 10\,\text{ms}^{-1}.$$

6. The density of atmosphere varies with height H above the ground as per relation $D = D_0 e^{-aH}$ where a is a constant, then the pressure at a height h is

(a) $\frac{D_0 g e^{-aH}}{h}$ (b) $\frac{D_0 g e^{-ah}}{a}$

(c) $\frac{D_0 g e^{aH}}{a}$ (d) $\frac{D_0 g e^{ah}}{a}$

Solution (b) $P = \int_h^0 Dg\,dh$

$$= -D_0 g \int_0^h e^{-aH} dh = \frac{D_0 g e^{-ah}}{a}.$$

7. A barometer kept in a stationary elevator reads h cm of Hg column. The air pressure in the elevator read by the barometer when it is accelerating down with an acceleration 'a' is

(a) $\rho g h$ (b) $\rho(g + a)h$

(c) $\rho(g - a)h$ (d) zero

Solution (c) In the elevator moving down net acceleration is $(g - a)$

∴ Pressure $P = \rho(g - a)h$

8. A cubical block of wood 10 cm on a side floats at the interface of oil and water as shown in Fig. 12.27. The density of oil is $0.6\,\text{g cm}^{-3}$. The mass of the block is

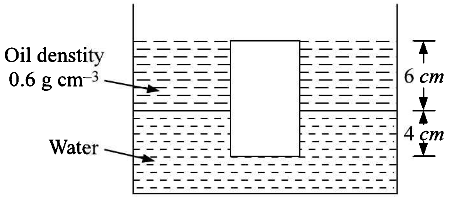


Fig. 12.27

(a) 706 g (b) 607 g

(c) 760 g (d) 670 g

Solution Volume of block = 1000 cc, 400 cc is in water and 600 cc is in oil

weight of block = weight of oil displaced + weight of water displaced

$$mg = 600(.6)g + 400(1)g = 760\,g$$

or $m = 760\,\text{g}.$

9. A certain block weighs 15 N in air and 12 N in water. When immersed in another liquid it weighs 13 N. Find specific gravity of the liquid

(a) 12/13 (c) 12/15

(c) 13/15 (d) 2/3

Solution (d) $\frac{\rho_{\text{liquid}}}{\rho_{\text{water}}} = \frac{\text{thrust in liquid}}{\text{thrust in water}} = \frac{15-13}{15-12} = \frac{2}{3}.$

10. When equal volumes of two metals are mixed together, the specific gravity of alloy is 4. When equal masses of the same metals are mixed together the specific gravity of the alloy is 3. Calculate the specific gravity of each metal.

(a) 2, 6 (b) 3, 5

(c) 4, 2 (d) 3, 4

Solution (a) When equal volumes are mixed

$$\rho = \frac{V\rho_1 + V\rho_2}{V + V} = \frac{\rho_1 + \rho_2}{2} = 4$$

When equal masses are mixed

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2} = 3$$

Solving these equations we get $\rho_1 = 2$, $\rho_2 = 6$

11. A u-tube of uniform cross-section contains mercury. When 20 cm of water is poured into one limb of the tube. How high will the mercury rise in the other limb?

(a) 1.46 cm (b) 1.73 cm

(c) 0.73 cm (d) none of these

Solution (c) When H_2O is added in one limb, the mercury level goes down by x and in the other limb rises by x

$$2xAg\,\rho_{\text{meraury}} = 20 \times \rho_{\text{water}} \times A \times g$$

or $$x = \frac{20 \times \rho_{\text{water}}}{2\,\rho_{\text{mercury}}} = \frac{10}{13.6}$$

$\simeq 0.73\,\text{cm}.$

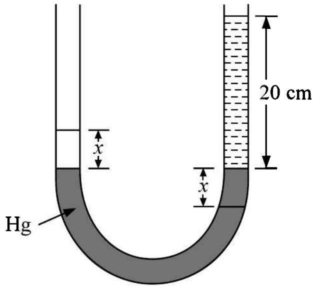


Fig. 12.28

12. To what height should a cylindrical vessel be filled with homogeneous liquid to make the force with which it presses on the side of the vessel to be equal to the force exerted by the liquid on the bottom?

(a) $r/2$ (b) $2r/3$

(c) $3r/4$ (d) r

Solution (d) Let the cylinder has radius r, density of the liquid ρ and filled to a height h. If P_0 is the atmospheric pressure

Pressure exerted by the liquid on the bottom = $\rho g h$

$$F_{\text{bottom}} = \rho g h\,(\pi r^2)$$

Pressure at depth y due to liquid only = $y\rho g$

Force on the curved surface of strip dy

$dF_C = y\rho g\,(2\pi r\, dy)$

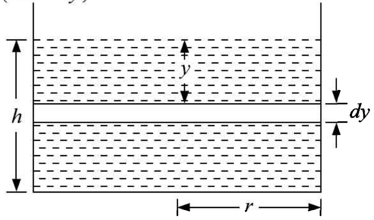


Fig. 12.29

$$F_C = 2\pi r\rho g \int_0^h y\,dy = 2\pi r\rho g\frac{h^2}{2}$$

But $F_C = F_{\text{bottom}} = \pi r^2\rho g h \Rightarrow h = r.$

13. Fig. 12.30 shows the action of a siphon. The liquid flowing through has density $1.5\,\text{g cm}^{-3}$. Find the pressure difference between points A and D and B and C

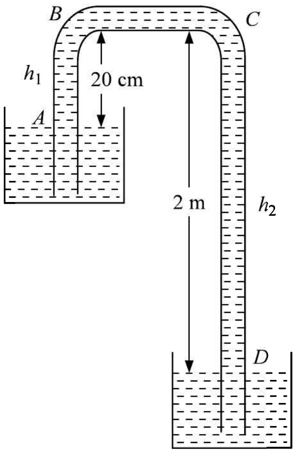


Fig. 12.30

Solution A and D are open ends, therefore,

$$P_A = P_D = P_{\text{atm}}$$

Hence $P_A - P_D = 0$

$P_B = P_{\text{atm}} - \rho g h_1$ similarly $P_C = P_{\text{atm}} - \rho g h_2$

$$\therefore \quad P_B - P_C = \rho g(h_2 - h_1)$$
$$= 1.5 \times (1.8) \times 9.8 \times 10^3$$
$$= 2.65 \times 10^4\,\text{Nm}^{-2}.$$

14. A soft polythene bag weighs W when empty. It is filled with air of density ρ and weighed in air then its weight is ... if volume of the bag is V.

(a) $W + V\rho g$ (b) W

(c) $W - V\rho g$ (d) none of these

Solution (b) When an air filled bag is weighed in air, it loses weight equal to the volume displaced. Therefore, its weight $W + V\rho g - V\rho g = W$.

15. A sphere has a cavity. It weighs 12 kg in air and 10 kg in water. Density of the material is $8\,\text{g cm}^{-3}$. The volume of the cavity is

(a) 2000 cc (b) 1000 cc

(c) 1500 cc (d) 500 cc

Solution (d) Weight lost in water = 2 kg, therefore, volume of sphere = 2000 cc

If V is the volume $(2000 - V)8 = 12000$ or $8V = 4000$

$V = 500$ cc.

16. A gold ornament weighs 200 g in air and 188 g in water. It has copper as impurity. If density of Gold is $19.6\,\text{g cm}^{-3}$ and density of copper $9.3\,\text{g cm}^{-3}$. The contents of copper in the ornament is

(a) 4.18 g (b) 3.78 g

(c) 1.28 g (d) 3.28 g

Solution (d) Volume of ornament = 12 cc (=weight lost). Let x g be contents of copper then $\frac{x}{9.3} + \frac{200 - x}{19.6} = 12$ on solving $x = 3.28$ g.

17. A beaker is exactly full of water with an ice piece floating. The ice piece has a lead piece in it. When ice melts then

(a) level remains unchanged

(b) water overflows

(c) level falls

(d) none of these

Solution (c) Since the average density of ice with lead piece increases, it displaces more water.

18. By what fraction an ice cube will float in water of density $1.03\,\text{g cm}^{-3}$. Density of ice = $0.92\,\text{g cm}^{-3}$.

(a) $0.92l$ (b) $0.89l$

(c) $0.86l$ (d) none of these

Solution (b) $0.92l\,Ag = 1.03hAg$

(using, the weight of ice = weight of liquid displaced)

$$\therefore \quad h = \frac{0.92l}{1.03} = 0.89l.$$

19. A piece of metal floats on Hg. The coefficient of expansion of metal and Hg are γ_1 and γ_2 respectively. If the temperature of both Hg and metal are increased by an amount ΔT, by what factor the fraction of the volume of metal submerged in mercury changes?

(a) $(\gamma_2 - \gamma_1)\Delta T$ (b) $\left(\frac{\gamma_2 + \gamma_1}{2}\right)\Delta T$

(c) $\frac{2\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\Delta T$ (d) $\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\Delta T$

Solution (a) $f_{\text{in}} = \frac{V_{\text{in}}}{V} = \frac{\rho}{\sigma}$ where ρ is density of metal and σ is density of Hg.

$$\frac{\Delta f}{f} = \frac{f'_{\text{in}} - f_{\text{in}}}{f_{\text{in}}} = \frac{f'_{\text{in}}}{f_{\text{in}}} - 1$$

$$= \frac{\frac{\rho}{\sigma}\left(\frac{1+\gamma_2\Delta T}{1+\gamma_1\Delta T}\right)}{\frac{\rho}{\sigma}} - 1 = (\gamma_2 - \gamma_1)\Delta T$$

(using Binomial theorem)

20. A glass beaker of interior volume 500 cc weighing 390 g just floats when half filled with water. Find the density of the material of glass.

(a) $3.2\,\text{g cm}^{-3}$ (b) $2.8\,\text{g cm}^{-3}$

(c) $3.0\,\text{g cm}^{-3}$ (d) $3.4\,\text{g cm}^{-3}$

Solution (b) mass of beaker + mass of water = V. σ (σ = 1 g/cc)

$390 + 250 = V \times 1$ or $V = 640$ cc

$$\rho_{\text{glass}} = \frac{390}{640 - 500} = \frac{390}{140} = 2.8\,\text{g cm}^{-3}.$$

21. A horizontal pipeline carries water in a streamline flow. At a point along the pipe where the cross-sectional area is $10\,\text{cm}^2$, the velocity of water is 1 ms^{-1} and the pressure is 2000 Pa. The pressure at a point of cross-sectional area 5 cm^2 is

(a) 1000 Pa (b) 1500 Pa
(c) 500 Pa (d) none of these

Solution (c) $A_1v_1 = A_2v_2$ or $V_2 = \frac{10}{5} \times 1 = 2\,ms^{-1}$

Apply Bernoulli's equation

$$P_1 + 1/2\rho v_1^2 = P_2 + 1/2\rho v_2^2$$

$$2000 + 1/2 \times 10^3(1)^2 = P_2 + 1/2 \times 10^3(2)^2$$

or $$P_2 = 500\,\text{Pa}.$$

22. The density of water > density of air. Even then clouds containing water float

(a) because clouds have large amount of air
(b) clouds actually come down with a very small terminal velocity
(c) clouds actually move up with a very small terminal velocity
(d) the droplets in clouds are in the form of bubbles

Solution (b) cloud contain very small droplets and terminal velocity $V_T \propto r^2$

Therefore, clouds come down with a very little speed and appear to float.

23. A spherical ball of radius 10^{-4}m and density 10^4 kgm^{-3} falls freely under gravity through a distance h before entering a tank of water. The velocity of the ball does not change after entering water. Find h. Viscosity of water 9.8×10^{-6} Poiseuille.

(a) 30.21 m (b) 20.41 m
(c) 40.41 m (d) 29.31 m

Solution (d) $V_T = \sqrt{2gh} = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$

or $$h = \left[\frac{2\times(10^{-4})^2[10^4 - 10^3]\times 9.8}{9\times 9.8\times 10^{-6}}\right]^2 \frac{1}{2\times 9.8} = 20.41\,\text{m}.$$

24. A Hg drop of radius 1 cm is splitted to 10^6 droplets of equal size. The energy required to split is ... if the surface tension of Hg is 35×10^{-3}Nm^{-1}

(a) 5.35×10^{-3}J (b) 4.35×10^{-3}J
(c) 4.35×10^{-3}cal (d) 5.35×10^{-3}cal

Solution (d) $\Delta E = 4\pi R^2T\,[n^{1/3} - 1]$

$$= 4 \times 3.14 \times 10^{-4} \times 35 \times 10^{-3}\,[99]$$

$$= 4.356 \times 10^{-3}\text{J}.$$

25. A gardening pipe having an internal diameter D is connected to a sprinkler having n holes each of diameter d. If the water in the pipe has speed v, the speed of water leaving the sprinkler is

(a) $\frac{D^2}{nd^2}v$ (b) $\frac{nD^2}{d^2}v$
(c) $\frac{nd^2}{D^2}v$ (d) $\frac{d^2}{nD^2}v$

Solution (a) $\frac{\pi D^2}{4}v = \frac{n\pi d^2}{4}v'$ or $v' = \frac{D^2}{nd^2}v$.

26. A vessel has liquid of density $\rho = ax + b$ where $a = 4 \times 10^3$ kgm^{-4} and $b = 10^3$kgm^{-3}. A solid of cylindrical shape of length 3/4 m and density $1/2 \times 10^3$ kgm^{-3} floats on it. The immersed length of the cylinder is

(a) 0.41 m (b) 0.14 m
(c) 0.61 m (d) 0.34 m

Solution (b) Weight of the cylinder = weight of the liquid displaced. Let A be the area of cross-section.

$$Al\sigma g = \int_0^x Adx\rho g = \int_0^x Ag\,(ax + b)\,dx$$

or $$l\sigma = \frac{ax^2}{2} + bx \quad \text{or} \quad \frac{3}{4}\times\frac{1}{2} \times 10^3$$

$$= \frac{4\times 10^3}{2}x^2 + 10^3x$$

Solving for x, $x = 0.14$ m.

27. Water from a tap emerges vertically downwards with an initial speed of 1 ms^{-1}. The cross-sectional area of the tap is $10^{-4}m^2$. Assuming pressure to be constant throughout the stream of water and flow to be steady the cross-sectional area 0.15 m below the tap is

(a) $10^{-4}m^2$ (b) $10^{-5}m^2$
(c) $0.5 \times 10^{-4}m^2$ (d) $0.2 \times 10^{-4}m^2$

Solution (c) $A_1v_1 = A_2v_2$ or $v_2 = \frac{A_1}{A_2}v_1$

$$v_2^2 = v_1^2 + 2gh \quad \text{and} \quad A_2 = A_1\frac{v_1}{v_2}$$

or $$A_2 = \frac{A_1v_1}{\sqrt{v_1^2 + 2gh}} = \frac{10^{-4}\times 1}{\sqrt{1 + 2\times 10\times .15}}$$

$$= 5 \times 10^{-5}m^2.$$

28. A wooden block with a coin placed on its top, floats in water as shown in Fig. 12.31. The distance l and h are illustrated. If the coin falls into the water the

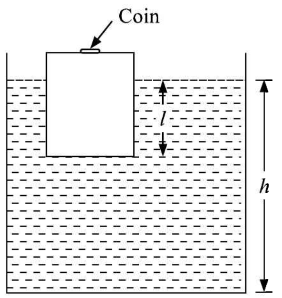


Fig. 12.31

(a) l decreases and h increases
(b) l increases and h decreases
(c) both l and h increase
(d) both l and h decrease

Solution (d) When the coin falls, it occupies less volume, therefore, h decreases. l decreases as density of wood decreases.

29. Radius of a soap bubble is increased from r to $2r$. Work done in this process is ——— if S is surface tension

(a) $12\pi r^2S$
(b) $24\pi r^2S$
(c) $36\pi r^2s$
(d) $48\pi r^2S$

Solution (b) Work done $W = 8\pi S[(2r)^2 - r^2] = 24\pi r^2S$.

Problems for Practice

1. A mercury barometer reads 75cm. If tube be inclined by 60° from the vertical, the length of mercury in the tube will be
   (a) $\frac{75\sqrt{3}}{2}$ (b) 100 cm
   (c) 37.5cm (d) 150 cm
2. An ice cube is floating in water contained in a beaker. When the ice melts, what will happen to the level of water in the beaker?
   (a) it will fall
   (b) it will rise
   (c) it will remain same
   (d) first it will fall and then rise to the same level as before.
3. A steel ball is floating in a trough of mercury. If we fill the empty part of the trough with water, what will happen to the steel ball?
   (a) it will move down
   (b) it will execute vertical oscillations
   (c) it will continue in its position
   (d) it will move up
4. A wooden block is floating in a trough of water. If the trough falls freely, the upward thrust on the wooden block will be
   (a) equal to the weight of the block in air
   (b) same as before
   (c) zero
   (d) more than earlier
5. By what fraction an ice cube will float in sea water of density 1.2gcm^{-3}. Density of ice = 0.9g cm^{-3}
   (a) 1/4 (b) 3/4 (c) 0.9 (d) 2/3
6. A tank is filled with water and an orifice is made in the wall so that the range x of water rushing out is maximum. If H is the height of water above the orifice in the tank, then
   (a) $3x = H$ (b) $2x = H$
   (c) $4x = H$ (d) $x = 2H$
7. A body of mass 0.5kg is attached to a thread and it just floats in a liquid. The tension in the thread is
   (a) less than 0.5kg wt
   (b) 0.5kg wt
   (c) more than 0.5kg wt
   (d) zero
8. A hose can deliver 0.5m^3 of water per minute from a hole of diameter 2cm. If the jet is placed horizontally against a wall and if the water that hits the wall drops down vertically, the force on the wall is
   (a) 98N (b) 196N (c) 220N (d) 284N
9. Two holes are made in the side of the tank such that the jets of water flowing out of them meet at the same point on a surface at the level of the bottom. If one hole is at the height h above the bottom, then what will be distance of the other hole from the top?
   (a) $h/4$ (b) $h/3$ (c) $h/2$ (d) h
10. Two circular metal plates of radius 1m and 2m are placed horizontally in a liquid at rest at the same depth. The ratio of thrusts on them is
    (a) 4 : 1 (b) 1 : 2 (c) 1 : 1 (d) 1 : 4
11. A vessel contains liquid of density ρ as shown in Fig. 12.32. The gauge pressure at the point P is

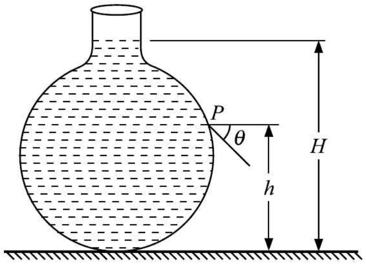


Fig. 12.32

   (a) $h\rho g$ (b) $(H - h)\,\rho g\cos\theta$
   (c) $H\rho g$ (d) $(H - h)\rho g$
12. A U-tube is partially filled with water. Oil which does not mix with water is poured into one side until water rises by 25cm on the other side. If the density of oil be 0.8, the oil level will stand higher then the water level by
    (a) 31.25cm (b) 20cm
    (c) 12.50cm (d) 6.25cm
13. As a bubble comes from the bottom of a lake to the top its radius should
    (a) zero (b) increase
    (c) decrease (d) remain constant
14. An inverted (bell) lying at the bottom of a lake 47.6m deep has 50cm^3 of air trapped in it. The bell is brought to the surface of the lake. The volume of the trapped air will be (atmospheric pressure = 70cm of Hg and density of Hg = 13.6gm cm^{-3}.
    (a) 22cm^3 (b) 250cm^3
    (c) 300cm^3 (d) 350cm^3
15. Which of the following would a hydrogen balloon find easiest to lift?
    (a) 10 kg cotton (b) 10 kg card board
    (c) 10 kg steel (d) all of them
16. A piston of cross-sectional area 100cm^2 is used in hydraulic press to exert a force 10^{7}dynes on the water. The cross-sectional area of the other piston which supports an object having a mass 2000kg is
    (a) 2×10^{10}cm^2 (b) 100cm^2
    (c) 2×10^4cm^2 (d) 10^9cm^2
17. A fisherman hooks an old log of wood of weight 12N and volume 1000cm^3. He pulls the log half way out of water. The tension in the string at this instant is
    (a) 7N (b) 8N (c) 10N (d) 12N
18. An iceberg is floating partially immersed in sea water. If the density of sea water is 1.03 gcm^{-3} and that of ice is 0.92gcm^{-3}. The fraction of the total volume of iceberg above the level of sea water is
    (a) 89% (b) 34% (c) 11% (d) 8%
19. A beaker containing water weights 100g. It is placed on the pan of a balance and a piece of metal weighing 70g and having a volume of 10cm^3 is placed inside the water in the beaker. The weight of the beaker and the metal would be
    (a) 30g (b) 100g (c) 160g (d) 170g
20. A weightless rubber balloon has 100g of water in it. Its weight in water will be
    (a) zero (b) 50g (c) 100g (d) 200g
21. A boat 3m long and 2m wide is floating in a lake. When a man climbs over it, it sinks 1cm further into water. The mass of the man is
    (a) 60kg (b) 64kg (c) 70kg (d) 72kg

22. A vessel contains oil (density $0.8\,\text{gcm}^{-3}$) over mercury (density $13.6\,\text{gcm}^{-3}$). A homogeneous sphere floats with half its volume immersed in mercury and the other half in oil. The density of the material of the sphere in gcm^{-3} is

(a) 12.8 (b) 7.2 (c) 6.4 (d) 3.3

23. A raft of wood (density $600\,\text{kgm}^{-3}$) of mass 120 kg floats in water. How much weight can be put on the raft to make it just sink.

(a) 80 kg (b) 40 kg
(c) 200 kg (d) 120 kg

24. Water is flowing through a pipe of uniform cross section under constant pressure, at some place the pipe becomes narrow, the pressure of water at this place

(a) depends on several factor
(b) increases
(c) decreases
(d) remain unchanged

25. The Fig. 12.33 represents vertical sections of four wings moving horizontally in air. In which case the force acts upwards?

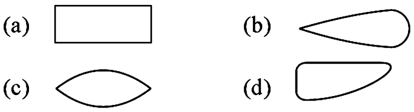

(a) (b) (c) (d)

Fig. 12.33

26. A cylinder is filled with a nonviscous liquid of density d to height h_0 and a hole is made at height h_1 from the bottom of the cylinder. The velocity of the liquid coming out of the hole is

(a) $\sqrt{2g(h_0 - h_1)}$ (b) $\sqrt{gd\,h_1}$
(c) $\sqrt{dgh_0}$ (d) $\sqrt{2h_0 g}$

27. A rectangular vessel when full of water takes 10 minutes to be emptied through an orifice in its bottom. How much time will it take to be emptied when half filled with water?

(a) 3 minutes (b) 5 minutes
(c) 7 minutes (d) 9 minutes

28. A block of wood floats in water with (4/5)th of its volume submerged, but it just floats in another liquid A. What is the density of liquid A?

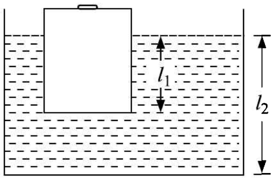


Fig. 12.34

(a) $750\,\text{kgm}^{-3}$ (b) $800\,\text{kgm}^{-3}$
(c) $1200\,\text{kgm}^{-3}$ (d) $1250\,\text{kgm}^{-3}$

29. An ice cube floats in water in a vessel as shown in the Fig. 12.34. Ametal piece is kept over it. If metal piece drops into water then

(a) l_1 decreases, l_2 increases
(b) l_2 decreases, l_1 remains unchanged
(c) l_1 and l_2 both increase
(d) l_1 and l_2 both decrease

30. Liquid is flowing steadily through a cylindrical tube. Velocity of the liquid in the stream line along the axis of the cylinder is $4\,\text{cm s}^{-1}$. The velocity of the liquid layer in contact with the tube is

(a) $4\,\text{cm s}^{-1}$ (b) zero
(c) $0 < v < 4\,\text{cm s}^{-1}$ (d) $1\,\text{cm s}^{-1}$

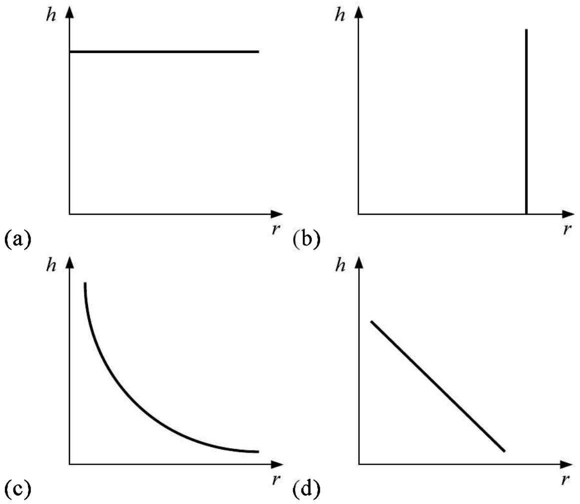
31. Angle between the viscous force and the direction of flow of the liquid is

(a) $\pi/4$ (b) $\pi/2$ (c) zero (d) none of these

32. Two glass balls of radii r and $2r$ are dropped in air. The terminal velocity of the ball with radius r is $1\,\text{cm s}^{-1}$, that of the other will be

(a) $0.5\,\text{cm s}^{-1}$ (b) $4\,\text{cm s}^{-1}$
(c) $1\,\text{cm s}^{-1}$ (d) $2\,\text{cm s}^{-1}$

33. The correct curve between the height or depression h of liquid in a capillary tube and its radius is



(a) (b) (c) (d)

Fig. 12.35

34. Water rises in a capillary tube of diameter 0.2×10^{-2}m, up to a height of 1.5 cm. The surface tension of water is

(a) $43.3 \times 10^{-3}\,\text{Nm}^{-1}$ (b) $73.5 \times 10^{-3}\,\text{Nm}^{-1}$
(c) $73.5 \times 10^{-3}\,\text{dyne cm}^{-1}$ (d) $35.7 \times 10^{-3}\,\text{Nm}^{-1}$

35. Two soap bubbles of radii 4 cm and 5 cm coalesce to form a single bubble. The radius of curvature of this combined bubble will be

(a) $\sqrt{14}$ cm (b) 1 cm
(c) 1.25 cm (d) 20 cm

36. A wire of length l experiences force due to surface tension and sinks as the force is half its weight. The wire is stretched to double its length. If placed in same liquid now

(a) it will sink
(b) it will be suspended in between top and bottom.
(c) it will just float
(d) condition cannot be predicted.

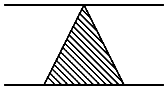
37. A spherical drop of liquid of radius R breaks into 8 equal drops. If the surface tension of liquid is T, then the work done in the process will be

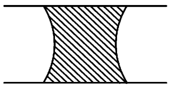
(a) $4\pi R^2 T$ (b) $3\pi R^2 T$
(c) $\pi R^2 T$ (d) $2\pi R^2 T$

38. A rectangular film of liquid is 5 cm long and 3 cm wide. If the work done in increasing its area to 6 cm × 5 cm is 3×10^{-4} Joule then the surface tension of liquid will be

(a) $0.1\,\text{Jm}^{-2}$ (b) $1\,\text{Jm}^{-2}$
(c) $10\,\text{Joule/m}^2$ (d) $72\,\text{Jm}^{-2}$

39. The correct shape of a liquid drop enclosed between two glass plates will be

(a)


(b)


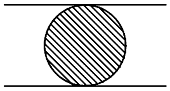
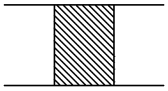
(c)  (d) 

Fig. 12.36

40. There is a hole of radius r in a cylindrical glass pot. To what depth in the sea can it be immersed so that water may not enter it? (surface tension of water is T)

(a) T/rgd (b) $rgd/2$ (c) $2T/r$ (d) $\frac{2T}{rgd}$

41. The surface tension of soap solution is T. The work done in increasing the diameter of soap bubble from D to $2D$ will be

(a) $3\pi D^2T$ (b) $2\pi D^2T$
(c) $8\pi D^2T$ (d) $4\pi D^2T$

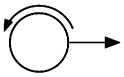
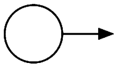
42. The ring of radius 1 m is lying on the surface of liquid. It is lifted from the liquid surface by a force of 4 Newton in such a way that the liquid film in it remains intact. The surface tension of the liquid is

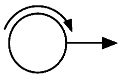
(a) $1/3\pi\,\text{Nm}^{-1}$ (b) $4/\pi\,\text{Nm}^{-1}$
(c) $2/\pi\,\text{Nm}^{-1}$ (d) $1/\pi\,\text{Nm}^{-1}$

43. When a liquid drop of radius R splits into n smaller drops each of radius r then the value of n will be

(a) $\sqrt{R/r}$ (b) $(rR)^{1/3}$
(c) $(r/R)^{1/3}$ (d) $(R/r)^3$

44. To get the maximum flight, a ball must be thrown as

(a)  (b) 

(c)  (d) none of these

45. The surface tension of cold water as compared to that of hot water is

(a) $4T/r$ (b) more (c) less (d) $2T/r$

46. A certain mass of metal is moulded in different shapes by means of a sieve. The surface area of the resulting surface will be manimum when it is a

(a) cylinder (b) cone
(c) ring (d) sphere

47. If a capillary tube is fitted 45° and 60° to the vertical then the ratio of lengths l_1 and l_2 of liquid columns in it will be

(a) $\sqrt{2} : 1$ (b) $2 : 1$
(c) $1 : 2$ (d) $1 : \sqrt{2}$

48. A metallic wire of density d is lying horizontal on the surface of water. The maximum length of wire so that it may not sink, will be

(a) $\sqrt{\frac{2Tg}{\pi d}}$ (b) of any length
(c) $\sqrt{\frac{2Tg}{\pi dg}}$ (d) $\sqrt{\frac{2\pi T}{dg}}$

49. A ring of radius r and weight W is lying on a liquid surface. If the surface tension of the liquid is T, then the minimum force required to be applied in order to lift the ring up

(a) $2W$ (b) $W + 2\pi rT$
(c) $W + 4\pi rT$ (d) W

50. A wire of mass 0.3 g is lying horizontal on the surface of water. The maximum length of wire so that it may not sink, will be ($T = 70 \times 10^{-3}\,\text{Nm}^{-1}$)

(a) 21 m (b) 2.1 m
(c) 0.21 m (d) 0.021 m

51. The work done in blowing a spherical soap bubble of diameter 2 cm will be ... if the surface tension of soap solution is $2 \times 10^{-2}\text{Nm}^{-1}$

(a) 50.2 J (b) 50.2×10^{-6}erg
(c) 50.2×10^{-6} J (d) zero

52. Bernoulli's principle is based on the conservation of

(a) momentum (b) mass
(c) energy (d) none of these

53. For which pair, out of the following, is the angle of contact maximum

(a) water and glass
(b) mercury and glass
(c) water and chromium
(d) water and silver

54. The maximum value of angle of contact can be

(a) 0° (b) 180° (c) 360° (d) 90°

55. The potential energy of molecules on the free surface of liquid is

(a) mgR (b) maximum
(c) infinite (d) minimum

56. A needle keeps floating on water surface. The reason for this is

(a) adhesive force (b) size of needle
(c) surface tension (d) cohesive force

57. A soap bubble is charged, then it

(a) contracts (b) expands
(c) unchanged (d) becomes erroneous

58. A tank has a hole made at its bottom. The time needed to empty the tank from level h_1 to h_2 will be proportional to

(a) $h_1 + h_2$ (b) $\sqrt{h_1} + \sqrt{h_2}$
(c) $h_1 - h_2$ (d) $\sqrt{h_1} - \sqrt{h_2}$

59. The surface tension can be expressed in

(a) Nm^{-1} (b) $\text{N}^{-1}\text{m}^{-1}$ (c) N^{-1}m (d) N m

60. A small spherical solid ball is falling down in a viscous liquid. Its velocity in the viscous liquid is best represen-ted by the curve

(a) A (b) B
(c) C (d) D

61. On what principle, out of the following, is the paint gun based?

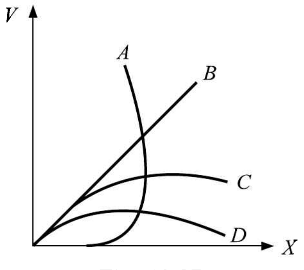


Fig. 12.37

(a) Newton's law of motion
(b) Archimede's principle
(c) Boyle's law
(d) Bernoulli's principle

62. Two soap bubbles of radius r_1 and r_2 combine isothermally, the new radius is

(a) $r_1 + r_2$ (b) $\sqrt{r_1^2 + r_2^2}$
(c) $\frac{r_1r_2}{r_1 + r_2}$ (d) $\frac{r_1r_2}{r_2 - r_1}$

63. Two soap bubbles A and B are connected at the ends of a tube having a valve V in between. When the valve is opened.

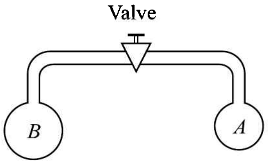


Fig. 12.38

(a) A and B acquire equal size
(b) B gain size
(c) A gains size and B loses.
(d) A and B both lose size
(e) A and B both gain size.

64. Two thin wooden sticks are floating on the surface of water close to each other. A hot needle touches the water between them. How will the sticks react to it? They will

(a) move erratically
(b) stay as before
(c) come closer
(d) move apart

ANSWERS

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (d)
11. (a)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (d)	**20.** (a)
21. (a)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (b)	**29.** (d)	**30.** (b)
31. (d)	**32.** (b)	**33.** (c)	**34.** (b)	**35.** (d)	**36.** (c)	**37.** (a)	**38.** (a)	**39.** (b)	**40.** (d)
41. (a)	**42.** (d)	**43.** (d)	**44.** (c)	**45.** (b)	**46.** (d)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (d)
51. (c)	**52.** (c)	**53.** (c)	**54.** (b)	**55.** (b)	**56.** (c)	**57.** (b)	**58.** (d)	**59.** (a)	**60.** (c)
61. (d)	**62.** (b)	**63.** (b)	**64.** (d)						

Kinetic Theory of Gases, Calorimetry and Specific Heat of Gases

13

BRIEF REVIEW OF THE CONCEPTS

Relation between different temperature scales

$$\frac{C}{100} = \frac{F-32}{180} = \frac{R}{80} = \frac{R_a - 460}{212}$$

$C \rightarrow$ °C (degree Celius)
$F \rightarrow$ °F (degree Fahrenheit)
$R \rightarrow$ Reaumer
$R_a \rightarrow$ Rankine
Triple point of water 273.16K or 0.16°C
K = °C + 273
Temperature of human body = 37°C or 98.4°F
$-40°C = -40°F$; $574.25 K = 574.25°F$

Production and measurement of very low temperatures is called **cryogenics** while measurement of very high temperatures is called **pyrometry**

Ideal gas equation $PV = nRT$
n = number of moles
$R = 8.314\,\text{J mol}^{-1}\text{K}^{-1}$

Ideal gas is a gas which always obeys this equation.

Force exerted by the gas

$$F = \frac{m}{3L} \Sigma v^2$$

Pressure $$P = \frac{F}{L^2} = \frac{1}{3}\rho V_{\text{rms}}^2$$

$$PV = \frac{1}{3} \text{M} V_{\text{rms}}^2$$

$$PV = \frac{1}{2} \text{Nm} V_{\text{rms}}^2$$

where N = total number of molecules/atoms

$$v_{\text{rms}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3PV}{M}} = \sqrt{\frac{3RT}{M}}$$

$M \rightarrow$ molar mass

Number of collisions exerted by a gas/unit area of the wall surface per unit time $N = \dfrac{n v_{\text{average}}}{4}$

where $v_{\text{average}} = \sqrt{\dfrac{8kT}{\pi m}} = \sqrt{\dfrac{8RT}{\pi M}}$ m is mass of a single molecule.

Relative number of gas molecules traversing distance x without collision is $N = N_0 e^{-x/\lambda}$ where $\lambda = \dfrac{1}{\sqrt{2}\pi d^2 n}$ is mean free path, n = number of molecules per unit volume and d is effective diameter of the molecule.

Kinetic energy $\text{K} = \dfrac{1}{2} MV^2$; $\dfrac{P}{P_{\text{triple point}}} = \dfrac{T}{273.16}$

Boyle's law $P \propto \dfrac{1}{V}$ or PV = constant

Charles's law of pressure

$$P \propto v_{\text{rms}}^2 \quad \text{or} \quad P \propto T; \quad v_{\text{rms}}^2 \propto T$$

Avogadro's law At the same temperature and pressure, equal volumes of all gases contain equal number of molecules.

Graham's law of diffusion When two gases at the same pressure and temperature are allowed to diffuse into each other, the rate of diffusion of each gas is inversely proportional to the square root of the density of the gas.

$$\frac{r_1}{r_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

Diflusion coefficient $D = \dfrac{1}{3}\lambda\, v_{\text{average}}$

Dalton's law of partial pressure The pressure exerted by a mixture of several gases equals the sum of the pressures exerted by each gas occupying the same volume as that of the mixture.

$$P = P_1 + P_2 + \ldots$$

$$v_{\text{average}} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$$

$$v_{\text{most probable}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}}$$

Maxwell's speed distribution

$$dN = 4\pi N \left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{\frac{-mv^2}{2kT}} dv$$

Van der waal's equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \text{ (for one mole)}$$

or $$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \text{ for } n \text{ moles}$$

$$V_{\text{critical}} = 3b,\quad P_{\text{critical}} = \frac{a}{27b^2},$$

$$T_{\text{critical}} = \frac{8a}{27Rb} = \frac{8}{27}T_B$$

where T_B is Boyle's temperature and $T_B = \frac{a}{Rb}$

The amount of heat needed to raise the temperature of 1g of water by 1°C (14.5°C to 15.5°C) at a pressure of 1 atm is called a calorie.

Specific heat capacity (c) It is the amount of heat required to raise the temperature of a unit mass of substance by 1°C

$$c = \frac{\Delta Q}{m\Delta\theta} \qquad \Delta\theta \rightarrow \text{rise in temperature.}$$

Latent heat It is the amount of heat required to change the state of unit mass of a substance (from liquid to vapour or solid to liquid) without changing the temperature. It is of two types: Latent heat of fusion (solid → liquid) and latent heat of vaporization (liquid → vapour).

Latent heat of fusion of ice (water) = 80 cal g^{-1}

$$L = \frac{\Delta Q}{M}$$

Latent heat of vaporization of water = 537 cal g^{-1} practically taken to be 540 cal g^{-1}.

Heat of sublimation Amount of heat required to convert a solid (unit mass) directly to vapour (gas).

Hoar frost is reverse of sublimation. Freezing of clouds is an example.

Reglation is the melting of ice under pressure and its resolidification when pressure is removed.

Variation of melting point in solids and boiling point in liquids (T_K) is given by

$$\frac{dP}{dt} = \frac{JL}{T_K(V_{\text{final}} - V_{\text{initial}})}$$

Mechanical equivalent of heat 1 calorie = 4.186 J (practically taken to be 4.2 J).

Law of calorimetry If no heat is wasted to the surroundings then heat gained = heat lost when two bodies are in contact.

Thermal capacity or water equivalent $W = mc$

Specific heat of gases is of two types: specific heat of gases at constant volume (S_V) and specific heat of gases at constant pressure (S_P).

$$S_V = \left[\frac{\Delta Q}{m\Delta T}\right]_{\text{constant volume}}$$

and molar specific heat at constant volume

$$C_v = \left(\frac{\Delta Q}{n\Delta T}\right)_{\text{constant volume}}$$

$$S_P = \left[\frac{\Delta Q}{m\Delta T}\right]_{\text{constant pressure}}$$

and molar specific heat at constant pressure is

$$C_P = \left(\frac{\Delta Q}{n\Delta T}\right)_{\text{constant pressure}}.$$

Mayer formula $C_P - C_V = R$

Note that C_V (adiabatic process) = 0

C_V (isothermal process) = ∞

Law of equipartition of energy Each degree of freedom in a gas contributes equal energy (average) $\frac{1}{2}kT$/molecule or $\frac{1}{2}RT$ per mole.

For a monatomic gas Number of degrees of freedom = 3 (all translational).

$$C_V = \frac{3}{2}R;\quad C_P = \frac{5}{2}R \text{ and } \gamma = \frac{C_P}{C_V} = \frac{5}{3} = 1.67$$

For a diatomic gas Number of degrees of freedom = 5 (3 translation + 2 rotation), assuming molecules do not vibrate.

$$C_V = \frac{5}{2}R,\quad C_P = \frac{7}{2}R \quad \gamma = 7/5 = 1.4$$

For polyatomic gases Number of degrees of freedom = 6 (3 translation, 2 rotation and 1 vibration)

$$C_V = 3R,\ C_P = 4R\ \gamma = \frac{C_P}{C_V} = \frac{4}{3}$$

There are certain gases which do vibrate. In such cases, number of degrees of freedom = 7.

$$C_V = \frac{7}{2}R,\quad C_P = \frac{9}{2}R,\quad \frac{C_P}{C_v} = \frac{9}{7} = 1.29$$

If all the degrees of freedom (translational, rotational and vibrational) are excited then for an N-atomic molecule (volume or network), number of degrees of freedom = $(6N - 3)$. For an N-atomic linear molecule number of degrees of freedom = $(6N - 5)$

Dulong Petit's law Specific heat of solids at constant volume = $3R$ or 6 $\text{cal mol}^{-1}\,{}^\circ\text{C}^{-1}$. It is valid at high temperature. It is based on the fact that there are 3N vibrational states for N molecules.

For a mixture of gases

$$\gamma_{\text{mix}} = \frac{C_{P\text{mix}}}{C_{V\text{mix}}}$$

$$C_{V\text{mix}} = \frac{n_1C_{V1} + n_2C_{V2}}{n_1 + n_2};\quad C_{P\text{mix}} = C_{V\text{mix}} + R$$

Specific heat of H_2 is maximum [3.5 $\text{cal g}^{-1}\,{}^\circ\text{C}^{-1}$]. This is followed by water [1 $\text{cal g}^{-1}\,{}^\circ\text{C}^{-1}$]. It is minimum for Radon and Actinium [0.22 $\text{cal g}^{-1}\,{}^\circ\text{C}^{-1}$].

At low temperature, specific heat $C \propto T^3$ (in super- conducting range) and at high temperature $C \propto T$.

SHORT-CUTS AND POINTS TO NOTE

(i) No gas in real practice is ideal. However, gases like H_2, N_2, O_2 and He may be considered ideal as these cannot be liquified easily. An ideal gas will follow $PV = nRT$ strictly. No molecular forces are present.

Therefore $\frac{\partial U}{\partial V} = 0$

(ii) In linear expansion $\Delta L = \alpha L\Delta T$

In superficial (area of the surface) expansion $\Delta S = \beta S\Delta T$

In cubical expansion $\Delta V = \gamma V\Delta T$

Note that $\beta = 2\alpha$

$\gamma = 3\alpha$

Variation of density with temperature

$$\rho(T) = \rho_0(1 - \gamma\Delta T)$$

Thermal stress = $Y\alpha\Delta T$

where Y is Young's modulus.

Thermal Force = stress × Area = $Y\alpha\Delta TA$

A pendulum clock becomes slower in summer and faster in winter. Change in time $\Delta T = \frac{1}{2}\alpha\Delta\theta(T)$

(iii) Water shows an anomalous behaviour between 0 and 4°C.

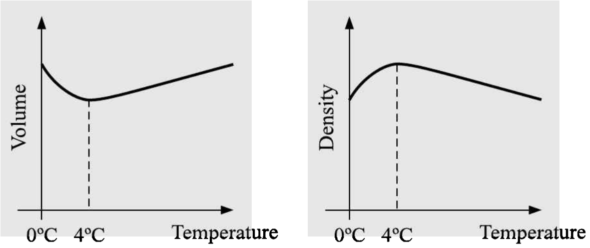


Fig. 13.1

It is due to anomalous behaviour of water that aquatic animals survive even when the upper surface of water in a lake freezes. High density water remains at 4°C.

(iv) When a liquid expands, since it is contained in a container, therefore, we actually measure apparent expansion coefficient γ_{app}. If γ_r is real expansion coefficient then $\gamma_r = \gamma_a + 3\alpha_{\text{container}}$

If $\gamma_r > \gamma_{\text{container}}$ then $\gamma_{\text{app}} > 0$ (level of the liquid rises)

If $\gamma_r < \gamma_{\text{container}}$ then $\gamma_{\text{app}} < 0$ (level of the liquid falls on heating)

If $\gamma_r = \gamma_{\text{container}}$ then $\gamma_{\text{app}} = 0$ (level unchanged)

$\gamma = \frac{1}{273}$ per degree Celsius for gases

(v) Substances like ice do not expand on heating (melting). Rather they expand on cooling in a specific range.

(vi) 1.0 g of steam at 100°C melts 8 g of ice at 0°C.

(vii) Thermal capacity = water equivalent = mc (units JK^{-1} or $cal^{-1}°C$). Heat required when a substance changes state without changing temperature.

$$\Delta Q = mL$$

L is called latent heat

According to calorimetry heat gained by a substance = heat lost by the other when the two are in contact. It is assumed that no heat is wasted to surroundings.

Specific heat of water = 1 cal g^{-1} $°C^{-1}$

Specific heat of ice = 0.5 cal g^{-1} $°C^{-1}$

Heat gained by $-\theta_1°C_{\text{ice}}$

$$= m_{\text{ice}}C_{\text{ice}}\theta_1 + m_{\text{ice}}L + m_{\text{ice}}C_w\Delta\theta$$

Heat gained by $0°C_{\text{ice}} = m_{\text{ice}}L + m_{\text{ice}}C_w\Delta\theta$

Heat lost by water = $m_wC_w\ \Delta\theta$

(viii) $v_{\text{rms}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3PV}{M}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3kT}{m}}$

where M is molar mass and m is mass of a molecule/atom.

R is gas constant and k is Boltzmann's constant.

$R = 8.314\,JK^{-1}mol^{-1}$

$k = 1.386 \times 10^{-23}$ erg $°C^{-1}$

Average velocity $v_{\text{average}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8kT}{\pi m}}$

Most probable velocity

$$v_{\text{most probable}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2kT}{m}}$$

(ix) Molar specific heat of gases $C_V = \frac{3}{2}R$, $C_P = \frac{5}{2}R$, $\gamma = \frac{5}{3}$ for monatomic gases

For diatomic gases $C_V = \frac{5}{2}R$, $C_P = \frac{7}{2}R$, $\gamma = \frac{7}{5}$

For polyatomic gases $C_V = 3R$, $C_P = 4R$, $\gamma = \frac{4}{3}$

(x) Dulong Petit's law for solids is valid at high temperatures

$$C_V = 3R = 6\,cal^{-1}mol°C^{-1}$$

(xi) Specific heat of lighter elements is higher than heavier elements and vice versa. Specific heat of the same substance in different states (solid, liquid and vapour) is different. For example, specific heat of water is 1 $calg^{-1}°C^{-1}$, that of ice is 0.5 $calg^{-1}°C^{-1}$.

(xii) With rise in temperature, the weight of a body will increase due to decreasing upthrust.

(xiii) Area of volume expansion of an isotropic body is independent of size and shape of the hole/cavity inside it.

(xiv) For ionic solids linear expansion coefficient is about 10 times more than that of non-ionic substances.

(xv) Boiling point of a liquid rises with pressure while melting point falls with increasing pressure. Melting points of wax and sulphur (expand on melting) rise with increase in pressure. Impurities also increase boiling point and lower melting point. For example, ice + salt forms freezing mixture.

(xvi) Saturated vapours do not obey gas laws. However, they obey Dalton's law of partial pressure.

(xvii) $\frac{V_{\text{rms}}}{V_{\text{sound}}} = \sqrt{\frac{3}{\gamma}} = \sqrt{\frac{2+f}{3f}}$ where f is number of degrees of freedom and $f = (3N - I)$ where N is number of particles having I independent relations.

(xviii) KE of monoatomic gas = $\frac{3}{2}$ RT (1 mole)

KE of diatomic gas = $\frac{5}{2}$ RT (1 mole)

KE of polyatomic gas = 3 RT (one mole)

$\text{KE} = \frac{m}{2}(v_{\text{rms}})^2$; KE per unit volume

$\frac{\text{KE}}{\text{vol.}} = \frac{\rho}{2}(v_{\text{rms}})^2$ $\rho \rightarrow$ density of the gas

Mean free path $\lambda = \frac{1}{\sqrt{2}\pi d^2 n}$ where $d \rightarrow$ diameter

$\lambda = \frac{kT}{4\pi\sqrt{2}r^2 p}$ where p is pressure and T is temperature in k

⚠ CAUTION

(i) Considering all gases have same specific heat.

- Monoatomic gases have

$$C_V = \frac{3}{2}R,\ C_P = \frac{5}{2}R$$

Diatomic gases have

$$C_V = \frac{5}{2}R,\ C_P = \frac{7}{2}R \text{ (no vibration)}$$

Polyatomic gases have $C_V = 3R$, $C_P = 4R$.

(ii) Considering $nC_V\Delta T$ is change in external energy.

- $nC_V\Delta T$ forms change in internal energy.

(iii) Considering substances only expand on heating

- Materials like silicon, germanium, selenium and cobalt and so on have negative expansion coefficient. Water has negative expansion coefficient between 0 and 4°C.

(iv) Considering that freezing point of ice = melting point of ice = 0°C

- Freezing point of ice is –4°C.

(v) Considering specific heat of ice, water and steam are equal.

- Specific heat of ice = $0.5\,\text{calg}^{-1}{}^\circ\text{C}^{-1}$ and specific heat of water = $1\,\text{calg}^{-1}{}^\circ\text{C}^{-1}$.

  Specific heat of steam = $0.75\ \text{calg}^{-1}{}^\circ\text{C}^{-1}$.

(vi) Confusing molar specific heat and specific heat to be equal.

- Molar specific heat C = molar mass × specific heat = Mc

(vii) Considering that specific heat is +ve and finite.

- Specific heat can be positive, zero, infinite or negative

  Specific heat of saturated vapours is negative. In adiabatic process specific heat is zero and in isothermal process it is infinite.

(viii) Considering that increase in length of the pendulum increases time period, and therefore, the clock becomes fast.

- As time interval increases, it will take less oscillation and hence it will become slow.

(ix) Considering that boiling point and melting point of water are standard.

- Melting point and boiling point vary with pressure and impurities.

(x) Confusing the values of R and k.

- $R = 8.314\,\text{JK}^{-1}\text{mol}^{-1}$ or $R = 2\,\text{cal}$.

$$k = \frac{R}{\text{Avogadro number}} = 1.38 \times 10^{-23}\,\text{JK}^{-1}$$

(xi) Confusing SI and CGS units.

- It is advised to do questions on calorimetry in CGS as calculations become simple. If the final answer is in Joules, then convert calories into Joules by multiplying the result by 4.2.

Solved Problems

1. A vessel contains 1 mole of O_2 (molar mass 32 g) at a temperature T. The pressure is P. An identical vessel containing 1 mole of He gas (molar mass 4) at a temperature $2T$ has a pressure

(a) P	(b) $\frac{P}{8}$
(c) $2P$	(d) $8P$

Solution (c) $PV = nRT$

$\therefore$ $P_{\text{He}} = 2P$ as temperature of He is doubled.

2. 70 calorie of heat is required to raise the temperature of a diatomic gas at constant pressure from 30 to 35°C. The amount of heat required (in calorie) to raise the temperature of the same gas through the same range (30 to 35°C) at constant volume is

(a) 30	(b) 60
(c) 50	(d) 70

Solution (c) $\frac{C_P}{C_V} = \gamma = 1.4$

$$\frac{(\Delta Q)_P}{(\Delta Q)_V} = \frac{nC_P\Delta T}{nC_V\Delta T} = \frac{C_P}{C_V} = 1.4$$

$$\therefore \quad (\Delta Q)_V = \frac{(\Delta Q)_P}{1.4} = 50\,\text{cal}$$

3. A vessel contains 1 mole of O_2 and 1 mole of He. The value of γ for the mixture is

(a) 1.4	(b) 1.50
(c) 1.53	(d) none of these

Solution (b) $C_{V\text{mix}} = \dfrac{\frac{3}{2}R + \frac{5}{2}R}{2} = 2R$

$$C_{P\text{mix}} = 2R + R = 3R$$

$$\gamma_{\text{mix}} = \frac{C_P}{C_V} = \frac{3}{2}$$

4. Steam at 100°C is passed into a calorimeter of water equivalent 10 g containing 94 cc of H_2O and 10 g of ice at 0°C. The temperature of the calorimeter and contents rise by 5°C. The amount of steam passed is

(a) 1 g	(b) 2 g
(c) 3 g	(d) 4 g

Solution (b) Let m_S be the amount of steam in grams. Then

$$m_S L + m_S C_w (100 - 5) = W_{\text{cal}} (5 - 0) + 10 \times 80 + (104)(5 - 0)$$

$$m_S (540) + m_S (95) = 10 (5) + 800 + 420$$

$635\ m_S = 1270$ or $m_S = 2\text{g}$.

5. 10 g of ice is added in 40 g of water at 15°C. The temperature of the mixture is

(a) 0	(b) 3°C
(c) 12°C	(d) 8°C

Solution (a) Heat required to melt the ice = mL = 10 × 80 = 800 cal.

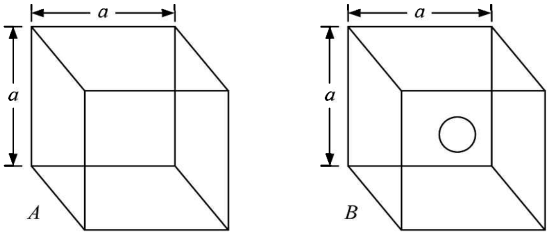
Maximum heat which can be supplied by hot water $= mC\Delta T = 40 \times 1 \times 15 = 600$ cal

As heat supplied < heat required to melt ice,

$\therefore$ Temperature of mixture the will be 0°C

(As even whole of the ice will not melt).

6. *A* and *B* are made up of an isotropic medium. Both *A* and *B* are of equal volume. Body *B* has cavity as shown in Fig. 13.2(b). Which of the following statements is true?



(a) (b)

Fig. 13.2

(a) expansion in volume of *A* > expansion in *B*
(b) expansion in volume of *B* > expansion in *A*
(c) expansion in *A* = expansion in *B*
(d) none of these

Solution (c) Thermal expansion of isotropic bodies is independent of shape, size and availability of hole/cavity.

7. If an annular disc of radii r_1 and r_2 is heated, then

(a) r_1 increases, r_2 decreases
(b) r_2 increases, r_1 decreases
(c) both r_1 and r_2 increase
(d) r_2 increases, r_1 remains unchanged.

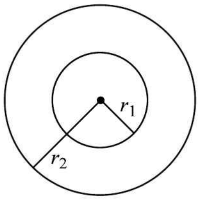


Fig. 13.3

Solution (c) The hole also expands.

8. An isotropic solid has linear expansion (coefficient of α_x, α_y and α_z for three rectangular axes in a solid. The coefficient of cubical expansion is

(a) $\alpha_x\,\alpha_y\,\alpha_z$ (b) $\dfrac{\alpha_x}{\alpha_y+\alpha_z}$

(c) $\alpha_x + \alpha_y + \alpha_z$ (d) $\alpha_x^2 + \alpha_y^2 + \alpha_z^2$

Solution (c) $V(T) = V_0\,(1 + \gamma\Delta T)$...(1)

$$V(T) = L_x(1 + \alpha_x\Delta T)\,L_y(1 + \alpha_y\Delta T) \times L_z(1 + \alpha_z\Delta T)$$
$$= L_xL_yL_z(1 + \alpha_x\Delta T + \alpha_y\Delta T + \alpha_z\Delta T) \qquad ...(2)$$

Neglecting square and higher power terms of α_x, α_y, α_z or $\alpha_x\alpha_y$,..., and $\alpha_x\alpha_y\alpha_z$.

Comparing (1) and (2) we get

$$\gamma = \alpha_x + \alpha_y + \alpha_z.$$

9. Three rods of equal length *l* are joined to form an equilateral ΔABC. *O* is the midpoint of *BC*. Distance *OA* remains same for small change in temperature. If the coefficient of linear expansion for *AB* and *AC* is α_2 and for *BC* is α_1, then

(a) $\alpha_2 = 3\alpha_1$ (b) $\alpha_z = 4\alpha_1$

(c) $\alpha_1 = 3\alpha_2$ (d) $\alpha_1 = 4\alpha_2$

Solution (c) $AO^2 = AB^2 - BO^2$

$$= l^2 - \left(\frac{l}{2}\right)^2$$

$$l^2 - \left(\frac{l}{2}\right)^2 = l^2(1 + \alpha_2\Delta T)^2 - \left(\frac{l}{2}\right)^2(1 + \alpha_1\Delta T)^2$$

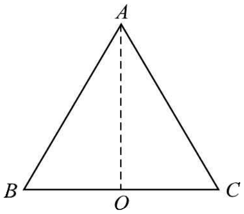


Fig. 13.4

$$l^2 - \frac{l^2}{4} = l^2(1 + 2\alpha_2\Delta T) - \frac{l^2}{4}(1 - 2\alpha_1\Delta T)$$

(Apply binomial theorem)

or $\alpha_2\Delta T - \dfrac{\alpha_1}{4}\Delta T = 0$ or $\alpha_1 = 4\alpha_2$

10. Which of the substances *A*, *B* or *C* has the highest specific heat? The temperature *vs* time graph is shown.

(a) *A*
(b) *B*
(c) *C*
(d) All have equal specific heat

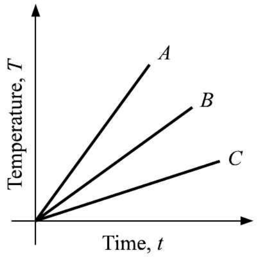


Fig. 13.5

Solution (c) Substances having more specific heat take longer time to get heated to a higher temperature and longer time to get cooled.

11. A thin wire of length *l* when heated to a certain temperature increases its length by 1%. *A* sheet of the same material of area $2l \times l$ is heated to the same temperature. The percentage increase in area will be

(a) 3% (b) 2.5%
(c) 2% (d) 1.5%

Solution (c) Because $\beta = 2\alpha$.

12. 10^{23} molecules of a gas strike a target of area 1 m^2 at angle 45° to normal and rebound elastically with speed 1 kms^{-1}. The impulse normal to wall per molecule is

(a) 4.7×10^{-24} kg ms^{-1} (b) 7.4×10^{-24} kg ms^{-1}
(c) 3.32×10^{-24} kg ms^{-1} (d) 2.33 kg ms^{-1}

Given: mass of a molecule = 3.32×10^{-27} kg

Solution (a) Change in momentum = 2 *mv* cos 45

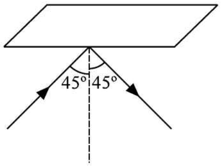


Fig. 13.6

$= 2 \times 3.32 \times 10^{-27} \times 10^{3} \times \frac{1}{\sqrt{2}}$

$= 4.7 \times 10^{-24}$ kgms^{-1}.

13. A cylinder has an alloy (piston) at a temperature of 20°C. There is all round clearance of 0.5 mm between piston and cylinder wall when the internal diameter of the cylinder is exactly 10 cm. The temperature at which it will exactly fit into the cylinder is

(a) 220°C (b) 250°C
(c) 270°C (d) 290°C

Given expansion coefficient of alloy is 1.6×10^{-5}/°C and expansion coefficient of cylinder is 1.2×10^{-5}/°C.

Solution (c) Total clearance = 0.05 mm × 2 = 0.1 mm

$d\,(\alpha_2 - \alpha_1)\Delta T = 0.1$ mm $= 0.01$ cm

$10\,(0.4 \times 10^{-5})\Delta T = 0.01$ or $\Delta T = 250$

or $T = 250 + 20 = 270$°C

14. The temperature of an ideal gas is increased from 120 K to 480 K. If at 120 K the rms velocity of the gas molecules is v, at 480 K it becomes

(a) $4v$ (b) $2v$
(c) $v/2$ (d) $v/4$

Solution (b) $\frac{v_1}{v_2} = \frac{\sqrt{\frac{3RT_1}{M}}}{\sqrt{\frac{3RT_2}{M}}} = \sqrt{\frac{T_1}{T_2}} \quad \therefore \quad v_2 = 2v$

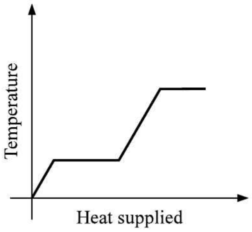
15. The average energy and the rms speed of molecules in a sample of oxygen gas at 300 K are 6.21×10^{-21} J and 484 ms^{-1} respectively. The corresponding values of 600 K are nearly

(a) 12.42×10^{-21} J, 968 ms^{-1}
(b) 8.78×10^{-21} J, 684 ms^{-1}
(c) 6.21×10^{-21} J, 968 ms^{-1}
(d) 12.42×10^{-21} J, 684 ms^{-1}

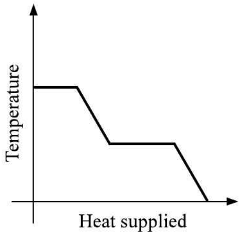
Solution (d) Average energy $\propto T$ and $v_{\text{rms}} \propto \sqrt{T}$

16. A block of ice at –10°C is slowly heated and converted to steam at 100°C. Which of the following curves represents the phenomenon qualitatively?

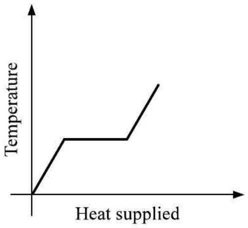
(a)



(b)



(c)



(d)

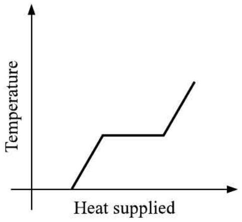


Fig. 13.7

Solution (a) Initially, on heating temperature rises from–10°C to 0°C. Then ice melts and temperature does not rise. After the whole ice has melted, temperature begins to rise until it reaches 100°C. Then it becomes constant, as at the boiling point temperature will not rise.

17. An ideal gas is taken through a cycle $A \rightarrow B \rightarrow C \rightarrow A$ as shown in Fig. 13.8. If the net heat supplied in the cycle is 5 J, then work done by the gas in the process $C \rightarrow A$ is

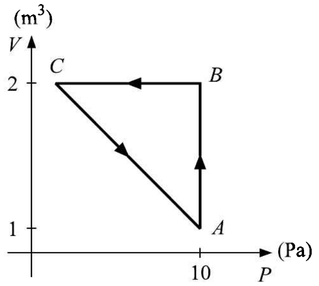


Fig. 13.8

(a) –5 J (b) –10 J
(c) –15 J (d) –20 J

Solution (a) Work done = area under the curve gives 10 J

$5 = W_{CA} + 10$ or $W_{CA} = -5$ J

18. Two gases having same pressure P and volume V are mixed at a temperature T. If the mixture is at a temperature T and occupies the same volume V then pressure of the mixture would be

(a) P (b) $2P$
(c) $\frac{P}{2}$ (d) $3P$

Solution (b) If the temperature is constant, then $P_1V_1 = P_2V_2$

$\therefore \quad P_2 = 2P$.

19. A and B are two gases. $\frac{T_A}{M_A} = 4\,\frac{T_B}{M_B}$ where T is the temperature and M is the molecular mass. If C_A and C_B are rms speeds, then $\frac{C_A}{C_B}$ will be

(a) 2 (b) 4
(c) 0.5 (d) 0.25

Solution (a) $\frac{C_A}{C_B} = \frac{\sqrt{\frac{3RT_A}{M_A}}}{\sqrt{\frac{3RT_B}{M_B}}} = 2$

20. The *P-T* diagram for an ideal gas is shown in Fig. Where AC is an adiabatic process. The correspond-ing *PV* diagram is

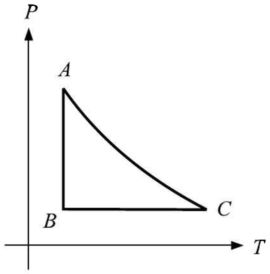
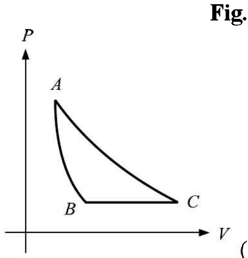
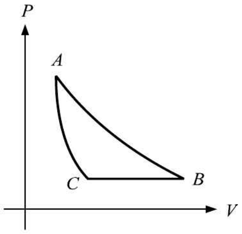
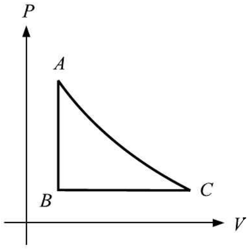
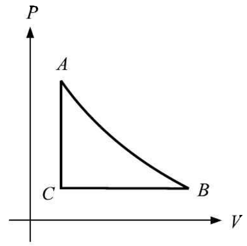


Fig. 13.9

(a)


(b)


(c)


(d)


Solution (b) Process *A* to *B* is isothermal. Then $P \propto \frac{1}{V}$.
Process $B \rightarrow C$ is isobaric and $C \rightarrow A$ adiabatic. Slope of adiabatic > slope of isothermal.

21. The temperature of the sun, if pressure is 1.4×10^9 atm, density is $1.4\,\text{gcm}^{-3}$ and average molecular weight is 2, will be

[Given $R = 8.4\,\text{J mol}^{-1}\text{K}^{-1}$]

(a) 1.2×10^7K (b) 2.4×10^7K
(c) 0.4×10^7K (d) 0.2×10^7K

Solution (b) $PV = nRT$ $\quad n = \frac{m}{M}$ and $\rho = \frac{m}{V}$

or $T = \frac{PV}{nR} = \frac{PM}{\rho R} = \frac{1.4\times10^9\times1.01\times10^5\times2\times10^{-3}}{1.4\times10^3\times8.4}$

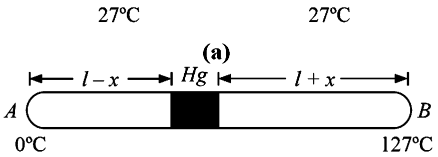
$= 2.4 \times 10^7$K

22. A glass tube sealed at both ends is 1 m long. It lies horizontally with the middle 10 cm containing Hg. The two ends of the tube equal in length contain air at 27°C and pressure 76 cm of Hg. The temperature at one end is kept 0°C and at the other end it is 127°C Neglect the change in length of Hg column. Then the change in length on two sides is

(a) 12.3 cm (b) 10.311 cm
(c) 9.9 cm (d) 8.49 cm

Solution (d) Initially $l = 45$ cm $\quad (2l + 10 = 100$ cm$)$

$P_1 = P_2 = P$ (say) ...(1)



(a)

(b)

Fig. 13.10

Applying gas law at end *A*, assume *A* is area of cross-section

$$\frac{45AP}{300} = \frac{(45-x)AP_1}{273} \quad ...(2)$$

At end *B*, $\frac{45AP}{300} = \frac{(45+x)AP_2}{400}$ ...(3)

From (1), (2) and (3)

$$\frac{(45-x)}{273} = \frac{45+x}{400} = 8.49\,\text{cm}$$

23. A thin tube of uniform cross-section is sealed at both ends. It lies horizontally. The middle 5 cm contains Hg and two equal ends contain air at the same pressure P_0. When the tube is held at an angle of 60° with the vertical, the length of the air column above and below the Hg are 46 cm and 44.5 cm. Calculate pressure P_0 in cm of Hg. Assume temperature of the system constant.

(a) 55 cm of Hg (b) 65 cm of Hg
(c) 70.4 cm of Hg (d) 75.4 cm of Hg

Solution (d) $2L + 5 = 46 + 5 + 44.5 \Rightarrow L = 45.25$ cm

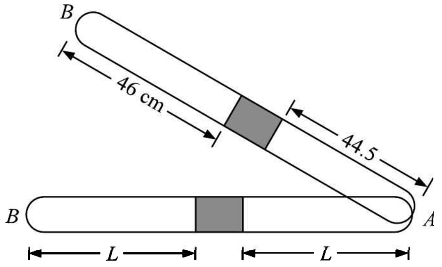


Fig. 13.11

In case (ii) $P_A = P_B + 5\cos 60$

or $P_A - P_B = 2.5$ cm of Hg

$P_A\,(44.5) = P_B(46)$

or $\left(\frac{46}{44.5} - 1\right) P_B = 2.5$ or $P_B = \frac{2.5\times44.5}{1.5}$

$P_0(45.25) = P_B \times 46$

Thus $P_0 = P_B \frac{46}{45.25} = \frac{2.5}{1.5}\times\frac{44.5\times46}{45.25}$

$= 75.4$ cm of Hg

24. Find the amount of work done to increase the temperature of one mole of an ideal gas by 30°C if it is expanding under the condition $V \propto T^{2/3}$.

(a) 166.2 J (b) 136.2 J
(c) 126.2 J (d) none of these

Solution (a) $PV = RT$ for 1 mole

$$W = \int P dV = \int \frac{RT}{V} dV$$

$V = CT^{2/3}$

$\therefore \quad dV = \frac{2}{3} C T^{-1/3} dT$

or $\dfrac{dV}{V} = \dfrac{2}{3}\dfrac{dT}{T}$

$\therefore \quad W = \int_{T_1}^{T_2} RT\left(\dfrac{2}{3}\right)\dfrac{dT}{T} = \dfrac{2}{3}R(T_2 - T_1) = 166.2\,\text{J}$

25. At what temperature the average translational KE of the molecules of a gas will become equal to the KE of an electron accelerated from rest through $1V$ potential difference?

(a) 10^4K (b) 2.34×10^4K

(c) 7.73×10^3K (d) none of these

Solution (c) $\dfrac{3}{2}kT = 1eV = 1.6 \times 10^{-19}$ J

or $T = \dfrac{2\times1.6\times10^{-19}}{3\times1.38\times10^{-23}} = 7730$K

26. 3 mole of H_2 is mixed with 1 mole of Ne. The specific heat at constant pressure is

(a) $9R/4$ (b) $13R/4$

(c) $9R/2$ (d) $13R/2$

Solution (b) $C_{V\text{mix}} = \dfrac{n_1C_{V1} + n_2C_{V2}}{n_1 + n_2}$

$= \dfrac{\frac{5}{2}R\times3 + \frac{3}{2}R\times1}{4} = \dfrac{18R}{8} = \dfrac{9R}{4}$

$C_{P\text{mix}} = C_{V\text{mix}} + R = \dfrac{13R}{4}$

27. The specific heat of Ar at constant volume is 0.075 kcal kg^{-1}K^{-1}. Calculate the atomic weight (R = 2 cal mol^{-1}K^{-1}).

(a) 40 (b) 40.4

(c) 40.2 (d) 40.8

Solution (a) $M = \dfrac{C_V}{S_V} = \dfrac{\frac{3\times2}{2}}{0.075} = 40$

$\because \quad C_V = \dfrac{3}{2}R$

28. Two rods one of Al and other of steel having initial lengths l_1 and l_2 are connected together to form a single rod of length $l_1 + l_2$. The coefficient of linear expansion of aluminium and steel are α_A and α_S respectively. If the length of each rod increases by same amount when the temperature is raised by t°C then find the relation $\dfrac{l_1}{l_1 + l_2}$.

(a) $\dfrac{\alpha_A}{\alpha_A + \alpha_S}$ (b) $\dfrac{\alpha_S}{\alpha_A}$

(c) $\dfrac{\alpha_A}{\alpha_S}$ (d) $\dfrac{\alpha_S}{\alpha_A + \alpha_S}$

Solution (d) $l_2/l_1 = \alpha_A/\alpha_S$

or $1 + \dfrac{l_2}{l_1} = 1 + \dfrac{\alpha_A}{\alpha_S}$ or $\dfrac{l_1}{l_1 + l_2} = \dfrac{\alpha_S}{\alpha_A + \alpha_S}$

29. 2 kg of ice at –20°C is mixed with 5 kg of H_2O at 20°C in an insulating vessel having negligible heat capacity. Calculate the final mass of water left in the container. Given specific heats of water and ice are 1 kcal kg^{-1}°C^{-1} and 0.5 kcal kg^{-1}°C^{-1} and latent heat of fusion of ice is 80 k cal kg^{-1}.

(a) 7 kg (b) 6 kg

(c) 4 kg (d) 2 kg

Solution (b) $m'_{\text{ice}} L + m_{\text{ice}}\, 20\,(0.5) = m_W\,(1)\,(20)$

$m'_{\text{ice}}\, 80 = 5 \times 20 - 2 \times (20) \times 0.5 = 80$

$\therefore \quad m'_{\text{ice}} = 1$ kg

Hence total amount of water = 6 kg.

30. An electricaly heating coil is placed in a calorimeter containing 360 g of H_2O at 10°C. The coil consumes energy at the rate of 90 W. The water equivalent of calorimeter and the coil is 40 g. The temperature of water after 10 minutes will be

(a) 42.14°C (b) 32.14°C

(c) 22.14°C (d) 52.14°C

Solution (a) $Q = P.t = (m + W)C\,\Delta T$

$\dfrac{90\times600}{4.2} = (360 + 40)\Delta T$ or $\Delta T = 32.14$°C

or $T = 10 + 32.14 = 42.14$°C

31. A bimetallic strip is formed out of two identical strips, one of Cu and the other of brass. The coefficients of linear expansion of the two metals are α_C and α_B. If on heating the temperature of the strip goes up by ΔT and the strip bends to form an arc of radius R, then R is

(a) proportional to ΔT

(b) inversely proportional to ΔT

(c) proportional to $|\alpha_B - \alpha_C|$

(d) inversely proportional to $|\alpha_B - \alpha_C|$

Solution (b), (d) $l_B = l_0(1 + \alpha_B\Delta T)$; $l_C = l_0(1 + \alpha_C\Delta T)$

$l_C = R\theta$; $l_B + (R + d)\theta$

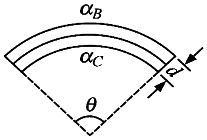


Fig. 13.12

Thus $\dfrac{R+d}{R} = \dfrac{1+\alpha_B\Delta T}{1+\alpha_C\Delta T}$

or $1 + \dfrac{d}{R} = 1 + (\alpha_B - \alpha_C)\Delta T$

(using binomial theorem)

or $R = \dfrac{d}{(\alpha_B - \alpha_C)\Delta T}$

$\therefore$ the correct choices are (b) and (d).

Problems for Practice

1. The internal energy of 1 mol of an ideal gas depends on
(a) only volume
(b) Only temperature
(c) only pressure
(d) Temperature and pressure

2. The temperature of a gas is increased from 27°C to 127°C. The ratio of its mean kinetic energies will be
(a) 10/9 (b) 9/16 (c) 4/3 (d) 3/4

3. A vessel of volume 4 litres contains a mixture of 8 g of O_2, 14 g of N_2 and 22 g of CO_2 at 27°C. The pressure exerted by the mixture is
(a) $5 \times 10^6 N/m^2$ (b) $6 \times 10^3 N/m^2$
(c) 10 atmosphere (d) $7.79 \times 10^5 N/m^2$

4. Equal masses of N_2 and O_2 gases are filled in vessel *A* and *B*. The volume of vessel *B* is double of *A*. The ratio of pressure in vessel *A* and *B* will be
(a) 16 : 7 (b) 16 : 14 (c) 32 : 7 (d) 32 : 28

5. The mean kinetic energy of a gas molecule at 27°C is 6.21×10^{-21} Joule. Its value at 227°C will be
(a) 12.35×10^{-21} Joule (b) 11.35×10^{-21} Joule
(c) 10.35×10^{-21} Joule (d) 9.35×10^{-21} Joule

6. The value of γ of triatomic gas (linear arrangement) molecules is
(a) 5/3 (b) 7/5 (c) 8/6 (d) 9/7

7. The correct relation between V_{rms}, V_{av} and V_{mp} is
(a) $V_{rms} > V_{mp} > V_{av}$
(b) $V_{rms} < V_{av} < V_{mp}$
(c) $V_{rms} > V_{av} > V_{mp}$
(d) $V_{rms} < V_{av} > V_{mp}$

8. One mol of a monoatomic gas is mixed with one mol of a diatomic gas. The molar specific heat of mixture at constant volume will be
(a) $R/2$ (b) R (c) $2R$ (d) $3R$

9. The value of C_V for 1 mol of polyatomic gas is
(f = number of degrees of freedom)
(a) $\frac{fR}{2T}$ (b) $\frac{fR}{2}$ (c) $\frac{fRT}{2}$ (d) $2fRT$

10. The pressure of a gas in a container is 10^{-11} pascal at 27°C. The number of molecules per unit volume of vessel will be
(a) $6 \times 10^{23} cm^{-3}$ (b) $2.68 \times 10^{19} cm^{-3}$
(c) $2.5 \times 10^6 cm^{-3}$ (d) $2400 cm^{-3}$

11. The value of γ for gas X is 1.66, then X is
(a) Ne (b) O_3 (c) N_2 (d) H_2

12. The mass of O_2 molecules is 16 times that of H_2 molecules. The rms velocity of O_2 molecules at room temperature is C_{rms}. The rms velocity of H_2 molecule at the same temperature will be
(a) $16\, C_{rms}$ (b) $4\, C_{rms}$
(c) $\frac{C_{rms}}{4}$ (d) $\frac{C_{rms}}{16}$

13. The amount of heat required to increase the temperature of 1 mol of a triatomic gas (non-linear) at constant volume is *n* time the amount of heat required for 1 mol of monatomic gas. The value of *n* will be
(a) 1 (b) 1.3 (c) 2 (d) 2.5

14. At what temperature will the mean molecular energy of a perfect gas will be one-third of its value at 27°C?
(a) 10°C (b) 10^1K (c) 10^2K (d) 10^3K

15. In the gas equation $PV = RT$, V is the volume of
(a) 1 mol of gas (b) 1 g of gas
(c) gas (d) 1 litre of gas

16. The mean kinetic energy of gas molecules is zero at
(a) 0°C (b) –273°C (c) 100K (d) 100°C

17. The speed of sound in a gas is V. If the rms velocity of gas molecules is C_{rms} then the value of $\frac{V}{C_{rms}}$ will be
(a) $3/\gamma$ (b) $\gamma/3$ (c) $\sqrt{3/\gamma}$ (d) $\sqrt{\gamma/3}$

18. The temperature at which the rms speed of gas molecules becomes double its value at 0°C is
(a) 819°C (b) 760°C (c) 273°C (d) 224°C

19. One mol of a gas at NTP is suddenly expanded to three times its initial volume. If $C_V = 2R$, the ratio of initial to final pressure of gas will be
(a) 5 (b) 4 (c) 3 (d) 2

20. The internal energy of a monoatomic ideal gas is
(a) only kinetic
(b) only potential
(c) party kinetic and partly potential
(d) none

21. If the number of gas molecules in a cubical vessel is increased from N to 3N, then its pressure and total energy will become
(a) four times (b) three times
(c) double (d) half

22. Which of the following curves is not correct at constant temperature?

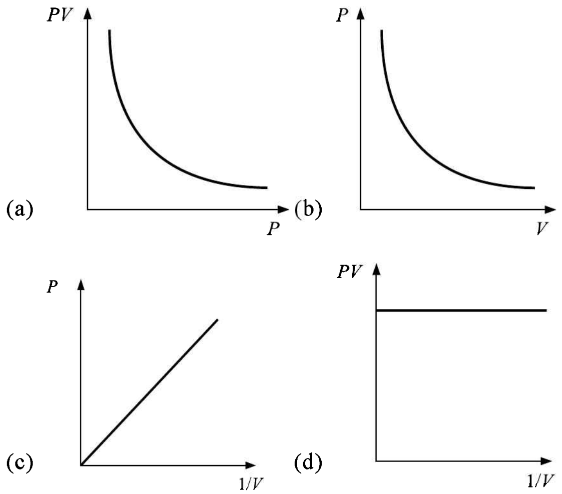


Fig. 13.13

23. The Graham's law of diffusion is
(a) $\frac{C_1}{C_2} = \frac{d_2}{d_1}$ (b) $\frac{C_1^2}{C_2^2} = \sqrt{\frac{d_2}{d_1}}$
(c) $\frac{C_1}{C_2} = \sqrt{\frac{d_2}{d_1}}$ (d) $\frac{C_1}{C_2} = \sqrt{\frac{d_1}{d_2}}$

24. The correct curve between V/T and $1/V$ for a gas at constant pressure is

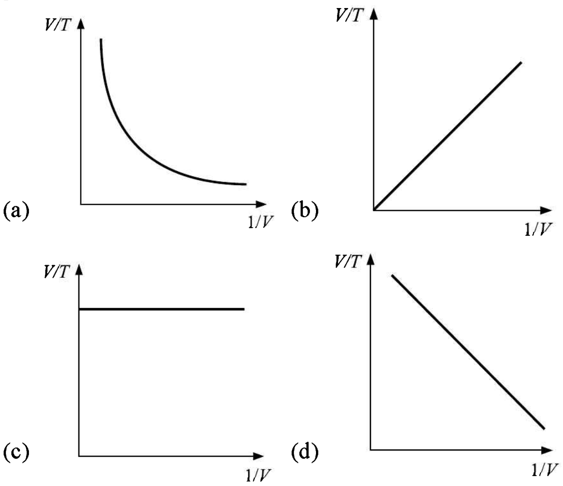


Fig. 13.14

25. If the mass of molecules of a gas in a closed vessel is halved and the speed doubled, then the ratio of initial to final pressure will be
(a) 4 : 1 (b) 1 : 4 (c) 1 : 2 (d) 2 : 1

26. If the total number of molecules in a gas is N then the number of molecules moving in negative X-direction will be
(a) N/6 (b) N/4 (c) N/3 (d) N

27. The temperature below which a gas can be liquified by increasing its pressure is known as
(a) zero (b) neutral temperature
(c) critical temperature (d) Boyle temperature

28. The expression for mean free path is
(a) $\lambda = \dfrac{kT}{\sqrt{2}\pi d^2 P}$ (b) $\lambda = \dfrac{\pi dP}{kT}$
(c) $\lambda = \dfrac{\pi d^2 P}{kT}$ (d) $\lambda = \dfrac{kT}{\pi dP}$

29. If the absolute temperature of a gas is tripled, then the rms velocity of gas molecules will become
(a) 1/3 (b) $\sqrt{3}$ times
(c) 3 times (d) 9 times

30. The correct expression for pressure exerted by a gas on wall of a container is
(a) $P = \dfrac{mn}{3l^3}\sqrt{c^2}$ (b) $P = \dfrac{mnc^2}{3l^3}$
(c) $P = \dfrac{3l^2c^2}{mn}$ (d) $P = \dfrac{mc^2}{3l^2}$

31. A gas is filled in a container at any temperature and at pressure 76 cm of Hg. If at the same temperature the mass of gas is increased by 50% then the resultant pressure will be
(a) 38 cm of Hg (b) 76 cm of Hg
(c) 114 cm of Hg (d) 152 cm of Hg

32. If the critical temperature of a gas is 100K then its Boyle temperature will be
(a) 33.3K (b) 103K
(c) 337K (d) 500K

33. The critical temperature for CO_2 is
(a) 31K (b) 31.1K
(c) 31.1°C (d) 31.3F

34. At what temperature the linear kinetic energy of a gas molecule will be equal to that of an electron accelerated through a potential difference of 10 volt?
(a) 273K (b) 19×10^3K
(c) 38.65×10^3K (d) 11.3×10^3K

35. Which of the following expression is not correct for rms velocity?
(a) $\sqrt{\dfrac{3\rho}{P}}$ (b) $\sqrt{\dfrac{3RT}{M}}$
(c) $\sqrt{\dfrac{3PV}{M}}$ (d) $\sqrt{\dfrac{2E_k}{M}}$

36. How many times is the forbidden volume (b) as compared to actual volume of gas molecules?
(a) double (b) 3 times
(c) 4 times (d) 8 times

37. At what temperature the kinetic energy of gas molecules will be double of its value at 27°C?
(a) 54°C (b) 108°C (c) 300°C (d) 327°C

38. If the pressure of a gas is increased then its mean free path becomes
(a) zero (b) less (c) more (d) ∞

39. On which of the following the kinetic theory of gases is not applicable?
(a) on free electron gas
(b) on bound electrons
(c) on water vapour
(d) on smoke particles

40. The mean molecular energy of a gas at 300K will be
(a) 2.6×10^{-20} Joule (b) 6.2×10^{-21} Joule
(c) 6.2×10^{-20} Joule (d) 6.2×10^{20} Joule

41. In outer space there are 10 molecules per cm^3 on an average and the temperature there is 3K. The average pressure of this light gas is
(a) $10^5 Nm^{-2}$ (b) $5 \times 10^{-14} Nm^{-2}$
(c) $0.4 \times 10^{-16} Nm^{-2}$ (d) $4.14 \times 10^{-16} Nm^{-2}$

42. The velocities of three molecules A, B and C of a gas are 1, $\sqrt{3}$ and $\sqrt{5}$ ms^{-1}. The value of their rms velocity will be
(a) $1.73 ms^{-1}$ (b) $2 ms^{-1}$
(c) $4 ms^{-1}$ (d) $9 ms^{-1}$

43. If the mean kinetic energy per unit volume of a gas is n times its pressure, then the value of n is
(a) 4.5 (b) 3.5 (c) 2.5 (d) 1.5

44. The volume of 0.1 mol of gas at NTP is
(a) 0.22 litre (b) 2.24 litre
(c) 1 litre (d) 22.4 litre

45. When a molecule moving with velocity u collides normally with the wall of the container, then the change in its velocity and momentum will be
(a) $-u$ and mu (b) $2u$ and $2mu$
(c) $2u$ and mu (d) u and $-mu$

46. Heat required to melt 1 g of ice is 80 cal. A man melts 60 g of ice by chewing in one minute. His power is
(a) 1.33 W (b) 0.75 W
(c) 336 W (d) 4800 W

47. Taking the unit of work as Joule and the unit of amount of heat as Kcal, the magnitude of Joule's mechanical equivalence of heat is

(a) 1 (b) 4.2×10^7
(c) 4.2 (d) 4.2×10^3

48. The temperature of 5 moles of a gas which was held at constant volume was changed from 100° to 120°C. The change in the internal energy of the gas was found to be 80 Joule, the total heat capacity of the gas at constant volume will be equal to
(a) $0.4\,JK^{-1}$ (b) $4\,JK^{-1}$
(c) $0.8\,JK^{-1}$ (d) $8\,JK^{-1}$

49. The rms speed of He gas atom is 5/7 of the rms speed of H_2 gas molecules. If the temperature of H_2 gas be 0°C, then the temperature of He will be approximately
(a) 273°C (b) 100K (c) 0°C (d) 0K

50. If 10 g of ice at –10°C is added to 40g of water at 15°C, the temperature of the mixture is
(a) 3.75°C (b) 0°C
(c) 3°C (d) –2°C

51. A cylinder contains 2 kg of air at a pressure of 10^5 Pa. If 2 kg more air is pumped into it, keeping the temperature constant, the pressure will be
(a) 10^{10} Pa (b) 2×10^5 Pa
(c) 10^5 Pa (d) 0.5×10^5 Pa

52. 10 g of steam passes over an ice block. What amount of ice will melt?
(a) 8 g (b) 18 g (c) 45 g (d) 80 g

53. When a gas filled in a closed vessel is heated through 1°C, its pressure increases by 0.4%. The initial temperature of the gas was
(a) 25°C (b) 250°C (c) 250K (d) 2500K

54. An inverted vessel (bell) lying at the bottom of a lake, 50.6 m deep has 50 cc of air trapped in it. The bell is brought to the surface of lake. The volume of the trapped air will now be
(a) 200 cc (b) 250 cc
(c) 300 cc (d) 350 cc

55. A mixture of two gases *X* and *Y* is enclosed at constant temperature. The relative molecular mass of *X*, which is diatomic, is 8 times that of *Y*, which is monatomic. The ratio v_{rms} of *Y* molecules to that of molecules of *X* is
(a) 8 (b) 4 (c) $2\sqrt{2}$ (d) 2

56. 50 g of ice at –5°C is added to 200 g of water at 40°C in a calorimeter. The water equivalent of calorimeter is 50 g of water. The temperature of the mixture is
(a) 9.58°C (b) 12.58°C
(c) 19.58°C (d) 29.58°C

57. The triple point of water is
(a) 273K (b) 0K
(c) 273.16K (d) 0°C

58. Compared to a burn due to water at 100°C, a burn due to steam at 100°C is
(a) more dangerous
(b) less dangerous
(c) equally dangerous
(d) none of these

59. The pressure of a gas kept in an isothermal container is 200 KPa. If half of the gas is removed from it, the pressure will be
(a) 800 KPa (b) 400 KPa
(c) 200 KPa (d) 100 KPa

60. According to the Boltzmann's law of equipartition of energy, the energy per degree of freedom and at a temperature *T*K is
(a) (3/2) *KT* (b) (2/3) *KT*
(c) *KT* (d) 1/2*KT*

61. One mole of a gas at a pressure 2 Pa and temperature 27°C is heated till both pressure and volume are doubled. What is the temperature of the gas?
(a) 1200K (b) 900K
(c) 600K (d) 300K

62. What is number of degrees of freedom of an ideal diatomic molecule at ordinary temperature?
(a) 7 (b) 6 (c) 5 (d) 3

63. To raise the temperature of 100 g of ice at 0°C to 10°C by a heater of 420 W the time required is
(a) 90 min (b) 90 seconds
(c) 21.2 min (d) 21.2 seconds

64. A man is climbing up a spiral type staircase. His degrees of freedom are
(a) 1 (b) 2 (c) 3 (d) more than 3

65. The law of equipartition of energy was given by
(a) Claussius (b) Maxwell
(c) Boltzmann (d) Carnot

66. The law of equipartition of energy is applicable to the system whose constituents are
(a) in random motion
(b) in orderly motion
(c) at rest
(d) moving with constant speed

67. A system consists of N particles, which have independent K relations among one another. The number of degrees of freedom of the system is given by
(a) 3NK (b) 3N/K
(c) 3N/K (d) 3N – K

68. The dimension of universal gas constant *R* are
(a) $M^2L^2T^{-2}$ (b) $ML^2T^{-2}\theta^{-1}$
(c) $M^2L^2T^{-2}\theta^{-2}$ (d) $MLT^{-2}\theta^{-2}$

69. The rms velocity of air at NTP will be ——— if density of air is $1.29\,kg/m^3$
(a) $0.485\,ms^{-1}$ (b) $0.485 \times 10^2\,ms^{-1}$
(c) $4.85 \times 10^2\,ms^{-1}$ (d) $10^4\,ms^{-1}$

70. The temperature, pressure and volume of two gases *X* and *Y* are *T*, *P* and *V* respectively. When the gases are mixed then the volume and temperature of the mixture becomes *V* and *T* respectively. The pressure and mass of the mixture will be
(a) *P* and *M* (b) *P* and 2*M*
(c) 2*P* and 2*M* (d) 2*P* and

71. The specific heat of a monoatomic gas at constant volume is $0.075\ kcal\,kg^{-1}K^{-1}$. Its atomic weight will be
(a) 10 (b) 30
(c) 40 (d) 90

72. The specific heat of ——— is negative
(a) CO_2
(b) Ne
(c) saturated vapoures
(d) none

ANSWERS

1. (b)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (d)
11. (a)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (b)
31. (c)	**32.** (c)	**33.** (c)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (d)	**38.** (b)	**39.** (b)	**40.** (b)
41. (d)	**42.** (a)	**43.** (d)	**44.** (b)	**45.** (b)	**46.** (c)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (b)
51. (b)	**52.** (d)	**53.** (c)	**54.** (c)	**55.** (c)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (d)	**60.** (d)
61. (a)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (a)	**66.** (a)	**67.** (d)	**66.** (b)	**69.** (c)	**70.** (c)
71. (c)	**72.** (c)								

Thermodynamics

14

BRIEF REVIEW OF THE CONCEPTS

First law of Thermodynamics $\Delta Q = \Delta U + \Delta W$ is called the first law of thermodynamics.

Work done by a gas = $P\Delta V$ or $W = \int_{V_1}^{V_2} PdV$

The first law denies the possibility of creating or destroying energy.

Thermal Processes In general thermal processes may be of three types: (a) reversible, (b) irreversible and (c) cyclic. A reversible process means if a process takes up the path *AB* then on reversing the conditions it comes back by *BA*. A thermal process however cannot be reversible. It could be reversible if the change is extremely small (infinitesimally small).

In irreversible process one will not reach back to *A* if the process *AB* has occurred.

In a cyclic process, if the process takes the path *AxB*, it returns via *ByA*.

Thermal processes may be cyclic or irreversible. Change in internal energy in a cyclic process is zero.

Hence $\Delta Q = \Delta W$

We can divide these processes as

(a) isobaric (b) isochoric

(c) isothermal (d) adiabatic

In isobaric process pressure remains constant and work done

$$W = P\Delta V = P(V_2 - V_1)$$

$$\therefore \quad dQ = dU + pdV$$

In isochoric process volume remains constant. Therefore $dV = 0$

Hence work done is zero.

$$\therefore \quad \Delta Q = \Delta U$$

In isothermal process the temperature remains constant. Melting and boiling are examples. Specific heat in isothermal process is ∞.

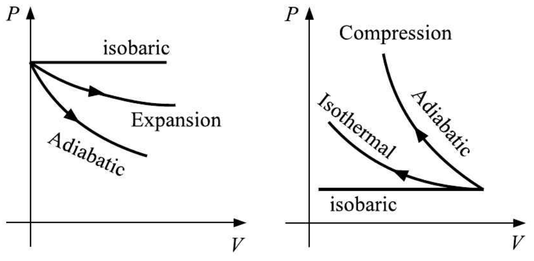
Work done,

$$W = \int pdV = n\mathrm{RT}\int_{V_1}^{V_2}\frac{\mathrm{dV}}{\mathrm{V}} = nRT\log_e\frac{V_2}{V_1}$$

$$= 2.303\ nRT\log\frac{V_2}{V_1} = 2.303\ nRT\log\frac{P_1}{P_2}$$

Isothermal elasticity = P (Bulk modulus)

In an adiabatic process heat is neither allowed to enter nor allowed to escape the system. Specific heat in an adiabatic process is zero.



(a) (b)

Fig. 14.1

Since $dQ = 0$

$\therefore \quad dU = -pdV$

In an adiabatic process

(i) PV^{γ} = constant (ii) $P^{1-\gamma}T^{\gamma}$ = constant

(iii) $TV^{\gamma-1}$ = constant

Note: during expansion

$W_{\text{isobaric}} > W_{\text{isothermal}} > W_{\text{adiabatic}}$ and during compression

$W_{\text{adiabatic}} > W_{\text{isothermal}} > W_{\text{isobaric}}$

Work done in an adiabatic process,

$$W = \frac{P_1V_1 - P_2V_2}{\gamma - 1} = \frac{nR(T_1 - T_2)}{\gamma - 1} \quad \text{where } \gamma = \frac{C_P}{C_V}$$

Adiabatic elasticity (Bulk modulus) = γP

Second law of thermodynamics The second law denies the possibility of utilization of heat out of a single body. The definitions of the second law of thermodynamics are:

(a) It is impossible to construct an engine which operating in a cycle will produce no effect other than the extraction of heat from a reservoir and the performance of an equivalent amount of work. (Kelvin Planck statement)

(b) Heat cannot flow itself from a colder to a hotter body.

(c) It is impossible to have a process in which the entropy of an isolated system is decreased.

Adiabatic → Thermally insulating

Diathermic → Thermally conducting

Heat Engine A heat engine takes a heat Q_1 from the furnace and rejects Q_2 to the heat sink and does a work $W = Q_2 - Q_1$

Thus efficiency of an engine $\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$

or $\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$

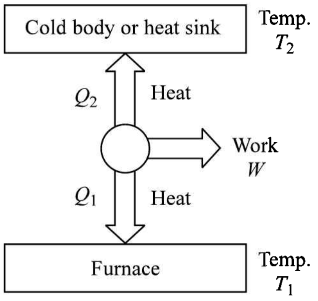


Fig. 14.2

Entropy $dS = \frac{dQ}{T}$ or $S_2 - S_1 = \int \frac{dQ}{T}$

Note that T is not differentiable. Entropy is a measure of randomness or disorder in a system.

Clausius inequality $\oint \frac{dQ}{T} \leq 0$

or $\Delta s \geq \int \frac{dQ}{T}$ or $dQ = Tds \geq dU + pdV$

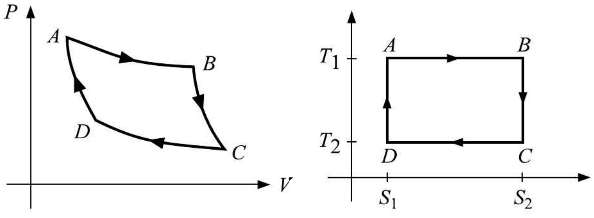
Relation between entropy and statistical weight Ω (thermodynamic property)

$$S = k \log_e \Omega$$

where k is Boltzmann's constant.

Amount of heat required to form a unit area of the liquid surface layer during the isothermal increase of its surface $H = -T\frac{d\sigma}{dT}$ where σ is surface tension.

Carnot's theorem All reversible engines operating between the same two temperatures have equal efficiency and no engine operating between the same two temperatures can have an efficiency greater than this.



(a) (b)

Fig. 14.3

According to Carnot's theorem, maximum efficiency

$$\eta = 1 - \frac{T_2}{T_1}$$

Since T_2 cannot be zero (as 0K cannot be obtained), therefore, efficiency cannot be 1.

Refrigerator or heat pump A heat engine takes heat from a hot body, converts part of it into work and rejects rest to cold body. The reverse operation is done by a refrigerator (or heat pump). It takes an amount Q_2 of heat from a cold body, an amount of work W is done on it by the surrounding and a total heat $Q_1 = Q_2 + W$ is supplied to hot body as illustrated in Fig. 14.4.

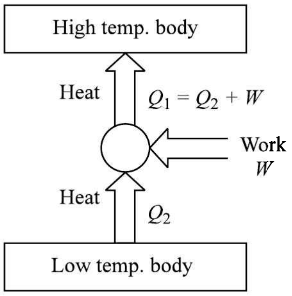


Fig. 14.4

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

$$\frac{Q_2 + W}{Q_2} = \frac{T_1}{T_2}$$

or $$W = Q_2\left(\frac{T_1}{T_2} - 1\right)$$

This leads to another statement of second law:

It is not possible to design a refrigerator which works in a cyclic process and whose only result is to transfer heat from a colder body to a hotter body. This is the Claussius statement of the second law of thermodynamics.

Coefficient of performance,

$$K = \frac{\text{heat extracted}}{\text{work done}} = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

In a perfect refrigerator K = ∞

that is, $Q_1 = Q_2$ or $W = 0$

SHORT-CUTS AND POINTS TO NOTE

(i) According to the first law of thermodynamics, total energy is conserved, that is, the first law denies the possibility of creating or destroying energy. Thus

$$\underset{\text{heat supplied}}{\Delta Q} = \underset{\text{change in internal energy}}{\Delta U} + \underset{\text{work done}}{W}$$

or $dQ = dU + PdV$

(ii) Processes may be reversible, irreversible or cyclic. Thermal processes cannot be completely reversible.

(iii) Thermal processes may be isochoric, isobaric, isothermal or adiabatic.

In an isobaric process, work done
$W = P\Delta V = P(V_2 - V_1)$; specific heat $= C_P$

In an isochoric process, work done
$W = 0$ (as $\Delta V = 0$).

Hence $dQ = dU$, specific heat C_V

In an isothermal process, work done

$W = n\text{R}T \log_e \frac{V_2}{V_1} = n\text{R}T \log_e \frac{P_1}{P_2}$,

specific heat = ∞

In an adiabatic process, work done

$W = \frac{n\text{R}(T_1 - T_2)}{\gamma - 1} = \frac{P_1V_1 - P_2V_2}{\gamma - 1}$,

specific heat = 0

PV^γ = constant, $P^{1-\gamma}T^\gamma$ = constant and $TV^{\gamma-1}$ = constant

(iv) Throttling process A fluid originally at high pressure seeps through a porous wall or needle-like narrow opening into a region of constant lower pressure work done

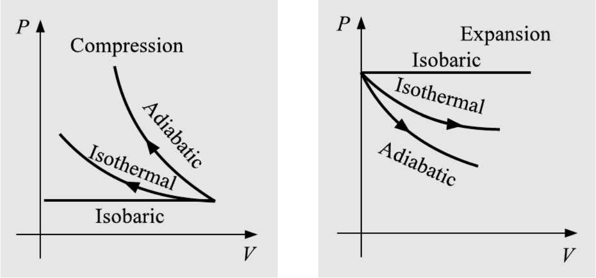
$W = P_2V_2 - P_1V_1$

Since the process is adiabatic, therefore

$\Delta U = U_2 - U_1 = -(P_2V_2 - P_1V_1)$

The sum $U + PV$ is called enthalpy. Throttling process plays an important role in refrigeration.

(v) The slope of an adiabatic process is higher than isothermal change.



(a) **(b)**

Fig. 14.5

(vi) Work done is positive if the arrow is clockwise and negative if the arrow is anticlockwise.

(vii) Second law of thermodynamics The various definitions are

(1) It is impossible to construct an engine which operating in a cycle will produce no effect other than the extraction of heat from a reservoir and the performance of an equivalent amount of work (Kelvin Planck Statement)

(2) Heat cannot flow by itself from a colder to a hotter body.

(3) It is impossible to have a process in which the entropy of an isolated system is decreased.

(4) It is not possible to design a refrigerator which works in a cyclic process and whose only result is to transfer heat from a body to a hotter body. (Claussius statement)

(viii) Entropy is a measure of randomness or disorder in a system.

$$dS = \frac{dQ}{T}$$

Note that T is not a differentiable quantity.

$S = k \log_e \omega$

$$\Delta S = k \log_e \frac{\omega_2}{\omega_1}$$

is the microscopic form of entropy. Where k is Boltzmann's constant ω represents the member of possible microscopic states in a given macroscopic state.

(ix) Thermal equilibrium If two systems have the same temperature they are said to be in thermal equilibrium.

(x) Thermodynamic equilibrium A process is said to be in thermodynamic equilibrium when it is in the state of thermal, mechanical and chemical equilibrium.

Mechanical equilibrium means $\Sigma F = 0$, $\Sigma\tau = 0$ (torque)

Chemical equilibrium means the concentration of reactants and products remains constant.

(xi) Thermodynamic variables P, V, T, and so on which form the equation of state are called thermodynamic variables.

(xii) Heat engine take up heat from a hotter body, convert partly into work and reject rest of the energy to a cold body (heat sink). Efficiency of a heat engine

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

(xiii) Carnot suggested that even an idealized engine cannot have efficiency 1. He considered a cyclic process consisting of four processes

(1) isothermal expansion
(2) adiabatic expansion
(3) isothermal compression
(4) adiabatic compression

Efficiency of Carnot engine

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$$

T_2 → temperature of heat sink (colder body)

T_1 → temperature of furnace or hot body

Since T_2 cannot be 0 K, therefore efficiency cannot be 1.

(xiv) Refrigerator or heat pump is reciprocal of heat engine

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \quad \text{or} \quad \frac{Q_2 + W}{Q_2} = \frac{T_1}{T_2}$$

or $$W = Q_2\left(\frac{T_1}{T_2} - 1\right)$$

Performance coefficient $K = \dfrac{Q_2}{W} = \dfrac{Q_2}{Q_1 - Q_2}$

(xv) In perfect refrigerator $K = \infty$, that is, $Q_1 = Q_2$

Area under $P - V$ graph is work done.

(xvi) Molar heat capacity in a polytropic process PV^k = constant $(k \neq \gamma)$

$$C = \frac{R}{\gamma - 1} - \frac{R}{k - 1} = \frac{(k - \gamma)R}{(k - 1)(\gamma - 1)}$$

(xvii) For a van der waal's gas $U = C_VT - \dfrac{a}{V_M}$ for one mole

⚠ CAUTION

(i) Not understanding the difference between C_p and C_V.

- Specific heat at constant volume C_V forms internal energy

$$\Delta U = nC_V\,\Delta T$$

When volume is constant work done

$W = PdV = 0 \quad \therefore \quad dQ = dU$

When C_p is being used $dQ = nC_p\,\Delta T$ and work done

$W = PdV = nRdT$

(ii) Not understanding whether work is positive or negative.

- When there is expansion of the gas or when the piston moves in the forward direction then work is positive. When there is compression or when the piston moves in a backward direction work done is negative. Alternatively, if the arrow is clockwise, work done by the gas is positive (+ve). If the arrow is anticlockwise, work is done on the gas, and is negative (–ve).

(iii) Not remembering the three relations of adiabatic process.

- In an adiabatic process

(1) PV^{γ} = constant or $P_1V_1^{\gamma} = P_2V_2^{\gamma}$

(2) $P^{1-\gamma}T^{\gamma}$ = constant or $P_1^{1-\gamma}T_1^{\gamma} = P_2^{1-\gamma}T_2^{\gamma}$

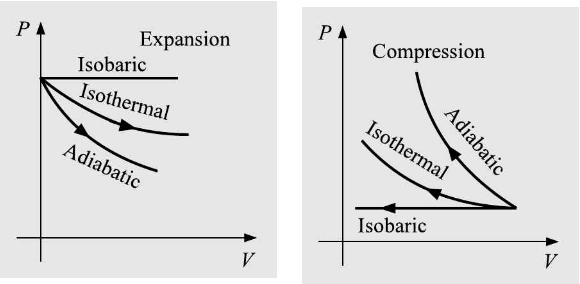
(3) $TV^{\gamma-1}$=constant or $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

(iv) Confusing when to take the process as adiabatic if not mentioned directly in the problem.

- ■ When the change is sudden or abrupt, the process is adiabatic.

(v) Confusing between adiabatic and isothermal expansion or compression.

- ■ In isothermal expansion or compression the slope is not large while in adiabatic expansion or compression the slope is large (See Fig. 14.6).



(a) (b)

Fig. 14.6

Note: During expansion more work is done in isothermal process and during compression more work is done in an adiabatic process, that is, during expansion $W_{\text{isobaric}} > W_{\text{isothermal}} > W_{\text{adiabatic}}$ and during compression $W_{\text{adiabatic}} > W_{\text{isothermal}} > W_{\text{isobaric}}$.

(vi) Not remembering the work done in various processes.

- ■ Work done in isochoric process, $W_{\text{isochoric}} = 0$

$$W_{\text{isobaric}} = P\,\Delta V = P(V_2 - V_1) = nR(\Delta T) = nR(T_2 - T_1)$$

$$W_{\text{isothermal}} = 2.303\, nRT \log_{10}\frac{V_2}{V_1} = 2.303\, nRT \log_{10}\frac{P_1}{P_2}$$

$$W_{\text{adiabatic}} = \frac{P_1V_1 - P_2V_2}{\gamma - 1} = \frac{nR(T_1 - T_2)}{\gamma - 1}$$

(vii) Considering that efficiency of an engine can be 1 (ideally or theoretically).

- ■ Efficiency of an engine cannot be 1. It is always < 1.

  According to Carnot's theorem $\eta = 1 - \frac{T_2}{T_1}$

  $\eta \to 1$ if $T_2 \to 0$. As 0K or absolute 0 cannot be achieved, therefore, $\eta \neq 1$.

(viii) Considering that total heat energy can be converted into mechanical work just like mechanical work which can be completely converted to heat.

- ■ Mechanical work can be converted to heat. But the whole of the heat cannot be converted into work.

(ix) Considering all engines give efficiency like Carnot engine.

- ■ Carnot is a theoretical idealized engine. Practically heat engines give efficiency much less than that given by Carnot engine.

(x) Confusing between first law and second law of thermodynamics.

- ■ The first law is based on conservation of energy. The second law states that no heat can flow by itself from a cold body to a hot body.

(xi) Not recalling a polytropic process.

- ■ In a polytropic process PV^k = constant and k is different from γ. Molar specific heat in polytropic process is $C = \frac{R}{\gamma - 1} - \frac{R}{k - 1}$

  However, $C_V = \frac{R}{\gamma - 1}$

(xii) Thinking that temperature may be taken in °C.

- ■ Use temperature in Kelvin (K)

Solved Problems

1. Calculate the work done by the gas in the state diagram shown.

(a) 30 J
(b) 20 J
(c) –20 J
(d) –10 J

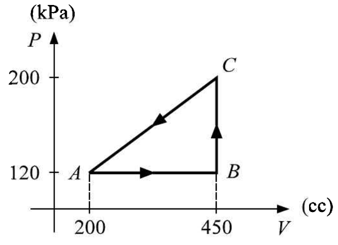


Fig. 14.7

Solution Work done

= Area under the P–V curve

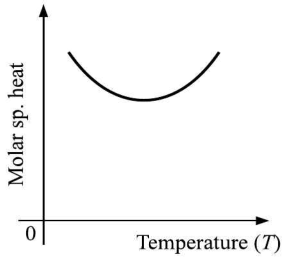
$$W = (80\text{ kPa})\,(250 \times 10^{-6}) \times \frac{1}{2} = 10\text{ J}$$

Since the arrow is anticlockwise,

∴ work done = –10 J

2. Variation of molar specific heat of a metal with temperature is best depicted by

(a)



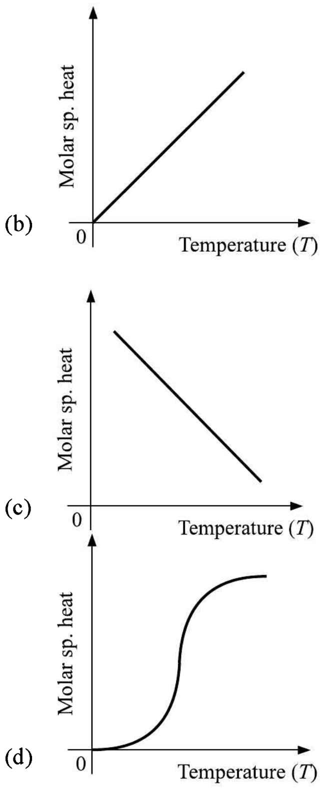


Fig. 14.8

Solution (d) For $T \to 0$, that is, at low temperature, molar specific heat $\propto T^3$ and at high temperature it becomes constant.

3. 1 g of H_2O changes from liquid to vapour phase at constant pressure at 1 atm. The volume increases from 1 cc to 1671 cc. The heat of vaporization at this pressure is 540 cal/g. The increase in internal energy of water is

(a) 2099 J (b) 3000 J
(c) 992 J (d) 2122 J

Solution (a) $W = P(dV) = 1.01 \times 10^5(1671 - 1) \times 10^{-6}$

$$= 167\text{ J}$$

$$\Delta Q = \Delta U + \Delta W$$

or $$\Delta U = \Delta Q - \Delta W$$

$$= mL - 167 = 540 \times 4.2 - 167 = 2099\text{ J}$$

4. A gas mixture consists of 2 moles of oxygen and 4 moles of Ar at temperature T. Neglecting all vibrational modes, the total internal energy of the system is

(a) 4RT (b) 15RT
(c) 9RT (d) 11RT

Solution (d) $u = n\,\frac{F}{2}RT = 2 \times \frac{5}{2}RT + 4 \times \frac{3}{2}RT = 11RT$

5. A tyre pumped to a pressure 3.375 atm at 27°C suddenly bursts. What is the final temperature ($\gamma = 1.5$)?

(a) 27°C (b) –27°C
(c) 0°C (d) –73°C

Solution (d) $T_1^\gamma P_1^{1-\gamma} = T_2^\gamma P_2^{1-\gamma}$

or $$\left(\frac{T_1}{T_2}\right)^\gamma = \left(\frac{P_1}{P_2}\right)^{\gamma-1} \text{ or } \left(\frac{300}{T_2}\right)^{3/2} = \left(\frac{3.375}{1}\right)^{3/2-1}$$

or $$T_2 = \frac{300}{(3.375)^{1/3}} = 200\text{K} = -73°\text{C}$$

6. A sound wave passing through air at NTP produces a pressure of 0.001 dyne/cm^2 during a compression. The corresponding change in temperature (given $\gamma = 1.5$ and assume gas to be ideal) is

(a) 8.97×10^{-4}K (b) 8.97×10^{-6}K
(c) 8.97×10^{-8}K (d) none of these

Solution (c) $T^\gamma P^{1-\gamma}$ = constant. Differentiating

$$\gamma T^{\gamma-1}\,dT\,P^{1-\gamma} + T^\gamma(1-\gamma)\,P^{-\gamma}\,dP = 0$$

or $$dT = \frac{(\gamma-1)T}{\gamma P}dP$$

or $$dT = \left(\frac{1.5-1}{1.5}\right)\left(\frac{273}{76 \times 13.6 \times 981} \times 0.001\right)$$

$$= 8.97 \times 10^{-8}\text{K}$$

7. When a system is taken from state 1 to 2 along the path 1a2 it absorbs 50 cal of heat and work done is 20 cal. Along the path 1b2, Q = 36 cal. What is the work done along 1b2?

(a) 56 cal (b) 66 cal
(c) 16 cal (d) 6 cal

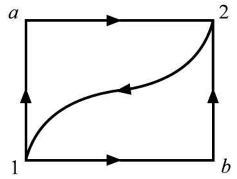


Fig. 14.9

Solution (d) $dQ = du + dW$ or $Q = (u_2 - u_1) + W$

$$W = Q_{1b2} - (u_2 - u_1) \text{ or } Q_{1a2} - W = u_2 - u_1$$

$$= 36 - 30 = 6\text{ cal} \quad \text{or } u_2 - u_1 = 50 - 20 = 30\text{ cal}$$

8. 1 g mole of an ideal gas at STP is subjected to a reversible adiabatic expansion to double its volume. Find the change in internal energy ($\gamma = 1.4$).

(a) 1169.5 J (b) 769.5 J
(c) 1369.5 J (d) 969.5 J

Solution (c) Use $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

or $$T_2 = \frac{T_1V_1^{\gamma-1}}{V_2^{\gamma-1}} = \frac{273}{(2)^{0.4}} = 207\text{K}$$

Change in internal energy

$$\Delta u = \frac{R}{(\gamma-1)}(T_1 - T_2)$$

$$= \frac{8.31(273-207)}{1.4-1} = 1369.5\text{ J}$$

9. A gram mole of a gas at 127°C expands isothermally until its volume is doubled. Find the amount of work done.

(a) 238 cal (b) 548 cal
(c) 548 J (d) 238 J

Solution (b) $W = 2.303\text{RT}\log(V_2/V_1)$

$$= 2.303 \times 8.311 \times 400 \times \log 2$$

$$= 2310.1\text{ J} = 548\text{ cal.}$$

10. Find the work required to compress adiabatically 1 g of air initially at NTP to half its volume. Density of air at NTP = 0.001129 gcm^{-3} and $\frac{C_P}{C_V} = 1.4$.

(a) 62.64 J (b) 32.64 J
(c) –32.64 J (d) –62.64 J

Solution (d) $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

or $$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = 273\ (2)^{0.4} = 360\text{K}$$

$$V = \frac{1\text{g}}{0.00129}\text{ cc}$$

We find R for 1g of air using $PV = RT$

$$R = \frac{76\times13.6\times981}{273\times0.00129} = 2.88\times10^6$$

$$W = \frac{R}{\gamma-1}(T_1 - T_2) = \frac{2.88\times10^6}{0.4}(273 - 360)$$

$= -62.64\times10^7$ erg.

$= -62.64$ J

11. A Carnot engine has the same efficiency between (i) 100K and 500K and (ii) T and 900K. Find T.

(a) 200K
(b) 190K
(c) 180K
(d) none of these

Solution (c) $\eta = 1 - \frac{T_2}{T_1}$ or $\frac{T_2}{T_1} = \frac{T_2'}{T_1'}$ or $\frac{100}{500} = \frac{T}{900}$

$T = 180\text{K}$

12. A reversible engine takes in heat from a reservoir of heat at 527°C and gives it to the sink at 127°C. How many calorie/s it shall take from the reservoir to do a work of 750W.

(a) $257\,\text{cal s}^{-1}$
(b) $357\,\text{cal s}^{-1}$
(c) $1500\,\text{cal s}^{-1}$
(d) none of these

Solution $\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{400}{800} = \frac{1}{2} = \frac{W}{Q_1}$

or $$Q_1 = 2W = \frac{2\times750}{4.2} = 357.1\,\text{cal s}^{-1}.$$

13. A Carnot engine has efficiency 40% (heat sink 27°C). To increase efficiency by 10%, the temperature be increased by

(a) 15.7K
(b) 25.7K
(c) 50.7K
(d) 35.7K

Solution $\eta = 40\% = \frac{2}{5}$ $\quad \eta = 1 - \frac{T_2}{T_1}$ or $\frac{T_2}{T_1} = \frac{3}{5}$

$\therefore \quad T_1 = 300\times\frac{5}{3} = 500\text{K}$

new efficiency $= 40 + 40\times\frac{10}{100} = 44\%$; $0.44 = 1 - \frac{300}{T_1'}$

or $T_1' = 535.7\text{K}$

$\therefore$ Temperature of heat source be raised by 35.7K

14. Two engines are working in such a way that sink of one is source of the other. Their efficiencies are equal. Find the temperature of the sink of first if its source temperature is 927°C and temperature of sink of the second is 27°C.

(a) 327K
(b) 327°C
(c) 600°C
(d) none of these

Solution (b) $\eta = 1 - T_2/T_1 = 1 - T_3/T_2$ or $T_2^2 = T_1T_3$

or $$T_2 = \sqrt{1200\times300} = 600\text{K} = 327°\text{C}$$

15. An ideal gas expands according to the law $PV^{3/2}$ = constant. We conclude

(a) The adiabatic exponent of the gas $K = 1.5$
(b) The molar heat capacity $C = C_V - 2R$
(c) Temperature increases during the process
(d) Such a gas is not feasible

Solution (b) Molar heat capacity

$$C = C_V + \frac{R}{1-K} = C_V + \frac{R}{1-3/2} = C_V - 2R$$

16. The ratio of work done by an ideal diatomic gas to the heat supplied by the gas in an isobaric process is

(a) 5/7
(b) 3/7
(c) 2/7
(d) 5/3

Solution (c) $\Delta U = nC_V\Delta T = n5RT/2$

$\Delta Q = nC_P\Delta T = n7R\Delta T/2$

$$W = \Delta Q - \Delta U = \frac{n7}{2}R\Delta T - \frac{n5}{2}R\Delta T = nR\Delta T$$

$W/Q = 2/7$

17. A monoatomic gas is supplied heat Q very slowly keeping the pressure constant. The work done by the gas is

(a) $2Q/5$
(b) $3Q/5$
(c) $Q/5$
(d) $2Q/3$

Solution (d) For monoatomic gas

$\Delta U/Q = 1/3$ or $\Delta U = Q/3$

From the first law of thermodynamics

$Q = \Delta U + W$

$\therefore \quad W = 2Q/3$

18. Which of the following parameters does not characterize the thermodynamic state of matter?

(a) work
(b) pressure
(c) temperature
(d) volume

Solution (a) P, V and T are thermodynamic variables.

19. A Carnot engine takes 3×10^6cal of heat from a reservoir at 627°C, and gives it to a sink at 27°C. The work done by the engine is

(a) 8.4×10^6J
(b) 16.8×10^6J
(c) zero
(d) 4.2×10^6J

Solution (a) $\eta = 1 - T_2/T_1 = 1 - 300/900 = W/Q_1$

or $\quad W = 2Q/3 = 2\times10^6\text{cal} = 8.4\times10^6\text{J}$

20. An ideal gas heat engine operated between 227°C to 127°C in a Carnot cycle. It absorbs 6Kcal at the higher temperature. The amount of heat (in kcal) converted to work is equal to

(a) 1.2
(b) 4.8
(c) 3.5
(d) 1.6

Solution (a) $\eta = 1 - T_2/T_1 = 1 - 400/500 = Q/Q_1$

or $\quad W = Q_1/5 = 1.2\text{Kcal}$

21. The efficiency of a Carnot engine operating between reservoirs maintained at 27°C and –123°C is

(a) 0.75
(b) 0.4
(c) 0.25
(d) 0.5

Solution (d) $\eta = 1 - T_2/T_1 = 1 - 150/300 = 1/2$

22. Calculate the change in entropy when 1g of ice at 0°C is heated to form water at 40°C

(a) 0.28 cal/°C
(b) 1.411 cal/°C
(c) 0.41 cal/°C
(d) none of these

Solution (c) $\Delta S = \underset{\text{To melt ice}}{\Delta S_1} + \underset{\text{To rise the temperature}}{\Delta S_2}$

$$= \frac{mL}{T} + mC\int_{T_1}^{T_2}\frac{dT}{T}$$

$$= \frac{1\times 80}{273} + 1 \times 1 \times 2.303 \log \frac{313}{273}$$

$$= 0.28 + 0.1366 = 0.42 \text{ cal°C}^{-1}$$

23. Calculate the change in entropy of n moles of a perfect gas when its temperature changes from T_1 to T_2 while its volume changes from V_1 to V_2

Solution $dQ = TdS = du + PdV$

or $dS = \frac{nC_V dT}{T} + \frac{PdV}{T}$

Since $PV = nRT$, therefore,

$$\frac{P}{T} = \frac{n\text{R}}{V}$$

Thus $dS = nC_V \frac{dT}{T} + nR\frac{dV}{V}$

Integrating $S_2 - S_1 = nC_V \log_e \frac{T_2}{T_1} + n\text{R} \log_e \frac{V_2}{V_1}$

Problems for Practice

1. The amount of work done by the gas system in increasing the volume of 10 mols of an ideal gas from one litre to 20 litre at 0°C will be

(a) zero (b) 3.49 Joule
(c) 3.49×10^4 Joule (d) 6.79×10^4 Joule

2. The minimum number of thermodynamic parameters required to specify the state of gas system is

(a) 1 (b) 2 (c) 3 (d) ∞

3. If C_P and C_V are the molar specific heats of a gas at constant pressure and volume respectively then the ratio of adiabatic and isothermal modulii of elasticity will be

(a) $\frac{C_P - C_V}{C_P}$ (b) $C_P\,C_V$
(c) $\frac{C_V}{C_P}$ (d) $\frac{C_P}{C_V}$

4. The internal energy of a compressed real gas, as compared to that of the normal gas at the same temperature, is

(a) less
(b) more
(c) sometimes less, sometimes more
(d) none of these

5. A system is given 400 calories of heat and 1,000 Joule of work is done by the system, then the change in internal energy of the system will be

(a) –860 Joule (b) 680 erg
(c) 680 Joule (d) 860 Joule

6. In a certain process 500 calories of heat is given to a system and the system does 100 Joule of work. The increase in internal energy of the system is

(a) 40 calorie (b) 82 calorie
(c) 1993 Joule (d) 2193 Joule

7. 11 g of carbon dioxide is heated at constant pressure from 27°C to 227°C. The amount of heat transferred to carbon dioxide will be

(a) 2200 calorie (b) 350 calorie
(c) 220 calorie (d) 110 calorie

8. The specific heat of a gas at constant pressure as compared to that at constant volume is

(a) less (b) equal (c) more (d) constant

9. An air bubble of volume 15 cm³ is formed at a depth of 50 m in a lake. If the temperature of the bubble while rising remains constant then the volume of bubble at the surface will be ($g = 10\,\text{ms}^{-2}$ and atmospheric pressure $= 1.0 \times 10^5$ Pa)

(a) 100 cm³ (b) 90 cm³
(c) 80 cm² (d) 40 cm²

10. The ratio of the slopes of adiabatic and isothermal curves is

(a) γ^{-2} (b) $1/\gamma$ (c) γ^3 (d) γ

11. Equal volumes of monoatomic and diatomic gases of same initial temperature and pressure are mixed. The ratio of the specific heats of the mixture (C_p/C_V) will be

(a) 1.53 (b) 1.52 (c) 1.5 (d) 1

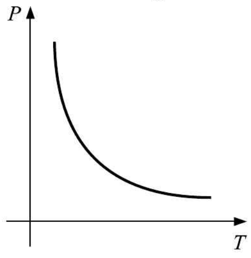
12. For a thermodynamic process $\delta Q = -50$ calorie and $W = -20$ calorie. If the initial internal energy is –30 calorie then, final internal energy will be

(a) –100 calorie (b) – 60 calorie
(c) 100 calorie (d) 191.20 calorie

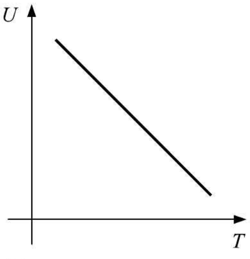
13. The change in internal energy of two mols of a gas during adiabatic expansion is found to be –100 Joule. The work done during the process is

(a) –100 Joule (b) 0
(c) 100 Joule (d) 200 Joule

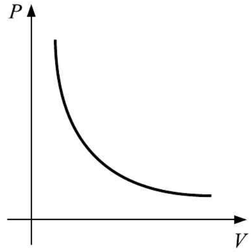
14. Out of the following, the indicator diagram is



(a)



(b)



(c)

(d)

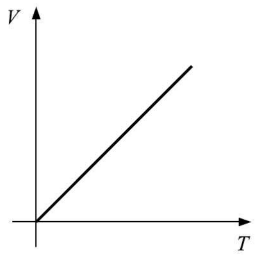


Fig. 14.10

15. The amount of heat required to raise the temperature of 100 g water from 20°C to 40°C will be
(a) zero (b) 100 calorie
(c) 2000 calorie (d) 4000 calorie

16. A liquid boils at such a temperature at which the saturated vapour pressure, as compared to atmospheric pressure, is
(a) one-third (b) equal
(c) half (d) double

17. The initial pressure of a gas is P. It is kept in an insulated container and suddenly its volume is reduced to one-third. Its final pressure will be
(a) $-3^{\gamma}P$ (b) $P/(3)^{\gamma}$ (c) $P/3$ (d) $3P$

18. The work done in Fig. 14.11 is

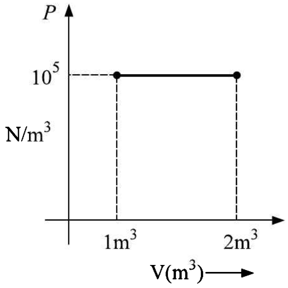


Fig. 14.11

(a) 3×10^5 J (b) 2×10^5 J
(c) 10^5 J (d) zero

19. 1 g of ice at 0°C is converted to steam at 100°C. The amount of heat required will be
(a) 12000 calorie (b) 756 calorie
(c) 716 calorie (d) 430 calorie

20. The heat capacity of a material depends upon
(a) density of matter
(b) specific heat of matter
(c) temperature of matter
(d) structure of matter

21. The isothermal bulk modulus of elasticity of a gas is $1.5 \times 10^5 \text{Nm}^{-2}$. Its adiabatic bulk modulus of elasticity will be (if $\gamma = 1.4$)
(a) $3 \times 10^5 \text{Nm}^{-2}$ (b) $2.1 \times 10^5 \text{Nm}^{-2}$
(c) $1.5 \times 10^5 \text{Nm}^{-2}$ (d) ∞

22. In changing the state of a system from state A to state B adiabatically the work done on the system is 322 Joule. If 100 calories of heat are given to the system in bringing it from state A to state B, then the work done on the system in this process will be
(a) 15.9 Joule (b) 38.2 Joule
(c) 98 Joule (d) 15.9 calorie

23. The indicator diagrams representing maximum and minimum amounts of work done are respectively
(a) (a) and (b) (b) (b) and (c)
(c) (b) and (d) (d) (c) and (d)

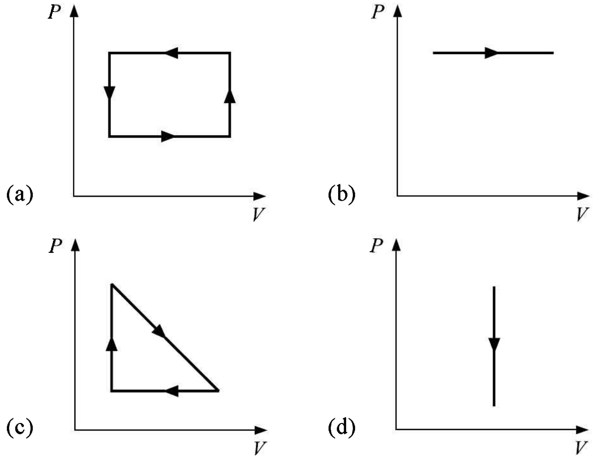


Fig. 14.12

24. Two samples of a gas A and B, initially at same temperature and pressure, are compressed to half their initial volume, A isothermally and B adiabatically. The final pressure in the two cases is related as
(a) $A = B$ (b) $A > B$
(c) $A < B$ (d) $A^2 = B$

25. A piece of ice at 0°C is dropped into water at 0°C. Then ice will
(a) melt
(b) be converted to water
(c) not melt
(d) partially melt

26. Four curves A, B, C and D are drawn for given mass a gas (Fig. 14.13). The curves which represent adiabatic and isothermal expansion are respectively

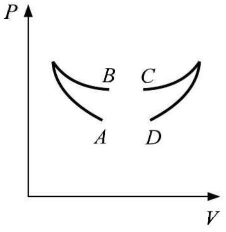


Fig. 14.13

(a) A and B
(b) C and D
(c) B and A
(d) D and C

27. How much work can be done by 250 calories of heat?
(a) zero (b) 1045 erg
(c) 1045 watt (d) 1050 Joule

28. In the gas equation $PV = RT$, V represents the volume of
(a) 1 mol of gas
(b) 1 g of gas
(c) 1 litre of gas
(d) any mass of gas

29. If, in defining the specific heat, temperature is represented in °F instead of °C then the value of specific heat will
(a) be converted to heat capacity

(b) remain unchanged
(c) decrease
(d) increase

30. The specific heat of an ideal gas varies as
(a) T^3 (b) T^2 (c) T^1 (d) T^0

31. When an ideal diatomic gas is heated at constant pressure then what fraction of heat given is used to increase internal energy of gas?
(a) 2/5 (b) 3/5 (c) 3/7 (d) 5/7

32. When the temperature of a gas in a vessel is increased by 1°C then its pressure is increased by 0.5%. The initial temperature is
(a) 100K (b) 200K (c) 273K (d) 300K

33. The internal energy of air in a room of volume $50\,m^3$ at atmospheric pressure will be
(a) 2.5×10^7 erg (b) 2.5×10^7 Joule
(c) 5.25×10^7 Joule (d) 1.25×10^7 Joule

34. One mol of helium is heated at 0°C and constant pressure. How much heat is required to increase its volume threefold?
(a) 2730 calorie
(b) 273 calorie
(c) 27.30 calorie
(d) 2.730 calorie

35. The pressure and volume of a gas are P and V respectively. If it is compressed suddenly to 1/32 of its initial volume then its final pressure will be
(a) $P/128$ (b) $P/32$ (c) $128P$ (d) $32P$

36. The net amount of work done in the following indicator diagram is

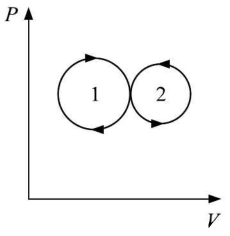


Fig. 14.14

(a) zero
(b) positive
(c) negative
(d) infinite

37. The volume of a gas is reduced to 1/4 of its initial volume adiabatically at 27°C. The final temperature of the gas (if γ = 1.4) will be
(a) $27 \times (4)^{0.4}$K (b) $300 \times (1/4)^{0.4}$K
(c) $100 \times (4)^{0.4}$K (d) $300 \times (4)^{0.4}$K

38. The concept of temperature is related to
(a) zeroeth law of thermodynamics
(b) first law of thermodynamics
(c) second law of thermodynamics
(d) third law of thermodynamics

39. When a liquid is heated, retaining its liquid state, then its molecules gain
(a) kinetic energy
(b) potential energy
(c) heat energy
(d) both kinetic and potential energy

40. A system absorbs 10^3 calories of heat and the system does 1677.3 Joule work. The internal energy of the system increases by 2515 Joule. The value of J is
(a) 4.19 Joule/cal (b) 4.18 cal/Joule
(c) 42 Joule/cal (d) 420 Joule/cal

41. The relation between P and T for monoatomic gas during adiabatic process is $P \propto T^c$. The value of c is
(a) 3/5 (b) 2/5 (c) 5/3 (d) 5/2

42. A player gets 400 kilocalories energy daily from the food. His power will be
(a) zero (b) 1.93 watt
(c) 19.3 watt (d) 193.5 watt

43. The thermodynamic scale of temperature was given by
(a) Dewar (b) Fahrenheit
(c) Kelvin (d) Carnot

44. $1\,m^3$ of a gas is compressed suddenly at atmospheric pressure and temperature 27°C such that its temperature becomes 627°C. The final pressure of the gas (if γ = 1.5) will be
(a) $2.7 \times 10^5 Nm^{-2}$ (b) $7.2 \times 10^5 Nm^{-2}$
(c) $27 \times 10^5 Nm^{-2}$ (d) $27 \times 10^6 Nm^{-2}$

45. The volume of $1\,m^3$ of gas is doubled at atmospheric pressure. The work done at constant pressure will be
(a) zero (b) 10^5 calorie
(c) 10^5 Joule (d) 10^5 erg

46. If the volume of a gas is decreased by 10% during isothermal process then its pressure will
(a) decrease by 10%
(b) increase by 10%
(c) decrease by 11.11%
(d) increase by 11.11%

47. At the boiling of water the saturated vapour pressure will be (in mm of Hg)
(a) 750 (b) 760 (c) 850 (d) 860

48. The ratio of the latent heat of steam to latent heat of ice is
(a) 4/9 (b) 9/4 (c) 4/27 (d) 27/4

49. The maximum efficiency of an engine operating between the temperature 400°C and 60°C is
(a) 55% (b) 75%
(c) 95% (d) none of these

50. A Carnot engine works between ice point and steam point. Its efficiency will be
(a) 85.42% (b) 71.23%
(c) 53.36% (d) 26.81%

51. If the temperature of the sink is absolute zero, the efficiency of the heat engine should be
(a) 100% (b) 50% (c) zero (d) none of these

52. When the temperature difference between the source and the sink increases, the efficiency of the heat engine will
(a) increase
(b) decrease
(c) is not affected
(d) may increase or decrease depending upon the nature of the working substance

53. A Carnot engine can be 100% efficient if its sink is at
(a) 0K (b) 0°C (c) 0°F (d) 273K

54. Which of the following is the best container for gas during adiabatic process?
(a) wood vessel (b) thermos flask
(c) copper vessel (d) glass vessel

55. In which of the following process the system always returns to the original thermodynamic state?
(a) isobaric (b) cyclic
(c) isothermal (d) adiabatic

56. A Carnot engine has an efficiency of 50% when its sink is at a temperature of 27°C. The temperature of the source is
(a) 300°C (b) 327°C (c) 373°C (d) 273°C

57. Figure 14.15 shows four indicator diagrams. In which case the work done is maximum?’

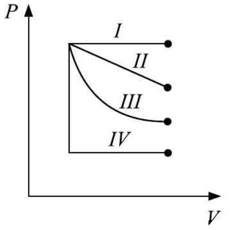


Fig. 14.15

(a) IV (b) II
(c) III (d) I

58. One mole of a monoatomic gas and one mole of a diatomic gas are mixed together. What is the molar specific heat at constant volume for the mixture?
(a) $5/2R$ (b) $2R$ (c) $3/2R$ (d) $3R$

59. What is the value of dp/p for adiabatic expansion of the gas?
(a) $\gamma dV/V$ (b) $-dV/V$
(c) dV/V (d) $-\gamma dV/V$

60. Which of the following has higher efficiency? An engine working between the temperatures
(a) 40K and 20K (b) 60K and 40K
(c) 80K and 60K (d) 100K and 80K

61. The temperature of the source of a Carnot heat engine is 0°C and that of sink is –39°C. The efficiency of the heat engine is
(a) 39% (b) 14.3% (c) zero (d) none of these

62. Work done during isothermal expansion depends on change in
(a) volume (b) pressure
(c) both (a) and (b) (d) none of these

63. For an engine operating between the temperatures t_1°C and t_2°C, the efficiency will be

(a) $\frac{t_1 - t_2}{t_1}$ (b) $\frac{t_1 - t_2}{t_1 + 273}$

(c) $\frac{t_2 + 273}{t_1 + 273}$ (d) $\frac{t_2}{t_1}$

64. For 100% efficiency of a Carnot engine the temperature of the sink should be
(a) 273°C (b) 0°C
(c) –273°C (d) none of these

65. When 1 mole of a monoatomic gas expands at constant pressure the ratio of the heat supplied that increases the internal energy of the gas and that used in expansion is
(a) 2/3 (b) 3/2 (c) 0 (d) ∞

66. The efficiency of the heat engine working between 327°C and 27°C is to be increased by 10%. The temperature of the source should be increased by
(a) 52°C (b) 67°C (c) 37°C (d) 77°C

67. A Carnot engine operates with a source at 500K and sink at 375K. If the engine consumes 600K cal of heat in one cycle, the heat rejected to the sink per cycle is
(a) 550Kcal (b) 450Kcal
(c) 350Kcal (d) 250Kcal

68. The change in which of the following solely determines the work done by a gas during adiabatic process?
(a) temperature (b) pressure
(c) volume (d) none of these

69. A Carnot engine, whose source is at 400K, takes 200cal of heat and rejects 150cal to the sink. What is temperature of the sink?
(a) 300K (b) 400K (c) 800K (d) none of these

70. A gas at pressure $6 \times 10^5 Nm^{-2}$ and volume $1 m^3$ expands to $3 m^3$ and its pressure falls to $4 \times 10^5 Nm^{-2}$. Given that the indicator diagram is a straight line, the work done on the system is
(a) 12×10^5 J (b) 6×10^5 J
(c) 4×10^5 J (d) 3×10^5 J

71. During an adiabatic expansion of 5moles of gas, the internal energy decreases by 75J. The work done during the process is
(a) –75J (b) zero (c) 15J (d) 75J

72. A monoatomic gas expands isobarically. The percentage of heat supplied that increases the thermal energy and that involved in doing work for expansion is
(a) 40 : 60 (b) 60 : 40
(c) 50 : 50 (d) none of these

73. How many dead centres are there in one cycle of steam engine?
(a) 4 (b) 3 (c) 2 (d) 1

74. For adiabatic expansion of a monoatomic perfect gas, the volume increases by 2.4%. What is the percentage decrease in pressure?
(a) 2.4% (b) 4.0% (c) 4.8% (d) 7.1%

75. Figure 14.16 represents two processes *a* and *b* for a given sample of gas. Let ΔQ_1 and ΔQ_2 be the heat absorbed by the systems in the two cases respectively. Which of the following relation is correct?

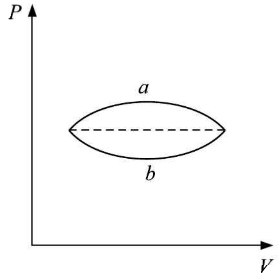
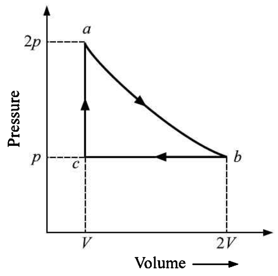


Fig. 14.16

(a) $\Delta Q_1 = \Delta Q_2$ (b) $\Delta Q_1 > \Delta Q_2$
(c) $\Delta Q_1 \leq \Delta Q_2$ (d) $\Delta Q_1 < \Delta Q_2$

76. A cylinder contains helium at 2.5 atmosphere pressure. Another identical cylinder contains argon at 1.5 atmosphere pressure at the same temperature. If both the gases are filled in any one of the cylinders, the pressure of the mixture will be
(a) 1.5 atm (b) 2.5 atm
(c) 4 atm (d) none of these

77. Figure 14.17 shows a cyclic process *abca* for one mole of an ideal gas. If *ab* is isothermal process, then which of the following is the *P–T* diagram for the cyclic process?



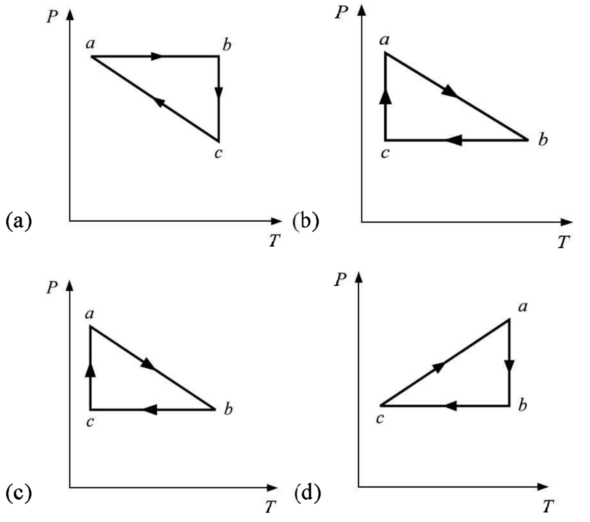


Fig. 14.17

78. A gas is enclosed in a vessel of volume 1000 cc at a pressure of 72.6 cm of Hg. It is being evacuated with the help of a piston pump, which expels 10% gas in each stroke. The pressure after the second stroke will be nearest to

(a) 60.0 cm (b) 55 cm
(c) 66 cm (d) 50 cm

79. An ideal gas heat engine operates in Carnot cycle between 272°C and 127°C. It absorbs 6.0×10^4 cal at the higher temperature. The amount of heat converted into work is equal to

(a) 1.2×10^4 cal
(b) 1.6×10^4 cal
(c) 3.5×10^4 cal
(d) 4.8×10^4 cal

80. In free expansion of a gas the internal energy of the system

(a) increases (b) decreases
(c) is unchanged (d) changes

81. A Carnot engine whose sink is at a temperature of 300 K has an efficiency of 40%. By how much should the temperature of the source be increased so as to increase the efficiency to 60%?

(a) 250 K (b) 275 K
(c) 325 K (d) 380 K

82. During an adiabatic expansion of 2 moles of a gas, the change in internal energy was found to be equal to –200 J. The work done during the process will be equal to

(a) –100 Joule (b) zero
(c) 100 Joule (d) 200 Joule

83. Which of the following represents an isothermal expansion?

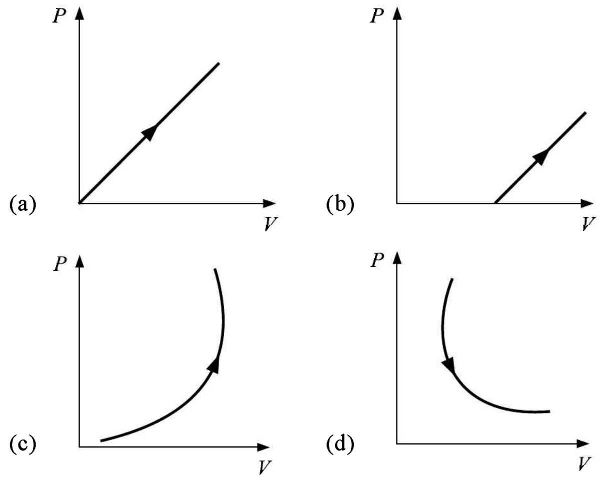


Fig. 14.18

84. Two perfect gases having masses m_1 and m_2 at temperatures T_1 and T_2 are mixed without any loss of internal kinetic energy of the molecules. The molecular weights of the gases are M_1 and M_2. What is the final temperature of the mixture?

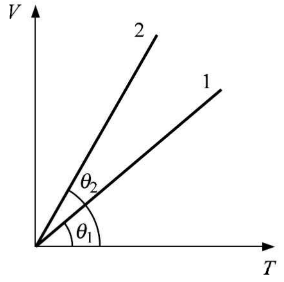


Fig. 14.19

(a) $\dfrac{\dfrac{M_1}{m_1}T_1 + \dfrac{M_2}{m_2}T_2}{\dfrac{M_1}{m_1} + \dfrac{M_2}{m_2}}$ (b) $\dfrac{\dfrac{m_1}{M_1}T_1 + \dfrac{m_2}{M_2}T_2}{\dfrac{m_1}{M_1} + \dfrac{m_2}{M_2}}$

(c) $\dfrac{M_1T_1 + M_2T_2}{M_1 + M_2}$ (d) $\dfrac{m_1T_1 + m_2T_2}{m_1 + m_2}$

85. In the following V–T diagram, what is the relation between P_1 and P_2?

(a) $P_2 = P_1$ (b) $P_2 > P_1$
(c) $P_2 < P_1$ (d) cannot be predicted

ANSWERS

1. (d)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (d)
11. (c)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (b)
21. (b)	**22.** (c)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (a)	**27.** (d)	**28.** (a)	**29.** (c)	**30.** (d)
31. (d)	**32.** (b)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (a)
41. (d)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (b)	**47.** (b)	**48.** (d)	**49.** (d)	**50.** (d)
51. (a)	**52.** (a)	**53.** (a)	**54.** (b)	**55.** (b)	**56.** (b)	**57.** (d)	**58.** (b)	**59.** (d)	**60.** (a)
61. (b)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (b)	**66.** (b)	**67.** (b)	**68.** (a)	**69.** (a)	**70.** (c)
71. (d)	**72.** (b)	**73.** (c)	**74.** (b)	**75.** (b)	**76.** (c)	**77.** (d)	**78.** (a)	**79.** (b)	**80.** (c)
81. (a)	**82.** (d)	**83.** (d)	**84.** (b)	**85.** (c)					

Heat Transfer Processes 15

BRIEF REVIEW OF THE CONCEPTS

Heat can be transferred from one place to another by three different methods: conduction, convection and radiation. Conduction usually occurs in solids, convection in liquids and gases. No medium is required for radiation.

Thermal conduction If A is area of cross-section of a conductor, l its length, K thermal conductivity, T_1 and T_2 temperatures at two ends, then rate of transfer of heat $\frac{dQ}{dt}$ or thermal current is given by

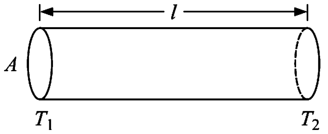


Fig. 15.1

$$\frac{dQ}{dt} = \frac{KA(T_2 - T_1)}{l} = \frac{-KA\,dT}{dx}$$

Temperature gradient $\frac{dT}{dx}$ is negative in the direction of heat flow. Comparing it with Ohm's law in electricity $i = V/R$

$$i_{\text{thermal}} = \frac{dQ}{dt},\ V_{\text{thermal}} = T_1 - T_2 \text{ and } R_{\text{thermal}} = \frac{l}{KA}$$

Laws of resistances in series and parallel as in electricity are valid in thermal resistances also. It is believed that current carriers (free electrons) and heat carriers are same because all electrical conductors are also thermal conductors. In general metals are better thermal conductors than liquids and gases as metals have large number of free electrons.

Thermometric Conductivity (D) It is the ratio of thermal conductivity to thermal capacity per unit volume. Thus **thermometric conductivity** or **diffusivity** is

$$D = \frac{K}{\rho C} \quad \text{where } K \to \text{thermal conductivity}$$
$$\rho \to \text{density and } C \to \text{specific heat}$$

Thermal Conductivity K of gases $K = \frac{1}{3} v_{av} \lambda \rho\, C_V = D\rho C_V$ where $D = \frac{1}{3} v_{av}\lambda$ is diffusion coefficient and $\lambda = \frac{1}{\sqrt{2}\pi d^2 n}$ is mean free path.

$$\begin{cases} d \to \text{effective diameter of a molecule} \\ n \to \text{number of molecules / volume.} \end{cases}$$

Wiedemann–Franz law Wiedemann and Franz have shown that at a given temperature T, the ratio of thermal conducitivity (K) to electrical conductivity (σ) is constant. That is,

$$K/\sigma T = \text{constant}$$

Ingen's–Housz's experiment Ingen Housz showed that if the number of identical rods of different metals are coated with wax and one of their ends is put in boiling water, then in steady state the square of length of the bar over which wax melts is directly proportional to the thermal conductivity of the metal. That is,

$$\frac{K}{L^2} = \text{constant}$$

Thermal resistances in series

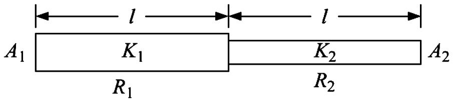


Fig. 15.2

$$R = R_1 + R_2$$

$$\frac{l_1 + l_2}{K_{\text{eff}} A_{\text{eff}}} = \frac{l_1}{K_1 A_1} + \frac{l_2}{K_2 A_2}$$

if $A_1 = A_2$ and $l_1 = l_2$

then $$\frac{2}{K_{\text{eff}}} = \frac{1}{K_1} + \frac{1}{K_2}$$

Thermal resistances in parallel

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

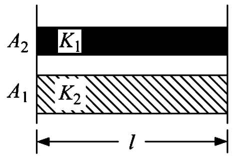


Fig. 15.3

$$\frac{K_{\text{eff}} A_{\text{eff}}}{l} = \frac{K_1 A_1}{l} + \frac{K_2 A_2}{l}$$

or $$K_{\text{eff}} A_{\text{eff}} = K_1 A_1 + K_2 A_2$$

and $$K_{\text{eff}} = \frac{K_1 + K_2}{2}$$

If the area of cross-section is equal then

$$A_{\text{eff}} = A_1 + A_2 = 2A$$

Convection It is the process in which heat is transferred from one place to the other by the actual movement of heated substance (usually fluid). Convection requires medium.

Lapse rate (the temperature falls by 6°C/km as we move up) is an example of natural convection. Blood circulation, which helps in maintaining the temperature of the body, is an examaple of forced convection.

Radiation The process by which heat is transferred directly from one body to another, without requiring a medium is called radiation. It is an electromagnetic radiation of wavelength 1 mm to 10^{-7} m. Velocity is equal to the speed of light. Radiation can be detected and measured by thermocouple, thermopile or radiometer, bolometer, and pyrometer.

Black body A black body is capable of emitting or absorbing radiation of all possible wavelengths. Initially it was thought that only a body which absorbs all possible radiation is a black body. Later on emission characteristics were also included.

The blackest body on the earth is lamp black. It is 98% black.

Prevost's black body or Fery Platinum black body is shown in Fig. 15.4

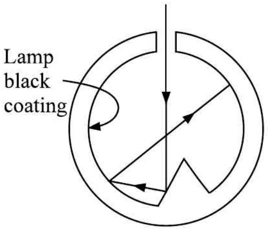


Fig. 15.4

Absorptive power The absorptive power of a substance is defined as the ratio of the radiant energy absorbed by it in a given time to the total radiant energy incident on it in the same time.

Spectral absorptive power

$$a = \int_0^{\infty} a_\lambda d\lambda \qquad \text{unit}(\text{Wm}^{-2})$$

For a perfectly black body absorptive power is 1.

Emissive power (*e*) Radiant energy emitted per unit area of the surface. However, if we consider emissive power of a surface for a particular wavelength, it is called spectral emissive power. Spectral emissive power

$$e = \int_0^{\infty} e_\lambda \, d\lambda \qquad \text{units} = \text{Wm}^{-2}$$

It is maximum for a black body and minimum for a white shining and smooth body.

Kirchoff's law According to this law, the ratio of emissive power to absorptive power is same for all surfaces at the same temperature and is equal to the emissive power of a perfectly black body at that temperature. Thus for a body having e and a as emissive and absorptive power

$$\frac{e}{a} = \frac{E}{A} = \frac{E}{1} = E$$

where E = emissive power of a black body

It implies a good absorber is also a good emitter (radiator).

Fraunhofer lines These are the dark lines in the spectrum of the sun and are explained on the basis of Kirchoff's Law. White light emitted from core (photosphere) of the sun when passes through its atmosphere (chromosphere) radiations of those wavelengths will be absorbed by the gases present there, which they usually emit (in emission spectrum) resulting in dark lines in the spectrum of sun.

Stefan's law Radiant energy emitted per unit area per second (or emissive power or intensity) of a black body is directly proportional to the fourth power of temperature.

$$E_R \propto T^4 \quad \text{or} \quad E_R = \sigma T^4$$

If the body is not perfectly black then

$$E_R = e\sigma T^4 \quad \text{where } \sigma = 5.67 \times 10^{-8}\text{Wm}^{-2}\text{K}^{-4}$$

Energy radiated per second or radiant power

$$P_R = eA\sigma T^4 \quad \text{or} \quad \int R\lambda d\lambda \, \alpha \, T^4$$

Cooling by radiation If a body is at temperature T in an environment of temperature T_0 ($< T$), then body loses energy by emitting radiations at a rate

$$P_1 = eA\sigma T^4$$

and it receives energy by absorbing radiation at a rate

$$P_2 = eA\sigma T_0^4.$$

So the net rate of loss is $P = P_1 - P_2$

or $\qquad P = eA\sigma(T^4 - T_0^4)$

When a body cools by radiation, then rate of cooling depends upon the following factors:

(a) **Nature of the radiating surface, that is, emissivity** Greater the emissivity faster will be the cooling.

(b) **Area of the radiating surface** More the surface area of the radiating surface, faster will be the cooling.

(c) **Mass of the radiating surface** Greater the mass of the radiating body slower will be the cooling.

(d) **Specific heat of the radiating body** More the specific heat slower will be the cooling.

(e) **Temperature of radiating body** Higher the temperature faster will be the cooling.

(f) **Temperature of the surrounding** Lesser the temperature of the surrounding faster is the cooling.

Newton's law of cooling The rate of cooling is proportional to the temperature difference between body and the surroundings provided the temperature difference is not very large from the surroundings. That is,

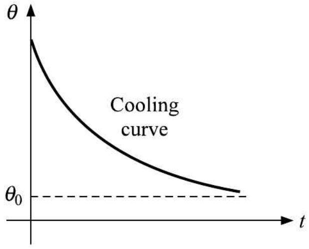


Fig. 15.5

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

or

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta - \theta_0} = \int_0^t -K dt$$

$$\log_e \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = -Kt$$

Fig. 15.5 shows the temperature vs time curve.

If time intervals are equal and successive then

$$\frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = \frac{\theta_3 - \theta_0}{\theta_1 - \theta_0}$$

Assuming temperature at $t = 0$ is θ_1, after first interval it is θ_2 and after second equal interval of time it is θ_3.

Solar constant The sun is a perfectly black body as it emits all possible radiations ($e = 1$). Solar constant S is defined as the intensity of solar radiation at the surface of the earth. That is,

$$S = \frac{P_R}{4\pi r^2} = \frac{4\pi R^2 \sigma T^4}{4\pi r^2}$$

$r = 1.5 \times 10^{11}$ km

(distance between the sun and the earth)

$R = 7 \times 10^{8}$ km (radius of the sun)

$$S = \frac{2\,\text{cal}}{\text{cm}^2\,\text{min}} = 1388\,\text{W/m}^2$$

Wien's displacement law When a body is heated, it emits all possible radiations. However, intensity of different wavelengths is different. According to Wien's law the product of wavelength (corresponding to maximum intensity of radiation) and temperature of the body in Kelvin is constant

$$\lambda_m T = b \text{ (constant)}$$

$$b = 2.89 \times 10^{-3}\,\text{m–K}$$

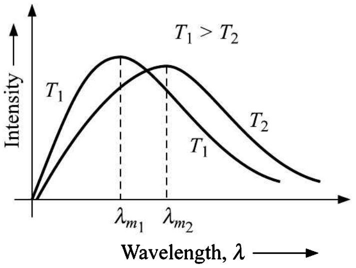


Fig. 15.6

Planck's law Planck assumed that electromagnetic radiations are not emitted or absorbed continuously but in discrete packets of energy called quanta or photons. The energy associated with each photon is

$$E = h\nu$$

where h is Planck's constant (=6.626×10^{-34} J – s). On the basis of quantum theory Planck showed that

$$E_R(\lambda) = \frac{2\pi hc^2}{\lambda^5}\,\frac{1}{\left[e^{\frac{hc}{\lambda kT}} - 1\right]}$$

SHORT-CUTS AND POINTS TO NOTE

(i) Thermal conductivity

$$\frac{dQ}{dt} = \frac{KA(T_2 - T_1)}{l} = -KA\,\frac{dT}{dx}$$

$$i_{\text{thermal}} = \frac{dQ}{dt},\; R_{\text{thermal}} = \frac{l}{KA},\; V_{\text{thermal}} = (T_2 - T_1)$$

Laws of resistances in series and parallel and Kirchoff's junction law are valid even in thermal conductance.

(ii) Thermometric conductivity (D)

$D = \frac{K}{\rho C}$ where K → thermal conductivity

ρ → density

C → specific heat

If σ is electrical conductivity then according to Weidemann–Franz Law

$$\frac{K}{\sigma T} = \text{constant}$$

(iii) Growth of Ice in a pond

$$\frac{dy}{dt} = \frac{K\theta}{\rho L} \times \frac{1}{y} \quad \text{or} \quad t = \frac{\rho L}{K\theta}\,\frac{y^2}{2}\,\frac{1}{2}$$

where L is latent heat of fusion of ice

The ratio of times for thickness of ice 0 to y : y to $2y$: $2y$ to $3y$: : 1 : 3 : 5.

$$\frac{dQ}{dt} = K\,\frac{4\pi r_1 r_2}{(r_2 - r_1)}(\theta_1 - \theta_2) \text{ in a hollow sphere}$$

and $\theta = \theta_1 - \frac{b\ (r - a)}{r\ (r_2 - r_1)}(\theta_1 - \theta_2)$ (temperature at any point distant r from the centre)

In a cylinder of radius r_1 and r_2 and length l

$$\frac{dQ}{dt} = \frac{2\pi K l(\theta_1 - \theta_2)}{\log_e \frac{r_2}{r_1}}$$

and temperature at any point distant r from the centre

$$\theta = \theta_1 - \frac{\log_e \frac{r}{r_1}}{\log_e \frac{r_2}{r_1}}(\theta_1 - \theta_2)$$

(iv) Convection is natural and forced

It occurs in fluids $\left|\frac{dQ}{dt}\right|_{\text{convection}} = hAd\theta$

Lapse rate — temperature falls by 6°C/km when we move up is an example of natural convection.

Forced convection of blood by heart (as a pump) in body helps to maintain the temperature of body at 37°C

(v) Radiation Heat transfer process without requiring medium. Heat radiations are em waves lying in *IR* and microwave range. Radiations exert pressure.

A black body is capable of emitting or absorbing all possible radiations. Therefore, the sun is a black body. We can say that the radiations of a black body are white (having all wavelengths).

Radiations are detected using bolometer, thermopile, radiometer, and so on.

(vi) Lamp black, the blackest body on earth, is 98% black.

(vii) Absorptive power

$$= \frac{\text{amount of radiation energy absorbed in a given time}}{\text{total radiation energy incident in that time}}$$

If spectral absorptivity is a_λ then total absorptivity

$$a = \int_0^\infty a_\lambda\, d\lambda$$

Emissive power is the Radiant energy emitted per unit area of the surface

$$e = \int_0^\infty e_\lambda\, d\lambda$$

Emissivity (ε) It is the ratio of emissive power of a substance with respect to a perfectly black body. That is,

$$\varepsilon = \frac{e_{\text{body}}}{E_{\text{black body}}}$$

For a perfectly black body absorptivity or emissivity is 1.

(viii) *Kirchhoff's Law* Ratio of emissivity to absorptivity of all bodies is fixed or constant and is equal to emissivity of a black body.

$$\frac{e}{a} = E \text{ (black body)}$$

It implies a good absorber is also a good emitter.

(ix) Stefan's Law $E_R = \sigma T^4$ for a perfectly black body and $E_R = e\sigma T^4$ for a body not perfectly black (e being emissivity)

$$\sigma = 5.67 \times 10^{-8} \text{W/m}^2$$

Radiant power $P_R = eA\sigma T^4$ or $\int E_R \, d\lambda \propto T^4$

Dimensional formula of $\sigma = [MT^{-3}\,\theta^{-4}]$

(x) *Newton's Law of Cooling*

$$\frac{d\theta}{dt} \propto \theta - \theta_0 \quad \text{or} \quad \int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta - \theta_0} = -K\int_0^t dt$$

$\log_e \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = -Kt$ (use this law if temperature at any time t is to be determined)

Use $\frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = \frac{\theta_3 - \theta_0}{\theta_2 - \theta_0}$

in order to find temperature in successive equal intervals. θ_1 is temperature at $t = 0$, θ_2 after time t and θ_3 after a total time $2t$ or next interval of t.

The law is valid if difference between temperature of the body and surroundings is not large (~30°C).

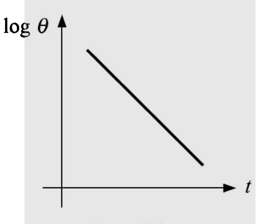


Fig. 15.7

(xi) Solar constant $S = \frac{P_R}{4\pi r^2} = \frac{4\pi R^2 \sigma T^4}{4\pi r^2} = 1388 \text{ W/m}^2$. For partical purpose, we can assume solar constant (s) to be 1400 W/m^2.

Temperature of sun = 6000K. Dimensional formula for $S = [MT^{-3}]$

(xii) Wein's displacement law $\lambda_m T = b = 2.89 \times 10^{-3}$m-K. Higher the temperature shorter is the wavelength of maximum intensity.

Use $\lambda_{m1} T_1 = \lambda_{m2} T_2$ if temperature or wavelength of a body is to be determined and that of the other is known; for example, for one star λ_{m1} and T_1 is known and λ_{m2} or T_2 is to be determined.

(xiii) Planck's Law is based on quantum nature of radiations. It assumes that discrete energy packets called photons are emitted or absorbed.

$$E = h\nu = h\frac{c}{\lambda} \text{ where } c \rightarrow \text{speed of light}$$

(xiv) Intensity $I \propto \frac{1}{d^2}$ for a point source and amplitude $A \propto \frac{1}{\sqrt{d}}$ for a cylindrical source.

CAUTION

(i) When you forget to take total length in case of series and total area in case of parallel while finding effective thermal conductivity

- In series $R = R_1 + R_2$

$$\frac{l_1 + l_2}{K_{\text{eff}} A_{\text{eff}}} = \frac{l_1}{K_1 A_1} + \frac{l_2}{K_2 A_2} \quad \text{and}$$

In parallel $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{K_{\text{eff}}(A_1 + A_2)}{l} = \frac{K_1 A_1}{l} + \frac{K_2 A_2}{l}$$

or $$K_{\text{eff}} = \frac{K_1 A_1 + K_2 A_2}{A_1 + A_2}.$$

(ii) Confusing how to find temperature of the junction (or at any other point) in cases like the one shown in Fig. 15.8.

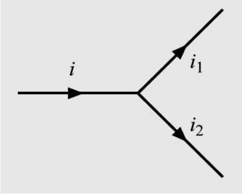


Fig. 15.8

- Apply Kirchchoff's law, that is, $i = i_1 + i_2$ and so on

(iii) Considering freezing of lake as a convection process.

- It is a conduction process and time taken to freeze a depth y is given by

$$t = \frac{\rho L}{K\theta} \frac{y^2}{2}$$

(iv) Considering that conduction does not occur in liquids or gases.

- It does occur but it is quite small as compared to solids.

(v) Considering any black colour object as a black body.

- Black body is one which is capable of emitting or absorbing all possible radiations. Thus even the sun is a black body.

(vi) Considering Stefan's law can be applied only to perfectly black bodies

- True, but the modified form of Stefan's law ($E_R = e\sigma T^4$, where e is emissivity of a body) can be applied to any body.

(vii) Considering that only black coloured objects absorb radiations.

- According to Kirchhoff's Law a good absorber is also a good emitter. Hence they emit radiations too.

(viii) Confusing emissive power and emissivity.

- Emissivity is emissive power of a body compared to a black body.

(ix) Not remembering value of σ or *b* (Wein's displacement Law)

- $\sigma = 5.67 \times 10^{-8} W/m^2$; $b = 2.89 \times 10^{-3} m - K$

(x) Considering that Newtons law of cooling can be applied for any temperature difference between hot body and surrounding.

- It is applicable only when difference between the temperature of the body and surroundings is not very large (~20 – 30°C).

Solved Problems

1. A man has a total surface area of 1.5 m². Find the total rate of radiation of energy from the body.

(a) 566 Js⁻¹ (b) 682 Js⁻¹
(c) 732 Js⁻¹ (d) 782 Js⁻¹

Solution (d) $P_R = \sigma A T^4$

$= 5.67 \times 10^{-8} \times 1.5 \times (310)^4 = 782\ Js^{-1}$

2. Three identical rods *A*, *B* and *C* of equal lengths and equal diameters are joined in series as shown in Fig. 15.9. Their thermal conductivities are 2K, K and K/2 respectively. Calculate the temperature at two junction points.

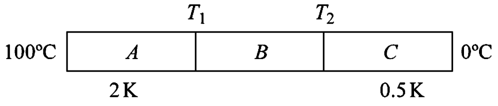


Fig. 15.9

(a) 85.7, 57.1°C (b) 80.85, 50.3°C
(c) 77.3, 48.3°C (d) 75.8, 49.3°C

Solution (a) $i_{th1} = i_{th2} = i_{th3}$

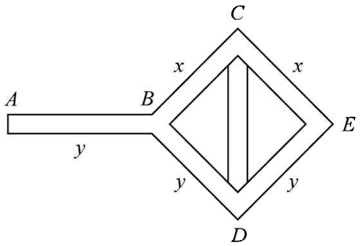
$$\frac{(100-T_1)2KA}{l} = \frac{(T_1-T_2)KA}{l} Q = \frac{(T_2-0)KA}{2l}$$

$$2(100 - T_1) = (T_1 - T_2) = \frac{T_2}{2} \qquad T_1 = \frac{3T_2}{2}$$

$$2(100 - T_1) = \frac{T_1}{3}$$

or $T_1 = \frac{600}{7} = 85.7°C$ and $T_2 = \frac{2T_1}{3} = 57.1°C$

3. Three rods of material *x* and three rods of material *y* are connected as shown in Fig. 15.10(a). All the rods are of identical length and cross-section If the end *A* is maintained at 60°C and the junction *E* at 10°C, find effective Thermal Resistance. Given length of each rod = *l*, area of cross-section = *A*, conductivity of *x* = K and conductivity of *y* = 2K.

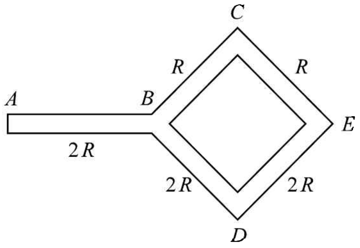


(a)

(a) $\frac{4l}{3KA}$ (b) $\frac{7l}{3KA}$

(c) $\frac{4KA}{3l}$ (d) $\frac{7KA}{3l}$

Solution (b) *BCED* part forms a Wheatstone bridge Therefore equivalent circuit is shown as Fig. 15.10(b).



(b)

Fig. 15.10

Net resistance is
$R_{eq} = R_{AB} + R_{BE}$,

$$R_{BE} = \frac{2R \times 4R}{2R+4R} = \frac{4}{3}R = R + \frac{4}{3}R = \frac{7}{3}R = \frac{7l}{3KA}$$

4. Two spheres of same material have radii 1 m and 4 m and temperature 4000 K and 2000 K respectively. The ratio of energy radiated per second is

(a) 1 (b) 2
(c) 4 (d) none of these

Solution (a) $\frac{P_1}{P_2} = \frac{e\sigma A_1 T_1^4}{e\sigma A_2 T_2^4} = \frac{(1)^2}{(4)^2}\frac{(4000)^4}{(2000)^4} = \frac{1}{1}$

5. The emissivity of a body of surface area 5 cm² and at temperature 727°C radiating 300 J of energy per minute is

(a) 0.48 (b) 0.38
(c) 0.28 (d) 0.18

Solution (d) $P = eA\sigma T^4$

$$e = \frac{P}{A\sigma T^4} = \frac{300/60}{5\times10^{-4}\times5.67\times10^{-8}(1000)^4}$$

$= 0.18$

6. A body cools in 7 minutes from 60°C to 40°C. The temperature after next 7 minutes will be ——— . Given temperature of surrounding is 10°C.

(a) 32°C (b) 38°C
(c) 22°C (d) none of these

Solution (d) $\frac{40-10}{60-10}=\frac{\theta_3-10}{40-10}$, that is, $\theta_3 = 28$

$$\left(\text{Using } \frac{\theta_2-\theta_0}{\theta_1-\theta_0}=\frac{\theta_3-\theta_0}{\theta_2-\theta_0}\right).$$

7. Bodies A and B have thermal emissivities of 0.01 and 0.81 respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. Find the temperature of B if the temperature of A is 5802 K.

(a) 1634 K (b) 1734 K
(c) 1934 K (d) none of these

Solution (c) Given $P = e_A\sigma AT_A^4 = e_B\sigma AT_B{}^4$

or $$T_B = \left(\frac{e_A}{e_B}\right)^{1/4} T_A = \left(\frac{0.01}{0.81}\right)^{1/4}(5802)$$

$$T_B = \frac{5802}{3} = 1934$$

8. A 10 cm long copper rod is welded to 20 cm long steel rod each having cross-section A. If their thermal conductivities are 386 $\text{Jm}^{-1}\text{s}^{-1}{}^\circ\text{C}^{-1}$ and 46 $\text{Jm}^{-1}\text{s}^{-1}{}^\circ\text{C}^{-1}$ the temperature of the junction will be ——— . (Given copper end is at 100°C and steel rod end is at 0°C.)

(a) 87.5°C (b) 79.3°C
(c) 75°C (d) 71.5°C

Solution (d) Let the junction temperature be T

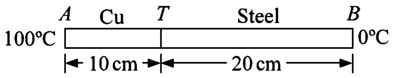


Fig. 15.11

$$\frac{(100-T)A\,386}{10}=\frac{(T-0)A\,46}{20}$$

or $$\frac{100-T}{T}=\frac{23}{386} \quad\text{or}\quad T = 71.63^\circ\text{C}$$

9. The temperature of a body falls from 40°C to 36°C in 5 minutes. The temperature of the body will become 32°C in

(a) less than 10 minutes (b) 10 minutes
(c) more than 10 minutes (d) none of these

Solution (c) Since $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

As $(\theta - \theta_0)$ will decrease, therefore it will take more time to cool.

10. A solid at temperature T_1 is kept in an evacuated chamber at temperature $T_2 > T_1$. The rate of growth of temperature is proportional to

(a) $T_2 - T_1$ (b) $T_2^2 - T_1^2$
(c) $T_2^3 - T_1{}^3$ (d) $T_2^4 - T_1^4$

Solution (d) $P_1 = Ae\sigma T_1^4$ (rate of loss of radiant energy)

$P_2 = Ae\sigma T_2^4$ (rate of gain of radiant energy)

Net rate of gain of radiant energy

$$(P_2 - P_1) \propto (T_2^4 - T_1^4)$$

11. Consider the radiations emitted by the human body. Which of the following statements is true?

(a) radiations lie in ultraviolet region
(b) radiations lie in infrared region
(c) radiations are emitted only during the day
(d) radiations are emitted during sammer and absorbed during winter

Solution (b) Heat radiations lie in infrared and microwave regions.

12. Two bodies are at temperature 27°C and 927°C. The heat energy radiated by them will be in the ratio

(a) 1 : 256 (b) 1 : 64
(c) 1 : 4 (d) 1 : 16

Solution (b) $\frac{H_1}{H_2}=\frac{T_1^4}{T_2^4}=\left(\frac{300}{1200}\right)^4=\frac{1}{64}$

13. The temperature of a black body increases from T to $2T$. The factor by which the rate of emission will increase is

(a) 2 (b) 4
(c) 8 (d) 16

Solution (d) $\frac{P_1}{P_2}=\left(\frac{T}{2T}\right)^2=\frac{1}{16}$

14. According to Newton's law of cooling, the rate of cooling of a body is proportional to $(\Delta\theta)^n$ where $\Delta\theta$ is the difference of temperature of the body and the surroundings and n is equal to

(a) 3 (b) 4
(c) 1 (d) 2

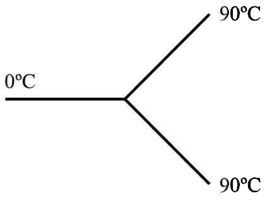
Solution (c) $\frac{d\theta}{dt} \propto (\theta - \theta_0)$ or $\frac{d\theta}{dt} \propto \Delta\theta$

15. A black body at a temperature 77°C radiates heat at a rate of 10 calcm^{-2} – s. The rate at which this body would radiate heat in units of calcm^{-2} – s at 427°C is closest to

(a) 40 (b) 160
(c) 200 (d) 400

Solution $\frac{P_1}{P_2}=\left(\frac{350}{700}\right)^4=\frac{10}{P_2}$ or $P_2 = 160$

16. Three rods made of the same material and having the same cross-section have been joined as shown in Fig. 15.12. Each rod is of the same length. The left and right ends are kept at 0°C and 90°C respectively. The temperature of the junction of the three rods is



(a)

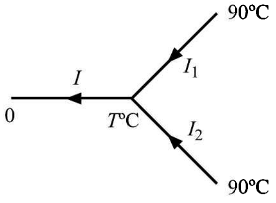
(a) 45°C (b) 60°C
(c) 30°C (d) 20°C

Solution $I = I_1 + I_2$

$$\frac{\text{kA}(90-T)}{l}+\frac{(90-T)\text{kA}}{l} = \frac{(T-0)\text{kA}}{l}$$

or $3T = 180^\circ\text{C}$

or $T = 60^\circ\text{C}$



(b)

Fig. 15.12

17. An ideal black body at room temperature is thrown into a furnace. It is observed that

(a) initially it is the darkest body and at the later times the brighest
(b) it is darkest body at all times
(c) it cannot be distinguished at all times
(d) initially it is the darkest and at later times it cannot be distinguished

Solution (a) According to Kirchoff's law good absorbers also are good emitters.

18. The thermal conductivity of a rod depends on

(a) length (b) mass
(c) area of cross-section (d) material of the rod

Solution (d)

19. The rate of heat flow through a cross-section of the rod shown in Fig. 15.13 is —— . ($\theta_2 > \theta_1$ and thermal conductivity of the material of the rod is K.)

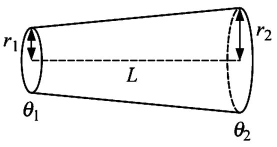


Fig. 15.13

(a) $\dfrac{K\pi r_1 r_2(\theta_2-\theta_1)}{L}$

(b) $\dfrac{K\pi(r_1+r_2)^2(\theta_2-\theta_1)}{4L}$

(c) $\dfrac{K\pi(r_2+r_1)^2(\theta_2-\theta_1)}{L}$

(d) $\dfrac{K\pi(r_2+r_1)^2(\theta_2-\theta_1)}{2L}$

Solution (a) $r_{\text{eff}} = \sqrt{r_1 r_2}$

$$\frac{dQ}{dt} = \frac{KA(\theta_2-\theta_1)}{L} = \frac{K\pi r_1 r_2(\theta_2-\theta_1)}{L}$$

20. Find the heat radiated per second by a body of surface area 12 cm^2 kept in thermal equilibrium in a room at temperature 20°C. The emissivity of the surface is 0.8 and $\sigma = 6 \times 10^{-8}$ Wm^{-2}K^{-4}.

(a) 4.2 J (b) 0.42 J
(c) 0.042 J (d) 42 J

Solution (b) $P = eA\sigma T^4$
$= 0.8 \times 12 \times 10^{-4} \times 6 \times 10^{-8} \times (293)^4$
$= 0.42$ J

21. Two copper spheres, one of large size and the other small, are heated to the same temperature. Which will cool first?

(a) bigger (b) smaller
(c) both in equal time
(d) insufficient data to reply

Solution (b) Small sphere (cooling $\propto \dfrac{1}{\text{mass}}$)

Problems for Practice

1. The temperature gradient in the earth's crust is 32°C per km and the mean conductivity of the rock is 0.008 CGS units. Considering the radius of the earth as 6000 km, the loss of heat by the earth everyday is about (in calories)
(a) 10^{10} (b) 10^{17} (c) 10^{27} (d) 10^{37}

2. A sphere and a cube, both made of copper, have equal volumes and are blackened. These are allowed to cool at same temperature and in same atmosphere. The ratio of their rates of loss of heat will be
(a) $4/3\pi : 1$ (b) $(\pi/6)^{1/3} : 1$
(c) $1 : 1$ (d) $(\pi/6)^{2/3} : 1$

3. The diameter of ball P is three times that of another ball Q of the same material. P and Q are heated to same temperature and allowed to cool up to same temperature. The relation between their rates of cooling will be
(a) $R_P = R_Q/3$ (b) $R_P = 3R_Q$
(c) $R_P = 9R_Q$ (d) $R_P = R_Q$

4. A sphere, a cube and a thin circular disc, all made of the same material, have same mass. Their initial temperature is 3 × 10^3°C. The body which cools rapidly is
(a) sphere
(b) cube
(c) both sphere and cube
(d) circular plate

5. The opposite faces of a cubical block of iron of cross-section 4 cm^2 are kept in contact with steam and melting ice. The quantity of ice melting at the end of 10 minutes will be (Given $k_{\text{iron}} = 0.2$ CGS units)
(a) 700 g (b) 500 g
(c) 300 g (d) 100 g

6. A wall has two layers A and B, each made of different materials. Both the layers have same thickness. The thermal conductivity of A is twice that of B. Under thermal equilibrium, the temperature difference across the wall is 36°C. The temperature across the layer A is
(a) 6°C (b) 12°C (c) 18°C (d) 24°C

7. The ratio of masses of two metal spheres A and B is 8 : 1. If their temperatures are 2000 K and 1000 K respectively, then the ratio of the rates of their energy emission will be
(a) 4 : 1 (b) 16 : 1 (c) 64 : 1 (d) 128 : 1

8. The temperature of a black body corresponding to which it will emit energy at the rate of 1 watt/cm^2 will be
(a) 650 K (b) 450 K (c) 350 K (d) 250 K

9. Two rods of the same material have diameters in the ratio of 1 : 2 and lengths in the ratio of 2 : 1. If the temperature difference between their ends is the same, the ratio of heats conducted by them in a given time will be
(a) 1 : 4 (b) 1 : 8 (c) 4 : 1 (d) 8 : 1

10. If a piece of metal is heated to an absolute temperature T and then put in an enclosure at absolute temperature t, then the heat generated in the atmosphere will be proportional to
(a) $(T - t)^4$ (b) $T^4 - t^4$
(c) $T^2 - t^2$ (d) $T^3 - t^3$

11. If the pressure of a gas is doubled then its thermal conductivity will
(a) remain constant (b) decrease
(d) increase (d) decrease exponentially

12. The colour of a distant star in the sky is an indication of its
(a) size (b) temperature
(c) distance (d) frequency

13. Two bodies A and B are kept in an evacuated chamber at 27°C. The temperature of A and B are 327°C and 427°C respectively. The ratio of rates of loss of heat from A and B will be
(a) 0.23 (b) 0.52 (c) 1.52 (d) 2.52

14. The ratio of amplitudes of radiation emitted by a cylindrical source at distances $2r$ and $18r$ from its axis will be
(a) 1 : 3 (b) 2 : 1 (c) 1 : 1 (d) 3 : 1

15. A hot body will emit radiations more rapidly if its surface is
(a) black and polished (b) white and polished
(c) black and rough (d) white and rough

16. If the amount of heat incident upon a body is X calorie and it absorbs Y calorie out of it, then the coefficient of absorption will be
(a) $X + Y$ (b) XY (c) Y/X (d) X/Y

17. There is a small hole in a hollow container. At what temperature should it be maintained in order that it emits one calorie of energy per second per meter^2?
(a) 10 K (b) 100 K (c) 200 K (d) 500 K

18. In which part of the electromagnetic spectrum the heat radiations lie
(a) visible (b) violet
(c) ultraviolet (d) infrared

19. The correct relation between the intensity of radiations (I) and distance (d) from the point source is
(a) $I \propto 1/d$ (b) $I \propto d$
(c) $I \propto 1/d^2$ (d) $I \propto d^2$

20. When a yellow piece of glass is heated in dark room, then it emits light of
(a) yellow colour (b) red colour
(c) blue colour (d) green colour

21. If the temperature of a black body is increased by 50% then the amount of radiation emitted by it will
(a) decrease by 50% (b) increase by 50%
(c) increase by 400% (d) decrease by 400%

22. A black body at temperature T radiates energy at the rate of E Wm^{-2} in reducing its temperature to $T/2$ the rate of energy radiation will be
(a) $E/16$ (b) $E/4$ (c) $E/2$ (d) $< E$

23. A piece of red glass is heated till it starts shining in a dark room. The colour of this shining glass will be
(a) violet (b) orange (c) green (d) red

24. The radiation force due to source of power P on a perfectly reflecting surface will be
(a) $P/3c$ (b) $2P/c$ (c) $P/2c$ (d) P/c

25. The value of solar constant is approximately
(a) 1340 watt/m^{-2} (b) 430 watt/m^2
(c) 340 watt m^2 (d) 1388 watt/m^{-2}

26. The material of prism used for obtaining spectrum of heat radiations is
(a) rock salt (b) quartz
(c) flint Glass (d) crown glass

27. Fraunhoffer lines are the result of which of the following phenomenon of radiation?
(a) scattering (b) compression
(c) emission (d) absorption

28. On which of the following laws, the constant volume thermometer works
(a) Gay Lussac's law (b) Dalton's law
(c) Boyle's law (d) Charles's law

29. Two rods of equal length and diameter but of thermal conductivities 2 and 3 units respectively are joined in series. The thermal conductivity of the combination is
(a) 6 (b) 5 (c) 1 (d) none of these

30. Ice starts forming on the surface of lake and takes 8 hours to form a layer of 1 cm thick. How much time will it take to increase the thickness of the layer to 2 cm?
(a) 8 hours
(b) less than and hours
(c) between 8 and 16 hours
(d) more than 16 hours

31. The amount of thermal radiations emitted from one square centimetre area of a black body in one second when at a temperature of 1000 K is
(a) 5.67 J (b) 56.7 J (c) 567 J (d) 5670 J

32. How is the velocity of thermal radiations (v) related to the velocity of light (c)?
(a) $v < c$
(b) $v > c$
(c) $v = c$
(d) the relation depends upon the wavelength of the radiations

33. The Kirchoff's law leads to the conclusion that the good radiators of thermal radiations are
(a) good absorbers (b) bad absorbers
(c) thermal insulators (d) none of these

34. The rate of loss of heat by radiation from a body at 400°C is R. The radiation from it when the temperature rises to 800°C is
(a) $16R$ (b) $4R$
(c) $2R$ (d) none of these

35. Given that p Joules of heat is incident on a body and out of it q Joules is reflected and transmitted by it. The absorption coefficient of the body is
(a) $(q - p)/p$ (b) q/p
(c) $(p - q)/p$ (d) p/q

36. A graph is drawn between λ and E_λ. The area A under the graph is related to the absolute temperature as
(a) $A \propto T^{-4}$ (b) $A \propto T^{-2}$
(c) $A \propto T^2$ (d) $A \propto T^4$

37. Corresponding to a given temperature, there is a wavelength λ_m, for which the intensity of heat radiations is
(a) maximum (b) constant
(c) zero (d) minimum

38. On which one of the factors do the nature of the thermal radiation depends inside an enclosure?
(a) size of the enclosure
(b) temperature
(c) nature of the walls
(d) colour of the walls

39. Thermal radiations are similar to
(a) α rays (b) X-rays
(c) cathode rays (d) none of these

40. A hot and a cold body are kept in vacuum separated from each other. Which of the following causes decrease in temperature of the hot body?
(a) conduction (b) radiations
(c) convection
(d) the temperature of both the bodies remains unchanged

41. Who explained the Fraunhofer lines in the spectrum of solar radiations?
(a) Wein (b) Fraunhofer
(c) Stefan (d) Kirchoff

42. The top of a lake is frozen. The air in contact with the surface of lake is at –15°C. Then the maximum temperature of the water in contact with the lower surface of ice will be
(a) –7.5°C (b) –4°C
(c) 0°C (d) 4°C

43. A slab consists of two parallel layers of two different materials of same thickness having thermal conductivities K_1 and K_2. The equivalent thermal conductivity of the slab is
(a) $\dfrac{K_1+K_2}{K_1K_2}$ (b) $\dfrac{2K_1K_2}{K_1+K_2}$
(c) $\dfrac{(K_1+K_2)}{2}$ (d) $K_1 + K_2$

44. In the steady state, the two ends of a metre rod are at 30°C and 20°C. The temperature at the 60th cm is
(a) 22°C (b) 23°C (c) 24°C (d) 25°C

45. Two stars A and B radiate maximum energy at 3600Å and 4800Å respectively. Then the ratio of absolute temperature of A and B is
(a) 256 : 81 (b) 81 : 256
(c) 4 : 3 (d) 3 : 4

46. A bucket full of hot water is kept in a room. It cools from 75°C to 70°C in T_1 minutes, from 70°C to 65°C in T_2 minutes and from 65°C to 60°C in T_3 minutes. Which of the following relation is correct?
(a) $T_1 > T_2 > T_3$
(b) $T_1 < T_2 < T_3$
(c) $T_1 < T_2 > T_3$
(d) $T_1 = T_2 = T_3$

47. The spectral energy distribution of the sun (temperature 6050K) has a maximum at 4753Å. The temperature of a star for which this maximum is at 9506Å is
(a) 24200K (b) 12100K
(c) 6050K (d) 3025K

48. The temperature of a piece of metal is raised from 27°C to 51.2°C. The rate at which the metal radiates energy increases nearly
(a) 1.36 times (b) 2 times
(c) 4 times (d) 8 times

49. If K denotes coefficient of thermal conductivity, d the density and C the specific heat, the unit of X, where $X = K/dC$, will be
(a) cm sec (b) $\text{cm}^2\,\text{sec}^{-2}$
(c) cm sec^{-2} (d) $\text{cm}^2\,\text{sec}^{-1}$

50. A 40 watt bulb converts 6% of its power to red light (wavelength 6500Å). The number of red light photons emitted by the bulb per second is
(a) 100 (b) 4×10^{18}
(c) 8×10^{18} (d) 13×10^{18}

51. The energy emitted by a black body at 727°C is E. If the temperature of the body is increased to 227°C, then the emitted energy will become
(a) 13 times (b) 2.27 times
(c) 1.9 times (d) 3.9 times

52. A metallic sphere cools from 50°C to 40°C in 300 seconds. If the room temperature is 20°C then its temperature in the next 5 minutes will be
(a) 30°C (b) 33.3°C (c) 36°C (d) 38°C

53. Two stars X and Y emit yellow and blue lights. Out of these whose temperature will be more?
(a) that of Y
(b) that of X
(c) that of both
(d) sometimes X and sometimes Y

54. A body cools from 62°C to 50°C in 10 minutes. If the temperature of the surroundings is 26°C then the temperature of the body after next 10 minutes will be
(a) 42°C (b) 44°C (c) 46°C (d) 48°C

55. An ideal black body emits maximum intensity of radiation of wavelength 5000Å at temperature 1227°C. If its temperature is increased by 10^3°C then the maximum emission wavelength will be
(a) 5000Å (b) 4000Å
(c) 3500Å (d) 3000Å

56. The maximum emission wavelength at temperature 2000K is 4μm. The maximum wavelength corresponding to temperature 2400K will be
(a) 0.66×10^{-6}m
(b) 1m
(c) 3.33×10^{-6}m
(d) 10^{-6}m

57. If the rates of cooling of two bodies are same then for which body the rate of fall of temperature will be more? For the body whose thermal capacity is
(a) less (b) more
(c) infinity (d) any value

58. If the temperature of a body increases by 2% then energy radiated will increase by
(a) 2% (b) 8% (c) 4% (d) 16%

59. If the rate of emission of heat radiation of an ideal black body is made sixteen times then its final temperature will become
(a) half (b) doubled
(c) 4 times (d) 8 times

60. Out of the following, which body is not an ideal black body?
(a) Wien's black body
(b) Ferry's black body
(c) coal
(d) sun

ANSWERS

1. (b)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (b)
11. (a)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (c)
21. (c)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (d)	**28.** (a)	**29.** (d)	**30.** (d)
31. (a)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (b)
41. (d)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (b)	**47.** (d)	**48.** (a)	**49.** (d)	**50.** (c)
51. (b)	**52.** (b)	**53.** (a)	**54.** (a)	**55.** (d)	**56.** (c)	**57.** (a)	**58.** (b)	**59.** (b)	**60.** (c)

Simple Harmonic Motion

16

BRIEF REVIEW OF THE CONCEPTS

Harmonic or periodic motion

If a moving body repeats its motion after regular intervals of time the motion is said to be harmonic or periodic. The time interval after which it repeats the motion is called **time period.** If the body moves to and fro on the same path, it is said to be oscillating. In simple harmonic motion the particle moves in a straight line and the acceleration of the particle is always directed towards a fixed point on the line. This fixed point is called **centre of oscillation.** The acceleration in SHM is given by

$$a = -\omega^2 x \quad \text{or} \quad F = -m\omega^2 x = -kx$$

$k = m\omega^2$ is called the force constant or spring constant.

The force which brings the particle back towards the equilibrium (or mean) position is called **restoring force**.

SHM may be thought of as the projection of uniform circular motion along a diameter.

$$x = r\cos\omega t$$

$$y = r\sin\omega t$$

$$a = -\omega^2 x$$

$$\text{or} \quad \frac{d^2x}{dt^2} = -\omega^2 x$$

gives the solution $x = x_0 \sin\omega t$ when the particle starts from mean position,

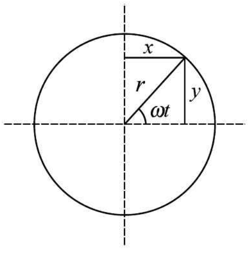


Fig. 16.1

$x = x_0 \cos\omega t$ when the particle starts from extreme position

$$\left.\begin{aligned} x &= x_0 \sin(\omega t \pm \phi) \\ x &= x_0 \cos(\omega t \pm \phi) \end{aligned}\right\} \text{When it starts in between mean and extreme positions.}$$

$$x = x_0 e^{\pm i\omega t}$$

where x is called instantaneous displacement, x_0 the amplitude (maximum displacement) ϕ initial phase angle or epoch or angle of repose and ω, the angular frequency. Linear frequency $f = \dfrac{\omega}{2\pi}$ and $T = \dfrac{1}{f}$ is time period.

Velocity of the particle executing SHM

Assume $x = x_0 \sin\omega t$, then

when the time is given $v = \dfrac{dx}{dt} = x_0\omega\cos\omega t$

When displacement is given $v = x_0\omega\sqrt{1-\sin^2\omega t} = \omega\sqrt{x_0^2 - x^2}$.

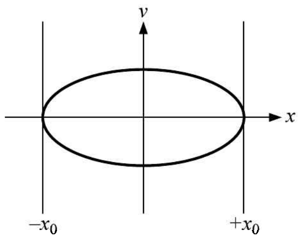


Fig. 16.2 Velocity-displacement graph

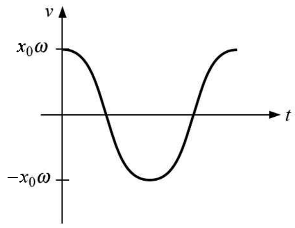
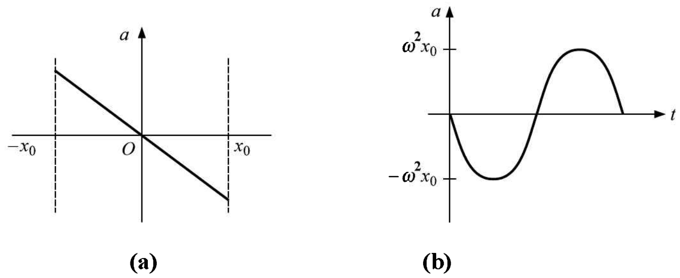


Fig. 16.3 Velocity-time relationship

$$a = -\omega^2 x \quad \text{or} \quad a = -\omega^2 x_0 \sin\omega t$$



(a) **(b)**

Fig. 16.4 Acceleration-displacement and acceleration-time graphs

Note: The graph between velocity and acceleration is an ellipse and the velocity displacement graph is an ellipse.

Kinetic energy (KE) of a particle executing

$$\text{SHM} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(x_0^2 - x^2).$$

Potential energy (PE) of a particle executing SHM $= \dfrac{1}{2} m\omega^2 x^2$.

Total Energy (E) = KE + PE $= \dfrac{1}{2} m\omega^2 x_0^2$ is conserved.

Note that velocity is maximum at mean position ($v_{\max} = x_0\omega$) and minimum at extreme position ($v_{\min} = 0$).

Acceleration is maximum at extreme position ($a_{\max} = \omega^2 x_0$) and is minimum at mean position ($a_{\min} = 0$).

Kinetic energy (KE) is maximum at mean position $KE_{max} = \frac{1}{2} m x_0^2 \omega^2$ and minimum at extreme position $KE_{min} = 0$.

Potential energy (PE) is maximum at extreme position $= \frac{1}{2} m x_0^2 \omega^2$ and is minimum at mean position = 0.

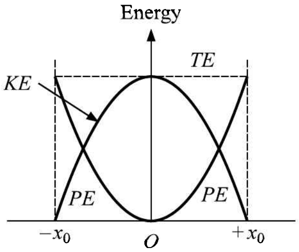


Fig. 16.5

Note that in SHM acceleration lags behind displacement by π but leads velocity by π/2.

In SHM the velocity–displacement curve is an ellipse.

$x = x_0 \sin \omega t$ ...(1)

$v = x_0 \omega \cos \omega t$ ...(2)

Square and add (1) and (2)

$$\frac{x^2}{x_0^2} + \frac{v^2}{x_0^2 \omega^2} = 1$$

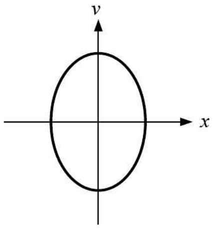


Fig. 16.6

In SHM acceleration–velocity curve is an ellipse

$a = -x_0 \omega^2 \sin \omega t$ ...(1)

$v = x_0 \omega \cos \omega t$ ...(2)

$$\frac{a^2}{x_0^2 \omega^4} + \frac{v^2}{x_0^2 \omega^2} = 1$$

If a tunnel is dug in the earth diametrically or along a chord then $T = 2\pi\sqrt{\frac{R}{g}}$ = 84 min 36 s for a particle released in the tunnel.

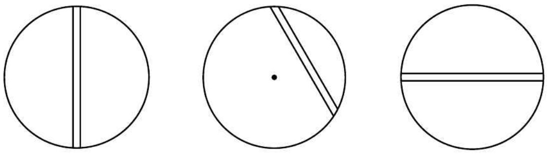


Fig. 16.7

If a ball is dropped from a height h in the tunnel the motion is no longer SHM but oscillatory and time period of the oscillation is

$$T = 4\sqrt{\frac{2h}{g}} + 4\sqrt{\frac{R}{g}} \sin^{-1}\left(\frac{R}{R+2h}\right).$$

If a point charge q is tunnelled in a uniformly charged sphere of charge Q and radius R then

$$T = 2\pi\sqrt{\frac{4\pi\varepsilon_0 R^3 m}{Qq}}. \quad \text{(See Fig. 16.8)}$$

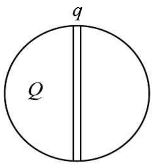


Fig. 16.8

Angular SHM A body free to rotate about a given axis can make angular oscillations when it is slightly pushed aside and released. The angular oscillations are called angular SHM if

(a) There is a point (mean position) where the resultant torque on the body is zero (θ = 0).

(b) The body is displaced through an angle from the mean position, a resultant torque acts which is proportional to the angle displaced.

(c) The nature of the torque (clockwise or anti-clockwise) is to bring the body towards mean position.

$$\tau = -k\theta$$

that is, $\alpha I = -k\theta$ or $\alpha = -\frac{k}{I}\theta$

$\theta = \theta_0 \sin \omega t$ if it starts from mean position

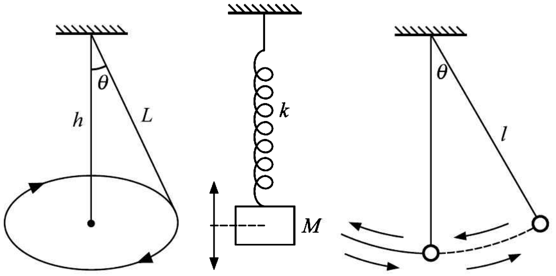
$\theta = \theta_0 \cos \omega t$ if it starts from extreme position

$$\frac{d\theta}{dt} = \Omega = \theta_0 \omega \cos \omega t = \omega\sqrt{\theta_0^2 - \theta^2}$$

or $\omega = \sqrt{\frac{k}{I}}$ or $T = 2\pi\sqrt{\frac{I}{k}}$

$$PE = \frac{1}{2}k\theta^2 = \frac{1}{2}I\omega^2\theta^2$$

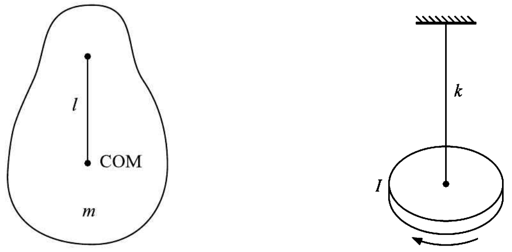
$$KE = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}I\theta_0^2\omega^2\cos^2\omega t.$$



(a) (b) (c)

Fig. 16.9

Conical pendulum	**Spring pendulum**	**Simple pendulum**
$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$ $= 2\pi\sqrt{\frac{h}{g}}$	$T = 2\pi\sqrt{\frac{M}{k}}$	$T = 2\pi\sqrt{\frac{l}{g}}$ if θ is infinitesimally small and $T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}\right)$ if θ is finite and is equal to θ_0.



(a) (b)

Fig. 16.10

Physical or compound pendulum **Torsional pendulum**

$$T = 2\pi\sqrt{\frac{I}{mgl}} \qquad T = 2\pi\sqrt{\frac{I}{k}}$$

If time period of a simple pendulum is 2 s, it is called second's pendulum.

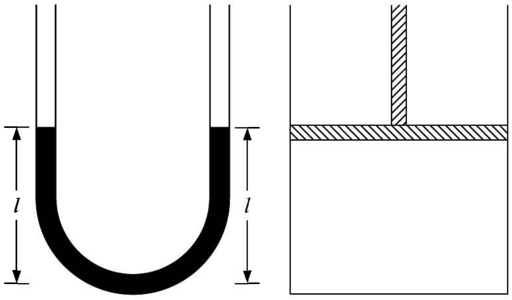
Longest time period $T = 2\pi\sqrt{\frac{R}{g}}$ = 84 min 36 s for an infinitely long simple pendulum.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l}+\frac{1}{R}\right)}} \text{ for a long pendulum}$$

$$T = 2\pi\sqrt{\frac{R}{2g}} \text{ if } l = R = 60 \text{ min or 1h}$$

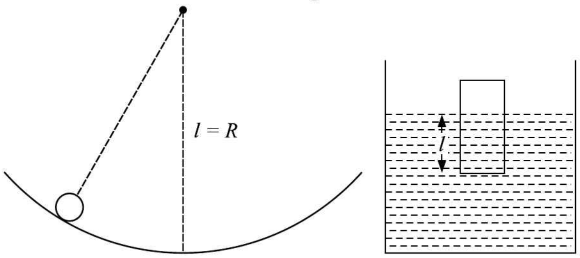
If the SHM occurs because of acceleration due to gravity then $T = 2\pi\sqrt{\frac{l}{g}}$.

Some examples of this SHM are liquid in a *u*-tube, vertical cylinder/piston, motion of a ball in bowl/concave mirror and a floating cylinder.



(a) (b)

Fig. 16.11



(c) (d)

Fig. 16.12

Effect of Temperature on Time period of a Simple pendulum

$$\frac{T}{T_0} = \left[1 + \frac{\alpha\Delta\theta}{2}\right]$$

$\Delta\theta$ = rise in temperature and α = linear coefficient of expansion.

$$\frac{\Delta T}{T_0} = \frac{\alpha\Delta\theta}{2}$$

If the suspended wire stretches due to elasticity

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{Mg}{2\pi r^2 Y}\right] \quad \text{or} \quad \Delta T = \frac{T\,Mg}{2\pi r^2 Y}.$$

If the damping is negligible and effect of upthrust in a liquid is considered then

$$\text{where } T = 2\pi\sqrt{\frac{l}{g\left(1-\frac{\sigma}{\rho}\right)}} \text{ and } g' = g\left(1-\frac{\sigma}{\rho}\right)$$

σ is density of liquid and ρ density of bob of the pendulum.

If a carriage is moving upwards with acceleration '*a*' carrying the pendulum

$$T = 2\pi\sqrt{\frac{l}{g+a}}.$$

If the carriage moves downwards with acceleration '*a*' carrying the pendulum

$$T = 2\pi\sqrt{\frac{l}{g-a}}.$$

If the carriage moves horizontally with acceleration '*a*'

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a^2}}}.$$

If the carriage is in circular motion with velocity *v*, radius *R*

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+\left(\frac{v^2}{R}\right)^2}}}.$$

If the pendulum has a charged bob and is placed in a uniform electric field such that:

(a) *q*E opposes *g*

$$T = 2\pi\sqrt{\frac{l}{g-\frac{qE}{m}}}.$$

(b) *q*E in the direction of *g*

$$T = 2\pi\sqrt{\frac{l}{g+\frac{qE}{m}}}.$$

(c) *q*E perpendicular to *g*

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+\frac{q^2E^2}{m^2}}}}.$$

For springs in parallel

$$k_{\text{eff}} = k_1 + k_2;\; T = 2\pi\sqrt{\frac{M}{k_1+k_2}}.$$

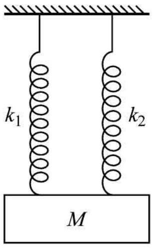
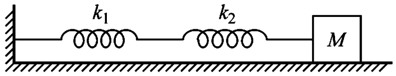


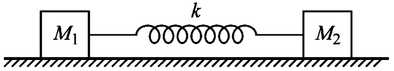
Fig. 16.13

For springs in series $k_{\text{eff}} = \dfrac{k_1k_2}{k_1+k_2}$; $T = 2\pi\sqrt{\dfrac{M}{\dfrac{k_1k_2}{k_1+k_2}}}$.



(a)

If the spring is attached to masses on both sides



(b)

Fig. 16.14

$$T = 2\pi\sqrt{\frac{\mu}{k}},\quad \mu = \frac{m_1m_2}{m_1+m_2}\ \text{is reduced mass.}$$

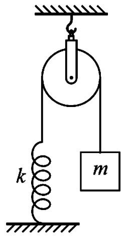


Fig. 16.15

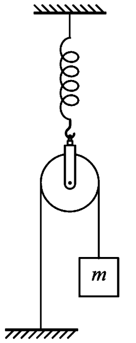


Fig. 16.16

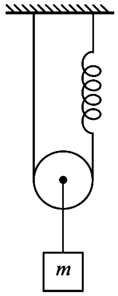


Fig. 16.17

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad T = 2\pi\sqrt{\frac{4m}{k}} = 4\pi\sqrt{\frac{m}{k}} \qquad T = 2\pi\sqrt{\frac{m}{4k}}$$

Composition of two SHMs in same direction

$x_1 = x_{01}\sin\omega t \quad x_2 = x_{02}\sin(\omega t + \alpha)$

$x = x_0\sin(\omega t + \phi)$ is resultant.

That is, $x = x_1 + x_2 = x_{01}\sin\omega t + x_{02}\sin(\omega t + \alpha)$.

$$x_0 = \sqrt{x_{01}^2 + x_{02}^2 + 2x_{01}x_{02}\cos\alpha}$$

$$\tan\phi = \frac{x_{02}\sin\alpha}{x_{01} + x_{02}\cos\alpha}$$

Note that SHMs are added like vectors

Composition of two SHMs in two perpendicular direction give rise to lissajous figures

$x = x_0\sin\omega t$ or $\sin\omega t = \dfrac{x}{x_0}$ and $\cos\omega t = \sqrt{1-\dfrac{x^2}{x_0^2}}$

$y = y_0\sin(\omega t + \phi)$

$y = y_0\sin\omega t\cos\phi + y_0\cos\omega t\sin\phi$

$$= y_0\frac{x}{x_0}\cos\phi + y_0\sqrt{1-\frac{x^2}{x_0^2}}\sin\phi$$

or $$\frac{y}{y_0} = \frac{x}{x_0}\cos\phi + \sqrt{1-\frac{x^2}{x_0^2}}\sin\phi$$

or $$\left(\frac{y}{y_0} - \frac{x}{x_0}\cos\phi\right)^2 = \left(1-\frac{x^2}{x_0^2}\right)\sin^2\phi$$

or $$\frac{y^2}{y_0^2} + \frac{x^2}{x_0^2} - \frac{2xy}{x_0y_0}\cos\phi = \sin^2\phi$$

if $$\phi = 0,\ \frac{y^2}{y_0^2} + \frac{x^2}{x_0^2} - \frac{2xy}{x_0y_0} = 0$$

or $$\left(\frac{y}{y_0} - \frac{x}{x_0}\right)^2 = 0 \text{ or } y = \frac{y_0}{x_0}x$$

that is, straight line is obtained.

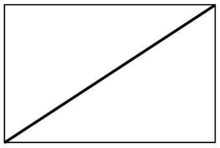


Fig. 16.18

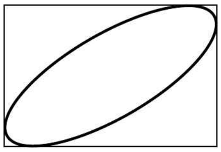


Fig. 16.19

If $0 < \phi < \pi/2$, for example, $\phi = \pi/4$ oblique ellipse is obtained.

If $\phi = \pi/2$, ellipse is obtained and if $x_0 = y_0$ then circle is obtained.

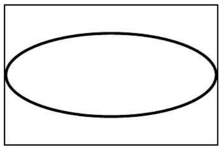


Fig. 16.20

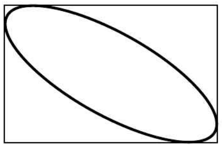


Fig. 16.21

If $\pi/2 < \phi < \pi$, for example, $\phi = 3\pi/4$ oblique ellipse results.

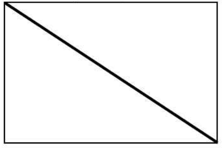


Fig. 16.22

If $\phi = \pi$ straight line is obtained.

Lissajous figures If the frequency of SHM in x and y-directions are different then

$$\frac{\omega_x}{\omega_y} = \frac{n_y}{n_x} = \frac{\text{number of times it touches } y\text{-axis}}{\text{number of times it touches } x\text{-axis}}$$

here $\dfrac{\omega_x}{\omega_y} = \dfrac{2}{1}$ $\qquad$ $\dfrac{\omega_x}{\omega_y} = \dfrac{2.5}{1}$

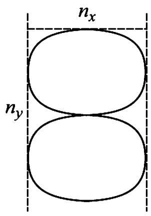


Fig. 16.23

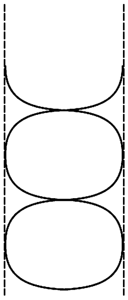


Fig. 16.24

Damped harmonic motion

$$\frac{md^2x}{dt^2} + r\frac{dx}{dt} + kx = 0$$

or $$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

or $$\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega^2x = 0$$

$b = \dfrac{r}{2m}$ is called damping coefficient.

Solution to the equation is

$$x = \frac{x_0}{2}e^{-bt}\left[\left(1+\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{+t\sqrt{b^2-\omega^2}} + \left(1-\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{-t\sqrt{b^2-\omega^2}}\right]$$

Note that $x_0' = x_0e^{-bt}$ is the amplitude at any time t.

If $\dfrac{r}{2m} > \sqrt{\dfrac{k}{m}}$ motion is non-oscillatory and overdamped.

If $\dfrac{r}{2m} = \sqrt{\dfrac{k}{m}}$ motion is critically damped.

If $\dfrac{r}{2m} < \sqrt{\dfrac{k}{m}}$ damped oscillatory motion results.

If $r = 0$ undamped oscillations result.

Oscillations may be of four types

(a) free or natural or fundamental frequency
(b) forced
(c) resonant
(d) damped

Free or natural vibrations depend upon dimensions and nature of the material (elastic constants).

If a periodic force of **frequency other than** the material's natural frequency is applied then **forced vibrations** result. For example, if $y = y_0\sin\omega t$ was the equation of SHM of a particle and a periodic force $p\sin\omega_1 t$ is applied then $\omega \neq \omega_1$ then $y = y_0\sin\omega t + p\sin\omega_1 t$ represents forced vibration.

The resultant frequency is different from the natural frequency of oscillation.

Resonant oscillations are a certain type of forced vibrations. If frequency of applied force is equal to the natural frequency of the source

that is, $y = y_0\sin\omega t + p\sin\omega t = (y_0 + p)\sin\omega t$

that is, amplitude increases or intensity increases with resonance.

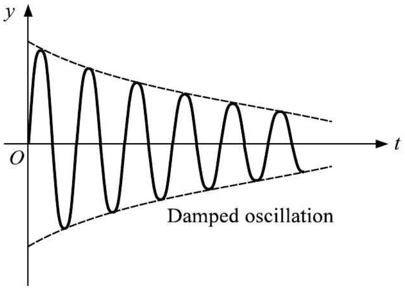


Fig. 16.25

In **damped oscillations** amplitude of the vibration falls with time as shown.

Amplitude at any instant is given by

$$y = y_0\,e^{-bt}$$

where y_0 is amplitude of first vibration and y is amplitude at time t and b is damping coefficient.

SHORT-CUTS AND POINTS TO NOTE

(i) Periodic motion may also be called isochronous. Complex periodic motion may be expressed as a series of simple harmonic motions using Fourier theorem.

(ii) The motion is SHM if $a = -\omega^2x$ or $F = -kx$ (linear SHM) or $\alpha = -\omega^2\theta$ where θ is angular displacement (angular SHM).

(iii) Solutions to the equation $\dfrac{d^2x}{dt^2} = -\omega^2x$ are

$x = x_0\sin\omega t$ (use this equation if motion begins from mean position that is, at $t = 0$, particle is at $x = 0$),

$x = x_0\cos\omega t$ (use this equation if at $t = 0$, the particle is at $x = x_0$ or extreme position) and

$$x = x_0\cos(\omega t \pm \phi) \quad \text{or} \quad x = x_0\sin(\omega t \pm \phi)$$

if the particle at $t = 0$ is somewhere in between mean and extreme position.

(iv) $v = \dfrac{dx}{dt} = x_0\omega\cos\omega t$ if particle at $t = 0$ is at mean position use this equation if time is given in the problem. (differentiate appropriate equation suitable in the problem)

$$v = \omega\sqrt{x_0^2 - x^2}$$

if displacement is given. Displacement is always measured from mean position. $v_{max} = x_0\omega$ (at mean position) and $v_{min} = 0$ at extreme position. Velocity–displacement graph is an ellipse.

(v) $a = -\omega^2x$ if displacement is given and $a = -\omega^2x_0\sin\omega t$ if time is given, we have assumed $x = x_0\sin\omega t$. Use appropriate equation otherwise.

$a_{max} = \omega^2x_0$ at extreme position. Acceleration–velocity graph is an ellipse.

$a_{min} = 0$ at mean position

(vi) $KE = \dfrac{1}{2}m\,\omega^2(x_0^2 - x^2)$.

Kinetic energy is maximum at mean position and minimum at extreme position. The frequency of kinetic energy is twice the frequency of SHM or natural frequency. Kinetic energy at extreme positions is zero.

(vii) $PE = \dfrac{1}{2}m\omega^2x^2$.

Potential energy is maximum at extreme positions and minimum at mean position. In certain cases minimum potential energy may not be zero. For instance, if a pendulum is at a height h then at mean position, potential energy is not equal to zero. The frequency of potential energy is twice the natural frequency of SHM.

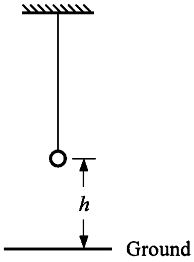


Fig. 16.26

(viii) If a tunnel is dug in the earth diametrically or along a chord then $T = 2\pi\sqrt{\frac{R}{g}}$ for a particle released along the tunnel.

However, if a ball is dropped from a height h in the tunnel the motion is no longer SHM but it becomes oscillatory. The time period of oscillations is

$$T = 4\sqrt{\frac{2h}{g}} + 4\sqrt{\frac{R}{g}}\sin^{-1}\left(\frac{h}{R+2h}\right).$$

(ix) If a charged particle q is released in a tunnel in a charged sphere of charge Q and radius R then

$$T = 2\pi\sqrt{\frac{4\pi\varepsilon_0 R^3 m}{Qq}}.$$

(x) If a dipole of moment p is suspended in a uniform electric field then time period of oscillation

$T = 2\pi\sqrt{\frac{I}{pE}}$ where I is moment of inertia.

(xi) If a magnetic dipole of moment M is oscillated in a uniform magnetic field of induction B then time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{MB}}.$$

(xii) For a simple pendulum

$T = 2\pi\sqrt{\frac{l}{g}}$ if θ is infinitesimally small.

For a long pendulum $T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}\right)$ if θ is finite and equal to θ_0.

If pendulum has bob of radius, r $T = 2\pi\sqrt{\frac{l^2 + 2/5r^2}{lg}}$

Time period of long pendulum $T = 2\pi\sqrt{\frac{1}{\left(\frac{1}{l}+\frac{1}{R}\right)g}}$. If $l \to \infty$ then $T = 2\pi\sqrt{\frac{R}{g}}$ = 84 min 36 s.

In a u tube $T = 2\pi\sqrt{\frac{l}{g}}$. If l is the length of the liquid column in one side $T = 2\pi\sqrt{\frac{l}{2g}}$, If l is considered as the length of total liquid column.

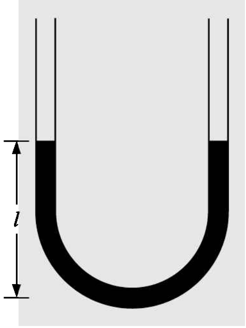


Fig. 16.27

If a cylinder/piston system is placed vertically $T = 2\pi\sqrt{\frac{l}{g}}$. (Fig. 16.28)

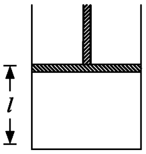


Fig. 16.28

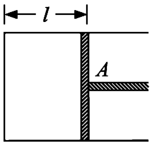


Fig. 16.29

If a cylinder/piston system has a gas of bulk modulus B then for Fig. 16.29

$$T = 2\pi\sqrt{\frac{MV_0}{A^2B}} = 2\pi\sqrt{\frac{mV_0}{A^2P}} = 2\pi\sqrt{\frac{mh}{AP}}$$

$$= 2\pi\sqrt{\frac{mV_0}{A^2\gamma P}} \text{ where } \gamma = \frac{C_P}{C_V}$$

where A is area of cross-section

$$T = 2\pi\sqrt{\frac{l}{g+a}}$$

if the pendulum is in a carriage moving up with an acceleration 'a'.

$T = 2\pi\sqrt{\frac{l}{g-a}}$ if the pendulum is in a carriage moving down with an acceleration 'a'.

$T = 2\pi\sqrt{\frac{l}{(g^2+a^2)^{1/2}}}$ if the acceleration of the carriage is horizontal

$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+\left(\frac{v^2}{R}\right)^2}}}$ if the carriage moves in a circle of radius R with velocity v, that is, it is in a merry-go-round.

$T = 2\pi\sqrt{\frac{R-r}{g}}$ if a ball of radius r oscillates in a concave mirror of radius R or in a hemispherical bowl.

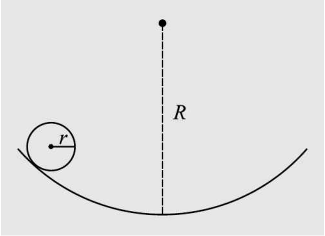


Fig. 16.30

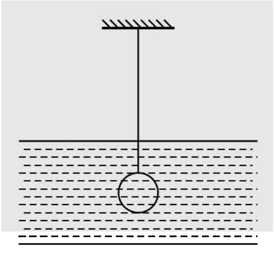


Fig. 16.31

$T = 2\pi\sqrt{\dfrac{l}{g\left(1-\dfrac{\sigma}{\rho}\right)}}$ if the bob of the pendulum is immersed in a non-viscous liquid of density σ.

$T = 2\pi\sqrt{\dfrac{l}{g-\dfrac{qE}{m}}}$ if electric force opposes gravitation.

$T = 2\pi\sqrt{\dfrac{l}{\sqrt{g^2+\dfrac{q^2E^2}{m^2}}}}$ if qE is perpendicular to g.

(xiii) $T = 2\pi\sqrt{\dfrac{M}{k_1+k_2}}$ in a case when springs are light and in parallel.

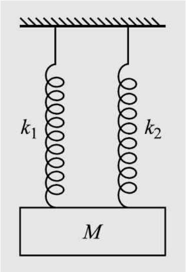


Fig. 16.32

$T = 2\pi\sqrt{\dfrac{M(k_1+k_2)}{k_1k_2}}$ when light springs are in series.

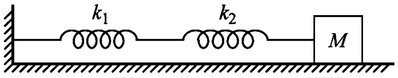


Fig. 16.33

If spring has mass M_s then

$$T = 2\pi\sqrt{\dfrac{M+\dfrac{M_S}{3}}{k}}$$

If spring has masses on two sides

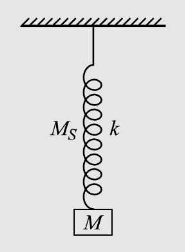


Fig. 16.34

$$T = 2\pi\sqrt{\dfrac{m_1m_2}{(m_1+m_2)k}}.$$

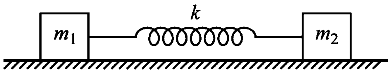


Fig. 16.35

(xiv) $f = \dfrac{1}{T}$ (linear frequency) and $\omega = \dfrac{2\pi}{T}$.

(xv)

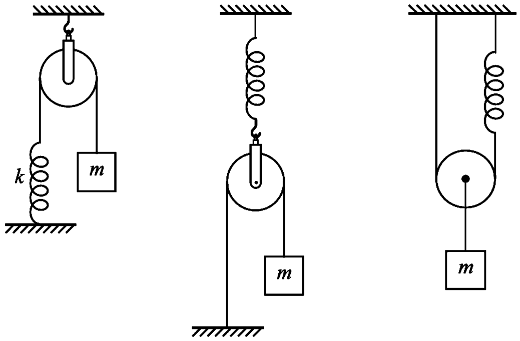


Fig. 16.36

$T = 2\pi\sqrt{\dfrac{m}{k}}$ $\quad$ $T = 4\pi\sqrt{\dfrac{m}{k}}$ $\quad$ $T = 2\pi\sqrt{\dfrac{m}{4k}}$

(xvi) The SHMs are added on the same direction like vectors (parallelogram law).

(xvii) When two SHMs are perpendicular their resultant is a straight line if phase shift between the two is zero or 180° otherwise ellipse is formed.

(xviii) If the frequency of SHM in x and y direction varies then $\dfrac{\omega_x}{\omega_y} = \dfrac{n_y}{n_x}$

(xix) If $b = \dfrac{r}{2m} > \omega\left(=\sqrt{\dfrac{k}{m}}\right)$ motion is non-oscillatory and over-damped.

If $\dfrac{r}{2m} = \sqrt{\dfrac{k}{m}}$ $(=\omega)$ motion is critically damped.

If $\dfrac{r}{2m} < \sqrt{\dfrac{k}{m}}$ damped oscillatory motion occurs.

If $r = 0$ undamped oscillations result.

(xx) Oscillations are of four types: free, forced, resonant and damped. Resonant oscillations are a certain type of forced oscillations where the frequencies of the source and force are equal.

In damped oscillations amplitude at any instant is obtained by $x_0' = x_0e^{-bt}$ where x_0 is amplitude of first oscillation.

(xxi) In compound or physical pendulum $T = 2\pi\sqrt{\dfrac{I}{mgl}}$.

(xxii) In torsional pendulum $T = 2\pi\sqrt{\dfrac{I}{k}}$.

(xxiii) In conical pendulum $T = 2\pi\sqrt{\dfrac{L\cos\theta}{g}} = 2\pi\sqrt{\dfrac{h}{g}}$.

(xxiv) Quality factor $Q = 2\pi\dfrac{\text{average energy stored}}{\text{energy loss in one cycle}}$

$= \omega_0\tau$ where τ is relaxation time $\dfrac{m}{b}$. (for energy)

Relaxation time for velocity $= \dfrac{2m}{b}$.

⚠ CAUTION

(i) Considering every vibratory motion as SHM

- Only those vibratory motions in which $a = -\omega^2 x$ or $\alpha = -\omega^2\theta$ are SHM. Remember in SHMs the amplitude is infinitesimally small ($\theta \to 0$) and force is directed towards the mean or equilibrium position.

(ii) Considering that decreasing amplitude with time in simple pendulum also decreases the time period.

- Time period remains unchanged.

(iii) Considering amplitude and span of SHM are identical.

- Span of SHM = 2 × amplitude of SHM

(iv) Considering ϕ as net phase in $x = x_0\sin(\omega t + \phi)$.

- ϕ is only initial phase at $t = 0$. Net phase at any instant is $(\omega t + \phi)$. Also note that ϕ should be small if the motion is to be SHM.

(v) Considering that the time periods in case of spring pendulum on different planes like the one on inclined plane, other on vertical and yet another on horizontal plane are different.

- Time period in a spring is independent of g.

(vi) Considering total length as length in compound or physical pendulums.

- The length is distance between COM and point of suspension.

(vii) Considering that at half the amplitude the time would be one-eighth of the time period.

- $t = \frac{T}{12}$ if the particle starts from mean position and $t = \frac{T}{6}$ if the particle starts from extreme position and it covers half the amplitude.

(viii) Considering that spring constant $k \propto l$.

- Spring constant is inversely related to length $\left(k \propto \frac{1}{l}\right)$.

(ix) Considering total energy as sum of kinetic energy and potential energy.

- Total energy is sum of KE + PE + resting energy.

  Also note that average KE = average PE = $\frac{1}{4}m\omega^2 x_0^2$ with respect to time.

  Average KE = 2 average PE with respect to position.

  Average KE = $\frac{1}{3}m\omega^2 x_0^2$; average PE = $\frac{1}{6}m\omega^2 x_0^2$.

(x) Considering that motion in v-tube is alike motion in u-tube.

- In the v-tube as shown in figure

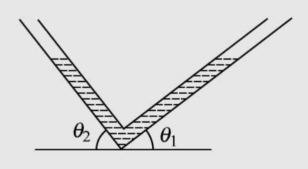


Fig. 16.37

$$T = 2\pi\sqrt{\frac{m}{Adg(\sin\theta_1 + \sin\theta_2)}}$$

where d is density of the liquid.

(xi) Considering that time period of simple pendulum depends upon mass or amplitude.

- $T = 2\pi\sqrt{\frac{l}{g}}$ clearly shows it depends upon length and acceleration

(xii) Considering mass of the spring does not affect time period.

- If mass of spring is M_S then net mass is $m + \frac{M_S}{3}$

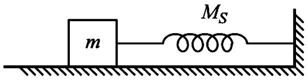


Fig. 16.38

(xiii) Considering we cannot find frequency of oscillation using energy conservation.

- In SHM total energy is conserved. Therefore $\frac{dE}{dt} = 0$

(xiv) Considering that only one restoring force will act on SHM

- There can be more than one restoring forces and total restoring force $\vec{F} = \vec{F_1} + \vec{F_2} + \ldots$.

Solved Problems

1. A particle is executing SHM of amplitude 10 cm. Its time period of oscillation is π seconds. The velocity of the particle when it is 2 cm from extreme position is

(a) 10 cms^{-1} (b) 12 cms^{-1}

(c) $16\sqrt{16}$ cms^{-1} (d) none of these

displacement = 8 cm.

Solution (b) $v = \omega\sqrt{x_0^2 - x^2} = \frac{2\pi}{\pi}\sqrt{10^2 - 8^2} = 12\ \text{cms}^{-1}$

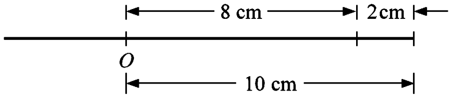


Fig. 16.39

2. The bob of a pendulum is a hollow cylinder. It has a hole at the bottom. The cylinder is filled with sand and set into oscillation. As the sand comes out

(a) the time period of oscillations remains unchanged
(b) the time period of oscillations increases with time
(c) the time period first increases and then decreases
(d) the time period first decreases and then increases

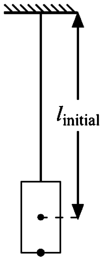


Fig. 16.40

Solution (c) Since the length of the pendulum is measured from *COM* as the sand leaves the pendulum, the length increases until it is half-filled. When the *COM* moves upwards length again decreases.

3. In the system shown a long uniform rod is attached at one end to a spring of spring constant k and the other end is hinged. It is displaced slightly and allowed to oscillate. The time period of oscillations is

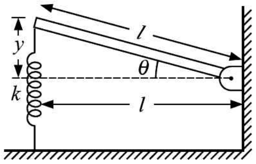


Fig. 16.41

(a) $2\pi\sqrt{\frac{M}{k}}$ (b) $2\pi\sqrt{\frac{M}{2k}}$

(c) $2\pi\sqrt{\frac{M}{3k}}$ (d) none of these

Solution (c) $\tau = kyl = kl^2\theta \quad \tau = \frac{Ml^2}{3}\alpha$

therefore, $\frac{Ml^2}{3}\alpha = -kl^2\theta$

or $\alpha = \frac{-3kl^2}{Ml^2}\theta \quad \omega^2 = \frac{3k}{M}$

$$T = 2\pi\sqrt{\frac{M}{3k}}.$$

4. A solid ball of mass m is allowed to fall from a height h to a pan suspended with a spring of spring constant k. Assume the ball does not rebound and pan is massless then amplitude of the oscillation is

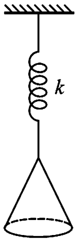


Fig. 16.42

(a) $\frac{mg}{k}$ (b) $\frac{mg}{k}+\left(\frac{2hk}{mg}\right)^{1/2}$

(c) $mg\sqrt{1+\frac{1+2hk}{mg}}$ (d) $\frac{mg}{k}\left[1+\sqrt{1+\frac{2hk}{mg}}\right]$

Solution (d) $mg(h+x) = \frac{1}{2}kx^2$

or $x^2 - \left(\frac{2mg}{k}\right)x - \frac{2mgh}{k} = 0$

or $x = \frac{mg}{k} + \frac{mg}{k}\sqrt{1+\frac{2hk}{mg}}$

5. A particle is executing SHM. Which of the graphs correctly represents this motion.

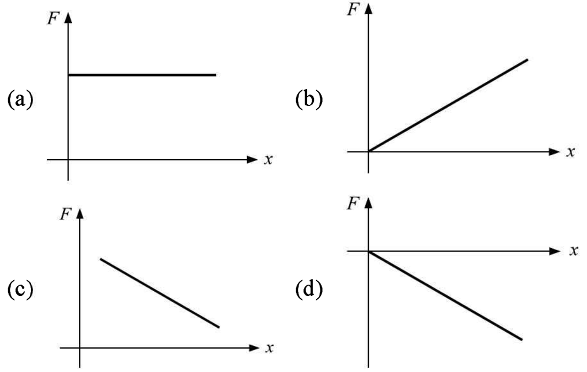


Fig. 16.43

Solution (d) Since $F = -kx$ represents SHM.

6. A hydrogen atom has a mass 1.68×10^{-24}g. When attached to a certain massive molecule, it oscillates as a classical oscillator with a frequency 10^{14} Hz and with an amplitude 10^{-9}cm. Calculate the force acting on the hydrogen atom.

(a) 6.63×10^{-9}N (b) 3.32×10^{-9}N
(c) 1.16×10^{-9}N (d) none of these

Solution (a) $T = 2\pi\sqrt{\frac{m}{k}}$ or $k = \frac{4\pi^2 m}{T^2} = 4\pi^2 mf^2$

$F = kx = 4\pi^2 mf^2 x$

$= 4\times\pi^2\times1.68\times10^{-27}\times(10^{14})^2\times10^{-11}$

$= 6.63\times10^{-9}$N

7. A common hydrometer has 1.00 and 0.80 specific gravity marks 4 cm apart. The time period of its oscillation when it floats in water and slightly displaced vertically is

(a) 0.4 s (b) 0.8 s
(c) 0.2 s (d) 4 m

Solution (b) from the principle of floatation

$\rho_{\text{water}}\, lAg = \rho_{\text{Liq}}(l+4)\,Ag$

or $1lAg = 0.8\,(l+4)\,Ag$

or $l = 16$ cm

$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{0.16}{10}} = 0.8$ s

8. A wrist watch keeps correct time at equator, it is taken at poles then

(a) it gains time (b) it loses time
(c) it keeps correct time (d) none of these

Solution (c) It is either quartz watch or spring watch. In both cases 'g' has no a role to play.

9. A particle moves on the x-axis according to the equation $x = A + B\sin\omega t$. The motion is SHM with amplitude

(a) A (b) $A + B$
(c) B (d) $\sqrt{A^2+B^2}$

Solution (c) determine the amplitude from $y = y_0 \sin \omega t$. that is, y_0 is amplitude. Hence in our problem amplitude is B.

10. The average energy in one time period of a particle executing SHM is

(a) $\frac{1}{2} m\omega^2 x_0^2$
(b) $\frac{1}{4} m\omega^2 x_0^2$
(c) $m\omega^2 x_0^2$
(d) zero

Solution (a)
$$E_{av} = KE_{av} + PE_{av}$$
$$= \frac{1}{4} m\omega^2 x_0^2 + \frac{1}{4} m\omega^2 x_0^2$$
$$= \frac{1}{2} m\omega^2 x_0^2$$

11. A particle executes SHM of frequency f. The frequency of its kinetic energy is

(a) f
(b) $\frac{f}{2}$
(c) $2f$
(d) zero

Solution (c)
$$KE = \frac{1}{2} m\omega^2 (x_0^2 - x^2)$$
$$= \frac{1}{2} m\omega^2 (x_0^2 - x_0^2 \sin^2 \omega t)$$
$$= \frac{1}{2} m\omega^2 x_0^2 \left[\frac{1+\cos 2\omega t}{2}\right]$$

Note the frequency is 2ω.

12. A pendulum clock keeps correct time at 30° latitude. If it is taken at poles

(a) it keeps correct time
(b) it gains time
(c) it loses time
(d) none of these

Solution (b) $T = 2\pi\sqrt{\frac{l}{g}}$, at poles g increases, and hence T decreases, therefore, it will gain time.

13. The displacement of the particle executing SHM in one complete period is

(a) x_0
(b) $2x_0$
(c) $4x_0$
(d) zero

Solution (d) as it reaches the original point.

14. A student says he has applied a force $F = -k\sqrt{x}$. He refuses to tell whether k is a constant or not. The particle executes SHM. Assume he has worked only in the $+x$ direction and no other force acts. Then

(a) as x increases k increases
(b) as x increases k decreases
(c) as x increases k remains constant
(d) the motion cannot be SHM

Solution (a) Match $F = -k_{\text{new}} x$ with $F = -k\sqrt{x}$.
k_{new} is a function of $\sqrt{x}$. Therefore k should increase as x increases.

15. A particle moves in a circular path with a uniform speed. Its motion is

(a) periodic
(b) oscillatory
(c) simple harmonic
(d) angular SHM

Solution (a)

16. Which of the following is not zero in an SHM?

(a) $\vec{F}\times\vec{a}$
(b) $\vec{v}\times\vec{r}$
(c) $\vec{a}\times\vec{r}$
(d) $\vec{F}\times\vec{r}$

Solution (b) because $\vec{v}$ and $\vec{r}$ have a phase shift of $\pi/2$. Others have a phase shift of 0 or π and $\sin 0 = 0 = \sin\pi$

17. A particle moves on the $x - y$ plane according to the equation $\vec{r} = (\hat{i} + 2\hat{j})\, A \cos\omega t$. The motion of the particle is not

(a) on a straight line
(b) on an ellipse
(c) simple harmonic
(d) periodic

Solution (b) because phase difference between x and y direction SHMs are zero.

Problems for Practice

1. Two simple pendulums of lengths 1 m and 16 m respectively are both given small displacements in the same direction at the same instant. They will again be in phase after the shorter pendulum has completed n oscillations where n is
(a) 1/3 (b) 1/4 (c) 4 (d) 5

2. A simple spring has length l and force constant k. It is cut into two springs of length l_1 and l_2 such that $l_1 = nl_2$ (n = an integer) the force constant of spring of length l_1 is
(a) $k(1 + n)$
(b) $k/(n + 1)$
(c) $k(1 + n)/n$
(d) k

3. The kinetic energy and potential energy of a particle executing SHM will be equal when displacement (amplitude = a) is
(a) $a\sqrt{2/3}$
(b) $a/2$
(c) $a/\sqrt{2}$
(d) $a\sqrt{2}$

4. If a particle performs SHM with a frequency v, then its K.E. will oscillate with a frequency
(a) zero (b) $v/2$ (c) v (d) $2v$

5. The displacement y in centimetres is given in terms of time t s by the equation:
$y = 3\sin 314t + 4\cos 314t$,
then the amplitude of SHM is
(a) 3 cm (b) 4 cm (c) 5 cm (d) 7 cm

6. A body of mass m is suspended from rubber and with force constant k. The maximum distance over which the body can be pulled down for the body's oscillation to remain harmonic is
(a) $mg/2k$
(b) $2k/mg$
(c) $2mg/k$
(d) mg/k

7. Length of second's pendulum is decreased by 1% then the gain or loss in time per day will be nearly
(a) 0.44 s (b) 4.4 s (c) 44 s (d) 440 s

8. Two pendulums oscillate with a constant phase difference of 90°, and same amplitude. The maximum velocity of one is v. The maximum velocity of other will be
(a) $\sqrt{2v}$ (b) $v\sqrt{2}$ (c) v (d) $2v$

9. What is the length of second's pendulum where g is $980\,\text{cms}^{-2}$?
(a) 102.4 cm (b) 99.2 cm
(c) 88 cm (d) 78 cm

10. The displacement of a particle in SHM in one time period is
(a) zero (b) a (c) $2a$ (d) $4a$

11. For a particle executing SHM having amplitude 'a' the speed of the particle is one half of its maximum speed when its displacement from the mean position is
(a) $a/2$ (b) a (c) $a\frac{\sqrt{3}}{2}$ (d) $2a$

12. A spring pendulum is suspended from the top of a car. If the car accelerates on a horizontal road, the frequency of oscillation will
(a) be zero (b) remain same
(c) increase (d) decrease

13. The length of the seconds pendulum on the surface of earth is 1 m. Its length on the surface of moon, where g is one-sixth the value of g on the surface of earth is
(a) 1/36 m (b) 1/6 m
(c) 6 m (d) 36 m

14. The phase angle between the projections of uniform circular motion on two mutually perpendicular diameter is
(a) π (b) $3\pi/4$ (c) $\pi/2$ (d) zero

15. When particle oscillates simple harmonically its potential energy varies periodically. If frequency of the particle is n, the frequency of the potential energy is
(a) $n/2$ (b) n (c) $2n$ (d) $4n$

16. A simple pendulum performs SHM about $x = 0$ with an amplitude A and time period T. The speed of the pendulum at $x = A/2$ will be
(a) $3\pi^2 A/T$ (b) $\pi A\frac{\sqrt{3}}{T}$
(c) $\pi\frac{A}{T}$ (d) $\pi A\frac{\sqrt{3}}{2T}$

17. The potential energy of a particle with displacement x is $U(x)$. The motion is simple harmonic. If k is a positive constant then
(a) $U = kx$ (b) $U = k$
(c) $U = -kx^2/2$ (d) $U = Kx^2$

18. A bob is suspended by a string of length l. The minimum horizontal velocity imparted to the ball for reaching it to the height of suspension is
(a) $\sqrt{l/g}$ (b) $\sqrt{2gl}$ (c) $\sqrt{g/l}$ (d) $2\sqrt{gl}$

19. The angle between the instantaneous velocity and the acceleration of a particle executing SHM is
(a) zero or π (b) $\pi/2$
(c) zero (d) π

20. The frequency of SHM is 100 Hz. Its time period is
(a) 0.01 s (b) 0.1 s (c) 1 s (d) 100 s

21. The displacement y of a particle executing periodic motion is given by
$y = 4\cos^2(t/2)\sin(100t)$.
This expression may be considered to be a result of the super position of
(a) 2 (b) 3 (c) 4 (d) 5

22. Two bodies M and N of equal masses are suspended from two separate massless spring of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal the ratio of the amplitude of M to that of N is
(a) k_2/k_1 (b) $\sqrt{k_2/k_1}$
(c) k_1/k_2 (d) $\sqrt{k_1/k_2}$

23. If E is the total energy of a particle executing SHM and 'A' is the amplitude of the vibratory motion, the E and 'A' are related as
(a) $E \propto A^2$ (b) $E \propto 1/A^2$
(c) $E \propto A$ (d) $E \propto 1/A$

24. A particle executes SHM with a time period of 2 s and amplitude 5 cm. Maximum magnitude of its velocity is
(a) $10\pi\,\text{cm s}^{-1}$ (b) $20\pi\,\text{cm s}^{-1}$
(c) $2.5\pi\,\text{cm s}^{-1}$ (d) $5\pi\,\text{cm s}^{-1}$

25. The dimensional formula for amplitude of SHM is
(a) MLT (b) $M^\circ L^\circ T^\circ$
(c) $M^\circ L T^\circ$ (d) MLT°

26. A simple pendulum is attached to the roof of a lift. Its time period of oscillation, when the lift is stationary is 0.5 s. Its frequency of oscillation when the lift falls freely will be
(a) infinite (b) zero
(c) 2 Hz (d) 20 Hz

27. A spring of force constant k is cut into two equal parts, which are then joined parallel to each other. The force constant of the combination will be
(a) $4k$ (b) $2k$ (c) k (d) $k/2$

28. A particle is placed on a plank undergoing SHM of frequency $3/\pi$ Hz. The maximum amplitude of the plank so that the particle does not leave the plank will be
(a) $\frac{5}{18}$ m (b) $\frac{5}{8}$ m (c) $\frac{2}{9}$ m (d) none of these

29. The intensities of two notes are equal. If frequency of one note is one-fourth that of the other then the ratio of their amplitudes is
(a) 16 (b) 4 (c) 2 (d) 1

30. A person wearing a wrist watch that keeps correct time at the equator goes to N-pole. His watch will
(a) keep correct time (b) gain time
(c) loose time (d) cannot say

31. Which of the following is not essential for the free oscillation of a mass attached to a spring
(a) Elasticity (b) Gravity
(c) Inertia (d) Restoring force

32. A pendulum suspended from the ceiling of the train beats seconds when the train is at rest. What will be the time period of the pendulum if the train accelerates at $10\,\text{ms}^{-2}$?. Take $g = 10\,\text{ms}^{-2}$
(a) $(2/\sqrt{2})$ s (b) 2 s
(c) $2\sqrt{2}$ s (d) None of these

33. Which of the following quantities connected with SHM do not vary periodically?
(a) Total energy (b) Velocity
(c) Displacement (d) Acceleration

34. A mass m is suspended from a spring of force constant k. The angular frequency of oscillation of the spring will be
(a) k/m (b) $\sqrt{m/k}$
(c) m/k (d) $\sqrt{k/m}$

35. What is the number of degrees of freedom of an oscillating simple pendulum?
(a) more than three (b) 3
(c) 2 (d) 1

36. The graph between restoring force and time in case of SHM is a
(a) parabola (b) sine curve
(c) straight line (d) circle

37. In SHM, which of the following quantities does not vary as per nature of the sine curve
(a) acceleration (b) time period
(c) displacement (d) velocity

38. Two particles P and Q describe SHM of same amplitude a and frequency v along the same straight line. The maximum distance between the two particles is $a\sqrt{2}$. The initial phase difference between the particle is
(a) $\pi/3$ (b) $\pi/2$ (c) $\pi/6$ (d) zero

39. A particle is moving on a circle with uniform speed. Its motion is
(a) aperiodic motion (b) periodic and SHM
(c) periodic but not SHM
(d) none of these

40. Assuming the surfaces to be smooth The time period of oscillation of mass M when slightly displaced is

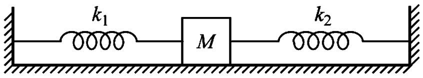


Fig. 16.44

(a) $2\pi\sqrt{\dfrac{k_1+k_2}{m}}$ (b) $2\pi\sqrt{\dfrac{k_1-k_2}{m(k_1+k_2)}}$
(c) $2\pi\sqrt{\dfrac{m}{k_1+k_2}}$ (d) $2\pi\sqrt{\dfrac{m}{k_1-k_2}}$

41. The mass and radius of a planet are double that of the earth. The time period of a pendulum on that planet which is a seconds pendulum on earth, will be
(a) $\dfrac{1}{\sqrt{2}}$ s (b) 0.5 s
(c) $2\sqrt{2}$ s (d) 2 s

42. The work done by a simple pendulum in one completed oscillation is
(a) equal to Ek (b) equal to U
(c) zero (d) equal to $U + Ek$

43. A particle is moving such that its acceleration is represented by the equation $a = -bx$, where x is its displacement from mean position and b is a constant. Its time period will be
(a) $2\pi/\sqrt{b}$ (b) $2\pi/b$
(c) $2\pi\sqrt{b}$ (d) $2\sqrt{\dfrac{\pi}{b}}$

44. The displacement of a particle executing SHM is half its amplitude. The fraction of its kinetic energy will be
(a) 2/3 (b) 3/4
(c) 1/3 (d) 1/2

45. The phase difference between the velocity and displacement of a particle executing SHM is
(a) $\pi/2$ radian (b) π radian
(c) 2π radian (d) zero

46. The ratio of the maximum velocity and maximum displacement of a particle executing SHM isequal to
(a) n (b) g
(c) T (d) ω

47. The physical quantity conserved in simple harmonic motion is
(a) time period (b) total energy
(c) potential energy (d) kinetic energy

48. The correct graph between the restoring force and displacement of a harmonic oscillator is

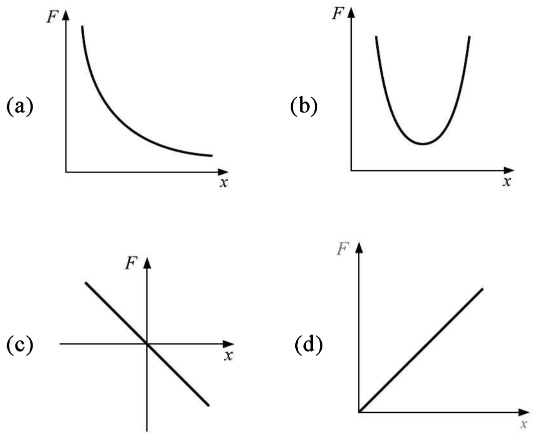


Fig. 16.45

49. The bob of a simple pendulum consists of a sphere filled with mercury. If a small quantity of mercury is taken out, then the period of pendulum will
(a) become erroneous (b) decrease
(c) increase (d) remain unchanged

50. The time period of a second's pendulum is 2 s. The mass of the spherical bob is 50 g and is empty. If it is replaced by another solid bob of same radius but mass 100 g then its time period will be
(a) 8 s (b) 4 s
(c) 2 s (d) 1 s

51. The amplitude and time period of simple harmonic oscillator are a and T respectively. The time taken by it in displacing from $x = 0$ to $x = a/2$ will be
(a) T (b) $T/2$ (c) $T/4$ (d) $T/6$

52. The time period of the hour hand of a watch is
(a) 24 h (b) 12 h (c) 1 h (d) 1 min.

53. A swing consists of a rod of mass M and length l suspended with two massless strings of length l each. The time period of the swing is

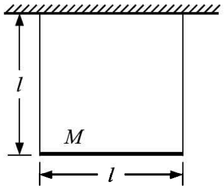


Fig. 16.46

(a) $2\pi\sqrt{\frac{l}{g}}$ (b) $2\pi\sqrt{\frac{13l}{12g}}$

(c) $2\pi\sqrt{\frac{l}{2g}}$ (d) $2\pi\sqrt{\frac{2l}{g}}$

54. A disc of mass M and radius R is suspended from one point on its periphery as shown below. It is given a little displacement to oscillate. The angular frequency of oscillations is

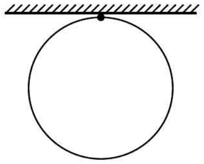


Fig. 16.47

(a) $\sqrt{\frac{3g}{2R}}$ (b) $\sqrt{\frac{g}{R}}$

(c) $\sqrt{\frac{2g}{3R}}$ (d) $\sqrt{\frac{2R}{3g}}$

55. A simple pendulum is released when $\theta = \pi/6$. The time period of oscillation is

(a) $2\pi\sqrt{\frac{l}{g}}$ (b) $2\pi\sqrt{\frac{l}{g}}\left(\frac{293}{288}\right)$

(c) $2\pi\sqrt{\frac{l}{g}}\left(\frac{288}{293}\right)$ (d) none of these

56. A mass m is suspended from a spring. Its frequency of oscillation is f. The spring is cut into two equal halves and the same mass is suspended from one of the two pieces of the spring. The frequency of oscillation of the mass will be

(a) $\sqrt{2}f$ (b) $2f$ (c) $f/2$ (d) f

57. Which of the following characteristics must remain constant for undamped oscillations of the particle.

(a) acceleration (b) phase

(c) amplitude (d) velocity

58. Identical springs of spring constant K are connected in series and parallel combinations. A mass m is suspended from them. The ratio of their frequencies of vertical oscillations will be

(a) 1 : 4 (b) 1 : 2 (c) 4 : 1 (d) 2 : 1

59. A disc of *MOI*, I is suspended through a wire of torsional rigidity G. The disc undergoes torsional oscillation. The time period of oscillation is

(a) $2\pi\sqrt{\frac{l}{g}}$ (b) $2\pi\sqrt{\frac{G}{I}}$

(c) $2\pi\sqrt{\frac{I}{G}}$ (d) $2\pi\sqrt{\frac{I}{mgl}}$

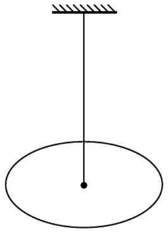


Fig. 16.48

60. The maximum displacement of a particle executing SHM from its mean position is 2 cm and its time period is 1 s. The equation of its displacement will be

(a) $x = 2\sin 4\pi t$ (b) $x = 2\sin 2\pi t$

(c) $x = \sin 2\pi t$ (d) $x = 4\sin 2\pi t$

61. The maximum value of the time period of a simple pendulum is

(a) 84.6 min (b) 1 year

(c) 1 day (d) 12 h

62. If a simple pendulum of length l has maximum angular displacement θ, then the maximum kinetic energy of the bob of mass m is

(a) $\frac{1}{2}m\,(l/g)$ (b) $mgl\,(1 - \cos\theta)$

(c) $(mgl\sin\theta)/2$ (d) $mg/2l$

63. A simple pendulum is carried at a depth of 1 km below sea level. It becomes

(a) slow (b) unchanged

(c) fast (d) none of these

64. The ratio of the kinetic energy E_k and potential energy of a particle executing SHM, at a distance x from mean position will be

(a) $\frac{x^2}{x^2 - a^2}$ (b) $\frac{x^2 - a^2}{x^2}$

(c) $\frac{x^2}{a^2 - x^2}$ (d) $\frac{a^2 - x^2}{x^2}$

65. Some gas is filled in a cylinder with a piston. The mass of the piston is M and it can easily slide. The axis of the cylinder is kept horizontal. If the piston executes SHM on displacing it from the mean position, then time period will be

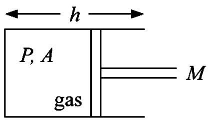


Fig. 16.49

(a) $T = 2\pi\sqrt{\frac{Mh}{PA}}$ (b) $T = 2\pi.\sqrt{\frac{M}{PAh}}$

(c) $T = 2\pi\sqrt{MAPh}$ (d) $T = 2\pi\sqrt{\frac{MA}{Ph}}$

66. The curve between the acceleration and velocity of a particle executing SHM is a an

(a) ellipse (b) circle

(c) parabola (d) straight line

67. On increasing the length of a second's pendulum by Z%, its time period will

(a) increase by 2Z%

(b) decrease by Z%

(c) decrease by 2Z%

(d) increase by 0.5Z%

68. A clock purchased in 1942 loses 1 min in 1 day. Its time period must have become

(a) extremely small

(b) extremely large

(c) shorter

(d) longer

69. The displacement of a particle executing SHM at any instant t is $x = 0.01\ \sin 100(t + 0.05)$ then its time period will be

(a) 0.06 s (b) 0.2 s
(c) 0.1 s (d) 0.02 s

70. Three identical springs, each of spring constant k are connected to a mass m as shown in figure. Then time period of oscillation if the mass M is displaced by x along one of the springs is

(a) $T = 2\pi\sqrt{\frac{2m}{k}}$ (b) $T = 2\pi\sqrt{\frac{m}{2k}}$

(c) $T = 2\pi\sqrt{\frac{m}{k}}$ (d) none of these

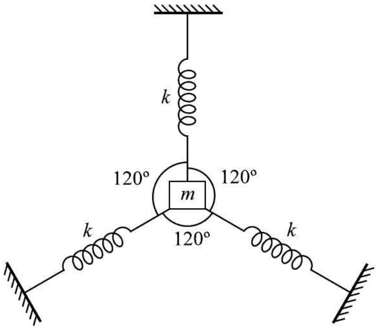


Fig. 16.50

ANSWERS

1. (c)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (c)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (a)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (a)
21. (b)	**22.** (b)	**23.** (a)	**24.** (d)	**25.** (c)	**26.** (b)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (a)
31. (b)	**32.** (d)	**33.** (a)	**34.** (d)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (c)
41. (c)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (a)	**46.** (d)	**47.** (b)	**48.** (c)	**49.** (c)	**50.** (c)
51. (d)	**52.** (b)	**53.** (a)	**54.** (c)	**55.** (b)	**56.** (a)	**57.** (c)	**58.** (b)	**59.** (c)	**60.** (b)
61. (a)	**62.** (b)	**63.** (a)	**64.** (d)	**65.** (a)	**66.** (a)	**67.** (d)	**68.** (d)	**69.** (a)	**70.** (b)

Wave Motion and Waves in a String

17

BRIEF REVIEW OF THE CONCEPTS

A wave is the disturbance which propagates energy from one place to the other without transporting matter. It is spread over a region without clear-cut boundaries. It is not localized.

Diffraction differentiates between particle nature and wave nature. It is a convincing proof of wave nature.

Mechanical waves require a medium to propagate while other waves like electromagnetic and matter waves do not require any medium to propagate. Therefore, mechanical waves are also called **elastic waves.**

Shock waves are a kind of pulse propagation and are mathematically expressed as

$$y = \frac{a}{b + (x \mp vt)^2}.$$

Shock waves are produced during earthquakes volcanic eruptions, bomb blasts and during a sonic boom.

$$y = y_0(\omega t - kx)$$

is the wave propagating along positive x direction.

Plane progressive wave is given by

$$y = y_0 \sin(\omega t - kx)$$

where k is called propagation constant or wave number, ω is called angular frequency, y_0 amplitude and y instantaneous displacement. Such a wave is called a displacement wave.

$k = \frac{2\pi}{\lambda}$ where λ is wavelength, $(\omega t - kx)$ is the **phase** at any instant. When path difference $\Delta x = \lambda$, then phase shift $\Delta\phi = 2\pi$. In general $k\Delta x = \Delta\phi$.

A wave can have two types of velocities.

Wave velocity or phase velocity and group velocity or particle velocity.

Wave velocity $v = \frac{dx}{dt} = \frac{\omega}{k} = f\lambda$

In a dispersive medium wave travels with a group velocity

$$v_{\text{group}} = v - \lambda \frac{dv}{d\lambda}$$

This is the case for electromagnetic waves. For example, in water and glass and so on different wavelength travel with different velocities.

Particle velocity $v_{\text{Particle}} = \frac{dy}{dt} = -\frac{dx}{dt} \times \frac{dy}{dx} = -v(\text{slope})$

= –wave velocity × slope at that point

A plane progressive wave mechanical or electromagnetic may be expressed in one of the following forms

$$y = y_0 \sin(\omega t - kx)$$

$$y = y_0 \sin \omega \left(t - \frac{x}{v}\right)$$

$$y = y_0 \sin 2\pi \left(ft - \frac{x}{\lambda}\right)$$

$$y = y_0 \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$y = y_0 \sin k(vt - x)$$

- If the sign between t and x is negative then the wave propagates in positive x direction.
- When a wave passes from one medium to the other its frequency does not change.

Velocity of wave in a string $v = \sqrt{\frac{T}{\mu}}$

where T is tension in the string and

μ is mass/length of the string.

Frequency of wave in a string

If vibrating in p-loops (transverse),

fundamental frequency $f = \frac{p}{2l}\sqrt{\frac{T}{\mu}}$

Longitudinal waves in a string have p–loops $f = \frac{p}{l}\sqrt{\frac{T}{\mu}}$.

Average power transmitted along the string

$$P_{\text{average}} = \frac{1}{2}\frac{\omega^2 x_0^2 F}{v} = 2\pi^2 \mu x_0^2 f^2 v$$

Average intensity

$$I_{\text{average}} = 2\pi^2 \rho x_0^2 f^2 v$$

Interference of waves in the same direction

If $y_1 = y_{01}\sin(kx - \omega t)$ and

$y_2 = y_{02}\sin(\omega t - kx + \phi)$

then $y = y_1 + y_2$ and $y = y_0 \sin(\omega t - kx + \delta)$

Apply vector laws

$$y_0 = \sqrt{y_{01}^2 + y_{02}^2 + 2y_{01}y_{02}\cos\phi}$$

$$\tan\delta = \frac{y_{02}\sin\phi}{y_{01} + y_{02}\cos\phi}.$$

y_0 will be maximum when $\cos\phi = 1$ or $\phi = 0$ or $2n\pi$ where $n = 0, 1, 2$ ...

y_0 will be minimum when $\cos\phi = -1$

$$\cos\phi = (2n + 1)\pi$$

when y_0 is maximum the **constructive interference** is said to take place as sound intensity will be maximum and the path difference is $n\lambda$.

$$y_{0(\max)} = y_{01} + y_{02};\ y_{0\min} = y_{01} - y_{02}$$

y_0 is minimum when phase difference is an odd multiple of π or path difference is an odd multiple of half the wave length.

$$\frac{I_{\max}}{I_{\min}} = \frac{(y_{01} + y_{02})^2}{(y_{01} - y_{02})^2}$$

when intensity is minimum **destructive interference** occurs.

If the reflection occurs from a denser medium, it introduces a phase shift of π radian or 180° between incident and reflected wave

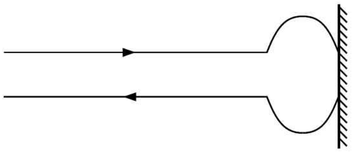


Fig. 17.1

Note that if the wave is travelling in a string which is a combination of two mediums then, if the wave travels from the lighter to the denser string it is reflected out of phase (or 180°) from the junction but is transmited in phase.

If a wave propagates from denser to lighter string phase is shifted neither for the reflected nor for the transmitted wave as illustrated in figure.

Standing waves or stationary waves are produced when two waves having the same amplitude and same frequency superpose while travelling in opposite directions.

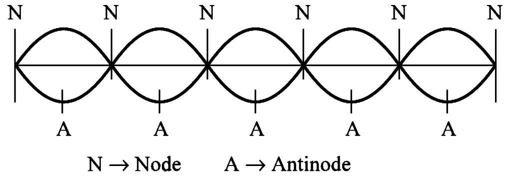


Fig. 17.2

that is, $y_1 = y_0\sin(kx - \omega t)$

and $y_2 = y_0\sin(kx + \omega t)$ superpose.

Thus, $y = y_1 + y_2 = (2y_0\sin kx)\cos\omega t.$

Normally a wave and its reflected wave superpose to produce standing wave. The points of maximum amplitude are called antinodes and minimum amplitude are called nodes. Separation between two consecutive nodes or antinodes is $\lambda/2$. Separation between a node and an antinode is $\lambda/4$. Amplitude of standing wave = $2y_0$, that is, twice the amplitude of the original travelling wave.

***Notes*:**

1. In a travelling wave the disturbance produced in a region propagates with a definite velocity and in a *standing wave* it is confined to the region where it is produced.
2. In a travelling wave the motion of all the particles is similar. In a *standing wave* different particles move with different amplitude.
3. In a *standing wave particles at node* always remain at *rest*. In a travelling wave there is no such particle which remains at rest always.
4. In a *standing* wave all the *particles cross* their *mean positions together*. In a travelling wave there is no instant when all the particles are at the mean position.
5. In a *standing wave* all the *particles* move in *phase*. In a travelling wave the phases of neighbouring particles are always different.
6. In a travelling wave energy from one region of space is transferred to the other region of space. In a *standing wave* the energy is always *confined* in that region.

Standing wave ratio (SWR) $\dfrac{y_{0\max}}{y_{0\min}} = \dfrac{y_{01} + y_{02}}{y_{01} - y_{02}}$

For a progressive wave, SWR = 1 (as $y_{02} = 0$). For standing wave SWR = ∞

In standing waves $\dfrac{d^2y}{dx^2} = \dfrac{1}{v^2}\dfrac{d^2y}{dt^2}$.

The amplitude of the wave $y_0 = 2y_0\sin kx$ is a periodic function of position (and not of time as in beats).

If a string vibrates in a single loop, the mode is fundamental. There are two nodes and one antinode and frequency is f. If there are n loops we say the string is vibrating with nth harmonic or $(n - 1)$th overtone and there will be n antinodes and $n + 1$ nodes. Frequency in this case will be nf.

That is, in a string fixed at both ends all integral multiples of fundamental frequency are allowed and $f = \dfrac{n}{2l}\sqrt{\dfrac{T}{\mu}}$.

Vibrations of Strings Fixed at One End

Note that at the open end an antinode will be formed and at the fixed end a node will be formed.

- Only odd multiple of frequencies are allowed.
- n^{th} harmonic = $(n - 1)$th overtone.
- Fundamental frequency is also called note or first harmonic.
- Octave is the tone whose frequency is double the fundamental frequency

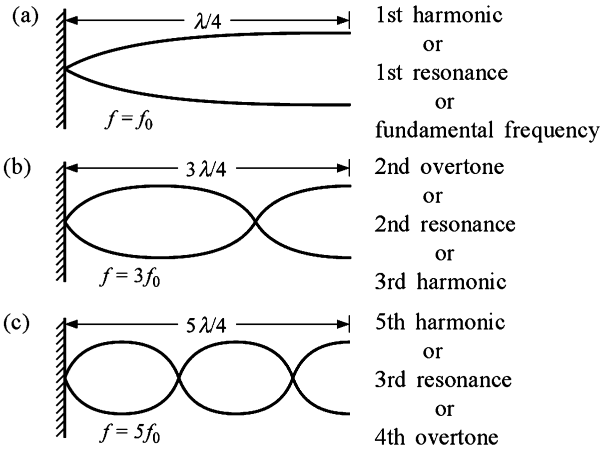
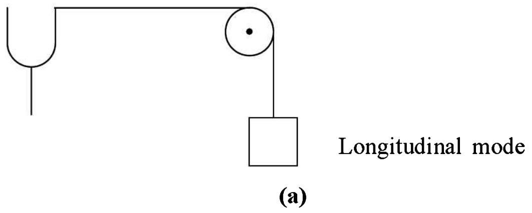


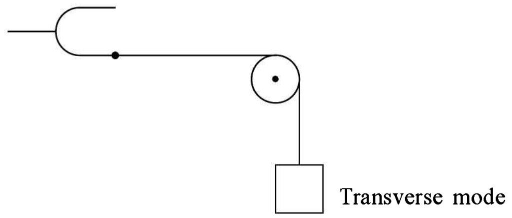
Fig. 17.3

Melde's experiment

$f = \dfrac{p}{l}\sqrt{\dfrac{T}{\mu}}$; p = number of loops



$f = \frac{p}{2l}\sqrt{\frac{T}{\mu}}$ p = number of loops



(b)

Fig. 17.4

$$p_{\text{longitudinal}} = \frac{p_{\text{transverse}}}{2}$$

Velocity of a wave on the surface of a liquid is

$$v_s = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}}$$

where T is surface tension and ρ is density of the liquid.

Velocity of Torsional waves in a rod is

$$V_T = \sqrt{\frac{\eta}{\rho}}$$

SHORT-CUTS AND POINTS TO NOTE

(i) To produce longitudinal waves the medium should possess bulk modulus of elasticity.

(ii) To produce transverse wave the medium must possess shear modulus of elasticity.

(iii) $y = y_0 \sin(\omega t - kx + \phi)$ is the equation of a plane progressive wave in positive x direction. ϕ is initial phase angle or epoch. Normally we write a simplified equation, $y = y_0 \sin(\omega t - kx)$. A more general equation of wave is $y = y_0 e^{\pm i(\omega t - kx)}$.

(iv) $k = \frac{2\pi}{\lambda}$ is propagation vector or wave number.

(v) Wave velocity is $\frac{dx}{dt} = \frac{\omega}{k} = f\lambda$ and may be called phase velocity. In a dispersive medium waves travel with group velocity v_g given by $v_g = v - \frac{\lambda dv}{d\lambda}$. In dispersive mediums waves of different wavelengths travel with different velocity if they are electromagnetic in nature.

(vi) Particle velocity $= \frac{dy}{dt} = \frac{-dx}{dt} \times \frac{dy}{dx} = -v_{\text{wave}}(\text{slope})$

Maximum particle velocity $= y_0\omega$

(vii) Frequency of the wave does not vary when a wave passes from one medium to the other.

(viii) Power (average) transmitted along the string is

$$P_{av} = \frac{\omega^2 x_0^2 F}{2v} = 2\pi^2 \mu x_0^2 f^2 v$$

Intensity $= 2\pi^2 \rho x_0^2 f^2 v$ where μ = mass/length and ρ = density of the medium.

(ix) Interference of waves travelling in the same direction is obtained using vector laws

$$y_0 = \sqrt{y_{01}^2 + y_{02}^2 + 2y_{01}y_{02}\cos\phi}$$

and $\tan\delta = \frac{y_{02}\sin\phi}{y_{01} + y_{02}\cos\phi}$

if $y_1 = y_{01}\sin(kx - \omega t)$

and $y_2 = y_{02}\sin(\omega t - kx + \phi)$ interfere.

(x) $\frac{I_{\max}}{I_{\min}} = \frac{(y_{01} + y_{02})^2}{(y_{01} - y_{02})^2}$.

Maximum intensity is obtained when phase shift is zero or path difference is an integral multiple of wavelength. Minimum intensity or destructive interference occurs when phase shift is an odd integral multiple of π radian or odd integral multiple of half the wavelength.

(xi) Reflection from a denser medium causes a phase shift of 180° and reflection from rarer or lighter medium occurs without change of phase in the string.

(xii) Standing waves are of two types – longitudinal standing waves and transverse standing waves. In strings both can occur. In pipes only longitudinal standing waves are seen.

(xiii) Standing waves result when two waves having same amplitude and same frequency travelling in opposite directions superpose.

(xiv) $y = 2y_0 \sin kx \cos \omega t$ represents a stationary wave in strings. Note that $(2y_0 \sin kx)$ shows amplitude and is a function of distance.

At certain places, amplitude is maximum (antinodes) and at other places amplitude is zero (nodes). Separation between consecutive nodes or antinodes is $\lambda/2$. Distance between a node and consecutive antinode is $\lambda/4$.

(xv) $\frac{v^2 d^2y}{dx^2} = \frac{d^2y}{dt^2}$ is the differential form of wave equation.

(xvi) $y = 2y_0 \cos kx \sin \omega t$ represents stationary wave equation which fits in closed or open pipes.

(xvii) $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$ is used for fundamental mode in strings vibrating with transverse stationary waves.

(xviii) In a sonometer, transverse stationary waves are produced.

$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$ can be used to find frequency if the string is vibrating in n loops.

(xix) In Melde's experiment, if the transverse mode is used $f = \frac{p}{2l}\sqrt{\frac{T}{\mu}}$ and $f = \frac{p}{l}\sqrt{\frac{T}{\mu}}$ for longitudinal mode.

(xx) Only transverse waves and not longitudinal waves can be polarized.

(xxi) Sound waves being pressure waves are longitudinal.

(xxii) If the disturbance produced is always along a fixed direction. The wave is linearly polarised in that direction. For example $y = y_0 \sin(\omega t - kx)$ is linearly polarised in y direction. Linearly polarized waves are also called plane polarised waves.

(xxiii) If each particle of a string moves in a small circle as the wave passes through it then the wave is circularly polarised. If each particle moves in an ellipse it is elliptically polarized and if each particle is randomly displaced, it is unpolarized.

(xxiv) A circularly polarised or unpolarized wave passing through a slit does not show change in intensity as the slit is rotated in its plane. But the transmitted wave becomes linearly polarised in the direction parallel to the slit.

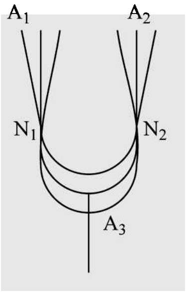


Fig. 17.5

(xxv) Number of nodes and antinodes in a tuning fork when vibrating in fundamental mode as illustrated in figure are two and three respectively and A_3 is longitudinal antinode while A_1 and A_2 are transverse antinodes. If the tuning fork vibrates in *n*th harmonic then
number of nodes = $2n$
number of antinodes = $(2n + 1)$

CAUTION

(i) Considering that all waves require a medium.
- Only mechanical waves require an elastic medium

(ii) Considering that both longitudinal and transverse waves can be produced in any medium.
- Transverse waves require medium that shall possess shear modulus of elasticity. Therefore transverse waves cannot be produced in gases.

(iii) Considering that waves could be only longitudinal or transverse.
- Waves could be combination of both. For example ripples, in water, seismic waves during earthquakes.

(iv) Considering that medium is also transported along with energy during propagation of a wave.
- Only energy is transported and not the medium during propagation of the wave.

(v) Considering only functions like $y_0\sin(\omega t - kx)$ or $y_0\cos(\omega t - kx)$ can represent a wave.
- Functions like
$$y = y_0\sin(\omega t - kx),\ y = y_0\cos(\omega t - kx)$$
and $y = y_0 + \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t$
also represent waves which are complex periodic waves. However, plane progressive harmonic waves can be represented as
$$y = y_{01}\sin(\omega t - kx) + y_{02}\sin(\omega t - kx)$$
$$y = y_0\sin(\omega t - kx) \text{ or } y = y_0\cos(\omega t - kx).$$

(vi) Confusion about the formula to be applied in strings to calculate frequency
- Use $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$ when vibrating in fundamental transverse mode and
$f = \frac{1}{l}\sqrt{\frac{T}{\mu}}$ when vibrating in fundamental longitudinal mode.

(vii) Confusion about the modes of production of transverse and longitudinal wave.

(viii) Considering that velocity of a wave is same at every point in the string.
- Velocity of wave in vertical strings is different at different points.

Solved Problems

1. The displacement wave in a string is $y = (3\text{cm})\sin 6.28(0.5x - 50t)$ where x is in centimetres and t in seconds. The velocity and wavelength of the wave is
(a) 2 cm, 100 cms^{-1} (b) 10 cm, 50 cms^{-1}
(c) 20 cm, 2 ms^{-1} (d) 2 m, 100 ms^{-1}

Solution (a) $k = \frac{2\pi}{\lambda}$

or $\lambda = \frac{2\pi}{k} = \frac{6.28}{6.28(0.5)} = 2\text{cm}$

$$v = \frac{\omega}{k} = \frac{50\times 6.28}{0.5\times 6.28} = 100\text{cms}^{-1}.$$

2. The equation of a wave is $10\sin(6.28x - 314t)$ where x is in centimetres and t is in seconds. The maximum velocity of the particle is
(a) 62.8 cm s^{-1} (b) 3140 ms^{-1}
(c) 50 cms^{-1} (d) 31.4 m s^{-1}

Solution (d) $y_{\max} = \omega y_0 = 314(10$ cm/s or $31.4\text{ms}^{-1})$.

3. The speed of a transverse wave travelling on a wire having a length 50 cm and mass 50 g is 80 ms^{-1}. The area of cross-section of the wire is 1 mm^2 and its Young's modulus is $16 \times 10^{11}\text{Nm}^{-2}$. Find the extension of the wire over natural length.
(a) 2 cm (b) 2 mm
(c) 0.2 mm (d) 0.02 m m

Solution (d) $v = \sqrt{T/\mu}$

or $T = v^2\mu = (80)^2\left(\frac{5}{0.5}\times 10^{-3}\right) = 64\text{N}$ and $Y = \frac{F/A}{\Delta l/l}$

or $\Delta l = \frac{Fl}{AY} = \frac{64\times 0.5}{10^{-6}\times 16\times 10^{11}} = 2 \times 10^{-5}\text{m}.$

4. Which of the following waves is progressing in the y direction?
(a) $x = x_0\cos(\omega t - ky)$ (b) $y = y_0\cos(\omega t - kx)$
(c) $y = y_0\cos kx \sin\omega t$ (d) $y = y_0\sin kx \cos\omega t$

Solution (a) The wave $x = x_0\cos(\omega t - ky)$ travels along y direction.

5. Velocity of sound in air is 332 m/s. Its velocity in vacuum is
   (a) $>332\,ms^{-1}$ (b) $3 \times 10^8 ms^{-1}$
   (c) $332\,ms^{-1}$ (d) none of these

Solution (d) None of these as velocity is zero as sound waves require medium.

6. A cork floating in a calm lake is executing SHM of frequency f. When a boat passes close to the cork then the
   (a) frequency becomes greater than f
   (b) frequency becomes less than f
   (c) frequency remains constant
   (d) none of these

Solution (c) Frequency remains constant and velocity will vary, that is, wavelength will vary.

7. Two waves of equal amplitude x_0 and equal frequency travel in the same direction in a medium. The amplitude of the resultant wave is
   (a) 0 (b) x_0
   (c) $2x_0$ (d) between 0 and $2x_0$

Solution (d) use $x_0' = \sqrt{x_{01}^2 + x_{02}^2 + 2x_{01}x_{02}\cos\theta}$

$\because \theta = 0$ and $x_{01} = x_{02} = x_0$

$x_0' = 2x_0$.

8. The fundamental frequency of a string is proportional to
   (a) inverse of the length (b) the diameter
   (c) tension (d) density

Solution (a) $f \propto 1/l$

9. A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope.
   (a) 0.06 m (b) 0.12 m
   (c) 0.09 m (d) none of these

Solution (b) $v = \sqrt{T/\mu}$

$$\frac{v_{top}}{v_{bottom}} = \sqrt{\frac{T_T}{T_B}} = \sqrt{\frac{(6+2)g}{2g}} = 2$$

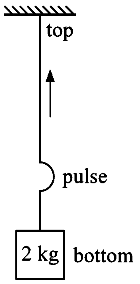


Fig. 17.6

$$\frac{f\lambda_{Top}}{f\lambda_{Bottom}} = 2 \text{ as frequency does not change}$$

$\therefore\ \lambda_{top} = \lambda_{bott} \times 2 = 0.12\,m$.

10. A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling. The speed of transverse waves in the rope at a point 0.5 m from the lower end is
    (a) $2.211\,ms^{-1}$ (b) $4.21\,ms^{-1}$
    (c) $7.21\,ms^{-1}$ (d) $3.31\,ms^{-1}$

Solution (a) $T = \frac{M}{L}(x)\,g$

and $v = \sqrt{\frac{\frac{M}{L}(x)\,g}{M/L}} = \sqrt{gx} = \sqrt{9.8 \times 0.5} = 2.21\,m/s$

11. The equations of motion of two waves propagating in the same direction is given by

    $y_1 = A\sin(\omega t - kx)$

    and $y_2 = A\sin(\omega t - kx - \theta)$.

    The amplitude of the medium particle will be
    (a) $\sqrt{2}A\cos\theta$ (b) $2A\cos\theta$
    (c) $\sqrt{2}A\cos\theta/2$ (d) $2A\cos\theta/2$

Solution (d) $y_0 = \sqrt{A^2 + A^2 + 2A(A\cos\phi)}$

$= A\sqrt{2(1+\cos\theta)} = 2A\cos\theta/2$.

12. The displacement y of a wave travelling in x direction is given by

    $$y = 10^{-1}\sin\left(600t - 2x + \frac{\pi}{3}\right)\text{m}$$

    where x is expressed in metres and t in seconds. The speed of the wave motion in metre per second is
    (a) 600 (b) 1200
    (c) 200 (d) 300

Solution (d) $v = \frac{\omega}{k} = \frac{600}{2} = 300\,ms^{-1}$.

13. A metal wire of linear mass density 9.8 g/m is stretched with a tension of 10 kgf the poles of a permanent magnet and it vibrates in resonance when carrying an ac of frequency n. The frequency n is
    (a) 100 Hz (b) 200 Hz
    (c) 25 Hz (d) 50 Hz

Solution (d) $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}} = \frac{1}{2}\sqrt{\frac{10 \times 9.8}{9.8 \times 10^{-3}}}$

$= \frac{10^2}{2} = 50\,\text{Hz}$.

14. The equation of a progressive wave is

    $$y = 8\sin\left[\pi\left(\frac{t}{10} - \frac{x}{4}\right) + \frac{\pi}{3}\right].$$

    The wavelength of the wave is
    (a) 8 m (b) 4 m
    (c) 2 m (d) 10 m

Solution (a) $\frac{2\pi}{\lambda} = k$ or $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/4} = 8$

15. The equation of a stationary wave is $y = \sin\frac{\pi x}{3}\cos 10\pi t$ where x and y are in centimetres and t in seconds. The separation between two consecutive nodes is
    (a) 1.5 cm (b) 6.0 cm
    (c) 3.0 cm (d) 18 cm

Solution (c) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/3} = 6\,\text{cm}$.

Separation between two consecutive nodes $= \lambda/2 = 3\,\text{cm}$.

16. If the amplitude of velocity of a particle acted by a force $F = F_0 \cos \omega t$ along x-axis is given by

$$v_0 = \frac{1}{(a\omega^2 - b\omega + c)^{1/2}} \text{ where } b^2 > 4ac .$$

The frequency of resonance is:
(a) $\omega = b/a$ (b) $b/2a$
(c) a/b (d) $a/2b$

Solution (b) For resonance $v_0 \to \infty$ (max)

$\therefore$ $(a\omega^2 - b\omega + c)^{1/2}$ should be minimum

or $$\frac{d}{d\omega}(a\omega^2 - b\omega + c) = 0$$

or $$2a\omega - b = 0 \text{ or } \omega = \frac{b}{2a}$$

17. An observer on the sea shore observes 54 waves reaching the coast per minute. If the wavelength is 10 m. The velocity is
(a) 9 ms^{-1} (b) 54 ms^{-1}
(c) 18 ms^{-1} (d) 36 ms^{-1}

Solution $f = \frac{54}{60} = \frac{9}{10}$ Hz $\quad v = f\lambda = \frac{9}{10} \times 10 = 9$ ms^{-1}.

18. A light pointer fixed to one prong of a tunning fork touches a vertical smoked plate. The fork is set to vibration and the plate is allowed to fall freely. Eight complete waves are counted when the plate falls through 10 cm. The frequency of the tunning fork is
(a) 112 Hz (b) 14 Hz
(c) 28 Hz (d) 56 Hz

Solution (d) $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 0.1}{9.8}} = \frac{1}{7}$ s.

$$f = \frac{\text{number of waves}}{\text{time}} = \frac{8}{1/7} = 56 \text{Hz}.$$

19. A progressive wave of frequency 500 Hz is travelling with a velocity 360 ms^{-1}. How far are two points 60° out of phase?
(a) 0.06 m (b) 0.12 m
(c) 0.18 m (d) 0.24 m

Solution (b) $\lambda = \frac{v}{f} = \frac{360}{500} = 0.72$ m $\quad \Delta\phi = \frac{2\pi}{\lambda}(\Delta x)$

or $$\Delta x = \frac{\Delta\phi\lambda}{2\pi} = \frac{\pi/3\,(0.72)}{2\pi} = 0.12 \text{ m}$$

20. Two blocks each having a mass 3.2 kg are connected by a wire *CD* and the system is suspended from the ceiling by another wire *AB* as shown in figure. The linear mass density of *AB* is 10 gm^{-1} and that of the *CD* is 8 gm^{-1}. The speed of the transverse wave pulse produced in *AB* and *CD* is

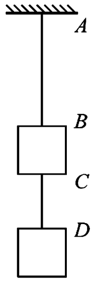


Fig. 17.7

(a) 80 ms^{-1}, 40 ms^{-1} (b) 40 ms^{-1}, 80 ms^{-1}
(c) 80 ms^{-1}, 63 ms^{-1} (d) none of these

Solution (c) $v = \sqrt{\frac{T}{\mu}} \Rightarrow v_{AB} = \sqrt{\frac{6.4 \times 10}{10 \times 10^{-3}}} = 80$ ms^{-1}

$$v_{CD} = \sqrt{\frac{3.2 \times 10}{8 \times 10^{-3}}} = 63 \text{ ms}^{-1}.$$

21. A transverse wave described by $y = 0.02 \sin(x + 30t)$ propagates on a stretched string of linear density 12 gm^{-1}. The tension in the string is
(a) 2.16 N (b) 1.08 N
(c) 0.108 N (d) 0.0108 N

Solution (c) $v = \omega/k = 30/1 = 30$ ms^{-1}

$$T = v^2\mu = (30)^2 \times 12 \times 10^{-3} = 0.108 \text{N}.$$

22. A circular loop of the string rotates about its axis on a frictionless horizontal plane at a uniform rate so that the tangential speed of any particle of the string is v. If a small transverse disturbance is produced at a point of the loop. The speed (relative to the string) at which the disturbance will travel is
(a) v (b) $v/2$
(c) $2v$ (d) $v/4$

Solution (a)

Problems for Practice

1. What happens when a sound wave is reflected from the boundary of a denser medium? The compression of the incident wave is returned as a
(a) rarefaction (b) crest
(c) trough (d) compression

2. The velocity of sound in vacuum is
(a) 332 ms^{-1} (b) 288 ms^{-1}
(c) 330 ms^{-1} (d) zero

3. The frequency of a man's voice is 300 Hz. If velocity of sound waves is 336 ms^{-1}, the wavelength of the sound is
(a) 1.12 m (b) 300 × 336 m
(c) 330/336 m (d) none of these

4. The angle between particle velocity and wave velocity in transverse waves is
(a) π (b) $\pi/2$
(c) $\pi/4$ (d) zero

5. Longitudinal waves cannot travel through
(a) liquids (b) gases
(c) vacuum (d) solids

6. In case of the electromagnetic waves the angle between the electric and magnetic field vectors is
(a) π
(b) $\pi/2$
(c) zero
(d) none of these

7. A wave of length 2m is superposed on its reflected wave to form a stationary wave. A node is located at $x = 3$m, the next node will be located at $x =$

(a) 4m (b) 3.75m

(c) 3.50m (d) 3.25m

8. Velocity of sound in the atmosphere of a planet is 500ms^{-1}. The minimum distance between the source of sound and the obstacle for an echo heard is

(a) 17m (b) 50m (c) 25m (d) 20m

9. If F is restoring force, k is force constant and y is displacement, which of the following expressions represent the equation of simple harmonic motion?

(a) $F = -ky$ (b) $F = \sqrt{ky}$

(c) $F = ky$ (d) none of these

10. The number of beats produced per second by two tuning forks when sounded together is 4. One of them has a frequency of 250Hz. The frequency of the other cannot be less than

(a) 254Hz (b) 252Hz

(c) 248Hz (d) 246Hz

11. A bomb explodes on the moon. How long will it take for the sound to reach the earth?

(a) 1day (b) 1000s (c) 10s (d) none of these

12. Two simple harmonic waves having the same amplitude and frequency with zero phase difference superimpose at right angles to each other. The resultant motion will be

(a) linear (b) elliptical

(c) circular (d) none of these

13. A property of the progressive wave that does not depend upon other characteristics mentioned below is

(a) wavelength (b) amplitude

(c) frequency (d) wave velocity

14. Two waves of same frequency but amplitudes equal to a and $2a$ travelling in the same direction superimpose out of phase. The resultant amplitude will be

(a) $\sqrt{a^2 + 2a^2}$ (b) $3a$

(c) $2a$ (d) a

15. The oscillators that can be described in terms of sine or cosine functions are called

(a) simple harmonic (b) natural

(c) sympathetic (d) free

16. The distance between two consecutive antinodes is 0.5m. The distance travelled by the wave in half the time period is

(a) 2m (b) 1m (c) 0.5m (d) 0.25m

17. Which of the following expressions is that of a simple harmonic progressive wave

(a) $A\sin(\omega t - kx)$ (b) $A\sin\omega t$

(c) $A\sin\omega t\cos kx$ (d) $A\cos kx$

18. A wave of frequency 400Hz has a velocity of 320ms^{-1}. The distance between the particles differing in phase by 90° is

(a) 80cm (b) 60cm (c) 40cm (d) 20cm

19. The ratio of intensities of two waves is 1 : 16. The ratio of their amplitudes is

(a) 16/17 (b) 1/16 (c) 1/4 (d) 1/2

20. Two waves each of loudness L superimpose to produce beats. The maximum loudness of the beats will be

(a) $4L$ (b) $2L$ (c) L (d) none of these

21. Two waves of intensities I and $4I$ superimpose. The minimum and maximum intensities will respectively be

(a) I, $9I$ (b) $3I$, $5I$ (c) I, $5I$ (d) none of these

22. The velocity of sound in oxygen at NTP is v. The velocity of sound in hydrogen at NTP will be

(a) $2\sqrt{2}v$ (b) $2v$ (c) $4v$ (d) none of these

23. The isothermal elasticity of a medium is E_i and the adiabatic elasticity in E_a. The velocity of the sound in the medium is proportional to

(a) $\sqrt{E_i}$ (b) E_a (c) $\sqrt{E_a}$ (d) E_i

24. The velocity of sound in air is v and the root mean square velocity of the molecules is c. Then v/c =

(a) $\gamma/3$ (b) $\gamma/\sqrt{3}$ (c) $\sqrt{\gamma}/3$ (d) $\sqrt{\gamma/3}$

25. The velocity of sound at 0°C is 332ms^{-1}. At what temperature will it be 664ms^{-1}?

(a) 273°C (b) 546°C (c) 819°C (d) 1092°C

26. Velocity of hydrogen at NTP is v. What will be the velocity of sound in a mixture of hydrogen and oxygen in the ratio 4 : 1 at NTP is

(a) v (b) $2v$ (c) $v/2$ (d) $v/4$

27. A sound wave is travelling in a medium in which the velocity is v. It is incident on the second medium in which the velocity of the wave is $2v$. What should be the minimum angle of incidence on the first medium, so that the wave fails to cross the surface of separation of the two media

(a) 60° (b) 45° (c) 30° (d) 15°

28. Beats are produced because of the superposition of two progressive notes. Maximum loudness at the waxing is n times the loudness of either notes. What is the value of n?

(a) 4 (b) 2 (c) $\sqrt{2}$ (d) 1

29. The first resonance length in a closed organ pipe is 50cm. Then the second resonance length will be

(a) 50cm (b) 100cm

(c) 150cm (d) 200cm

30. Which type of oscillations give rise to resonance?

(a) damped (b) free

(c) forced (d) all of these

31. At what temperature the speed of sound in air will be 1.5 times its value at 27°C in air?

(a) 102°C (b) 204°K (c) 204°C (d) 402°C

32. The ratio of speeds of sound in hydrogen gas and oxygen gas at same temperature will be

(a) 8 : 1 (b) 4 : 1 (c) 1 : 8 (d) 1 : 4

33. The distance between a node and an anti-node is

(a) 2λ (b) λ (c) $\lambda/2$ (d) $\lambda/4$

34. The speed of a supersonic wave, as compared to that of sound is

(a) less (b) more (c) equal (d) 1/10

35. The increase in the speed of sound, on increasing the temperature of the medium by 10°C, will be

(a) 600ms^{-1} (b) 6ms^{-1}

(c) 0.61ms^{-1} (d) 60ms^{-1}

36. The velocity of sound in air is 350ms^{-1}. The fundamental frequency of an open pipe of length 50cm will be

(a) 700s^{-1} (b) 350s^{-1}

(c) 175s^{-1} (d) 50s^{-1}

37. The fundamental frequency of an open organ pipe is n. The pipe is vertically immersed in water such that half of its length is submerged. The fundamental frequency of air column in this position will be

(a) $n/3$ (b) $n/2$ (c) n (d) $2n$

38. If the ratio of amplitudes of two waves at any point in the medium is 1 : 3, then the ratio of maximum and minimum intensifies because of their superposition will be
(a) 2 : 1 (b) 3 : 1 (c) 4 : 1 (d) 2 : 3

39. The phase difference between the particles vibrating between two consecutive nodes is
(a) zero (b) $\pi/2$ (c) π (d) 2π

40. The frequency of an open organ pipe is n. If one end is closed then its fundamental frequency will be
(a) $n/2$ (b) $3n/4$ (c) n (d) $2n$

41. Two sound waves of equal intensity I superimpose at point P in 90° out of phase. The resultant intensity at point P will be
(a) $4I$ (b) $\sqrt{2}I$ (c) $2I$ (d) I

42. The equation of a wave propagating in a string is $y = 2\cos\pi(100t - x)$. Its wavelength will be
(a) 2 cm (b) 5 cm (c) 2 m (d) 50 cm

43. On vibrating a tuning fork of frequency 256 Hz with another fork A, six beats per second are heard. On loading A, again six beats per second are heard. The frequency of A will be
(a) 244 Hz (b) 250 Hz
(c) 262 Hz (d) 268 Hz

44. The ratio of frequencies in a stretched string is
(a) 1 : 2 : 3 (b) 1 : 3 : 5
(c) 2 : 4 : 6 (d) 3 : 2 : 1

45. The property of a medium necessary for wave propagation is its
(a) elasticity (b) low resistance
(c) inertia (d) all of above

46. The ratio (v) of velocities of sound in dry air and humid air is
(a) $v < 1$ (b) $v > 1$ (c) $v = 1$ (d) zero

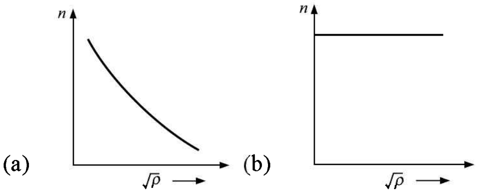
47. The waves propagating on water surface are
(a) ultrasonic (b) longitudinal
(c) unaudible (d) transverse

48. A tuning fork produces four beats per second with 49 cm and 50 cm lengths of a stretched wire of a sonometer. The frequency of the fork is
(a) 196 Hz (b) 296 Hz
(c) 396 Hz (d) 693 Hz

49. In Melde's experiment, eight loops are formed with a tension of 0.75 N. If the tension is increased to four times then the number of loops produced will be
(a) 2 (b) 4 (c) 8 (d) 16

50. The third harmonic in an open organ pipe is known as
(a) fundamental frequency
(b) second overtone
(c) third overtone
(d) first overtone

51. The correct graph between the frequency n and square root of density (ρ) of a wire, keeping its length, radius and tension constant, is



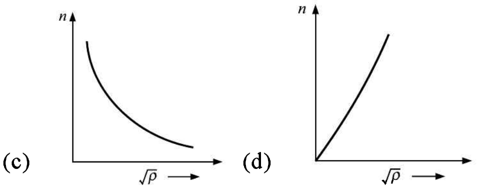


Fig. 17.8

52. P is the junction of two wires A and B. B is made of steel and is thicker while A is made of aluminium and is thinner as shown. If a wave pulse as shown in the figure approaches P, the reflected and transmitted waves from P are respectively

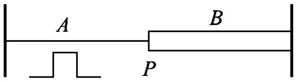
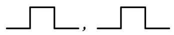


Fig. 17.9

(a)

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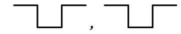
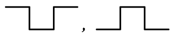
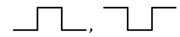

(c)

(d)


53. In Kundt's tube, when waves of frequency 10^3 Hz are produced the distance between five consecutive nodes is 82.5 cm. The speed of sound in gas filled in the tube will be
(a) 660 ms^{-1} (b) 330 ms^{-1}
(c) 230 ms^{-1} (d) 100 ms^{-1}

54. A resonance tube is resonated with tuning fork of frequency 256 Hz. If the length of resonating air columns are 32 cm and 100 cm, then end correction will be
(a) 1 cm (b) 2 cm (c) 4 cm (d) 6 cm

55. On decreasing the temperature the frequency of an organ pipe becomes
(a) less (b) more (c) equal (d) infinity

56. When a sound wave is reflected from a rigid support then its path changes by
(a) $\lambda/2$ (b) λ (c) π (d) 2π

57. The vibration in the stem of tuning fork are
(a) transverse (b) longitudinal
(c) both (d) none of these

58. The ratio of the velocity of body and velocity of sound is known as
(a) laplace number (b) positive integer
(c) stable number (d) mach number

59. For constructive interference, the path difference between two waves must be
(a) $(2n + 1)\lambda/2$ (b) $(2n + 1)\lambda$
(c) $n\lambda/2$ (d) $n\lambda$

60. The frequencies of two sound sources are 256 Hz and 260 Hz respectively. The beat frequencies produced by them will be
(a) 0.025 s^{-1} (b) 2.5 s^{-1}
(c) 4.00 s^{-1} (d) 25 s^{-1}

61. On sounding a string and a tuning fork simultaneously, six beats per second are produced. When the length of the string is 95 cm or 100 cm. The frequency of the fork is
(a) 117 Hz (b) 234 Hz
(c) 432 Hz (d) 702 Hz

62. The resultant amplitude, when two waves of same frequency but with amplitudes a_1 and a_2 superimpose with a phase difference of $\pi/2$ will be
(a) $a_1^2 + a_2^2$ (b) $\sqrt{a_1^2 + a_2^2}$
(c) $a_1 - a_2$ (d) $a_1 + a_2$

63. In a stationary wave the distance between consecutive antinodes is 25 cm. If the wave velocity is 300 ms^{-1} then the frequency of wave will be
(a) 150 Hz (b) 300 Hz
(c) 600 Hz (d) 750 Hz

64. The intensity of sound after passing through a slab decreases by 20%. On passing through two such slabs, the intensity will decrease by
(a) 30% (b) 36% (c) 40% (d) 50%

65. Two waves travel in the mutually opposite directions in a medium. When superimposed, the phenomenon observed is
(a) stationary waves (b) harmonic nodes
(c) beats (d) resonance

66. When two plane progressive waves travelling in same direction superpose over each other, the velocity of resultant wave will
(a) increase (b) remain unchanged
(c) be zero (d) decrease

67. The fundamental frequency in a stretched string is 100 Hz. To double the frequency, the tension in it must be changed to
(a) $T_2 = 2T_1$ (b) $T_2 = 4T_1$
(c) $T_2 = T_1$ (d) $T_2 = T_1/4$

68. The frequency of an open pipe is 300 Hz. The first overtone of this pipe is the same as the second overtone of a closed pipe. The length of the closed organ pipe is
(a) 11 cm (b) 21 cm (c) 42 cm (d) 84 cm

69. In the equation of the motion of a particle $y = 0.5 \sin(0.3t + 0.1)$, the initial phase of motion is
(a) $(0.3t + 0.1)$ (b) 0.3
(c) $0.3t$ (d) 0.1

70. Two waves of same amplitude and same frequency reach a point in a medium simultaneously. The phase difference between them for resultant amplitude to be zero, will be
(a) 4π (b) 2π (c) π (d) 0°

71. Under similar conditions of temperature and pressure, the velocity of sound is maximum in
(a) CO_2 (b) H_2 (c) N_2 (d) O_2

72. The sound box of a sonometer increases
(a) speed of sound
(b) sound intensity
(c) wavelength of sound
(d) sound frequency

73. The loudness of sound depends on
(a) amplitude (b) wavelength
(c) pitch (c) speed

74. The ratio of intensities of two sound waves is 4 : 9. The ratio of their amplitudes will
(a) 9 : 4 (b) 4 : 9 (c) 2 : 3 (d) 3 : 2

75. The speed of sound in air is 320 ms^{-1}. The length of a closed pipe is 1 m. Neglecting the end correction, the resonant frequency for the pipe will be
(a) 80 Hz (b) 240 Hz
(c) 320 Hz (d) 400 Hz

76. On increasing the distance between the source of sound and observer thrice, the intensity of sound becomes
(a) 3 (b) 1/3 (c) 9 (d) 1/9

77. The velocity of sound in air at room temperature is 340 m/s and density of air is 1.2 kgm^{-3}. The value of atmospheric pressure in terms of the height of mercury column in metre will be
(a) 0.75 (b) 7.5 (c) 75 (d) 750

78. In Melde's experiment the string vibrates in seven segments under tension of 9 gm-wt. If the string is to be vibrated in three segments then the tension required will be
(a) 1.4 gm-wt (b) 13 gm-wt
(c) 49 gm-wt (d) 61 gm-wt

79. The length of strings of a cello is 0.8 m. In order to change the pitch in frequency ratio 5/4, their length should be decreased by
(a) 0.08 m (b) 0.2 m
(c) 0.13 m (d) 0.16 m

80. Two waves of same frequency but of amplitudes a and $2a$ respectively superimpose over each other. The intensity at a point where the phase difference is $3\pi/2$, will be
(a) a (b) $3a^2$ (c) $5a^2$ (d) $9a^2$

ANSWERS

1. (d)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (d)
11. (d)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (d)	**19.** (c)	**20.** (a)
21. (a)	**22.** (c)	**23.** (c)	**24.** (d)	**25.** (c)	**26.** (c)	**27.** (c)	**28.** (a)	**29.** (c)	**30.** (c)
31. (d)	**32.** (b)	**33.** (d)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (c)	**38.** (c)	**39.** (a)	**40.** (a)
41. (b)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (d)	**46.** (a)	**47.** (d)	**48.** (c)	**49.** (b)	**50.** (b)
51. (c)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (a)	**57.** (b)	**58.** (d)	**59.** (d)	**60.** (c)
61. (b)	**62.** (b)	**63.** (c)	**64.** (b)	**65.** (a)	**66.** (b)	**67.** (b)	**68.** (c)	**69.** (d)	**70.** (c)
71. (b)	**72.** (b)	**73.** (a)	**74.** (c)	**75.** (a)	**76.** (d)	**77.** (a)	**78.** (c)	**79.** (d)	**80.** (c)

18 Sound Waves

BRIEF REVIEW OF THE CONCEPTS

A sound wave is pressure wave given by $p = p_0 \sin(\omega t - kx)$. It is longitudinal in nature and consists of alternate compressions and rarefactions. To human ear only the frequency range 20–20000 Hz is audible. These limits are subjective and may vary slightly from person to person

If $y = y_0 \sin(\omega t - kx)$ is displacement wave,

then change in volume

$$dV = A dy = \frac{A dy}{dx} dx = Ay_0(-k) \cos(\omega t - kx) dx.$$

Volumetric strain

$$\frac{dV}{V} = \frac{Ay_0\left(-\frac{\omega}{v}\right)\cos(\omega t - kx)\, dx}{A dx} \qquad \left\{\begin{array}{l} \because \frac{\omega}{k} = v \\ \therefore\ k = \frac{\omega}{v} \end{array}\right.$$

$$p = -\frac{B\Delta V}{V} = \frac{B.y_0\omega}{v} \cos(\omega t - kx)$$

where B is bulk modulus.

Comparing with $p = p_0 \cos(\omega t - kx)$

we get $p_0 = \frac{By_0\omega}{v} = Bky_0$

$P_{\max}$(inner ear) = 0.4 Pa

$P_{\min}$(inner ear) = 3×10^{-5} Pa

Also note that there exists a phase shift of 90° between displacement and pressure wave

General formula $v = \sqrt{\frac{E}{\rho}}$ where E is elastic constant

Speed of the Sound Wave

Newton's formula $v = \sqrt{\frac{P}{\rho}}$

Newton considered the change to be isothermal.

Laplace's correction Laplace considered adiabatic change and derived

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad \text{(It gives correct results)}$$

where $\gamma = \frac{C_P}{C_V}$.

$$v = \sqrt{\frac{\gamma RT}{M}} \qquad v = \sqrt{\frac{B}{\rho}}$$

where M is molecular mass of the gas and B is bulk modulus.

In solids we may write

$$v = \sqrt{\frac{Y}{\rho}} \quad \text{where } Y \text{ is Young's modulus}$$

For transverse waves in solids (bulk material)

$$v = \sqrt{\frac{B + \eta/3}{\rho}} \quad \text{where } \eta \text{ is shear modulus.}$$

Velocity of sound in inner ear $V_{\text{inner ear}} = 1500\ \text{ms}^{-1}$ maximum displacement of fluid in inner ear = 4.4×10^{-11}m.

Effect of temperature $v \propto \sqrt{T}$ where T is temperature in Kelvin.

$$\frac{v}{v_0} = \sqrt{\frac{T}{273}} = \sqrt{1 + \frac{t}{273}}$$

where t is temperature in celsius (°C)

when temperature rises by 1°C velocity of sound increases by 0.61 ms^{-1}.

Intensity $I = 2\pi^2 \rho y_0^2 f^2 v$

$$I = \frac{2\pi^2 B y_0^2 f^2}{v} = \frac{p_0^2 v}{2B} = \frac{p_0^2}{2\rho v}$$

Intensity $I \propto \frac{1}{r^2}$ (for an isotropic source)

where r is the distance between the source and observer.

Effect of Pressure Velocity of sound is not affected by pressure.

Effect of density $v \propto \frac{1}{\sqrt{\rho}}$ where ρ is density. With increase in humidity density of air decreases and hence speed of sound increases.

Appearance of sound in human air is characterised by three parameters—pitch, loudness and quality.

Pitch is related to frequency. Higher the pitch sweeter is the sound. Children and ladies speak at higher pitch as compared to men, therefore, their sound appears sweeter. Higher the frequency higher is the pitch.

Loudness is correlated with sound level. Human ear can hear a minimum intensity

$I_0 = 10^{-12}\text{W/m}^2$

whispering → 10 dB normal talk → 60 dB

maximum tolerable sound → 120 dB

sound level in dB $S_L = 10\log_{10}\left(\frac{I}{I_0}\right)$

Even at 80 dB (heard continuously for sometime) headache begins. At 130 dB person may become temporarily insane.

Quality No source of sound generates a single frequency. For example even a tuning fork marked 288 Hz will not produce only fundamental frequency of 288 Hz but also produces along with it, integral multiple of frequencies like 2 × 288 = 576 Hz , 3 × 288 = 864 Hz and so on and so forth. The difference in sound of a **tabla** and a **mridung** being played at same frequencies is due to number of harmonics produced and their amplitudes.

Remember that speech ends upto 3 kHz. Rest frequency range upto 20 kHz are only higher harmonics and are used in music. The higher harmonics are particularly pleasant to the ear. A noise has frequencies that do not bear any well-defined relationship among themselves.

Interference of sound waves If $p_1 = p_{01}\sin(\omega t - kx)$ and $p_2 = p_{02}\sin(\omega t - kx + \delta)$ interfere we assume the sources are coherent (say two tuning forks of same frequency) then

$$p = p_1 + p_2 = p_0\sin(\omega t - kx + \phi)$$

$$p_0 = \sqrt{p_{01}^2 + p_{02}^2 + 2p_{01}p_{02}\cos\delta}$$

and $$\tan\phi = \frac{p_{02}\sin\delta}{p_{01} + p_{02}\cos\delta}$$

If phase difference $\delta = k\Delta x = 0$ or $2n\pi$ then intensity will be maximum and constructive interference results. Path difference in such cases is an integral multiple of wavelength.

If phase difference $\delta = k\Delta x = (2n + 1)\pi$ then intensity will be minimum and destructive interference results. In such cases path difference $\Delta x = (2n + 1)\frac{\lambda}{2}$.

$$\frac{I_{\max}}{I_{\min}} = \frac{(p_{01} + p_{02})^2}{(p_{01} - p_{02})^2} = \frac{(y_{01} + y_{02})^2}{(y_{01} - y_{02})^2}$$

Quinke's tube is used to demonstrate interference of sound.

If path difference is Δx then phase difference

$$\delta = k\Delta x = \frac{2\pi\Delta x}{\lambda}.$$

Reflection of sound wave can cause: (a) echo (b) longitudinal standing waves.

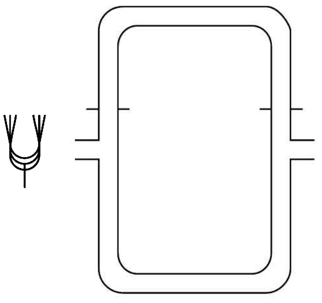


Fig. 18.1

Echo is produced when the reflected wave (sound) is heard again by the producer or by others also.

Echo is produced when a minimum distance between the source and the reflector is 16.6 m as demonstrated in the Fig. 18.2. Distance covered by sound to reach the producer is

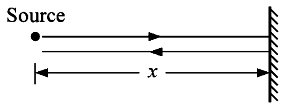


Fig. 18.2

$$2x = vt$$

$$x = \frac{vt}{2} = \frac{332}{2} \times \frac{1}{10} = 16.6\,\text{m} \simeq 55\,\text{ft}.$$

We take $t = \frac{1}{10}$ s because this is the minimum time between two syllables being heard clearly.

Echo can be heard in a smaller room provided it is empty and windows and doors are closed.

To find the distance in echo production use $x = \frac{vt}{2}$.

Standing waves

(a) Standing waves in closed pipes are with reference to displacement waves. For pressure waves position of nodes and antinodes will interchange. Same is true for Fig. 18.4, that is, open pipes.

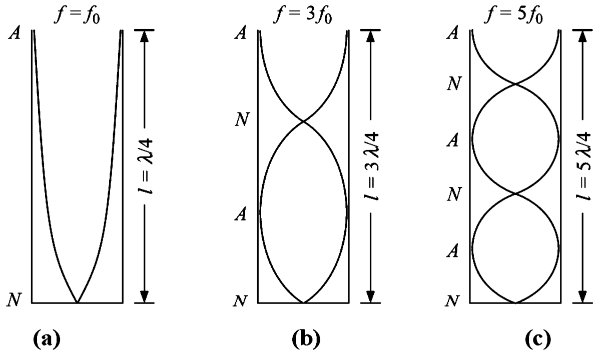


Fig. 18.3

1st resonance	2nd resonance	3rd resonance
or	or	or
1st harmonic	3rd harmonic	5th harmonic
or	or	or
fundamental frequency	2nd harmonic of colsed pipe	3rd harmonic of closed pipe

In closed pipes resonance occurs at

$$l = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots,$$ that is, at odd multiple of λ/4.

Only odd integral multiples of fundamental frequencies f_0, $3f_0$, $5f_0$ and so on are allowed.

Note that at the open end an antinode occurs and at closed end a node occurs.

(b) Open pipes

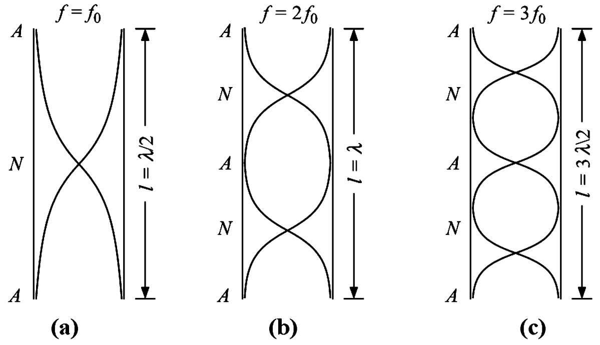


Fig. 18.4

fundamental frequency	2nd resonance	3rd resonance
or	or	or
1st resonance	2nd harmonic	3rd harmonic
or	or	or
1st harmonic	1st overtone	2nd overtone

In open pipes resonance occurs at

$l = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$, that is, all integral multiples of $\lambda/2$.

All integral multiple of fundamental frequencies f_0, $2f_0$, $3f_0$ ... are allowed or all harmonics are allowed

End correction $l_1 + 0.3d = \lambda/4$ for 1st resonance. where, d is diameter of the pipe

$$l_2 + 0.3d = \frac{3\lambda}{4}$$

$$(l_2 - l_1) = \frac{\lambda}{2}$$

and $$v = 2(l_2 - l_1)f \quad (= f\lambda).$$

In Kundt's tube heaps of lycopodium powder/sand and so on are collected at nodes.

$\therefore$ separation between two heaps is equal to $\frac{\lambda}{2}$.

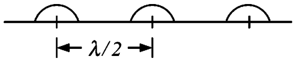


Fig. 18.5

Beats Periodic increase and fall in the intensity of sound is called beats. Beats are produced when two sources of sound of nearly same frequency are sounded together. Beat frequency

$$n = |f_2 - f_1|$$
$$n \leq 10\,\text{Hz}$$

Beats can also be produced by superposition of tones. We illustrate it by an example. Assume two sources of sound of frequencies 200 Hz and 404 Hz are sounded together [as $f_2 - f_1 >> 10$ no beat should have been heard] then 4 beats/s are heard. It is because of the fact that

$$404 - 2(200) = 4 \text{ beats/s are produced.}$$

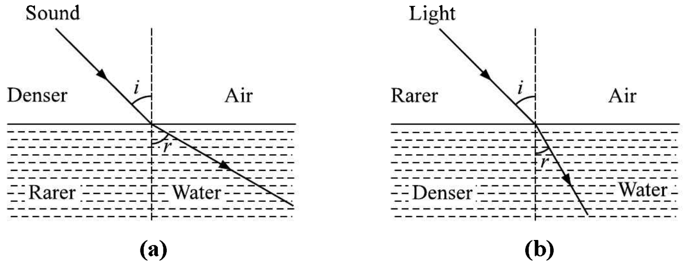
That is, fundamental frequency of 404 Hz superposes with 2nd harmonic of 200 Hz wave to produce 4 beats/s.

Note: Beat is interference in the time regime while generally known interference is superposition in distance regime.

Refraction of sound

As solids are most elastic and gases are least elastic

or $$E_{\text{solid}} > E_{\text{liquid}} > E_{\text{gas}}$$



(a) (b)

Fig. 18.6

Therefore, velocities are maximum in solids.

$$v_{\text{solid}} > v_{\text{liquid}} > v_{\text{gas}}.$$

There could be few exceptions. For example, in vulcanized rubber velocity of sound is less than that of gases. In alcohol also velocity of sound is less than that of gases. Velocity of sound is virtually independent of frequency variations.

For light, water is denser as the speed of light decreases. In case of sound water is rarer as speed of sound increases as illustrated in Fig. 18.6.

However $\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$ is valid even for sound.

Diffraction Bending of waves from an obstacle or an opening is called diffraction. Diffraction is a characteristic property of wave motion. All types of waves are diffracted. Wave nature of even electrons was demonstrated by Davison and Germer by diffraction of electrons. Diffraction of sound is more pronounced as the wavelength is large. Therefore, it can be diffracted from any material object.

Doppler effect When there is relative motion between the source and the listener the apparent frequency changes. This change in apparent frequency because of relative motion is called Doppler effect.

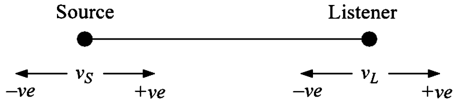


Fig. 18.7

Let v be the velocity of sound, v_S velocity of the source, v_L velocity of the listener then

$$f_{\text{app}} = \frac{v - v_L}{v - v_S} f$$

where f_{app} is the apparent frequency as heard by the listener and f is the frequency of the source.

The above formula is written keeping in view the positive and negative sign to be assigned for v_S and v_L as shown in Fig. 18.7.

If any of the two is at rest that particular velocity becomes zero in the above formula. Thus this formula may be applied to all cases.

If the source or listener moves with a velocity greater than velocity of sound then Doppler effect cannot be applied.

When the source of sound goes past the observer (stationary) the change in frequency is

$$\Delta f = \frac{2vv_S f}{v^2 - v_S^2}$$

If the observer goes past a stationary source then change in frequency

$$\Delta f = \frac{2v_L}{v} f.$$

Doppler effect in light is $\frac{\Delta\lambda}{\lambda} = \frac{\Delta_f}{f} = \frac{v}{c}$

SHORT-CUTS AND POINTS TO NOTE

(i) $\frac{v_{\text{sound}}}{v_{\text{rms}}\ (\text{of a gas})} = \sqrt{\frac{\gamma}{3}}$ where $\gamma = \frac{C_P}{C_V}$.

(ii) Speed of sound

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

where M is molecular mass of the gas and ρ its density.

In solids $v = \sqrt{\frac{Y}{\rho}}$ (if rod or string or long rail) where Y is Young's modulus.

$v = \sqrt{\frac{B}{\rho}}$ in bulk of material. Where B is bulk modulus

$v = \sqrt{\frac{B + \eta/3}{\rho}}$ in bulk of material for transverse mechanical waves where η is shear modulus.

(iii) $v = \frac{\omega}{k} = f\lambda$.

(iv) There is a phase shift of 90° between pressure and displacement wave.

(v) $p = p_0 \sin(\omega t - kx)$ is the pressure wave or sound wave.

$p_0 = \frac{B y_0 \omega}{v}$ where, B is bulk modulus, y_0 is amplitude of displacement wave.

(vi) Effect of temperature $v \propto \sqrt{T}$

or $\frac{v}{v_0} = \sqrt{\frac{T(\text{K})}{273}} = \sqrt{1 + \frac{t°\text{C}}{273}}$

$v_0 = 330\,\text{ms}^{-1}$ at 0°C

(vii) Velocity of sound in a medium is independent of wavelength or frequency. Frequency of a tuning fork in falling plate method is $f = m\sqrt{\frac{g}{d_2 - d_1}}$ where m is complete number of waves used and d_2 and d_1 are consecutive distance for m waves. In strobscopic method $f = mp$ where m is number of holes on the plate and p is angular frequency in revolution per second.

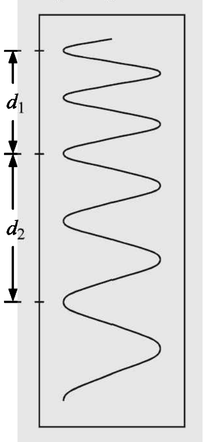


Fig. 18.8

(viii) Velocity of sound is independent of pressure. But it varies with density $v \propto 1/\sqrt{\rho}$. Velocity of sound is maximum in rainy season.

(ix) Intensity of sound $I = 2\pi^2 \rho y_0^2 f^2 v$

$$I = 2\pi^2 \rho y_0^2 f^2 v = \frac{2\pi^2 B y_0^2 f^2}{v} = \frac{p_0^2 v}{2B} = \frac{p_0^2}{2\rho v}.$$

(x) Pitch is related to frequency. Higher the pitch, higher is the frequency. Children and ladies speak at higher pitch compared to men. Higher frequency or higher pitch sound is more sweet.

(xi) Loudness is correlated with sound level. Minimum intensity that is audible to human ear is 10^{-12}Wm^{-2} and pressure 3×10^{-5}Pa

Sound level in dB $S_L = 10 \log_{10}(I/I_0)$.

80 dB sound level can cause headache if heard continuously for some time.

Sound level ≥ 130 dB may make a person temporarily insane. Maximum tolerable sound is 120 dB. Normal talking level is 60 dB.

(xii) Quality of sound is related to number of harmonics produced and their amplitude by a source. It is due to quality of sound that we can recognize a person by his/her voice. Even an instrument being played can be judged.

(xiii) Interference of sound in time frame produces beats, that is, if two sources having frequencies nearly equal superpose then periodic increase and fall in the intensity of sound is heard. This is called beats.

Beat frequency $n = |f_1 - f_2| \leq 10$ if they are to be heard.

If a tuning fork is filed its frequency slightly increases and if a tuning fork is waxed, its frequency slightly decreases. More than 10 beats/s cannot be heard. However, beats > 10 can be produced.

Tuning fork frequency $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$.

(xiv) Interference in space (x) is termed by interference and gives intensity of sound varying with distance. Sound intensity is maximum if phase shift is an integral multiple of 2π or path difference is $n\lambda$. Intensity is minimum when phase difference is $(2n + 1)\pi$ or path difference is $(2n + 1)\frac{\lambda}{2}$.

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{p_{01} + p_{02}}{p_{01} - p_{02}}\right)^2 = \left(\frac{y_{01} + y_{02}}{y_{01} - y_{02}}\right)^2$$

(xv) Quinke's tube is used to study interference of sound.

(xvi) Reflection of sound from a general obstacle may result in an echo. For echo to be produced separation between source and obstacle should be 16.6 m or 55 ft. Though echo can be produced because of multiple reflection in a closed and empty room. Felts, cushion and curtains and so on are absorbers of sound. A window (or opening) is the best absorber of sound. Human beings also absorb sound.

(xvii) Reflection of sound wave in organ pipes produce standing waves. A flute may be used both as closed and an open pipe. If all the holes are closed it acts like closed pipe. If specific holes are closed, we can vary the length of the pipe and hence frequency varies. In closed pipes resonance occurs at

$$l = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$$

Only odd multiple of frequencies are allowed in closed pipes.

(xviii) In open pipes all integral multiple of fundamental frequency are allowed. Resonance occurs when

$$l = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots, \text{ that is, } l = \frac{n\lambda}{2} \quad (n = 1, 2, 3, \ldots)$$

(xix) End correction is required in resonance tube experiment.

$$l_1 + 0.3d = \frac{\lambda}{4} \quad \text{for first resonance.}$$

and $$l_2 + 0.3d = \frac{3\lambda}{4} \quad \text{for second resonance.}$$

$$v = 2f(l_2 - l_1).$$

(xx) Separation between two consecutive nodes or antinodes is $\lambda/2$ and separation between a node and an antinode is $\lambda/4$.

(xxi) Refraction of sound occurs when sound wave travels from one medium to another. Normally velocity of sound follows the trend

$$v_{\text{solid}} > v_{\text{liquid}} > v_{\text{gas}}.$$

For light, glass or water is denser than air. But for sound, glass or water is rarer than air as velocity of sound is more in these materials.

$\mu = \frac{\sin i}{\sin r} = \frac{v_1}{v_2}$, that is, Snell's law is valid.

(xxii) Diffraction of sound is more pronounced than light because wavelength of sound is large. The diffraction occurs from any obstacle or a hole. Diffraction is a specific characteristic of wave.

(xxiii) Doppler effect is the apparent change in frequency of sound appearing to the listener because of motion between source and listener.

$$f_{\text{app}} = \left(\frac{v - v_L}{v - v_S}\right) f \quad \text{can be applied}$$

If wind of velocity v_w blows in the direction of sound then change v to $v + v_w$ or $v - v_w$ depending upon wind is blowing in same or opposite direction.

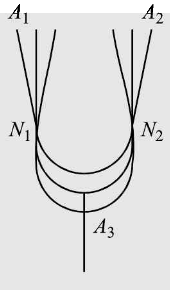
$$f_{\text{app}} = \frac{(v + v_w - v_L) f}{(v + v_w - v_s)}.$$

(xxiv) If the source or listener moves with a speed greater than the speed of sound then Doppler effect cannot be applied.

(xxv) Music is formed only with vowels. Octave (1 : 2) majortone (8 : 9), minortone (9 : 10) and semitone (15 : 16).

(xxvi) Mach number $= \frac{\text{Velocity of a body}}{\text{Velocity of sound}} = \frac{v_{\text{body}}}{330}$

(xxvii) Number of nodes = $2n$ and number of antinodes = $2n + 1$ when a tuning fork vibrates in nth harmonic.



Tuning fork vibrating in 1st harmonic

Fig. 18.9

CAUTION

(i) Not applying end correction in resonant pipes.

- Apply an end correction equal to $0.3d$ where d is diameter of the pipe using

$\lambda/4 = l_1 + 0.3d$ and $\frac{3\lambda}{4} = l_2 + 0.3d$ use $\frac{\lambda}{2} = l_2 - l_1.$

(ii) Confusing formulae for open and closed pipes.

- In closed pipes only odd integral multiple of fundamental frequency are allowed.

  length of the pipe $l = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$

  for 1st, 2nd, 3rd, harmonic.

  In open pipes all harmonics are allowed and length of the pipe is

$$l = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

(iii) Considering that a vibrating source always produces sound.

- Sound frequency lies between 20 and 20000 Hz. Frequencies less than 20 Hz are called infrasonic and are not audible to human ear. Frequencies > 20 kHz are inaudible and termed as ultrasonic. Bodies/waves having velocity > 330 ms^{-1} (velocity of sound) are termed as supersonic.

(iv) Considering intensity of sound and loudness as identical terms.

- Loudness is related to level of sound. It is measured in dB.

  Sound level $S_L = 10 \log_e \frac{I}{I_0}$

  where $I_0 = 10^{-12}$ Wm^{-2}

  is the minimum intensity audible to human ear. Min. Pressure audible by human ear is 3×10^{-5} Pa.

(v) Considering that a source/musical instrument of same frequency will have same number of harmonics.

- Number of harmonics and their amplitudes are different and form quality of sound.

(vi) Assuming that frequency, wavelength and velocity all change when a wave passes from one medium to another.

- Frequency does not vary. Also note that unlike light waves, sound waves with different wavelengths pass through a medium with same velocity.

(vii) Considering that doppler effect can always be applied if there is a relative motion between source and listener.

- You cannot apply Doppler effect if the velocity of source/listener is larger than speed of sound.

(viii) Confusing between wave number and velocity amplitude and acceleration amplitude.

- Wave number or propagation constant $k = \frac{2\pi}{\lambda}$.

Velocity amplitude $v_0 = \frac{2\pi y_0}{T} = k y_0 \omega$

Acceleration amplitude

$$a_0 = \frac{2\pi^2 y_0}{T^2} = \frac{2\pi^2 y_0}{T^2} \text{ where } T \rightarrow \text{time period}$$

(ix) Considering that there is no exception in the rule $v_{solid} > v_{liquid} > v_{gas}$.

- ■ In vulcanised rubber the velocity of sound < velocity of sound in hydrogen. In alcohol also velocity of sound < velocity of sound in hydrogen.

(x) Not remembering value of γ for monoatomic, diatomic or polyatomic gases

- ■ Values of $\gamma = \frac{5}{3}$ for monoatomic, $\gamma = 1.4$ for diatomic and $\gamma = \frac{4}{3}$ for polyatomic gases.

  Use $v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$

(xi) Confusing how to use the formula

$$v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma P}{\rho}} \text{ in a mixture of gases.}$$

- ■ Use $\gamma_{av} = \frac{n_1\gamma_1 + n_2\gamma_2}{n_1 + n_2}$;

  $$M_{av} = \frac{n_1M_1 + n_2M_2}{n_1 + n_2}$$

(xii) Considering that mediums which are denser for light are also denser for sound waves from refractive index point of view.

- ■ Velocity of sound waves $v_{solid} > v_{liquid} > v_{gas}$.

  Therefore gases appear to have higher refractive index than liquids or solids.

  However, $\frac{v_1}{v_2} = \frac{\sin i}{\sin r} = \mu$ is valid.

Solved Problems

1. Sound signal is sent through a composite tube as shown in the Fig. 18.10. The radius of the semicircle is r. Speed of sound in air is v. The source of sound is capable to generate frequencies in the range f_1 to $f_2 (f_2 > f_1)$. If n is an integer then frequency for maximum intensity is given by

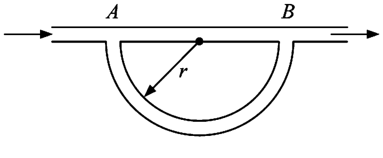


Fig. 18.10

(a) $\frac{nv}{r}$ (b) $\frac{nv}{r(\pi - 2)}$

(c) $\frac{nv}{\pi r}$ (d) $\frac{nv}{(r-2)\pi}$

Solution (b) path difference $\pi r - 2r = n\lambda$

or $r(\pi - 2) = \frac{nv}{f}$ thus $f = \frac{nv}{r(\pi - 2)}$.

2. Two tuning forks when sounded together produce 6 beats/s. The first fork has the frequency 3% higher than a standard one and the second has the frequency 2% less than the standard fork. The frequencies for the forks are

(a) 126.3, 120.3 Hz (b) 162.7, 156.7 Hz

(c) 136.2, 130.2 Hz (d) 123.6, 117.6 Hz

Solution (d) $\left(f + \frac{3f}{100}\right) - \left(f - \frac{2f}{100}\right) = 6$

$\Rightarrow 5f = 600$ or $f = 120$ Hz

$f + \frac{3f}{100} = 123.6$ Hz,

$f - \frac{2f}{100} = 120 - \frac{2 \times 120}{100} = 117.6$ Hz.

3. A and B are two wave trains shown in the Fig. 18.11, the ratio of intensity of A to B is

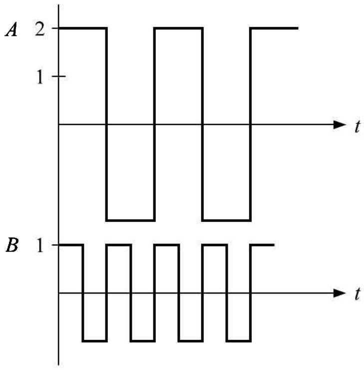


Fig. 18.11

(a) 1 (b) 2

(c) 4 (d) 8

Solution (a) $I \propto x_0^2$ and $I \propto f^2$

In A $x_{0A} = 2x_0$ and $f_{0A} = f_0$

In B $x_{0B} = x_0$ and $f_{0B} = 2f_0$ $\therefore \frac{I_A}{I_B} = 1$.

4. The fundamental frequency of a closed organ pipe is same as the first overtone frequency of the open pipe. The length of open pipe is 50 cm. The length of closed pipe is

(a) 25 cm (b) 100 cm

(c) 200 cm (d) 12.5 cm

Solution (d) $f_{0(open)} = 330/2l = 330$ Hz

$f_1 = 2 \times 330 = 660$ Hz

$f_{0(closed)} = 330/4l = 660$

or $8l = 1$ m, that is, $l = 12.5$ cm.

5. Sound waves from a tuning fork F reach a point P by two separate routes FAP and FBP. FBP is 12 cm larger than FAP. There is silence at P. If the separation becomes 24 cm, the sound becomes maximum at P and at 36 cm there is again silence and so on. The least frequency of tuning fork is

(a) 1537 Hz (b) 1735 Hz
(c) 1375 Hz (d) 1400 Hz

Solution (c) $\frac{\lambda}{2} = 12\text{ cm}$ or $\lambda = 24\text{ cm}$

$$f = \frac{330}{0.24} = 1375\text{ Hz}.$$

6. A sound source emits sound waves in a uniform medium. If energy density is E and maximum speed of the particles of the medium is v_{max}. The plot between E and v_{max} is best represented by

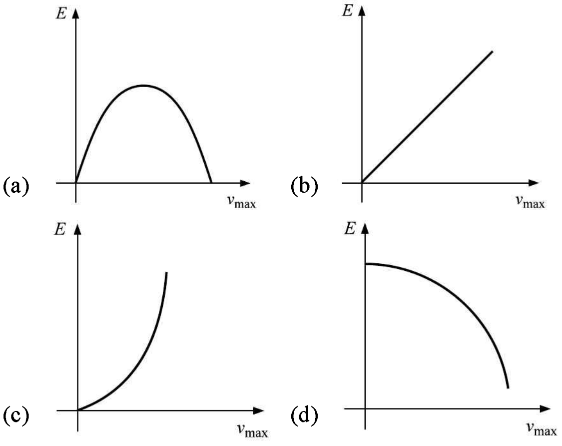
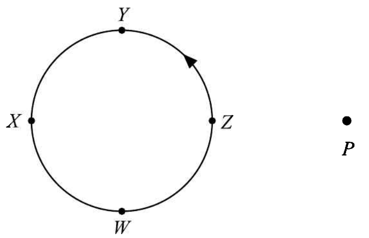


Fig. 18.12

Solution (a) Energy density $= I/v = 2\pi^2\rho f^2 x_0^2$

$v_{max} = \omega x_0 = 2\pi f x_0$, that is, $E \propto V^2_{max}$.

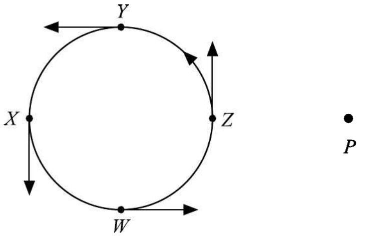
7. A sound source rotates anticlock wise with an angular velocity ω. Radius of the circle is R. A person is at P. The maximum frequency is heared when position of the source is at



(a)

(a) Y (b) X
(c) Z (d) W

Solution (d) Note from Fig. 18.13(b) that velocity at W is towards the listener. Hence according to Doppler's effect maximum frequency is heared at W.



(b)

Fig. 18.13

8. The intensity level 1 m away from a source is 60 dB. If there is no loss of sound power in air. Threshold intensity of hearing is 10^{-12}Wm^{-2}. Then intensity level at 2000 cm from the source is

(a) 45 dB (b) 34 dB
(c) 35 dB (d) 64 dB

Solution (b) $I \propto 1/r^2$ and $\Delta I = I_1 - I_2$

In $dB = 10 \log I_1/I_0 - 10 \log I_2/I_0$

$= 10 \log I_1/I_2 = 10 \log 400 = 26.02\text{ dB}$

∴ intensity level at 2000 cm away is 60 – 26 = 34 dB

9. Three tuning forks of frequency 400 Hz, 401 Hz and 402 Hz are sounded simultaneously. The number of beats heard per second are

(a) 1 (b) 2
(c) 3 (d) none of these

Solution (a) See from Fig. 18.14 that 2 Hz and 1 Hz are sounded together giving 1 beats/s.

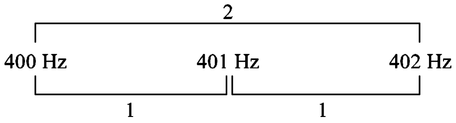


Fig. 18.14

10. Two open pipes of length 50 cm and 51 cm produce 6 beats when sounded together, find the speed of sound.

(a) 330 ms^{-1} (b) 316 ms^{-1}
(c) 306 ms^{-1} (d) 360 ms^{-1}

Solution (c) $f_1 - f_2 = 6$ or $\frac{v}{2l_1} - \frac{v}{2l_2} = 6$

$$\frac{v}{2(0.5)} - \frac{v}{2(0.51)} = 6 \text{ or } v = 306\text{ ms}^{-1}.$$

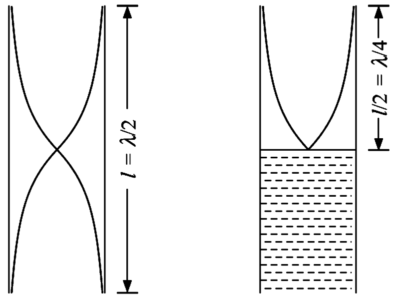
11. If fundamental frequency of an open pipe is f_0. Its fundamental frequency when it is half-filled with water is

(a) f_0 (b) $f_0/2$
(c) $2f_0$ (d) none of these

Solution (a) See the situation shown in the Fig. 18.15(b). When the pipe is half-filled with water it becomes a closed pipe and the length

$$l/2 = \lambda/4 \text{ or } \lambda = 2l$$

same wavelength existed in open pipe. Therefore frequency remains unchanged as $f = v/\lambda$.



(a) **(b)**

Fig. 18.15

12. In the experiment for determination of the speed of sound in air using resonance tube method. The length of air column that resonates with fundamental mode with a tuning fork is 0.1 m. When its length is changed to 0.35 m, the same tuning fork resonates with a length 0.35 m in first over tone. The end correction is

(a) 0.012 m (b) 0.025 m
(c) 0.05 m (d) 0.0024 m

Solution (b) $l_1 + 0.3d = \frac{\lambda}{4}$, $l_2 + 0.3d = \frac{3\lambda}{4}$;

$$\frac{\lambda}{2} = l_2 - l_1 = 0.25\,\text{m}$$

$$0.1 + 0.3d = 0.125 \quad \text{or} \quad 0.3d = 0.025\,\text{m}.$$

13. An observer moves towards a stationary source of sound with one-fifth of the speed of sound. The wavelength and frequency of the source emitted are λ and f respectively. The apparent frequency and wavelength recorded by the observer are

(a) $0.85f$, 0.8λ (b) $1.2f$, 1.2λ
(c) $1.2f$, λ (d) f, 1.2λ

Solution (c) $f_{\text{app}} = \frac{v + v/5}{v} f = 1.2f$ wavelength remains unchanged.

14. An air column closed at one end and open at the other end resonates with a tuning fork when 45 and 99 cm of length. The wavelength of the sound in air column is

(a) 36 cm (b) 54 cm
(c) 108 cm (d) 180 cm

Solution (c) $\frac{\lambda}{2} = 99 - 45 = 54\,\text{cm} \quad \therefore \quad \lambda = 108\,\text{cm}.$

15. The frequency of a tuning fork is 384 Hz and velocity of sound in air is 352 ms^{-1}. How far sound has travelled when fork completes 36 vibration

(a) 33 m (b) 16.5 m
(c) 11 m (d) 22 m

Solution (a) $x = v.t = 352 \times \frac{36}{384} = 33\,\text{m}.$

16. A sound source is falling under gravity. At some time $t = 0$ the detector lies vertically below source at a height H as shown in Fig. 18.16. If v is velocity of sound and f_0 is frequency of the source then the apparent frequency recorded after $t = 2$ second is

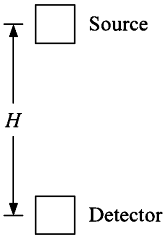


Fig. 18.16

(a) f_0 (b) $f_0 \frac{(v+2g)}{v}$
(c) $f_0 \frac{(v-2g)}{v}$ (d) $f_0 \left(\frac{v}{v-2g}\right)$

Solution (d) $v_S = 0 + g(2) = 2g$

and $f_{\text{app}} = f_0 \frac{v}{v - v_S} = f_0 \left(\frac{v}{v-2g}\right).$

17. An open pipe is suddenly closed at one end. As a result the frequency of third harmonic of the closed pipe is found to be higher by 100 Hz. The fundamental frequency of open pipe is

(a) 200 Hz (b) 30 Hz
(c) 240 Hz (d) 480 Hz

Solution $f_{0(\text{closed})} = v/\lambda = v/4l$

third harmonic of closed pipe $= 3f_{0(\text{closed})} = 3v/4l$

$3v/4l - v/2l = 100$ or $v/4l = 100$ $f_{0(\text{open})} = v/2l$

$v/2l = 200.$

18. As a wave propagates

(a) the wave intensity remains constant for a plane wave
(b) the wave intensity decreases as the inverse of the distance from source for a spherical wave
(c) the wave intensity falls as the inverse square of the distance from a spherical wave
(d) total intensity of the spherical wave over the spherical surface centred at the source remains constant at all times.

Solution (a), (c) and (d).

19. Two monatomic ideal gases 1 and 2 of molecular masses m_1 and m_2 respectively are enclosed in separate containers kept at the same temperature. The ratio of the speed of sound in gas 1 to gas 2 is given by

(a) $\frac{m_1}{m_2}$ (b) $\sqrt{\frac{m_1}{m_2}}$
(c) $\frac{m_2}{m_1}$ (d) $\sqrt{\frac{m_2}{m_1}}$

Solution (d) As $v = \sqrt{\frac{\gamma RT}{M}} \quad \therefore \quad \frac{v_1}{v_2} = \sqrt{\frac{m_2}{m_1}}.$

20. A siren placed at a railway platform is emitting sound of frequency 5 kHz. A passenger sitting in a moving train A records a frequency of 5.5 kHz when the train approaches the siren. During his return journey in a different train B he records the frequency of 6 kHz while approaching the same siren. The ratio of velocity of train B to train A is

(a) 242/252 (b) 5/6
(c) 2 (d) 11/6

Solution (c) $\left(\frac{v + v_{L1}}{v}\right)5 = 5.5$, $\left(\frac{v + v_{L2}}{v}\right)5 = 6$

or $\frac{5v_{L1}}{v} = 0.5$ or $\frac{5v_{L2}}{v} = 1$ or $\frac{v_{L2}}{v_{L1}} = 2$

21. A piezo electric quartz crystal of thickness 0.005 m is vibrating in resonate conditions. Calculate the fundamental frequency f_0 for quartz

$$Y = 8 \times 10^{10}\,\text{Nm}^{-2} \text{ and } \rho = 2.65 \times 10^3\,\text{kgm}^{-3}$$

(a) 5.5 MHz (b) 55 MHz
(c) 0.55 MHz (d) 5.5 kHz

Solution (c) $v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{8\times10^{10}}{2.69\times10^3}} = 5.5 \times 10^3\,\text{ms}^{-1}$;

$$f = \frac{v}{\lambda} = \frac{5.5\times10^3}{2\times0.005} = 5.5 \times 10^5\,\text{Hz}.$$

22. Calculate the ratio of speed of sound wave in Neon to that in H_2O vapours at any temperature.

(a) $\frac{9}{8}$ (b) $\frac{3}{2\sqrt{2}}$
(c) 3/2 (d) 8/9

Solution (b) $\frac{v_{\text{Ne}}}{v_{\text{H}_2\text{O}}} = \sqrt{\frac{\gamma_{\text{Ne}} M_{\text{H}_2\text{O}}}{M_{\text{Ne}} \gamma_{\text{H}_2\text{O}}}} = \sqrt{\frac{5/3\times18}{4/3\times20}} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}.$

23. Find the speed of sound in a mixture of 1 mole of He and 2 mole of O_2 at 27°C

(a) $480\,ms^{-1}$ (b) $621\,ms^{-1}$

(c) $401\,ms^{-1}$ (d) $601\,ms^{-1}$

Solution (c) $M_{mix} = \frac{n_1M_1 + n_2M_2}{n_1 + n_2}$

$$= \frac{1\times4+2\times32}{1+2} = \frac{68}{3}$$

$$C_{V(mixtue)} = \frac{n_1C_{V1}+n_2C_{V2}}{n_1+n_2} = \frac{\left(1\times=\frac{3}{2}+2\times\frac{5}{2}\right)R}{1+2} = \frac{13}{6}R$$

$$C_{P(mix)} = C_V + R = \frac{19}{6}R; \quad \frac{C_P}{C_V} = \frac{19}{13}R$$

$$v = \sqrt{\frac{19}{13}\times\frac{8.31\times300}{\frac{68}{3}\times10^{-3}}} = 400.9\,ms^{-1}.$$

24. The velocity of sound is v_S in air. If density of air is increased twice then the new velocity of sound will be

(a) v_S (b) $\frac{v_S}{\sqrt{2}}$

(c) $\sqrt{2}v_S$ (d) $\frac{3}{2}v_S$

Solution (b) $v = \sqrt{\frac{\gamma P}{\rho}}$, that is, $\frac{v'_s}{v_S} = \sqrt{\frac{\rho}{2\rho}} \Rightarrow v'_s = \frac{v_S}{\sqrt{2}}$.

25. Two radio stations broadcast their programmes at the same amplitude A and at slightly different frequencies ω_1 and ω_2 respectively where $\omega_2 - \omega_1 = 1\,kHz$. A detector receives the signals from the two stations simultaneously. It can only detect signals of intensity $> 2A^2$. Find the interval between successive maxima of the intensity of the signal received by the detector.

(a) $2 \times 10^{-3}s$ (b) $4 \times 10^{-3}s$

(c) $1.5 \times 10^{-3}s$ (d) $10^{-3}s$

Solution (d) $y_1 = A\sin 2\pi\omega_1 t$ and $y_2 = A\sin 2\pi\omega_2 t$

$$y = y_1 + y_2 = A\sin 2\pi\omega_1 t + A\sin 2\pi\omega_2 t$$

$$= 2A\sin 2\pi\frac{(\omega_2+\omega_1)}{2}t \qquad \cos 2\pi\frac{(\omega_2-\omega_1)}{2}t$$

$$A' = 2A\cos 2\pi\frac{(\omega_2-\omega_1)}{2}t$$

$$= 2A\cos\pi(\omega_2-\omega_1)t$$

$$I \propto A'^2 = 4A^2\cos^2\pi(\omega_2-\omega_1)t$$

For I to be maximum $\cos\pi(\omega_2-\omega_1)t = \pm1$

or $\pi(\omega_2-\omega_1)t = 0, \pi, 2\pi, \ldots$

$$T = t_2 - t_1 = \frac{1}{\omega_2-\omega_1} = 10^{-3}s.$$

26. Which of the following will pair up to produce stationary wave

(1) $Z_1 = A\cos(kx - \omega t)$ (2) $Z_2 = A\cos(kx + \omega t)$

(3) $Z_3 = A\cos(ky - \omega t)$ (4) $Z_4 = A\cos(kz + \omega t)$

(a) 1 and 2 (b) 2 and 3

(c) 3 and 4 (d) 1 and 3

Solution (a) The waves must be travelling in opposite directions and have same amplitude and same frequency.

27. A quartz crystal is used to produce ultrasonic. The frequency will be inversely related to

(a) Young's modulus (b) thickness

(c) density (d) length

Solution (b) $f \propto 1/t$.

Problems for Practice

1. When both source and listener move in the same direction with a velocity equal to half the velocity of sound, the change in frequency of the sound as detected by the listener is

(a) 50% (b) 25% (c) zero (d) none of these

2. The wavelength of the sound produced by a source is 0.8 m. If the source moves towards the stationary listener at $32\,ms^{-1}$, what will be apparent wavelength of the sound? The velocity of sound is $320\,ms^{-1}$

(a) 0.80 m (b) 0.72 m

(c) 0.40 m (d) 0.32 m

3. The wavelength of light received from a galaxy is 10% greater than that received from an identical source on the earth. The velocity of the galaxy relative to the earth is

(a) $3 \times 10^6 ms^{-1}$ (b) $3 \times 10^5 ms^{-1}$

(c) $3 \times 10^8 ms^{-1}$ (d) $3 \times 10^7 ms^{-1}$

4. The velocity of sound in air is $330\,ms^{-1}$. To increase the apparent frequency of the sound by 50%, the source should move towards the stationary source with a velocity equal to

(a) $110\,ms^{-1}$ (b) $105\,ms^{-1}$

(c) $220\,ms^{-1}$ (d) $330\,ms^{-1}$

5. A source of sound moves towards a stationary listener with the velocity of sound. If the actual frequency of the sound produced by the source be f, then change in frequency will be

(a) f (b) $f/2$ (c) $f/4$ (d) none of these

6. A train is approaching the platform with a speed of $4\,ms^{-1}$. Another train is leaving the platform with the same speed. The velocity of sound is $320\,ms^{-1}$. If both the trains sound their whistles at frequency 230 Hz, the number of beats heard per second will be

(a) 10 (b) 8 (c) 7 (d) 6

7. A man runs towards a source of sound at $10\,ms^{-1}$. The frequency of the sound produced by the source is 400 Hz and heard is 410 Hz. The speed of the sound perceived by the man will be

(a) $330\,ms^{-1}$ (b) $400\,ms^{-1}$

(c) $350\,ms^{-1}$ (d) $340\,ms^{-1}$

8. A pendulum vibrates with a time period of 1 s. The sound produced by it is in the
(a) audible range (b) infrasonic range
(c) ultrasonic range (d) super sonic range

9. Which of the following characteristics successively increases in the musical scale?
(a) quality (b) pitch
(c) loudness (d) none of these

10. To change the quality of sound produced by an instrument, we need to vary the
(a) number of overtones
(b) pitch
(c) loudness
(d) amplitude

11. Two sound waves given by $y_1 = 5\sin(300\pi t)$ and $y_2 = 4\sin(302\pi t)$ superimpose. The ratio of the maximum to minimum intensity of the sound waves will be
(a) 302/300 (b) 81
(c) 9 (d) 5/4

12. The pressure of air increases by 100 mm of Hg the temperature decreases by 1°C. What will be the change in the speed of sound in air?
(a) 61 ms^{-1} (b) 61 mm^{-1}
(c) 61 cms^{-1} (d) none of the above

13. A sound wave propagating in air may be treated either as a displacement wave or a pressure wave. What is the phase difference between the displacement and pressure wave?
(a) 180° (b) 90° (c) 45° (d) zero

14. If A is the amplitude of sound wave after covering a distance r, then
(a) $A \propto 1/r$ (b) $A \propto r^2$
(c) $A \propto 1/r^2$ (d) $A \propto r$

15. If the loudness changes from 30 dB to 60 dB. What is the ratio of the intensities in two cases?
(a) 10,000 (b) 1000
(c) 100 (d) 10

16. The power of a loud speaker is increased from 20 W to 400 W. What is the power increase as compared to the original value?
(a) 13 dB (b) 7 dB (c) 4 dB (d) 2 dB

17. What is the ratio of the speed of sound in neon and water vapour at the same temperature. It is nearest to
(a) 2.5 (b) 2 (c) 1.5 (d) 1

18. The wavelength of a sound wave is reduced by 50%. Then the percentage change in its frequency will be
(a) 100% (b) 200%
(c) 400% (d) 800%

19. A resonance tube of length 1 m is resonated with a tuning fork of frequency 300 Hz. If the velocity of sound in air is 300 ms^{-1} then the number of harmonics produced in the tube will be
(a) 1 (b) 2 (c) 3 (d) 4

20. The velocity of sound in dry air at 0°C and at 74 cm of Hg pressure is 332 ms^{-1}, then the velocity of sound at 50°C and 77.5 cm an pressure in ms^{-1} will be
(a) 322.7 (b) 347.1 (c) 352.4 (d) 361.1

21. The frequency of sound in oxygen at room temperature is v. The frequency of sound in a mixture of oxygen and hydrogen at the same temperature will be
(a) uncertain (b) equal to v
(c) less than v (d) more than v

22. Beats are the result of
(a) constructive and destructive interference in time
(b) constructive and destructive interference in space
(c) destructive interference in space
(d) constructive interference in space

23. In Kundt's tube experiment wavelength in the metallic rod and air are 80 cm and 16 cm respectively. If the velocity of sound in air is 300 ms^{-1} then the velocity of sound in rod will be
(a) 80 ms^{-1} (b) 3.75 ms^{-1}
(c) 240 ms^{-1} (d) 1500 ms^{-1}

24. The velocity of sound in air is 332 ms^{-1}. The length of a closed pipe whose frequency of second overtone is 332 Hz, will be
(a) 0.51 m (b) 0.75 m
(c) 1.25 m (d) 1.75 m

25. A closed organ pipe, of length 1.2 m and filled with a gas, is resonated in its fundamental mode with a fork. Another open pipe of same length but filled with air resonates with the same fork. The room temperature is 40°C. If the speed of sound in air at 40°C is 360 ms^{-1}, then the speed of sound in gas at 40°C will be
(a) 341.5 ms^{-1} (b) 637 ms^{-1}
(c) 633 ms^{-1} (d) 720 ms^{-1}

26. The velocity of sound in dry air at 0°C and 74 cm pressure is 332 ms^{-1} then the velocity of sound at 50°C and 77.5 cm pressure in ms^{-1} will be
(a) 322.7 (b) 347.1 (c) 352.4 (d) 361.1

27. In Kundt's tube experiment the metallic rod executes
(a) transverse vibration
(b) longitudinal vibrations
(c) both
(d) none of these

28. Five beats per second are produced on vibrating two closed organ pipes simultaneously. If the ratio of their lengths is 21/20, then their frequencies will be
(a) 105 Hz and 100 Hz (b) 105 Hz and 110 Hz
(c) 100 Hz and 105 Hz (d) 110 Hz and 105 Hz

29. If the adiabatic constant for helium and hydrogen gases at the same temperature are 5/3 and 7/5 respectively, then the ratio of velocity of sound in these gases will be
(a) 42 : 5 (c) 5 : $\sqrt{42}$
(c) $\sqrt{42}$: 5 (d) 5 : 42

30. A source of sound is emitting sound waves in all directions in an absorptionless medium. This source is at distance of 4 m and 16 m from points x and y respectively. The ratio of amplitudes of waves at points x and y will be
(a) 2 : 4 (b) 4 : 1 (c) 4 : 2 (d) 1 : 4

31. Two open pipes of length L are vibrated simultaneously. If length of one of the pipes is reduced by y, then the number of beats heard per second will nearly be (if the velocity of sound is v and $y < L$)
(a) $\frac{vy}{2L}$ (b) $\frac{2L^2}{Vy}$
(c) $\frac{vy}{2L^2}$ (d) $\frac{vy}{L^2}$

32. Two waves of wavelength 1.00 m and 1.01 m produce 10 beats in 3 s in a gas. The speed of sound in the gas will be
(a) 316.6 ms^{-1} (b) 336.6 ms^{-1}
(c) 356.6 ms^{-1} (d) 396.6 ms^{-1}

33. A source of sound of frequency 90 Hz is moving towards an observer with a velocity one-tenth the velocity of sound. The frequency heard by the observer will be
(a) 50 Hz (b) 100 Hz
(c) 200 Hz (d) 300 Hz

34. A source of sound of frequency 512 Hz is moving towards a wall with velocity v equal to that of second. An observer is standing between the source and the wall, then he will listen
(a) no beats s^{-1} (b) 3 beats s^{-1}
(c) 6 beats s^{-1} (d) 12 beats s^{-1}

35. An engine blowing whistle, is approaching a stationary observer with a velocity of 110 ms^{-1}. The ratio of frequencies heard by the observer while engine approaching and receding away from him will be (if $v = 330\,ms^{-1}$)
(a) 1 : 4 (b) 4 : 1 (c) 2 : 1 (d) 1 : 2

36. Earth is moving towards a stationary star with a velocity 100 kms^{-1}. If the wavelength of light emitted by the star is 5000 Å, then the apparent change in wavelength observed by the observer on earth will be
(a) 0.67 Å (b) 1.67 Å
(c) 16.7 Å (d) 167 Å

37. An observer measures speed of light to be c when he is stationary with respect to the source. If the observer moves with velocity v towards the source then the velocity of light observed will be
(a) $c - v$ (b) $c + v$
(c) $\sqrt{1-v^2/c^2}$ (d) c

38. A whistle is revolved with high speed in a horizontal circle of radius R. To an observer at the centre of the circle the frequency of the whistle will appear to be
(a) decreasing (b) increasing
(c) both (d) constant

39. A source of sound is emitting a wave of wavelength 40 cm in air. If this source starts moving towards east with a velocity one-fourth the velocity of sound then the apparent wavelength of sound in a direction opposite to that of source will be
(a) 20 cm (b) 50 cm (c) 80 cm (d) 100 cm

40. A siren is producing sound of frequency 930 Hz. This siren is moving away from an observer towards a wall with, velocity of 20 ms^{-1}. The frequency of sound directly coming from the siren will be
(a) 882 Hz (b) 1000 Hz
(c) 930 Hz (d) 977 Hz

41. The apparent wavelength of light from a star moving away from earth is observed to be 0.01% more than its real wavelength. The velocity of star is
(a) 120 kms^{-1} (b) 90 kms^{-1}
(c) 60 kms^{-1} (d) 30 kms^{-1}

42. A star is receding away from earth with a velocity of $10^5\,ms^{-1}$. If the wavelength of its spectral line is 5700 Å then Doppler shift will be
(a) 0.2 Å (b) 1.9 Å
(c) 20 Å (d) 200 Å

43. The wavelength of H_α line in hydrogen spectrum was found 6563 Å in the laboratory. If the wavelength of same line in the spectrum of a milky way is observed to be 6586 Å then the recessional velocity of the milky way will be
(a) $0.105 \times 10^6\,ms^{-1}$ (b) $1.05 \times 10^6\,ms^{-1}$
(c) 10.5 ms^{-1} (d) none of these

44. If a soldier jumps from an aeroplane moving with a constant horizontal velocity, then the ratio of the frequency of aeroplane sound heard by him v' and real frequency v will be
(a) 1 : 4 (b) 2 : 1 (c) 1 : 2 (d) 1 : 1

45. A rocket is receding away from earth with velocity 0.2 c. The rocket emits signal of frequency 4×10^7 Hz. The apparent frequency of the signal produced by the rocket observed by the observer on earth will be
(a) 3×10^6 Hz (b) 4×10^6 Hz
(c) 3.2×10^7 Hz (d) 5×10^7 Hz

46. A spectral line is obtained from a gas discharge tube at 5000 Å. If the rms velocity of gas molecules is $10^5\,ms^{-1}$ then the width of spectral line will be
(a) 3.3 Å (b) 4.8 Å (d) 7.2 Å (d) 9.1 Å

47. When an observer is approaching a stationary source with a velocity v_0 then the apparent change in frequency observed by him will be
(a) $\frac{v}{v+v_0}n$ (b) $\frac{v}{v_0}n$
(c) $\frac{v+v_0}{v}n$ (d) $\frac{v_0}{v}n$

48. A car is moving towards a person. The person observes a change of 2.5% in the frequency of its horn. If the velocity of sound is 320 ms^{-1} then the velocity of car is
(a) 6 ms^{-1} (b) 8 ms^{-1}
(c) 7.5 ms^{-1} (b) 800 ms^{-1}

49. A whistle produces 256 waves per second. If the velocity of whistle is towards the observer and its magnitude is one-third the velocity of sound in air then the number of waves received by the observer per second will be
(a) 192 (b) 200 (c) 300 (d) 384

50. When a source moves away from observer then apparent change in frequency is Δn_1. When an observer approaches the stationary source with same velocity v then change in frequency is Δn_2 then
(a) $\Delta n_1 = \Delta n_2$ (b) $\Delta n_1 > \Delta n_2$
(c) $\Delta n_1 < \Delta n_2$ (d) none of these

51. A SONAR inside sea works at 40 kHz. A submarine is approaching it with a velocity 360 Kmh^{-1}. If the speed of sound in water is 1450 ms^{-1} then the apparent frequency of waves after reflection from submarine will be
(a) 11.5 kHz (b) 36.8 kHz
(c) 45.9 kHz (d) 98.6 kHz

52. When a source of sound approaches a stationary observer with velocity v_s then the apparent frequency observed by the observer will be (v = velocity of sound)
(a) $\frac{v_s}{v-v_s}n$ (b) $\frac{v-v_s}{v}n$
(c) $\frac{v+v_s}{v_s}n$ (d) $\frac{v_s}{v-v_s}n$

53. A supersonic jet is moving with a velocity twice that of sound, the angle of conical wave front produced by the jet will be
(a) 120° (b) 90° (c) 60° (d) 30°

54. The frequency of radar waves is 7.8×10^9 Hz. The frequency of these waves after reflection from aeroplane is observed to have increased by 2.7×10^4 Hz. The velocity of aeroplane in $kmhr^{-1}$ will be
(a) 1.872×10^3 (b) 2.6×10^3
(c) 3.1×10^3 (d) 7.398×10^3

55. A boy blowing a whistle, is running away from a wall towards an observer with a speed of 1 ms^{-1}. The frequency of whistle is 680 Hz. The number of beats heard per second by the observer will be (given $v = 340\,ms^{-1}$)
(a) zero (b) 2 (c) 4 (d) 8

56. An ultrasonic scanner is used in a hospital to detect tumour in tissue. The working frequency of the scanner is 4.2 mega Hz. The velocity of sound in the tissue is 1.7 kms^{-1}. The wavelength of sound in the tissue is nearest to
(a) 4×10^{-3} m (b) 8×10^{-3} m
(c) 4×10^{-4} m (d) 8×10^{-4} m

57. In a Kundtz tube experiment the heaps of lycopodium powder are collected at 20 cm separations. The frequency of tuning fork used is
(a) 660 Hz (b) 825 Hz
(c) 775 Hz (d) 915 Hz

58. When a sound wave of frequency 300 Hz passes through a medium, the maximum displacement of a particle of the medium is 0.1 cm. The maximum velocity of the particle is equal to
(a) 30 cms^{-1} (b) 30π cms^{-1}
(c) 60 cms^{-1} (d) 60π cms^{-1}

59. Which of the following is mechanical wave?
(a) light waves (b) sound wave
(c) X-rays (d) radio waves

60. Which of the following properties of sound is not affected by change in the temperature of air?
(a) wavelength (b) intensity
(c) amplitude (d) frequency

61. A sound wave is represented by $y = a \sin(1000\pi t - 3x)$. The distance between two points having a phase difference of $\pi/3$ is
(a) $5\pi/18$ (b) $2\pi/9$
(c) $\pi/18$ (d) $\pi/9$

62. If a tuning fork sends a wave $5\sin\left(600\omega t - \dfrac{\pi}{0.6}x\right)$ then the amplitude of the intensity heard is

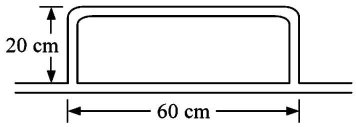


Fig. 18.17

(a) 5 (b) $5\sqrt{2}$ (c) $5\sqrt{3}$ (d) none of these

63. Surface waves strike the rock with their crests 160 m apart. The velocity of the waves is 40 ms^{-1}. the time interval between two crests striking the rock is
(a) 8 s (b) 4 s (c) 2 s (d) 1 s

64. Sound waves in air differ from the electromagnetic waves in that they cannot be
(a) diffracted (b) polarised
(c) reflected (d) refracted

65. When a stone is dropped on the surface of still water the waves produced are
(a) stationary (b) transverse
(b) longitudinal (d) none of these

66. Velocity of sound in a gas is proportional to
(a) square root of isothermal elasticity
(b) isothermal elasticity
(c) square root of adiabatic elasticity
(d) adiabatic elasticity

67. The velocity of sound in a gas is 300 ms^{-1}. The root mean square velocity of the molecules is of the order of
(a) 4 ms^{-1} (b) 40 ms^{-1}
(c) 400 ms^{-1} (d) 4000 ms^{-1}

68. A man standing unsymmetrically between two parallel cliffs, claps his hands and starts hearing a series of echoes at intervals of 1 s. If the speed of sound in air is 340 ms^{-1}, then the distance between the two parallel cliffs, is
(a) 170 m (b) 340 m (c) 510 m (d) 680 m

69. Compressional wave impulses are sent to the bottom of sea from a ship and the echo is heard after 4 s. If bulk modulus of water is $2 \times 10^9\,Nm^{-2}$ and the mean temperature is 4°C, then depth of the sea is
(a) 2000×10^3 m (b) 2828 m
(c) 1414 m (d) 707 m

70. Which of the following cannot produce ultrasonics?
(a) galton whistle
(b) quartz crystal
(c) magnetostriction effect
(d) quincke's tube

71. Velocity of sound in oxygen at NTP is v. The velocity of sound in helium at NTP should be
(a) $2v$ (b) $4v$ (c) $2\sqrt{2}\,v$ (d) none of these

72. A stone is dropped into a well and the sound of impact of stone with the water is heard after 2.056 s of the release of stone from the top. If the acceleration due to gravity is 980 cms^{-2} and the velocity of sound in air is 350 ms^{-1}, then the depth of the well is
(a) 7 m (b) 19.6 m
(c) 700 m (d) 1960 m

73. When you speak to your friend, which of the following quantities have a unique value in the sound produced?
(a) amplitude
(b) wave velocity
(c) frequency
(d) wavelength

74. Bullet train in Japan travels with 1.2 Mach. A man standing on the platform will hear the frequency as if the train blows a whistle of 800 Hz while approaching the station.
(a) 800 Hz (b) 960 Hz
(c) 1960 Hz (d) none of these

75. A rope of length l and mass m hangs freely from the ceiling. The velocity of transverse wave as a function of position x along the rope is proportional to
(a) $1/\sqrt{x}$ (b) $\sqrt{x}$ (c) x (d) x^0

76. A sound wave of frequency 500 Hz covers a distance of 1000 m in 5 s between points x and y. Then the number of waves between x and y are
(a) 5000 (b) 2500
(c) 100 (d) 500

77. A string has a mass per unit length of 10^{-6} $kgcm^{-1}$. The equation of simple harmonic wave produced in it is $y = 0.2 \sin(2x + 80t)$. The tension in the string is
(a) 0.0016 N (b) 0.16 N
(c) 16 N (d) 1.6 N

78. The velocity of sound in air is 332 ms^{-1}. If the air consists of nitrogen and oxygen in the ratio 4 : 1, the velocity of sound in oxygen is
(a) 278 ms^{-1} (b) 315 ms^{-1}
(c) 372 ms^{-1} (d) 418 ms^{-1}

79. A jet aeroplane is flying at supersonic speed. When the sound of the jet appears to be coming vertically downwards, the angle of sight of the aeroplane with the horizontal cannot be

(a) 60° (b) 40°
(c) 30° (d) 25°

80. The first loud sound is heard in a resonance tube when the air column is 26.2 cm and second load sound is heard when the air column is 80 cm. The diameter of the tube is

(a) 0.6 cm (b) 1.2 cm
(c) 2.0 cm (d) 1.6 cm

ANSWERS

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (b)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (d)	**18.** (a)	**19.** (b)	**20.** (d)
21. (d)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (d)	**26.** (d)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (b)
31. (c)	**32.** (b)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (d)	**38.** (d)	**39.** (b)	**40.** (a)
41. (d)	**42.** (b)	**43.** (b)	**44.** (d)	**45.** (c)	**46.** (a)	**47.** (d)	**48.** (b)	**49.** (d)	**50.** (b)
51. (c)	**52.** (d)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (b)	**58.** (d)	**59.** (b)	**60.** (d)
61. (d)	**62.** (a)	**63.** (b)	**64.** (b)	**65.** (b)	**66.** (c)	**67.** (c)	**68.** (c)	**69.** (b)	**70.** (d)
71. (c)	**72.** (b)	**73.** (b)	**74.** (a)	**75.** (b)	**76.** (b)	**77.** (b)	**78.** (b)	**79.** (a)	**80.** (c)

Electrostatics 19

BRIEF REVIEW OF THE CONCEPTS

Charge The fundamental property of matter with which it exerts coulomb force. Charge is of two types, positive and negative. Like charges repel and unlike charges attract. A charged particle can even attract an uncharged particle. Charges can be detected using Gold Leaf Electroscope (GLE). The unit of charge is coulomb (C). $\frac{e}{m}$ is called specific charge.

Charge is conserved (a) In an isolated system charge can neither be created nor be destroyed (b) Total charge of the universe is constant (c) Charge can be created or destroyed but in equal and opposite pairs, for example, a γ ray of energy $\geq 1.02\,\text{MeV}$ can be converted to electron and positron, that is,

$$E_\gamma(\geq 1.02\,\text{MeV}) \rightarrow \underset{\text{electron}}{e^-} + \underset{\text{positron}}{e^+}.$$

This process is called *pair production*. Even electron and positron combine to form γ ray (charge distruction)

$$e^- + e^+ \rightarrow \gamma(E_\gamma = 1.02\,\text{MeV}).$$

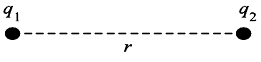
This process is called *Pair annihilation.*

Charge is quantised A charged body will have charge $Q = \pm ne$ where $n = 1, 2, 3, \ldots$ and e is charge on an electron. A body is charged by the transfer of electrons. If it gains electrons it is negatively charged. If it loses electrons it is positively charged. Though there are particles called *quarks* which may have charge $\frac{e}{3}$ or $\frac{2e}{3}$. Since these are generated during disintegration of nucleus (neutron, proton and so on). Therefore, these cannot be transferred. Charge on an electron = 1.6×10^{-19} coulomb (C).

$$1\text{ esu} = \frac{1}{3\times10^9}\,\text{C} \qquad 1\,\text{emu} = 10\,\text{coulomb}$$

A body can be charged (a) by rubbing, for example, glass rod with silk cloth (b) by induction (c) by physical contact of uncharged body with a charged body. (d) a capacitor can be charged with a battery.

Coulomb's Law $F \propto q_1q_2$



or $$F = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} \text{ in free space.}$$

{ε_0 is permittivity of free space}

$$F = \frac{q_1q_2}{4\pi\varepsilon_0\varepsilon_r r^2}$$

in a medium of relative permittivity ε_r

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\text{Nm}^2\text{C}^{-2}$$

and $$\varepsilon_0 = 8.85 \times 10^{-12}\text{C}^2\text{N}^{-1}\text{m}^{-2}$$

$$\varepsilon_r = \frac{\varepsilon_{\text{medium}}}{\varepsilon_0}$$

where ε_r is dimensionless

Coulomb's law is valid if $r \geq 10^{-15}$m and charges are point charges or spheres.

Dielectric is an insulator. It is of two types polar or non-polar. Polar dielectrics are those which have permanent electric dipole moments like water. Dielectric constant or relative permittivity for water is 80. It is ∞ for metals (infinite).

Electric field is the region or space around a given charge in which its influence can be detected.

Electric field intensity or electric field strength is the force experienced by a unit positive charge at that point in the electric field of a given charge. Its unit is N/C or V/m.

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{F}{q}$$

In vector form $= \dfrac{Q\vec{r}}{4\pi\varepsilon_0 r^3}$

There are three $\vec{E}$ types of electric vectors namely E, P and D

where $D = \varepsilon_0 E + P$ $\quad D \rightarrow$ Electric displacement

$P = \varepsilon_0(K - 1)E$ $\quad P \rightarrow$ Polarising vector

For vacuum $K = 1$

Electrets do not follow $P = \varepsilon_0(K - 1)E$

Further $\quad E = \dfrac{-dV}{dr}$

where V is electric potential.

For equipotential surface $E = 0$

If charge is not a point charge then the linear charge density $\lambda = \frac{Q}{l}$, surface charge density $\sigma = \frac{Q}{\text{Area}}$ or volume charge density $\rho = \frac{Q}{\text{Volume}}$ is determined. A small length dx (for linear charge), a small area *ds* (for surface charge density) is considered in order to find a point charge. We need to write electric force or electric field with it and integrate for total force/field. Electric field obeys superposition principle.

Electric lines of force are imaginary lines originating from positive charge and terminating at negative charge such that, tangent at any point gives the direction of electric field. No two electric lines of force intersect each other.

Electric flux The electric lines of force passing through any area in an electric field is known as electric flux. $\phi = \int E \cdot dS$. If E and S are mutually perpendicular then $\phi = 0$. The unit of electric flux is Nm^2C^{-1} and dimensions are $ML^3T^{-3}A^{-1}$. It is a scalar quantity.

Electric Potential (*V*) The amount of work done to bring a unit positive charge from infinity to that point against the electric field of a given charge without changing kinetic energy. $V = \int_{\infty}^{r} -E \cdot dx$ $= \frac{Q}{4\pi\varepsilon_0 r}$. It is a scalar quantiy and its unit is volt. 1 volt $= \frac{1J}{1C}$

The dimensions are $M^1L^2T^{-3}A^{-1}$.

Potential difference

$$\Delta V = V_2 - V_1 = \int_{r_1}^{r_2} -E \cdot dr = \frac{Q}{4\pi\varepsilon_0 r}\Bigg|_{r_1}^{r_2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right].$$

Equipotential surface is the surface, where the potential is equal at every point. For a point charge, a sphere will be equipotential surface with charge at the centre of the sphere. Equipotential surface for a long line charge is cylinder with line charge along axial line. For a dipole, equipotential surface is a plane passing through equatorial line.

The work done in carrying a charge from one point to another in an equipotential surface is zero.

The electric lines of force are always perpendicular to the equipotential surface.

Every conductor is an equipotential surface as electric field E is perpendicular to it.

Electric field along the equipotential surface is zero. These surfaces do not intersect each other.

$$\vec{E} = -\nabla V$$

Electric field and surface charge density are maximum at pointed ends.

Electric field intensity due to a shell

$$E_{\text{inside}} = 0 \qquad x < R$$

$$E_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad x = R$$

$$E_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 x^2} \qquad x > R$$

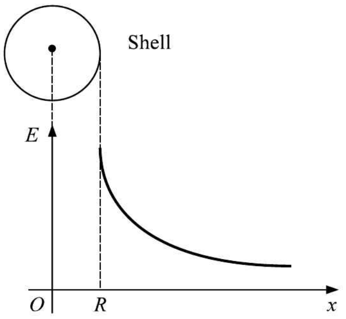


Fig. 19.1

Electric potential due to a shell

$$V_{\text{inside}} = \frac{Q}{4\pi\varepsilon_0 R} = V_{\text{surface}} \qquad x < R$$

$$V_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R} \qquad x = R$$

$$V_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 x} \qquad x > R$$

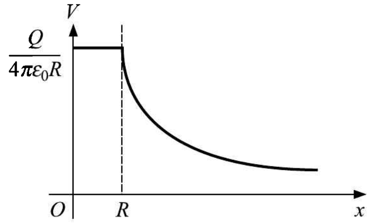


Fig. 19.2

Electric field due to a finite line charge on perpendicular bisector

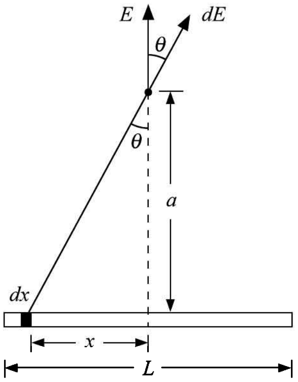


Fig. 19.3

$$E = \frac{Q}{2\pi\varepsilon_0 a\sqrt{L^2 + 4a^2}}$$

Electric field intensity due to a ring of radius *r* at a distance *x* on the axial line

$$E = \frac{Qx}{4\pi\varepsilon_0 (x^2 + r^2)^{3/2}}$$

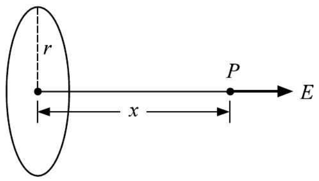


Fig. 19.4

At the centre of the ring $E = 0$

Electric field is maximum if $x = \frac{r}{\sqrt{2}}1$

Electric potential at the point P is $\frac{Q}{4\pi\varepsilon_0\sqrt{x^2 + r^2}}$

Electric field intensity due to a disc of radius *r* having surface charge density σ at a point *P*, distant *x* on the axial line is

$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{x^2 + r^2}}\right] = \frac{\sigma}{2\varepsilon_0} \text{ if } r \to \infty$$

or $x \to 0$, that is, at the centre of the disc

Electric potential $V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{x^2 + r^2} - x\right]$

Dipole moment $\vec{P} = q(2l)$. The direction is along negative to positive side.

Electric field intensity due to a dipole

(a) Along axial line

$$E_{\text{axial}} = \frac{2px}{2\pi\varepsilon_0(x^2 - l^2)^2}$$

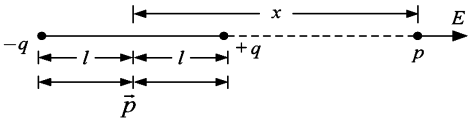


Fig. 19.5

for a short dipole $x >> l$ $E_{\text{axial}} = \frac{2p}{4\pi\varepsilon_0 x^3}$

(b) **Along equatorial line**

$$E_{\text{equatorial}} = \frac{p}{4\pi\varepsilon_0(x^2 + l^2)^{3/2}}$$

Note: The direction of electric field is antiparallel to dipole moment p. For a short dipole.

$$E_{\text{equatorial}} = \frac{p}{4\pi\varepsilon_0 x^3}$$

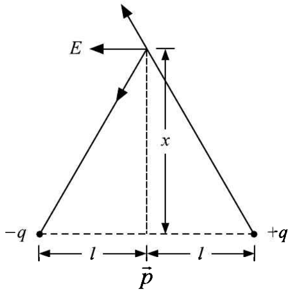


Fig. 19.6

(c) At any point due to a short dipole

$$E_{\text{any point}} = \frac{p}{4\pi\varepsilon_0 x^3}\sqrt{3\cos^2\theta + 1}\,;\ \tan\alpha = \frac{\tan\theta}{2}$$

$$E_x = \frac{-\partial V}{\partial x} \qquad E_y = \frac{-1}{x}\frac{\partial V}{\partial \theta}$$

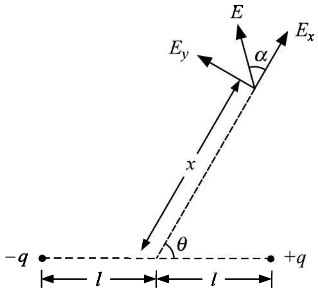


Fig. 19.7

Special cases if $\theta = 0$, $E = E_{\text{axial}} = \frac{2p}{4\pi\varepsilon_0 x^3}$ due to a short dipole.

If $\theta = 90°, E = E_{\text{equatorial}} = \frac{p}{4\pi\varepsilon_0 x^3}$ due to a short dipole.

Potential due to a dipole

(a) Along axial line

$$V_{\text{axial}} = \frac{p}{4\pi\varepsilon_0(x^2 - l^2)} = \frac{p}{4\pi\varepsilon_0 x^2}$$

due to a short dipole.

(b) Along equatorial line

$$V_{\text{equatorial}} = 0$$

(c) At any point

$$V_{\text{any point}} = \frac{p\cos\theta}{4\pi\varepsilon_0 x^2} \quad \text{due to a short dipole.}$$

Torque experienced by a dipole when placed in a uniform electric field

$\Sigma F = 0;\quad \tau = \mathrm{d}\vec{P}\times\vec{E} = PE\sin\theta$

Special cases if $\theta = 0$ or $180°$, $\tau = 0$

If $\theta = 90°$, τ is maximum $= pE$

Equilibrium is stable if $\theta = 0°$ and unstable if $\theta = 180°$.

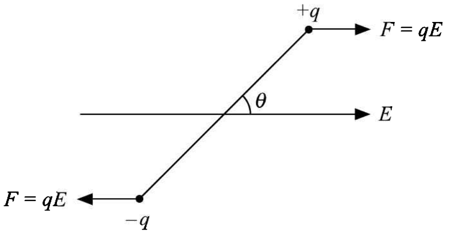


Fig. 19.8

Work done $W = \int_{\theta_1}^{\theta_2} \tau \cdot d\theta = pE(\cos\theta_1 - \cos\theta_2)$

If $\theta_1 = 0$, $\theta_2 = 180°$ $W = 2pE$;

If $\theta_1 = 0$, $\theta_2 = 90°$ $W = pE$

Potential energy $= -pE\cos\theta$

If electric field is nonuniform

$\Sigma F \neq 0 \qquad \tau \neq 0$

$$\vec{F} = \vec{p}\times\frac{d\vec{E}}{dx}$$

Potential energy (*PE*) It is the amount of work done to bring a charge q from infinity to that point against the electric field of a given charge without changing its kinetic energy.

$$PE = \frac{qQ}{4\pi\varepsilon_0 r} = qV$$

Work done $W = \Delta PE = PE_{\text{final}} - PE_{\text{initial}}$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

Force on a surface charge The repulsive force acting on an element of a charged surface due to rest of the charged surface is called electric force on a charged conducting surface.

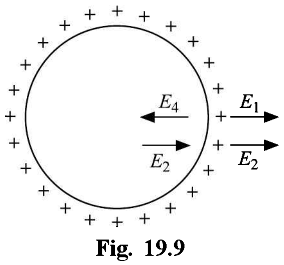


Fig. 19.9

$E = E_1 + E_2 = \dfrac{\sigma}{\varepsilon_0}$ where E_1 is the electric field due to small element and E_2 is due to rest of the charge.

$\because \; E_1 = E_2$

Therefore, electric field intensity near a charged surface $= \dfrac{\sigma}{2\varepsilon_0}$ and force $dF = \dfrac{\sigma^2}{2\varepsilon_0} ds$

Therefore, $F = \oint \dfrac{\sigma^2}{2\varepsilon_0} ds$

and pressure $P_{\text{elect.}} = \dfrac{dF}{ds} = \dfrac{\sigma^2}{2\varepsilon_0} = \dfrac{\varepsilon_0 E^2}{2}$

In case of a soap bubble

$$P_{\text{in}} - P_{\text{out}} = P_{\text{excess}} = P_{ST} - P_{\text{elect.}} = \frac{4T}{r} - \frac{\sigma^2}{2\varepsilon_0}$$

$$= \frac{4T}{r} - \frac{q^2}{2A^2\varepsilon_0} = \frac{4T}{r} - \frac{q^2}{32\pi^2 r^4 \varepsilon_0}$$

If air pressure inside and outside the bubble are equal then

$$P_{\text{in}} = P_{\text{out}} \quad \text{or} \quad \frac{4T}{r} = \frac{q^2}{32\pi^2 r^4 \varepsilon_o}$$

Energy density = Energy per unit volume

$$u = \frac{\varepsilon_0 E^2}{2} = \frac{\sigma^2}{2\varepsilon_0}$$

Total energy $U_{\text{tot}} = \int \dfrac{\varepsilon_0 E^2}{2} dV$

Charged liquid drop If n identical drops each of radius r and charge q are joined to form a big drop of radius R and charge Q

then $R = n^{1/3} r \quad Q_{\text{big}} = nq_{\text{small}}$

$E_{\text{big}} = n^{1/3} E_{\text{small}}$; $V_{\text{big}} = n^{2/3} V_{\text{small}}$; $\sigma_{\text{big}} = n^{1/3} \sigma_{\text{small}}$

Equilibrium for a drop of charge q, mass m to be balanced

$$E = \frac{mg}{q}$$

Equilibrium is of three types

Fig. 19.10

(a) stable (b) unstable (c) neutral

Stable equilibrium $F = \dfrac{-dU}{dr} = 0$

For stable equilibrium $F = 0$; $U(PE)$ = is minimum

This is possible if at extreme ends charges are similar and in between (where equilibrium is found) charge is opposite in nature.

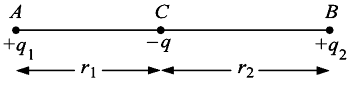


Fig. 19.11

For stable equilibrium $\Sigma F = 0$ at A, B or C.

For q to be in equilibrium $\dfrac{q_1}{q_2} = \dfrac{r_1^2}{r_2^2}$ or $\dfrac{r_1}{r_2} = \dfrac{\sqrt{q_1}}{\sqrt{q_2}}$

For q_2 to be in equilibrium

$$\frac{q}{q_1} = \frac{r_2^2}{(r_1 + r_2)^2}\pi \quad \text{or} \quad \frac{r_2}{r_1 + r_2} = \frac{\sqrt{q}}{\sqrt{q_1}}$$

For q_1 to be in equilibrium

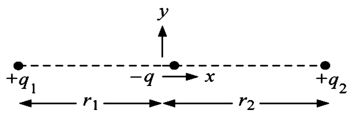
$$\frac{q}{q_2} = \frac{r_1^2}{(r_1 + r_2)^2}$$

or $\dfrac{r_1}{r_1 + r_2} = \sqrt{\dfrac{q}{q_1}}$

A particle in stable equilibrium will execute SHM if disturbed slightly along x or y direction.

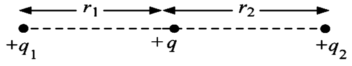
If x and y are large, motion is oscillatory but not SHM.

Unstable equilibrium For unstable equilibrium $\Sigma F = 0$, $U(PE)$ is maximum.



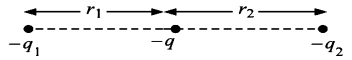
Stable equilibrium

(a)



Unstable equilibrium

(b)



Unstable equilibrium

(c)

Fig. 19.12

This is possible if charges are similar q will be in equilibrium if

$$\frac{q_1}{q_2} = \frac{r_1^2}{r_2^2}$$

or $\dfrac{r_1}{r_2} = \sqrt{\dfrac{q_1}{q_2}}$

q_1 and q_2 cannot be in stable equilibrium.

Particle will not execute SHM if slightly displaced along x or y direction rather it may move linearly.

Charged particle in motion

$$F = qE$$

$$\therefore \quad ma = qE$$

$$\text{or} \quad a = \frac{qE}{m}$$

Velocity v after travelling a distance d

$$v = \sqrt{\frac{2qEd}{m}} \quad (\text{Apply } v^2 = 2as)$$

Velocity after time t if it starts from rest.

$$v = at = \frac{qEt}{m}$$

SHORT-CUTS AND POINTS TO NOTE

(i) Coulomb force $\vec{F} = \dfrac{q_1 q_2 \vec{r}}{4\pi\varepsilon_0 r^3}$ in vacuum is valid if

(a) Charge is a point charge (b) separation between the charges $> 10^{-15}$ m. If the charge is distributed, consider a point charge on it using linear charge density λ (if charge is a line charge), surface charge density σ (if charge is distributed on a surface) and volume charge density ρ (if charge is distributed on the volume). Apply the law for a small element and integrate to find total force or electric field as the case may be.

(ii) The force is mutual, that is, $\vec{F}_{12} = -\vec{F}_{21}$

(iii) Newton's third law is inapplicable in certain cases with reference to electrostatics or electrodynamics. For instance, if a charge q_1 is placed inside a conducting shell while charge q_2 is outside the shell as shown in the Fig. 19.13 then the force of q_1 on $q_2 \neq 0$ while force of q_2 on q_1 is zero.

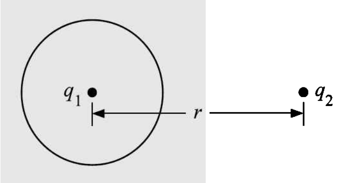


Fig. 19.13

$$\vec{F} = \frac{q_1 q_2 \vec{r}}{4\pi\varepsilon_0 \varepsilon_r r^3}$$ in any other medium.

(iv) If there is more than one medium as shown the dielectric slab of dielectric constant *k*, thickness *t* is placed in between two charges. Then to solve the problem, find equivalent of distance *t* in vacuum, that is, equivalent distance is $t\sqrt{k}$ in vacuum.

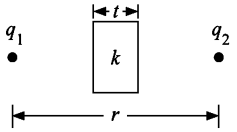


Fig. 19.14

Hence the net distance between the charges is

$$r' = r - t + t\sqrt{k}$$

$$\therefore \quad F = \frac{q_1 q_2}{4\pi\varepsilon_0 (r - t + \sqrt{k}t)^2}$$

Note that the effective distance in vacuum for a dielectric slab of thickness *t* and dielectric constant *k* is $t_{\text{eff}} = \sqrt{k}\, t$.

(v) The electric field intensity or electric force is a vector quantity. Therefore, exploit vector algebra to solve problems.

(vi) Electric field intensity due to a point charge

$$E = \frac{F}{q} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

(vii) Electric field intensity inside a hollow body (conducting) is zero irrespective of its shape.

(viii) Electric field intensity due to a shell (spherical)

$$E_{\text{in}} = 0; \quad E_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R^2}; \quad E_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 x^2}$$

(ix) Electric field intensity due to a dipole $E \propto \dfrac{1}{x^3}$

$$E_{\text{axial}} = \frac{2px}{4\pi\varepsilon_0 (x^2 - l^2)^2}$$

$= \dfrac{2p}{4\pi\varepsilon_0 x^3}$. For a short dipole along axial line or end on position.

Electric field is parallel to dipole moment

$$E_{\text{equaltorial}} = \frac{p}{4\pi\varepsilon_0 (x^2 + l^2)^{3/2}}$$

$= \dfrac{\pi}{4\pi\varepsilon_0 x^3}$. For a short dipole along equatorial line or broad side-on position.

$E_{\text{equatorial}}$ is antiparallel to dipole moment $\vec{p}$

$$E_{\text{any point}} = \frac{p}{4\pi\varepsilon_0 x^3}\sqrt{3\cos^2\theta + 1}$$

for a short dipole and angle $\tan\alpha = \tan\theta/2$.

(x) Electric field intensity due to a ring at any point on axial line

$$E_{\text{ring}} = \frac{Qx}{4\pi\varepsilon_0 (x^2 + r^2)^{3/2}}$$

Electric field strength is zero at the centre of the ring. It is maximum when $x = \dfrac{r}{\sqrt{2}}$.

Electric field due to a disc of radius *r*, along the axial line at any point

$$E_{\text{disc}} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{x^2 + r^2}}\right]$$

(xi) Torque due to a dipole

If electric field is uniform. $\tau = p \times E$ and $\Sigma F = 0$.

Torque is maximum if $\theta = 90°$.

(xii) Work done

$W = \int_{\theta_1}^{\theta_2} \tau \cdot d\theta = pE(\cos\theta_1 - \cos\theta_2)$ work done is maximum if angle of twist is 0° to 180°.

(xiii) If electric field is non-uniform then force $\vec{F}$ is given by

$$\vec{F} = \vec{p} \times \frac{d\vec{E}}{dx}$$

There is linear motion along with rotational motion. To balance a torque, torque is needed and to balance a force, force is required. Hence both force and torque are required to balance a dipole in a non-uniform field.

(xiv) $V = -\int E \cdot dl \quad \oint E \cdot dl = 0$, for a closed loop because electrostatic force is conservative

If $|E_1| = |E_2|$ then resolve the vectors. We get magnitude and direction, for example, in the Fig. 19.15.

$$E = 2E_1 \cos\theta = \frac{2q}{4\pi\varepsilon_0 l^2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}q}{4\pi\varepsilon_0 l^2}$$

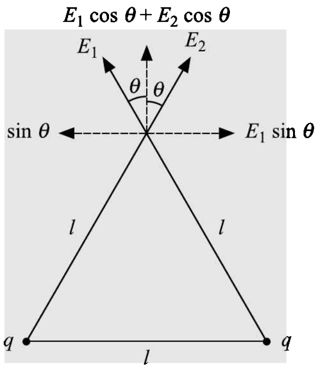


Fig. 19.15

(xv) $E = -\dfrac{dV}{dx}$ suggests that if V = constant, $E = 0$;

$E = 0$ if V is maximum and; $E = 0$ if V is mimimum.

(xvi) It is possible that $E = 0$, $V \neq 0$ (inside the shell or a hollow conductor)

and $E \neq 0$, $V = 0$ (in case of a dipole along the equatorial line)

Also if $Q = 0$, $E = 0$, $V = 0$

(xvii) A moving charge in a dielectric generates electric field and magnetic field while current in a conductor generates only magnetic field. In a conductor, $E_{\text{inside}} = 0$.

(xviii) Electric potential

$$V = -\int_{\infty}^{r} E \cdot dx = \frac{q}{4\pi\varepsilon_0 r}$$

(potential difference) $\Delta V = -\int_{r_1}^{r_2} E \cdot dx$

$$= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r_2} - \frac{1}{r_1}\right\}$$

For three dimensional electric field

$$V = -\left[\int_{\infty}^{x} E_x \cdot dx + \int_{\infty}^{y} E_y \cdot dy + \int_{\infty}^{z} E_z \cdot dz\right]$$

(xix) Electric potential due to a shell

$$V_{\text{in}} = V_{\text{sur}} = \frac{Q}{4\pi\varepsilon_0 R}; \quad V_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 x} \; x > R$$

(xx) Electric potential due to a dipole

$$V_{\text{axial}} = \frac{p}{4\pi\varepsilon_0 (x^2 - l^2)}; \quad V_{\text{equatorial}} = 0$$

$V_{\text{any point}} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 x^2}$ due to a short dipole.

(xxi) Potential energy

$$PE = -\int_{\infty}^{r} F \cdot dx = \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = qV$$

Change in potential energy

$$\Delta PE = -\int_{r_1}^{r_2} F \cdot dx = \theta \Delta V = q\,(V_2 - V_1)$$

$$= \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \text{Work done}$$

(xxii) For equipotential surface $\Delta PE = 0$ or no work is done when, charge moves from one point to the other.

To find the velocity gained/lost by a charged particle, use $\Delta KE = \Delta PE$

(xxiii) Acceleration of a charged particle under electric field

$a = \dfrac{qE}{m}$.

Apply equations of mechanics $v = u + at$, $s = ut + \dfrac{1}{2}at^2$, $v^2 + u^2 = 2as$ for further processing.

If acceleration is in y-direction while initial motion was along x-direction, then $v_y = a_y t = \dfrac{qE}{m}t$

and $v_{\text{net}} = \sqrt{v_x^2 + \left(\dfrac{qE}{m}t\right)^2}$

and $\tan\beta = \dfrac{v_y}{v_x} = \dfrac{qEt}{mv_x}$ and $t = \dfrac{l}{v_x}$.

for a particle projected in the electric field as shown in Fig. 19.16.

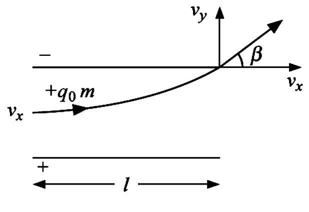


Fig. 19.16

(xxiv) If charges are similar at extreme position and an opposite charge is placed in between the equilibrium is stable. If all the charges are similar equilibrium is unstable.

(xxv) If n drops each of radius r and charge q coalesce to form a big drop then $R_{\text{big}} = n^{1/3}r$, $q_{\text{big}} = nq$, $V_{\text{big}} = n^{2/3}V_{\text{small}}$. $C_{\text{big}} = n^{1/3}C_{\text{small}}$.

⚠ CAUTION

(i) Adding electric field intensity or force algebraically.

- ■ They are vectors hence vector algebra be applied.

(ii) Considering potential is also vector because $V = \int -E \cdot dx$

- ■ Potential or PE are scalar and are added algebraically.

(iii) Not remembering to put vector sign while finding electric field from potential because $E = -\dfrac{dV}{dr}$.

- ■ Actually $\vec{E} = -\left[\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right]V$

(iv) Not finding net electric field or net potential when group of charges are given in the problem.

- ■ Must find the equivalent electric field strength or electric potential as per problem when a group of charges are given. For finding electric field strength, vector algebra be used. Apply superposition theorem for electric force/field. For finding potential use algebraic rules.

(v) Ignoring the directions of velocities and accelerations. For instance, a charged particle is initially moving in x-direction with speed u but due to electric field acceleration is formed in y- direction. Applying equation $v = u + at$ or $s = ut + \frac{1}{2}at^2$ is not correct.

- Use $v = u\hat{i} + at\hat{j}$ so that $|v| = \sqrt{u^2 + a^2t^2}$

and $\beta = \tan^{-1}(u/at)$.

(vi) Not recalling that $E_{\text{inside}} = 0$ for a shell or for a hollow metallic body.

- $E_{\text{inside}} = 0$ for a shell, for a hollow conductor and solid conductor. If, however, charge is distributed uniformly on a solid sphere then $E_{\text{inside}} \neq 0$. Rather, $E_{\text{inside}} = \dfrac{Qx}{4\pi\varepsilon_0 R^3}$, that is, $E_{\text{inside}} \propto x$.

(vii) Not remembering $V_{\text{in}} = V_{\text{sur}} = \dfrac{Q}{4\pi\varepsilon_0 R}$ in case of a shell.

Interpreting $V = 0$ because $E = 0$ is superfluous.

- If $E = 0$ and $Q = 0$ only then $V = 0$.

  Even $V = 0$, $E \neq 0$ occurs along equatorial line in a dipole.

  In a shell when $E = 0$; $V \neq 0$.

(viii) Not recalling that work done on an equipotential surface is zero.

- Because electrostatic force is conservative, therefore, $W = q(V_1 - V_2) = 0$ on equipotential surface. Moreover, $\oint E \cdot dl = 0$.

However, work will be done if a charge moves from one equipotential surface to another equipotential surface.

(ix) The notion that electric lines of force can never form a closed loop.

- In case of electromagnetic induction these lines form a closed loop.

(x) The notion that similar charges only repel.

- If one charge is very large as compared to another charge then even similar charges or like charges attract. The reason is that due to induction an opposite charge appears.

(xi) Considering that in case of electrostatics work done depends upon path followed.

- Work done is independent of path as electrostatic force is conservative.

(xii) Considering that a small sphere (~mm) or edge of a pin can hold a large charge.

- Due to carona discharge, charge leaks from sharp corners or edges.

(xiii) Considering equipotential surfaces can intersect.

- They cannot intersect otherwise electric field will have two directions at the point of intersection as electric field is perpendicular to the surface.

(xiv) Considering that a positively charged insulated body has always positive potential.

- It may have negative potential if placed in the electric field generated by strong negative charge.

(xv) Considering that a charged metal plate has uniformly distributed charge.

- A charged metal plate has maximum charge density at the corners of the plate and minimum at flat portion.

Solved Problems

1. What uniform electric field be applied so that a charged particle starts moving with same speed to its left after 2.65 μs

(a) 2.35 N/C toward left

(b) 0.235 NC^{-1} toward left

(c) 23.5 NC^{-1} towards left

(d) 2.35 NC^{-1} towards right

Solution (c) $v = u + at$

$$-1.5\hat{i} = 1.5\hat{i} + a(2.65 \times 10^{-6})$$

or $$a = -\frac{3 \times 10^6 \times 10^3 \hat{i}}{2.65}$$

Using $qE = ma$

$$E = \frac{ma}{q} = -\frac{6.64 \times 10^{-27} \times 3 \times 10^6 \times 10^3 \hat{i}}{2 \times 1.6 \times 10^{-19} \times 2.65}$$

$$= -23.5\hat{i}\ \text{N/C}$$

i.e. 23.5 N/C toward left.

2. A charge $+q$ is placed at $(a, 0, 0)$ and another $+q$ charge is placed at $(-a, 0, 0)$. A charge $-q$, is placed at the origin if it is slightly displaced along y-axis:

(a) it will move away

(b) it will oscillate but not SHM

(c) it will execute SHM

(d) it will stand at the displaced position

Solution (c)

3. Assuming mass m of the charged particle in Question 2, find time period of oscillation.

(a) $2\pi\sqrt{\dfrac{q_1 q}{4\pi\varepsilon_0 a^3 m}}$

(b) $2\pi\sqrt{\dfrac{4\pi\varepsilon_0 a^3 m}{q_1 q}}$

(c) $\dfrac{\pi}{2}\sqrt{\dfrac{q_1 q}{4\pi\varepsilon_0 a^3 m}}$

(d) none of these

Solution $E = 2E_1 \cos\theta = \dfrac{q}{4\pi\varepsilon_0 (x^2 + a^2)} \dfrac{x}{\sqrt{x^2 + a^2}}$

$$-q_1 E = F = mf = \frac{-q_1 q x}{4\pi\varepsilon_0 (x^2 + a^2)^{3/2}}$$

as $x << a$ neglecting x^2 as compared to a^2

acceleration $f = -\dfrac{q_1 q x}{4\pi\varepsilon_0 a^3 m}$

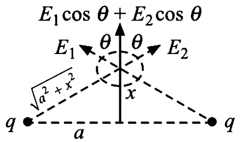


Fig. 19.17

$$\omega = \sqrt{-\frac{q_1 q}{4\pi\varepsilon_0 a^3 m}}$$

or $$T = 2\pi\sqrt{-\frac{4\pi\varepsilon_0 a^3 m}{q_1 q}}$$

4. As per diagram, a charge q is placed at the origin O. Work done by a charge $-Q$, in taking it from $A(0, a)$ to $B(a, 0)$ along the path AB

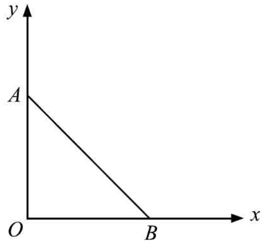


Fig. 19.18

(a) zero (b) $\sqrt{2}{}_2a\left(\frac{qQ}{4\pi\varepsilon_0 a^2}\right)$

(c) $\left(\frac{-qQ}{4\pi\varepsilon_0 a^2}\right)\sqrt{2}\,a$ (d) $\left(\frac{qQ}{4\pi\varepsilon_0 a^2}\right)\frac{a}{\sqrt{2}}$

Solution (a) A and B are at same potential

$\therefore\ W = 0$

5. Two charges q_1 and q_2 are placed 30 cm apart as shown in Fig 19.19. A third charge q_3 is moved along the arc of a circle of radius 40 cm from C to D. The change in potential energy of the system is $\frac{q_3 k}{4\pi\varepsilon_0}$ where k is

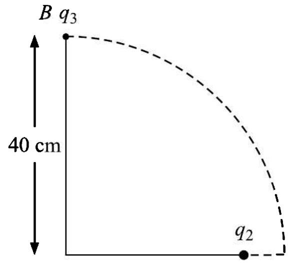


Fig. 19.19

(a) $8q_1$ (b) $6q_1$

(c) $8q_2$ (d) $6q_2$

Solution (c) $\Delta PE = \frac{q_2 q_3}{4\pi\varepsilon_0}\left[\frac{1}{0.1} - \frac{1}{0.5}\right] = \frac{8q_2 q_3}{4\pi\varepsilon_0}$

6. Two point charges $+8q$ and $-2q$ are located at $x = 0$ and $x = L$ respectively. The location of a point on the x-axis at which the net electric field due to these two point charges is zero.

(a) $2L$ (b) $L/4$

(c) $8L$ (d) $4L$

Solution (a) $-\frac{2q}{4\pi\varepsilon_0(x-L)^2} + \frac{8q}{4\pi\varepsilon_0 x^2} = 0$ or $x = 2L$

7. Two thin wires rings each having a radius R are placed at a distance d apart with their axis coinciding. The charges on the two rings are $+q$ and $-q$. The potential difference between the rings are:

(a) $\frac{QR}{4\pi\varepsilon_0 d^2}$ (b) $\frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+d^2}}\right]$

(c) $\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+d^2}}\right]$ (d) zero

Solution (b) $V_1 = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+d^2}}\right]$

and $V_2 = \frac{-q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+d^2}}\right]$

$\Delta V = V_1 - V_2$

8. A uniform electric field and a uniform magnetic field are acting along the same direction in a certain region. If an electron is projected along the direction of the fields with a certain velocity then

(a) its velocity decreases
(b) its velocity increases
(c) it will turn towards right of its motion
(d) it will turn towards left of direction of motion

Solution (a) $F = -eE + e(\vec{v}\times\vec{B}) = -eE$

and $v = v_0 - \frac{eE}{m}t$

9. Four point positive charges of same magnitude Q are placed at the four corners of a rigid square frame a shown in Fig. 19.20. The plane of the frame is perpendicular to z-axis. If a negative charge $-q$ is placed at a distance z away from the above frame ($z << L$) then

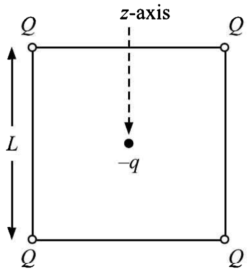


Fig. 19.20

(a) negative charge oscillates along the z-axis
(b) it moves away from the frame
(c) it moves slowly towards the frame and stays in the plane of the frame
(d) it passes through the frame only once.

Solution (a) Because the resultant force acts as restoring force.

10. The work done in carrying a charge q once round a circle of radius r with a charge Q at the centre is

(a) $\frac{qQ}{4\pi\varepsilon_0 r}$ (b) $\frac{qQ}{4\pi\varepsilon_0^2 r^2}$

(c) $\frac{qQ}{4\pi\varepsilon_0 r^2}$ (d) none of these

Solution (d) $W = 0\ \because$ Electrostatic force is conservative.

11. Two small spheres each of mass m and charge q are tied from the same rigid support with the help of silk threads of length L. They make angle θ with the vertical as shown in the Fig. 19.21. If length L is decreased then angle θ with the vertical

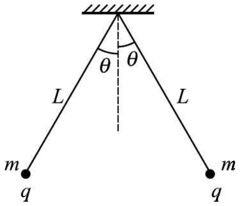


Fig. 19.21

(a) increases (b) decreases
(c) unaffected (d) connot say

Solution (a) θ is related inversely to length L.

12. Two small spheres each of radius 1 mm are kept 10 cm apart. Assuming each proton has a charge $+e$ and each electron has a charge 0.1% less than $+e$ then find the force between the two spheres. Density of copper is 8.9 g cm^{-3} and atomic mass number is

(a) 1.2×10^2N (b) 1.2×10^{-2}N
(c) 1.2×10^3N (d) 1.2×10^{-4}N

Solution (d) Mass of 1 mm radius sphere

$$= \frac{qQ}{4\pi\varepsilon_0 r^2} A 63.5 \times \frac{4}{3}\pi(0.1)^3 = 0.2667\,\text{g}$$

Charge on the sphere

$$\frac{0.2667}{63.5} \times 6.023 \times 10^{23} \times 29 \times \frac{0.1}{100} \times 1.6 \times 10^{-19} = 11.6\,\text{C}$$

$$F = \frac{11.6 \times 11.6 \times 9 \times 10^9}{(0.1)^2} = 1.21 \times 10^{14}\,\text{N}$$

13. Find the minimum force between the two electrons of the nucleus. Assume radius of the nucleus = 6.8 Å

(a) 12 N (b) 1.2 N
(c) 0.12 N (d) 0.012 N

Solution (b) Force will be minimum when electrons are diametrically opposite.

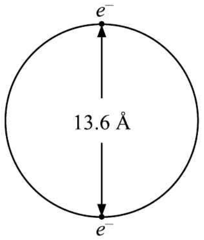


Fig. 19.22

$$\therefore F_{\min} = \frac{1.6 \times 1.6 \times 10^{-38} \times 9 \times 10^9}{(13.6)^2 \times 10^{-20}} = 1.2\,\text{N}$$

14. A line charge of length l and charge Q uniformly distributed over the whole length is placed a distance r from one edge from a point charge q as shown. Find the force on the point charge

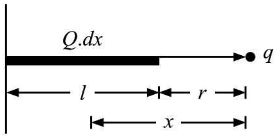


Fig. 19.23

(a) $\frac{qQ}{4\pi\varepsilon_0 r(r+l)}$ (b) $\frac{qQ}{4\pi\varepsilon_0 (r+r/2)^2}$

(c) $\frac{qQ}{4\pi\varepsilon_0}\left[\frac{1}{(r)^2} - \frac{1}{(r+l)^2}\right]$ (d) none of these

Solution (a) Consider a small element dx of the line charge at a distance x from the point charge.

$$\text{Force} \quad dF = \frac{qQ\,dx}{L 4\pi\varepsilon_0 x^2}$$

$$F = \frac{qQ}{4\pi\varepsilon_0 l}\int_r^{r+l}\frac{dx}{x^2} = \frac{qQ}{4\pi\varepsilon_0 l}\left[\frac{1}{r} - \frac{1}{r+l}\right]$$

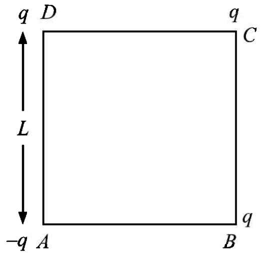
$$= \frac{qQ}{4\pi\varepsilon_0 r(r+l)}$$

15. When two charges are equal q each, for they exert on each other is F. When one of the charge is doubled the $2q$ charge exerts a force $2F$ on charge q. The force exerted by q on $2q$ is

(a) F (b) $F/2$
(c) $F/4$ (d) $2F$
(e) $4F$

Solution (d) Force is mutual.

16. *ABCD* is a square frame of side l. The force at B if charges as shown in Fig. 19.24 (a) are placed at the corners of the square.

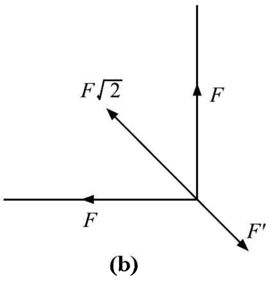


(a)
Fig. 19.24

(a) $\frac{q^2(2\sqrt{2}-1)}{4\pi\varepsilon_0 2l^2}$ (b) $\frac{q^2(2\sqrt{2}+1)}{4\pi\varepsilon_0 l^2}$

(c) $\frac{q^2(2\sqrt{2}-1)}{4\pi\varepsilon_0 l^2}$ (d) $\frac{q^2(2\sqrt{2}+1)}{4\pi\varepsilon_0 2l^2}$

Solution (a) As illustrated in Fig. 19.24 (b). The net force at B is



(b)
Fig. 19.24

$$F\sqrt{2} - F' = \frac{q^2\sqrt{2}}{2\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0(\sqrt{2}l)^2} = \frac{q^2(2\sqrt{2}-1)}{4\pi\varepsilon_0 2l^2}$$

17. Charge Q is distributed uniformly on length l of a wire. It is bent in the from of a circumarticular. Find the electric field at the centre of the ring.

Solution (d) Consider two small elements of length dl each. Charge dq on each element

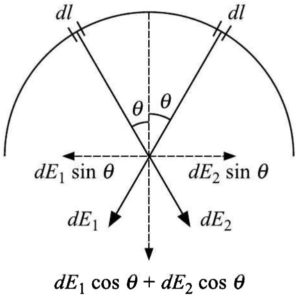


Fig. 19.25

$$dq = \frac{Q}{l}dl = \frac{Q(rd\theta)}{\pi r} = \frac{Qd\theta}{\pi}$$

∵ $|dE_1| = |dE_2|$, resolve dE_1 and dE_2 their sin components cancel out.

$$dE = 2dE_1\cos\theta = \frac{2Qd\theta}{\pi 4\pi\varepsilon_0 r^2}\cos\theta$$

and
$$E = \frac{2Q}{4\pi^2\varepsilon_0 r^2}\int_0^{\pi/2}\cos\theta\, d\theta$$

or
$$E = \frac{2Q}{4\pi^2\varepsilon_0\left(\frac{l}{\pi}\right)^2}[\sin 90 - \sin 0] = \frac{Q}{2\varepsilon_0 l^2}$$

18. Two charged particles each of mass 5g and charge $2\,\mu C$ are suspended as shown in Fig. 19.25. The system is taken in a Seattleite. The force between the charges is

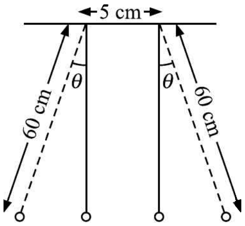


Fig. 19.26

(a) 23×10^{-3}N (b) 2.3×10^{-3}N
(c) 0.23×10^{-3}N (d) None of these

Solution (a) $F = \frac{q^2}{4\pi\varepsilon_0 d^2} = \frac{2\times 2\times 10^{-12}\times 9\times 10^9}{(1.25)^2}$

$= 23 \times 10^{-3}$N

[Here $d = (21 + 5)$cm $= 125$cm as there is no gravity. Therefore electrostatic force will push them away.]

Problems for Practice

1. The length of each side of a cubical closed surface is α. If charge q is situated on one of the vertices of the cube then the flux passing through each face of the cube will be

(a) $\frac{q}{6\varepsilon_0}$ (b) $\frac{q}{24\varepsilon_0}$ (c) $\frac{q}{8\varepsilon_0}$ (d) zero

2. A charged conductor has charge on its
(a) outside surface (b) surrounding
(c) middle point (d) inner surface

3. The laws of forces that govern the force between two electric charges were discovered by
(a) Faraday (b) Ampere
(c) ohm (d) coulomb

4. A charge Q is placed on to two opposite corners of a square. A charge q is placed at each of other two corners. Given that resultant electric force on Q is zero, then Q is equal to

(a) $(2\sqrt{2})/q$ (b) $-q/(2\sqrt{2})$

(c) $(2\sqrt{2})q$ (d) $(-2\sqrt{2})q$

5. Let us suppose that earth (radius 6400km) had a net charge equivalent to one electron per m^2 of its surface area. Its potential in volts will be
(a) −1.2 (b) −0.12 (c) 0.12 (d) 1.2

6. A charge Q is divided into two parts. The two charges kept at a distance apart have a maximum columbian repulsion. Then the ratio of Q and one of the parts is given by
(a) 1 : 4 (b) 1 : 2 (c) 2 : 1 (d) 4 : 1

7. In comparison with the electrostatic force between two electrons, the electrostatic force between two protons is
(a) zero (b) smaller (c) same (d) greater

8. A positively charged ball hangs from a long silk thread. We put a positive test charge q_0 at a point and measure F/q_0, then it can be predicted that field E

(a) $>F/q_0$ (b) $<F/q_0$
(c) is equal to F/q_0 (d) none of these

9. An electron of mass m and charge e is accelerated from rest through a potential difference V in vacuum. Its final speed will be

(a) $\sqrt{\frac{2eV}{m}}$ (b) $\sqrt{\frac{eV}{m}}$

(c) $\frac{eV}{2m}$ (d) $\frac{eV}{m}$

10. A helium ion and a hydrogen ion are accelerated from rest through a potential difference of V to velocities v_{He} and v_H respectively. If helium has lost one electron, the ratio of v_{He}/v_H is
(a) 1/4 (b) 1/2 (c) 1 (d) $\sqrt{2}$

11. A charge q is placed at the centre of the line joining two equal charges Q. The system of the three charges will be in equilibrium if q is equal to
(a) $-(Q/4)$ (b) $-(Q/2)$
(c) $(Q/2)$ (d) $(Q/4)$

12. A proton has a mass 1.67×10^{-27} kg and charge $+1.6 \times 10^{-19}$ C. If the proton is being accelerated through a potential difference of one millions volts, then the K.E. is

(a) 1.6×10^{-25} (b) 3.2×10^{-13} J
(c) 1.6×10^{-15} J (d) 1.6×10^{-13} J

13. An electron having charge $-e$ located at A, in the presence of a point charge $+q$ located at O, is moved to the point B such that OAB forms an equilateral triangle. The work done in the process is equal to

(a) $-eq/AB$ (b) eq/AB
(c) q/AB (d) zero

14. The electric flux ϕ through a hemispherical surface of radius R, placed in a uniform electric field of intensity E parallel to the axis of its circular plane is

(a) $(4/3)\pi R^3 E$ (b) $2\pi R^2 E$
(c) $\pi R^2 E$ (d) $2\pi RE$

15. A charge of 6.76 μC in an electric field is acted upon by a force of 2.5 N. The potential gradient at this point is

(a) 3.71×10^{15} Vm^{-1} (b) 3.71×10^{12} Vm^{-1}
(c) 3.71×10^{10} Vm^{-1} (d) 3.71×10^{5} Vm^{-1}

16. Two identical thin rings each of radius R are coaxially placed at a distance R apart. If Q_1 and Q_2 are respectively the charges uniformly spread on the two rings, the work done in moving a charge q from the centre of one ring to the other is

(a) $\dfrac{q(Q_1/Q_2)(\sqrt{2}+1)}{\sqrt{2}\,4\pi\varepsilon_0 R}$

(b) $\dfrac{q(Q_1-Q_2)(\sqrt{2}-1)}{\sqrt{2}\,4\pi\varepsilon_0 R}$

(c) $\dfrac{q\sqrt{2}(Q_1+Q_2)}{4\pi\varepsilon_0 R}$

(d) zero

17. Two pith balls each of mass 1 g and carrying a charge 1 μ C are attached to the ends of silk threads 1 m long, the other ends of which are attached to some fixed point, in a gravity free space. The force between them is

(a) 9.8×10^{-12} N (b) 9.0×10^{-6} N
(c) 4.5×10^{-6} N (d) 2.25×10^{-3} N

18. In Fig. 19.27, there is a charge $+q$ at the centre of the circle $ABCDE$. Another charge is placed at A. The work done in carrying a charge placed at A, is

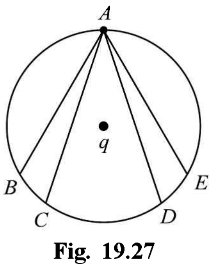


Fig. 19.27

(a) minimum along AE
(b) minimum along AB
(c) finite and equal along any of the paths AB, AC, AD and AE
(d) zero along any of the paths AB, AC, AD and AE

19. Two equal metal balls are charged to 10 and 20 units of electricity. They are brought in contact with each other and then again separated to the original distance. The ratio of forces between the two balls before and after contact is

(a) 3 : 2 (b) 1 : 8
(c) 2 : 3 (d) 8 : 9

20. No current flows between two charged bodies when connected if they have same

(a) capacity (b) charge
(c) potential (d) none of the above

21. The circular lines shown in the diagram below are centred at the stationary charged particle. They could represent

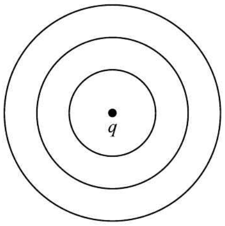


Fig. 19.28

(a) electric field lines
(b) electric equipotential lines
(c) magnetic field lines
(d) gravitational field lines

22. The dielectric strength of air is 3.0×10^{6} NC^{-1}. The largest charge that a 0.30 cm radius metal sphere can hold without sparking is

(a) 9 nC (b) 8.2 nC (c) 6 nC (d) 3 nC

23. A ring of radius R carries a uniformly distributed charge $+Q$. A point charge q is placed on the axis of the ring and released from rest. The force experienced by the particle varies with distance from the centre as

(a) $x/(R^2+x^2)^{3/2}$ (b) $1/\sqrt{x}$
(c) $1/x^3$ (d) $1/x^2$

24. An electron is accelerated through a potential difference of 500 volt. The velocity acquired by the electron is

(a) $(2/3) \times 10^{7}$ ms^{-1} (b) $(1/6) \times 10^{7}$ ms^{-1}
(c) $(1/3) \times 10^{7}$ ms^{-1} (d) none of the above

25. An electrical charge of 2 μ C is placed at the point (1, 2, 3). At the point (2, 3, 4) the electric field and potential will be

(a) 6×10^{3} NC^{-1} and 6×10^{3} JC^{-1}
(b) 6000 NC^{-1} and $6000\sqrt{3}$ JC^{-1}
(c) 6×10^{3} NC^{-1} and $3\sqrt{3}$ JC^{-1}
(d) none of the above

26. The charge per unit length for a very long straight wire is λ. The electric field at points near the wire (but outside it) and far from the ends varies with distance r as

(a) r (b) $1/r$ (c) $1/r^2$ (d) $1/r^3$

27. A particle of charge $+q$ and mass m moving under the influence of a uniform electric field $E\hat{i}$ and a uniform magnetic field $B\hat{k}$ follows a trajectory from P to Q as shown in the Fig. 19.29. The velocities at P and Q are v_i and $-v_j$ respectively. Which of the following statement (s) is/are correct?

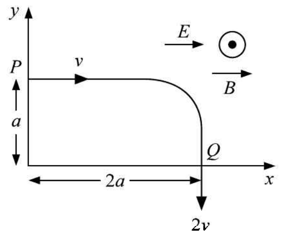


Fig. 19.29

(a) $E = \frac{3}{4}\left(\frac{mv^2}{qa}\right)$

(b) rate of work done by electric field at P is zero

(c) rate of work done by the electric field at P is $\frac{3}{4}\left(\frac{mv^3}{a}\right)$

(d) rate of work done by both the fields at Q is zero

28. A charged particle of mass m and charge q is released from rest in an electric field of constant magnitude E. The K.E. of the particle after time t is

(a) $\frac{Eq^2m}{2t^2}$ (b) $\frac{Eqm}{2t}$

(c) $\frac{2E^2t^2}{mq}$ (d) $\frac{E^2q^2t^2}{2m}$

29. Four charges are arranged at the corners of square $ABCD$ as shown in Fig. 19.30. Force on charge kept at the centre is

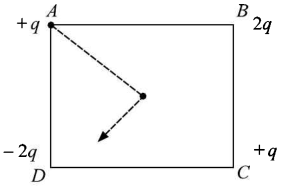


Fig. 19.30

(a) perpendicular to side AB
(b) along diagonal AC
(c) along diagonal BD
(d) zero

30. A solid conducting sphere having a charge Q is surrounded by an uncharged concentric conducting hollow spherical shell. Let the potential difference between the surface of the solid sphere and that of the outer surface of the hollow shell be V. If the shell is now given a charge of $-3Q$ the new potential difference between the same two surfaces is

(a) $-2V$ (b) V (c) $2V$ (d) $4V$

31. A uniform electric field having a magnitude E_0 and direction along the positive x-axis exists. If the potential V is zero at $x = 0$, then its value at $x = +x$ will be

(a) $V(x) = x^2E_0$ (b) $V(x) = -x^2E_0$

(c) $V(x) = -xE_0$ (d) $V(x) = xE_0$

32. A given charge is situated at a certain distance from an electric dipole in the end on position, experiences a force F. If the distance of the charge is doubled, the force acting on the charge will be

(a) $F/8$ (b) $F/4$ (c) $F/2$ (d) $2F$

33. Eight charged water drops each with a radius of 1 mm and a charge of 10^{-10} coulumb merge into a single drop. The potential of this single drop is

(a) 36 V (b) 1000 V

(c) 3600 V (d) 8000 V

34. $ABCD$ is a square of 1 metre side of a non-conducting material. Four metallic spheres of 4, 5, 8 and 10 cm diameters are placed at the four corners. All of them are connected by a fine metallic wire and a charge of 540 units is imparted to the system. The potential at the centre of the square is

(a) $\frac{540\sqrt{2}}{400}$ (b) $\frac{540\sqrt{2}}{200}$

(c) $\frac{540\sqrt{2}}{100}$ (d) $\frac{540\sqrt{2}}{10}$

35. The electric potential at a point situated at a distance r on the axis of a short electric dipole of moment p will be $1/4(\pi\varepsilon_0)$ times

(a) p/r^3 (b) p/r^2

(c) p/r (d) none of the above

36. Two concentric thin, metallic spheres of radii R_1 and R_2 ($R_1 > R_2$) bear charges Q_1 and Q_2 respectively. Then the potential at a radius r between R_1 and R_2 will be $1/4(\pi\varepsilon_0)$ times

(a) $\frac{Q_1+Q_2}{4}$ (b) $\frac{Q_1}{R_1}+\frac{Q_2}{r}$

(c) $\frac{Q_1}{R_1}+\frac{Q_2}{R_2}$ (d) $\frac{Q_1}{R_2}+\frac{Q_2}{R_1}$

37. An electric dipole of moment p is kept along an electric field E. The work done in rotating it from an equilibrium position by an angle θ is,

(a) $PE(1 - \cos\theta)$ (b) $PE(1 - \sin\theta)$

(c) $PE\cos\theta$ (d) $PE\sin\theta$

38. A body has a charge of one coulumb. The number of excess (or lesser) electrons on it from its normal state will be

(a) ∞ (b) 1.6×10^{-19}

(c) 1.6×10^{19} (d) 6.25×10^{18}

39. Which of the following is a vector quantity?
(a) electric field intensity
(b) charge density
(c) elecric charge
(d) electric potential

40. In electrostatics
(a) all the Newton's law are valid
(b) newton's first law may not be valid in certain cases
(c) Newton's second law may not be valid in certain cases
(d) Newton's third law may not be valid in certain cases

41. Two spheres A and B of radii 17 cm each and having charges 1 and 2 coulombs respectively are separated by a distance of 80 cm. The electric field at a point on the line joining the centres of two spheres is approximately zero at a distance of 33 cm from the sphere A. The electric potential at this point is

(a) 1.2×10^{11} V (b) 6.56×10^{10} V

(c) 2.03×10^{9} V (d) 8.12×10^{7} V

42. For a system of four charges shown in the Fig. 19.31 (arranged at the four corners of a square of side $2a$), the resulting electrostatic force on the charge at the origin O will make an angle θ with Oy' axis where $\tan\theta$ is

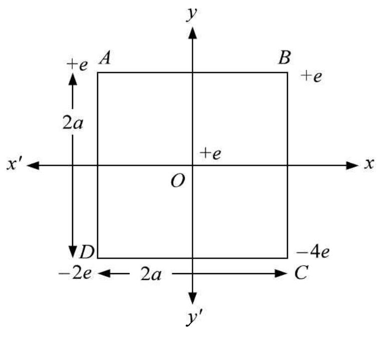


Fig. 19.31

(a) 1/4 (b) 1/3

(c) 1/2 (d) 1

43. The force between two charged particles is given by $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{r^2}$, where the symbols have their usual meaning. The dimensions of ε_0 in free space in *MKSQ* system are

(a) $M^{-1}L^2T^2Q^4$ (b) $ML^{-3}T^4Q^3$

(c) $M^{-1}L^{-3}T^4Q^2$ (d) $M^{-1}L^{-3}T^2Q^2$

44. The breakdown voltage for a spherical conductor in air varies with the radius r of a sphere as r^n. The value of n is equal to

(a) –2 (b) –1 (c) 1 (d) 2

45. Two conducting sphere of radii r_1 and r_2 have same electric field near their surfaces. The ratio of their electric potential is

(a) r_1/r_2 (b) r_2^2/r_1^2 (c) r_2/r_1 (d) r_1^2/r_2^2

46. The correct relation between induced charge on dielectric surface and the free charge q is

(a) $q = q'\left(1-\frac{1}{k}\right)$ (b) $q' = qk$

(c) $q' = q\left(1-\frac{1}{k}\right)$ (c) $q' = \frac{k}{q}$

47. Out of the following which one is not the unit of electric field intensity?

(a) $\frac{\text{Stat-volt}}{\text{cm}}$ (b) $\frac{\text{Coulumb}}{\text{Newton}}$

(c) $\frac{\text{Volt}}{\text{m}}$ (d) $\frac{\text{Newton}}{\text{coulumb}}$

48. What will be the potential on the surface of gold nucleus if its radius is 6.6×10^{-15} m and $Z = 79$?

(a) 0.7×10^7 V (b) 1.7×10^7 V

(c) 3.14×10^7 V (d) zero

49. Two similar sphere's, carrying unequal and opposite charges, are situated 90 cm apart. These are mutually brought in contact with each other and then again placed at same distance apart. Now they repel each other with a force of 0.025 N. The final charge on them will be

(a) 1.5 μ C (b) 3 μ C

(c) 1.5 coulumb (d) 3 coulumb

50. A point charge of 1 μ C is situated 0.5 m above a square of side 1 m. The electric flux emanating out of the square surface will be

(a) 10^7 V-m (b) 10^6 V-m

(c) 1.88×10^4 V-m (d) zero

51. The number of electron removed from a body in order to produce a positive charge of 8×10^{-19} coulumb on it, will be

(a) 9 (b) 7 (c) 5 (d) 3

52. Two parallel conducting plates, each of area 10^4 cm², are given equal and opposite charge of 8.8×10^{-6} coulumb. The dielectric medium is partially filled between the plates. If the electric field inside the dielctric is 1.4×10^5 Vm⁻¹ then the electric field in vacuum will

(a) 10^5 Vm^{-1} (b) 9.94×10^5 Vm^{-1}

(c) 1.94×10^5 Vm^{-1} (d) zero

53. The potential on the surface of a charged metallic sphere of radius 5 cm is 10 volt. The potential at the centre of the sphere will

(a) zero (b) 5 V (c) 10 V (d) 20 V

54. An electron and a proton are lying in a uniform electric field. The ratio of their acceleration will be

(a) $\frac{m_e}{m_p}$ (b) $\frac{m_p}{m_e}$ (c) 1 (d) zero

55. The *V-r* curve for a uniformly charged non-conducting sphere will be

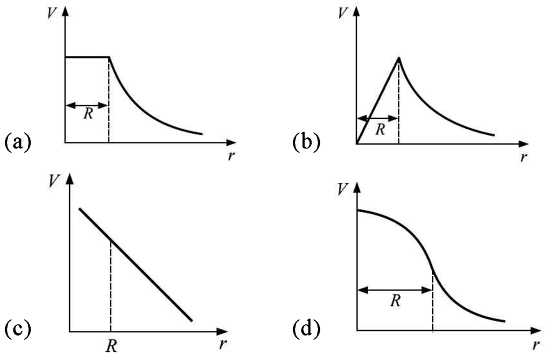


Fig. 19.32

56. The intensity of electric field of 10 NC⁻¹ is directed at an angle of 60° from the normal to rectangular surface 2 m long and 4 m wide. The electric flux coming out of it will be

(a) zero (b) 20 Nm²C⁻¹

(c) 40 Nm²C⁻¹ (d) 80 Nm²C⁻¹

57. A current of 1 ampere is flowing in a conductor. The number of electrons flowing per second will be

(a) 124×10^{18} (b) 6.2×10^{19}

(c) 6.2×10^{18} (d) 1.6×10^{-19}

58. The distance between two parallel equipotential surfaces *A* and *B* is r. The work done in carrying a charge q from surface *A* to surface *B* will be

(a) $W = \frac{-Kq}{r}$ (b) $W = \frac{Kq}{r}$

(c) $W = \frac{Kq}{r^2}$ (d) $W = 0$

(c) F_e and F_g (d) F_e and F_n

ANSWERS

1. (b)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (b)
11. (a)	**12.** (d)	**13.** (d)	**14.** (c)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (d)	**19.** (d)	**20.** (c)
21. (b)	**22.** (d)	**23.** (a)	**24.** (d)	**25.** (b)	**26.** (b)	**27.** (a,c,d)	**28.** (d)	**29.** (c)	**30.** (b)
31. (c)	**32.** (a)	**33.** (c)	**34.** (c)	**35.** (b)	**36.** (b)	**37.** (a)	**38.** (d)	**39.** (a)	**40.** (d)
41. (b)	**42.** (c)	**43.** (c)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (c)
51. (c)	**52.** (b)	**53.** (c)	**54.** (b)	**55.** (d)	**56.** (c)	**57.** (c)	**58.** (d)		

20

Gauss's Law

BRIEF REVIEW OF THE CONCEPTS

Gauss's law The closed integral of electric field intensity is equal to $\frac{q}{\varepsilon_0}$ where q is charge enclosed in the closed surface. In other words total electric flux through a closed surface enclosing a charge q is given by $\oint E \cdot dS = \frac{q}{\varepsilon_0}$.

- If E is at right angle to the surface area A at all points and has the same magnitude at all points of the surface then $E_\perp = E$ and $\int E_\perp dA = EA$.
- If E is parallel to the surface on all points then $E_\perp = 0$. Hence integral is also zero.
- If $E = 0$ at all points on a surface then $\phi = 0$.
- The surface need not be a real physical surface, it can be an hypothetical one.
- Electric field in $\oint E \cdot dS$ is complete electric field. It may be partly due to charge with in the surface and partly due to charge outside the surface. However, if there is no charge enclosed in the Gaussian surface $E_\perp$ will be zero and hence $\oint E \cdot dS = 0$.
- While evaluating $\oint E \cdot dS$, the field should lie on the surface and there should be enough symmetry to evaluate the integral.

Electric field due to a long thread (Line charge) having linear charge density λ is

$$E = \frac{\lambda}{2\pi\varepsilon_0 y} = \frac{18 \times 10^9 \lambda}{y}$$

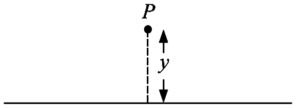


Fig. 20.1

Electric field due to a uniformly charged sphere of radius R having charge Q

$$E_{\text{inside}} = \frac{Qx}{4\pi\varepsilon_0 R^3} \qquad x < R$$

$$E_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R^3} \qquad x = R$$

$$E_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 x^2} \qquad x > R$$

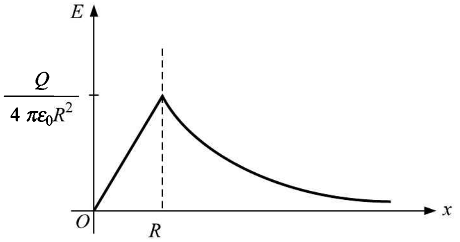


Fig. 20.2

Potential due to a uniformly charged sphere

$$V_{\text{inside}} = \frac{Q}{4\pi\varepsilon_0 R} + \int_R^x -\frac{Qx}{4\pi\varepsilon_0 R^3}\,dx \qquad x < R$$

$$V_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R} \qquad x = R$$

$$V_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 R} \qquad x > R$$

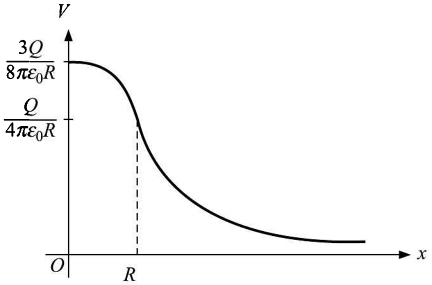


Fig. 20.3

Electric field due to a thin plane sheet (long) of charge density σ

$$E = \frac{\sigma}{2\varepsilon_0}$$

Electric field due to a charged surface having surface charge density σ

$$E = \frac{\sigma}{\varepsilon_0}$$

Electric field due to a conducting plate $E = \sigma / 2\varepsilon_0$

Electric field due to a non-conducting plate $E = \sigma / \varepsilon_0$

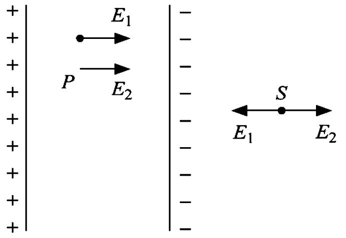


Fig. 20.4

Electric field between two oppositely charged sheets at any point is $E_{in} = \frac{\sigma}{\varepsilon_0}$ ($= E_1 + E_2$) assuming equal surface charge density, for example, in a capacitor. Electric field intensity is zero at any point outside the plates as $E_{net} = E_1 - E_2 = 0$.

SHORT-CUTS AND POINTS TO NOTE

(i) Electric flux through symmetrical surfaces placed inside the uniform electric field is zero. For example, for a cylinder (solid or hollow) placed in a uniform field $\phi = 0$. Hence no charge is stored.

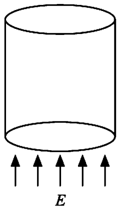


Fig. 20.5

(ii) $\oint E \cdot dS = \frac{\sigma}{\varepsilon_0}$ where Q is charge enclosed in the surface.

The electric field here is perpendicular to the surface so that it is parallel to the surface vector dS otherwise dot product will be zero.

(iii) Electric field intensity due to a long line charge of linear charge density λ is

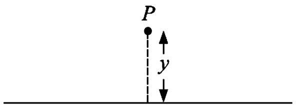


Fig. 20.6

$$E = \frac{\lambda}{2\pi\varepsilon_0 y} = \frac{18\times10^9\lambda}{y}$$

Electric field due to a long line charge of linear charge density λ at point P distant y from the end A is $\frac{Qx}{4\pi\varepsilon_0 R^3}$ and A makes an angle of 45° with the vertical.

(iv) Electric field due to a uniformly charged sphere of radius R having charge Q

$$E_{inside} = \frac{Qx}{4\pi\varepsilon_0 R^3} \qquad x < R$$

$$E_{surface} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad x = R$$

$$E_{outside} = \frac{Q}{4\pi\varepsilon_0 x^2} \qquad x > R$$

(v) Electric potential due to a uniformly charged sphere

$$V_{inside} = \frac{Q}{4\pi\varepsilon_0 R} + \int_R^x -\frac{Qx}{4\pi\varepsilon_0 R^3}\,dx \qquad x < R$$

$$V_{surface} = \frac{Q}{4\pi\varepsilon_0 R} \qquad x = R$$

$$V_{outside} = \frac{Q}{4\pi\varepsilon_0 x} \qquad x > R$$

(vi) Electric field due to a long plane sheet of charge density σ

$$E = \frac{\sigma}{2\varepsilon_0}$$

Thin sheet/nonconducting sheet on which charge is distributed on both sides gives this result.

(vii) If the sheet is thick and conducting then charge will remain on one side

In such a case $E = \frac{\sigma}{\varepsilon_0}$

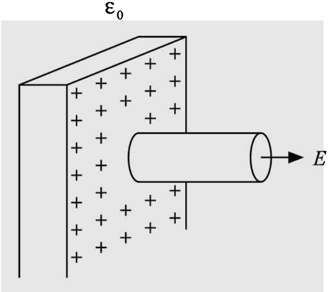


Fig. 20.7

(viii) In a capacitor sheets are thin therefore, elecric field inside them is given by

$$E = E_1 + E_2$$

$$= \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = .\,\frac{\sigma}{\varepsilon_0}$$

(ix) In a long cylinder of radius R, with charge per unit length λ

$$E_{outside} = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad \text{for } r \geq R$$

$$E_{inside} = 0 \qquad \text{for } r < R$$

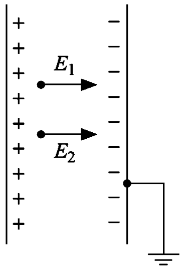


Fig. 20.8

CAUTION

(i) Considering that any electric field in $\oint E \cdot dS$ will form electric flux.

- In the Fig. 20.9 $E_\perp$ and $E_\parallel$ two fields are shown. $E_\parallel$ will not form any flux as $\vec{ds}$ and $\vec{E}$ are perpendicular. Only $E_\perp$ which is parallel to ds will form the flux.

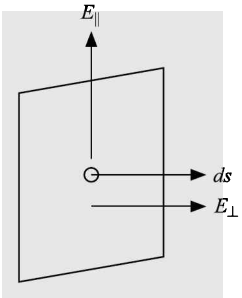


Fig. 20.9

(ii) Considering that in a sphere electric field inside the sphere ($x < R$) is zero.

- If the sphere is conducting, that is, charged at outside surface only then $E_{in}(x < R) = 0$.

If the charge is uniformly distributed over the whole volume as in insulated spheres then

$$E_{\text{inside}}\ (x < R) = \frac{Qx}{4\pi\varepsilon_0 R^3}.$$

(iii) Considering $E = 0$ if $V = 0$ or vice versa.

- Inside a shell or conducting sphere $E_{\text{in}} = 0$ but $V_{\text{in}} \neq 0$ and along the equatorial line in a dipole $E \neq 0$ but $V = 0$.

(iv) Considering that $E_{\text{in}} = 0$ only in a shell (sphere).

- $E_{\text{in}} = 0$ in any type of hollow metallic body. $E_{\text{in}} = 0$ even in a long cylinder.

(v) Considering $V_{\text{in}} = \frac{Q}{4\pi\varepsilon_0 R}$ in a sphere of charge Q and radius R.

- If the charge is uniformly distributed over the outside surface (in a hollow sphere or in a solid conducting sphere). Then $E_{\text{in}} = 0$ and $V_{\text{in}} = \frac{Q}{4\pi\varepsilon_0 R}$. But if the charge is distributed throughout the sphere in its volume then

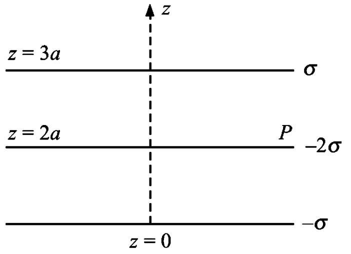
$$V_{\text{in}} = \frac{Q}{4\pi\varepsilon_0 R} + \int_R^x \frac{-Qx}{4\pi\varepsilon_0 R^3}\,dx.$$

Solved Problems

1. Three infinitely charged sheets are kept parallel to $x - y$ plane having charge densities as shown in the Fig. 20.10(a), (b). The electric field at P is

(a) $-\frac{4\sigma}{\varepsilon_0}\hat{k}$ (d) $\frac{4\sigma}{\varepsilon_0}\hat{k}$

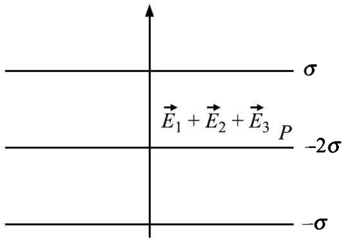
(c) $\frac{-2\sigma}{\varepsilon_0}\hat{k}$ (d) $\frac{2\sigma}{\varepsilon_0}\hat{k}$



(a)

Solution (c) $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$

$$= \frac{\sigma}{2\varepsilon_0}(-\hat{k}) + \left(\frac{2\sigma}{2\varepsilon_0}\right)\hat{k} + \left(\frac{-\sigma}{2\varepsilon_0}\right)\hat{k} = -\frac{2\sigma}{\varepsilon_0}\hat{k}.$$



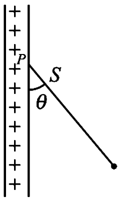
(b)

Fig. 20.10

2. A charged ball hangs from a silk thread of lenght l. It makes an angle θ with a large charged conducting sheet P as shown in Fig. 20.11(a). The surface charge density σ of the sheet is proportional to

(a) cos θ (b) cot θ

(c) sin θ (d) tan θ



(a)

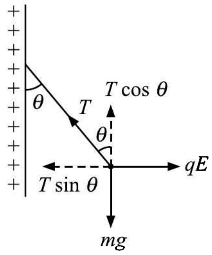
Solution (d) Resolve T;

$T \cos\theta = mg$ ...(1)

$T \cos\theta = qE = \frac{q\sigma}{2E_0}$ ...(2)

Divide eq. (2) by (1)

$\tan\theta = \frac{q\sigma}{2\varepsilon_0 g}$, that is, $\sigma \propto \tan\theta$



(b)

Fig. 20.11

3. Two infinitely long parallel conducting plates having surface charge densities +σ and –σ respectively are separated by a small distance. The medium between the plates is vacuum. If ε_0 is the dielectric permittivity of vacuum then the electric field in the region between the plates is

(a) $0\,\text{Vm}^{-1}$ (b) $\frac{\sigma}{\varepsilon_0}\,\text{Vm}^{-1}$

(c) $\frac{\sigma}{\varepsilon_0}\,\text{Vm}^{-1}$ (d) $\frac{2\sigma}{\varepsilon_0}\,\text{Vm}^{-1}$

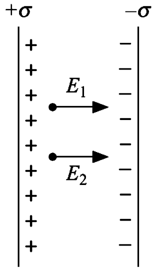


Fig. 20.12

Solution (c) $E = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$ (See Fig.)

4. A charge Q is distributed uniformly in a sphere (solid). Then the electric field at any point r where $r < R$ (R is radius of the sphere) varies as

(a) $r^{1/2}$ (b) r^{-1}

(c) r (d) r^{-2}

Solution (c) $E = \frac{Qr}{4\pi\varepsilon_0 R^3}$ i.e. $E \propto r$

5. An elliptical cavity is carved out in a perfect conductor. A positive charge $\boldsymbol{q}$ is placed at the centre of the cavity. The points $\boldsymbol{A}$ and $\boldsymbol{B}$ are shown in Fig. 20.13. Then

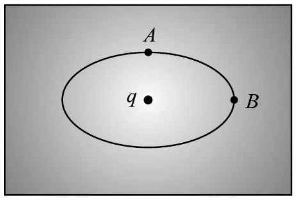


Fig. 20.13

(a) electric field near $\boldsymbol{A}$ in the cavity = electric field near $\boldsymbol{B}$ in the cavity.
(b) charge density at $\boldsymbol{A}$ = charge density at $\boldsymbol{B}$.
(c) potential at $\boldsymbol{A}$ = potential at $\boldsymbol{B}$
(d) total electric field flux through the surface of the cavity $= \dfrac{q}{\varepsilon_0}$.

Solution (c) and (d) for (c) Because A and B lie on the same conductor, therefore, potential at each point is equal. For (d) Gauss law.

6. A solid conducting sphere having a charge $\boldsymbol{Q}$ is surrounded by an uncharged concentric conducting hollow spherical shell. Let the potential difference between the surface of the solid sphere and that of the outer surface of the hollow shell be $\boldsymbol{V}$. If the shell is now given a charge $-3\boldsymbol{Q}$, the new potential difference between the same two surfaces is

(a) V	(b) $2V$
(c) $4V$	(d) $-2V$

Solution (a) Because the potential difference between solid sphere and hollow shell depends on the radii of two spheres and charge on the inner sphere. Since the two values have not changed, therefore, potential difference does not change.

7. A metallic solid sphere is placed in a uniform electric field. The lines of force follow the path(s) shown in Fig. 20.14 as

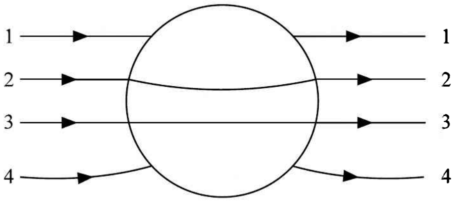


Fig. 20.14

(a) 1	(b) 2
(c) 3	(d) 4

Solution (d) $\boldsymbol{E}_{\text{in}} = 0$ and electric field lines are perpendicular to the equipotential surface.

8. Two conducting plates $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel. $\boldsymbol{A}$ is given a charge $\boldsymbol{Q}_1$ and $\boldsymbol{B}$ is given a charge $\boldsymbol{Q}_2$. The charge on inner side of $\boldsymbol{B}$ is

(a) $\dfrac{Q_1 - Q_2}{2}$	(b) $\dfrac{-(Q_1 - Q_2)}{2}$
(c) $\dfrac{Q_1 + Q_2}{2}$	(d) $\dfrac{-(Q_1 + Q_2)}{2}$

Solution (b) Electric field inside the conductor at point $\boldsymbol{P} = 0$

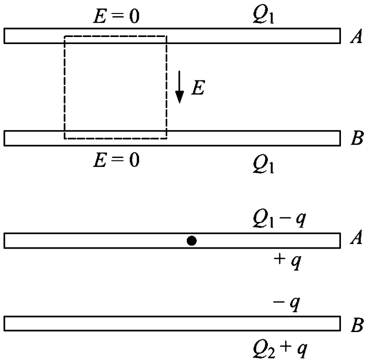


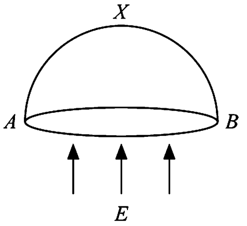
Fig. 20.15

$$\therefore \quad \frac{Q_1 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0}1 - \frac{Q_2 + q}{2A\varepsilon_0} = 0$$

or $\quad \boldsymbol{Q}_1 - \boldsymbol{q} - (\boldsymbol{Q}_2 + \boldsymbol{q}) = 0$

or $\quad \boldsymbol{q} = \dfrac{Q_1 - Q_2}{2}$.

9. A hemisphere of radius $\boldsymbol{r}$ is placed in a uniform electric field of strength $\boldsymbol{E}$. The electric flux through the hemisphere is



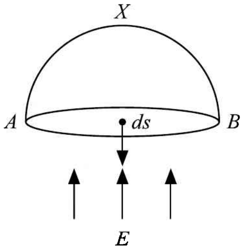
(a)

(a) $2E\pi r^2$	(b) $-E\pi r^2$
(c) $-2E\pi r^2$	(d) zero

Solution (b) $\phi = \oint E \cdot dS$

$$= \int_A^B E \cdot dS + \int_B^X E \cdot dS + \int_X^A E \cdot dS$$

$$= -E\pi r^2 + E\pi r^2 - E\pi r^2 = -E\pi R^2$$



(b)

Fig. 20.16

Short cut $\boldsymbol{AXB}$ is symmetrical surface $\therefore$ Electric flux due to this part is zero. However, electric flux due to $\boldsymbol{AB}$ part is $-\boldsymbol{E\pi R}^2$.

10. A positively charged sphere suspended with a silk thread is slowly pushed in a metal bucket. After its insertion the lid is closed. What will be the electric field intensity inside when the sphere has touched the bucket σ is the surface charge density of sphere.

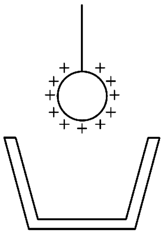


Fig. 20.17

(a) zero (b) $\sigma/2\varepsilon_0$
(c) σ/ε_0 (d) none of these

Solution (a) The charge of sphere will be taken by the bucket and appear on its outer surface only.

$\therefore$ $E_{in} = 0$.

11. An electric field in a region is $800\sqrt{x}\,\hat{i}$. The charge contained in a cubical volume bounded by the surface $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$ and $z = a$

(a) $\dfrac{800\,a^{7/2}}{7/2}\varepsilon_0$ (b) $800(a)^{5/2}\varepsilon_0$
(c) $800\,a^2\varepsilon_0$ (d) $800\sqrt{a}\,\varepsilon_0$

Solution (b) $\phi = Ea^2$ and $Q = \phi\varepsilon_0 = Ea^2\varepsilon_0 = 800(a)^{5/2}\varepsilon_0$

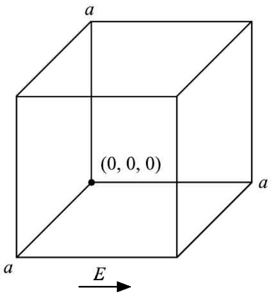


Fig. 20.18

12. A dipole is placed in a shell as shown. Find the electric flux emerging out of the shell and in a hypothetical sphere of radius r as shown in Fig. 20.19.

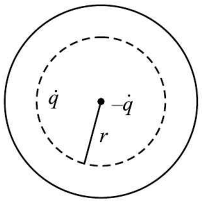


Fig. 20.19

(a) $\dfrac{2q}{\varepsilon_0}, 0$ (b) $\dfrac{q}{\varepsilon_0}, \dfrac{-q}{\varepsilon_0}$
(c) $\dfrac{q}{\varepsilon_0}, \dfrac{q}{\varepsilon_0}$ (d) 0, 0

Solution (d)

13. The concentric shells as shown enclose charge q and $-q$. The electric flux from the shell of radius R is

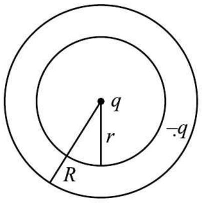


Fig. 20.20

(a) $\dfrac{-q}{\varepsilon_0}$ (b) $\dfrac{q}{\varepsilon_0}$
(c) $\dfrac{2q}{\varepsilon_0}$ (d) 0

Solution (d)

Problems for Practice

1. A metallic particle having no net charge is placed near a finite metal plate carrying a positive charge. The electric force on the particle will be
(a) towards the plate (b) away from the plate
(c) parallel to the plate (d) zero

2. A thin metallic spherical shell contains a charge Q on it. A point charge q is placed at the centre and another charge q_1 is placed outside it as shown in Fig. 20.21. All the three charges are positive. The force on the charge at the centre is

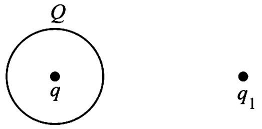


Fig. 20.21

(a) towards left (b) towards right
(c) upwards (d) zero

3. Electric charges are distributed in a small volume. The flux of the electric field through a spherical surface of radius 10 cm surrounding the total charge is 25 V-m. The flux over concentric sphere of radius 20 cm will be
(a) 25 Vm (b) 50 Vm
(c) 100 Vm (d) 200 Vm

4. A charge q is placed at the centre of the open end of a cylindrical vessel. The flux of the electric field through the surface of the vessel is

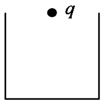


Fig. 20.22

(a) zero (b) $\dfrac{q}{\varepsilon_0}$ (c) $\dfrac{q}{\varepsilon_0}$ (d) $\dfrac{2q}{\varepsilon_0}$

5. Mark the correct options
   (a) Gauss's law is valid only for symmetrical charge distribution
   (b) Gauss's law is valid only for charges placed in vacuum
   (c) the electric field calculated by Gauss's law is the field due to the charges inside the Gaussian surface
   (d) the flux of the electric field through a closed surface due to all the charges is equal to the flux due to the charges enclosed by the surface

6. A positive point charge Q is brought near an isolated metal cube
   (a) the cube becomes negatively charged
   (b) the cube becomes positively charged
   (c) the interior becomes positively charged and the surface becomes negatively charged
   (d) the interior remains charge free and the surface gets non-uniform charge distribution.

7. The electric field in a region is given by $\vec{E} = \frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j}$ with $E_0 = 2 \times 10^3 NC^{-1}$. Find the flux of this field through a rectangular surface of area $0.2 m^2$ parallel to the Y-Z plane.
   (a) 320 Nm^2C^{-1} (b) 240 Nm^2C^{-1}
   (c) 400 Nm^2C^{-1} (d) none of these

8. A charge Q is uniformly distributed over a rod of length l. Consider a hypothetical cube of edge l with the centre of the cube at one end of the rod. The minimum possible flux of the electric field through the entire surface of the cube is
   (a) $\frac{Q}{\varepsilon_0}$ (b) $\frac{Q}{2\varepsilon_0}$ (c) $\frac{Q}{8\varepsilon_0}$ (d) $\frac{Q}{6\varepsilon_0}$

9. The electric field in a region is
$$E = \frac{5\times10^3(NC^{-1}cm^{-1})x}{2}\hat{i}.$$
   The charge contained inside a cubical volume bounded by the surfaces $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ is (where x, y, z are in cm)
   (a) $2.21 \times 10^{-12}C$ (b) $4.42 \times 10^{-12}C$
   (c) $2.21 \times 10^{-8}C$ (d) $4.42 \times 10^{-8}C$

10. A charge Q is placed at the centre of a cube. The flux through the six surfaces of the cube is
   (a) $\frac{Q}{\varepsilon_0}$ (b) $\frac{6Q}{\varepsilon_0}$ (c) $\frac{Q}{6\varepsilon_0}$ (d) $\frac{Q}{3\varepsilon_0}$

11. A charge q is placed at a distance $a/2$ above the centre of a horizontal square surface of edge a as shown in Fig. 20.23. The electric flux through the square surface is

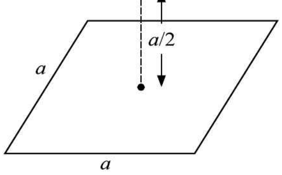


Fig. 20.23

   (a) $Q/2\varepsilon_0$ (b) Q/ε_0 (c) $Q/6\varepsilon_0$ (d) $Q/8\varepsilon_0$

12. A spherical volume has a uniformly distributed charge density $2 \times 10^{-4} Cm^{-3}$. The electric field at a point inside the volume at a distance 4.0 cm from the centre is
   (a) $3.15 \times 10^5 NC^{-1}$ (b) $2.1 \times 10^5 NC^{-1}$
   (c) $6.2 \times 10^5 NC^{-1}$ (d) none of these

13. The radius of gold nucleus is about 7×10^{-15} m ($Z = 79$). The electric field at the mid-point of the radius assuming charge is uniformly distributed is
   (a) $1.16 \times 10^{19} NC^{-1}$ (b) $1.16 \times 10^{21} NC^{-1}$
   (c) $2.32 \times 10^{21} NC^{-1}$ (d) $2.32 \times 10^{19} NC^{-1}$

14. A charge q is uniformly distributed in the hollow sphere of radii r_1 and r_2 ($r_2 > r_1$). The electric field at a point P distance x from the centre for $r_1 < x < r_2$ is
   (a) $\frac{Q(x)}{4\pi\varepsilon_0(r_2^3 - r_1^3)}$ (b) $\frac{Q(x^3 - r_1^3)}{4\pi\varepsilon_0(r_2^3 - r_1^3)}$
   (c) $\frac{Q(x^3 - r_1^3)}{4\pi\varepsilon_0 x^2(r_2^3 - r_1^3)}$ (d) $\frac{Qr_1^3}{4\pi\varepsilon_0 x^2(r_3^3 - r_1^3)}$

15. The electric field at a point 5 cm from a long line charge of density $2.5 \times 10^{-6} Cm^{-1}$ is
   (a) $9 \times 10^3 NC^{-1}$ (b) $9 \times 10^4 NC^{-1}$
   (c) $9 \times 10^5 NC^{-1}$ (d) $9 \times 10^6 NC^{-1}$

16. A long cylindrical wire carries a linear density of $3 \times 10^{-8} cm^{-1}$. An electron revolves around it in a circular path under the influence of the attractive force. KE of the electron is
   (a) $1.44 \times 10^{-17} J$ (b) $2.88 \times 10^{-17} J$
   (c) $4.32 \times 10^{-17} J$ (d) $8.64 \times 10^{-17} J$

17. A long cylinder contains uniformly distributed charge density ρ. The electric field at a point P inside the cylinder at a distance x from the axis is
   (a) $\frac{\rho x}{\varepsilon_0}$ (b) $\frac{\rho x}{2\varepsilon_0}$ (c) $\frac{\rho x}{4\varepsilon_0}$ (d) none of these

18. Two conducting plates A and B having large surface area A (on one side) are placed parallel to each other as shown in Fig. 20.24. The plate X is given a charge Q whereas the other is neutral. The electric field strength in between the two plates at any point is

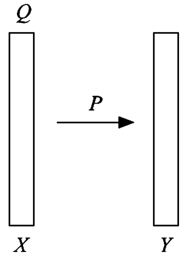


Fig. 20.24

   (a) $\frac{Q}{A\varepsilon_0}$ (b) $\frac{Q}{2A\varepsilon_0}$
   (c) $\frac{2Q}{A\varepsilon_0}$ (d) $\frac{Q}{3A\varepsilon_0}$

19. A non-conducting sheet of large surface area and thickness d contains uniform charge density ρ. The electric field at a point P inside the plane at a distance x from the central plane $0 < x < d$
   (a) $\frac{\rho x}{2\varepsilon_0}$ (b) $\frac{\rho d}{2\varepsilon_0}$ (c) $\frac{\rho x}{\varepsilon_0}$ (d) $\frac{\rho x}{2\varepsilon_0}$

20. Three identical metal plates with large surface area are kept parallel to each other as shown in Fig. 20.25 The charges given are also shown. The charge appearing on the outer surface of extreme right plate is

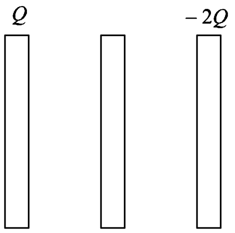


Fig. 20.25

(a) $\frac{Q}{2}$ (b) $\frac{-Q}{2}$ (c) Q (d) $-Q$

21. A hemisphere of radius R is placed in a uniform electric field as shown in Fig. 20.26. The electric flux through the hemisphere is

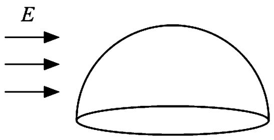


Fig. 20.26

(a) $-E(3\pi R^2)$ (b) $-E(2\pi R^2)$

(c) $-E\pi R^2$ (d) $E(2\pi R^2)$

22. A charged particle having a charge -2×10^{-6}C is placed close to the non-conducting plate having a surface charge density 4 × 10^{-6}Cm^{-2}. The force of attraction between the particle and the plate is nearly

(a) 0.9N (b) 0.71N

(c) 0.62N (d) 0.45N

23. A, B and C are three concentric spherical shells of radius a, b and c respectively. The charges on shells A and C are q and $-q$ respectively and shell B is earthed as shown in Fig. 20.27. The charge on the surface of B is

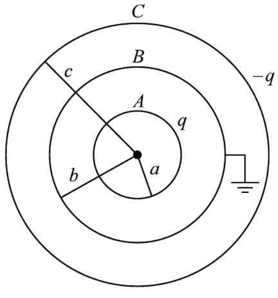


Fig. 20.27

(a) zero (b) a/cq

(c) a/bq (d) b/cq

24. An electric dipole having charges q and $-q$ separated by l is placed in front of a long line charge of linear density $+\lambda$ as shown in Fig. 20.28. The force experienced by the dipole is

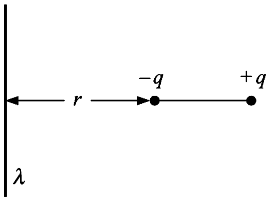


Fig. 20.28

(a) $\frac{q\lambda}{2\pi\varepsilon_0 r}$ (b) $\frac{q\lambda}{2\pi\varepsilon_0(r+l)}$

(c) $\frac{q\lambda}{2\pi\varepsilon_0\left(r+\frac{l}{2}\right)}$ (d) $\frac{q\lambda l}{2\pi\varepsilon_0 r(r+l)}$

25. The electric field in a region is radially outwards and has a magnitude $E = Kr$. The charge contained in a sphere of radius a is

(a) $K4\pi\varepsilon_0 a^2$ (b) $K\frac{4}{3}\pi\varepsilon_0 a^3$

(c) $K4\pi\varepsilon_0 a^3$ (d) none of these

26. A charge q is placed at the vertex A of a cube of edge 'a'. Find the flux through the surface $ABCD$.

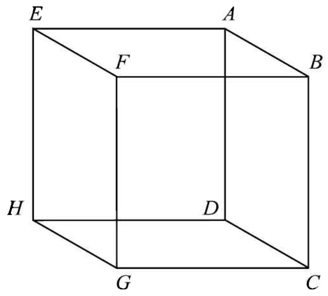


Fig. 20.29

(a) $\frac{q}{2\varepsilon_0}$ (b) $\frac{q}{6\varepsilon_0}$

(c) $\frac{q}{12\varepsilon_0}$ (d) $\frac{q}{24\varepsilon_0}$

ANSWERS

1. (a)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (b)
21. (c)	**22.** (d)	**23.** (d)	**24.** (d)	**25.** (c)	**26.** (d)				

Capacitors 21

BRIEF REVIEW OF THE CONCEPTS

Capacitor A device to store charge or electrostatic energy is called a capacitor.

Capacitance It is the capacity of a capacitor to store charge. In a capacitor $Q \propto V$ or $Q = CV$; C is called the capacitance

$$C = (M^{-1}L^{-2}T^{4}A^{2})$$

According to shapes, capacitors may be of three types: spherical, parallel plate and cylindrical.

Unit of capacitance is Farad $1\,\text{F} = \dfrac{1\text{C}}{1\text{V}}$

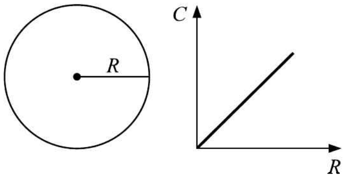
1F is a very big unit. Therefore, μF or *n*F or μμF (*p*F) and so on are used.

Spherical capacitors may be of two types

(a) isolated spherical capacitor

(b) concentric spherical capacitor

(a) Isolated spherical capacitor is a single sphere. Its capacitance is given by $C = 4\pi\varepsilon_0 R$ where R is radius of the sphere.



(a) (b)

Fig. 21.1

(b) Two concentric spherical shells or the inner one may be solid.

$$C = 4\pi\varepsilon_0 \frac{(R_2R_1)}{(R_2 - R_1)}$$

If a dielectric of strength K is introduced between R_1 and R_2.

$$C = 4\pi\varepsilon_0 K \frac{R_2R_1}{R_2 - R_1}$$

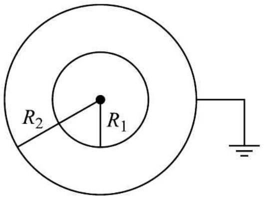


Fig. 21.2

Parallel plate capacitor If A is area of each plate and d is the separation between two plates then

$$C = \frac{\varepsilon_0 A}{d} \text{ with free space as dielectric}$$

$$C = \frac{K\varepsilon_0 A}{d} \text{ if a dielectric of strength } K \text{ is added}$$

If the dielectric slab has thickness t ($t < d$) then

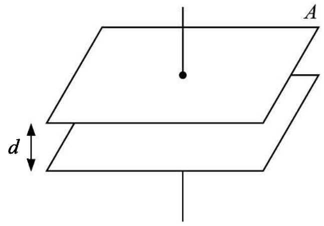


Fig. 21.3

$$C = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{k}\right)}$$

If a dielectric of strength k is introduced in between, electrolytic capacitors may have high values and go upto mF.

Capacitance of a cylindrical capacitor (as shown in Fig. 21.4)

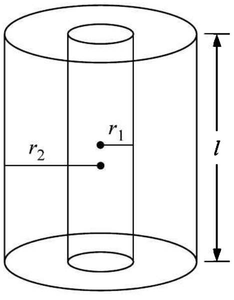


Fig. 21.4

$$C = \frac{2\pi\varepsilon_0 l}{\log_e \frac{r_2}{r_1}}$$

If the space between two cylinders is filled with a dielectric of strenght k then

$$C = \frac{2\pi\varepsilon_0 kl}{\log_e \frac{r_2}{r_1}}$$

Magnitude of induced charge $Q_p = Q\left[1-\frac{1}{k}\right]$

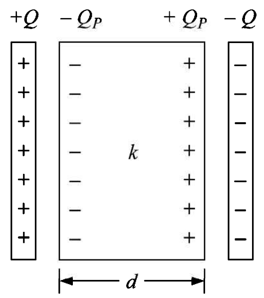


Fig. 21.5

Force between the plates of a capacitor (attractive force)

$$F = \frac{Q^2}{2A\varepsilon_0}$$

Energy stored (electrostatic) in a capacitor

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{QV}{2}$$

Energy stored per unit volume = $\frac{1}{2}\varepsilon_0 E^2$

where E is electric field intensity. The capacitance of a variable tuning capacitor (used for tuning radio) having n plates is

$C = \frac{(n-1)A\,\varepsilon_0}{d}$ where d is the separation between each plate.

If dielectrics are added in the manner shown, then the net capacitance from equivalent circuit is a parallel combination of C_1, C_2 and C_3. Hence

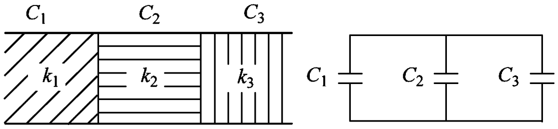


Fig. 21.6

$$C_1 = \frac{\varepsilon_0 k_1 A/3}{d},\ C_2 = \frac{\varepsilon_0 k_2 A/3}{d},\ C_3 = \frac{\varepsilon_0 k_3 A/3}{d}$$

If the dielectrics are arranged as shown in Fig. 21.7 then from equivalent circuit, it is evident that the net capacitance is a series combination of C_1, C_2 and C_3

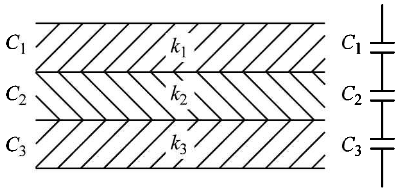


Fig. 21.7

$$C_1 = \frac{A\varepsilon_0 k_1}{t_1};\ C_2 = \frac{A\varepsilon_0 k_2}{t_2};\ C_3 = \frac{A\varepsilon_0 k_3}{t_3}$$

$$C_{eq} = \frac{\varepsilon_0 A}{t_1/k_1 + t_2/k_2 + t_3/k_3}$$

Capacitors in series If C_1, C_2, ... C_n are in series then

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

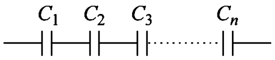


Fig. 21.8

For n equal capacitors to be in series each of capacitance C, then $C_{eq} = \frac{C}{n}$.

For two capacitors C_1 and C_2 to be in series $C_{eq} = \frac{C_1C_2}{C_1+C_2}$.

Capacitors in parallel If C_1, C_2, ... C_n are in parallel then $C_{eq} = C_1 + C_2 + \ldots + C_n$.

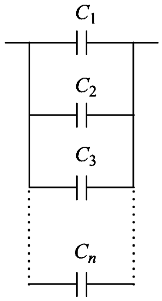


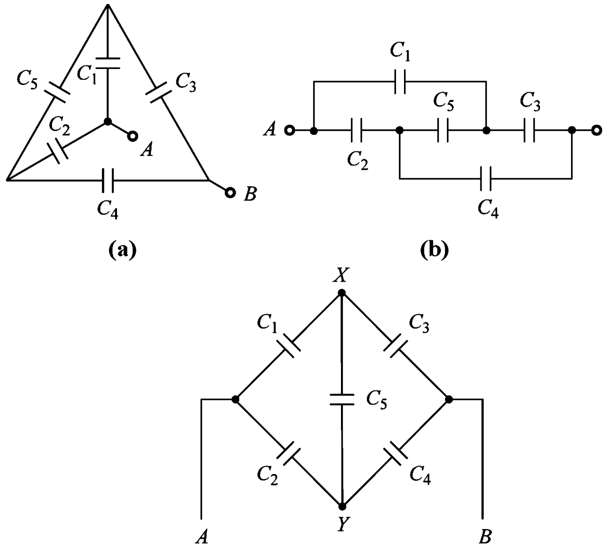
Fig. 21.9

If n equal capacitors are in parallel each of capacitance C then $C_{eq} = nC$.

Four methods to simplify capacitance networks

(a) series/parallel method
(b) wheatstone bridge method
(c) charge distribution method
(d) star/delta network

Wheatstone bridge cases



(a) (b)

(c)

Fig. 21.10

If $\frac{C_1}{C_2} = \frac{C_3}{C_4}$ then eliminate C_5.

If in a wheatstone bridge each capacitor is C then $C_{eq} = C$.

Charge distribution cases

Apply charge distribution if the circuit is symmetrical. In symmetrical circuits, charge entering a branch = charge leaving an identical branch.

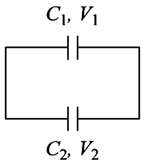


Fig. 21.11

If two capacitors C_1 and C_2 charged to V_1 and V_2 are joined together then common potential is

$$V_{\text{common}} = \frac{V_1C_1 + V_2C_2}{C_1 + C_2} = \frac{Q_1 + Q_2}{C_1 + C_2}$$

Charge on capacitors after joining $\frac{Q_1'}{Q_2'} = \frac{C_1}{C_2}$

$$Q_1' = \frac{(Q_1 + Q_2)C_1}{(C_1 + C_2)}, \quad Q_2' = \frac{(Q_1 + Q_2)C_2}{C_1 + C_2}$$

Loss in energy when two capacitors C_1 and C_2 charged to V_1 and V_2 are joined together is

$$\Delta E = \frac{C_1C_2}{2(C_1 + C_2)}(V_1 - V_2)^2$$

If a dielectric slab in a capacitor is being introduced in the rigidly held plates then the force required to insert the slab is

$$F = \frac{1}{2}V^2\frac{dC}{dx}$$

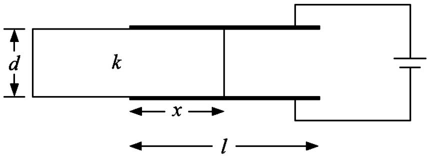


Fig. 21.12

Growth transient or charging of a capacitor If Q is charge at any instant across the capacitor then

$$Q = Q_0(1 - e^{-t/RC}) \quad \text{where} \quad Q = CV_0$$

$$V_R = V_0e^{-t/RC} \qquad I = \frac{dQ}{dt} = \frac{dQ}{dt}A\frac{Q_0}{RC}e^{-t/RC}$$

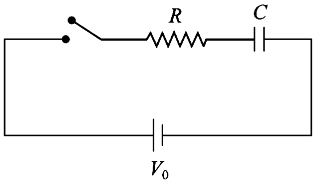


Fig. 21.13

Time constant τ is that time during which a capacitor charges to 63% of its maximum value of charge.

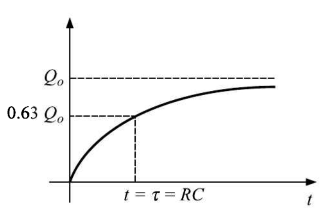


Fig. 21.14

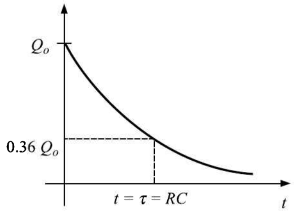


Fig. 21.15

Discharging of a capacitor (decay transient)

$$Q = Q_0\ e^{-t/RC};\ V_R = V_0(1 - e^{-t/RC})$$

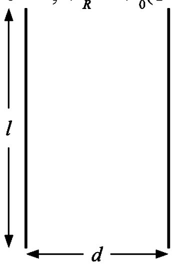


Fig. 21.16

Time constant $\tau = RC$ is defined as the time during which capacitor discharges to 36% of maximum charge.

Capacitance of a transmission line

$$C = \frac{\pi\varepsilon_0 kl}{\log_e \frac{d}{r}}$$

where r is radius of either of the wire.

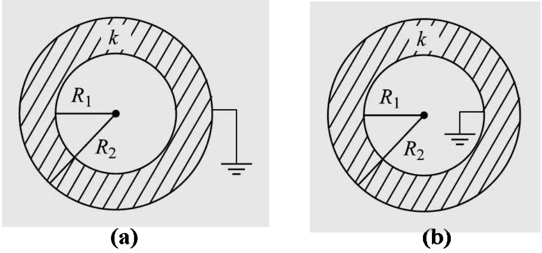
Note: If n identical drops each of radius r and charge q combine to form a big drop of radius R then charge on big drop $Q_{\text{big}} = nq$

$$C_{\text{big}} = n^{1/3}\ C_{\text{small}} \quad \text{and} \quad V_{\text{big}} = n^{2/3}V_{\text{small}};\quad R = n^{1/3}r$$

SHORT-CUTS AND POINTS TO NOTE

(i) If the outer surface is grounded

$$C = \frac{4\pi k\varepsilon_0(R_1R_2)}{(R_2 - R_1)}$$



(a) (b)

Fig. 21.17

If inner sphere is grounded

$$C_{eq} = \frac{4\pi\ k\varepsilon_0 R_1R_2}{(R_2 - R_1)} + 4\pi\varepsilon_0 R_2$$

Because $\frac{4\pi\varepsilon_0 kR_1R_2}{(R_2 - R_1)}$ and $4\pi\varepsilon_0 R_2$ of an isolated spherical capacitor of an outer shell become in parallel.

(ii) Potential drop V_1 and V_2 across capacitors connected in sereis as shown in Fig. 21.18

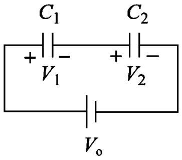


Fig. 21.18

$$V_1 = \frac{V_0C_2}{C_1 + C_2} \quad \text{and} \quad V_2 = \frac{V_0C_1}{C_1 + C_2}.$$

(iii) If two capacitors C_1 and C_2 are in series

$$C_{eq} = \frac{C_1C_2}{C_1 + C_2}.$$

(iv) In series, charge remains same and in parallel potential difference remains same.

(v) If two spheres of radius R_1 and R_2 charged to Q_1 and Q_2 are joined by a wire then common potential

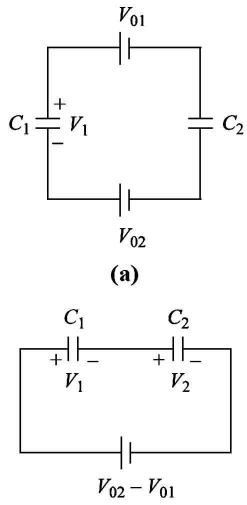
$$V_{\text{common}} = \frac{Q_1 + Q_2}{4\pi\varepsilon_0(R_1 + R_2)}$$

$$Q_1' \text{(after joining)} = \frac{(Q_1 + Q_2)(R_1)}{(R_1 + R_2)}$$

and
$$Q_2' = \frac{(Q_1 + Q_2)R_2}{(R_1 + R_2)}.$$

(vi) If n equal capacitors are in series $C_{eq} = \frac{C}{n}$ and if n equal capacitors are in parallel $C_{eq} = nC$.

(vii) In questions like those shown in Fig. 21.19(a) reduce it to the problem as shown in Fig. 21.19(b).



(a)

(b)

Fig. 21.19

(If the positive terminal of one battery is connected to the negative terminal of the other battery then they are in series hence use $V = V_{01} + V_{02}$ and if the positive terminal of one battery is connected to the positive terminal of the other battery then use $V_{\text{net}} = V_{02} - V_{01}$ or $V_{01} - V_{02}$, which ever is greater and keep the positive or negative sign of the greater emf battery.

Now
$$V_1' = \frac{C_2}{C_1 + C_2}(V_{02} - V_{01})$$

and
$$V_2' = \frac{C_1}{C_1 + C_2}(V_{02} - V_{01}).$$

(viii) If n identical plates each of area A and the separation between two consecutive plates being d are alternately connected as shown in Fig. 21.20. Then $C_{eq} = (n - 1)\,A\varepsilon_0/d$

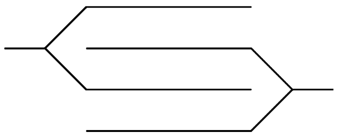


Fig. 21.20

(ix) If a metal plate of thickness t is introduced in between parallel plates of separation d and the area of plates is A then new capacitance is

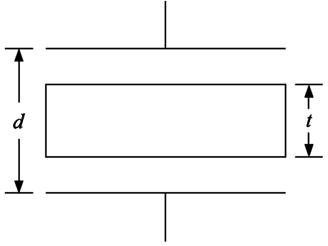


Fig. 21.21

$$C' = \frac{A\varepsilon_0}{(d - t)}$$

(x) If large number of identical capacitors of rating C/V are available and you wish to make a capacitor of rating C'/nV then n capacitors will be required in series. Each row of n capacitors in series has $C_{eq} = \frac{C}{n}$. To make C' we need $m = \frac{C'}{C/n}$ such rows in parallel. Therefore, $m \times n$ capacitors will be required.

(xi) For the network shown below

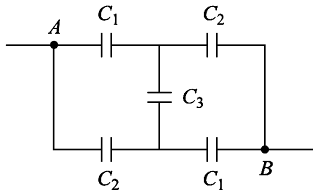


Fig. 21.22

$$C_{AB} = \frac{2C_1C_2 + C_3C_1 + C_3C_2}{C_1 + C_2 + 2C_3}$$

(xii) If capacitance C is connnected along each side of a skeleton cube then the equivalent capacitance along the longest diagonal is $\frac{6}{5}C$, (a) along a face diagonal is $\frac{4}{3}C$, (b) along one side is $\frac{12}{7}C$.

(xiii) If one side of a skeleton cube is open then $C_{AB} = \frac{5}{7}C$.

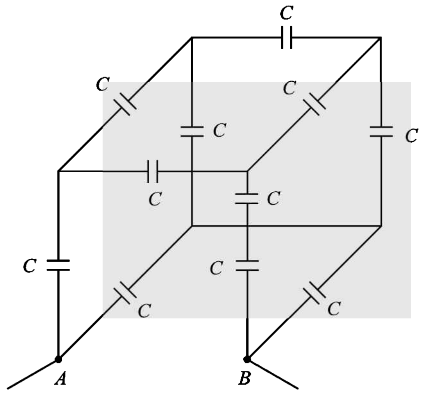


Fig. 21.23

CAUTION

(i) Considering spherical charges as ordinary charges when their sizes are different and they are joined.

- Consider them as capacitors. Find common potential if they are joined and then proceed.

(ii) Confusing that voltage will remain constant if the capacitor is charged and distance between the plates is changed.

- If the battery is removed then the charge will remain conserved and potential drop across the capacitor will change as the capacitance has been changed. If, however, battery remains connected when the separation between the plates is varied then the voltage remains unchanged and is equal to the emf of the battery.

(iii) Confusing that current passes through the capacitor.

- Steady or DC current does not pass through capacitor in steady state. However, during transient, that is, at the time of switch operating ON or OFF, current passes for a short interval according to the equation $\frac{d}{dt}[Q(t)] = i(t)$

  If charging $Q = Q_0(1 - e^{-t/RC})$; If discharing $Q = Q_0 e^{-t/RC}$.

  AC current passes through the capacitor as it is a function of time.

(iv) Confusing that capacitors are added in series or parallel like resistors

- In series $\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots$ is used. If there are two capacitors use $C_S = \frac{C_1 C_2}{C_1 + C_2}$

  In parallel $C_P = C_1 + C_2 + \ldots$

(v) Confusion in series and parallel cases.

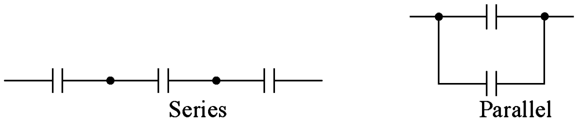

Series Parallel

Fig. 21.24

- Note that in series only one end of a capacitor is connected to one end of the other capacitor. In parallel both ends of the capacitors are joined with two ends of other capacitors as illustrated above.

(iv) Confusing in cases like shown in Fig. 21.25 whether capacitors are in series or parallel.

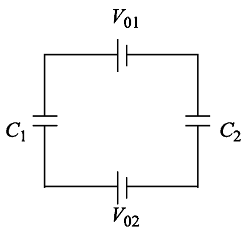


Fig. 21.25

- Capacitors are in series in this case as battery is another element present and capacitors are not joined directly end to end.

(vii) Confusing about wheatstone bridge

- Note that wheatstone bridge is formed if $\frac{C_1}{C_2} = \frac{C_3}{C_4}$ and C_5 is connected between the terminals where we are not intersted to find equivalent capacitance. In such a situation C_5 is removed.

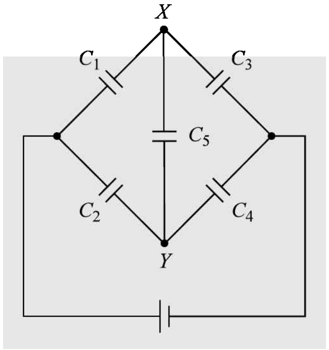


Fig. 21.26

(viii) Considering that a metal plate introduction between the plates of the capacitor has no effect on the capacitance of the capacitor.

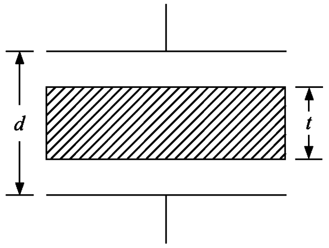


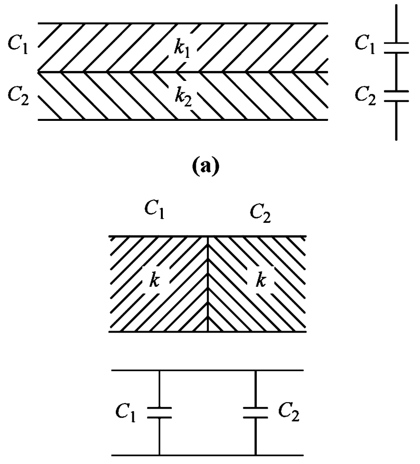
Fig. 21.27

- When a metal plate of thickness t is introduced the effective separation between the plates is reduced by t, that is,

  $d_{\text{effective}} = d - t$

(ix) Confusing whether new capacitors are in series or in parallel when dielectrics are added

- If dielectrics partition the space horizontally (Fig. 21.28(a)). They are in series. If they partition the space vertically then they are in parallel as shown in Fig. 21.28(b).



(a)

(b)

Fig. 21.28

(x) Not knowing the effect of rating of the capacitors.

- 10 μF/200 V capacitor would mean you cannot apply a voltage greater than 200 V across it and it cannot hold a charge greater than 2000 μC ($Q = CV$). Thus, if two capacitors of different ratings are connected in series you cannot give a charge more than the smaller of the two. for example, if 10 μF/200 V and 20 μF/600 V are connected in series then the charge cannot exceed 2000 μC.

(xi) When more than two plates are connected and you are not able to recognise their combination.

- Mark the plates 1, 2, 3, ... and see carefully how the plate numbers are connected and draw an equivalent circuit. Find equivalent capacitance using this equivalent circuit.

Solved Problems

1. A capacitor has charge 50 μC. When the gap between the plates is field with glass wool 120 μC charge flows through the battery the dielectric constant of glass wool is

(a) 3.4 (b) 1.4

(c) 2.4 (d) none of these

Solution (a) $k = \frac{Q'}{Q} = \frac{120+150}{50} = 3.4$

2. A charge of 1 μ C is given to one plate of a capacitor and a charge of 2 μC is given to the other plate of a 0.1 μF capacitor. Find the potential different across the two plates of a capacitor.

(a) 5 V (b) 10 V

(c) 15 V (d) 30 V

Solution (a) $Q_{\text{net}} = 2 - 1 = 1$ μ C. charge = 1/2 μC will appear on each side of the plate as illustrated shown in Fig. 21.29.

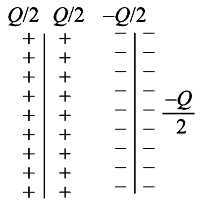


Fig. 21.29

$$\therefore \quad V = \frac{Q_{\text{net}}/2}{C} = \frac{0.5}{0.1} = 5\text{V.}$$

3. A large conducting plane has surface charge density 10^{-4} C/m². Find the electrostatic energy stored in a cubical volume of side 1 cm in front of the plane.

(a) 1.4 J (b) 2.8 J

(c) 5.6 J (d) none of these

Solution (c) $U = \frac{1}{2}\varepsilon_0 E^2$ (Vol.)

$$= \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 \text{(Volume)} = \frac{\sigma^2}{2\varepsilon_0} \text{(Volume)}$$

$$= \frac{(10^{-4})^2 \times 10^{-6}}{2 \times 8.85 \times 10^{-12}} = \frac{100}{2 \times 8.89}\frac{1}{C_{eq}} = 5.6\text{ J}$$

4. In the network shown $C_1 = C$, $C_2 = 2C$, $C_3 = 3C$, $C_4 = 4C$. Find the ratio of charge C_2 to C_4.

(a) 4/7 (b) 22/3

(c) 7/4 (d) 3/22

Solution (d) C_2, C_1 and C_3 are in series

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{2C} + \frac{1}{3C} = \frac{11}{6C}$$

or

$$C_{eq} = \frac{6C}{11}$$

$$\frac{Q_2}{Q_4} = \frac{C_{eq}V_0}{C_4V_0}$$

$$= \frac{6/11}{4} = \frac{3}{22}.$$

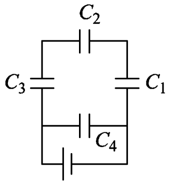


Fig. 21.30

5. A fully charged capacitor has a capacitor 'C'. It is discharged through a small coil of resistance wire embeded in a thermally insulated block of specific heat capacity S and mass m. If the temperature of the block is raised by ΔT, the potential difference V across the capacitance is

(a) $\sqrt{\frac{2mC\,\Delta T}{S}}$ (b) $\frac{mC\,\Delta T}{S}$

(c) $\frac{mS\Delta T}{C}$ (d) $\sqrt{\frac{2mS\,\Delta T}{S}}$

Solution (d) $\frac{1}{2}CV^2 = mS\Delta T$

or $V = \sqrt{\frac{2mS\Delta T}{C}}$

6. A parallel plate capacitor is made of n equally spaced plates connected alternately. If capacitance between two adjacent plates is C then the resultant capacitance is

(a) $(n - 1)C$ (b) $(n + 1)C$

(c) C (d) nC

Solution (a) $C_{\text{eq}} = (n - 1)C$ (one less than the number of plates)

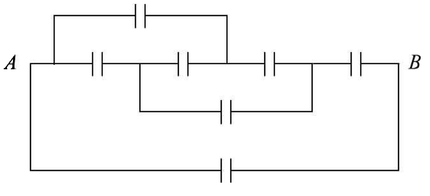
7. An air filled parallel plate capacitor has a capacity 2 pF. The separation between the plates is doubled and the inter space is filled with wax. if the capacity is increased to 6 pF, the dielectric constant of the wax is

(a) 2 (b) 4

(c) 3 (d) 6

Solution (d) $k = \frac{C''}{C'} = \frac{6\,pF}{C/2} = \frac{6\,pF}{1\,pF} = 6$

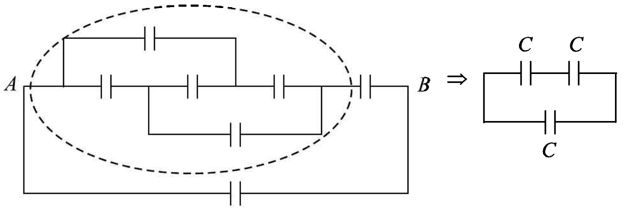
8. If each capacitor has capacitance C then find C_{AB}



(a)

(a) C (b) $C/2$
(c) $3C/2$ (d) none of these

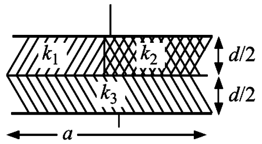
Solution (c) Look into equivalent Fig. 21.31(b) and (c). The dotted part is wheat stone bridge with C_{eq} = C then further equivalent circuit is shown in fig.



(b) (c)

Fig. 21.31

9. Square plates of sides 'a' are filled with dielctrics of strength k_1, k_2 and k_3 as shown in Fig. 21.32. Find C_{eq}

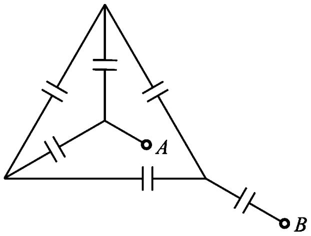


(a)

(a) $\dfrac{\varepsilon_0 a^2(k_1k_3+k_2k_3)}{d(k_1+k_2+k_3)}$ (b) $\dfrac{2\varepsilon_0 a^2(k_1k_3+k_2k_3)}{d(k_1+k_2+k_3)}$

(c) $\dfrac{2\varepsilon_0 a^2(k_1k_3+k_2k_3)}{d(2k_1+k_2+k_3)}$ (d) None of these

Solution (b) The equivalent capacitance circuit is where



(a)

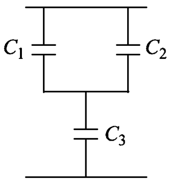
$$C_1 = \frac{\varepsilon_0 k_1 a^2}{d}, \quad C_2 = \frac{\varepsilon_0 k_1 a^2}{d}$$

and $$C_3 = \frac{2\varepsilon_0 k_3 a^2}{d}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1+C_2} + \frac{1}{C_3}$$

$$= \frac{d}{\varepsilon_0 a^2(k_1+k_2)} + \frac{d}{\varepsilon_0 a^2 2k_3}$$

$$C_{eq} = \frac{2\varepsilon_0 a^2(k_1+k_2)k_3}{(k_1+k_2+2k_3)d}$$



(b)

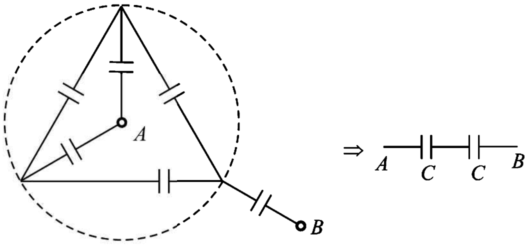
Fig. 21.32

10. Each capacitor has capacitance C in the Fig. 21.33(a) shown. Find C_{AB}

(a) C (b) $2C$
(c) $C/2$ (d) $3C/2$

Solution (c) Note that dotted part in the circuit is a wheat stone bridge with $C_{eq} = C$.

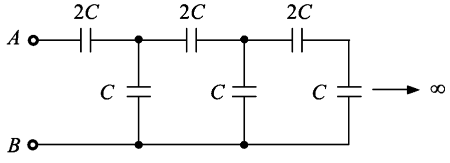
$\therefore$ $C_{AB} = C/2$ from Fig. 21.33(c).



(b) (c)

Fig. 21.33

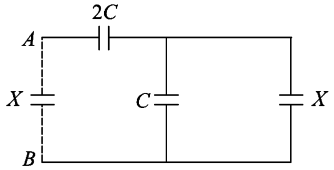
11. Find C_{AB} in the infinite network shown in Fig. 21.34(a).



(a)

(a) C (b) $-2C$
(c) $2C$ (d) $-C$

Solution (a) Let X be the equivalent capacitance. If one more network is added capacitance remains unchanged. Thus from equivalent circuit of Fig. 21.34(b)



(b)

Fig. 21.34

$$X = \frac{(C+X)2C}{X+3C}$$

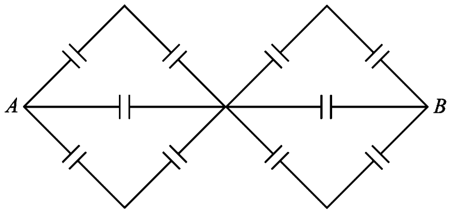
or $X^2 + 3CX = 2C^2 + 2CX$

or $X^2 + CX - 2C^2 = 0$

or $(X+2C)(X-C) = 0 \Rightarrow X = C,$

$X \neq -2C$ $\because$ Capacitance can not be negative.

12. Find C_{AB} if each capacitor is C

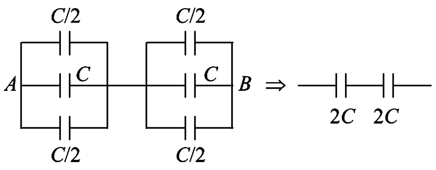


(a)

(a) $3C$ (b) $2C$
(c) C (d) $C/2$

Solution (c) The equivalent circuit is shown in fig. 12.35(b) and (c) from Fig. (c)

$$C_{eq} = \frac{2C}{2} = C.$$



(b)

Fig. 21.35

13. In the circuit shown in Fig. 21.36 charge on 1μF and 3μF capacitors respectively is

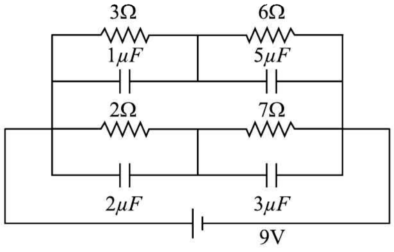


Fig. 21.36

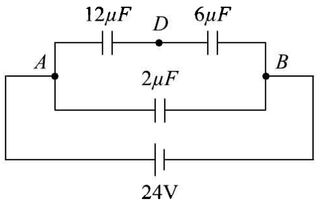
(a) 7 μ C, 3 μC (b) 3 μC, 3 μC
(c) 7 μC, 21 μC (d) 3 μC, 21 μC

Solution (d) $I = \frac{9V}{9\Omega} = 1A.$

Potential drop across 1 μF capacitor = 1 × 3 = 3 V
Potential drop across 3 μF capacitor = 1 × 7 = 7 V
Charge on 1 μF capacitor $Q_1 = CV = 1 \times 3 = 3$ μ C
Similarly charge on 3μF capacitor $Q_2 = 7 \times 3 = 21$ μC

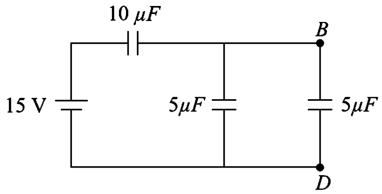
14. Find the potential at D taking potential at B to be zero.
(a) 24 V (b) 8 V
(c) 12 V (d) 16 V

Solution (d) $V_2 = \frac{C_1V_0}{C_1 + C_2} = \frac{12\times24}{18} = 16\text{V}.$



21.37

15. Find potential drop across BD in the given Fig. 21.38(a)

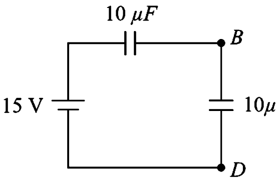


(a)

(a) 5 V (b) 7.5 V
(c) 10 V (d) none of these

Solution (b) See equivalent circuit of Fig. 21.38(b)

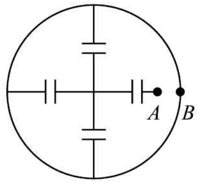
$$V_{BD} = \frac{C_1}{C_1 + C_2}V_0 = \frac{10\times15}{20} = 7.5\text{V}$$



(b)

Fig. 21.38

16. If each capacitor is C find C_{AB} in the given circuit

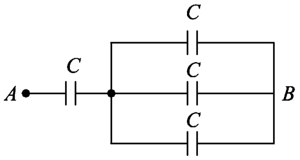


(a)

(a) $\frac{3C}{4}$ (b) 4C
(c) $\frac{C}{4}$ (d) $\frac{C}{2}$

Solution (a) From equivalent circuit of Fig. 21.39(b)

$$C_{AB} = \frac{C\times3C}{C+3C} =$$



(b)

21.39

17. Each capacitor in the circuit is 4μF, when the switch S is closed how much charge will flow through AB

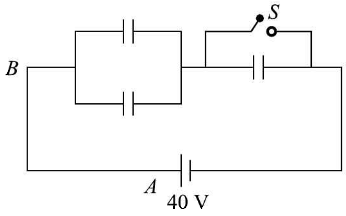


Fig. 21.40

(a) 320μ C (b) 213μC
(c) 107μC (d) none of these

Solution (b) **Case (i).** Switch is open

$$Q = \frac{8\times4}{12}\times40 = \frac{320}{3}\ \mu\text{C}$$

Case (ii). Switch is closed $Q = 8 \times 40 = 320$μ C

Charge flown through AB = $320 - \frac{320}{3} = \frac{640}{3}$ μ C

18. A parallel plate capacitor has plate area 100 cm^2 and separation between the plates is 1 cm. A glass plate ($\varepsilon_r = 6$) of thickness 6 mm and an ebonite plate ($\varepsilon_r = 4$) of thickness 4 mm are inserted. Find C_{eq}
(a) 10^{-10}F (b) 44.25 pF
(c) 4.425 nF (d) 44.25 nF

Solution (b) $\dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} = \dfrac{1}{\varepsilon_0 A}\left[\dfrac{d_1}{k_1} + \dfrac{d_2}{k_2}\right]$

$$\frac{1}{C_{eq}} = \frac{1}{8.85\times10^{-12}\times10^{-2}}\left[\frac{6\times10^{-3}}{6} + \frac{4\times10^{-3}}{4}\right]$$

$$C_{eq} = \frac{8.85\times10^{-12}\times10^{-2}}{2\times10^{-3}} = 4.425 \times 10^{-11}F$$

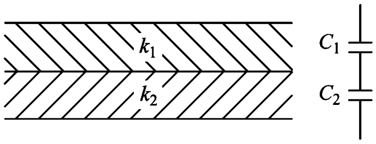


Fig. 21.41

19. A 5 μF capacitor is charged to 12 V. The positive plate of the capacitor is connected to the negative terminal of a 12 V battery and vice versa. Find the heat developed in the connecting wires.

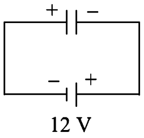


Fig. 21.42

(a) 72 μJ (b) 720 μJ
(c) 1.44 μJ (d) 144 μJ

Solution (c) $E = \dfrac{1}{2}CV^2 = \dfrac{1}{2}\times5\times10^{-6}(24)^2 = 1.44\,\text{mJ}.$

20. Consider the assembly of 3 shells (conducting and concentric) of radii *a*, *b* and *c* as shown in Fig 21.43. Find the capacitance between *A* and *B*.

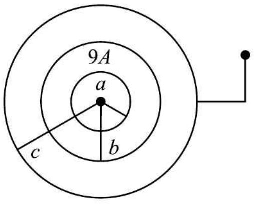


Fig. 21.43

(a) $4\pi\varepsilon_0\left[\dfrac{ba}{b-a} + \dfrac{bc}{c-b}\right]$

(b) $\dfrac{4\pi\varepsilon_0(ba)(bc)}{b(b-a)+(c-b)c}$

(c) $\dfrac{4\pi\varepsilon_0 ca}{c-a}$

(d) none of these

Solution (c) Presence of a thin sheet between parallel plates does not affect the capacitance.

Hence $C = \dfrac{4\pi\varepsilon_0 ca}{c-a}$.

21. A parallel plate capacitor with plate area 100 cm² and separation between the plate 5 mm is connected across a 24 V battery. The force of attraction between the plates is of the order of

(a) 10^{-6}N (b) 10^{-8}N
(c) 10^{-4}N (d) 10^{-7}N

Solution (a) $F = \dfrac{Q^2}{2A\varepsilon_0} = \dfrac{(CV)^2}{2A\varepsilon_0} = \dfrac{\left(\dfrac{A\varepsilon_0}{d}\right)^2 V^2}{2A\varepsilon_0}$

$$= \frac{A\varepsilon_0 V^2}{2d^2} = \frac{10^{-2}\times8.85\times10^{-12}\times24^2}{2\times25\times10^{-6}}$$

$= 1.08 \times 10^{-6}$N

22. A capacitor 10 μF charged to 50 V is joined to another uncharged 50 μF capacitor. Find the loss in energy.

(a) 1.04×10^{-4}J (b) 4.01×10^{-4}J
(c) 6.25×10^{-4}J (d) 1.64×10^{-4}J

Solution (a) Energy loss $= \dfrac{C_1C_2}{2(C_1+C_2)}(V_1 - V_2)^2$

$$= \frac{10\times50\times10^{-12}}{2(10+50)\times10^{-6}}(50-0)^2$$

$= 1.04 \times 10^{-4}$J

23. Two spheres of radius 5 cm and 10 cm both charged to 120μC are joined by a metal wire and then metal wire is removed. What is the charge on each after removal of the wire.

(a) 120μ C, 120μC (b) 80μC, 160 μC
(c) 100μC, 140μC (d) None of these

Solution (b) $Q_1' = \dfrac{(Q_1+Q_2)r_1}{r_1+r_2} = \dfrac{240\times5}{15} = 80\mu C$

$Q_2' = 240 - 80 = 160\,\mu C$

24. In the Fig. 21.44 shown the potential drops across 3μF capacitor when switch *S* is open and switch *S* is closed is

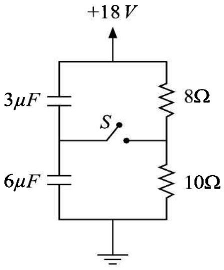


Fig. 21.44

(a) 9V, 8V (b) 9V, 9V
(c) 6V, 8V (d) 12V, 8V

Solution (d) When switch is open 18 V is applied across 6 μF and 3 μF capacitor

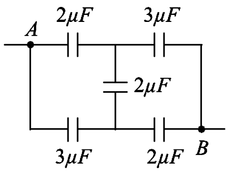


Fig. 21.45

$$V_1 = \frac{18\times6}{6+3} = 12V$$

when the switch is closed potential drop across 8Ω resistor is the potential drop across 3 μF capacitor i.e., 8V.

25. Find the net capacitances between A and B

(a) $\frac{5}{2}\ \mu\text{F}$ (b) $\frac{2}{5}\ \mu\text{F}$

(c) $\frac{9}{22}\ \mu\text{F}$ (d) $\frac{22}{9}\ \mu\text{F}$

Solution (d) $C_{AB} = \dfrac{2C_1C_2 + C_1C_3 + C_2C_3}{C_1 + C_2 + 2C_3}$

$$= \frac{2\times2\times3 + 2\times2 + 3\times2}{2 + 3 + 2\times2} = \frac{22}{9}\ \mu\text{F}$$

26. A 10 μF/400 V and a 4 μF/100 V are connected in series. Find the maximum potential which can be applied.

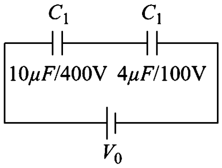


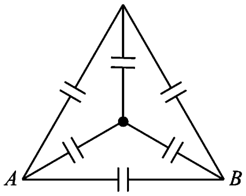
Fig. 21.46

(a) 100 V (b) 500 V
(c) 400 V (d) 140 V
(e) none of these

Solution (d) In series charge remains same. The maximum charge which can be applied is 400μC (maximum rating of 4 μF/100 V) capacitor. Then potential drop across C_1 is $V_1 = \dfrac{400\ \mu\text{C}}{10\ \mu\text{F}} = 40\text{V}$.

$\therefore$ Maximum potential or emf which can be applied is 100 + 40 = 140 V

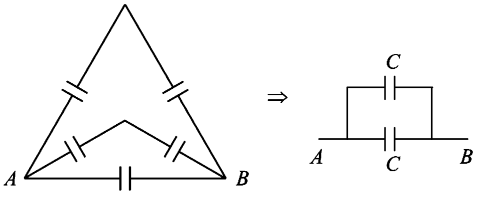
27. Each side of a tetrahedral has a capacitor of capacitance C. Find the capacitance between a side.



(a)

(a) $C/2$ (b) $2C$
(c) C (d) $C/3$

Solution (b) The equivalent circuit of Fig. 21.47(a) is shown in Fig. (b) and Fig. (c) respectively.



(b) **(c)**

Fig. 21.47

$\therefore \quad C_{AB} = 2C$

28. The switch S is kept closed for a long time. It is opened at $t = 0$. Find the current in R_1 at $t = 1$ ms.

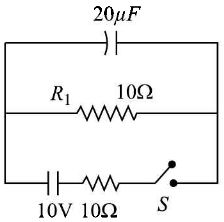


Fig. 21.48

(a) 11.2 mA
(b) 12.4 mA
(c) 13.4 mA
(d) 14.4 mA

Solution (c) $Q = Q_0 e^{-t/RC}$

and $$\frac{dQ}{dt} = i = \frac{Q_0}{R_C} e^{-t/RC}$$

$$= \frac{20\times20\times10^{-6}}{10\times20\times10^{-6}} e^{-5}1$$

$$= 2(0.0067) = 13.4\ \text{mA}$$

Problems for Practice

1. Three condensers C_1, C_2 and C_3 are connected to a 100 volt. DC source as shown in the Fig. 21.49. If the charges stored on the plates of C_1, C_2, and C_3 are q_a, q_b and q_c, q_d and q_e, q_f respectively, then

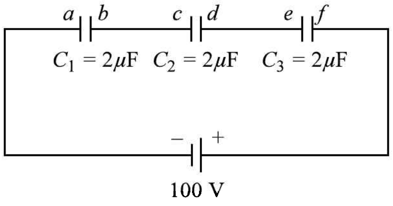


Fig. 21.49

(a) $q_a + q_c + q_e = 50$ coulomb
(b) $q_b = q_d = q_f$
(c) $q_b + q_d + q_f = 100/9$ coulomb
(d) $q_b + q_d + q_f = 0$

2. 64 water drops having equal charges combine to form one bigger drop. The capacitance of bigger drop, as compared to that of smaller drop will be
(a) 4 times (b) 8 times
(c) 16 times (d) 64 times

3. Two parallel plate condensers of capacitance of 20 μF and 30 μF are charged to the potential of 30 V and 20 V respectively. If likely charged plates are connected together then the common potential difference will be
(a) 10 V (b) 24 V (c) 50 V (d) 100 V

4. The energy acquired by a charged particle of 4 μC when it is accelerated through a potential difference of 8 volt will be
(a) 3.2×10^{-7} (b) 3.2×10^{-5}
(c) 2×10^{-6} J (d) 2×10^{-5}

5. The effective capacitance between the points x and y in the will be
(a) 1 μF (b) 1.5 μF
(c) 2 μF (d) 4 μF

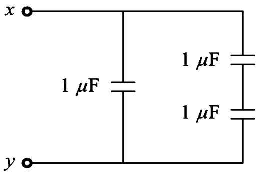


Fig. 21.50

6. The equivalent capacitance in the adjoining diagram will be

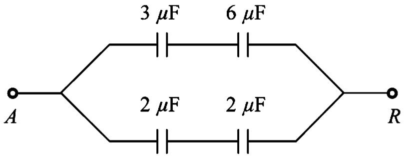


Fig. 21.51

(a) 13 μF (b) 36/13 μF
(c) 7 mF (d) 3 μF

7. The capacitance of a parallel plate capacitor is 4 μF. If a dielectric material of dielectric constant 16 is placed between the plates then the new capacitance will be
(a) 1/64 μF (b) 0.25 μF
(c) 64 μF (d) 40 μF

8. A conducting hollow sphere of radius 0.1 m is given a charge of 10 μC. The electric potential on the surface of sphere will be
(a) zero (b) 3×10^5 V
(c) 9×10^5 V (d) 9×10^9 V

9. Three condensers of capacity 2 μF, 4 μF and 8 μF respectively, are first connected in series and then connected in parallel. The ratio of equivalent capacitances in two cases will be
(a) 7 : 3 (b) 49 : 4 (c) 3 : 7 (d) 4 : 49

10. In the following figure the effective area of each plate of a mica condenser is A and distance between two consecutive plates is d. If the dielectric constant of mica is k, then the capacitance of the condenser will be

Fig. 21.52

(a) $\frac{\varepsilon_0 kA}{2d}$ (b) $\frac{2\varepsilon_0 kA}{d}$

(c) $\frac{2\varepsilon_0 kA}{d}$ (d) $\frac{4\varepsilon_0 kA}{d}$

11. Which material sheet should be placed between the plates of a parallel plate condenser in order to increase its capacitance?
(a) mica (b) copper (c) tin (d) iron

12. Three condensers each of capacitance 2 F, are connected in series. The resultant capacitance will be
(a) 6 F (b) 5 F (c) 2/3 F (d) 3/2 F

13. What will be the area of pieces of paper in order to make a paper condenser of capacitance 0.04 μF, if the dielectric constant of paper is 2.5 and its thickness is 0.025 mm?
(a) 1 m^2 (b) $2 \times 10^{-3} m^2$
(c) $4.51 \times 10^{-3} m^2$ (d) $10^{-3} m^2$

14. When two condensers of capacitance 1 μF and 2 μF are connected is series then the effective capacitance will be
(a) 2/3 μF (b) 3/2 μF (c) 3 μF (d) 4 μF

15. The equivalent capacitance between the points X and Y in the Fig. 21.53 will be

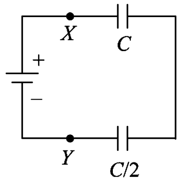


Fig. 21.53

(a) $2C/3$ (b) $C/3$
(c) $3C/2$ (d) $3C$

16. The capacitance of a parallel plate capacitor in air is 2 μF. If dielectric medium is placed between the plates then the potential difference reduces to 1/6 of the original value. The dielectric constant of the medium is
(a) 6 (b) 3 (c) 2.2 (d) 4.4

17. A condenser is charged to a potential difference of 200 volts as a result of which it gains charge of 0.1 coulomb. When it is discharged then the energy released will be
(a) 1 J (b) 2 J (c) 10 J (d) 20 J

18. The equivalent capacitance in the Fig. 21.54 will be

Fig. 21.54

(a) $\frac{\varepsilon_0 A}{3d}$ (b) $\frac{\varepsilon_0 A}{2d}$ (c) $\frac{5\varepsilon_0 A}{d}$ (d) $\frac{\varepsilon_0 A}{d}$

19. If the diameter of earth is 128×10^2 km, then its capacitance will be
(a) 711 μF (b) 331 μF
(c) 211 μF (d) 111 μF

20. The charge on the capacitor in the Fig. 21.55 is

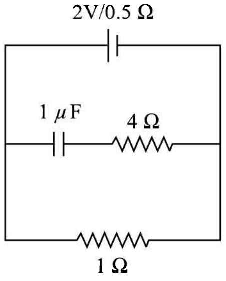


Fig. 21.55

(a) 2 μC (b) 2/3 μC
(c) 4/3 μC (d) zero

21. The capacitance of a condenser is 20 μF and it is charged to a potential of 2000 V. The energy stored in it will be
(a) zero (b) 40 J (c) 80 J (d) 120 J

22. A 100 μF capacitor is charged to 200 volt. It is discharged through a 2 ohm resistance. The amount of heat generated will be
(a) 0.4 J (b) 0.2 J (c) 2 J (d) 4 J

23. The capacitance of a charged condenser is C and energy stored on account of charge on it is U, then the quantity of charge on the conductor will be
(a) $\sqrt{2UC}$ (b) $\sqrt{\frac{UC}{2}}$
(c) $2\,UC$ (d) zero

24. Two charged conducting spheres are joined by a conducting wire then
(a) nothing will be conserved
(b) the total energy will be conserved
(c) the total charge will be conserved
(d) the total charge and energy will be conserved

25. The net charge on a condenser is
(a) infinity (b) $q/2$
(c) $2q$ (d) zero

26. The capacitance of a spherical conductor of radius r is proportional to
(a) $1/r$ (b) r (c) $1/r^2$ (d) r^2

27. The energy stored between the plates of a condenser is *not* represented by
(a) $U = \frac{CV^2}{2}$ (b) $U = 2qV$
(c) $U = \frac{q^2}{2C}$ (d) $U = \frac{q^2}{2C}$

28. The capacitance of a parallel plate air condensor is 10 μF. Two dielectric media of equal size of relative permittivity 2 and 4 are filled in it. The effective capacitance of the system will be

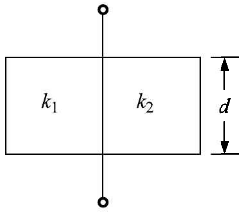


Fig. 21.56

(a) 10 μF (b) 20 μF
(c) 30 μF (d) 40 μF

29. Farad is not equivalent to
(a) CV^2 (b) J/V^2 (c) Q^2/J (d) Q/V

30. Two spherical shells of radii r_1 and r_2 carrying charge q_1 and q_2 are connected by a wire as shown in the Fig. 21.57. The equivalent capacitance of the system is

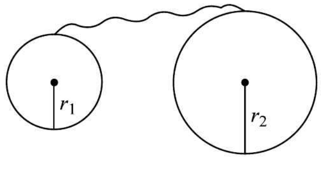


Fig. 21.57

(a) $4\pi\varepsilon_0 \frac{r_1 r_2}{r_2 - r_1}$ (b) $4\pi\varepsilon_0 \frac{r_1 r_2}{r_1 + r_2}$
(c) $4\pi\varepsilon_0(r_1 + r_2)$ (d) $4\pi\varepsilon_0(r_2 - r_1)$

31. The electric energy density between the plates of charged condenser is
(a) $q/2\varepsilon_0 A^2$ (b) $q/2\varepsilon_0 A$
(c) $q^2/(2\varepsilon_0 A^2)$ (d) none of the above

32. The potential difference between the plates of a condenser of capacitance 0.5 μF is 100 volt. It is connected to an uncharged condenser of capacity 0.2 μF by a copper wire. The loss of energy in this process will be
(a) 0 J (b) 0.5×10^{-3} J
(c) 0.7×10^{-3} J (d) 10^{-3} J

33. The capacitance of a conducting metallic sphere will be 1 μF if its radius is nearly
(a) 1.11 cm (b) 10 cm
(c) 1.11 m (d) 9 km

34. A capacitor of capacitance C is connected to battery of emf V_0. Without removing the battery, a dielectric of strength ε_r is inserted between the parallel plates of the capacitor C, then the charge on the capacitor is
(a) CV_0 (b) $\varepsilon_r CV_0$
(c) $\frac{CV_0}{\varepsilon_r}$ (d) none of these

35. Three capacitors of capacity 6 μF, 3 μF and 9 μF are connected as shown in the Fig. 21.58. The potential difference between the plates of 9 μF capacitor will be

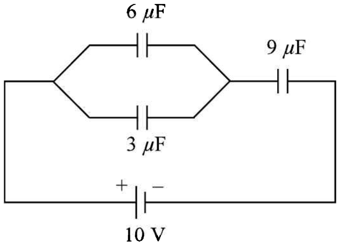


Fig. 21.58

(a) 2 V (b) 4 V (c) 5 V (d) 6 V

36. When dielectric medium of constant k is filled between the plates of a charged parallel plate condenser, then the energy stored becomes, as compared to its previous value
(a) k^{-2} times (b) k^2 times
(c) k^{-1} times (d) k times

37. The potential difference across the plates of 4.5 μF capacitor in the Fig. 21.60 will be

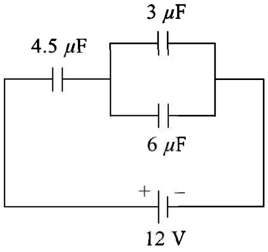


Fig. 21.59

(a) 83 V (b) 8 V
(c) 4 V (d) 6 V

38. When a thin mica sheet is placed between the plates of a condenser then the amount of charge, as compared to its previous value, on its plates will become

(a) unchanged (b) zero
(c) less (d) more

39. The equivalent capacitance between points A and B is

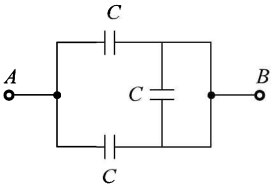


Fig. 21.60

(a) $3C/2$ (b) $C/3$
(c) $C/2$ (d) $2C$

40. A $10\,\mu F$ condenser is charged to a potential of 100 volt. It is now connected to another uncharged condenser. The common potential reached is 40 volt. The capacitance of second condenser is
(a) $2\,\mu F$ (b) $10\,\mu F$ (c) $15\,\mu F$ (d) $22\,\mu F$

41. The minimum number of condensers each of capacitance of $2\,\mu F$, in order to obtain resultant capacitance of $5\,\mu F$ will be
(a) 4 (b) 10 (c) 5 (d) 6

42. The equivalent capacitance across the larger diagonal of a skeleton cube containing capacitance C in each arm will be ———. The equivalent capacitance of Fig. 21.61 is ———.

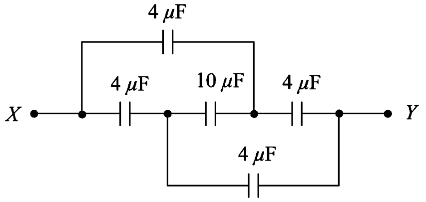


Fig. 21.61

(a) $4C/3$, $4\,\mu F$ (b) $12C/7$, $2\,\mu F$
(c) $7C/12$, $8.3\,\mu F$ (d) $6C/5$, $4\mu F$

43. The distance between the plates of a circular parallel plate condenser of diameter 40 mm, in order to make its capacitance equal to that of a metallic sphere of radius 1 m, will be
(a) 0.01 mm (b) 0.1 mm
(c) 1 mm (d) 10 mm

44. A battery of 100 V is connected to series combination of two identical parallel plate condensers. If dielectric of constant 4 is slipped between the plates of second condenser, then the potential difference on the condensers will respectively become
(a) 80 V, 20 V (b) 75 V, 25 V
(c) 50 V, 80 V (d) 20 V, 80 V

45. The intensity of an electric field between the plates of a charged condenser of plate area A will be
(a) $A/(q\varepsilon_0)$ (b) qA/ε_0
(c) $q/(\varepsilon_0 A)$ (d) none of the above

46. In the below (Fig. 21.62) circuit, the capacitance between the points A and B will be

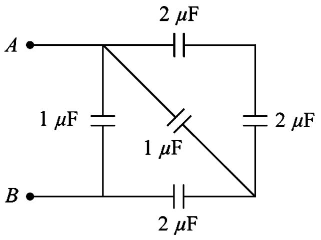


Fig. 21.62

(a) $1\,\mu F$ (b) $2\,\mu F$ (c) $3\,\mu F$ (d) $4\,\mu F$

47. The area of each plate of a parallel plate capacitor is $2\,m^2$. The space between the plates is filled with materials of dielectric constants 2, 3 and 6 and their thickness are 0.4 mm, 0.6mm and 1.2 mm respectively. The capacitance of the capacitor will be
(a) $8.94 \times 10^{-4} F$ (b) $6.94 \times 10^{-7} F$
(c) $2.94 \times 10^{-8} F$ (d) $10^{-8} F$

48. In the following circuit (Fig. 21.63) the potentials at points A and B will be respectively

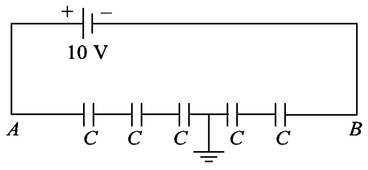


Fig. 21.63

(a) 10 V, 0 V (b) 5 V, –5 V
(c) 4 V, –6 V (d) 6 V, –4 V

49. A parallel plate condenser with plate separation d is charged with the help of a battery so that V_0 energy is stored in the system. The battery is now removed. A plate of dielectric constant k and thickness d is placed between the plates of condenser. The new energy of the system will be
(a) $V_0 k^{-2}$ (b) $k^2 V_0$ (c) $V_0 k^{-1}$ (d) kV_0

50. In the adjoining Fig. 21.64 a section of a complicated circuit is shown in which E = 10 volt, $C_1 = 2\,\mu F$, $C_2 = 3\,\mu F$ and $(V_B - V_A)$ = 10 volt. The potential on C_1 will be

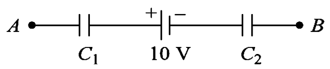


Fig. 21.64

(a) 0 volt (b) 4 volt
(c) 12 volt (d) 16 volt

51. The equivalent capacity between the points A and B in the Fig. 21.65 is

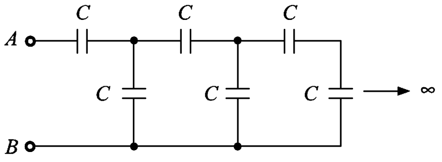


Fig. 21.65

(a) $\dfrac{C+C\sqrt{5}}{2}$ (b) $\dfrac{-C+C\sqrt{5}}{2}$
(c) $\dfrac{-C-C\sqrt{5}}{2}$ (d) zero

52. A parallel plate condenser is connected to a battery of emf 4 volt. If a plate of dielectric constant 8 is inserted into it, then the potential difference on the condenser will be
(a) 32 V (b) 4 V (c) 1/2 V (d) 2 V

53. The force of attraction between the plates of a charged condenser is
(a) $q^2/(2\varepsilon_0 A)$ (b) $q^2/(2\varepsilon_0 A^2)$
(c) $q/(2\varepsilon_0 A^2)$ (d) none of the above

54. On increasing the plate separation of a charged condenser its energy
(a) remains unchanged
(b) decreases
(c) increases
(d) none of the above

55. Two condensers, each of capacitance 1 μF, are connected in parallel. These are charged by a DC source of 200 volt. The total energy of their charges in Joule will be
(a) 0.06 (b) 0.04
(c) 0.02 (d) 0.01

56. A parallel plate condenser of capacitance C is connected to a battery and is charged to potential V. Another condenser of capacity 2C is connected to another battery and is charged to potential 2V. The charging batteries are removed and now the condensers are connected in parallel in such a way that the positive plate of one is connected to negative plate of another. The final energy of this system is
(a) $25CV^2/6$ (b) $9CV^2/2$
(c) $3CV^2/2$ (d) zero

57. A 4 μF capacitor is charged to 50 V and another capacitor of 2 μF is charged to 100 V. The two condensers are connected such that their likely charged plates are connected together. The total energy of the system before and after joining will be in multiple of 10^{-2} J.
(a) 3.0 and 2.67 (b) 2.67 and 3.0
(c) 1.5 and 1.33 (d) 1.33 and 1.5

58. A condenser of capacitance C_1 is charged to V_0 volt. The energy stored in it is U_0. It is connected in parallel to another uncharged condenser of capacitance C_2. The energy loss in the process is
(a) $\dfrac{C_1C_2U_0}{2(C_1+C_2)}$ (b) $\left(\dfrac{C_1-C_2}{C_1+C_2}\right)^2 U_0$
(c) $\dfrac{C_1U_0}{C_1+C_2}$ (d) $\dfrac{C_2U_0}{C_1+C_2}$

59. The plates of a parallel plate condenser are being moved away with velocity *v*. If the plate separation at any instant of time is *d* then the rate of change of capacitance with time is proportional to
(a) d^2 (b) d (c) d^{-2} (d) d^{-1}

60. A 1 mm thick paper of dielectric constant 4 lies between the plates of a parallel plate capacitor. It is charged to 100 volt. The intensity of electric field between the plates of the condenser will be
(a) 100 (b) 100000
(c) 400000 (d) 25000

61. The potential difference V_{AB} in the Fig. 21. 66 shwon is
(a) $\dfrac{(C_1-C_2)V_0}{C_1+C_2}$ (b) $\dfrac{(C_2-C_1)V_0}{C_1+C_2}$
(c) $\dfrac{C_2V_0}{C_1+C_2}$ (d) $\dfrac{C_1C_2V_0}{C_2(C_1+C_2)}$

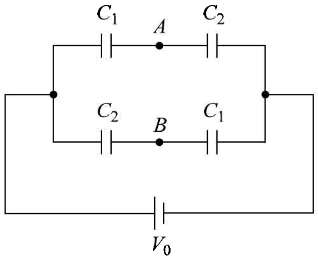


Fig. 21.66

62. The plates of a parallel plate capacitor are not exactly parallel. The surface charge density, therefore,
(a) is smallest where the plates are closest.
(b) is higher at the closer end
(c) will not be uniform
(d) Each plate will have the same potential at each point

63. Two capacitors 6μF/200V and 1μF/60V are connected in series The maximum emf which can be applied is
(a) 260 V (b) 200 V (c) 70 V (d) none of these

64. The plates of a capacitor are charged to a potential difference of 100 V and are then connected across a resistor. The potential difference across the capacitor decays exponentially with time. After 1 second the potential difference between the plates of the capacitor is 80 V. The fraction of the stored energy which has been dissipated is
(a) 1/5 (b) 1/25 (c) 9/25 (d) 16/25

65. An uncharged parallel plate capacitor having a dielectric of constant *k* is connected to a similar air filled capacitor charged to a potential *V*. The two share the charge and the common potential is V''. The dielectric constant *k* is
(a) $\dfrac{V-V''}{V''}$ (b) $\dfrac{V''-V}{V''}$
(c) $\dfrac{V''-V}{V''+V}$ (d) $\dfrac{V''-V}{V}$

ANSWERS

1. (b)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (c)
11. (a)	**12.** (c)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (c)
21. (b)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (c)
31. (c)	**32.** (c)	**33.** (d)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (b)	**38.** (a)	**39.** (d)	**40.** (c)
41. (a)	**42.** (d)	**43.** (b)	**44.** (a)	**45.** (c)	**46.** (b)	**47.** (c)	**48.** (d)	**49.** (c)	**50.** (c)
51. (b)	**52.** (b)	**53.** (a)	**54.** (c)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (d)	**59.** (c)	**60.** (d)
61. (a)	**62.** (b, c, d)	**63.** (c)	**64.** (c)	**65.** (a)					

Electricity

22

BRIEF REVIEW OF THE CONCEPTS

Electric current The time rate of change of charge is called current, that is,

$$I = dQ/dt$$

Its unit is ampere

DC current is scalar while AC current is a phasor (vector).

Cells and generators are common sources of electricity. Ideal voltage source has zero internal resistance and ideal current source has infinite resistance.

Electromotive force (*emf*) is the maximum potential difference between two points when current drawn is zero (open circuit). Therefore, emf is equal to open circuit voltage. Its unit is volt

$$1 \text{ volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}}$$

Emf of a cell depends upon the nature of electrolyte and nature of electrodes.

Potential (*V*) Amount of work done to bring a unit positive charge from infinity to that point against the electric field of a given charge without changing kinetic energy. (unit → volt). Practically potential cannot be measured as infinity is not defined. Hence we use potential difference.

Potential difference is the difference of potential between two points. Its unit is volt.

$$V = V_1 - V_2$$

Conduction in metals Current in a conductor or metal is given by

$$I = neav_d$$

where n is number electron density, e is charge on an elecrtron, a is area of cross-section and v_d is drift velocity.

Current density (*J*) Current per unit area is defined as current density

$$J = I/A = nev_d \text{ is a vector; } J = \sigma E \quad \sigma = ne\mu$$

where σ is conductivity and μ is mobility.

Drift velocity The average directed velocity along the conductor in the presnece of applied electric field is called drift velocity and is given by

$$v_d = \frac{eE\tau}{m}$$

where τ is relaxation time, E is electric field strength and m is mass of the electron.

Relaxation time The average time between two successive collisions of electrons.

Short circuit current is the maximum possible current when circuit resistance (external resistance) is zero. Thus, $I_{SC} = I_{\max} = \varepsilon/r$ where ε is emf and r is internal resistance of the battery.

Resistance R of a conductor $\propto l$ (length of the conductor)

$R \propto 1/A$ (Area of cross-section)

Thus $R = \rho l/A$ (where ρ is resistivity)

Resistances are of three types ohmic, non ohmic and negative. Ohmic resistances follow ohm's law $V = IR$ (that is, $V \propto I$). In nonohmic resistance V is not linearly related to I. Vacuum tubes and semiconductor are examples of non ohmic resistances. We find dynamic resistance for a non-ohmic device. In a negative resistance device $I \propto 1/V$. Examples of negative resistances are tunnel diode, tetrode and thyristor.

Resistivity or specific resistance (ρ) is the resistance of a unit cube of a material.

$$\rho = \frac{m}{ne^2\tau}$$

Its unit is ohm-m

Conductivity (σ) is reciprocal of resistivity, that is, $\sigma = 1/\rho$ Its unit is (ohm-m)$^{-1}$.

Resistance depends upon (a) collision between electrons and electrons (b) collision between electron and core (c) interaction between electrons and lattice vibration (d) trap centres.

Alloys have higher resistance due to more trap centres. Material used to make standard resistances is Manganin as it has high specific resistance and it varies negligibly with temperature.

Alloy used in making rheostat is Constantan

Alloy used in heaters (press, geyser, room heaters and so on) Nichrome.

Specific resistance depends upon the nature of the substance. It varies directly with temperature $\rho(T) = \rho_0(1 + \alpha T)$ where α is called thermal coefficient of resistivity. Resistivity varies inversely with pressure. It does not depend upon length and area of cross-section of the material.

Silver is the best conductor followed by copper and then aluminium. Hence copper is commonly used followed by aluminium.

Conductance $G = 1/R$. Its unit is ohm^{-1} or mho or Siemen (S), Superconductors have zero resistance.

Laws of resistance

In series $R_S = R_1 + R_2 + \ldots + R_n$

In parallel $\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$

$$R_P = \frac{R_1 R_2}{R_1 + R_2}$$

If there are n equal resistances in series $R_S = nR$

If there are n equal resistances in parallel $R_P = \frac{R}{n}$

Terminal potential V $= \varepsilon - Ir$ is the potential drop across a resistance or across the cell when current is passing.

r is internal resistance and ε is emf of the cell.

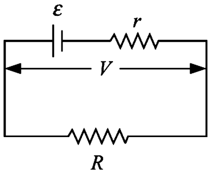


Fig. 22.1

Normally terminal potential is less than emf. But during charging of a cell terminal potential is greater than emf.

Cells in series If n identical cells each of emf ε are connected in series each having internal resistance r.

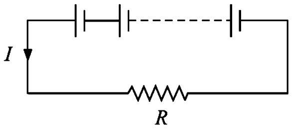


Fig. 22.2

Then $I = \frac{n\varepsilon}{R + nr}$ where R is external resistance

Cells in parallel If n cells (identical) are in parallel each having emf ε and internal resistance r.

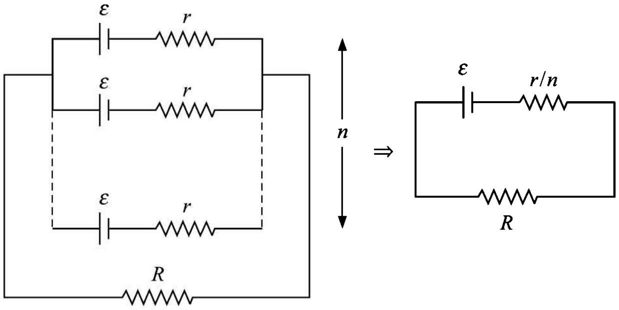
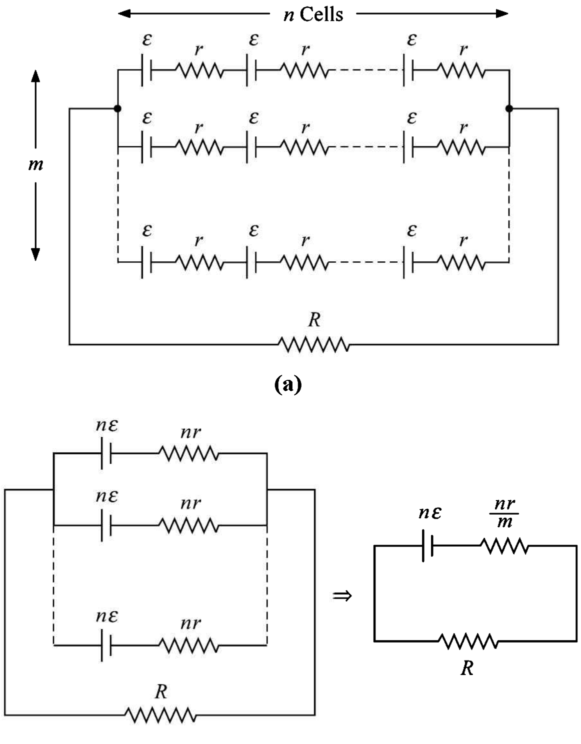


Fig. 22.3

Then $I = \frac{\varepsilon}{R + r/n}$

Cells in mixed grouping m rows of n cells each are connected to an external resistance R. Assuming all cells are identical, each having emf ε and internal resistance r as shown in Fig. 22.4.



(a)

(b) (c)

Fig. 22.4

$$I = \frac{n\varepsilon}{nr/m + R}$$

Maximum power is delivered by a cell/source if external resistance = internal resistance.

Maximum current is delivered when short circuited (external resistance = zero).

Kirchoff's current law Algebraic sum of all the currents entering at any instant in a junction is zero. In other words, sum of all the currents entering the junction is equal to sum of all the currents leaving the junction at that instant. It is based on charge conservation.

Loop law or Kirchoff's voltage law Algebraic sum of all the potential drops in a closed loop is zero. It is based on conservation of energy.

Wheatstone bridge bridge is said to be balanced if $V_X = V_Y$ or $I_G = 0$. When the bridge is balanced $\frac{P}{Q} = \frac{R}{S}$.

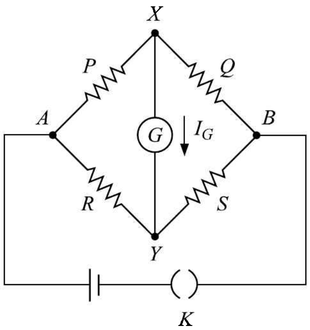


Fig. 22.5

Potentiometer The fall of potential along the length of a conductor of uniform area of cross-section and uniform density is proportional to its length when current I passes through it, provided physical conditions like temperature, pressure and so on remain unchanged.

Here potential gradient $k = V/l$.

To find emf by comparison method

$$\varepsilon_1/\varepsilon_2 = l_1/l_2$$

To find internal resistance of a cell

$$r = R\left(\frac{l_1 - l_2}{l_2}\right)$$

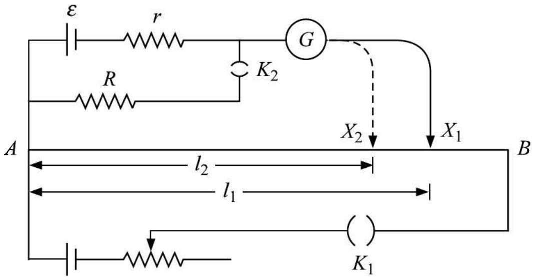


Fig. 22.6

where l_1 is the length when key k_2 is not introduced
l_2 is the length when key k_2 is also introduced
} These are the lengths at null point.

Meter bridge

$$\frac{P}{Q} = \frac{l}{(100 - l)}$$

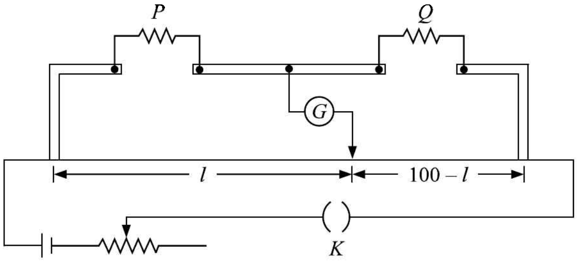


Fig. 22.7

SHORT-CUTS AND POINTS TO NOTE

(i) In current division rule current is divided in the inverse ratio of resistances

$$I_1 = \frac{IR_2}{R_1 + R_2}; \quad I_2 = \frac{IR_1}{R_1 + R_2}$$

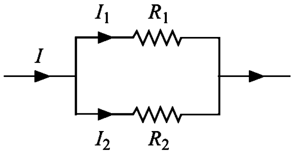


Fig. 22.8

(ii) Potential division rule ($V = IR$)

$$V_1 = \frac{V_0R_1}{R_1 + R_2 + R_3}, \quad V_2 = \frac{V_0R_2}{R_1 + R_2 + R_3},$$

$$V_3 = \frac{V_0R_3}{R_1 + R_2 + R_3}$$

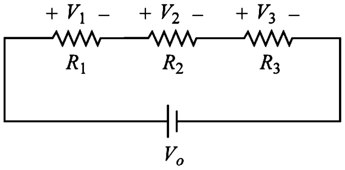
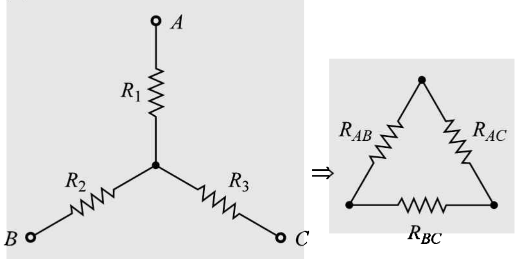


Fig. 22.9

(iii) In order to find equivalent resistance there are four methods to simplify resistance networks involving
(1) series/parallel method. (when visible clearly)
(2) wheatstone bridge method (when series/parallel method fails)
(3) current division method (for symmetrical circuits)
(4) star-delta method



(a) (b)

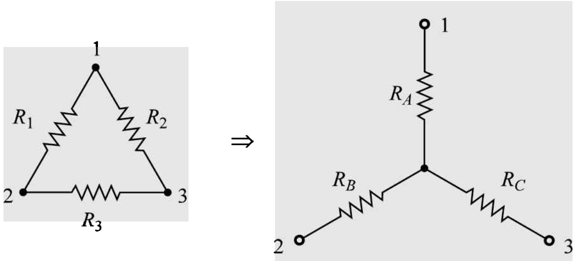
Fig. 22.10

$$R_{AB} = R_1 + R_2 + \frac{R_1R_2}{R_3}$$

$$R_{AC} = R_1 + R_3 + \frac{R_1R_3}{R_2}$$

$$R_{BC} = R_2 + R_3 + \frac{R_2R_3}{R_1}$$

Delta to star conversion



(a) (b)

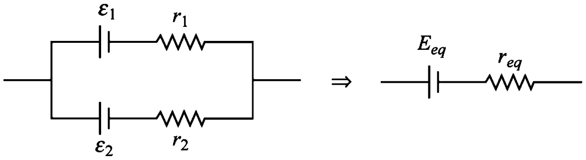
Fig. 22.11

$$\mathbf{R}_A = \frac{R_1R_2}{R_1 + R_2 + R_3}; \quad R_B = \frac{R_1R_3}{R_1 + R_2 + R_3}$$

$$R_C = \frac{R_2R_3}{R_1 + R_2 + R_3}$$

(iv) If two non-identical cells are in parallel then

$$E_{eq} = \frac{\varepsilon_1 r_2 + \varepsilon_2 r_1}{r_1 + r_2} \text{ and } r_{eq} = \frac{r_1 r_2}{r_1 + r_2}$$

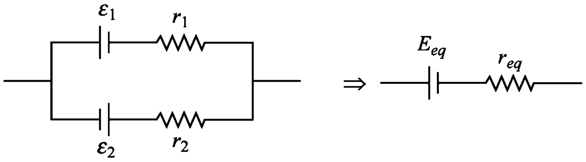


(a) **(b)**

Fig. 22.12

If cells are connected in opposite direction then

$$E_{eq} = \frac{\varepsilon_1 r_2 - \varepsilon_2 r_1}{r_1 + r_2} \text{ and } r_{eq} = \frac{r_1 r_2}{r_1 + r_2}$$



(a) **(b)**

Fig. 22.13

(v) If current meets the positive terminal at the enterance point then take ε_1 as positive.

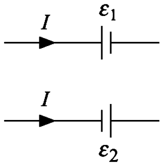


Fig. 22.14

(vi) If current enters the negative terminal then take ε_2 as negative.

(vii) To find I_1, I_2 and I_3 in circuits as shown below. Find V_{AB}

Then $I_1 = V_{AB}/R_1$; $I_2 = V_{AB}/R_2$ and $I_3 = V_{AB}/R_3$

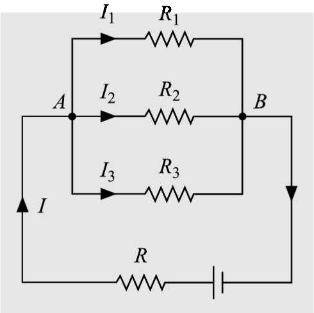


Fig. 22.15

(viii) If identical cells are connected in a loop in order then emf between any two points is zero.

(ix) If wheatstone bridge is made of identical resistances then equivalent resistance is equal to any one of them. Thus, in Fig. 22.16 $r_{AB} = r$

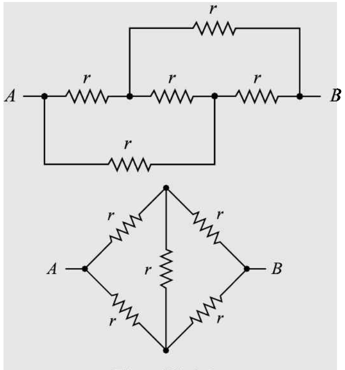


Fig. 22.16

(x) Normally a voltmeter is connected in parallel. But in order to find high resistance it may be used in series as shown in Fig. 22.17 If voltmeter reads V and its internal resistance is R. Then $V = \dfrac{V_0 R}{X + R}$ and hence X can be determined.

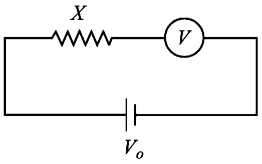


Fig. 22.17

(xi) Current in a branch is zero if potential at its end terminals are equal, that is, $I = 0$ if $V_1 = V_2$.

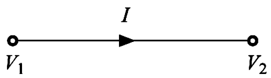


Fig. 22.18

(xii) If a branch contains a capacitor then the current through that branch is zero in steady state when DC current is applied.

(xiii) The equivalent resistance for the circuit shown in Fig. 22.19 is

$$R_{AB} = \frac{R_1(R_1 + 3R_2)}{(R_2 + 3R_1)}$$

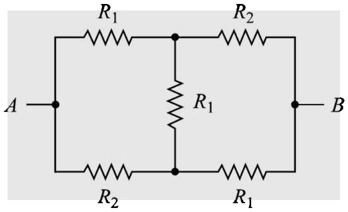


Fig. 22.19

(xiv) If n identical cells are to be connected in series and m of them are wrongly connected then the net emf E_{net} is given by

$E_{net} = n\varepsilon - 2m\varepsilon$ where ε is emf of each cell.

(xv) To convert a galvanometer (or coil) of maximum current (full scale current) I_g and internal resistance R_g into an ammeter, to measure current I, then a shunt S be connected in parallel

$$S = \frac{R_g I_g}{I - I_g}$$

(xvi) To convert a galvanometer of maximum current I_g and internal resistance R_g into a voltmeter, in order to measure V volts, a resistance R be connected in series given by

$$R = V/I_g - R_g$$

(xvii) An ideal ammeter has zero resistance and an ideal voltmeter has infinite resistance.

(xviii) If you short circuit two points the resistance across those two points will be zero irrespective of resistance already shown.

(xix) In parallel circuit the net resistance is smaller than the smallest in the circuit.

(xx) If a skeleton cube is made with 12 equal resistances/ wires each having resistance r then the net resistance across

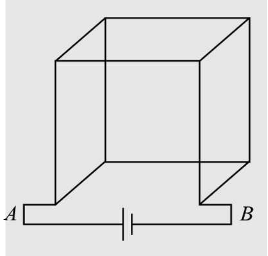


Fig. 22.20

(a) the longest diagonal is $5/6r$

(b) the diagonal of face is $3/4r$

(c) a side is $7/12r$

(d) the side open circuited $7/5r$

(xxi) Strictly speaking resistance varies non-linearly for metals as

$$R(T) = R_0(1 + \alpha T + \beta T^2 + ...)$$

(xxii) You can determine the temperature using wheatstone bridge

$$\frac{P}{R + \Delta R} = \frac{Q}{S(1 + \alpha \Delta T)}$$

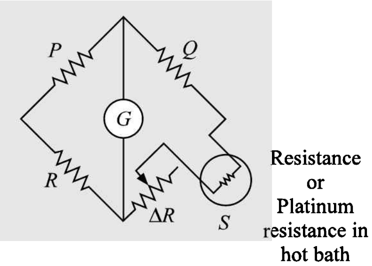


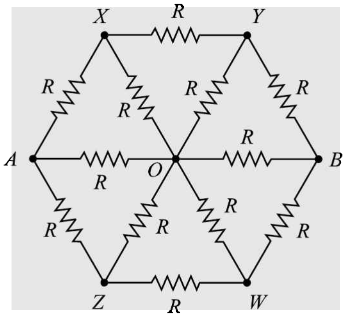
Fig. 22.21

if $P = Q$ then

$\Delta R = S\alpha\Delta T$ or $\Delta T = \dfrac{\Delta R}{S\alpha}$.

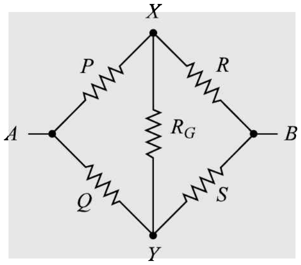
CAUTION

(i) Wrongly detecting wheatstone bridge.For example, considering any of $AXYO$; $XYBO$; $AOWZ$; and $OZWB$ as wheatstone bridge



(a)

- If A and B are point of interest where equivalent resistance is to be determined and in wheatstone brid-ge R_G is connected between XY points and $\dfrac{P}{Q} = \dfrac{R}{S}$ or $V_X = V_Y$ or no current passes through XY or R_G, then only remove resistance $\mathbf{R_G}$



(b)

Fig. 22.22

(ii) Adding ε_1 and ε_2 in order to find net emf when negative terminal of one is connected to negative terminal of another or positive terminal of one connected to positive terminal of another.

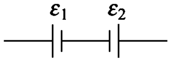


Fig. 22.23

- In such cases use $\varepsilon_1 - \varepsilon_2$ or $\varepsilon_2 - \varepsilon_1$, keeping in mind which is greater or the direction of current chosen.

(iii) Not applying current division when branching occurs

- Always stick to junction law in such cases.

(iv) Not considering voltmeter or ammeter resistance when given

- Since voltmeter is connected in parallel, its resistance drastically changes the resistance of the circuit.

(v) Not taking into account internal resistance of the cell

- When current is passing (circuit is not open) then terminal voltage is less than emf and is equal to $\varepsilon - Ir$ where ε is emf and r is internal resistance of the cell or battery.

(vi) Assuming DC current passes through capacitor during steady state

- $Q = CV$, $\dfrac{dQ}{dt} = I = \dfrac{CdV}{dt}$ makes clear that the current will pass through capacitor as long as V is varying with time, that is, either AC is connected or during transients (for a short period when switch is made ON or OFF).

(vii) Considering potentiometer wire has no resistance

- Its resistance must be taken into account and potential drop across the wire divided by total length of wire gives potential gradient (not just emf of the source connected divided by length of the wire).

(viii) Considering resistivity varies with length or area of cross-section

- Only resistance varies with length and not resistivity. Resistivity, however, is inversely related to pressure and directly to temperature.

(ix) Considering that when current through capacitor is zero, there will be no potential drop

- Though current through the capacitor is zero but potential drop occurs.

(x) Not understanding the meaning of a switch

- In Fig. 22. 24 the potential drop across *AB* is –6 V when the switch is open and 0 when the switch is closed. A close switch is equivalent to zero resistance (short circuit).

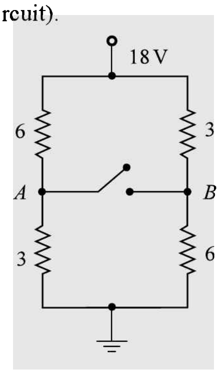


Fig. 22.24

Solved Problems

1. Find the drift velocity in Cu wire if it has 1A current through 2 mm² cross-section. Free electron density is 8.5×10^{22} cm^{-3}

(a) 0.36 mm s^{-1} (b) 0.36 cm s^{-1}
(c) 0.036 mms^{-1} (d) 0.036 cms^{-1}

Solution (c) $v_d = \frac{i}{neA}$

$$= \frac{1}{8.5\times10^{28}\times1.6\times10^{-19}\times2\times10^{-6}} = \frac{1\times10^{-3}}{27.2} = 0.036\,\text{mms}^{-1}$$

2. A 5Ω constantan wire is bent to form a ring. Find the resistance across the diameter of the wire.

(a) 2.5Ω (b) 1.25Ω
(c) 5Ω (d) 0.625Ω

Solution (b) $R \propto l$. Therefore 2.5 Ω resistances are in parallel across *AB*. Thus

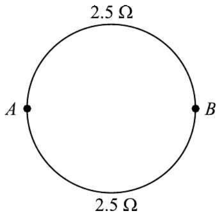
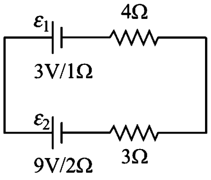


Fig. 22.25

$$R = \frac{2.5}{2} = 1.25\,\Omega$$

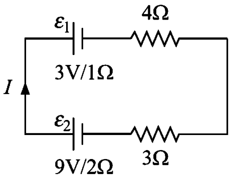
3. Find the terminal voltage across ε_1 and ε_2 in Fig. 22.26(a)



(a)

(a) 3.6V, 7.8V (b) 2.4 V, 7.8V
(c) 3.6V, 10.2V (d) 2.4 V, 10.2V

Solution (a) Current in the circuit 22.26(b)



(b)

Fig. 22.26

$$I = \frac{9-3}{10} = 0.6\text{A}$$

$$V_1 = \varepsilon_1 + Ir_1 = 3 + 0.6 \times 1 = 3.6\text{V};$$

$$V_2 = \varepsilon_2 - Ir_2 = 9 - 0.6 \times 2 = 7.8\,\text{V}$$

4. Find I_1 and I_2 in Fig. 22.27

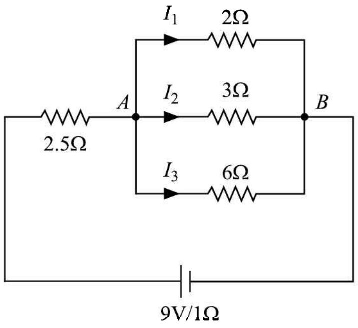


Fig. 22.27

(a) 2.0A, 4/3A (b) 1A, 2/3A
(c) 2A, 2/3A (d) 1A, 1/3A

Solution (b) $\dfrac{1}{R_{AB}} = \dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{6} = \dfrac{3+2+1}{6}$

and $I = \dfrac{9}{1+1+2.5} = 2A$

$$V_{AB} = I \cdot R_{AB} = 2 \times 1 = 2V$$

$$I_1 = \frac{2V}{2\Omega} = 1A;\ I_2 = \frac{2V}{3\Omega} = \frac{2}{3}A$$

5. Find the value of R so that no current passes through in the galvanometer whether switch S is closed or open.

(a) 4 Ω (b) 8 Ω
(c) 6 Ω (d) none of these

Solution (b) It is wheatstone bridge case. Therefore

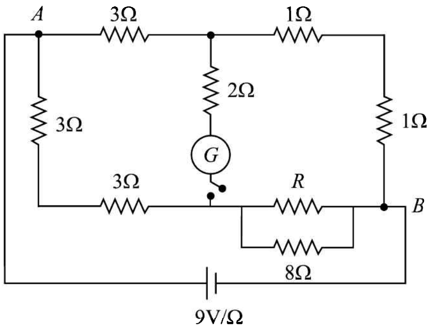
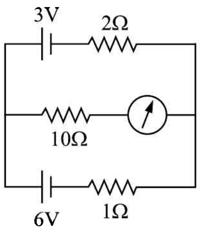


Fig. 22.28

$$\frac{3}{3+3} = \frac{1+1}{\dfrac{8R}{8+R}} \text{ or } \frac{8R}{8+R} = 4 \Rightarrow R = 8\,\Omega$$

6. The ammeter reading in the circuit of Fig. 22.29(a) is

(a) 15/32A (b) 14/33A
(c) 17/33A (d) 15/31A

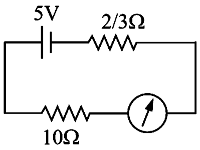


(a)

Solution (a) $E_{eq} = \dfrac{3\times1+6\times2}{1+2} = 5\text{V};$

$r_{eq} = \dfrac{2\times1}{2+1} = \dfrac{2}{3}\Omega$

From equivalent circuit of Fig. 22.29(b)

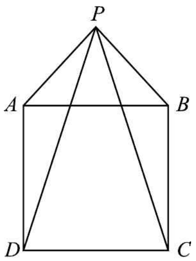


(b)

Fig. 22.29

$$I = \frac{5}{10+2/3} = \frac{15}{32}A$$

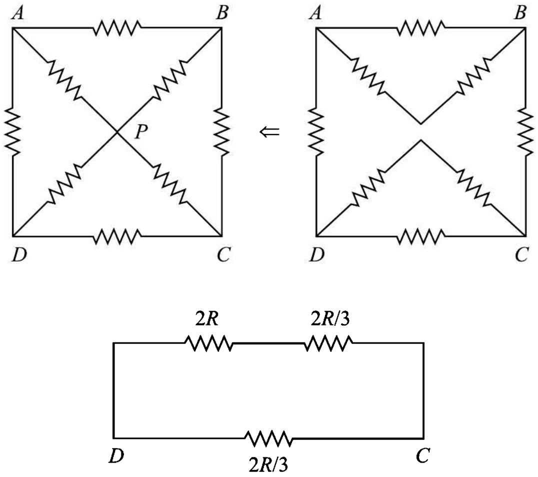
7. Find the equivalent resistance about any branch of the base of the square pyramid shown Assume resistance of each branch is R



(a)

(a) 7R/15
(b) 8R/15
(c) R/2
(d) none of these

Solution (b) From the equivalent circuit of Fig. 22.30(b)



(d)

Fig. 22.30

$$\mathbf{R}_{eq} = \frac{\dfrac{8R}{3}\times\dfrac{2R}{3}}{\dfrac{8R}{3}+\dfrac{2R}{3}} = \frac{16R}{30} = \frac{8R}{15}$$

8. A hollow cylinder of radius a and b is filled with a material of resistivity ρ. Find the current through ammeter

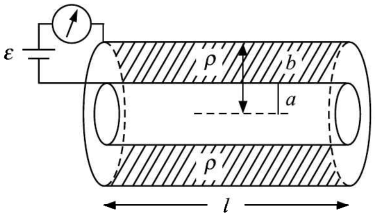


Fig. 22.31

(a) $\dfrac{\varepsilon\pi(b^2-a^2)}{\rho l}$ (b) $\dfrac{\varepsilon\pi l}{\left(\rho \log \dfrac{b}{a}\right)}$

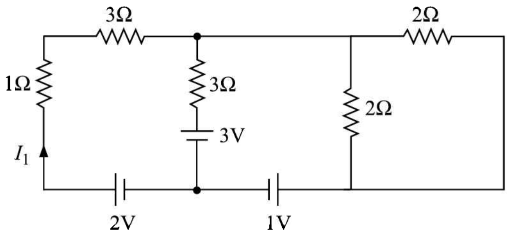
(c) $\dfrac{\varepsilon 2\pi l}{\rho \log_e \dfrac{b}{a}}$ (d) $\dfrac{\varepsilon 2\pi l}{\rho \log_e \dfrac{a}{b}}$

Solution (c) Assume a hypothetical cylinder of radius x and thickness dx then

$$\int dR = \int_a^b \rho \frac{dx}{2\pi x l}$$

$$R = \frac{\rho \log_e \frac{b}{a}}{2\pi l} \text{ and } I = \frac{\varepsilon}{R} = \frac{\varepsilon}{\rho}\frac{2\pi l}{\log_e \frac{b}{a}}$$

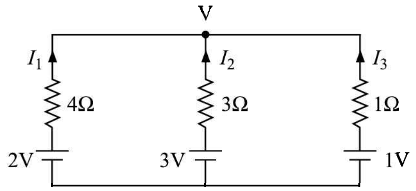
9. Find current I_1 in Fig. 22.32(a)



(a)

(a) 1/19A (b) 2/19A
(c) 3/19A (d) none of these

Solution (b) Draw equivalent circuit as shown in Fig. 22.32(b). Let node potential be V then applying KCl (Junction law)



(b)

Fig. 22.32

$$I_1 + I_2 + I_3 = 0 \text{ or } \frac{2-V}{4} + \frac{3-V}{3} + \frac{1-V}{1} = 0$$

or $6 - 3V + 12 + 4V + 12 - 12V = 0$ or $V = 30/10$ volt

$$I_1 = \frac{2 - 30/19}{4} = \frac{2}{19}\text{A}.$$

10. Find the potential drop across 4μF capacitor and 6Ω resistor in Fig. 22.33.

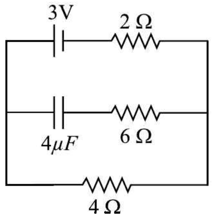


Fig. 22.33

(a) 0, 0 (b) 0, 3V
(c) 0, 2V (d) 3V, 2V

Solution (d) $I = \frac{3}{2+4} = \frac{1}{2} A$

Potential drop across 4Ω resistor is $V = 4 \times \frac{1}{2} = 2V$

The whole potential drop occurs across 4μF capacitor as current does not flow through the branch containing capacitor.

11. The temperature of a conductor is increased. The product of resistivity and conductivity
(a) increases
(b) decreases
(c) remains constant
(d) may increase or decrease

Solution (c)

12. Two non ideal batteries are connected in parallel. Then
(A) The equivalent *emf* is less than either of the two *emfs*.
(B) The equivalent internal resistance is less than either of the two internal resistance
(a) both A and B are correct
(b) only A is correct
(c) only B is correct
(d) both A and B are wrong

Solution (c)

13. A resistor connected to a battery is heated due to current through it. Which of the following quantity does not vary
(a) resistance
(b) drift velocity
(c) resistivity
(d) number of free electrons volume

Solution (d)

14. Find the electric field in the copper wire of area of cross-section 2 mm² carrying a current of 1 A. The resistivity of copper is 1.7×10^{-8} Ωm
(a) 4.25×10^{-3} V/m (b) 8.5×10^{-3} V/m
(c) 8.5 V/m (d) 8.5×10^{3} V/m

Solution (b) $J = \sigma E$

or $E = \frac{J}{\sigma} = J\rho = \frac{I\rho}{A}$

$$= \frac{1 \times 1.7 \times 10^{-8}}{2 \times 10^{-6}} = 8.5 \times 10^{-3}\text{V/m}$$

15. A high resistance voltmeter reads 1.52 V when switch S is open and 1.48 V when switch S is closed. The ammeter reading at this instant is 1A. Find r.

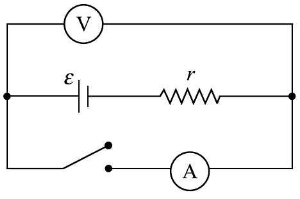
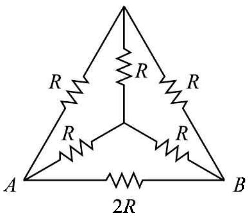


Fig. 22.34

(a) 0.2 Ω (b) 0.3 Ω
(c) 0.4 Ω (d) 0.8 Ω

Solution (c) $\varepsilon = 1.52V$ and $V = \varepsilon - Ir$
$1.48 = 1.52 - 1(r)$

16. Find the resistance across AB in Fig. 22.35(a)

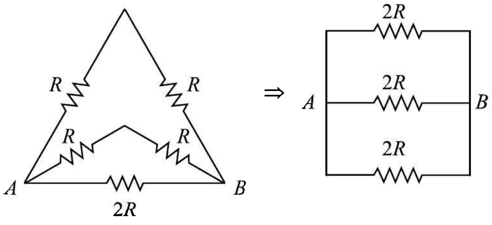


(a)

(a) R (b) $\frac{2}{3}R$

(c) $\frac{R}{3}$ (d) $\frac{4}{3}R$

Solution (b) Draw equivalent circuit of Fig. 22.35(b) then $R_{eq} = 2R/3$.



(b)
Fig. 22.35

17. When the current in a wire is 1A, the drift velocity is $1.2 \times 10^{-4}\,ms^{-1}$. The drift velocity when current becomes 5A is

(a) $1.2 \times 10^{-4}\,ms^{-1}$

(b) $3.6 \times 10^{-4}\,ms^{-1}$

(c) $6 \times 10^{-4}\,ms^{-1}$

(d) $4.8 \times 10^{-4}\,ms^{-1}$

Solution (c) $I \propto v_d$

∴ New drift velocity is $6 \times 10^{-4}\,ms^{-1}$.

18. An ideal voltmeter is connected in Fig. 22.36. The current in the circuit is

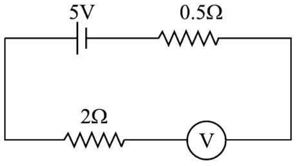


Fig. 22.36

(a) 2A (b) 2.5A

(c) ∞ (d) zero

Solution (d) Ideal voltmeter has infinite resistance. Therefore current will be zero.

19. In Fig. 22.37 *AB* is 300cm long wire having resistance 10Ω per meter. Rheostat is set at 20Ω. The balance point will be attained at

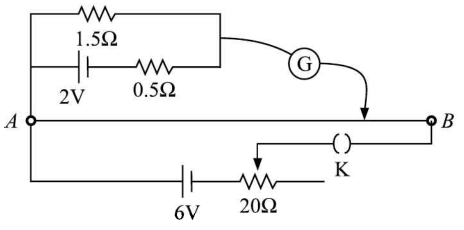


Fig. 22.37

(a) 1.0m (b) 1.25m

(c) 1.5m (d) cannot be determined

Solution (b) $V_{AB} = \frac{6 \times 30}{50} = 3.6\text{V}$.

Terminal voltage of cell $= \frac{2 \times 1.5}{2} = 1.5\text{V}$

Using $V = kl \Rightarrow 1.5 = \frac{3.6}{300} l$ or $l = 125\text{cm}$

20. Ohm's law can be applied to

(a) Ohmic devices (b) non-ohmic devices

(c) both (a) and (b) (d) none

Solution (c)

21. Which of the V-I graph obeys ohm's law?

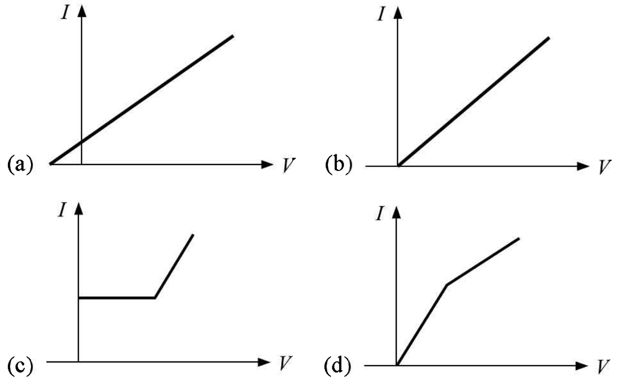


Fig. 22.38

Solution (b)

22. The V-I graph of a conductor at two different temperatures is shown in Fig. 22.39. The ratio of temperature T_1/T_2 is

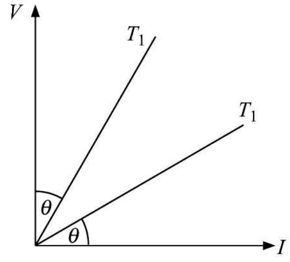


Fig. 22.39

(a) $\tan^2\theta$ (b) $\cot^2\theta$

(c) $\sec^2\theta$ (d) $\operatorname{cosec}^2\theta$

Solution (b)

23. Two cells of emfs 1.25V and 0.75V having equal internal resistance are connected in parallel. The effective emf is

(a) 0.75V (b) 1.25V

(c) 2.0V (d) 1.0V

(e) 0.5V

Solution (d)

24. A 250cm long wire has diameter 1 mm. It is connected to right gap of a slide wire bridge. When a 3Ω resistance is connected to left gap, the null point is obtained at 60cm. The specific resistance of the wire is

(a) $6.28 \times 10^{-6}\,\Omega\text{m}$ (b) $6.28 \times 10^{-5}\,\Omega\text{m}$

(c) $6.28 \times 10^{-8}\,\Omega\text{m}$ (d) $6.28 \times 10^{-7}\,\Omega\text{m}$

Solution (d) Using $\frac{R_1}{R_2} = \frac{l}{100 - l} \Rightarrow \frac{3}{R_2} = \frac{60}{40}$

∴ $R_2 = 2\Omega$

$$\rho = \frac{R \times \pi r^2}{l} = \frac{2 \times 3.14 \times (0.5 \times 10^{-3})^2}{2.5}$$

$\rho = 6.28 \times 10^{-7}\,\Omega\text{m}$

25. An ammeter reads 500mA. When a shunt of 0.1Ω is connected across the ammeter its reading drops to 50mA. The resistance of the ammeter is

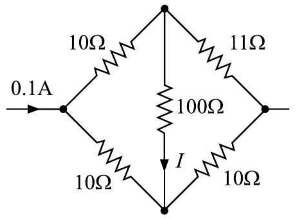
(a) 1 Ω (b) 1.1 Ω

(c) 0.9 Ω (d) none of these

Solution (c) In parallel voltage remains same.

$\therefore \quad 50R = 450 \times 0.1$ or $R = 0.9\,\Omega$

26. Find the current in 100 Ω resistance in Fig. 22.40



(a)

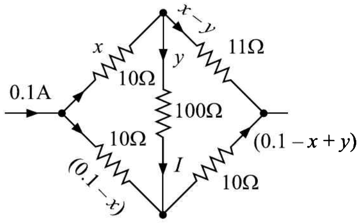
(a) 0.01A nearly

(b) 0.001A nearly

(c) 0.02A nearly

(d) 0.002A nearly

Solution (b) Solve using loop law



(b)

Fig. 22.40

$$10x + 100y = 10(0.1 - x)$$

or $$20x + 100y = 1 \quad \text{...(1)}$$

$$100y + 10(0.1 - x + y) = 11(x - y)$$

$$21x - 121y = 1 \quad \text{...(2)}$$

Solving eq. (1) and (2) for y we get y = 0.001A.

27. A plastic tube 25 m long and 4 cm in diameter is dipped into Ag solution depositing a silver layer 0.1 mm thick uniformly over its outer surface. Find the current if this coated tube is connected across a 12 V battery. $\rho_{Ag} = 1.47 \times 10^{-8}\,\Omega$-m

(a) 4.14A (b) 41.4A

(c) 414A (d) 414 mA

Solution (c) $R = \dfrac{\rho l}{\pi(r_2^2 - r_1^2)} = \dfrac{1.47 \times 10^{-8} \times 25}{3.14(2.01^2 - 2^2) \times 10^{-4}}$

$= \dfrac{1.47 \times 25 \times 10^{-4}}{3.14 \times 4.01 \times (0.01)} = 2.9 \times 10^{-2}\,\Omega$

$I = \dfrac{12}{2.9 \times 10^{-2}} = 4.14 \times 10^2 = 414\text{A}.$

28. Open circuit voltage of a source is 7.86 V and its short circuit current is 9.25A. Find the current when an external resistance of 2.4 Ω is connected

(a) 1.4A (b) 1.82A

(c) 2.01A (d) 2.4A

Solution (d) $r = \dfrac{7.86}{9.25} = 0.847\,\Omega$

$I = \dfrac{7.86}{2.4 + 0.847} A \dfrac{It}{e} = 2.4\text{A}.$

29. In the Fig. 22.41. Find I, R, ε_1 and ε_2

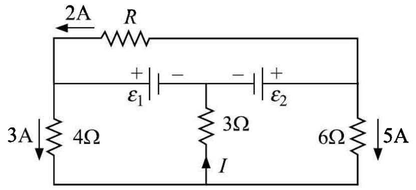


Fig. 22.41

(a) 8A, 9 Ω, 54 V, 36 V (b) 8A, 9 Ω, 36 V, 54 V

(c) 9A, 8 Ω, 36 V, 54 V (d) none of these

Solution (b) $2R + \varepsilon_1 - \varepsilon_2 = 0 \quad \text{...(1)}$

$4(3) + 3I = \varepsilon_1 \quad \text{...(2)}$

$5(6) + 3I = \varepsilon_2 \quad \text{...(3)}$

$2R + 4(3) = 5(6) \quad \text{...(4)}$

30. A current of 2A passes through a wire for 20 minutes. The number of electrons that crossed the cross-sections in this period is

(a) 1.5×10^{21} (b) 1.5×10^{20}

(c) 1.5×10^{22} (c) 1.5×10^{23}

Solution (c) $n = \dfrac{It}{e} = \dfrac{2 \times 20 \times 60}{1.6 \times 10^{-19}} = 15 \times 10^{21}$

31. To measure a small resistance ~$10^{-5}\,\Omega$, one should use

(a) wheat stone bridge (b) Post office box

(c) Wein's bridge (d) Carry foster bridge

Solution (d)

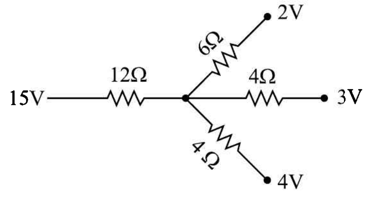
32. The free electron gas theory explains conduction in

(a) metals only (b) semiconductors only

(c) insulators only (d) All of these

Solution (a)

33. Find current through 12 Ω resistor in Fig. 22.42(a)

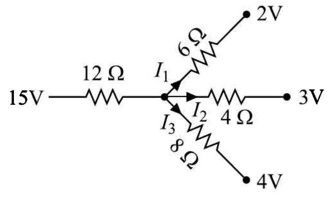


(a)

(a) 49/60A (b) 41/60A

(c) 21/40A (d) 23/40A

Solution (a) Let V_p be the potential at P then applying KCL at junction P.



(b)

Fig. 22.42

$I = I_1 + I_2 + I_3$

$$\frac{15 - V}{12} = \frac{V - 2}{6} + \frac{V - 3}{4} + \frac{V - 4}{8}$$

$15 - V = 2(V - 2) + 3(V - 3) + 1.5(V - 4)$

$7.5V = 39$ or $V = \frac{39}{7.5} = 5.2\,\text{V}$

and $I = \frac{15 - 5.2}{12} = \frac{4.9}{6}$

33. To terminate the network shown in Fig. 22.43, the resistance required is

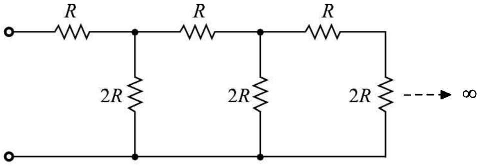


Fig. 22.43

(a) R (b) $2R$
(c) $3R$ (d) $R/2$

Solution (b) $2R$ (equal to equivalent resistance).

34. The ammeter in Fig. 22.44 will read

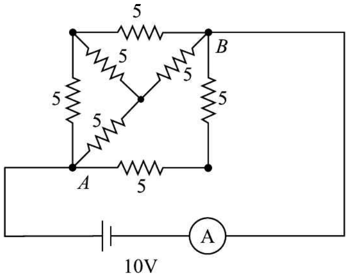


Fig. 22.44

(a) 3A (b) $\frac{10}{3}$ A
(c) 30A (d) $\frac{100}{3}$ A

Solution (a) $R_{AB} = \frac{10}{3}$ (use wheat stone bridge)

$$I = \frac{10}{10/3} = 3\text{A}.$$

35. Find X in the Fig. 22.45 so that galvanometer shows null deflection

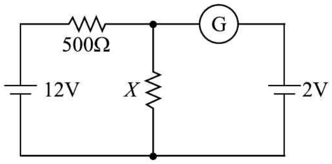


Fig. 22.45

(a) 100 Ω
(b) 400 Ω
(c) 200 Ω
(d) 250 Ω

Solution (a) Potential drop across X should be 2 V

$\therefore \quad 2 = \frac{12X}{X + 500}$

or $X = 100\,\Omega$.

Problems for Practice

1. Each of the resistance in the network shown in the figure below is equal to R. The resistance between the terminals A and B is

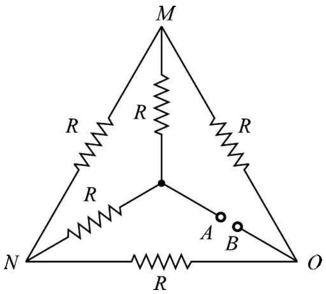


Fig. 22.46

(a) R (b) $5R$ (c) $3R$ (d) $5/3R$

2. Ampere-second stands for the unit of
(a) emf (b) energy
(c) charge (d) power

3. Kirchoff's second law is based on the law of conservation of
(a) momentum (b) charge
(c) energy (d) sum of mass and energy

4. A wire has a resistance 12 ohm. It is bent in the form of a circle. The effective resistance between the two points on any diameter of circle is
(a) 12 Ω (b) 6 Ω (c) 24 Ω (d) 3 Ω

5. The smallest resistance obtained by connecting 50 resistances of 1/4 ohm each is
(a) 200 Ω (b) 1/200 Ω
(c) 50/4 Ω (d) 4/50 Ω

6. A primary cell has emf 2 volt. When short circuited it gives a current of 4 ampere. Its internal resistance in ohms will be
(a) 8 (b) 5 (c) 2 (d) 0.5

7. Four cells of equal emf (E_m) and internal resistance (r) are connected in series. If one cell is connected wrongly, the equivalent emf and internal resistance will be
(a) $2E_m$ and $4r$ (b) $3E_m$ and $4r$
(c) $3E_m$ and $3r$ (d) $2E_m$ and $3r$

8. The current 'i' in the figure below is

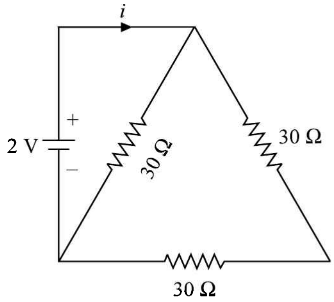


Fig. 22.47

(a) 1/5 A (b) 1/10A
(c) 1/15A (d) 1/45A

9. In the circuit shown in Fig. 22.48 the ammeter '*A*' reads 4 ampere and the voltmeter '*V*' reads 20 volt. The value of the resistance '*R*' is

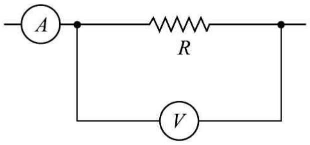


Fig. 22.48

(a) exactly 5 ohm (b) less than 5 ohm
(c) more than 5 ohm (d) none of these

10. A 50 volt battery is connected across a 10 ohm resistor. The current is 4.5 ampere. The internal resistance of battery will be

(a) 1.1 ohm (b) 0.5 ohm
(c) 5.0 ohm (d) zero

11. Two resistance R_1 and R_2 ($R_1 < R_2$) are connected in parallel. Which of the following is true for equivalent resistance R?

(a) $R < R_1$
(b) $R_1 < R < R_2$
(c) $R > (R_1 + R_2)$
(d) $R_2 < R < (R_1 + R_2)$

12. Three 2Ω resistors are arranged in a triangle. What is the resistance between any two corners?

(a) 4Ω (b) 3Ω
(c) 4/3Ω (d) 3/4Ω

13. Each cell has emf ε and internal resistance *r* in the Fig. 22.49. Find the current through resistance *R*.

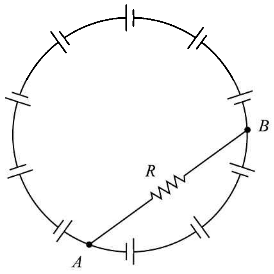


Fig. 22.49

(a) $\frac{4\varepsilon}{r}$ (b) $\frac{3\varepsilon}{r}$

(c) $\frac{\varepsilon}{r}$ (d) zero

14. In the Fig. 22.50 the equivalent resistance between point *B* and *C* is

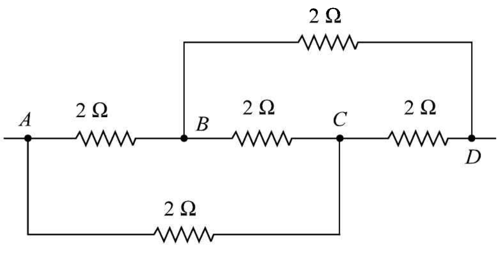


Fig. 22.50

(a) 1Ω (b) 1.25Ω
(c) 2Ω (d) 4Ω
(e) 8Ω

15. The equivalent resistance between point *A* and *B* in question number 14 above is

(a) 8Ω (b) 4Ω
(c) 2Ω (d) 1Ω
(e) 1.25Ω

16. In Fig. 22.51 the current in 2Ω resistor is

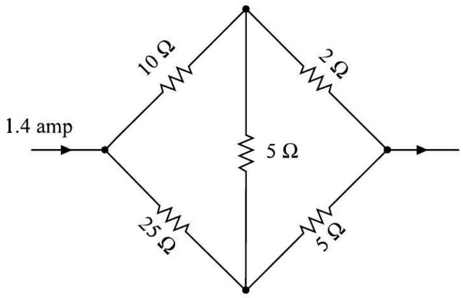


Fig. 22.51

(a) 0.4A (b) 1.0A
(d) 1.2A (d) 1.4A

17. Voltmeters V_1 and V_2 are connected in series across a DC line. V_1 reads 80 volt and has a per volt resistance of 200 ohm. V_2 has a total resistance of 32 kilo-ohm. The line voltage is

(a) 120 V (b) 160 V
(c) 220 V (d) 240 V

18. Two cells each of emf E_m and internal resistance '*r*' are connected in parallel across a resistor *R*. The power delivered to the resistor is maximum if

(a) $R = r/2$
(b) $R = r$
(c) $R = 2r$
(d) $R = 0$

19. A part of an electronic circuit is shown in Fig. 22.52. The current *i* in the branch *XY* is

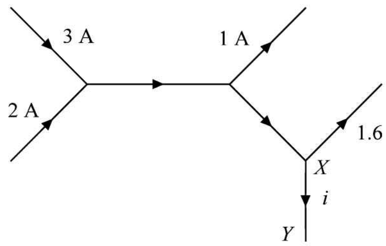


Fig. 22.52

(a) 3.4A (b) 2.4A (c) 1.4A (d) 1.0A

20. The heating wires of equal length are first connected in series and then in parallel. The ratio of heat produced in two cases is

(a) 4 : 1 (b) 1 : 4
(c) 2 : 1 (d) 1 : 2

21. In Fig. 22.53 each resistance is *R*. Find R_{AB} if *CD* is short circuited.

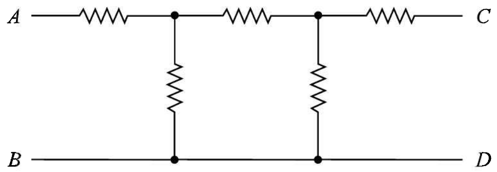


Fig. 22.53

(a) $3R/2$ (b) $5R/8$
(c) $8R/5$ (d) $2R$

22. In the circuit as shown in Fig. 22.54, potential at O, that is, V_0 is equal to

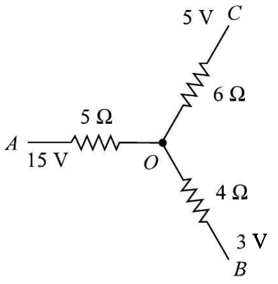


Fig. 22.54

(a) $\frac{375}{37}$ V (b) $\frac{275}{37}$ V

(c) $\frac{175}{27}$ V (d) $\frac{275}{27}$ V

23. A piece of copper wire is cut into ten equal parts. These parts are connected in parallel. The effective resistance of parallel combination will be equal to original resistance of the original wire multiplied by

(a) 0.01 (b) 0.1

(c) 1.0 (d) 10

24. A man wants to test his ammeter whose range is 0-1 ampere. He connects it to a 12 V storage battery and a carbon resistance of 48 ohm 0.5 watt. Which of the following is most likely to happen?

(a) the reading remains at 0.25A

(b) the reading remains at 0.25A but after some time resistor melts.

(c) the resistor burns immediately

(d) the reading stands at 0.5A.

25. A current of 2.0 ampere passes through a cell of emf 1.5 volt having internal resistance of 0.15 ohm. The potential difference measured in volt, across both the ends of the cell will be

(a) 1.00 (b) 1.20

(c) 1.35 (d) 1.50

26. A current of 2 ampere flows in a system of conductors shown in the figure below. The potential difference $(V_A - V_B)$ will be

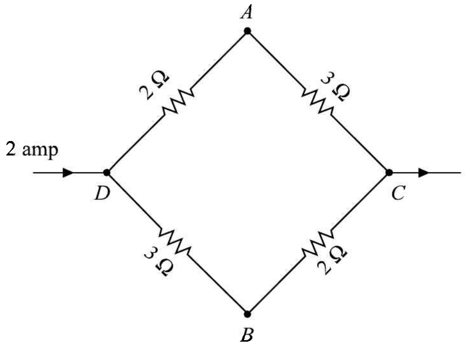


Fig. 22.55

(a) +1V (b) −1V

(c) +2V (d) −2V

27. The current in 10Ω resistance is

(a) 1/16 A along *AB* (b) 1/32 A along *BA*

(c) 1/16 A along *AB* (d) 1/16 A along *BA*

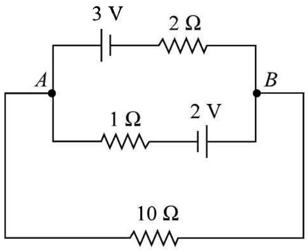


Fig. 22.56

28. A wired resistance is heated to temp-erature T_1 and T_2 and their *V-I* curves are plotted. Then T_1/T_2 is equal to

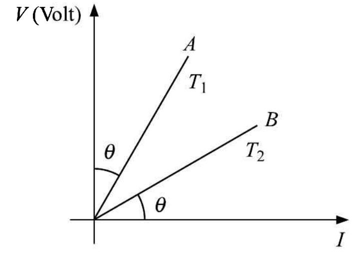


Fig. 22.57

(a) tan θ (b) $\cot^2\theta$ (c) $\tan^2\theta$ (d) cot θ

29. Three cells of emf 1.5 volts and internal resistance 1 ohm are connected in parallel. The combination of cells will have the emf

(a) 0.5 volt (b) 1.5 volt

(c) 3.0 volt (d) 4.5 volt

30. A 10 m long wire of resistance 20 Ω is connected in series with a battery of emf 3 V (negligible internal resistance) and resistance of 10 Ω. The potential gradient along the wire in volt per meter is

(a) 0.02 (b) 0.1 (c) 0.2 (d) 1.2

31. A voltmeter of resistance 1 kΩ/V reads 20V in the circuit shown below. The resistance X is

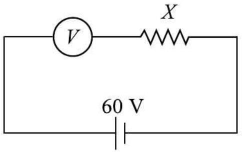


Fig. 22.58

(a) 10 kΩ (b) 20 kΩ

(c) 30 kΩ (d) 40 kΩ

32. The potential difference between the the points A and B in the Fig. 22.59 is

(a) 2/3 volt (b) 4/3 volt

(c) 2 volt (d) 8/9 volt

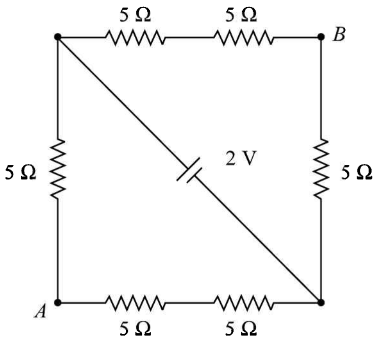


Fig. 22.59

33. In the measurement of resistance by a meter bridge, the known and unknown resistance are interchanged to eliminate
(a) index errors
(b) random errors
(c) end error
(d) error due to thermo-electric effect

34. The potential drop across 3 μF capacitor is

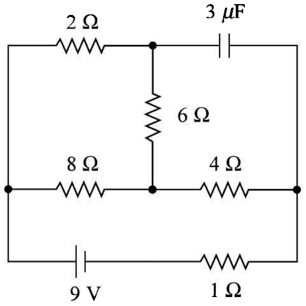


Fig. 22.60

(a) 4.4V (b) 7.0V (c) 6.4V (d) 5.6V

35. When a battery of emf E_m and internal resistance r is connected across some external resistance R, the power consumed in R is maximum when ratio r/R is
(a) 2 : 1 (b) 1 : 2 (c) 1 : 1/2 (d) 1 : 1

36. When cells are connected in parallel
(a) net emf becomes higher
(b) net emf becomes lower
(c) current capacity becomes higher
(d) current capacity becomes lower

37. What should be the value of R_3 so that thermal power generated in it is practicality independent of small variations in the resistance

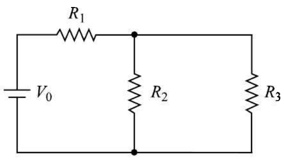


Fig. 22.61

(a) R_2 (b) $\frac{R_2}{R_1+R_2}$
(c) $\frac{R_1R_2}{R_1+R_2}$ (d) $\frac{R_1R_2}{R_1-R_2}$

38. The reading of ammeter in the circuit shown belowis

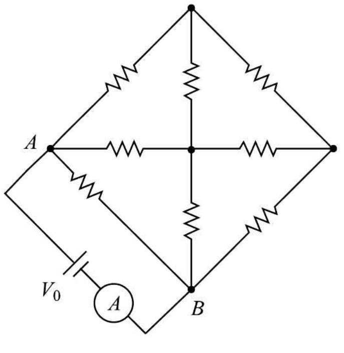


Fig. 22.62

(a) $\frac{15V_0}{8R}$ (b) $\frac{5V_0}{8R}$
(c) $\frac{10V_0}{8R}$ (d) none of these

39. Current I_3 in the given circuit [Fig. 22.63] is

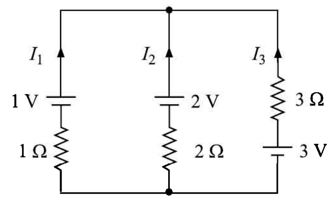


Fig. 22.63

(a) $\frac{5}{11}$A (b) $\frac{7}{11}$A (c) $\frac{2}{11}$A (d) none of these

40. A 90 W – 30 V bulb is to be lightened by a supply of 120 volt. For this a wire is to be connected in series with it. The resistance of the wire should be
(a) 10 Ω (b) 20 Ω (c) 30 Ω (d) 40 Ω

41. In order to obtain maximum current from a combination of cells, we must join the cells in
(a) series combination
(b) parallel combination
(c) mixed combination
(d) in any of the above combinations depending on the relative values of external and internal resistance

42. Two unequal resistances are connected parallel across a battery. Which of the following statement is true?
(a) same current will flow through both resistances
(b) current through smaller ressitance is higher
(c) current through larger resistance is higher
(d) current can be higher in any resistance depending on emf of the cell

43. When current is 4A in a conductor the drift velocity is v_d. When the current is 3A the drift velocity will be
(a) $3v_d/4$ (b) v_d (c) $4v_d/3$ (d) none of these

44. In the circuit shown below, the reading of ammeter when the key is opened

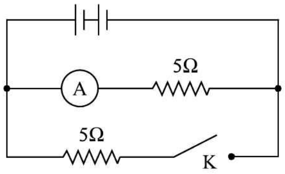


Fig. 22.64

(a) becomes 2 times (b) becomes 1/2
(c) becomes 1/4 (d) remains unchanged

45. A student wants to measure temperature using platinum simple resistance thermometer. The thermometer should be a part of
(a) wheatstone bridge
(b) wheatstone bridge with one arm having resistance box
(c) wheatstone bridge with a potential divider arrangement
(d) any of the (a), (b) or (c)

46. The null point in a potentiometer with a cell of emf E_m is obtained at a distance l on the wire. Then
(a) $E_m \propto 1$ (b) $E_m \propto 1/l$
(c) $E_m \propto l^2$ (d) $E_m \propto 1/l^2$

47. An ideal current source shall have resistance
(a) zero (b) finite but not zero
(c) infinite (d) depends upon the requirement

48. A potentiometer has a high sensitivity if
(a) resistivity of wire is higher
(b) the wire is of larger length
(c) battery in auxilliary circuit has higher emf
(c) none of the above

49. A 1m potentionetes wire having resistance $10\,\Omega$ is used to find the null point. At the null point condition l is equal to

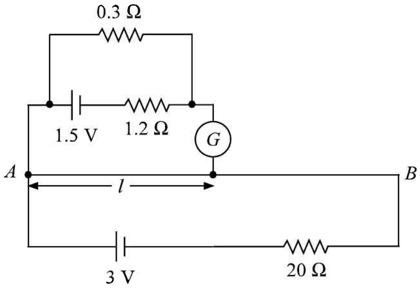


Fig. 22.65

(a) 37.5 cm (b) 30 cm
(c) 24 cm (d) 45 cm

50. In the given circuit [Fig. 22.66] the potential drop across XY is

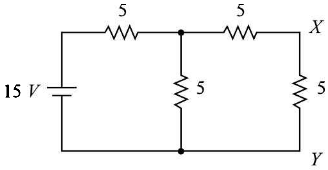


Fig. 22.66

(a) 10/3V (b) 3 V (c) 2 V (d) 5 V

51. A conical wire of length l has end diameters a and b as shown below. The specific resistance of the material is ρ, the resistance of the wire is

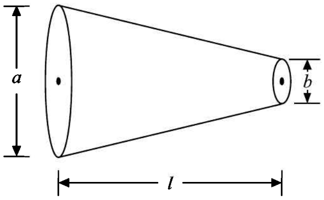


Fig. 22.67

(a) $\dfrac{4\rho l}{\pi ab}$ (b) $\dfrac{\rho l}{\pi(a^2+b^2)}$

(c) $\dfrac{16\rho l}{\pi(a+b)^2}$ (d) $\dfrac{4\rho l}{\pi(a-b)^2}$

52. An electron in the potentiometer wire experiences a force of 3.2×10^{-19}N. The length of the potentiometer wire is 4 m. The emf of the battery across the wire is

(a) 1.6 V (b) 3.2 V (c) 4.8 V (d) 8 V

53. In the circuit shown below, current drawn from the battery is 4 ampere. If $10\,\Omega$ resistance is replaced by $20\,\Omega$ resistance, the current drawn from the battery will be

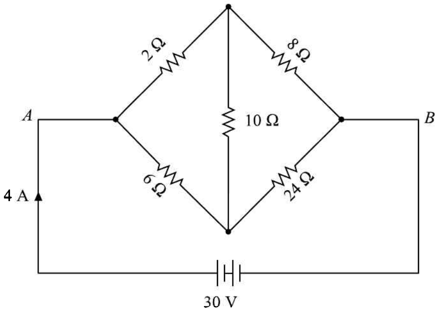


Fig. 22.68

(a) 1A (b) 2A (c) 4A (d) 8A

54. What is the current through the resistor R in the circuit shown below? The emf of each cell is E_m and internal resistance is r.

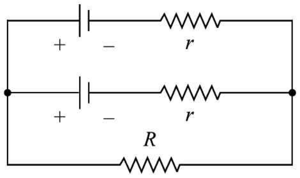


Fig. 22.69

(a) $\dfrac{E_m}{2R+r}$ (b) $\dfrac{E_m}{2r+R}$

(c) $\dfrac{2E_m}{R+2r}$ (d) $\dfrac{2E_m}{2R+r}$

55. When wheatstone bridge is balanced

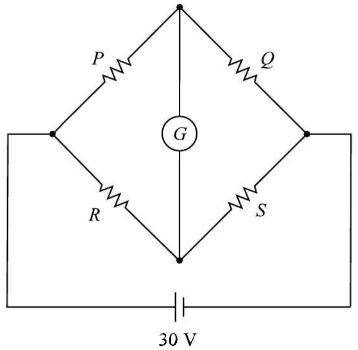


Fig. 22.70

(a) current in resistance P = current in resistance R
(b) current in resistance P = current in resistance Q
(c) current in resistance R = current in resistance S
(d) current through galvanometer = zero

56. Two identical cells when connected in series send a current of 10 ampere through an external resistance of $5\,\Omega$, but when connected in parallel, the current through same resistance is 8 ampere. The internal resistance of each cell is

(a) $1.5\,\Omega$ (b) $2.5\,\Omega$
(c) $10\,\Omega$ (d) zero

57. Post office box is a device for measuring the resistance, based on the principle of

(a) ohm meter (b) galvanometer
(c) meter bridge (d) potentiometer

58. A skeleton cube has resistance R in each branch. The resistance across one side is

(a) $4/3R$ (b) $7/12R$
(c) $5/6R$ (d) $5/6R$

59. Find the current through the resistor R in the circuit shown below

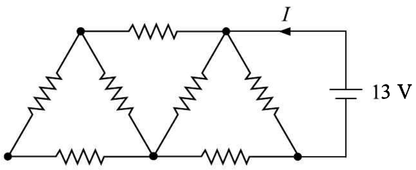


Fig. 22.71

(a) $\dfrac{E}{2R+3r}$ (b) $\dfrac{3E}{2R+3r}$

(c) $\dfrac{E}{2R+r}$ (d) none of these

60. When no resistance is connected across a cell, the potential difference between its terminals is E. What will be the potential drop across the terminals if an external resistance $R = r$ (r = internal resistance of cell) is connected across it?
(a) $E/4$ (b) $E/2$ (c) $2E$ (d) $4E$

61. A cell having emf of 3 volt and negligible internal resistance is connected across a series combination of three resistances of 3, 4 and 5 Ω. The potential difference across 4 Ω resistance is
(a) 0.5 V (b) 1.0 V (c) 1.5 V (d) 2.0 V

62. Five cells, each having emf 0.2 volt and internal resistance 1 Ω are connected in series across a resistance of 10 Ohm. The current flowing through external resistance is
(a) 1/5A (b) 1/3A (c) 1/10A (d) 1/15A

63. Each resistance of 7 Ω is connected in the circuit as shown below.

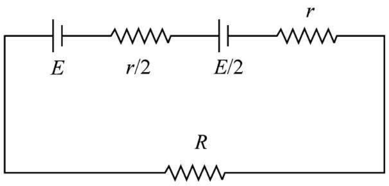


Fig. 22.72

(a) 1/3A (b) 3A (c) 4/3A (d) 8/3

64. A cell of emf E_m is connected across a resistance r. The potential difference between the terminals of the cell is found to be V. The internal resistance of the cell must be:
(a) $[2(E_m - V)V]/r$ (b) $[2(E_m - V)r]/E$
(c) $(E_m - V)r$ (d) $\dfrac{(E_m - V)\,r}{V}$

65. The equivalent resistance of a parallel combination of resistances is
(a) higher than the highest of component resistors
(b) less than the lowest of component resistors
(c) in between the lowest and highest of component resistors
(d) equal to sum of the component resistors

66. The current in 3 Ω and 5 Ω resistance are respectively
(a) $\frac{1}{5}$A, $\frac{1}{5}$A (b) 0, $\frac{1}{4}$A
(c) 0, $\frac{1}{5}$A (d) none of these

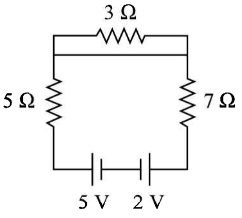


Fig. 22.73

67. A piece of wire is cut into four equal parts and the pieces are bundled together side by side to form a thicker wire. If the original resistance of the wire was R, the resistance of the bundle will be
(a) $R/32$ (b) $R/16$ (c) $R/8$ (d) $R/4$

68. Thermistor has
(a) positive resistance coefficient
(b) zero resistance coefficient
(c) negative resistance coefficient
(d) positive at low temperature and negative at high temperature

69. If σ_1, σ_2 and σ_3 are the conductances of three conductors, then their equivalent conductance, when they are joined in series, will be
(a) $\sigma_1 + \sigma_2 + \sigma_3$
(b) $(\sigma_1\sigma_2\sigma_3)/(\sigma_1 + \sigma_2 + \sigma_3)$
(c) $1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3$
(d) none of these

70. In a potentiometer, the length and resistance of the wire are 10 meter and 20 Ω respectively. The wire is connected in series with a resistance of 5 Ω and a battery of emf 5 volt and negligible internal resistance. The potential gradient along the wire in volt per meter is
(a) 0.1 (b) 0.2 (c) 0.3 (d) 0.4

71. An electrician has only two resistances. By joining them in series or in parallel combinations, he is able to obtain resistances of 3, 4, 12 and 16 Ω. The two resistances are
(a) 4 Ω and 12 Ω (b) 4 Ω and 16 Ω
(c) 7 Ω and 9 Ω (d) 6 Ω and 10 Ω

72. The potential gradient along the length of a uniform wire is 5 Vm^{-1}. There are two points on the wire at a distance of 20 cm and 40 cm from initial end of the wire. The potential difference between these points is
(a) 4 V (b) 2 V (c) 1 V (d) 3 V

ANSWERS

1. (a)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (a)
11. (a)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (e)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (b)	**20.** (b)
21. (c)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (b)	**30.** (c)
31. (d)	**32.** (a)	**33.** (c)	**34.** (b)	**35.** (d)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (a)	**40.** (c)
41. (d)	**42.** (b)	**43.** (a)	**44.** (d)	**45.** (c)	**46.** (a)	**47.** (c)	**48.** (b)	**49.** (b)	**50.** (b)
51. (a)	**52.** (d)	**53.** (c)	**54.** (d)	**55.** (b,c,d)	**56.** (b)	**57.** (c)	**58.** (b)	**59.** (b)	**60.** (b)
61. (b)	**62.** (d)	**63.** (b)	**64.** (d)	**65.** (b)	**66.** (b)	**67.** (b)	**68.** (c)	**69.** (d)	**70.** (d)
71. (a)	**72.** (c)								

Thermal and Chemical Effects of Current

23

BRIEF REVIEW OF THE CONCEPTS

THERMAL EFFECTS

Joule's law of heating Heat produced in a conductor when current I passes through it for a time t is $H = I^2Rt$ where R is resistance of the conductor. Heat produced in the conductor is independent of direction of the current.

Seebeck effect When two metallic strips made of two different metals and joined at the ends to form a loop as shown in Fig. 23.1 is called Thermocouple. If two junctions of a thermocouple are kept at different temperature, an electric current is induced in the loop. This effect is called seebeck effect and emf developed is known as seebeck emf or thermo emf.

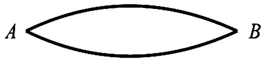


Fig. 23.1

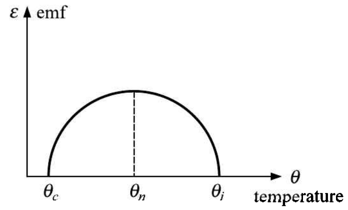


Fig. 23.2

The magnitude and the direction of the emf depends upon the metals and the temperatures of the hot and cold junctions.

Thermo emf induced is given by

$$\varepsilon = \alpha\theta + \frac{1}{2}\beta\theta^2$$

where θ is temperature difference between hot and cold junction (if $\theta_c = 0$ then θ is temperature of hot junction). Note the curve is parabolic.

θ_N is neutral temperature at which the emf is maximum. Moreover, at $\theta = \theta_N$

$$\frac{d\varepsilon}{d\theta} = 0 \quad \text{or} \quad \theta_N = -\frac{\alpha}{\beta}$$

Note that θ_N depends upon the nature of materials which form junction. θ_i is the inversion temperature at which the emf changes sign.

From Fig. 23.2 $\theta_i - \theta_N = \theta_N - \theta_C$

$$\text{or} \quad \theta_N = \frac{\theta_i - \theta_C}{2}$$

If $\theta_C = 0$ then $\theta_i = 2\theta_N$

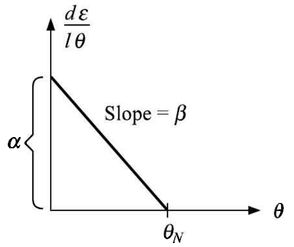


Fig. 23.3

There is a series of metals called thermoelectric series. The first and the last element of the series if used to form a thermocouple give maximum emf. The series is

Sb, Fe, Zn, Ag, Au, Mo, Cr, Sn, Pb, Hg, Mn, Cu, Co, Ni, Bi

If hot and cold junctions are interchanged then the direction of emf changes.

Peltier effect It is converse of seebeck effect. If current is passed through a thermocouple, one of the junction bcomes hot and the other gets cold. The heat liberated or absorbed at one of the junctions is proportional to charge transferred.

$$\text{Peltier emf } \pi = \frac{\Delta H}{\Delta Q}$$

Peltier coefficient is the amount of heat liberated or absorbed per second when 1A of current is passed through the thermocouple. The hot and cold junction will interchange if the direction of current is reversed. $\pi = TS = \frac{Td\varepsilon}{d\theta}$ where S is seebeck coefficient, that is, $S = \frac{d\varepsilon}{d\theta}$

Thomson effect Emf is developed between two parts of a single conductor if they are at different temperatures. This effect is called Thomson effect.

If dV is the potential difference between two points of a conductor then Thomson coefficient

$$\sigma = \frac{dV}{d\theta} = -T\frac{d^2\varepsilon}{d\theta^2} = -T\frac{dS}{d\theta}$$

$$\text{Seeback's coefficient } S = \frac{d\varepsilon}{d\theta}$$

If one part of conductor is at different potential than the other or the current is flowing, a temperature difference $d\theta$ will be developed across the two ends.

Applications of thermal effects

(a) Measurement of temperature (thermocouple thermometer and platinum resistance thermometer)
(b) detection of heat radiation
(c) refrigeration
(d) power generation (thermopile)

Power $P = I^2R$, use this formula when devices are in series
$P = V^2/R$, use this formula when devices are in parallel
$P = V{\cdot}I$, when potential drop across the device and current through it are known.

The SI unit of power is watt. (W)

Kilo watt-hour (kWh) or Board of trade unit or simply called unit = 3.6×10^6 J.

CHEMICAL EFFECTS

Electrolyte An ionic compound in aqueous or molten state is called electrolyte.

Voltameter the vessel in which electrolysis is carried out is termed as voltameter. It is also called coulombmeter.

Electrolysis On passing current through an electrolyte, chemical changes occur in the electrolyte and substances are liberated at the electrodes. This process is called electrolysis.

Faraday's laws

First law The mass of a substance liberated on an electrode is proportional to the current passed

$m \propto Q$ or $m = ZIt$ where $Z \rightarrow$ ece or electrochemical equivalent

$$Z = \frac{\text{chemical equivalent in gm or gram equivalent}}{\text{1 faraday}}$$

Faraday (F) It is the amount of charge on 1 mole of electrons.

1 F = 96485C. Practically we take 1 F = 96500 C.

Second law If same quantity of electricity (charge) is passed through different electrolytes, the masses of the substances deposited at the respective cathodes are directly proportional to the chemical equivalent

$$m \propto E \quad \text{or} \quad m_1/m_2 = E_1/E_2$$

One faraday of charge liberates 1 gram equivalent of any substance in electrolysis. Electrolytic cell or voltameter or coulombmeter measures the current more accurately.

$$I = m/(2t)$$

Applications of electrolysis

(a) purification of metals
(b) extraction of metals
(c) local aneasthesia
(d) electroplating
(e) anodising
(f) fabrication of capacitors
(g) thin film deposition
(h) rust proof coating
(i) measuring current
(j) callibrating ammeters

Back emf is the potential difference opposite to the external emf set up in the electrolyte due to the accumulation of ions around the insoluble electrodes. The back emf depends upon nature of electrodes and concentration of ions. On heating the rate of electrolysis increases as the rate of decomposition of ions increases.

SHORT-CUTS AND POINTS TO NOTE

(i) Joule's heat energy $H = I^2Rt$.

(ii) Power $P = V^2/R$. To find resistance of an electrical device use power and voltage of rating values. For example, a 100 W/220 V incandascent lamp will have resistance $R = \frac{220^2}{100} = 484\,\Omega$.

(iii) Maximum power is delivered by a cell/battery if internal resistance is equal to external resistance and maximum power $= \frac{E_m^2}{4r}$ where E_m is emf and r is internal resistance.

(iv) When a bulb glows its temperature is 2800 K - 3000 K.

(v) If electrical devices are connected in series then a device with lowest power rating will consume maximum power.

(vi) If electrical devices are connected in parallel then a device with maximum power rating will consume maximum power.

(vii) In a fuse safe current $I \propto r^{3/2}$. It is independent of length and r is radius of the wire.

(viii) Use $P = V^2/R$ in parallel. Thus, a device/wire of low resistance will consume more power.

(ix) Use $P = I^2R$ in series. Thus, a device/wire with more resistance will consume higher power.

(x) Heat produced in a circuit is minimum if it is a closed circuit with high circuit resistance.

(xi) The effective power in series combination

$$\frac{1}{P_{\text{series}}} = \frac{1}{P_1} + \frac{1}{P_2} + \ldots$$

(xii) The effective power in parallel combination

$$P_{\text{Parallel}} = P_1 + P_2 + \ldots$$

(xiii) Efficiency of a cell $\eta = \frac{R}{r+R}$ where R is external resistance and r is internal resistance.

(xiv) Maximum current through a battery or cell $I_{\text{max}} = I_{\text{short circuit}} = \frac{E_m}{r}$, where E_m is emf of the battery/cell and r its internal resistance.

(xv) Power from a battery to a load R $\quad P = \frac{E_m^2 R}{(R+r)^2}$.

(xvi) The electric power supplied by a cell/battery= Emf × current $= E_m I = \frac{V_m^2}{R+r}$.

(xvii) Standard cell is the one whose emf does not vary with temperature.

clarke cell $V_0 = 1.4328\,\text{V} - 1.19 \times 10^{-6}\,\text{V}$ at 15°C

weston cell $V_0 = 1.0184\,\text{V} - 4.06 \times 10^{-3}\,\text{V}$ at 20°C

are considered as standard cells. Out of the above two clarke cell has lesser temperature coefficient and its emf is constant to a better extent.

(xviii) Thomson coefficient of lead is zero.

(xix) Relation between seebeck coefficient S, Peltier coefficient π and Thomson coefficient σ is

$$S = \frac{d\varepsilon}{d\theta} = \alpha + \beta\theta$$

$$S = \frac{d\varepsilon}{d\theta} = \frac{\pi}{\theta}$$

$$\sigma = -\theta\frac{dS}{d\theta} = -\theta\frac{d^2\varepsilon}{d\theta^2}$$

Seebeck effect is the resultant of Peltier and Thomson effect.

(xx) In Peltier and Thomson's effect heat evolved or absorbed $\propto I$ while in Joules law heat generated $\propto I^2$.

(xxi) If the electrodes are insoluble in electrolyte they do not follow ohm's law rather show back emf.

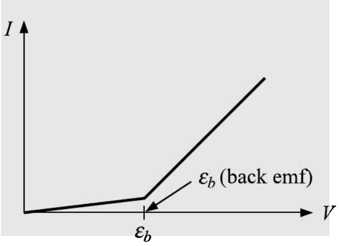


Fig. 23.4

(xxii) The temperature difference between the two junction of various thermocouples is equal (say θ)

Then $\varepsilon_{AC} = \varepsilon_{AB} + \varepsilon_{BC}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$.

CAUTION

(i) Assuming that seebeck effect, Peltier effect and Thomson effect do not depend upon direction like Joule's law.

- Seebeck, Peltier and Thomson effects are direction dependent. Reversing the condition (say hot and cold junction) in seebeck effect, reverses the direction of current. In Peltier effect, if the direction of current is reversed hot and cold junction are interchanged.

(ii) Assuming that like Joule's law heat produced in Peltier effect is also proportional to I^2.

- In Peltier or Thomson effect heat produced is directly proportional to current.

(iii) Not remembering which formula should be used in the problems related to power

- $P = \frac{V^2}{R}$ is used in parallel

  $P = I^2R$ is used in series

  $P = VI$ when voltage across the device and current through it are known.

(iv) Not remembering that whether the voltage applied or rating voltage of the device is, to be used to find resistance of the device.

- To find resistance of the device use rating voltage

$$R = \frac{V^2_{\text{rating}}}{P_{\text{rating}}}.$$

(v) Assuming that in series a device consumes same power as in parallel.

- In series a device of higher power rating consumes less power.

(vi) Not remembering the theorem of maximum power transfer.

- Maximum power is transferred when external resistance in the circuit is equal to internal resistance of the voltage source.

(vii) Confusion between equivalent weight and atomic weight.

- When 1 Faraday charge is passed (applied) then mass equal to equivalent weight is deposited on the cathode. If valency is 1 then equivalent weight and atomic weights are equal.

(viii) Considering that emf is always greater than potential drop.

- Potential drop across a battery, when it is charging, is more than its emf.

(ix) Lack of clarity on current division in a complex circuit.

- Apply current division wherever needed (when devices are in parallel).

Solved Problems

1. A 5 ampere fuse wire can withstand a maximum power of 1 watt in the circuit. The resistance of the fuse wire is

(a) 0.04 Ω (b) 0.2 Ω

(c) 5 Ω (d) 0.4 Ω

Solution (a) $I^2R = P$ or $R = 1/25$

2. Two sources of equal emf are connected to external resistance R. The internal resistances of the two sources are R_1 and R_2 ($R_2 > R_1$). If the potential diference across the source having internal resistance R_2 is zero then

(a) $R_1 = R_2 - R_1$ (b) $R = \frac{R_2(R_1 + R_2)}{R_2 - R_1}$

(c) $R = \frac{R_1R_2}{R_2 - R_1}$ (d) $R = \frac{R_1R_2}{R_2 + R_1}$

Solution (a) $I = \frac{2\varepsilon}{R_1 + R_2 + R}$. If $IR_2 = \varepsilon$ then potential drop across the source of resistance R_2 will be zero.

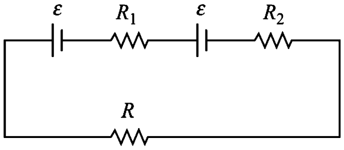


Fig. 23.5

$$\frac{2\varepsilon}{R_1 + R_2 + R}(R_2) = \varepsilon$$

or $2R_2 = R_1 + R_2 + R$ or $R = R_2 - R_1$

3. A heater coil is cut into two equal parts and only one part is now used in the heater. The heat generated will now be

(a) four times (b) doubled

(c) halved (d) one fourth

Solution (b) $\because H = \frac{V^2}{R}t$.

AS R is halved H is doubled.

4. The resistance of a hot tungsten filament is about 10 times the cold resistance. What will be the resistance of 100 W/200 V lamp when not in use?

(a) 20 Ω (b) 40 Ω

(c) 200 Ω (d) 400 Ω

Solution (b) $R = \frac{(200)^2}{100} = 400\,\Omega$ when operating.

$R(\text{cold}) = 400/10 = 40\,\Omega$

5. Two voltameters, one of copper another of silver are joined in parallel when a total charge q flows equal amount of metals are deposited. If ece of Cu and Ag are z_1 and z_2 respectively then charge flown through the silver voltameter is

(a) $\frac{q}{1 + z_2 / z_1}$ (b) $\frac{q}{1 + z_1 / z_2}$

(c) qz_2/z_1 (d) qz_1/z_2

Solution (a) $m = zq$

$\therefore \quad z_1q_1 = z_1q_2$ or $q_2 = \frac{z_1q_1}{z_2}$ but $q = q_1 + q_2$

$\therefore q_2 = z_1 / z_2 (q - q_2)$ or $q_2(1 + z_1 / z_2) = z_1/z_2q$

or $q_2 = \frac{q z_1}{z_1 + z_2}$ or $\frac{q}{1 + z_2 / z_1}$

6. An 1800 W toaster, a 1.3 kW frying pan are plugged into the same 20 A/120 V line then

(a) fuse will not blow (b) fuse will blow

(c) supply will spark (d) fuse is just saved

Solution (b) $I_{\text{toaster}} = \frac{1800}{120} = 15A$,

$I_{\text{frying pan}} = \frac{1300}{120} = 10.83A$

$I_{\text{net}} = 25.83A > 20A$

$\therefore$ Fuse will blow.

7. In the given circuit energy stored in capacitor is

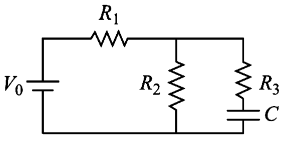


Fig. 23.6

(a) $\frac{CV_0^2}{2}$ (b) $\frac{C}{2}\left(\frac{V_0R_2}{R_1 + R_2}\right)^2$

(c) $\left(\frac{C(V_0R_3)}{2R_1 + \frac{R_2R_3}{R_2 + R_3}}\right)^z$ (d) $\frac{C}{2}\left(\frac{V_0R_1}{R_1 + R_2}\right)^2$

Solution (b) The potential drop across capacitor is same as that across R_2. Hence $E = \frac{C}{2}\left(\frac{V_0R_2}{R_1 + R_2}\right)^2$.

8. Two 32 W/100 V bulbs are connected (a) in series (b) in parallel to a 100 V supply the power consumed by each in two cases is

(a) 16 W, 64 W (b) 8 W, 32 W

(c) 32 W, 32 W (d) 16 W, 32 W

Solution (b) $\frac{1}{P_{\text{series}}} = \frac{1}{32} + \frac{1}{32}$ or $P_{\text{series}} = 16$ watt by two bulbs. As bulbs are identical they consume 8 W each. In parallel they consume 32 W each.

9. The power consumed in 10 Ω resistor in the given circuit is

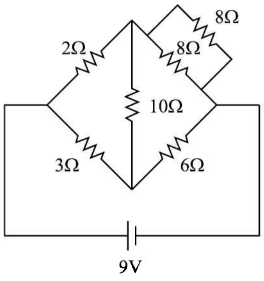


Fig. 23.7

(a) 10 W (b) 8 W

(c) >10 W (d) zero

Solution (d) As the bridge is balanced, no current will pass through 10 Ω resistance.

10. The wiring of a house has resistance 6 Ω. A 100Ω/220 Ω bulb is glowing in the bathroom. A Geyser of 1000W/220Ω is switched on. The drop in potential across the bulb is

(a) nin (b) 24 V

(c) 32 V (d) d12 V

Solution $R_{\text{bulb}} = \frac{220^2}{100} = 484\,\Omega$

$R_{\text{geyser}} = 48.4\,\Omega$

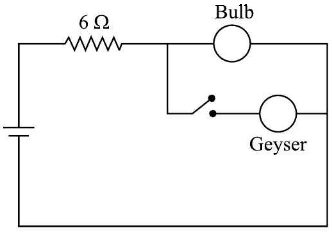


Fig. 23.8

(i) When only bulb is ON

$V_{\text{bulb}} = \frac{220 \times 484}{490} = 217.4\,\text{V}$

(ii) When geyser is also switched ON

$R_{\text{net}} = \frac{484 \times 48.4}{484 + 48.4} = 44\,\Omega$;

$V_{\text{bulb}} = \frac{220 \times 44}{50} = 193.6\,\text{V}$

$\Delta V = (217.4 - 193.6)V \simeq 24\,\text{V}$

11. A capacitor is charged to V_0 volts, a resistor R is connected across it. The heat produced in the resistor is

(a) CV_0^2 (b) $2CV_0^2$

(c) $CV_0^2/2$ (d) $4CV_0^2$

(e) $CV_0^2/4$

Solution (c) The energy stored in the capacitor will be converted to heat.

12. A silver, a copper and an iron wire of identical dimensions are connected in series with a voltage source. P_S, P_C and P_I are power consumed in three respectively then

(a) $P_S > P_C > P_I$ (b) $P_C > P_S > P_I$
(c) $P_I > P_S > P_C$ (d) $P_I > P_C > P_S$

Solution (d) Larger the resistance, more is the power consumed in series.

13. When the switch S is made ON, the bulb which glows brightest is

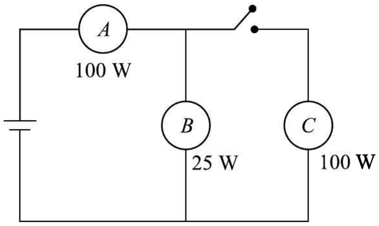


Fig. 23.9

(a) A (b) B
(c) C (d) A and C

Solution (a) Note the current will be divided amongst B and C. A gets maximum current.

14. A 10A fuse wire is used in a mains line in a house. In another hosue a 15A fuse wire is used. The ratio of their radii is

(a) $\sqrt{3/2}$ (b) $\sqrt{2/3}$
(c) $(2/3)^{3/2}$ (d) $(2/3)^{2/3}$
(e) none of these

Solution (d) $I_1/I_2 = (r_1/r_2)^{3/2}$

or $r_1/r_2 = (10/15)^{3/2} = (2/3)^{3/2}$.

15. A 3 Ω resistor as shown in fig. is dipped into a calorimeter containing H_2O. The thermal capacity of H_2O + calorimeter is 2000 J/K. If the circuit is active for 15 minutes find the rise in temp. of H_2O is

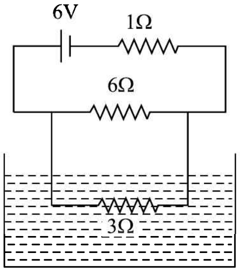


Fig. 23.10

(a) 2.4°C (b) 2.9°C
(c) 3.4°C (d) 1.9°C

Solution (a) $I = \dfrac{6}{R_{11}+1} = \dfrac{6}{2+1} = 2\text{A}$

$$R_{11} = \frac{6\times 3}{6+3} = 2\,\Omega$$

Current through 3 Ω resistor $I' = (I \times 6)/9 = 4/3\text{A}$

$$(\text{mC})\Delta T = I'^2 Rt$$

$$2000 \times \Delta T = \left(\frac{4}{3}\right)^2 \times 3 \times 15 \times 60$$

$$\Delta T = 2.4°\text{C}.$$

16. The potential difference across the terminals of a battery of emf 12 V and internal resistance 2 Ω is 10 V when it is connected to a Ag voltmeter. Find the Ag deposited in half an hour.

(a) 0.2 g (b) 0.1 g
(c) 0.02 g (d) 0.01 g

Solution (c) $I = (12 - 10)/2 = 1\text{A}$

$$m = ZIt = \frac{1.08}{96500} \times 1 \times 30 \times 60 = 0.02\,\text{g}$$

17. Find the charge required to liberate one atom of a divalent material in an electrolyte.

(a) 1.6×10^{-19}
(b) 3.2×10^{-19}C
(c) 4.8×10^{-19}C
(d) none of these

Solution (b) To neutralize divalent atom 3.2×10^{-19}C charge is required.

18. A plate of area 10 cm² is to be electroplated with Cu (density 9 g/CC) to a thickness of 10 μm on both sides using 12 V battery. Calculate the energy spent by the battery in this process. ECE of Cu is 3×10^{-7}kg/C.

(a) 1.8 kJ (b) 3.6 kJ
(c) 5.4 kJ (d) 7.2 kJ
(e) none of these

Solution (d) $m = \rho$ (vol.)

$$= 9 \times 10 \times 20 \times 10^{-4}$$
$$= 18 \times 10^{-2}\,\text{gram}$$
$$m = zQ$$
$$Q = \frac{m}{z} = \frac{18\times 10^{-2}}{3\times 10^{-4}} = 200\,\text{C}$$

Energy spent $E = VQ = 12 \times 600 = 7.2$ kJ.

19. A bulb of 100 W/200 V is connected to a 160 V supply find the power consumed.

(a) 32 W (b) 25 W
(c) 50 W (d) 64 W

Solution (d) $R = \dfrac{(200)^2}{100} = 400\,\Omega$

$$P_{\text{consumed}} = \frac{160\times 160}{400} = 64\,\text{W}$$

20. A kettle boils 1 kg of water in 16 min when one filament is used.If the 2nd filament is used it takes 8 minutes. If both the filaments are used (i) in parallel (ii) in series, time to boil respectively 1 kg of H_2O is

(a) 6 min, 18 min (b) 16/3 min, 18 min
(c) 16/3 min, 24 min (d) 6 min, 24 min

Solution (c) In Parallel $\dfrac{1}{t} = \dfrac{1}{t_1} + \dfrac{1}{t_2}$

or $t = \dfrac{16\times 8}{16+8} = \dfrac{16}{3}$ min

In series $t = t_1 + t_2 = 24$ min

21. An electric bulb rated 220 V/100 W will fuse if it consumes 121 W. What voltage fluctuations can it withstand?

(a) upto 230 V (b) upto 241 V
(c) upto 225 V (d) utpo 232 V

Solution (d) $R = \dfrac{(220)^2}{100} = 484\,\Omega$

Case (ii): $V^2 = 484 \times 121$ or $V = 232V$.

22. In an electrolysis experiment aftersome time, the battery connection is reversed then
(a) the electrolysis will stop
(b) the rate of liberation of material at electrodes will increase
(c) the rate of liberation will decrease
(d) rate of liberation will remain unchanged
(e) more heat will be produced at the electrons.

Solution (d)

23. Find the thermo emf developed in a Cu—Ag thermo couple when the junctions are kept at 0°C and 40°C.
[α for Cu and Ag is 2.76 and 2.5 μV/°C and β for both Cu andAg is 0.012 $\mu V/(°C)^2$
(a) 1.04 μV (b) 10.4 μV
(c) 210.4 μV (d) none of these

Solution (b) $\alpha_{net} = (2.76 - 2.50) \times 10^{-6}$

$\beta_{net} = 0$

$\therefore \quad \varepsilon = \alpha\theta = 0.26 \times 40\,\mu V = 10.4\,\mu V.$

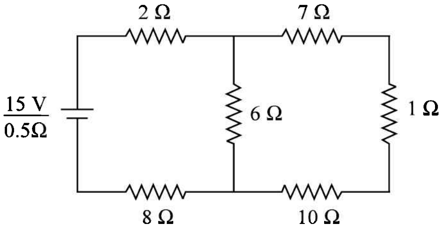
24. Find the time required to collect 11.2 *lt* of H_2 in a water volameter operating at 2A.
(a) 19 min (b) 24 min
(c) 29 min (d) 14.5 min

Solution (c) $\because$ 22.4 *lt* ≡ 2g

$$\therefore \quad 1\,lt = \frac{2}{22.4}\,g, \text{ using } m = ZIt$$

$$\frac{2}{22.4} = \frac{1}{96500} \times 5 \times t \quad \text{or} \quad t = 29\,\text{min}.$$

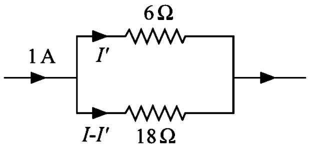
25. The power consumed by 6 Ω resistor in the given circuit is



(a)

(a) 4.611 W (b) 3.375 W
(c) 1.125 W (d) none of these

Solution (b) $I = \dfrac{15}{.5 + 10 + 4.5} = 1\text{A}$



(b)

Fig. 23.11

Using current division rule current in 6 Ω resistor

$$I' = \frac{18}{24} \times 1 = \frac{3}{4}\text{A}$$

$$P = I'^2R = \left(\frac{3}{4}\right)^2 \times 6 = 3.375\,\text{W}.$$

26. The thermo emf of a thermocouple is 25 μV/°C at room temperature. A galvanometer of 40 Ω resistance capable of detecting current as low as 10^{-5}A is connected with the thermocouple. The smallest temperature difference that can be detected by the system is
(a) 12°C (b) 8°C
(c) 20°C (d) 16°C

Solution (d) $\Delta\theta = \dfrac{40 \times 10^{-5}}{25 \times 10^{-6}} = 16°\text{C}.$

27. The negative Zn plate of a Daniel cell, sending a constant current through a circuit, decreases in mass by 0.13 gram in 30 minutes. If chemical equivalent of Zn and Cu are 32.5 and 31.5 respectively, the increase in mass of the copper pole is
(a) 0.141 g (b) 0.126 g
(c) 0.242 g (d) 0.180 g

Solution (b) $\dfrac{m_1}{m_2} = \dfrac{E_1}{E_2}$

$$m_{Cu} = \frac{31.5}{32.5} \times (.13) = 0.126\,\text{g}.$$

28. In a copper voltameter, the mass deposited in 30 minutes is .... if the current as shown in Fig. 23.12 passes through the voltameter.

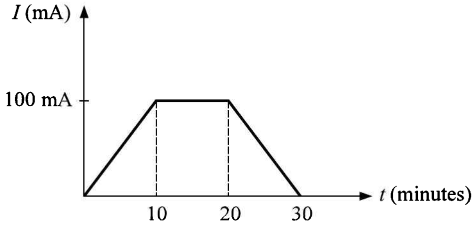


Fig. 23.12

(a) 0.078 g (b) 0.039 g
(c) 0.054 g (d) none of these

Solution (b) Q = area under graph

$= 0.1 \times 20 \times 60 = 120\,\text{C}.$

$$m = Z \cdot Q = \frac{31.5}{96500} \times 120 = 0.039\,\text{g}$$

29. In a thermocouple minimum current flows at
(a) neutral temperature
(b) inversion temperature
(c) half the neutral temperature
(d) $\dfrac{3}{2}$ of the neutral temperature

Solution (b) It is clear from Fig. 23.13, that at inversion temp emf = 0. $\therefore$ Current is also zero.

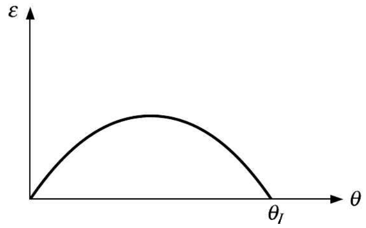


Fig. 23.13

30. Find the Peltier coefficient π in a thermocouple, if one junction is at 0°C and emf is given by $\varepsilon = \alpha\theta + \beta\theta^2$.
(a) $\theta(\alpha + 2\beta\theta)$ (b) $(273 - \theta)(\alpha + 2\beta)$
(c) $(\theta + 273)(\alpha + 2\beta)$ (d) $(\theta + 273)(\alpha + 2\beta\theta)$

Solution (d) $\pi = T\dfrac{\partial\varepsilon}{\partial\theta} = (\theta + 273)(\alpha + 2\beta\theta).$

Problems for Practice

1. A 12 V – 36 W bulb has to be used on a 20 V supply. The series resistance needed is
(a) 11 Ω (b) 6.7 Ω (c) 30 Ω (d) 2.70 Ω

2. If the two bulbs of 25 W and 100 W respectively, each rated at 220 volts are connected in series with the supply of 440 volts. Which of the bulb will fuse?
(a) 100 W bulb (b) 25 W bulb
(c) both of them (d) none of them

3. *A*, *B* and *C* are 100W, 100W and 25W bulbs respectively. Which bulb glows brightest. If *B* gets fused, which will glow brighter?

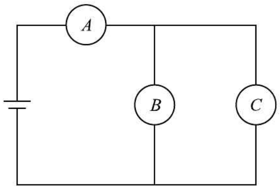


Fig. 23.14

(a) *A*, *A* (b) *A*, *C*
(c) *B*, *C* (d) *C*, *C*

4. A current *i* flows in a wire of circular cross-section with the free electrons travelling with a drift velocity v_d. The drift velocity of electrons when a current 2*i* flows in another wire of twice the radius and of the same material is
(a) $2v_d$ (b) v_d (c) $v_d/2$ (d) $\sqrt{2}\,v_d$

5. One kilowatt hour is equal to
(a) 10^3 J (b) 36×10^3 J
(c) 10^5 J (d) 36×10^5 J

6. An electric bulb is rated 220 volts and 100 watts. The power consumed by it when operated on 110 volt is
(a) 25W (b) 50W (c) 75W (d) 90W

7. Two electric bulbs have tungsten filaments of same length. If one of them give 60 W and other 100 W, then
(a) 100 W bulb has thicker filament
(b) 60 W bulb has thicker filament
(c) both have same thickness of filament
(d) it is impossible to get different wattage bulbs unless the lengths are different

8. What is immaterial for a electric fuse?
(a) its length
(b) its radius
(c) its specific resistance
(d) current flowing through it

9. If an electric current is passed through the nerve of a man, he
(a) begins to weep (b) begins to laugh
(c) is excited (d) becomes insensitive

10. The example of a non-ohmic resistance is
(a) diode (b) copper wire
(c) tungsten wire (d) carbon resistance

11. The temperature coefficient of resistance of a wire is 0.00125/°C. At 300 K its resistance is one ohm. The resistance of wire will be 2 ohm at
(a) 1154 K (b) 1127 K
(c) 1100 K (d) 1400 K

12. A copper wire and a steel wire of equal length and equal thickness are connected in series with a battery. Which will get more heated?
(a) copper (b) steel
(c) equally heated (d) insufficient data to reply.

13. If R_1 and R_2 are respectively the filament resistance of a 200 watt bulb and a 100 watt bulb designed to operate on the same voltage, then
(a) $R_1 = 2R_2$ (b) $R_1 = 4R_2$
(c) $R_2 = 2R_1$ (d) $R_2 = 4R_1$

14. Consider the following statements
(1) The product of a volt and a coulomb is a Joule
(2) The praduct of volt and watt is horse power
(3) The product of a volt and an ampere is joule/second
(4) Watt-hour can be measured in terms of electron-volt.
(a) All the four are correct
(b) (1) and (2) are correct
(c) (1), (3) and (4) are correct
(d) (2) and (4) are correct

15. The heater element in an electric iron is made of
(a) iron (b) nichrome
(c) Constantan (d) tungsten

16. The wiring of a house has resistance 6 Ω. A 100 W/220V bulb is glowing when a geyser of 1000W is switched on. The potential drop across the bulb becomes nearly equal to
(a) 120V (b) 162V (c) 172V (d) 194V

17. The fuse wire melting is related as (*r* is the radius of wire, *l* is the length of wire
(a) $\propto r^{3/2}$ (b) $\propto l$
(c) both (a) and (b) (d) $\propto t$

18. The heating element of an electric heater should be made of a material which should have
(a) high specific resistance and high melting point
(b) low specific resistance and low melting point
(c) high specific resistance and low melting point
(d) low specific resistance and high melting point

19. In 10 minutes 3000 coulomb of free electrons enter one end of a conductor and 3000 coulomb leave the other end. The current is
(a) 30A (b) 10A (c) 5A (d) zero

20. A mains line has rating 11000 watt. The maximum number of 200W bulbs (to glow fully) which can be connected with the line are
(a) 55 (b) 50 (c) 52 (d) none of these

21. The drift velocity of the electrons in ms^{-1} is of the order of
(a) 10^{-8}ms^{-1} (b) 10^{-4}ms^{-1}
(c) 10^{4}ms^{-1} (d) 10^{8}ms^{-1}

22. Copper and zinc voltameters are connected in series and current *i* is passed through them for a time *t*. If zinc is liberated in mkg then copper deposited is approximately
(a) mkg (b) 0.97 mkg
(c) 1.1 mkg (d) 2.3 mkg

23. The electric bulbs having resistance ratio 1 : 2 are connected in series, the power disposed in them have the ratio (connected to a constant voltage source)
(a) 1 : 2 (b) 2 : 1
(c) 1 : 4 (d) 1 : 1

24. The resistance of the heating element of an electric wire is 100 ohm. The resistance of heating element of another wire having twice the power rating for the same voltage is

(a) $25\,\Omega$ (b) $50\,\Omega$
(c) $100\,\Omega$ (d) $200\,\Omega$

25. If a copper wire is stretched to make it 0.1% longer, the percentage increase in resistance will be
(a) 0.1 (b) 0.2 (c) 1 (d) 2

26. Two heater wires of equal length are first connected in series and then in parallel. The ratio of heat produced in two cases will be
(a) 1:2 (b) 1:4
(c) 2:1 (d) 4:1

27. Two electric bulbs having resistance ratio 1 : 2 are connected in parallel to a constant voltage source. The powers dissipated in them have the ratio
(a) 1:2 (b) 2:1
(c) 1:4 (d) 1:1

28. A bulb is rated 100W/220V. It is operated at 150V then the resistance of the bulb is
(a) $484\,\Omega$ (b) $\geq 484\,\Omega$
(c) $\leq 484\,\Omega$ (d) cannot be said

29. Two bulbs take 50 watts each when connected in parallel to 100V source. The total power consumed by them when they are connected in series with the same source is
(a) 25W (b) 50W (c) 75W (d) 100W

30. A piece of copper and other of germanium are cooled from room temperature to 80K. The resistance of
(a) copper increases
(b) copper increases and germanium decreases
(c) copper decreases
(d) copper decreases and germanium increases

31. Three equal resistors connected in series across a source of emf together dissipate 10 watts of power. What should be power dissipated if the same resistors are connected in parallel across the same source of emf?
(a) 9W (b) 10 W (c) 90W (d) 100W

32. A certain piece of copper is to be shaped into a conductor of minimum resistance. Its length and diameter should be respectively
(a) l, d (b) $2l$, d (c) $l/2$, $2d$ (d) $2l$, $d/2$

33. 16 cells each of internal resistance 1Ω and emf 2V are to be grouped in such a way so that they supply maximum current to an external resistance of 4Ω. Then they should be connected in
(a) two rows of 8 cells in series
(b) 4 rows of 4 cells in series
(c) 8 rows of 2 cells in series
(d) 1 row of 16 cells in series

34. The power consumed by resistance C is 12W. What will be the power consumed by parallel combination of B and A?
(a) 24W (b) 20W
(c) 10W (d) 40W

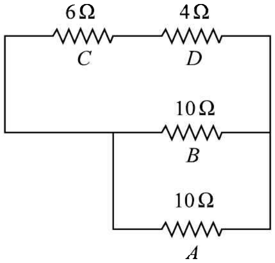


Fig. 23.15

35. According to Joule's law, if potential difference across a conductor having a material of specific resistance ρ remains constant, then the heat produced in the conductor is directly proportional to
(a) ρ (b) ρ^2
(c) $1/\sqrt{\rho}$ (d) $1/\rho$

36. How many calories of heat will approximately be developed in a 210 watt electric bulb in 5 minutes?
(a) 1050 cal (b) 15000 cal
(c) 63000 cal (d) 80000 cal
(e) none of these

37. You are given three bulbs of 25, 40 and 60 watt. Which of them has lowest resistance?
(a) 25 W bulb (b) 40 W bulb
(c) 60 W bulb (d) none of these

38. The length of a wire of cross-section $0.5\,mm^2$ and specific resistance $2.5 \times 10^{-7}\,\Omega m$ required to make a heater rated at 72W to operate from a constant supply of 12V is
(a) 4 cm (b) 4 m
(c) 40 cm (d) 40 m

39. A battery has an emf of 12V. When connected with $6\,\Omega$ and $12\,\Omega$ it dissipates equal power in the load. The internal resistance is nearly
(a) $8.48\,\Omega$ (b) $6.48\,\Omega$
(c) $12.48\,\Omega$ (d) $9.84\,\Omega$

40. Charge carriers responsible for conduction in an electrolyte are
(a) electrons only
(b) positive ions only
(c) negative ions only
(d) both positive and negative ions

41. The amount of ions liberated by 96500 C of charge when passed through the electrolyte is called
(a) chemical equivalent
(b) electrochemical equivalent
(c) gram equivalent
(d) none of these

42. How much electrical energy in kilowatt hours is consumed in operating ten 50W bulbs for 10 hours per day in a month of 30 days?
(a) 15 (b) 150 (c) 1500 (d) 15000

43. If 'I' be the safe current for the fuse wire of radius 'r' then
(a) $I \propto r^{1/2}$ (b) $I \propto r^2$
(c) $I \propto r^{3/2}$ (d) $I \propto r$

44. The mass of an element liberated during electrolysis due to the flow of 1A current for 1 second is called
(a) chemical equivalent
(b) equivalent weight
(c) electrochemical equivalent
(d) none of these

45. In the electrolysis of acidic water, the ratio of the number of hydrogen to that of oxygen atoms liberated is
(a) 4 : 1 (b) 1 : 4 (c) 2 : 1 (d) 1 : 2

46. The effective wattage of 60W and 40W lamps connected in series is equal to
(a) 24W (b) 20W
(c) 100W (d) 80W

47. The *ece* of copper is 3.3×10^{-7} kgcoloumb^{-1}. If 100 kWh energy is consumed at 33 volt in a copper voltameter, then the mass of copper liberated is
(a) 1 mg (b) 1 kg
(c) 3.3 kg (d) 3.6 kg

48. Chemical equivalent of a substance is W. To obtain its electrochemical equivalent we should multiply W with *ece* of
(a) oxygen (b) hydrogen
(c) carbon (d) none of these

49. There are two metallic wires having same material, same length but different radius, are connected in series across a battery the heat produced is H_1, but when connected in parallel, with the same battery, heat produced is H_2 in the same time. Then which of the following is correct
(a) $H_1 > H_2$ (b) $H_1 = H_2$
(c) $H_1 < H_2$ (d) none of these

50. In the question number 49 if on connecting two wires in series, heat produced in thinner wire is H_1 and that in thicker wire is H_2. Then which of the following is correct?
(a) $H_1 > H_2$ (b) $H_1 = H_2$
(c) $H_1 < H_2$ (d) none of these

51. In question number 49 if two wires are connected in parallel and heat produced in thinner and thicker wire is H_1 and H_2 respectively then
(a) $H_1 > H_2$ (b) $H_1 = H_2$
(c) $H_1 < H_2$ (d) none of these

52. When 1A of current is passed through $CuSO_4$ solution for 10 seconds, then the number of copper ions deposited at the cathode will be about
(a) 3.1×10^{19} (b) 6.2×10^{19}
(c) 4.8×10^{19} (d) 1.6×10^{19}

53. If same current is passed through copper and silver voltameters, then the rate of increase in weights of the cathode in two voltameters will be proportional to
(a) relative densities
(b) atomic numbers
(c) atomic masses
(d) none of these

54. When current is passed through a wire, heat is produced due to
(a) inter atomic collisions
(b) fall of electrons from higher energy orbits to lower energy orbits
(c) inter electron collisions
(d) collision of conduction electrons with atoms

55. There are two bulbs in a house. One glows brighter than the other. Which of the two bulbs has got a higher resistance?
(a) brighter bulb
(b) dim bulb
(c) both with same resistance
(d) brightness is not linked with resistance

56. Copper and iron thermocouple produces 0.63 V, iron and constantan thermocouple produces – 0.83 V when the temperature difference between the junctions of two thermocouples is equal. The emf produced in copper constantan thermocouple for same difference in temperature is
(a) – 0.2 V (b) – 0.5 V
(c) 1.46 V (d) none of these

57. The maximum current that can be allowed to pass through 100 W/250 V lamp is
(a) 0.25A (b) 0.40A
(c) 2.5A (d) 1.00A

58. Same current is passed through silver and zinc voltameters connected in series, for the same time. If the amount of silver liberated is x kg, then zinc liberated is nearly
(a) x kg (b) $x/2$ kg
(c) $x/3$ kg (d) $x/5$ kg

59. If emf in a thermocouple is $\varepsilon = \alpha T + \beta T^2$ then the neutral temperature of the thermocouple is
(a) $-\beta/(2\alpha)$ (b) $-2\beta/\alpha$
(c) $-\alpha/(2\beta)$ (d) $-2\alpha/\beta$

60. Hydrogen is liberated at the cathode when electric current is passed through the solution of
(a) sulphuric acid (b) copper sulphate
(c) sodium hydroxide (d) sugar

ANSWERS

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (a)
21. (b)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (d)
31. (c)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (d)	**36.** (c)	**37.** (c)	**38.** (b)	**39.** (a)	**40.** (d)
41. (c)	**42.** (a)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (a)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (a)
51. (c)	**52.** (a)	**53.** (d)	**54.** (d)	**55.** (b)	**56.** (a)	**57.** (b)	**58.** (c)	**59.** (c)	**60.** (a)

24 Magnetic Effects of Current

BRIEF REVIEW OF THE CONCEPTS

Magnetic field

If a charged particle having charge q, mass m enters a magnetic field B with a velocity v then the force experienced by the particle is

$$\vec{F} = q(\vec{v} \times \vec{B})$$

This principle is used in television receiver for deflection of electrons. The unit of magnetic field is

$$B = \text{Wbm}^{-2} \text{ or Tesla}$$

The CGS unit is Gauss = Maxwell/cm^2.

If electric and magnetic fields are present then the force experienced is called Lorentz force and is given by

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

J.J. Thomson used this method to find e/m.

If the velocity of a charged particle is always perpendicular to the magnetic field then it describes a circle of radius R such that

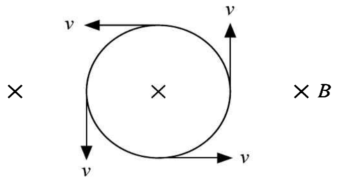


Fig. 24.1

$$R = \frac{mv}{qB}$$

Time period of revolution $T = \frac{2\pi R}{v} = \frac{2\pi m}{qB}$.

Cyclotron frequency $f = \frac{1}{T} = \frac{qB}{2\pi m}$.

If the charged particle enters obliquely in a magnetic field then the velocity can be resolved into x and y components. The v_y perpendicular component describes a circle and v_x parallel component describes linear motion. Therefore, the particle moves in a helix.

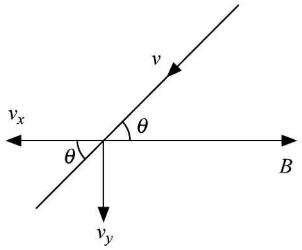


Fig. 24.2

The horizontal or linear distance moved in one complete revolution is called the pitch of the helix.

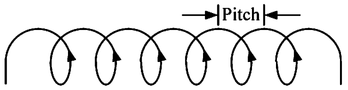


Fig. 24.3

Pitch of the helix $= v_xT = v_x\left(\frac{2\pi m}{qB}\right)$

Magnetic force due to a current carrying conductor

$$d\vec{F} = I\,d\vec{l} \times \vec{B}$$

The direction of force is determined using Fleming's left hand rule.

If the conductor is straight

$$\vec{F} = I\,\vec{l} \times \vec{B}$$

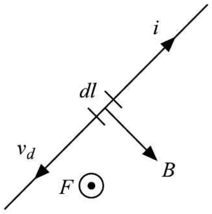


Fig. 24.4

otherwise integrate to find

$$\vec{F} = \int I\,d\vec{l} \times \vec{B}$$

Torque acting on a current carrying loop $\tau = I\vec{A} \times \vec{B}$

If there are n loops $\tau = nI\vec{A} \times \vec{B}$, where A is area vector $A = lb$

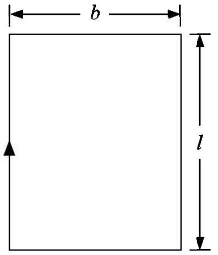


Fig. 24.5

$\tau = \vec{M} \times \vec{B}$; where $M = IA$ or nIA is magnetic dipole moment.

Biot Savart law

$$\vec{dB} = \frac{\mu_0 I dl \sin\theta}{4\pi r^2}$$

is the magnetic field due to a length *dl* of a conductor carrying current *I* at a distance *r* from the conductor.

$$dB = \frac{\mu_0 I dl \sin\theta}{4\pi r^2} \quad \text{where}$$

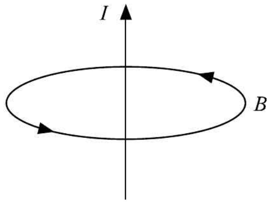


Fig. 24.6

$\mu_0 = 4\pi \times 10^{-7}\text{Wb(A-m)}^{-1}$

or Henerym^{-1}

The direction of magnetic field is illustrated in fig 24.6.

Magnetic field strength due to a straight finite current carrying conductor

$$B = \frac{\mu_0 I}{4\pi d}[\cos\theta_1 - \cos\theta_2]$$

$$= \frac{\mu_0 I}{4\pi d}[\sin\alpha - \sin\beta]$$

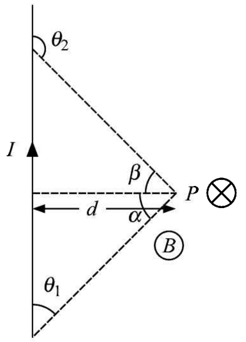


Fig. 24.7

The direction of magnetic field is given by right hand thumb rule.

Note: From Fig. 24.8 that magnetic field at *P* is perpendicular inwards to the plane of the paper and magnetic field at *S* is perpendicular outward to the plane of the paper.

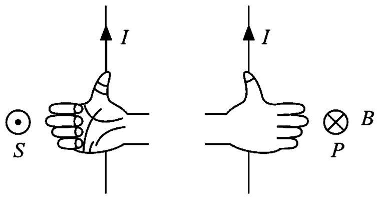
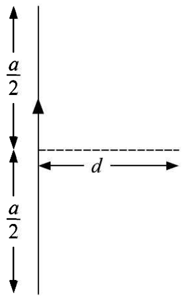


Fig. 24.8

Magnetic field at perpendicular bisector of a current carrying conductor (Fig. 24.9)

$$B = \frac{\mu_0 I a}{2\pi d\sqrt{a^2 + 4d^2}}$$



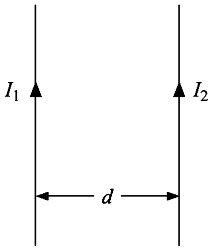


Fig. 24.9 Fig. 24.10

Magnetic field due to a long straight conductor

$$B = \frac{\mu_0 I}{2\pi d}$$

Force between parallel conductors (Fig. 24.10)

$$\frac{dF}{dl} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

The force is attractive if the currents are in the same direction and repulsive if the currents are in the opposite direction.

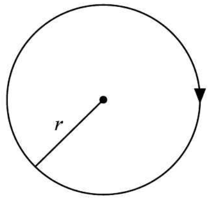


Fig. 24.11

Magnetic field at the centre of a circular loop carrying current

$$B = \frac{\mu_0 I}{2r}$$

The direction is perpendicular inwards if the current is clockwise and perpendicular outwards to the plane of paper if the current is anticlockwise.

Magnetic field due to a circular arc of radius *r* at the centre *P*

$$B = \frac{\mu_0 I(\alpha)}{4\pi r}$$

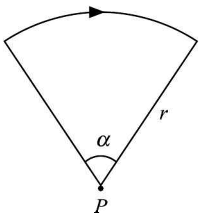


Fig. 24.12

Take α in radian

For a semicircular loop $\alpha = \pi$

$$B = \frac{\mu_0 I}{4r}$$

Magnetic field at any point on axial line

$$B = \frac{\mu_0 I r^2}{2(r^2 + x^2)^{3/2}}$$

Find the direction as for a circular coil.

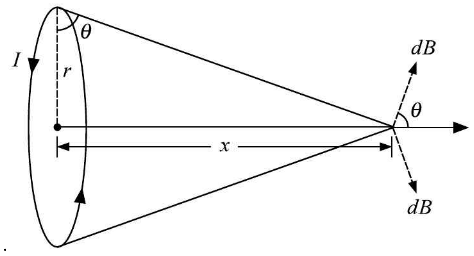


Fig. 24.13

Special case Magnetic field at the centre (Put $x = 0$)

$$B = \frac{\mu_0 I}{2r}$$

Ampere circuital law

$$\oint B \cdot dl = \mu_0 I$$

Magnetic field due to a long solenoid at the axis of a solenoid.

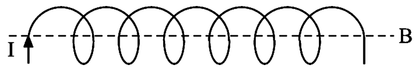


Fig. 24.14

$$B = n\mu_0 I$$

where n is number of turns per unit length.

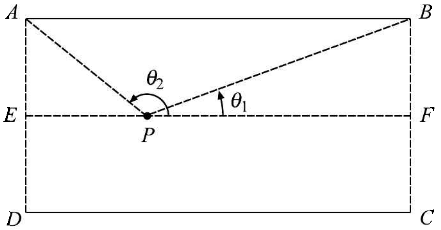


Fig. 24.15

Magnetic field at any point P in the solenoid

$$B_P = \frac{1}{2}\mu_0 nI\,(\cos\theta_1 - \cos\theta_2)$$

At point, F or E due to a long solenoid

$$B_F = B_E = \frac{\mu_0 nI}{2}$$

Magnetic field at any point P (acting tangentially) on a toroid

$$B = \frac{\mu_0 NI}{2\pi r}$$

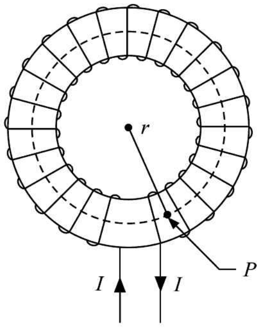


Fig. 24.16

where N = total number of turns

Magnetic field due to a moving charge

$$B = \frac{\mu_0 qv\sin\theta}{4\pi r^2}\,;\quad \vec{B} = \frac{\mu_0 q\,\vec{v}\times\vec{r}}{4\pi r^3}$$

Magnetic force between moving charges

$$F_{\text{mag}} = \frac{\mu_0 q_1 q_2 v_1 v_2}{4\pi r^2}$$

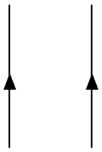


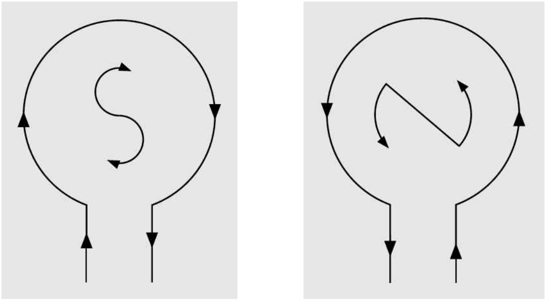
Fig. 24.17

If the charges are like and move in same direction then they repel each other because $F_{\text{elec.}} > F_{\text{mag}}$ which is repulsive though magnetic force is attractive.

The force will be attractive only if the charges are of opposite nature.

SHORT-CUTS AND POINTS TO NOTE

(i) (1) If in a coil the current is clockwise it acts as a S-pole.



(a) (b)

Fig. 24.18

(2) If the current is anticlockwise in the coil it acts a N-pole when seen from the face.

(ii) No magnetic field occurs on a point P on the current carrying conductor or at any point S or R which lies on the extended part of the conductor.

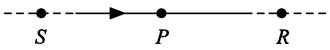


Fig. 24.19

(iii) Magnetic field intensity at the centre of a loop made with a uniform cross-section wire and uniform density is zero irrespective of its shape provided current enters from a point and leaves from another point on the conductor.

(iv) If magnetic field and electric field are perpendicular to each other and a charged particle enters perpendicular to both electric and magnetic fields such that it goes undeviated then.

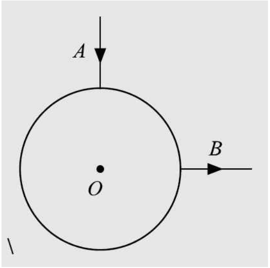


Fig. 24.20

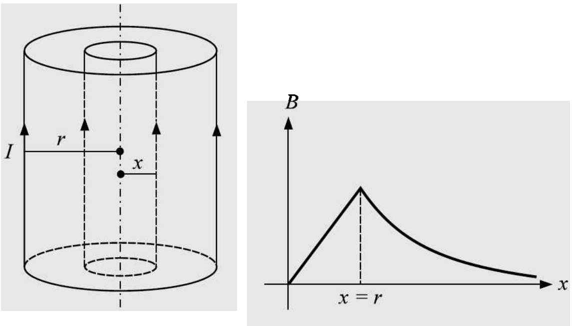
$$E = vB \text{ or } v = \frac{E}{B}\,.$$

(v) Magnetic field intensity in a thick current carrying conductor at any point $x < r$ (inside the conductor)

$$B_{\text{inside}} = \frac{\mu_0 I x}{2\pi r^2} \qquad x < r$$

$$B_{\text{surface}} = \frac{\mu_0 I}{2\pi r} \qquad x = r$$

$$B_{\text{out}} = \frac{\mu_0 I}{2\pi x} \qquad x > r$$



(a) **(b)**

Fig. 24.21

(vi) When current passes through a spring it shrinks because all the rings in it are carrying current in the same direction and are attracted towards one another.

(vii) Momentum of a charged particle in a cyclotron is $p = Bqr = \sqrt{2(KE)m}$ where (KE) is the kinetic energy of the particle.

(viii) Helmohltz coils If two coaxial coils of equal radii are placed at a distance equal to the radius of either of the coils and in which same current is flowing are known as helmohltz coils. The magnetic field intensity versus x graph is shown below. At points R and S the nature of curve becomes vice-versa to concave downwards or convex upwards (or points of unit slope) They are called points of inflexion.

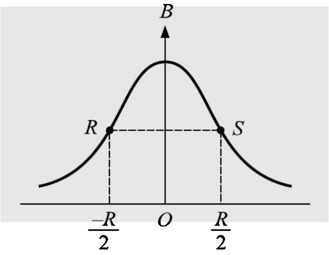


Fig. 24.22

(ix) In cyclotron when kinetic energy and magnetic fields are equal for two charged particles then $\frac{r_1}{r_2} = \frac{q_2}{q_1}\sqrt{\frac{m_1}{m_2}}$

If only magnetic field is same for two charged particles then $\frac{r_1}{r_2} = \frac{q_2}{q_1}\sqrt{\frac{m_1(KE_1)}{m_2(KE_2)}}$

where KE_1 and KE_2 are KE of the particles

(x) If charged particle having same charge and different mass enter a magnetic field with same velocities then

$$\frac{r_1}{r_2} = \frac{m_1}{m_2}$$

This arrangement is called mass spectrometer.

(xi) While finding force on a non-linear (curved part) use displacement as the length and then $F = IlB$, for example, in this problem force for curved part is $2IRB$.

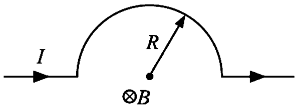


Fig. 24.23

CAUTION

(i) Considering physical length of the conductor is to be taken as $F = \int i\, dl \times B$

- The separation between end points be taken as total length.

(ii) Considering that magnetic field exists on a conductor carrying current or on its extended part.

- If the conductor is thin (as usually is the case) then the magnetic field does not exist on the conductor or its extended part.

(iii) Considering that the magnetic field is zero only along the axis of a hollow cylindrical conductor carrying current.

- Magnetic field is zero at all points inside a current carrying hollow cylindrical conductor.

(iv) While finding the direction of magnetic field using charged particles not taking into account in which direction current is flowing due to them

- Note that all laws are based on conventional current. Therefore, the direction of current due to the positive charge is same in which they are moving. The direction of current for negative charges is opposite to their direction of motion.

(v) While finding magnetic field due to a straight conductor not considering about the perpendicular distance.
For example, in the Fig. 24.24 taking perpendicular distance for straight part AB to be r.

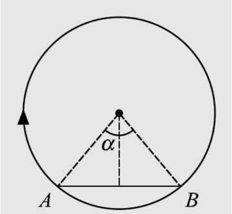


Fig. 24.24

- Take perpendicular distance $r\cos(\alpha/2)$, α being semi-minor angle by AB.

(vi) Confusing that the work is done due to magnetic force.

- Since force is perpendicular to both field B and velocity v (direction of motion), therefore, $W = \vec{F}.\vec{d} = 0$, that is, no work is done.

(vii) Considering that magnetic moment M shall depend upon the shape of the current carrying loop.

- Magnetic moment $M = nIA$ which is independent of the shape and is equal, if areas of various loops are equal and number of turns are equal.

(viii) Considering no force will act on current carrying conductors placed transverse to a long current carrying conductor as shown in Fig. 24.25.

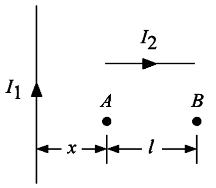


Fig. 24.25

- In such a case magnetic field at every point will change, therefore,

$$F = \frac{\mu_0 I_1 I_2}{2\pi}\int_x^{x+l}\frac{dy}{y} = \frac{\mu_0 I_1 I_2}{2\pi}\log_e\frac{x+l}{x}.$$

(ix) Considering that if the plane of a coil is parallel to the magnetic field the net force experienced by the coil is zero.

- If the magnetic field is uniform then the statement is correct. But if the magnetic field is non-uniform then along with the torque, magnetic force also exists.

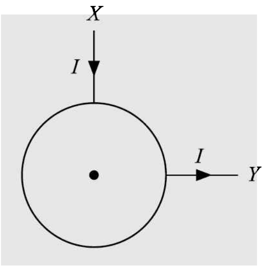


Fig. 24.26

(x) Considering magnetic field at the centre of a loop is always zero if the current enters and leaves or loop is connected to a battery.

- If the wire making loop is uniform, the statement is true. But if the wire is not uniform or has unequal density or made of different materials then the magnetic field will be non-zero at the centre.

(xi) Considering due to magnetic force speed varies

- Speed does not vary. Hence its *KE* does not vary.

Solved Problems

1. A magnetic needle is kept in a non uniform magnetic field. It experiences

(a) neither a force nor a torque
(b) a torque but not a force
(c) a force but not a torque
(d) a force and a torque

Solution (d)

2. A charged particle of mass *m* and charge *q* travels on a circular path of radius *r* that is perpendicular to the magnetic field *B*. The time taken by the particle to complete one revolution is

(a) $\frac{2\pi q^2 B}{m}$ (b) $\frac{2\pi mq}{B}$
(c) $\frac{2\pi m}{qB}$ (d) $\frac{2\pi qB}{m}$

Solution (c)

3. Two concentric coil each of radius 2π cm are placed at right angles to each other, 3*A* and 4*A* are the currents flowing in them respectively. Find magnetic induction in Wb/m^2 at the centre of the coils.

(a) 10^{-5} (b) 12×10^{-5}
(c) 7×10^{-5} (d) 5×10^{-5}

Solution (d) $B = \mu_0 I/2r$

Since the two coils are perpendicular so are the magnetic inductions.

$$\therefore \quad B_{\text{net}} = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2r}\sqrt{I_1^2 + I_2^2}$$

$$= \frac{4\pi\times10^{-7}}{2(2\pi\times10^{-2})}\sqrt{3^2+4^2}$$

$$= 5 \times 10^{-5}\,\text{Wb/m}^2$$

4. Two thin long parallel wires separated by a distance *d* carry a current *i* each in the same direction. They will

(a) repel each other with a force $\mu_0 i^2/2\pi d$
(b) attract each other with a force $\mu_0 i^2/2\pi d$
(c) repel each other with a force $\mu_0 i^2/2\pi\, d^2$
(d) attract each other with a force $\mu_0 i^2/2\pi\, d^2$

Solution (b) $\frac{dF}{dl} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\mu_0 i^2}{2\pi d}$

Since the currents are in same direction they attract each other.

5. A uniform electric field and a uniform magnetic field are acting along the same direction in a certain region. If an electron is projected along the direction of the fields with a certain velocity then

(a) its velocity will increase
(b) its velocity will decrease
(c) it will turn towards left of its motion
(d) it will turn towards right of its motion.

Solution (b) Since $F = q[\vec{E} + (\vec{v}\times\vec{B})]$

$$F_{\text{mag}} = q(\vec{v}\times\vec{B}) = 0$$

as $\vec{v}$ and $\vec{B}$ are parallel.

Since electron is moving along the field, force *qE* is repulsive and hence it will slow down.

6. A very long straight wire carries a current *I*. At the instant when a charge $+Q$ at point *P* has velocity $\vec{v}$, as shown, the force on charge is

(a) along *oy* (b) opposite to *oy*
(c) along *ox* (d) opposite to *ox*.

Solution (a) The mag field at P is inwards due to a straight long conductor. The Flemming's left hand rule gives the direction along oy.

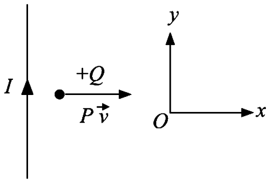


Fig. 24.27

7. An electron moves in a circular orbit with a uniform speed v. It produces a magnetic field B at the centre of the circle. The radius of the circle is proportional to

(a) $\sqrt{B/v}$ (b) B/v

(c) $\sqrt{v/B}$ (d) v/B

Solution (d) $r = mv/(qB)$ as m/q is constant.

$\therefore \quad r \propto v/B$.

8. A circular loop of wire 4 cm radius carries a current of 80A. Find the energy density at the centre of the loop.

(a) $\pi\,\text{J/m}^3$ (b) $2\pi\,\text{J/m}^3$

(c) $0.1\pi\,\text{J/m}^3$ (d) $0.2\pi\ \text{Jm}^{-3}$

Solution (d) $B = \dfrac{\mu_0 i}{2r}$

and energy density $u = \dfrac{B^2}{2\mu_0} = \dfrac{\mu_0^2 i^2}{4r^2(2\mu b)} = \dfrac{\mu_0 i^2}{8r^2}$

$$u = \frac{4\pi\times10^{-7}\times(80)^2}{8(4\times10^{-2})^2} = \frac{4\pi\times8\times0.8}{8\times16}$$

$= 0.2\pi = 0.628\,\text{Jm}^{-3}$.

9. An α-particle enters at the middle as shown in Fig. 24.28 with $10^5\,\text{ms}^{-1}$. In which direction it will bend?

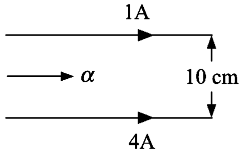


Fig. 24.28

(a) towards 1A wire

(b) towards 4A wire

(c) upward the plane of wire

(d) downwards the plane of wires

Solution (b)

10. A particle having mass m and charge q is released from the origin in a region in which electric field and magnetic field are given by $B = -B_0\hat{j}$ and $E = E_0\hat{k}$. Find the speed of the particle as a function of its z coordinate.

(a) $\sqrt{\dfrac{qEz}{m}}$ (b) $\sqrt{\dfrac{2(qvV + qE)z}{m}}$

(c) $\sqrt{\dfrac{(-qvB + qE)2z}{m}}$ (d) $\sqrt{\dfrac{2qEz}{m}}$

Solution (d) $v^2 = 2az$

$a = \dfrac{qE}{m}$ or $v = \sqrt{\dfrac{2qEz}{m}}$.

11. An electron has a speed $\sqrt{2}\times10^6\ \text{ms}^{-1}$ at A as shown. Find the direction and magni-tude of magnetic field so that electron reaches C following a semicircular path.

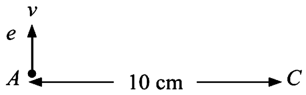


Fig. 24.29

(a) $1.6\times10^{-4}T$ ⊗ (b) $1.6\times10^{-4}T$ ○

(c) $3.6\times10^{-4}T$ ⊗ (d) none of these

Solution (a) $r = 5\,\text{cm} = \dfrac{mv}{qB}$

or $$B = \frac{mv}{er} = \frac{9\times10^{-31}\times\sqrt{2}\times10^6}{1.6\times10^{-19}\times5\times10^{-2}}$$

$$= \frac{9\sqrt{2}\times10^{-4}}{8} = 1.6\times10^{-4}T$$

Direction perpendicular inwards the plane of paper.

12. An electron in the neck of a TV picture tube is accelerated by a pot-difference 2 kV. Then it passes through a transverse magnetic field to produce a circular arc of radius 0.18 m. Find the magnetic field.

(a) $6.38\times10^{-4}T$ (b) $7.68\times10^{-4}T$

(c) $8.38\times10^{-4}T$ (d) $8.98\times10^{-4}T$

Solution (c) $r = \dfrac{mv}{qB}$

or $B = \dfrac{mv}{er} = \dfrac{\sqrt{2eVm}}{er} = \sqrt{\dfrac{2Vm}{er^2}}$

or $B = \sqrt{\dfrac{2\times10^3\times2\times9\times10^{-31}}{1.6\times10^{-19}\times324\times10^{-4}}}$

$= \sqrt{\dfrac{10^{-5}}{0.4\times36}} = \sqrt{\dfrac{10^{-5}}{14.4}} = 8.38\times10^{-4}T.$

13. Figure shows a thin 50 cm long rod rests on two metallic supports in a uniform magnetic field of 0.45 T. Find the maximum voltage which can be applied without breaking the circuit. Mass of the rod is 750 g. Take g = 10 ms²

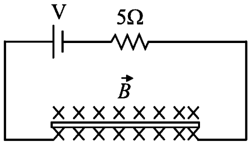


Fig. 24.30

(a) 83.3 V (b) 8.33 V

(c) 833 V (d) 0.833 V

Solution (c) $mg = IlB$ or $mg = \dfrac{V}{R}lB$

or $$V = \frac{mgR}{lB} = \frac{3\times10\times25}{4\times0.5\times0.45} = 833\,\text{V}$$

14. A wire along x-axis carries a current 3.5A. Find the force on 1 cm section of the wire exerted by

$$B = 0.74T\,\hat{j} - 0.36T\,\hat{k}$$

(a) $2.59\,\hat{k} + 1.26\hat{j}$ (b) $1.26\,\hat{k} - 2.59\hat{j}$

(c) $-2.59\,\hat{k} - 1.26\hat{j}$ (d) $-1.26\,\hat{k} - 2.59\hat{j}$

Solution (a) $F = I(\vec{l}\times\vec{B})$

$= 3.5[10^{-2}\hat{i}\times(0.74\,\hat{j} - 0.36\,\hat{k})]$

$= (2.59\,\hat{k} + 1.26\,\hat{j})\times10^{-2}$

15. An electron and a Li nucleus enter a magnetic field with same velocity. Find the ratio of number of revolutions per second of the two

(a) 2.44×10^3 (b) 4.24×10^3
(c) 3.24×10^3 (d) 5.42×10^3

Solution (b) $f = \dfrac{qB}{2\pi m}$

$$\therefore = \frac{f_e}{f_{Li}} = \frac{e/m_e}{3e/m_{Li}} = \frac{m_{Li}}{3m_e} = \frac{7\times1.6\times10^{-27}}{3\times9\times10^{-31}}$$

$$= \frac{11.2\times10^4}{27} = 4.24 \times 10^3$$

16. A thin uniform rod of negligible mass and length attached to the floor by a hinge P. The other end is connected to a spring of force constant k. Rod is in a uniform magnetic field B pointing inwards the plane of paper. A current I is passed through the rod. Find the torque acting on the rod due to magnetic force when the rod makes an angle 53° as shown.

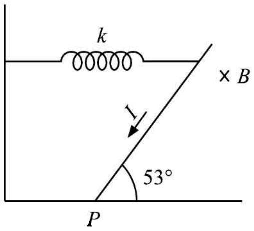


Fig. 24.31

(a) Il^2B (b) $Il^2B/2$
(c) $3Il^2B/5$ (d) $4Il^2B/5$

Solution (b) $F = kx = IlB$

$$\tau = \int d\tau = \int IlB\,dl = \frac{Il^2B}{2}.$$

17. An infinitely long wire carries a current (see fig. 24.32). Find the magnetic field at P

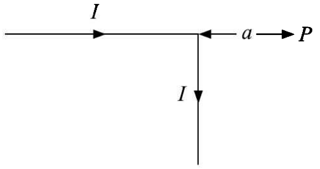


Fig. 24.32

(a) $\dfrac{\mu_0 I}{2\pi a}$ (b) $\dfrac{\sqrt{2}\mu_0 I}{4\pi a}$

(c) $\dfrac{\mu_0 I}{4\pi a}$ (d) $\dfrac{\mu_0 I}{4E\pi a}$

Solution (c) $B = B_1 + B_2$

$$B = 0 + \frac{\mu_0 i}{4\pi a}\,[\sin 90 + \sin 0] = \frac{\mu_0 I}{4\pi a}$$

18. A wire bent as shown in Fig. 24.33 carries a current I. Find the magnetic field at P.

(a) $\dfrac{\mu_0 I}{4R}$ (b) $\dfrac{3\mu_0 I}{2R}$

(c) $\dfrac{4\mu_0 I}{8R}$ (d) $\dfrac{\mu_0 I}{8R}$

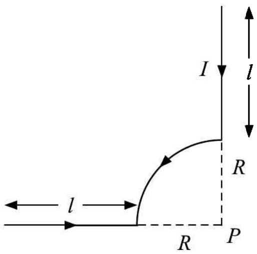
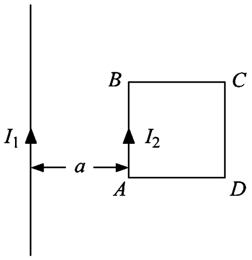


Fig. 24.33

Solution (d) $B = B_1 + B_2 + B_3$

$$= 0 + \frac{\mu_0 I}{2R}\left(\frac{\pi/2}{2\pi}\right) + 0 = \frac{\mu_0 I}{8R}$$

19. A square loop of side a is placed at a distance a away from a long wire carrying a current I_1. If the loop carries a current I_2 then the nature of the force and its amount is:



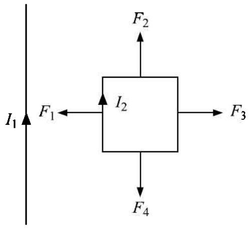
(a)

(a) $\dfrac{\mu_0 I_1 I_2}{2\pi a}$, attractive (b) $\dfrac{\mu_0 I_1 I_2}{4\pi}$, attractive

(c) $\dfrac{\mu_0 I_1 I_2}{4\pi}$, repulsive (d) $\dfrac{\mu_0 I_1 I_2}{4\pi a}$, repulsive

Solution (b) F2 and F4 cancel one another. F1 is attractive F3 is repulsive. But F1 > F3

∴ Force is attractive



(b)

Fig. 24.34

$$F_1 = \frac{\mu_0 I_1 I_2}{2\pi a}\,(a),$$

$$F_3 = \frac{\mu_0 I_1 I_2}{4\pi a}\,(a)$$

$$F_{net} = F_1 - F_3 = \frac{\mu_0 I_1 I_2}{4\pi}.$$

20. Fig. 24.35 shows a circular wire of radius a carrying a current i. The force of compression on the wire is

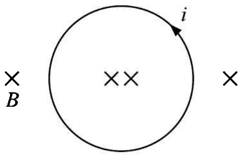


Fig. 24.35

(a) $2iaB$ (b) iaB
(c) $2\pi iaB$ (d) none of these

Solution (b) $dF = idlB$

$$F = \int_0^a idl B = iaB$$

($\because$ maximum shrink will be equal to the radius).

21. A square coil of edge l have n turns and carries a current i. It is placed on a smooth horizontal plate. A magnetic field B parallel to one edge is applied. The total mass of the coil is M. The minimum value of B for which the coil will tip over is.

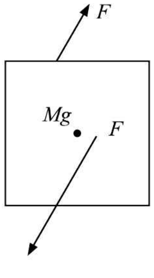


Fig. 24.36

(a) $\frac{Mg}{lin}$ (b) $\frac{Mg}{2lin}$
(c) $\frac{2Mg}{lin}$ (d) none of these

Solution (b) $F = linB \qquad \tau = F \cdot a = \frac{Mg}{lin}$

or $\quad Mg = 2linB$ or $B = \frac{Mg}{2lin}$.

22. A particle of mass M and charge Q moving with a velocity $\vec{v}$ describes a circular path of radius R when subjected to a uniform transverse magnetic field of induction B. The work done by the field when the particle completes one full circle is

(a) zero (b) $BQ2\pi R$
(c) $BQv(2\pi R)$ (d) $\left(\frac{Mv^2}{R}\right)(2\pi R)$

Solution (a) As displacement is zero.

23. A particle of charge $q = 16 \times 10^{-18}$C moving with $10\,\text{ms}^{-1}$ along x-axis enters a magnetic field of induction B along the y-axis and an electric field $10^4\,\text{Vm}^{-1}$ along negative z-direction. If the particle continues to move along x-axis then the strength of magnetic field is

(a) $10^5\,\text{Wbm}^{-2}$ (b) $10^{16}\,\text{Wbm}^{-2}$
(c) $10^{-3}\,\text{Wbm}^{-2}$ (d) $10^3\,\text{Wbm}^{-2}$

Solution (d) $v = \frac{E}{B} \Rightarrow \frac{10^4}{B} = 10 \therefore B = 10^3\,\text{Wbm}^{-2}$.

24. A conducting loop carrying a current I is placed in a uniform magnetic field pointing into the plane as shown in Fig. 24.37. The loop will have tendency to

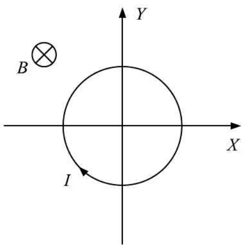


Fig. 24.37

(a) contract
(b) expand
(c) move towards positive x-axis
(d) move towards negative x-axis

Solution (b) Using Fleming left hand rule you find that the force is acting outwards.

25. In a square loop made with a wire of uniform cross-section current I enters from point A and leaves from point B. The magnetic field strength B at the centre of the square is

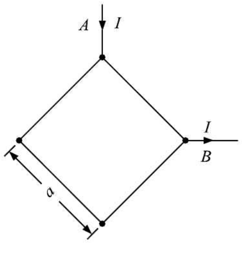


Fig. 24.38

(a) zero (b) $\frac{\mu_0 I 2\sqrt{2}}{4\pi a}$
(c) $\frac{4\sqrt{2}\mu_0 I}{4\pi a}$ (d) $\frac{2\sqrt{2}\mu_0 I}{4a}$

Solution (a) See shortcut (iii).

26. In the Fig. 24.39 shown below each battery has emf = 5 V. Then the magnetic field at P is

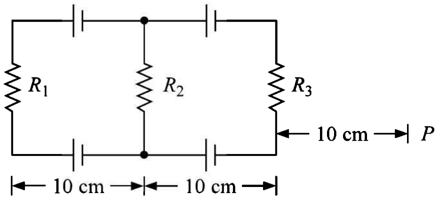


Fig. 24.39

(a) zero (b) $\frac{10\,\mu_0}{R_1(4\pi)(.2)}$
(c) $\frac{20\,\mu_0}{(R_1 + R_2)(.8\pi)}$ (d) none of these

Solution (a) Because current in the loop is zero.

27. The magnetic field strength at O due to current I in the figure below is.

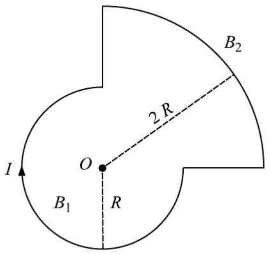


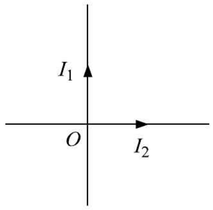
Fig. 24.40

(a) $\frac{7\mu_0 I}{16R}$ (b) $\frac{15\mu_0 I}{16R}$
(c) $\frac{11\mu_0 I}{32R}$ (d) $\frac{13\mu_0 I}{32R}$

Solution (a) $B = B_1 + B_2 = \frac{\mu_0 I}{2R}\left(\frac{3}{4}\right) + \frac{\mu_0 I}{4R}\left(\frac{1}{4}\right) = \frac{7\mu_0 I}{16R}$.

28. Two long wires carrying current are kept crossed (not joined at O). The locus where magnetic field is zero is

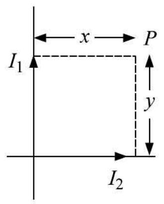
(a) $I_1 = \frac{x}{y} I_2$ (b) $I_1 = \frac{y}{x} I_2$
(c) $I_1 = I_2$ (d) $I_1 = -I_2$



(a)

Solution (a) Magnetic field could the zero in 1st or 3rd quadrant.

$$\frac{\mu_0 I_1}{2\pi x} = \frac{\mu_0 I_2}{2\pi y} \quad \text{or} \quad I_1 = \frac{x}{y} I_2.$$



(b)
Fig. 24.41

29. A long solenoid has magnetic field strength 3.14×10^{-2}T inside it when a current of 5A passes through it. The number of turns in 1 m of the solenoid is

(a) 1000 (b) 3000
(c) 5000 (d) 10000

Solution (c) $n = \frac{B}{\mu_0 I} = \frac{3.14 \times 10^{-2}}{4\pi \times 10^{-7} \times 5}$

$$= \frac{10^5}{20} = 5000.$$

Problems for Practice

1. The vector form of Biot Savart law is

(a) $\vec{dB} = \frac{Ki\,\vec{dl} \times \vec{r}}{r^2}$ (b) $\vec{dB} = \frac{Ki\,\vec{dl} \times \vec{r}}{r}$
(c) $\vec{dB} = \frac{Ki\,\vec{dl} \times \vec{r}}{r^3}$ (d) $\vec{dB} = \frac{Ki\,\vec{dl} \times \vec{r}}{r}$

2. The magnetic induction at the centre of a circular current carrying coil having n turns of radius a will be

(a) $\frac{2\mu_0 ni}{a}$ (b) $2 \times 10^{-7} \times \frac{\pi ni}{a}$
(c) zero (d) $\mu_0 ni$

3. The ratio of magnetic field on the axis of a circular current carrying coil of radius r to the magnetic field at its centre will be

(a) $\frac{1}{(1 + x^2/r^2)^{3/2}}$ (b) $\frac{1}{(1 + r^2/x^2)^3}$
(c) $\frac{1}{(1 + r^2/x^2)^2}$ (d) $\frac{1}{(1 + r^2/x^2)^{1/2}}$

4. The ratio of magnetic inductions at the centre of a circular coil of radius r and on its axis at a distance equal to its radius, will be

(a) $2\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$
(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

5. The phenomenon of production of magnetic field on passing an electric current in a straight conducting wire is based on the law of

(a) coulomb (b) oersted
(c) ampere (d) faraday

6. The radius of each of the coils of a Helmholtz galvanometer is 0.1 m and number of turns in each is 25. When a current is passed in it then the deflection of magnetic needle observed was 45°. If the horizontal component of earth's magnetic field is 0.314×10^{-4}Tesla, then the value of current will be

(a) 0.07A (b) 0.14A
(c) 0.28A (d) 0.42A

7. The magnetic induction due to a long straight conductor at a distance 2 cm from it is 10^{-6}Tesla. The current flowing in the conductor will be

(a) 10A (b) 1A (c) 0.1A (d) zero

8. A current of 50 ampere is flowing in a copper wire of diameter 0.5×10^{-2}m. The magnetic induction on its surface will be

(a) 4×10^{-2}T (b) 4×10^{3}T
(c) 4×10^{2}T (d) 4×10^{-3}T

9. The magnetic lines of force due to straight current carrying conductor are

(a) elliptical lines (b) circular lines
(c) straight lines (d) concentric lines

10. An electric current is flowing in a circular coil of radius r. At what distance from the centre on the axis of the coil will the magnetic field be 1/8th of its value at the centre?

(a) $r/\sqrt{3}$ (b) $\sqrt{3}r$
(c) $r/3$ (d) $3r$

11. Magnetic induction B exists as shown in Fig. 24.42 on the other side of line. A charged particle of charge q, mass m enters the magnetic field at 45°. The displacement of the particle when it emerges out of the magnetic field will be (velocity of the particle is v.)

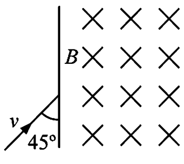


Fig. 24.42

(a) $\frac{\sqrt{2}mv}{qB}$ (b) $\frac{mv}{qB}$

(c) $\frac{mv}{\sqrt{2}qB}$ (d) $\frac{\sqrt{3}mv}{qB}$

12. A current i is flowing in a circular conductor of radius r. It is lying in a uniform magnetic field B such that its plane is normal to B. The magnetic force acting on the loop will be

(a) irB (b) $2\pi irB$ (c) πirB (d) zero

13. The magnetic induction at the centre of square frame of a conducting wire, when its diagonally opposite corners are connected to the battery, will be

(a) $\frac{4\mu_0 i}{\pi a}$ (b) $\frac{\mu_0}{\pi a}$ (c) $\frac{2\mu_0 i}{\pi a}$ (d) zero

14. A ring of radius r is carrying a current i. The magnetic field B is always perpendicular to the ring as shown below. The force on the ring is

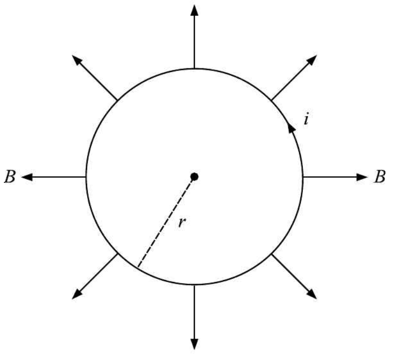


Fig. 24.43

(a) $2irB$ (b) $2\pi riB$

(c) irB (d) zero

15. Current i is flowing in a circular coil of radius r, then the magnetic induction at centre is B. If the current is doubled then the magnetic induction will be

(a) $2\sqrt{2}B$ (b) $2B$

(c) $4B$ (d) B

16. The length of a solenoid is 0.1 m and its diameter is very small. A wire is wound over it in two layers. The numbers of turns in the inner layer is 50 and that on the outer layer is 40. The strength of current flowing in two layers in the same direction is 3 ampere. The magnetic induction in the middle of the solenoid will be

(a) 3.4×10^3 Tesla (b) 3.4×10^{-3} Tesla

(c) 3.4×10^3 Gauss (d) 3.4×10^{-3} Gauss

17. The magnetic induction due to a current i passed in a straight conductor at a distance d from it is proportional to

(a) i/d (b) d^2 (c) i/d^2 (d) d

18. Uniform electric and magnetic fields are produced in the same direction. An electron moves in such a way that its velocity remains in the direction of electric field. The electron will

(a) get accelerated (b) turn towards right

(c) get decelerated (d) turn towards

19. The work done in deflecting a bar magnet of magnetic moment M through an angle α in a magnetic field H will be

(a) $MH(1 - \cos\alpha)$

(b) $MH \sin\alpha$

(c) $MH \cos\alpha$

(d) $MH[1 + \cos\alpha]$

20. Current i is flowing in a long straight conducting wire. The magnetic induction at a distance r from it is 0.4 Tesla, then its value at double the distance will be

(a) 0.1T (b) 0.8T (c) 1.6T (d) 0.2T

21. If the currents in two straight current carrying conductors, distant d apart, are i_1 and i_2 respectively in the same direction then they will

(a) rotate about a central axis

(b) attract each other

(c) repel each other

(d) neither attract nor repel each other

22. If the angular momentum of an electron is J then its magnetic moment will be

(a) $2m/eJ$ (b) $eJ/2m$ (c) $2Jem$ (d) eJ/m

23. The correct curve between the magnetic field B due to a long straight current carrying conductor and distance r from it will be

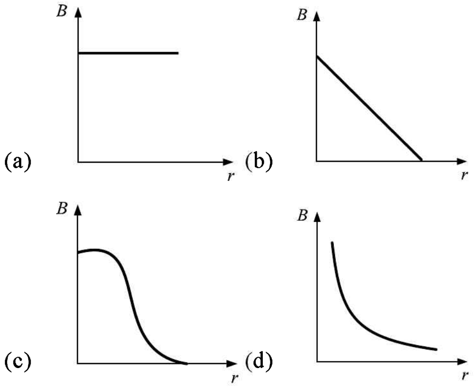


Fig. 24.45

24. A current of 0.1 ampere is flowing in an air cored solenoid of length 1 m and number of turns 10^3. The magnetic flux density inside it will be

(a) zero (b) $4\pi \times 10^{-5}$ T

(c) $4\pi \times 10^{-7}$ T (d) $4\pi \times 10^{-4}$ T

25. The distance between two thin long straight parallel conducting wires is b. On passing the same current i in them, the force per unit length between them will be

(a) zero (b) $\frac{\mu_0 i^2}{2\pi b}$

(c) $\frac{\mu_0 i}{2\pi b}$ (d) $\frac{\mu_0 i}{2\pi}$

26. A current of 10A is flowing in a wire of length 1.5 m, when it is placed in a uniform magnetic field of 2 Tesla, then a force of 15N acts on it. The angle between the magnetic field and the direction of current flow will be

(a) 90° (b) 45° (c) 60° (d) 30°

27. An electron is revolving in a circular path of radius 2.0×10^{-10} m with a uniform speed of 3×10^6 ms^{-1}. The magnetic induction at the centre of the circular path will be

(a) 0 (b) 1.2×10^{-6} T

(c) 0.6T (d) 1.2T

28. A charged particle having a charge q, is moving at right angles to a magnetic field. The quantity which varies is

(a) path of motion (b) kinetic energy

(c) speed (d) angular velocity

29. A wire is lying parallel to a square coil. Same current is flowing in same direction in both of them. The magnetic induction at any point P inside the coil will be

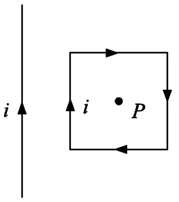


Fig. 24.45

(a) equal to that produced only by the coil
(b) more than that produced only by the coil
(c) zero
(d) less than that produced only by the coil

30. The velocities of two identical particles entering a uniform magnetic field are in the ratio 1 : 3. Their path becomes circular in the magnetic field. The ratio of the radii of their circular paths will be
(a) 3 : 1 (b) 9 : 1 (c) 1 : 9 (d) 1 : 3

31. The magnetic field inside a solenoid is
(a) zero (b) non-uniform
(c) infinite (d) uniform

32. The wall of a straight tube of infiinte length is thin. On passing current *i* through it, the value of magnetic induction inside the tube will be
(a) $2r/i$ (b) $2i\mu_0/r$ (c) $2i/r$ (d) zero

33. Which of the following quantities is not affected by a magnetic field?
(a) moving charge
(b) change in magnetic flux
(c) current flowing in a conductor
(d) stationary charge

34. Two parallel wires each of length 0.5 m, are lying at a distance of 1 m. If the current flowing in each wire is 1 ampere then the force between them will be
(a) 10^7N (b) 0.5N
(c) 10^{-7}N (d) 2×10^{-7}N

35. Find the magnetic field at *O*. The central part is a semicircle in the figure below and straight part is long.

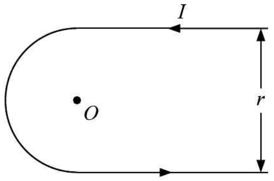


Fig. 24.46

(a) $B = \frac{\mu_0 i}{4r}$ (b) $\frac{\mu_0 i}{4r}(1+2\pi)$
(c) $\frac{\mu_0 i}{4\pi r}(\pi-2)$ (d) $\frac{\mu_0 i}{4\pi r}(\pi+2)$

36. A current *i* is flowing in a conductor as shown in the Fig. 24.47. The magnetic induction at point *O* will be

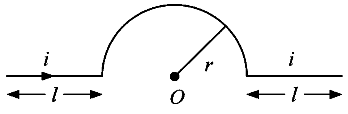


Fig. 24.47

(a) zero (b) $\alpha\mu_0 i/r$
(c) $\mu_0 i/4r$ (d) $\mu_0 i/r$

37. A long cylindrical wire of diameter *d* carrier a current *i*. The magnetic field at a distance $x < d/2$ will be

(a) $\frac{\mu_0 i x}{2\pi d^2}$ (b) $\frac{\mu_0 i x}{\pi d^2}$
(c) $\frac{\mu_0 i}{2\pi d}$ (d) $\frac{2\mu_0 i x}{\pi d^2}$

38. When a charged particle moves at right angles to a magnetic field then which of the following quantities change?
(a) momentum (b) speed
(c) energy (d) all of the above

39. Two particles *P* and *Q*, carrying same charge, after being accelerated through same potential difference, enters a uniform magnetic field. If these particles move in circular paths of radii r_1 and r_2 respectively, then the ratio of masses of *P* and *Q* will be

(a) $\sqrt{r_2/r_1}$ (b) r_1/r_2
(c) r_2/r_1 (d) $(r_1/r_2)^2$

40. A 0.5 m long straight wire in which a current of 1.2 A is flowing is kept at right angles to a uniform magnetic field of 2.0 Tesla. The force acting on the wire is
(a) 3N (b) 2.4N (c) 2N (d) 1.2N

41. A current of 2A exists in a square loop of side 10 cm. The magnetic field of the centre of the loop is
(a) 14.6 μT (b) 17.6 μT
(c) 20.4 μT (d) 22.6 μT

42. Consider a cable, the inner wire of which has radius *a* surrounded by an outer shell of inner and outer radii '*b*' and '*c*' raspectively. The inner wire carries a current of i_0. The outer shell carries the same current but in the opposite direction. Find the magnetic field at $a < x < b$

(a) $\frac{\mu_0 i_0}{2\pi x}$ (b) $\frac{\mu_0 i_0}{2\pi a} - \frac{\mu_0 i_0}{2\pi b}$
(c) $\frac{\mu_0 i_0 x}{2\pi(a^2+b^2)}$ (d) $\frac{\mu_0 i_0 x}{2\pi ab}$

43. A proton and an electron with same momenta, enter a magnetic field in a direction at right angles to the lines of force. If the radii of their circular paths are r_p and r_c respectively then the value of $r_p : r_c$ will be
(a) 4 : 1 (b) 1 : 2 (c) 2 : 1 (d) 1 : 1

44. A long straight wire carries a current *i*. A particle of charge +*q* and mass *m* is projected with a speed *v* from a distance x_0 as shown below. The minimum separation between the wire and the particle is

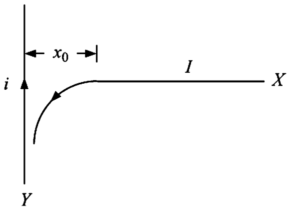


Fig. 24.48

(a) $x_0 e^{\frac{-2\pi mv}{\mu_0 qi}}$ (b) $x_0 e^{\frac{-mv\,2\pi x_0}{\mu_0 qi}}$
(c) $x_0 e^{\frac{-2\pi mv}{qi}}$ (d) zero

45. The ($\tau - \theta$) graph for a coil is

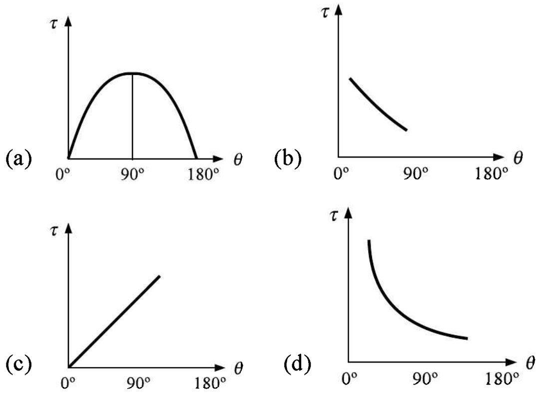


Fig. 24.49

46. The magnetic induction of centre O due to the arrangement shown in the figure below

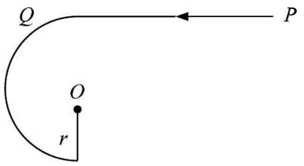


Fig. 24.50

(a) $\frac{\mu_0 i}{r}$ (b) $\frac{\mu_0 i}{4\pi r}$

(c) $\frac{\mu_0 i}{4\pi r}(1-\pi)$ (d) $\frac{\mu_0 i}{4\pi r}(1+\pi)$

47. A magnetic needle placed in a non-uniform magnetic field experience
(a) only torque
(b) only force
(c) force and torque
(d) neither force nor torque

48. An electron of energy 10 eV is revolving in a plane at right angles to a uniform magnetic field of 10^{-4} Tesla. The radius of circular path will be
(a) 105.76 m
(b) 10.57 cm
(c) 1.057 cm
(d) 0.10 cm

49. A magnetic field directed in north direction acts on an electron moving in east direction. The magnetic force on the electron will act
(a) vertically upward
(b) toward the east
(c) vertically downwards
(d) towards the north

50. The number of turns per unit length in a toroid is 10^3 and the current flowing in it is $1/4\pi$ ampere, then the magnetic induction produced in it is
(a) 10^{-2}T (b) 10^4T
(c) 10^3T (d) 10^{-4}T

51. On applying a uniform magnetic field on a current carrying coil, the coil rotates in such a way that its plane
(a) makes an angle 45° with the magnetic field
(b) becomes parallel to the magnetic field
(c) makes an angle of 45° with the magnetic field
(d) becomes perpendicular to the magnetic field

52. The use of Helmholtz coils is to produce
(a) non-uniform magnetic field
(b) varying magnetic field
(c) uniform magnetic field
(d) zero magnetic field

53. The work done by a normal magnetic field in revolving a charged particle q in a circular path will be
(a) $MB(1-\cos\theta)$ (b) $-MB$
(c) MB (d) zero

54. An α particle, a deutron and a proton are moving with same momentum in a uniform magnetic field. The ratio of their radii will be
(a) 1 : 1 : 1 (b) 4 : 2 : 1
(c) 2 : 2 : 4 (d) 1 : 2 : 2

55. Two wires carry currents of 100A and 200A respectively and they repel each other with a force of 0.4 Nm^{-1}. The distance between them will be
(a) 1 cm (b) 25 cm (c) 50 cm (d) 1 m

56. An electron is moving in a perpendicular magnetic field of strength 4×10^{-3} Tesla with a velocity of $4 \times 10^7 \text{ms}^{-1}$. The radius of electron path will be
(a) 0.056 m (b) 5.6 m
(c) 0.56 m (d) 56 m

57. A positive charge is moving towards an observer. The direction of magnetic induction will be
(a) towards right
(b) anti-clockwise
(c) towards left
(d) clockwise

58. The ratio of magnetic force (F_m) and electric force (F_e) acting on a moving charge is
(a) $(v/c)^2$ (b) v/c (c) $(c/v)^2$ (d) c/v

59. The value of B, at the points of inflexion in B-x curve is
(a) negative (b) maximum
(c) positive (d) constant

60. The correct expression for Lorentz force is
(a) $q(\vec{v}\times\vec{B})$ (b) $q[\vec{E}+(\vec{v}\times\vec{B})]$
(c) $q[\vec{E}+(\vec{B}\times\vec{v})]$ (d) $q\vec{E}$

61. A proton is moving with a velocity of $3 \times 10^7 \text{ms}^{-1}$ in the direction of a uniform magnetic field of 0.5 Tesla. The force acting on proton is
(a) zero (b) 2N (c) 4N (d) 6N

62. Gauss is the unit of
(a) M (b) m (c) B (d) H

63. A particle of mass m and charge q is projected into a region having a perpendicular magnetic field. The angle of deviation of the particle when it comes out of the magnetic field region of with $mv/2aB$
(a) $\pi/3$ (b) $\pi/2$
(c) $\pi/6$ (d) $2\pi/3$

64. Two parallel wires P and Q carry electric currents of 10A and 2A respectively in mutually opposite directions. The distance between the wires is 10 cm. If the wire P is of infinite length and wire Q is 2 m long, then the force acting will be
(a) zero
(b) 4×10^{-5}N
(c) 8×10^{-5}N
(d) 4×10^5N

65. In television picture tube the deflection system employs
(a) electric deflection
(b) magnetic deflection
(c) both electric and magnetic deflection
(d) none of these

66. An electron is moving vertically downards at any place. The direction of magnetic force acting on it due to horizontal component of earth's magnetic field will be
(a) towards west (b) towards south
(c) towards east (d) towards north

67. Consider the Fig. 24.51. The value of $\oint B.dl$ at P is and Q respectively,

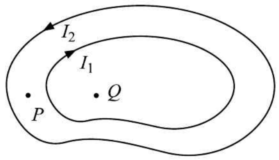


Fig. 24.51

ANSWERS

1. (c)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (d)	**9.** (b)	**10.** (b)
11. (a)	**12.** (d)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (d)
21. (b)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (b)	**26.** (d)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (d)
31. (d)	**32.** (d)	**33.** (d)	**34.** (c)	**35.** (d)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (d)	**40.** (d)
41. (d)	**42.** (a)	**43.** (d)	**44.** (a)	**45.** (a)	**46.** (d)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (d)
51. (d)	**52.** (c)	**53.** (d)	**54.** (d)	**55.** (a)	**56.** (a)	**57.** (b)	**58.** (b)	**59.** (d)	**60.** (b)
61. (a)	**62.** (c)	**63.** (c)	**64.** (c)	**65.** (b)	**66.** (a)	**67.** (a)			

Permanent Magnets

25

BRIEF REVIEW OF THE CONCEPTS

Magnetic dipole moment $(M) = ml$, where m is pole strength of the magnet also called magnetic charge. Magnetic poles are of two types: N-pole and S-pole.

$$M = ml = IA$$

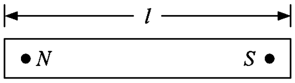


Fig. 25.1

where I is current and A is area. The unit of pole strength is A-m and unit of dipole moment is Am^2

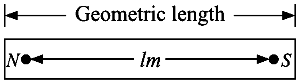


Fig. 25.2

$$\frac{\text{magnetic length}}{\text{geometric length}} = 0.84$$

Torque $\quad \tau = M \times B$

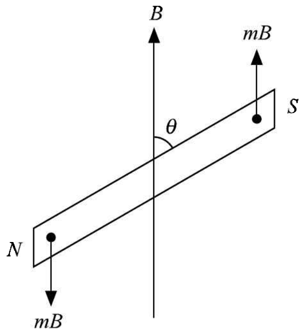


Fig. 25.3

$$F = mB$$

$$\oint B \cdot dS = 0 \quad \text{or} \quad \vec{\nabla} \cdot \vec{B} = 0$$

This is Gauss law and states that monopole cannot exist.

Magnetic field due to a bar magnet along axial line or on *end-on position.*

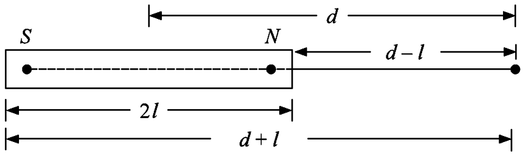


Fig. 25.4

$$B = \frac{\mu_0 2Md}{4\pi(d^2 - l^2)^2}$$

$$= \frac{2\mu_0 M}{4\pi d^3} \text{ due to a short magnetic dipole (or short magnet)}$$

Magnetic field due to a bar magnet along equatorial line or on broadside on position.

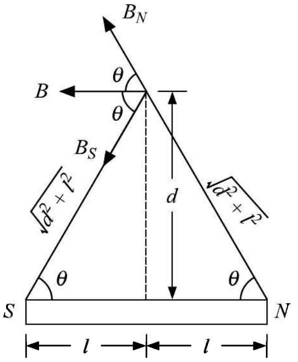


Fig. 25.5

$$B = \frac{\mu_0 M}{4\pi(d^2 + l^2)^{3/2}} \quad \text{(along } N \text{ to } S \text{ pole)}$$

$$= \frac{\mu_0 M}{4\pi d^3} \quad \text{(due to a short magnet)}$$

Dimensional formula of $B = [\mathrm{ML}^{\circ}\mathrm{T}^{-2}\mathrm{A}^{-1}]$.

Unit of magnetic field is Tesla (T) or $\mathrm{Wb/m}^2$ (SI) or Gauss (CGS).

Magnetic potential (V)

$$B = -\frac{dV}{dr} \quad \text{or} \quad V = \int_{\infty}^{r} -B \cdot dr = \frac{\mu_0 m}{4\pi r}.$$

Magnetic potential at any point P due to a bar magnetc

$$V = \frac{\mu_0 M \cos\theta}{4\pi(r^2 - l^2\cos\theta)}$$

$$= \frac{\mu_0 M \cos\theta}{4\pi r^2} \quad \text{(due to a short magnet)}$$

Special cases

(a) Magnetic potential along axial line (end-on position) $= \dfrac{\mu_0 M}{4\pi r^2}$ $(\theta = 0 \ \cos\theta = 1)$.

(b) Magnetic potential along equatorial line = 0.

Magnetic field strength due to a magnetic dipole (bar magnet) at any point P

$$B = \frac{\mu_0 M}{4\pi r^3}\sqrt{1 + 3\cos^2\theta}$$

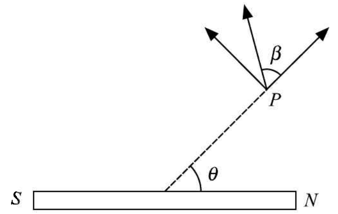


Fig. 25.6

$$\tan\beta = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2}.$$

Magnetic lines of force make a closed loop. They start from N-pole and end at S-pole outside the magnet and inside the magnet S-pole to N-pole as shown in the figure.

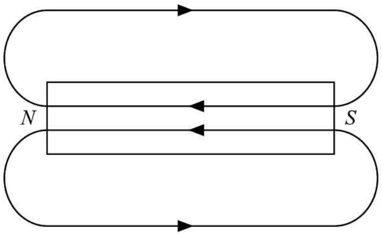


Fig. 25.7

Magnetizing field H $= B/\mu_0$ in vacuum. Dimensional formula is $[M^0L^{-1}T^0A^1]$.

The electric current enclosed in a closed path of unit length in a magnetic field is defined as magnetising field H, that is, $H = iL$ when $L = 1$ m, $H = i$.

Magnetic flux density or magnetic induction B

$$B = \phi/A \quad \text{or} \quad B = \phi \text{ if } A = 1\,\text{m}^2$$

That is, magnetic lines of force passing through unit normal area in a magnetic field is defined as magnetic induction.

The direction in which a current carrying conductor in a magnetic field experiences no force is the direction of magnetic induction.

Magnetic moment of an electron due to its orbital motion is M_{orbital} and due to its spin is M_{spin}.

$$M_{\text{orbital}} = -\left(\frac{eh}{4\pi m_l}\right).$$

$$M_{\text{spin}} = M_S = \frac{e}{m}S = \frac{eh}{2\pi m}$$

Thus, $M_S = 2M_L$.

where μ_B is Bohr magneton.

$$\mu_B = \frac{eh}{4\pi m} = 0.93 \times 10^{-23}\text{Am}^2.$$

About 90% of magnetic moment is due to spin motion and only 10% is due to orbital motion.

The magnetic dipole moment of earth is 8×10^{22} J/T. The magnetic axis makes 11.5° with axis of rotation of earth.

The point where the dipole axis cuts near N-pole and the other near S-pole is termed as the geomagnetic north pole and the geomagnetic south pole respectively.

Earth's magnetic field changes both in magnitude and direction with passage of time. It is believed that earth's magnetic field has reversed 171 times in the past 7.6×10^7 years. The latest reversal occured 10,000 years ago.

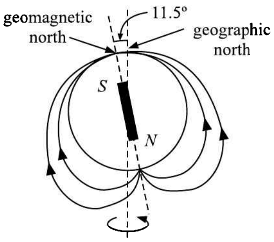


Fig. 25.8

Components of earth's magnetic field declination, dip and horizontal component of the field are three components of earth's magnetic field.

Declination (θ) Angle between the geographic meridian and the magnetic meridian is called declination (θ). The knowledge of declination fixes the vertical plane in which the earth's magnetic field lies.

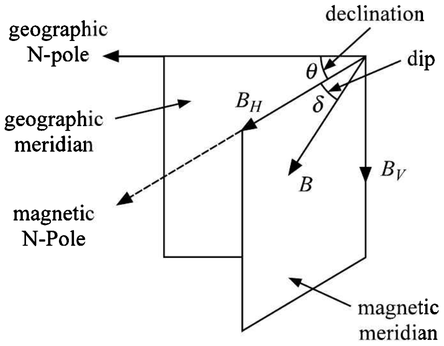


Fig. 25.9

Dip (δ) The angle made by the earth's magnetic field with the horizontal direction in the magnetic meridian is called the *inclination* or *dip*.

In the northern hemisphere north pole of compass needle (or magnetic needle) dips downwards and in the sourthern hemisphere S-pole of the compass needle dips downwards.

$B_H = B\cos\delta$ and $B_V = B\sin\delta$

$\tan\delta = B_V/B_H$

Magnetic elements help to understand the location of a place. This is why magnetic maps are used in navigation and aviation.

Isogonic lines are lines that join same declination in a magnetic map.

Isoclinc lines join same dip in magnetic maps.

Isodynamic lines join same B_H (horizontal component of earth's magnetic field) in magnetic maps.

Aclinic line is called the magnetic equator or line joining zero dip. At poles angle of dip is 90°.

Angle of dip is measured using dip circle. If dip circle is set in magnetic meridian then the mean of the reading of the ends of the magnetic needle on vertical scale is angle of dip.

If the dip circle is inclined at an angle θ with, magnetic meridian then $\tan\delta' \cos\theta = \tan\delta$ where δ' is apparent dip and δ is true dip.

If dip circle makes an angle θ with magnetic meridian and it is unknown, then rotate the dip circle by 90° and note another apparent dip δ''' at the rotated position then true dip is $\cot^2\delta = \cot^2\delta' + \cot^2\delta''$

Neutral points The points where magnetic field due to a magnet is equal and opposite to the earth's horizontal field. Compass needle can stay at any position at such a point.

When N-pole of the magnet points to the S-pole of the earth, neutral point occurs on end-on position or on the axial line.

When N-pole of the magnet points to the N-pole of the earth, neutral points occur on broad side on position or on equatorial line.

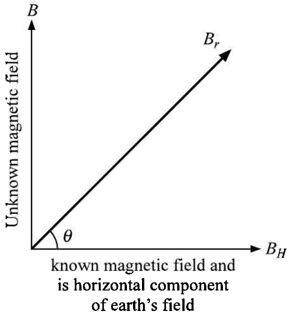


Fig. 25.10

Tangent law If two fields are perpendicular and one of them is known. Then the other can be determined.

From the Fig. 25.10, $B_H = B_r \cos\theta$ and $B = B_r \sin\theta$

Therefore $B = B_H \tan\theta$.

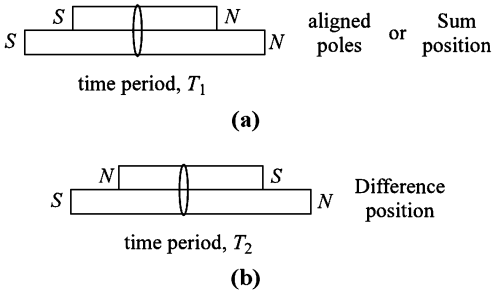
Tangent galvanometer Principle used is tangent law. When a current i is passed through the coil (of n turns) of the tangent galvanometer then $i = k\tan\theta$ or $k = \dfrac{2r\,B_H}{\mu_0 n}$

where, r is radius of the coil and k is reduction factor.

Sensitivity of the tangent galvanometer is maximum when $\theta = 45°$.

Comparison method

To find M (Magnetic dipole moment) When two magnets having magnetic dipole moments are aligned (sum position) time period of vibration is T_1 and, when N-pole of one is along S-pole of other (difference position) time period is T_2 then



(a)

(b)

Fig. 25.11

$$\frac{M_1}{M_2} = \frac{T_2^2 + T_1^2}{T_2^2 - T_1^2}$$

SHORT-CUTS AND POINTS TO NOTE

(i) Magnetic field intensity due to a magnetic pole of pole strength m at a distance r from it

$$B = \frac{\mu_0 m}{4\pi r^2} \quad \text{in SI system.}$$

$$= \frac{m}{r^2} \quad \text{in CGS system.}$$

(ii) Magnetic field intensity at any point due to a dipole (magnetic) of dipole moment M due to a short magnet

$$B = \frac{\mu_0 M}{4\pi r^3}\sqrt{3\cos^2\theta + 1} \quad \text{and} \quad \tan\beta = \frac{\tan\theta}{2}.$$

(iii) Torque experienced by a magnet suspended in magnetic field

$\tau = M \times B = MB\sin\theta$.

(iv) Work done $W = \int_0^{\theta} \tau \cdot d\theta = MB(1 - \cos\theta)$.

(v) Potential energy $U = -\vec{M}\cdot\vec{B}$ = work done.

Change in potential energy

$$U[\theta] - U(\theta_0) = MB(\cos\theta_0 - \cos\theta).$$

(vi) If a magnetic dipole is suspended in two mutually perpendicular magnetic fields then it orients itself making an angle θ with the horizontal magnetic field B_H.

Then $B/B_H = \tan\theta$ and is called tangent law, where B is magnetic field perpendicular to B_H.

(vii) If two magnets are lying mutually perpendicular then

$$M' = \sqrt{M_1^2 + M_2^2}.$$

$$M' = \sqrt{2}M \quad \text{if} \quad M_1 = M_2 = M.$$

(viii) Magnetomotive force

$$F_m = \oint H\cdot dl = \frac{1}{\mu_0}\oint B\cdot dl = \Sigma i$$

unit ampere turns

(ix) In the figure I_1, I_2 and I_3 are currents inside the loop and I_4 and I_5 are outside the loop then

$$\frac{1}{\mu_0}\oint B\cdot dl = I_1 + I_3 - I_2.$$

This is called the Ampere circuital law

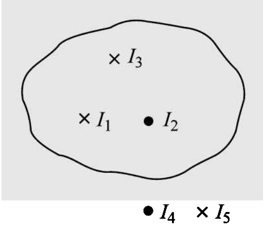


Fig. 25.12

(x) In tangent galvanometer $I = K\tan\theta$ $K = \dfrac{2B_H r}{\mu_0 n}$ is the reduction factor. Sensitivity of the galvanometer is maximum when $\theta = 45°$.

Note at $\theta = 45°$ $K = I$.

(xi) Vibration magnetometer

$$T = 2\pi\sqrt{\frac{I}{MB_H}}.$$

(xii) When a magnet is brought closer and its magnetic field is B'

then $$T = 2\pi\sqrt{\frac{I}{M(B_H \pm B')}}$$

if T_1 is time period of vibration in sum position and T_2 is time period of virbation in difference position then

$$\frac{M_1}{M_2} = \frac{T_2^2 + T_1^2}{T_2^2 - T_1^2}.$$

(xiii) If δ' is apparent dip when dip circle makes an angle θ with the magnetic meridian, then true dip δ is given by $\tan\delta = \tan\delta'\cos\theta$.

When θ is also unknown, rotate the dip circle by 90°, the new apparant dip at 90° rotation is δ'' then $\cot^2\delta = \cos^2\delta' + \cot^2\delta''$.

(xiv) For shielding a certain region from magentic field, surround the region in soft iron rings.

(xv) At poles total intensity = 0.66 Oersted and at equator it is 0.33 Oersted.

Thus $I_{\text{pole}} = 2\,I_{\text{equator}}$.

Total intensity of earth's field

$$I = I_0\sqrt{1+3\sin^2\lambda} \quad \text{where } I_0 = M/R^3.$$

At equator $\lambda = 0$, thus, $I = I_0$. At poles $I = 2I_0$. as $\lambda = 90°$.

In a limited region magnetic lines of force of earth's field are parallel, that is, we can consider uniform field.

CAUTION

(i) Considering that when a magnet is cut pole strength does not vary.

- If a magnet is cut in a vertical plane as illustrated then pole strength does not vary. However if we cut the magnet along its axis in the a horizontal plane then pole strength varies.

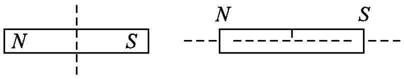


Fig. 25.13

(ii) Considering that physical length and magnetic length of a bar magnet are equal.

- Since poles lie slightly inside, magnetic length is less than physical length (= 0.84 l_{physical})

(iii) Considering that angle of dip can be obtained from dip circle irrespective of its orientation.

- Angle of dip is obtained only when dip circle is set in the magnetic meridian otherwise true dip is found using $\tan\delta = \tan\delta'\cos\theta$ or $\cot^2\delta = \cot^2\delta' + \cot^2\delta''$.

(iv) Not knowing tan A and tan B settings of Deflection magnetometer.

- tan A setting — Magnetic needle points N–S and arms and pointer in the E–W and pointer coincides with 0-0. tan B setting — Magnetic needle and Arms point N–S and pointer in the E–W and pointer coincides with 0-0.

(v) Considering magnetic intensity of earth is same at all points.

- Magnetic intersity changes with lattitude and is given by $I = I_0\sqrt{1+3\sin^2\lambda}$. At poles $I = 2I_0$ and at equator $I = I_0$. At poles the total intensity is 0.66 Oersted and at equator it is 0.33 Oersted.

(vi) Considering in vibration magnetometer, $\frac{M_1}{M_2} = \frac{T_1^2}{T_2^2}$ where T_1 and T_2 are time periods in sum and difference positions.

- $\frac{M_1}{M_2} = \frac{T_1^2+T_2^2}{T_2^2-T_1^2}$.

Solved Problems

1. A small magnet is suspended freely by a vertical string. The horizontal and vertical component of earth's field is 20 μT and 30 μT respectively find the orientation in which magnet will stay in equilibrium.

(a) $\tan^{-1}(3/2)$ (b) $\tan^{-1}(2/3)$

(c) $\sin^{-1}(2/3)$ (d) $\cos^{-1}(2/3)$

Solution (a) $\tan\theta = 30/20$

2. Two identicals short magnetic dipoles are placed as shown separated by a distance d. The magnetic field midway between the dipoles is

(a) $\frac{\mu_0 M\sqrt{2}}{4\pi d^3}$ (b) $\frac{\mu_0 M\sqrt{5}}{4\pi d^3}$

(c) $\frac{2\mu_0 M\sqrt{5}}{\pi d^3}$ (d) none of these

Solution (c) $B = \sqrt{B_1^2 + B_2^2}$

$$= \frac{\mu_0 M}{4\pi(d/2)^2}\sqrt{1^2+2^2}$$

$$= \frac{2\mu_0 m\sqrt{5}}{\pi d^3}$$

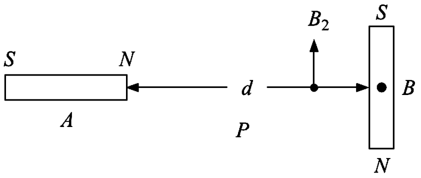


Fig. 25.14

3. A 10 cm long magnet has pole strength 12 Am. The magnetic field at a distance 20 cm from it is

(a) 2.4×10^{-5} T

(b) 3.4×10^{-5} T

(c) 3.8×10^{-5} T

(d) 3.0×10^{-5} T

Solution (b) $B = \frac{2\mu_0 Md}{4\pi(d^2-l^2)^2} = \frac{2\times 4\pi\times 10^{-7}\times(0.2)}{4\pi(0.2^2-0.05^2)^2}$

$$= 3.4\times 10^{-5}\,\text{T}$$

4. A bar magnet of dipole moment 10^4 J/T is free to rotate in a horizontal plane. A horizontal magnetic field 4×10^{-5} T exists in the space. Find the work done in rotating the magnet slowly from a direction parallel to field to a direction 60° from the field.

(a) 0.1 J (b) 0.2 J

(c) 0.4 J (d) 0.5 J

Solution (c) $W = MB(\cos\theta - \cos 60^\circ)$

$$= \frac{1}{2}MB = \frac{1}{2}\times 4\times 10^{-5}\times 10^{4} = 0.2\,\text{J}$$

5. A bar magnet of dipole moment M is bent to form a semicircle. Find new dipole moment.

(a) $2M/\pi$ (b) $\pi M/2$

(c) $M\pi$ (d) $2M$

Solution (a) $\pi r = l$ or $r = l/\pi$; new dipole moment

$$M' = m(2r) = \frac{m(2l)}{\pi} = 2M/\pi$$

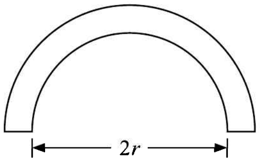


Fig. 25.15

6. Find the magnetic field due to a dipole of magnetic moment 1.2Am² at a point 1 m away from it and making an angle 60° with axis of the dipole

(a) 1.6×10^{-7}T, $\tan^{-1}\sqrt{3}$

(b) 1.6×10^{-7}T, 30°

(c) 1.6×10^{-7}T, $\tan^{-1}\sqrt{3}/2$

(d) none of these

Solution (c) $B = \frac{\mu_0 M}{4\pi q^3}\sqrt{3\cos^3\theta + 1}$

$$= \frac{1.2\times 10^{-7}\times 4\pi}{4\pi\times(1)^3}\sqrt{3(1/2)^2+1}$$

$$= 1.6 \times 10^{-7}\,\text{T}$$

$$\tan\beta = \frac{\tan\theta}{2} = \frac{\sqrt{3}}{2}.$$

7. Horizontal component of earth's field is 3×10^{-3}T and of dip is 53°. Find the magnetic field of earth at that place.

(a) 4×10^{-5}T (b) 3×10^{-5}T

(c) 5×10^{-5}T (d) 1×10^{-5}T

Solution (c) $B_A = B\cos\delta$

or $$B = \frac{B_H}{\cos\delta} = \frac{3\times 10^{-5}}{3/5} = 5 \times 10^{-5}\,\text{T}$$

8. At 45° to the magnetic meridian, apparent dip is 30°. Find true dip

(a) $\tan^{-1}1/\sqrt{2}$ (b) $\tan^{-1}\sqrt{6}$

(c) $\tan^{-1}\sqrt{2}$ (d) $\tan^{-1}1/\sqrt{6}$

Solution (d) $\tan\delta = \tan\delta' \cos 45 = \frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{6}}$.

9. A tangent galvanometer has 28 turns and diameter of the coil 22 cm. When a current of 0.2A is passed it gives a deflection of 45°. Evaluate horizontal component of the earth's field.

(a) 1.6×10^{-5}T (b) 2.6×10^{-5}T

(c) 1.6×10^{-6}T (d) 1.6×10^{-4}T

Solution (a) $i = k\tan\theta$

or $$i = \frac{2rB_H}{\mu_0 n}\tan\theta$$

or $$B_H = \frac{i\mu_0 x}{2r\tan\theta} = \frac{0.2\times 4\pi\times 10^{-7}\times 28}{2\times 22\times 10^{-2}} = 1.6 \times 10^{-5}\,\text{T}$$

10. When two magnets are placed 15 cm and 20 cm away from a deflection magnetometer on two arms, no deflection is observed. The ratio of magnetic dipole moments is

(a) 3/4 (b) 9/16

(c) 27/64 (d) 91/256

Solution (c) $\frac{M_1}{M_2} = \left(\frac{d_1}{d_2}\right)^3 = \left(\frac{15}{20}\right)^3 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$.

11. A compass needle oscillates 20 times per minute at a place where dip is 45° and 30 times per minute where the dip is 30°. Compare the total magnetic field of earth at two places.

(a) 1.51 (b) 1.83

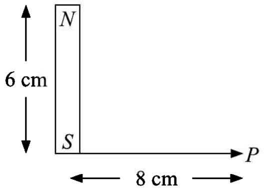
(c) 1.63 (d) 1.23

Solution (b) $\frac{B_{H_2}}{B_{H_1}} = \frac{T_1^2}{T_2^2}$

$$\Rightarrow \frac{B_2\cos\delta_2}{B_1\cos\delta_1} = \frac{T_1^2}{T_2^2} = \left(\frac{3}{2}\right)^2\frac{1/\sqrt{2}}{\sqrt{3}/2}$$

$$= \frac{9}{4}\times\sqrt{\frac{2}{3}} = \frac{3\sqrt{3}}{2\sqrt{2}} = 1.83$$

12. A bar magnet has pole strength 1Am and is placed as shown in Fig. 25.16. Find the magnetic field at P.

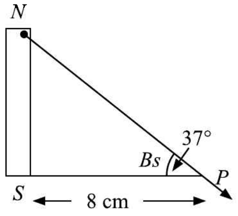


(a)

(a) 11 μT (b) 14 μT

(c) 19 μT (d) 22 μT

Solution (d) $B = \sqrt{B_S^2 + B_N^2 + 2B_SB_N\cos 143}$



(b)

Fig. 25.16

$$= \sqrt{B_S^2 + B_N^2 + 2B_SB_N\cos(90+53)}$$

$$= \frac{\mu_0 m}{4\pi}\sqrt{\left(\frac{1}{8\times 10^{-2}}\right)^4 + \left(\frac{1}{10\times 10^{-2}}\right)^4 - \frac{2\times 1}{(8\times 10^{-2})^2}\times\frac{1}{(10\times 10^{-2})^2}\sin 53}$$

$= 0.22 \times 10^{-4}\,\text{T} = 22\,\mu\text{T}$

13. A magnetic needle free to rotate in a fixed vertical plane stays at an angle 60° with the horizontal. If the dip at that place is 37°, find the angle of the fixed vertical plane with the meridian.

(a) $\theta = \cos^{-1}\frac{\sqrt{3}}{5}$ (b) $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

(c) $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\theta = \cos^{-1}\frac{2}{3}$

Solution (c) $\tan\delta_1 = \frac{B_V}{B_H\cos\theta} = \frac{\tan\delta}{\cos\theta}$

or $\cos\theta = \frac{\tan\delta}{\tan\delta_1} = \frac{\tan 37}{\tan 60} = \frac{3/4}{\sqrt{3}} = \frac{\sqrt{3}}{4}$

14. A dipole circle shows an apparent dip of 60° at a place where the true dip is 45°. If the dip circle is rotated by 90° what will be the apparent dip?

(a) $\cos^{-1}\sqrt{2}/3$ (b) $\tan^{-1}\sqrt{2}/3$

(c) $\sin^{-1}\sqrt{2}/3$ (d) $\cot^{-1}\sqrt{2}/3$

Solution (d) $\cos^2\delta_2 = \cot^2\delta - \cot^2\delta_1$

$= \cot^2 45 - \cot^2 60 = 1-\frac{1}{3}1 = 2/3$

$\delta_2 = \cot^{-1}\sqrt{2/3}$

15. A tangent galvanometer has a coil of 50 turns and a radius of 20 cm. The horizontal component of the earth's field is 3×10^{-5} TV. Find the current through the coil.

(a) 0.19A (b) 0.29A

(c) 0.39A (d) none of these

Solution (a) $\tan B = \frac{n\mu_0 I}{2r}$

$$I = \frac{2rB}{nm_0} = \frac{2(0.2)\times3\times10^{-5}}{50\times4\pi\times10^{-7}} = 0.19\text{A}.$$

16. A short magnet is executing oscillation in earth's horizontal magnetic field of 24 μT. The time for 20 vibrations is 2*s*. An upward electric current of 18A is established in a vertical wire placed 20 cm east of the magnet. The new time period is

(a) 0.1 s (b) 0.2 s

(c) 0.26 s (d) 0.16 s

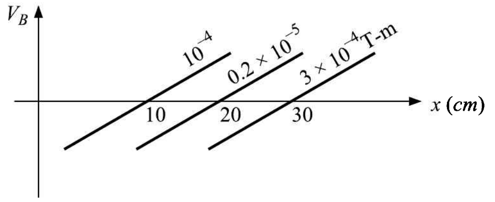
Solution (b) $B = \frac{\mu_0 I}{2\pi d} = \frac{4\pi\times10^{-7}\times18}{2\pi\times0.2} = 18\mu\text{T}$

South to North

$$\frac{T_{\text{new}}}{T} = \sqrt{\frac{B_H}{B_H - B}} = \sqrt{\frac{24}{24-18}} = 2$$

Thus $T_{\text{new}} = 2(0.1) = 0.2\,\text{s}$

17. Fig shows some of the equipotential surfaces of the magnetic scalar potential. The magnetic field at a point in the region is



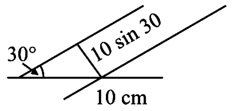
(a)

(a) 2×10^{-5}T (b) 10^{-5}T

(c) 10^{-4}T (d) 2×10^{-4}T

Solution (d) $B = -\frac{dV_B}{dx}$

$$= \frac{2\times10^{-5} - 1\times10^{-5}}{(0.1)\sin 30} = 2\times10^{-4}\text{T}$$



(b)

Fig. 25.17

18. A magnetic dipole of moment 0.96 Am² horizontally with the N-pole pointing south. The position of neutral point is .... if horizontal component of the earth's field is 24 μT.

(a) 10 cm (b) 8 cm

(c) 20 cm (d) 20 mm

Solution (c) $B_H = \frac{2\mu_0 M}{4\pi d^3}$

or $d^3 = \frac{2\mu_0 M}{4\pi B_H} = \frac{2\times4\pi\times10^{-7}\times0.96}{4\pi\times24\times10^{-6}}$

$= 8\times10^{-3}$ or $d = 0.2$ m.

19. If the magnitude of the earth's magnetic field at the equator is 3.3×10^{-5}T then find the magnitude of the field at the poles.

(a) 3.3×10^{-5}T (b) 6.6×10^{-5}T

(c) 8.4×10^{-5}T (d) none of these

Solution $B = B_0\sqrt{1+3\sin^2\lambda}$

$= 3.3\times10^{-5}\times2 = 6.6\times10^{-5}$T

Problems for Practice

1. A neutral point in a magnetic field is a point at which
   (a) the resultant magnetic intensity is zero
   (b) earth's field is zero
   (c) the magnetic field is zero
   (d) the magnetism is more
2. A current-carrying loop suspended freely in a uniform magnetic field will experience
   (a) torque only (b) beither torque nor force
   (c) force only (d) both
3. A bar magnet of magnetic dipole moment is cut into two halves by a horizontal plane. Which of the following statements is correct?
   (a) only pole strength becomes half
   (b) only magnetic dipole moment gets half
   (c) both dipole moment and pole strength become half
   (d) dipole moment remains unaltered.
4. A person is facing magnetic north. An electron in front of him flies horizontally towards the north and deflects towards east. He is in/at the
   (a) southern hemisphere
   (b) equator
   (c) northern hemisphere
   (d) none of these

5. The needle of the dip circle at a place stays at 30°. The dipcircle is inclined at an angle of 30° with the magnetic meridian. The true dip at the place is
(a) $\tan^{-1}\sqrt{3}$ (b) $\tan^{-1}\sqrt{3}/2$
(c) $\tan^{-1}1/2$ (d) $\tan^{-1}3/2$

6. The intensity of magnetic field at a distance d from an isolated pole of m units in air is
(a) md^2 (b) m/d (c) md (d) m/d^2

7. Isogonic lines on a magnetic map represent lines joining places with same
(a) B_H (b) dip
(c) zero dip (d) declination

8. If a bar magnet of magnetic moment m is deflected through an angle θ in a uniform magnetic field of induction B, the work done in doing so is
(a) $MB\sin\theta$ (b) MB
(c) $MB(1-\cos\theta)$ (d) $MB\cos\theta$

9. A current-carrying coil suspended freely in a uniform magnetic field will be in stable equalibrium, if the angle between its magnetic dipole moment vector and the magnetic field is
(a) 180° (b) zero (c) 45° (d) 90°

10. Sensitivity of tangent galvanometer is maximum when deflection is
(a) 45° (b) 90° (c) 30° (d) 04

11. Tan-A position means
(a) magnetic needle and arms point north-south and pointer east-west and coincides 0-0
(b) magnetic needle and pointer point north-south and arms are along east-west
(c) magnetic needle points north-south and arms and pointer east-west are pointer coincides 0-0
(d) magnetic needle points north-south arms and pointer east-west and pointer coincides 90-90°

12. Earth's magnetic field inside a closed iron box, as compared to that outside is
(a) same (b) less (c) more (d) zero

13. Reduction factor in a tangent galvanometer is related to number of turns as
(a) n (b) n^2 (c) $n^{-1/2}$ (d) n^{-1}

14. Gauss law in magnetism concludes that
(a) monopole do not exist
(b) magnetic flux can not be determined
(c) $\nabla \times B = 0$
(d) $\nabla \cdot B = 0$

15. The time period of vibration of a magnetic dipole is T. On bringing a bar magnet closer along the axis of vibration, the new time period T' will be
(a) $T' > T$ (b) $T' < T$
(c) $T' = T$ (d) depends which pole faces

16. When two magnets are placed 20 cm and 15 cm away on the two arms of a deflection magnetometer, it shows no deflection. The ratio of magnetic moments is
(a) $\frac{M_1}{M_2} = \frac{64}{27}$ (b) $\frac{M_1}{M_2} = \frac{4}{3}$
(c) $\frac{M_1}{M_2} = \frac{16}{9}$ (d) none of these

17. The angle of dip at a place where horizontal and vertical components of earth's magnetic field are equal, is
(a) 90° (b) 30° (c) 45° (d) 0°

18. Angle of dip at the magnetic equator is
(a) 0° (b) 45° (c) 90° (d) none of these

19. Magnetic field strength due to a short bar magnet on its axial line at a distance x is B. What is its value at the same distance on the equatorial line?
(a) $4B$ (b) $B/2$ (c) B (d) $2B$

20. The angle between the magnetic meridian and the geographical meridian is known as
(a) magnetic pole strength
(b) magnetic dip
(c) magnetic declination
(d) magnetic moment

21. A galvanometer gives full scale deflection when the current passed through it is 1 mA. Its resistance is 100 Ω. Without connecting additional resistance in series with it, it can be used as a voltmeter of range
(a) 1.000 V (b) 0.010 V
(c) 0.001 V (d) 0.100 V

22. The vertical component of earth's magnetic field is zero at
(a) magnetic equator (b) magnetic poles
(c) geographical poles (d) every where

23. A dip needle in a plane perpendicular to magnetic meridian will remain
(a) horizontal
(b) vertical
(c) inclined at 45° with horizontal
(d) in any direction

24. Which of the following is best suited to make the core of a transformers?
(a) soft iron (b) steel
(c) alinco (d) none of these

25. If dip circle is not set into magnetic meridian and the angle at which it is inclined to the magnetic meridian is unknown. Only δ' and δ'' are apparent dips at a place in which dip circle is kept in transverse positions the true dip δ is
(a) $\cot\delta = \cot\delta' + \cot\delta''$
(b) $\tan^2\delta' = \tan\delta' + \tan^2\delta''$
(c) $\cos^2\delta = \cos^2\delta' + \cos^2\delta''$
(d) $\cot^2\delta = \cot^2\delta' + \cot^2\delta''$

26. The unit of pole strength is
(a) Am² (b) Am⁻¹ (c) Am (d) Am⁻²

27. The magnetic moment of a dipole is ...... if l is magnetic length and m is pole strength.
(a) $M = 2ml$ (b) $M = ml$
(c) $2m/l$ (d) πml

28. The dipole moment of a magnet is M. Its time period of vibration is T. If the dipole moment of another magnet is $2M$ with same physical dimensions as the former. The time period of vibration will be
(a) $T/2$ (b) $T/\sqrt{2}$ (c) $\sqrt{2}T$ (d) $2T$

29. At a place the horizontal component of earth's field is B_H. The time period of vibration of a magnet is T. If we take it to a place where earth's horizontal component is $B_H/2$ then the time period will be
(a) T (b) $T/2$ (c) $T\sqrt{2}$ (d) $2T$

30. A coil (8 cm × 4 cm) carries a current 2A and has 200 turns. Find the magnetic dipole moment.
(a) 1.28 Am² (b) 1.28 Am²
(c) 0.64 Am² (d) 0.64 Am²

31. When *N*-pole points south of the earth the neutral points lie on
(a) axial line (b) equatorial line
(c) any of the lines (d) both axial and equatorial lines

32. Which of the following cannot be ferromagnetic?
(a) alloys (b) solids
(c) liquids (d) gases

33. Magnetic susceptibility χ of a paramagnetic material changes with absolute temperature T as
(a) $\chi \propto T^{-1}$ (b) χ = constant
(d) $\chi \propto e^T$ (d) $\chi \propto T$

34. Points A and B are situated on an axis perpendicular to a 2 cm long bar magnet at large distances x and $3x$ from its centre on opposite sides. The ratio of the magnetic field at A and B will be approximately equal to
(a) 27 : 1 (b) 2 : 9 (c) 9 : 1 (d) 1 : 9

35. Which of the following expressions is applicable to the moving coil galvanometer?
(a) $\vec{F}_m = q(\vec{V} \times \vec{B})$ (b) $B = B_0 \tan\theta$
(c) $\vec{i} = \vec{M} \times \vec{B} 1$ (d) none of these

36. The magnetic susceptibility of a paramagnetic substance is 3×10^{-4}. It is placed in a magnetising field of 4×10^3 Am^{-1}. The intensity of magnetisation will be
(a) 24Am^{-1} (b) 3×10^8Am^{-1}
(c) 12×10^8Am^{-1} (d) 1.2Am^{-1}

37. The value of relative magnetic permeability (μ_r) for ferromagnetic materials is
(a) $\mu_r = 1$ (b) $\mu_r >> 1$
(c) $\mu_r < 1$ (d) $\mu_r > 1$

38. A current of 1A is flowing in a coil of 10 turns and with radius 10 cm. Its magnetic moment will be
(a) 3140Am^2 (b) 100Am^2
(c) μ_0 Am^2 (d) 0.314Am^2

39. A bar magnet of length l and dipole moment M is bent to form a semicircle. The new dipole moment is
(a) $2M/2$ (b) M
(c) $\pi M/2$ (d) none of these

40. A current i is flowing in a conductor of length l. When it is bent in the form of a loop its magnetic moment will be
(a) $4\pi l^2 i$ (b) $il^2/4\pi$ (c) $4\pi/l^2 i$ (d) $l^2/4\pi$

41. Which of the following behaves as a bar magnet?
(a) a long wire carrying current
(b) a circular coil carrying current
(c) none of (a) and (b)
(d) both (a) and (b)

42. A rod of ferromagnetic material with dimensions 10 cm × 0.5 cm × 0.2 cm is placed in a magnetic field of strength 0.5×10^4Am^{-1} as a result of which a magnetic moment of 5Am^{-2} is produced in the rod. The value of magnetic induction will be
(a) 0.358 T (b) 0.54 T
(c) 6.28 T (d) 2.519 T

43. Angle of dip at a place is 60° and earth's magnetic field is $8\mu T$. Its horizontal component that is, B_H is
(a) $4\sqrt{3}\mu T$ (b) $4\mu T$
(c) $8\sqrt{3}/\mu T$ (d) $8\sqrt{3}\mu T$

44. The value of Bohr magneton is
(a) $e/4\pi m$ (b) $eh/(2\pi m)$
(c) $eh/(4\pi m)$ (d) $eh/(\pi m)$

45. Which of the following does not form magnetic element?
(a) B_H (b) B_V
(c) dip (d) declination

46. The value of magnetic susceptibility for paramagnetic substances is
(a) infinity (b) low positive
(c) low negative (d) zero

47. The resultant magnetic moment due to two current (i) carrying concentric coils of radius r, mutually perpendicular to each other will be
(a) $\sqrt{2}\, ir$ (b) $\sqrt{2}\, i\pi r^2$
(c) $2\pi r^2$ (d) $\sqrt{2}\, ir^2$

48. The lines joining same dip are called
(a) aclinic (b) isogonic
(c) isodynamic (d) isoclinic

49. The magnetic property inherent in all materials is
(a) paramagnetism (b) ferro magnetism
(c) diamagnetism (d) non-magnetism

50. The magnetic lines of force like electrical lines of force
(a) are closed (b) are not closed
(c) are open (d) are not open

51. Two magnetic lines of force
(a) never cut each other
(b) cut near the poles
(c) cut according to the position of magnet
(d) cut at a neutral point

52. The magnetic lines of force due to the horizontal component of earth's magnetic field will be
(a) circular (b) elliptical
(c) curved (d) horizontal and parallel

53. The magnetic susceptibility of a paramagnetic material at 73°C is 0.0075. Then its value at 173°C will be
(a) 0.0030 (b) 0.0075
(c) 0.0045 (d) 0.0015

54. In deflection magnetometer to find dipole moment M of a magnet, angle of deflection should be
(a) 0° (b) 90° (c) 45° (d) any angle

55. The dipole moment of a coil of area A and number of turns N and carrying current i will be
(a) iA/N (b) Ni/A
(c) NiA (d) N^2iA

56. The *M-T* curve for ferromagnetic material is

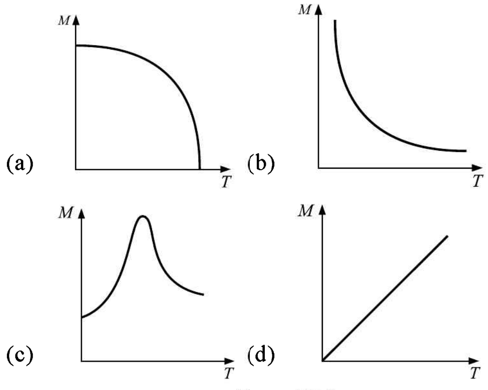


Fig. 25.18

57. 1 Weber is equivalent to
(a) 10^{-8} Maxwell (b) 10^{12} Maxwell
(c) 10^{8} Maxwell (d) 10^{4} Maxwell

58. The magnetic moment of a magnet of mass 75 g is 9×10^{-7} Am^2. If the density of the material of magnet is 7.5×10^3 kgm^{-3} then intensity of magnetisation will be
(a) 90 Am^{-1} (b) 0.09 Am^{-1}
(c) 0.9 Am^{-1} (d) 9 Am^{-1}

59. Soft iron is used for the construction of electromagnets because for iron
(a) area of hysteresis loop is more
(b) coercive force is high
(c) retentivity is high
(d) magnetic saturation limit is high and coercivity is low

60. The value of Curie temperature for Ni is nearly
(a) 631K (b) 770K (c) 890K (d) 1100K

61. The correct relation between B, H and I is
(a) $B = \mu_0 I \times \mu_0 H$ (b) $B = \frac{\mu_0 I}{H}$
(c) $B = \mu_0(I + H)$ (d) $B = \mu_0 I\text{-}H$

62. At Curie temperature the ferromagnetic materials get converted into
(a) non-magnetic material
(b) paramagnetic material
(c) diamagnetic material
(d) all of the above

63. A powerful magnet attracts
(a) all materials
(b) only iron and its ores and repels all the rest
(c) some materials and repels some other materials
(d) only iron and its ores but don't repel any material

64. The area of *I–H* curve and area of *B–H* curve are related as
(a) area of *I-H* curve < Area of *B-H* curve
(b) area of *I-H* curve > Area of *B-H* curve
(c) area of *I-H* curve = Area of *B-H* curve
(d) none of these

65. The value of earth's magnetic field is
(a) 5 G (b) 50 G (c) 100 G (d) 500 G

66. The intensity of magnetisation depends on magnetising field, for its low values
(a) $I \propto 1/H$ (b) $I \propto H$
(c) $I \propto 1/H^2$ (d) $I \propto H^2$

67. The volume susceptibility of a magnetic material is 30×10^{-4}. Its relative permeability will be
(a) 1.0003 (b) 29×10^{-4}
(c) 1.003 (d) 31×10^{-4}

68. The distance between pole pieces of a horse shoe magnet is 4 cm and the pole strength of each pole is 40 statamp-cm. The magnetic potential betwen the pole pieces will be
(a) 80 units (b) 20 units
(c) 10 units (d) zero

69. The areas of cross-section of three magnets of the same length are A, 2A and 6A. The ratio of their magnetic moments will be if their intensities are in the ratio 1 : 2 : 6.
(a) 36 : 4 : 1 (b) 6 : 2 : 1
(c) 1 : 4 : 36 (d) 1 : 2 : 6

70. A magnet is enclosed by an iron ring as shown in the figure. The magnetic lines of force will be

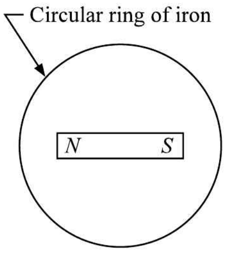


Fig. 25.19

(a) uniformly distributed
(b) dense in air
(c) unaffected
(d) dense in ring

71. The compass needle in the northern hemisphere shows
(a) north pole dip downwards
(b) needle straight
(c) south pole dip downwards
(d) none of these

72. When a big hole is made in a magnet, its magnetic moment becomes
(a) zero (b) more (c) less (d) same

73. The dimensionless quantity, out of the following, is
(a) μ/μ_0 (b) μ_r (c) χ (d) all of the above

74. The magnetism in the middle of a bar magnet is
(a) negative (b) maximum
(c) minimum (d) zero

75. If I is expressed in Wm^{-2} then the correct formula for χ will be
(a) $\chi = H/I$ (b) $\chi = I/\mu_0 H$
(c) $\chi = I/H$ (d) $\chi = \mu_0 H/I$

ANSWERS

1. (a)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (a)
11. (c)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (c)
21. (d)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (d)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (b)
31. (a)	**32.** (c), (d)	**33.** (a)	**34.** (a)	**35.** (c)	**36.** (d)	**37.** (b)	**38.** (d)	**39.** (a)	**40.** (b)
41. (b)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (b)	**46.** (b)	**47.** (b)	**48.** (d)	**49.** (c)	**50.** (d)
51. (a)	**52.** (d)	**53.** (a)	**54.** (c)	**55.** (c)	**56.** (a)	**57.** (c)	**58.** (b)	**59.** (d)	**60.** (a)
61. (c)	**62.** (b)	**63.** (b)	**64.** (b)	**65.** (d)	**66.** (c)	**67.** (c)	**68.** (d)	**69.** (c)	**70.** (d)
71. (a)	**72.** (d)	**73.** (d)	**74.** (d)	**75.** (b)					

26

Magnetic Properties of Substances

BRIEF REVIEW OF THE CONCEPTS

The motion of electron in its orbit acts like a current loop. This gives it a magnetic dipole moment. Besides this, electron has a spin angular momentum that contributes to dipole moment = 9.285 × 10^{-24} J/T. The nucleus may also have a magnetic moment but it is several thousand times smaller than the magnetic moment of an electron. The resultant magnetic moment of an atom is vector sum of all such magnetic moments.

Magnetization vector (I) $= \dfrac{\text{magnetic moment}}{\text{volume}} = \dfrac{\vec{M}}{V}$

$$= \frac{2ml}{2lA} = \frac{m}{A}$$

$$= \frac{\text{pole strength}}{\text{area}} \quad \text{unit Am}^{-1}$$

Magnetizing field intensity (H)

$$\vec{H} = \frac{B}{\mu_0} - I \quad \text{unit Am}^{-1} \quad \text{or} \quad B = \mu_0(H + I)$$

$$B = \mu_0 H\left(1 + \frac{I}{H}\right) = \mu_0 H\,(1 + \chi) = \mu_r \mu_0 H = \mu_m H$$

Subsceptibility $\chi = \dfrac{I}{H}$

where $\mu_r = 1 + \chi$ is called relative permeability.

$\mu_r = \dfrac{\mu_m}{\mu_0}$ where μ_m is peremeability of the medium.

Thus $\dfrac{B}{\mu_m} = H$

In vacuum or free space $B_0 = \mu_0 ni$

$$H = \frac{B_0}{\mu_0} = ni$$

$$I = \frac{B}{\mu_0} - H = \frac{B}{\mu_0} - ni\,.$$

Curie's Law As the temperature increases, randomization of individual atomic magnetic moments increases. The susceptibility of paramagnetic substances is inversely proportional to the absolute temperature.

$\chi \propto \dfrac{1}{T}$ or $\chi = \dfrac{C}{T}$ where C is called the Curie's constant.

Ferromagnetic materials when heated become paramagnetic beyond Curie temperature. Thus for ferromagnetic substances $\chi = \dfrac{C}{T - T_c}$. T_c is Curie temperature.

For iron it is 1043 K or 770°C.

For cobalt 1394 K or 1121°C.

For Nickel 631 K or 358°C.

For Gadolinium 317 K or 44°C.

CGS unit of B is Gauss (G) 1G = 10^{-4} T.

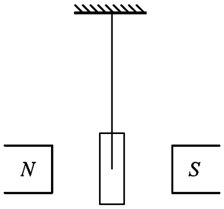
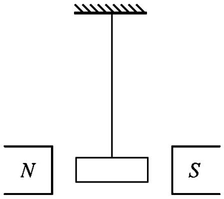
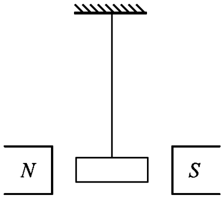
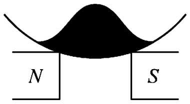
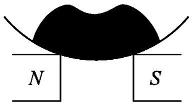
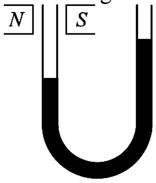
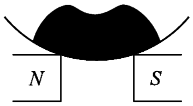
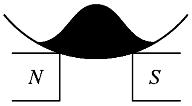
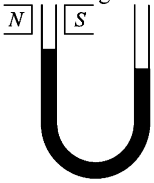
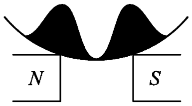
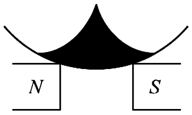
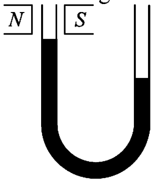
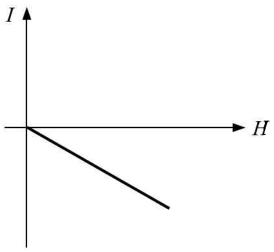
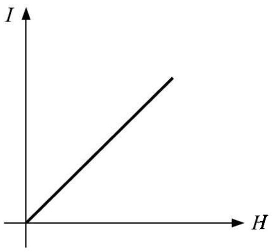
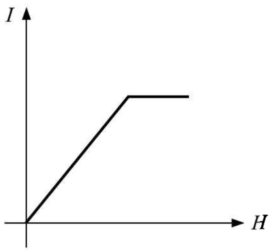
CGS unit of H is oersted.

$$1 \text{ oersted} = \frac{1\text{G}}{\mu_0} = \frac{10^{-4}\,\text{Am}^{-1}}{4\pi \text{ x } 10^{-7}} = 80\,\text{Am}^{-1}$$

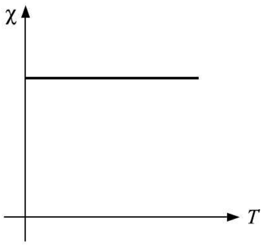
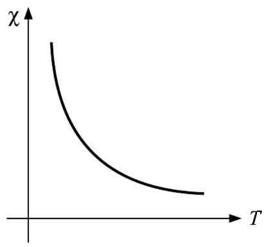
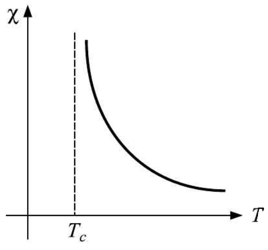
Neel's temperature ($\mathbf{T_N}$) is that temperature at which an antiferromagnetic substance becomes paramagnetic.

Table 26.1

Property	*Diamagnetic*	*Paramagnetic*	*Ferromagnetic*
Cause of magnetism	Orbital motion of electrons	Spin motion of electrons	Formation of domains
Behaviour in an external magnetic field	Feebly repelled	Feebly attracted	Strongly attracted
State of magnetisation	N [N S] S That is, weakly magnetised in the opposite direction to applied magnetic field	N [S N] S Weakly magnetised in the same direction as the applied field	N [S N] S Strongly magnetised in the same direction as the applied field

Property	*Diamagnetic*	*Paramagnetic*	*Ferromagnetic*
Alignment of a freely suspended material	 Align's at right angle to the field	 Align's in the direction of field	 Align's in the direction of field
Liquid or powder in a watch glass	 Weak field  Strong field  Level depressed in that limb	 Weak field  Strong field  Level slightly rises	 Weak field  Strong field  Level rises in that limb
Gas placed in magnetic field	Expands at right angle to the field	Expands in the direction of the field	Expands in the direction of the field
Magnetic induction	$B < B_0$	$B > B_0$	$B >> B_0$
Susceptibility (χ)	$\chi < 0$ (negative) χ does not depend upon temperature	$\chi \geq 0$ (positive but low) $\chi \propto \frac{1}{T}$ or $\chi = \frac{C}{T}$	$\chi >> 0$ ($10^2 - 10^3$) (positive and high) $\chi = -\frac{C}{T - T_C}$
Relative permeability μ_r	$\mu_r < 1$	$\mu_r > 1$	$\mu_r >> 1$ ($10^2 - 10^3$)
Magnetising vector (I)	In opposite direction to H, has a very low value	In the direction of H, has a low value	In the direction of H, has a very high value
I–H curve			
Magnetic dipole moment (M)	Very low and in opposite direction to H	Very low but in the direction of H	High in the direction of H
χ –T curve			

Continued on following page

Property	*Diamagnetic*	*Paramagnetic*	*Ferromagnetic*
			
Examples	Cu, Ag, Au, Zn, Bi, Sb, NaCl, H_2O, air, Ne, He	Al, Mn, Pt, Na, $CuCl_2$ O_2 and crown glass	Fe, Co, Ni, Gd, Fe_3O_4
Nature of effect	Distortion	Orientation	Hysteresis

Hysteresis

Magnetic saturation The state of magnetic material in which value of I or B becomes maximum.

Retentivity When applied magnetizing field is removed the magnetism B or I that remains in the material is called retentivity or remenant magnetism. In the figure $OX = OU$ = retentivity (Fig. 26.1).

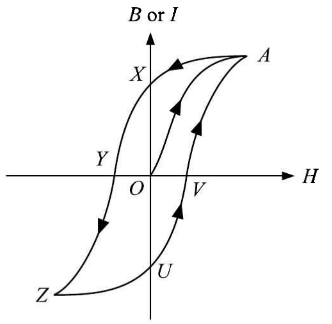


Fig. 26.1

Coercive force or Coercivity The magnetising force or H applied in negative direction to make retentivity zero is called coercivity. In the figure $OY = OV$ = coercivity.

Alnico or carbon steel are used to make permanent magnets. Alnico is an alloy of Fe, Al, Ni and Co. Their coercivity is high and retentivity is low.

Electromagnets are made from materials whose retentivity is high and coercivity is low. For example soft iron. Such materials are also used to make core of transformers, motors, dynamo and so on. Four per cent (Si) is added in soft iron core to further reduce the coercivity. For transformer core permalloy is preferred. For high frequency transformers μ-metal or radiometals are used to make core.

SHORT-CUTS AND POINTS TO NOTE

(i) In vacuum $H = \dfrac{B}{\mu_0}$

In medium $H = \dfrac{B}{\mu_m}$

$\mu_m = \mu_r \cdot \mu_0$

$\mu_r = 1 + \chi \quad \chi = \dfrac{I}{H}$

$\mu_m = \mu_0(1 + \chi)$

Magnetic energy density $U_B = \dfrac{B^2}{2\mu_0}$.

(ii) The magnetic potential due to a small magnet at a distance r (From Fig. 26.2)

$$V = \frac{\mu_0 M \cos\theta}{4\pi r^2}.$$

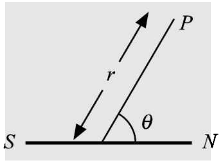


Fig. 26.2

(iii) The mutual interaction force between two small magnets of magnetic moment M_1 and M_2 is

$$F = \frac{\mu_0 6 M_1 M_2}{4\pi r^4}.$$

(iv) Domains in ferromagnets have dimensions ~10^{-5}m. Each domain contains 10^{17} to 10^{21} atoms whose spins are aligned.

The χ for ferromagnets is quite high ~10^2 to 10^3.

The χ for paramagnetic material lies between 0 and 1. It is positive but less than 1.

The χ for diamagnetic substances is negative but close to zero.

(v) μ_r for forromagnets ~10^2 to 10^3.

μ_r for paramagnets > 1

μ_r for diamagnets < 1

(vi) At Curie temperature ferromagnets become paramagnetic.

(vii) At Neel's temperature antiferromagnets become paramagnetic.

(viii) A bar magnet strongly attracts ferromagnets at poles but at the centre of the bar magnet attraction is nearly zero.

The tip of an alpin acts as a magnet because it has a single domain which is always aligned.

(ix) Materials showing high retentivity and low coercivity are suitable to make electromagnets, core of transformers, motors, generator and so on.

Materials with high retentivity and high coercivity are suitable to make permanent magnets.

(x) For high frequency transformer used in radio or TV, μ-metal, radio-metal or ferrites are used.

(xi) Super conductors are perfect diamagnets. This effect is called Miessner's effect. If superconductors are subjected to a magnetic field strength greater than a critical magnetic field strength they become normal conductor.

(xii) Processing frequency of proton in a magnetic field

$$\omega_p = \frac{MB}{L} \quad \text{or} \quad f_p = \frac{MB}{2\pi L}$$

where M is magnetic dipole moment and L is quantized spin angular momentum.

(xiii) Change in magnetic moment if a circulating electron is placed in a magnetic field of strength B is

$$\Delta M = \frac{1}{2} er^2 \Delta\omega = \frac{1}{2} er^2 \left(\frac{eB}{2m}\right) = \frac{e^2 r^2 B}{4m}.$$

Magnetic dipole moment $M = NiA = ef\pi\, r^2$.

CAUTION

(i) Not differentiating between magnetization vector and magnetising field intensity.

- Magnetization vector $I = \dfrac{\vec{M}}{V}$

$$= \frac{\text{dipole moment}}{\text{volume}}$$

$$= \frac{m}{A} = \frac{\text{pole strength}}{\text{area}}.$$

Magnetizing field intensity (H) $= \vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{I}$.

(ii) Confusing magnetic field intensity with magnetic induction.

- Magnetic induction is B and magnetic field intensity is H and $H = \dfrac{B}{\mu_0}$ in vacuum and $H = \dfrac{B}{\mu_m}$ in a medium.

(iii) Confusing Curie temperature with Neel's temperature.

- At Curie tempmerature a ferromagnetic substance changes to paramagnetic and at Neel's temperature an antiferromagnetic substance changes to paramagnetic.

(iv) Considering that substances (magnetic) behave alike in weak and strong fields.

- Their behaviours are opposite in weak and strong magnetic fields. See section 26.1 carefully.

(v) Considering magnetic induction increases in the medium as compared to free space in all types of substances.

- In paramagnetic and ferromagnetic substances it increases while in diamagnetic substances it decreases.

(vi) Considering susceptibility χ is always positive.

- For diamagnetic substance it is negative and for other substances it is positive. It is quite high for ferromagnetic substances.

(vii) Considering any ferromagnetic material is suitable to make permanent magnets.

- To make permanent magnets, the material shall have high coercivity. Alnico and carbon steel are used to make permanent magnets.

  To make electromagnets, the material shall have high retentivity and low coercivity. Soft iron is used for this purpose. If 4% silicon is added in soft iron it becomes ideal to be used for electromagnets.

Solved Problems

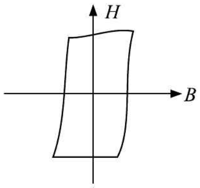
1. Find the power loss due to hysteresis in a transformer core of volume $0.01 m^3$ at $50Hz$. ?The area of the loop is $600 J/m^3$

(a) 300 W (b) 320 W
(c) 360 W (d) 400 W

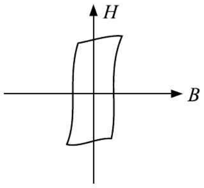
Solution (a) $P_h = AfV = 600 \times 50 \times 0.01 = 300\text{W}$

2. The material suitable to make memory core in computers has hysteresis curve

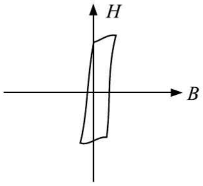
(a)



(b)



(c)



(d) none of these

Fig. 26.3

3. Low temperature ~0.1K can be achieved using demagnetization of

(a) Ferromagnetic salts
(b) Ferrimagnetic salts
(c) Antiferromagnetic salts
(d) Paramagnetic salts
(e) diamagnetic salts

Solution (d)

4. Which of the following shows paramagnetism beyond Neel's temperature

(a) Fe (b) Gd_5Si
(c) MnF_2 (d) He

Solution (c) Antiferromagnetic substances show paramagnetism beyond Neel's temperature

5. Self inductance of a coil is 10 mH. If a Si rod is inserted the self inductance

(a) increases
(b) decreases
(c) remains unchanged
(d) insufficient data to reply

Solution (b) Si is diamagnetic having $\mu_r < 1$ and $L' = \mu_0\mu_r \frac{N^2}{l} A$ decreases.

6. The relative permeability of ... is zero
   (a) He (b) diamond
   (c) graphite (d) superconductor

Solution (d) Only super conductors are perfect diamagnets.

7. When a core is added the self inductance of a coil becomes 50 mH from 10 mH. The susbtance is .... and its susceptibility is.......
   (a) ferrite, 5
   (b) ferrite, 4
   (c) paramagnetic, 4
   (d) ferromagnetic, 4

Solution (b) $\mu_r = L'/L = 5$

$\chi = \mu_r - 1 = 4$

8. A cylindrical iron core has cross-section 5 cm^2 and is inserted in a long solonoid having 2000 turns m^{-1} and carrying a current 2*A*. The magnetic field inside the core is 1.57 T. Neglecting the end effects, find the magnetisation *I* of the core.

Solution $H = ni = 2000 \times 2 = 4000\,\text{Am}^{-1}$

$B = \mu_0(H + I)$

or $I = \frac{B}{\mu_0} - H = \frac{1.57}{4\pi\times10^{-7}} - 4000 = 1.25 \times 10^6\,\text{Am}^{-1}$.

9. The magnetic energy density has the form
   (a) $\frac{B\times H}{2}$ (b) $\frac{B\cdot H}{2}$
   (c) $\frac{\nabla B\times H}{2}$ (d) $\frac{\nabla B\cdot H}{2}$

Solution (b) Magnetic energy density

$$\frac{B^2}{2\mu_0} = \frac{B\cdot\left(\frac{B}{\mu_0}\right)}{2} = \frac{B\cdot H}{2}.$$

10. The dipole moment associated with an iron atom in an iron bar is 1.8×10^{-23} Am^2. Assume the bar is 5 cm long and has cross-sectional area of 1 cm^2 and all iron atoms have their dipole moment aligned. The dipole moment of the bar is
    (a) 5.6Am^2 (b) 6.6Am^2
    (c) 7.6Am^2 (d) none of these

Solution (c) Net dipole moment = dipole moment on one atom × total number of atoms

$$= 1.8 \times 10^{-23} \times \frac{5\times1\times7.8\times6.023\times10^{23}}{56} = 7.6\,\text{Am}^2.$$

11. The magnetic dipole moment of the earth is 6.4×10^{21} Am^2. What current could be set up in a single turn wire going around the earth at its magnetic equator to generate the same dipole moment.
    (a) 20.48×10^{33}A
    (b) 10.24×10^{23}A
    (c) 5×10^{11}A
    (d) 5×10^{7}A

Solution (d) $I\pi R^2 = M$

$$I = \frac{M}{\pi R^2} = \frac{6.4\times10^{21}}{3.14\times(6.4\times10^6)^2} = 5 \times 10^7\text{A}.$$

12. $\oint H\cdot dl = ni$ is valid for
    (a) all currents (b) displacement current
    (c) true current (d) none of these

Solution (c) *H* is independent of core material.

13. The area under the hysteresis loop is proportional to
    (a) magnetic energy density
    (b) thermal energy per unit volume
    (c) electrical energy per unit volume
    (d) mechanical energy per unit volume

Solution (b)

14. Godalinium (Gd) belongs to .... magnetic substance
    (a) ferro (b) para (c) antiferro (d) ferri (e) dia

Solution (a)

15. An electron has 10^8 rotations per sec in its orbit of radius 0.53 Å. The magnetic dipole moment associated with it is
    (a) $>1\,\mu_B$ (b) $<1\,\mu_B$
    (c) $= \mu_B$ (d) cannot say
    where μ_B is Bohrmagneton.

Solution (b) $M = ef\pi r^2$

$= 1.6 \times 10^{-19} \times 10^8 \times \pi(0.53 \times 10^{-10})^2$

$= 1.28 \times 10^{-31}\,\text{Am}^2$

16. The material which shows the effect shown in Fig. 26.4 when placed in a uniform magnetic field is called

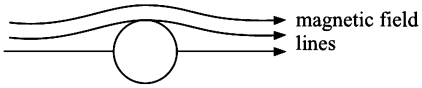


Fig. 26.4

   (a) paramagnetic (b) diamagnetic
   (c) ferromagnet (d) antiferromagnetic

Solution (b) diamagnetic materials show a feeble repulsion.

17. Which of the following represents magnetization in domains of a single crystal of Ni in strong magnetic field.

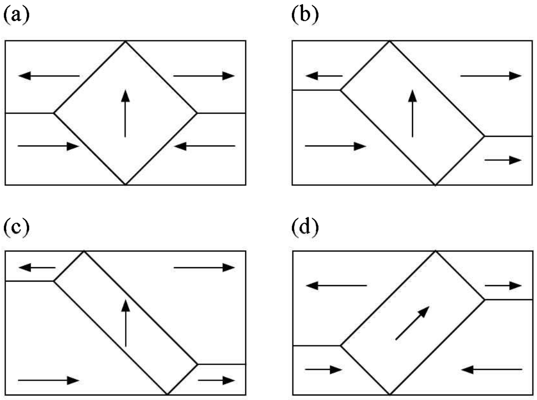


Fig. 26.5

Solution (c) In strong magnetic field domains in the applied direction of magnetic field grow.

Problems for Practice

1. The area of *I–H* curve and area of *B–H* curve are related as
(a) area of *I-H* curve < area of *B-H* curve
(b) area of *I-H* curve > area of *B-H* curve
(c) area of *I-H* curve = area of *B-H* curve
(d) none of these

2. The magnetic moment of diamagnetic material is
(a) 100Am^2 (b) zero
(c) infinity (d) none of these

3. The value of relative magnetic permeability (μ_r) for ferromagnetic material is
(a) $\mu_r >> 1$ (b) $\mu_r = 1$
(c) $\mu_r > 1$ (d) $\mu_r < 1$

4. The cause of paramagnetism is
(a) paired electrons and orbital motion of electrons
(b) electrons and orbital motion of electrons
(c) umpaired electrons
(d) electrons excess and spin motion of electrons

5. The area of *I–H* loop for soft iron, as compared that for steel is
(a) more (b) equal (c) less (d) none of these

6. The magnetic susceptibility for paramagnetic materials depends on temperature as
(a) $\chi \propto 1/T$ (b) $\chi \propto T^2$
(c) $\chi \propto 1/T^2$ (d) $\chi \propto T$

7. The intensity of magnetisation depends on magnetising field, for its low values, as
(a) $I \propto 1/H$ (b) $1 \propto H^2$
(c) $I \propto H$ (d) $I \propto 1/H^2$

8. The liquid in the watch glass in the following figure is

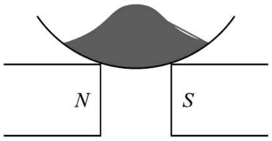


Fig. 26.6

(a) diamagnetic (b) paramagnetic
(c) ferromagnetic (d) low ferromagnetic

9. At Curie temperature the ferromagnetic materials get converted into a
(a) paramagnetic material
(b) diamagnetic material
(c) non-magnetic material
(d) none of these

10. Powerful permanent magnets are made of
(a) alnico (b) cobalt–steel
(c) cobalt (d) aluminium

11. The magnetic moment of paramagnetic materials is
(a) constant but low (b) negative
(c) infinity (d) zero

12. Which of the following statements is correct for diamagnetic materials?
(a) χ does not depend on the temperature
(b) $\mu_r < 1$
(c) χ is negative and low
(d) all of the above

13. The correct relation between *B*, *H* and *I* is
(a) $B = \mu_0 I/H$ (b) $B - \mu_0 I \times \mu_0 H$
(c) $B = \mu_0(I + H)$ (d) $B = \mu_0 I\text{-}H$

14. If a magnetic material is beaten with a hammer then its magnetism
(a) increases (b) decreases
(c) get spoiled (d) none of these

15. The value of magnetic susceptibility for paramagnetic substances is
(a) low positive (b) low negative
(c) zero (d) infinite

16. The magnetic susceptibility of a paramagnetic substance is 3×10^{-4}. It is placed in magnetising field of 4×10^3Am^{-1}. The intensity of magnetisation will be
(a) 1.2Am^{-1} (b) 24Am^{-1}
(c) 3×10^8Am^{-1} (d) 12×10^8Am^{-1}

17. The value of magnetic susceptibility for superconductors is
(a) 1 (b) –1
(c) 0 (d) ∞

18. The cause of diamagnetism is
(a) paired electrons
(b) orbital motion of electrons
(c) spin motion of electrons
(d) none of these

19. The correct relation between magnetic susceptibility and relative permeability is
(a) $\chi = \mu + 1$
(b) $\chi = \mu - 1$
(c) $\chi = \mu_r - 1$
(d) $\chi = \mu_r + 1$

20. The *M–T* curve for ferromagnetic material is

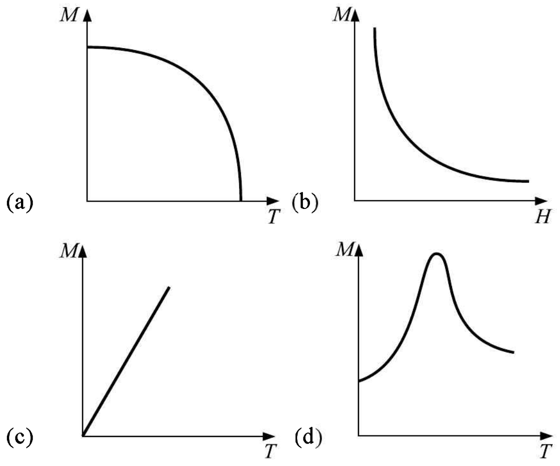


Fig. 26.7

21. If *I* is expressed in Wbm^{-2} then the correct formula for χ will be
(a) $\chi = \mu_0 H/I$ (b) $\chi = H/I$
(c) $\chi = I/\mu_0 H$ (d) $\chi = I/H$

22. The relative permeability of air is
(a) μ_0 (b) 1 (c) 0 (d) ∞

23. The correct expression for Curie-Weiss law is
(a) $\chi = C/T_C$ (b) $\chi = C(T - T_C)$
(c) $\chi = C/T$ (d) $\chi = C/(T - T_C)$

24. The correct I–H curve for paramagnetic materials is

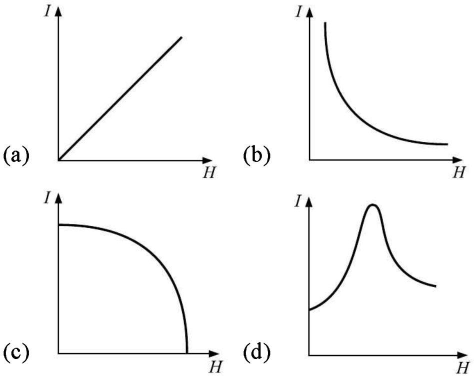


Fig. 26.8

25. If the magnetic susceptibility of a magnetic material is –1.004 then its nature will be
(a) ferromagnetic (b) antiferromagnetic
(c) diamagnetic (d) paramagnetic

26. If the relative permeability of a material is 0.9999 then its nature will be
(a) ferromagnetic (b) non-magnetic
(c) diamagnetic (d) paramagnetic

27. In a ferromagnetic material the volume of each domain and number of atoms in it, are respectively
(a) $10^{-3}\,m^3$ and 10^{-20} (b) 10^{-20} and $10^3\,m^3$
(c) $10^{-10}\,m^3$ and 10^{20} (d) 10^{20} and $10^{-3}\,m^3$

28. The magnetic property inherent in all materials is
(a) paramagnetism (b) non-magnetism
(c) ferromagnetism (d) diamagnetism

29. The dipole moment of each molecule of a paramagnetic gas is $1.5 \times 10^{-23}\,Am^2$. The temperature is 27°C and the number of molecules per unit volume in it is $2 \times 10^{26}\,m^{-3}$. Maximum possible intensity of magnetization in the gas will be (in Am^{-1})
(a) 4×10^{-3} (b) 6×10^{-4}
(c) 3×10^{3} (d) 5×10^{5}

30. A magnetizing field of $5000\,Am^{-1}$ produces a magnetic flux of $5 \times 10^{-5}\,Wb$ in an iron rod of $0.5\,cm^2$, then the permeability of the rod is (in $Henrym^{-1}$)
(a) 4×10^{-6} (b) 3×10^{-5}
(c) 2×10^{-4} (d) 1×10^{-3}

31. The magnetic induction along the axis of an air solenoid is 0.03 T. On placing an iron core inside the solenoid the magnetic induction becomes 15 T. The permeability of iron will be (in $Wb^{-1}A^{-1}m^{-1}$)
(a) 12.5×10^{-6} (b) 9.41×10^{-5}
(c) 6.28×10^{-4} (d) 3.14×10^{-3}

32. For permanent magnets the value of H is
(a) 1 (b) 2.5 (c) 0 (d) ∞

33. A magnetizing field of 5000 A/m produces a magnetic flux of $5 \times 10^{-5}\,Wb$ in an iron rod. If the area of cross-section of the rod is $0.5\,cm^2$, then the magnetic susceptiability of the rod will be
(a) 343.6 (b) 295.3 (c) 199.0 (d) 158.2

34. A paramagnetic material is placed in a magnetic field. Consider the following statements
(A) If the magentic field is increased, the magnetization is increased (B) If the temperature is increased, the magnitisation is increased
(a) both A and B are false
(b) B is true but A is false
(c) both A and B are true
(d) A is true but B is false

35. A paramagnetic material is kept in a magnetic field. The field is increased till the magnetisation becomes constant. If the temperature is now decreased, the magnetization
(a) remains constant
(b) decrease
(c) will increase
(d) both increases and decreases

36. When a ferromagnetic material goes through a hysteresis loop, the non-magnetic susceptibility is
(a) infinite (b) negative
(c) a fixed value (d) zero

37. The permanent magnetic moment of the atoms of a material is zero. The material
(a) must be ferromagnetic
(b) must be paramagnetic
(c) may be antiferromagnetic
(d) must be diamagnetic

38. A ferromagnetic material is placed in an exnternal magnetic field. The magnetic domains
(a) increase in size
(b) decrease in size
(c) may increase or decrease in size
(d) have no relation with the field

39. The magnetic susceptibility x of a paramagnetic materials varies with absolute temperature T as
(a) $\chi \propto T^{-1}$ (b) $\chi \propto e^T$
(c) $\chi \propto T$ (d) χ = constant

40. The correct curve between χ and $1/T$ for paramagnetic materials is

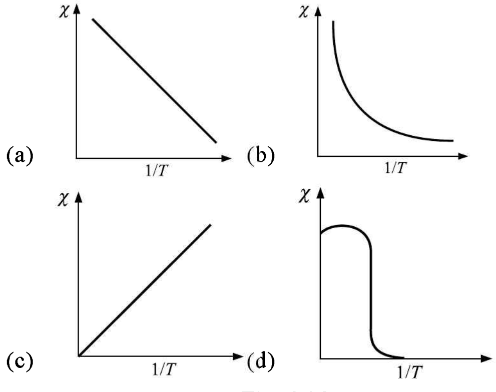


Fig. 26.9

41. The permeability of the feromagnetic substance is
(a) negative
(b) less than 1
(c) small but more than 1
(d) very large

42. What is the relation between pole strength q_m, area of cross-section A and the intensity of magnetisation M?
(a) $M = q_m/A^2$
(b) $M = q_mA^2$
(c) $M = q_m/A$
(d) $M = q_mA$

43. The dimension of $\sqrt{\mu\varepsilon}$ is the same as that of

(a) velocity (b) acceleration

(c) 1/velocity (d) 1/acceleration

44. Hysteresis cycle for the material of a permanent magnet should be

(a) short and wide (b) short and thin

(c) long and wide (d) long and thin

45. Which of the following is most suitable as the core of transformers?

(a) soft iron (b) alinco

(c) steel (d) none of these

46. To shield an instrument from an external magnetic field it may be placed in a cabinet made of

(a) metal (b) diamagnetic substance

(c) wood (d) ebonite

47. What is the relation between dipole moment p_m, volume of bar magnet V and intensity of magnetization M?

(a) $M = p_m V^2$ (b) $M = p_m / V^2$

(c) $M = p_m V$ (d) $M = p_m / V$

48. To which category of magnetic substances do the ferrite belong?

(a) ferromagnetic (b) ferrimagnetic

(c) paramagnetic (d) diamagnetic

49. Which of the following is not the charcteristic/use of ferrites?

(a) suitable for permanent magnets

(b) suitable as transformer core

(c) low permeability

(d) low electric conductivity

50. An air core coil has a value 10 μH. When a ferrite core is added in between, the inductance of the coil increases to 60 μH. The susceptibility of the core is

(a) 7 (b) 6

(c) 5 (d) none of these

ANSWERS

1. (b)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (a)
11. (a)	**12.** (d)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (a)
21. (c)	**22.** (b)	**23.** (d)	**24.** (a)	**25.** (c)	**26.** (c)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (c)
31. (c)	**32.** (c)	**33.** (d)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (d)	**38.** (c)	**39.** (a)	**40.** (c)
41. (d)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (c)

27 Electromagnetic Induction

BRIEF REVIEW OF THE CONCEPTS

Faraday's laws

$$\varepsilon = \frac{-d\phi}{dt} \quad \text{where } \phi = \int \vec{B} \cdot \vec{dS}.$$

SI unit of magnetic flux is Weber and the CGS unit of magnetic flux is Maxwell. 1 Weber = 10^8 Maxwell

Second law The emf lasts as long as the flux linked with the loop varies.

$i = \frac{\varepsilon}{R} = -\frac{1}{R}\frac{d\phi}{dt}$. The negative sign gives the direction of the current.

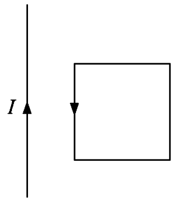


Fig. 27.1

Lenz's law The direction of the induced emf is such that it opposes the change that has induced it.

For example if a loop is placed as shown in the Fig. 27.1 (close to the current carrying conductor) and current is increasing then $\frac{d\phi}{dt}$ is positive and hence emf induced is negative. So that the current will be in the direction opposite to the direction of current carrying wire as shown in Fig. 27.1. If, however, the current I is decreasing in the conductor then $\frac{d\phi}{dt}$ is negative and $\varepsilon = -\frac{d\phi}{dt}$ is positive. The current will be in the same direction as illustrated in the Fig. 27.2.

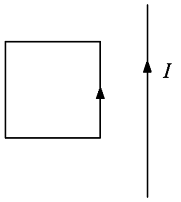


Fig. 27.2

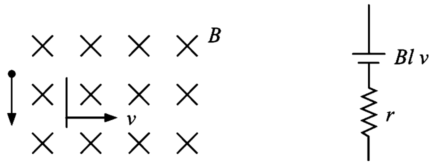
Note that the flux $\int B \cdot dS$ can be varied

(a) by keeping B constant with time and moving whole or part of the loop (that is, varying dS with time). The emf so produced is called motional emf.

(b) by keeping the loop at rest and changing the magnetic field. The emf so produced is called induced electric field.

(c) by combining (a) and (b), that is, moving loop (partly or wholly) as well as varying the magnetic field.

Motional emf $\varepsilon = Blv$ is the magnitude of emf generated when a conductor of length l is moved in a magnetic field of strength B. The equivalent circuit of emf is shown along with.



Equivalent circuit

Fig. 27.3

If a conductor of length l rotates with a velocity ω then magnitude of emf induced is $Bl^2\omega/2$.

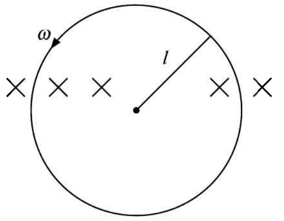


Fig. 27.4

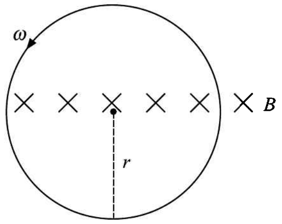


Fig. 27.5

If f is the frequency of rotation and $A = \pi l^2$ then magnitude of induced emf $\varepsilon = BAf$.

If a disc of radius r rotates with an angular velocity ω then magnitude of emf induced $= Br^2\omega/2$.

$\varepsilon = BAf = B\pi r^2 f$, if f is the frequency of rotation.

If the rod moves making an angle θ then

$\varepsilon = Blv \sin\theta$.

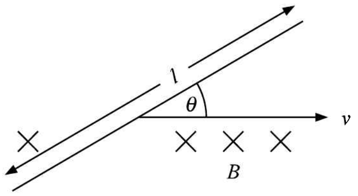


Fig. 27.6

The direction of induced current is given by Fleming's right hand rule:

Fleming's right hand rule If the thumb, the forefinger and the middle finger of the right hand are stretched in mutually perpendicular directions, the thumb gives the direction of motion of the conductor, the forefinger gives the direction of magnetic field and the middle finger the direction of induced current.

Induced electric field When a conducting loop is at rest and magnetic field varies with time, an electric field is generated that forces the electrons to move. The electric field produced by changing magnetic field is *non-electrostatic* and hence *non-conservative*. We call it *induced electric field.*

$$\varepsilon = \oint E \cdot dl = \int v \times B \cdot dl \text{ and } \varepsilon = -d\phi/dt.$$

Therefore, $\oint E \cdot dl = -d\phi/dt.$

Self inductance magnetic flux $\phi \propto i$ or $\phi = Li$ where L is called self inductance.

Unit of self inductance is Henry

Emf $\varepsilon = -d\phi/dt = -Ldi/dt.$

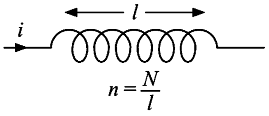
Thus if a time varying current i passes through a coil then an emf is induced in the coil equal to $-Ldi/dt$. SI unit of self inductance is Henry (H) or Weber/Ampere or $V\text{-}SA^{-1}$.

(a)

If the coil has N turns then flux through each turn is $\int B \cdot dS$.

Emf $\varepsilon = -\frac{N d}{dt}\int B \cdot dS$

Self inductance $L = \mu_0 n^2 A l$



(b)

Fig. 27.7

where n is number of turns per unit length, A area of cross-section l length of the coil.

$$L = \mu_r \mu_0 n^2 A l,$$

if core of relative permeability μ_r is introduced.

Growth of current in a series *RL* circuit Assume at $t = 0$, there is no emf across L when we make the switch *ON* in the circuit of series resistance R and inductor L as shown then for a short interval of time current varies. This is called transient. Since in this case current increases, it is therefore termed as *growth transient*.

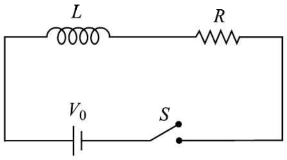


Fig. 27.8

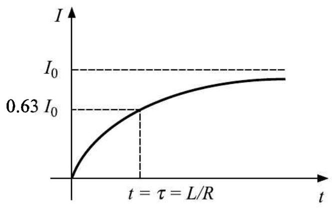


Fig. 28.9

$$i = \frac{V_0}{R}(1 - e^{-tRL})$$

$\frac{L}{R} = \tau$ is called time constant. It is the time in which the current increases to 63% of its maximum value $\left(I_0 = \frac{V_0}{R}\right)$.

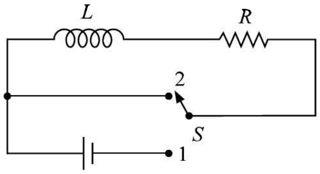


Fig. 27.10

$$i = I_0(1 - e^{-t/\tau}).$$

Decay of current At $t = 0$, the switch S is brought from position 1 to 2 (After the current has saturated to I_0). Then current starts decaying following the equation

$$i = \frac{V_0}{R}e^{-t/\tau} = I_0 e^{-t/\tau}.$$

At $t = \tau$ the current decay's to 36.6% of I_0. Energy stored in an inductor $U = \frac{Li^2}{2}$.

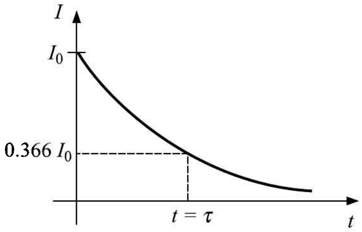


Fig. 27.11

Energy density in magnetic field $u = \frac{U}{\text{vol}} = \frac{B^2}{2\mu_0}$.

$$B = \mu_0 n i.$$

Mutual inductance Assume two closed circuits are placed close to each other. If a time varying current is passed through one of them emf or current is induced in the other coil also.

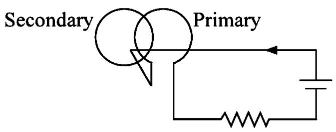


Fig. 27.12

Flux $\phi = Mi$

$\varepsilon = \frac{d\phi}{dt} = -\frac{Mdi}{dt}$ where, M is called mutual inductance.

If one coil is wound over the other and if n_1 is number of turns per unit length in the primary, N_2 total number of turns in the secondary then mutual inductance

$$M = \mu_0 n_1 N_2 A$$

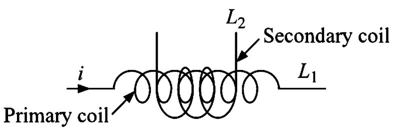


Fig. 27.13

where A is area of cross-section of the coil. If we use a core of relative permeability μ_r in the coil then

$$M = \mu_r \mu_0 n_1 N_2 A.$$

If L_1 and L_2 are self inductances of two coils then mutual inductance $M = k\sqrt{L_1 L_2}$ k is called coupling factor.

$$k \leq 1.$$

If one coil is wound over the other then $k = 1$. As the separation between the coil increases the value of M decreases. Inductances are added in series and parallel like resistors, that is,

$$L_{\text{series}} = L_1 + L_2 + \ldots$$

$$\frac{1}{L_{\text{parallel}}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots$$

Eddy current Consider a metal cube moved in a magnetic field. Assume large number of small loops in the cube as shown. When the cube enters the magnetic field, magnetic flux in these loops varies and therefore current is induced in various different directions. Such currents are called eddy currents.

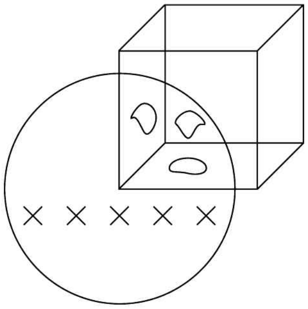


Fig. 27.14

Hall effect If i is the current in a strip of metal/semi conductor in the direction shown and B is a magnetic field then a Hall emf is developed in the transverse direction xy. The sign of emf will determine whether the charge causing the Hall voltage is positive or negative.

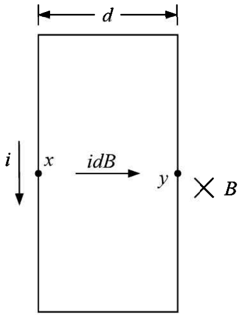


Fig. 27.15

$$E_H = \frac{V_{xy}}{d},\ E_H = -v_d \times B$$

$$\therefore \quad E_H = \frac{JB}{ne} \text{ or } J = nev_d \text{ or } v_d = \frac{J}{ne}$$

Even concentration of the carriers n can be determined.

Self inductance of a toroid $L = \dfrac{\mu_0 N^2 r}{2}$.

Dimensional formula is $ML^2T^{-2}A^{-2}$.

Poles of a coil can be found If the current is clockwise the pole will be S-pole and if the current is anticlockwise then the pole will be the N-pole.

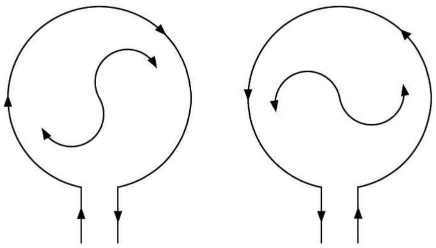
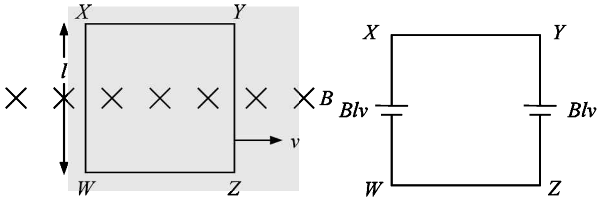


Fig. 27.16

SHORT-CUTS AND POINTS TO NOTE

(i) Time varying current or emf or voltage is AC. AC voltage and currents are phasors. Phasors are added like vectors. Therefore apply vector laws.

(ii) If a loop is moving with a velocity v in a magnetic field Fig. 27.17(a) such that it always remains in the field the net emf induced is zero as is clear from Fig. 27.17(b), the equivalent circuit of 27.17(a). Therefore, no current will flow. Also note that no emf is induced in xy and zw branch as no flux is linked with them.



(a) (b)

Fig. 27.17

(iii) Emf induced in a rotating disc of radius r is

$$\varepsilon = \frac{Br^2\omega}{2} = B\pi\, r^2 f \text{ where}$$

f is frequency of rotation and, ω is angular frequency.

(iv) If B changes with time as well as distance then the emfis

$$\varepsilon = A\left[\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial z}\right] \text{ where } A \text{ is area, } v \text{ is velocity.}$$

(v) When a conducting rod of length l moves in the magnetic field with a velocity v as shown in the Fig. 27.18 emf generated is Blv.

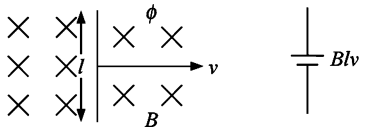


Fig. 27.18

The direction of current is determined using Fleming‘s right hand rule. If the rod makes an angle θ as shown then emf generated = $Blv \sin\theta$

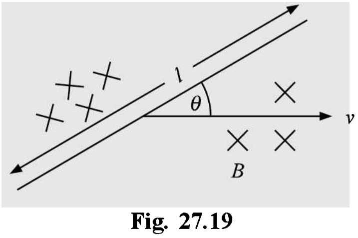


Fig. 27.19

(vi) If a rod of length l rotates with a velocity ω fixed at one end as shown in figure

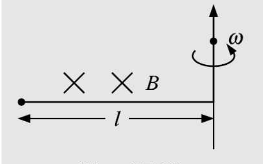


Fig. 27.20

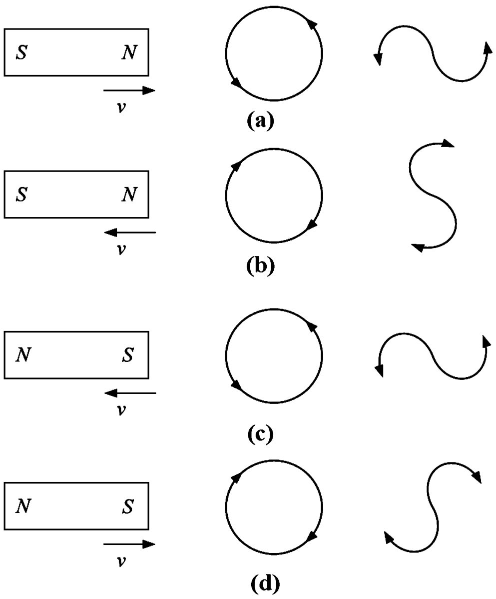
then emf $\varepsilon = \dfrac{Bl^2\omega}{2} = B\pi l^2 f$

where f is linear frequency.

(vii) Power spent in moving the conductor is

$$P = \frac{B^2 l^2 v^2}{R}.$$

(viii) Note the directions of current generated in the coil and pole. They behave for different motions of magnet.



(a)

(b)

(c)

(d)

Fig. 27.21

(ix) When current is increasing the current induced in the loop is in a direction opposite to the current in the conductor.

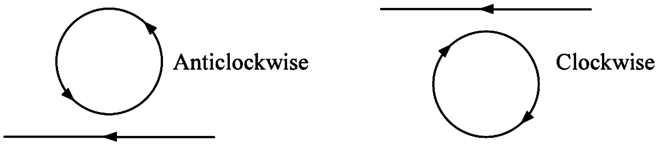


Fig. 27.22

(x) When the magnet is dropped into the coil or a metallic ring with north pole towards the coil, the current is induced in anticlockwise direction and acceleration of the magnet is less than '*g*' as the coil opposes the motion.

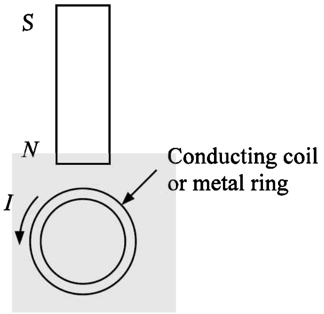


Fig. 27.23

(xi) Self inductance of two coaxial cylinders per metre (Fig. 27.24) is

$$L = \frac{\mu_0}{2\pi}\log_e\frac{r_2}{r_1} = \frac{2.303\mu_0}{2\pi}\log_{10}\frac{r_2}{r_1}.$$

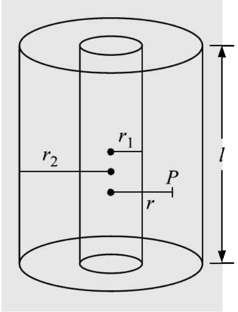


Fig. 27.24

(xii) Mutual inductance between two concentric coils having radii r_p (primary coil) and r_S (secondary coil) is

$$M = \frac{\pi\mu_0 N_P N_S r_S^2}{2r_p}$$

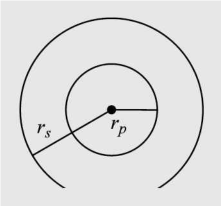


Fig. 27.25

where N_p and N_S are number of turns in primary and secondary coils.

(xiii) Coupling factor

$$K = \frac{M}{\sqrt{L_1 L_2}} = \frac{\phi_S}{\phi_P} = \frac{\text{flux linked with secondary}}{\text{flux linked with primary}}.$$

(xiv) Self inductance of a toroid $= L = \frac{\mu_0 N^2 A}{2\pi r} = \frac{\mu_0 N^2 r}{2}$.

(xv) Inductance in series if mutual inductance of the coils is also present is

$$L_{\text{eff}} = L_1 + L_2 + 2M.$$

(xvi) Inductances in parallel if mutual inductances of the coils is present is

$$L_{\text{eff}} = \frac{L_1 L_2 + M^2}{L_1 + L_2 + 2M}.$$

(xvii) Energy stored $U = \frac{LI^2}{2}$. This is stored in the form of magnetic energy.

Interaction energy of two currents is $U = L_{12} I_1 I_2$

(xviii) Displacement current density is

$$J_{\text{dis}} = \frac{1}{R}\frac{\partial B}{\partial t}.$$

(xix) Volume density of magnetic field energy is

$$U = \frac{B^2}{2\mu_0\mu_r} = \frac{B \cdot H}{2}$$

(xx) The time in which current grows to 63% $= \tau = \frac{L}{R}$.

The time in which current grows to 90% $= 2.303\tau$.

The time in which current grows to 95% $= 3\tau$.

The time in which current grows to 99% $= 5\tau$.

CAUTION

(i) Considering that AC voltage and currents are also scalar quantities like their DC counterparts.

- AC voltage and currents are phasors. Apply vector algebra. In EMI only AC voltage or current is generated.

(ii) Assuming that induced electric field (produced due to changing magnetic field) is a conservative field or electrostatic field.

- Induced electric field is non-elecrostatic and non-conservative. It is produced due to varying magnetic field. They together, that is, varying magnetic field and induced electric field produce electormagnetic waves.

(iii) Considering that current induced in electromagentic induction is like drift current (generated by battery).

- It is like a displacement current.

(iv) Considering that angle between the coils plays no role in determining mutual inductance M.

- If the angle between the axes of two closely placed coils is θ then $M \propto \cos\theta$, that is, if two coils are at right angles as shown in figure then $M = 0$. Hence $K = 0$.

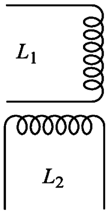


Fig. 27.26

(v) Considering that a cylinder or a wire does not have self inductance.

- A cylinder or a wire has self inductance though it is small. It becomes effective at very high frequencies. Self inductance $L = \frac{\mu_0 mr}{4\pi\rho l}$

  where m is mass, r radius, ρ density, l length.

(vi) Not remembering the Fleming's left hand rule (for direction of magnetic force) and Fleming's right hand rule (for direction of induced current).

- Remember that these rules are mirror images of each other and apply them correctly.

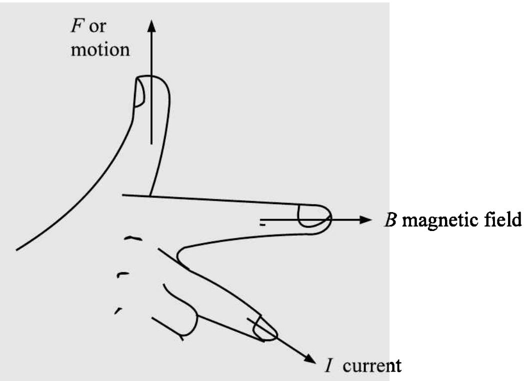


Fig. 27.27

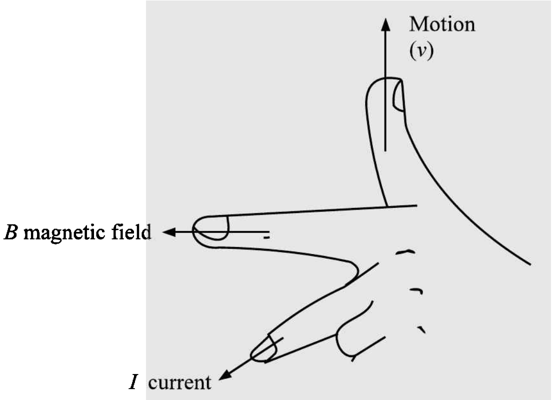


Fig. 27.28

(vii) If a magnet is falling along the axis of an incomplete ring, thinking that as induced current is zero, emf should also be zero.

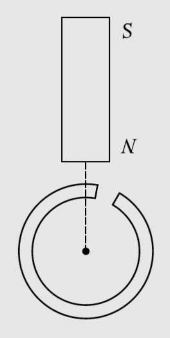


Fig. 27.29

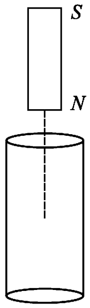


Fig. 27.30

- Note that induced current is zero as the path is not complete but induced emf is present. However, acceleration of falling magnet is g in this case as no current is present to oppose the motion.

(viii) If a magnet is falling along the axis of a long copper cylinder then considering (as in a solenoid) acceleration is less than g. Therefore, acceleration will also be less than g in the cylinder

- As the copper cylinder has nearly zero resistance the long copper cylinder opposes the magnet fully and $a = g - g = 0$.

(ix) Considering that there is no effect of temperature when the magnet is falling along the axis of a metal ring.

- The resistance is higher at high temperatures, therefore, current will be low and hence resistance will be lower. Magnet will fall faster. On the other hand if the temperature is low, resistance of the ring is less. Opposition to the motion of magnet is more. Therefore magnet falls slowly.

Solved Problems

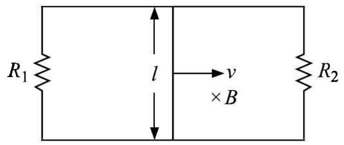
1. A rectangular loop with a slide wire of length l is kept in a uniform magnetic field as shown in Fig. 27.31(a). The resistance of slider is R. Neglecting self inductance of the loop find the current in the connector during its motion with a velocity v.

(a) $\dfrac{Blv}{R_1 + R_2 + R}$

(b) $\dfrac{Blv(R_1 + R_2)}{R(R_1 + R_2)}$

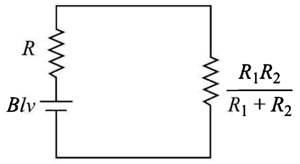
(c) $\dfrac{Blv(R_1 + R_2)}{RR_1 + RR_2 + R_1R_2}$

(d) $Blv\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R}\right)$



(a)

Solution (c) The equivalent circuit is shown in fig. 27.31(b) obviously



(b)

Fig. 27.31

$$I = \frac{Blv}{R+\frac{R_1R_2}{R_1+R_2}} = \frac{Blv(R_1+R_2)}{RR_1+RR_2+R_1R_2}$$

2. A square wire frame of side a is placed a distance b away from a straight conductor carrying current I. The frame has resistance R and self inductance L. The frame is rotated by 180° about OO' as shown in Fig. 27.32. Find the electric charge flown through the frame.

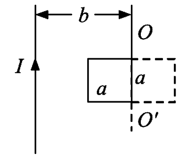


Fig. 27.32

(a) $\frac{\mu_0 ia[2a]}{2\pi Rb}$ (b) $\frac{\mu_0 i}{2\pi R}\log_e\frac{b+a}{b-a}$

(c) $\frac{\mu_0 i a}{2\pi R}\log\frac{b+a}{b-a}$ (d) none of these

Solution (c) $i = \frac{1}{R}\left[\frac{d\phi}{dt}+L\frac{di}{dt}\right]$

$$q = \int i\,dt = \frac{1}{R}[\Delta\phi+0] = \frac{\Delta\phi}{R}$$

$$= \frac{1}{R}\int_{b-a}^{b+a} Ba\,dx$$

$$= \frac{1}{R}\int_{b-a}^{b+a}\frac{\mu_0 i\,a\,dx}{2\pi x} = \frac{\mu_0 i a}{2\pi R}\log\frac{b+a}{b-a}.$$

3. One conducting U tube can slide inside the other as shown in fig. maintaining electric contacts between the tubes. The magnetic field B is perpendicular to the plane of the Fig. 27.33. If each tube moves towards the other at a constant speed v then the induced *emf* in terms of B, l and v where l is the width of each tube, will be

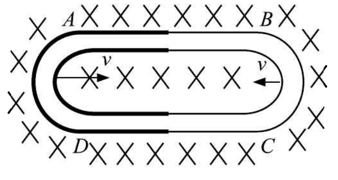


Fig. 27.33

(a) Blv (b) $-Blv$

(c) $2Blv$ (d) zero

Solution (c) $\left|\frac{d\phi}{dt}\right| = 2Blv$

4. A coil of inductance 300 mH and resistance 2 Ω is connected to a source of voltage 2V. the current reaches half of its steady state value in

(a) 0.1 s (b) 0.3 s

(c) 0.05 s (d) 0.15 s

Solution (a) $I = I_0(1-e^{-\frac{RT}{L}})$

$$t = \tau\log_e\left(\frac{I_0}{I_0-I}\right) = \frac{L}{R}\log_e 2$$

$$= 0.693\frac{L}{R} = 0.693\times\frac{a^3}{2} \simeq 0.1\text{ s}$$

5. As a result of change in magnetic flux linked to the closed loop shown in fig. 27.34, an emf V volt is induced in the loop. The work done in taking a charge Q coulomb once along the loop is

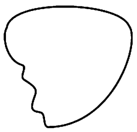


Fig. 27.34

(a) QV (b) $2QV$

(c) $QV/2$ (d) zero

Solution (a) QV because induced electric field so generated is non-conservative is $\oint E\cdot dl = V$.

6. A conducting ring of radius 1 m is placed in a uniform magnetic field of 0.01 T oscillating with frequency 100 Hz with its plane at right angle to B. What will be the induced electric field?

(a) π V/m (b) 2 V/m

(c) 100 Vm^{-1} (d) 62 Vm^{-1}

Solution (b) After every $T/2$ the field will change from B to $-B$ as illustrated in fig. 27.35 $\varepsilon = 2BAf$

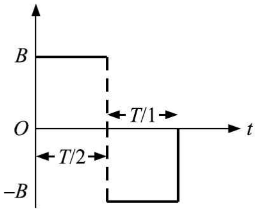


Fig. 27.35

$$\oint E\cdot dl = \varepsilon$$

or $$E = \frac{E}{2\pi R} = \frac{2B\pi R^2 f}{2\pi R}$$

$$= BRf = 0.01\times 1\times 200$$

$$= 2\text{ Vm}^{-1}$$

7. A magnet is made to oscillate with a particular frequency passing through a coil as shown in Fig. 27.36, the time variation of the magnitude of emf generated across the coil during one cycle is.

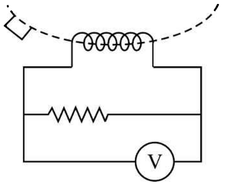
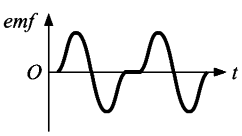
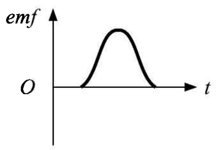
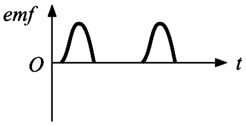
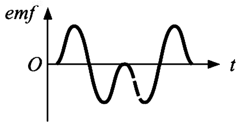


Fig. 27.36

(a) 

(b) 

(c) 

(d) 

Solution (a)

8. The induction coil works on the principle of
   (a) self induction
   (b) mutual induction
   (c) Amperes rule
   (d) Flaming's right hand rule

Solution (b)

9. The coil is wound on an iron core and looped back on itself so that core has two step of closely wound coils carrying current in opposite directions. The self inductance is

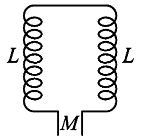


Fig. 27.37

(a) zero (b) $2L$
(c) $2L + M$ (d) $L + 2M$

Solution (a) $L_{eff} = L_1 + L_2 - 2M = L + L - 2\sqrt{LL} = 0$

10. The magnetic flux in a coil is $\phi = 12t^2 + 5t$ (ϕ is in milli weber and t in s)
   (a) 0 (b) 12.5 V
   (c) 0.15 V (d) 0.125 V

Solution (d) $\frac{d\phi}{dt} = (24t + 5) \times 10^{-3}\Big|_{t=5}$

$= (24 \times 5 + 5) \times 10^{-3} = 0.125\text{ V}$

11. A magnet falls with its S-pole along the axis of a ring, the current generated is .... and acceleration is ....

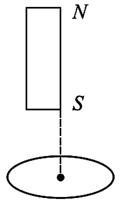


Fig. 27.38

(a) Clockwise, $>g$ (b) Clockwise, $<g$
(c) anticlockwise, $>g$ (d) anticlockwise, $<g$
(e) clockwise, $= g$.

Solution (b) South pole should be formed by the current in the ring. ∴ current is clockwise and south with repel south, hence $a < g$.

12. A long wire carries a current 5A. The energy stored in the magnetic field inside a volume 1 mm³ at a distance 10 cm from the wire is
   (a) $\pi/4 \times 10^{-13}$ J (b) $\pi/2 \times 10^{-13}$ J
   (c) $\pi \times 10^{-13}$ J (d) $\pi/8 \times 10^{-13}$ J

Solution (d) U (energy per unit volume)

$$= \frac{B^2}{2\mu_0} \text{ and energy } U = \frac{B^2}{2\mu_0} \times \text{vol.}$$

$$U = \left(\frac{\mu_0 I}{2\pi d}\right)^2 \times \frac{1}{2\mu_0} \times \text{vol.}$$

$$= \frac{\mu_0 I^2}{8\pi^2 d^2} \times \text{vol.}$$

$$= \frac{4\pi \times 10^{-7} \times 25 \times 10^{-9}}{8 \times 10(10^{-2})} = \frac{\pi}{8} \times 10^{-13}\text{ J}$$

13. If a Bismuth rod is introduced in the air core coil as shown then current in the coil

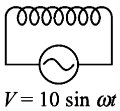


Fig. 27.39

(a) increases
(b) remains unchanged
(c) decreases
(d) none of these

Solution (a) L will decrease as Bi is diamagnetic

$$\therefore \quad I = \frac{V}{X_L} \text{ will increase.}$$

14. The voltmeter reading in the Fig. 27.40 shown is

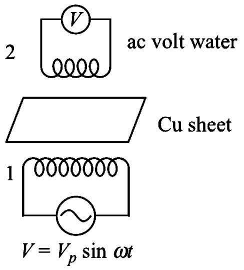


Fig. 27.40

(a) zero (b) $\frac{V_p}{\sqrt{2}}$
(c) $\frac{V_p}{2}1$ (d) none of these

Solution (a) Because Cu is diamagnetic, no magnetic flux will link to coil 2.

15. A satellite orbiting the earth at 400 km above the surface of the earth has a 2 m long antenna oriented perpendicular to the earth's surface. At the equator the earth's magnetic field is 8×10^{-5} T and is horizontal. Assuming the orbit to be circular, find emf induced across the ends of the antenna.
   (a) 1.3 V (b) 1.2 V
   (c) 1.0 V (d) 0.12 V
   (e) 0.13 V

Solution (b) $v_0 = \sqrt{(R+h)g'}$

$emf = Blv_0$

$= 8 \times 10^{-5} \times 2 \times 7.2 \times 10^3 = 1.2\text{ V}$

16. Assume a long solenoid is wound with 500 turns m⁻¹ and current is increasing at 100 As⁻¹. The cross-section of the coil has area 4 cm². Find the induced electric field with in the loop of radius 2 cm

(a) 2×10^{-4} Vm^{-1} (b) 4×10^{-4} Vm^{-1}

(c) 3×10^{-4} Vm^{-1} (d) none of these

Solution $\varepsilon = -\mu_0 nA\dfrac{dI}{dt}$ and $\oint E \cdot dl = \varepsilon$

$$= 4\pi \times 10^{-7} \times 500 \times 10^{-4} \times 100$$
$$= 25 \times 10^{-6}\text{V}$$

or $$E = \frac{\varepsilon}{2\pi r} = \frac{25\times10^{-6}}{2\pi\times2\times10^{-2}} = 2 \times 10^{-4}\text{V.}$$

17. A long solenoid of radius 2 cm has 100 turns/cm and is surrounded by a 100 turn coil of radius 4*t* cm having a total resistance 20 Ω. If current changes from 5A to –5A, find the charge through galvanometer.

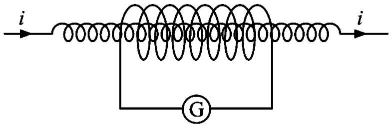


Fig. 27.41

(a) zero (b) 800 μC

(c) 400 μC (d) 600 μC

Solution (b) $\phi_{\text{perturn}} = B\pi r^2$

$$\varepsilon = \frac{d\phi}{dt} = N\pi r^2\frac{dB}{dt} = N\pi r^2\mu_0 n\frac{di}{dt}$$

$$I = \frac{\varepsilon}{R} \text{ and } \Delta Q = I\Delta t = \frac{N\pi r^2\mu_0 n\Delta i}{R}$$

$$= \frac{100\times\pi\times(2\times10^{-2})^2\times10^4\times4\pi\times10^{-7}\times10}{20}$$

$$= 8 \times 10^{-4}\text{C} = 800\,\mu\text{C}.$$

18. A rod of length *l* is moved with a velocity *v* in a magnetic field *B* as shown in Fig. 27.42 each the equivalent electrical circuit.

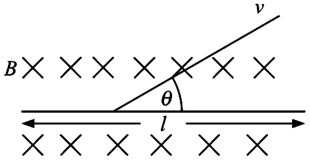
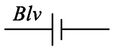
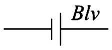
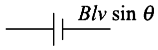
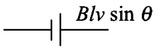


Fig. 27.42

(a) 

(b) 

(c) 

(d) 

Solution (c) The positive charge of the rod shift toward left due to $F = q(\vec{v}\times\vec{B})$

19. Two conducting circular loops of radii R_1 and R_2 ($R_1 >> R_2$) are placed in the same plane with their centres coinciding. Find the mutual inductance between them

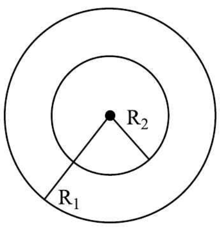


Fig. 27.43

(a) $\dfrac{\mu_0\pi R_1^2}{R_2}$ (b) $\dfrac{\mu_0\pi R_2^2}{R_1}$

(c) $\dfrac{\mu_0\pi R_1^2}{2R_2}$ (d) $\dfrac{\mu_0\pi R_2^2}{2R_1}$

Solution (d) Assume current *i* passes through outer loop then

$$B = \frac{\mu_0 i}{2R_1} \text{ and } \phi_{\text{2nd loop}} = \frac{\mu_0 i}{2R_1}\pi R_2^2$$

Using $\phi = Mi,\ M = \dfrac{\mu_0\pi R_2^2}{2R_1}$.

20. In a closed ring *A* and in an open ring *B* magnets are falling along the axis of the ring. The current generated in *A* and *B* have directions

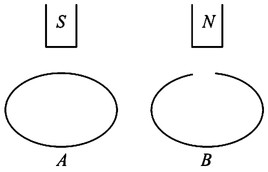


Fig. 27.44

(a) clockwise, anti-clockwise
(b) anticlockwise, clockwise
(c) clockwise, zero
(d) anti-clockwise, zero
(e) zero, zero

Solution (c) According to Lenz's law the current generated in *A* shall develop *S* pole to oppose the cause producing it., therefore current is clockwise. In *B* the circuit is open, therefore no current will flow.

21. A metallic wire bent into a right Δ*abc* moves with a uniform velocity *v* as shown in Fig. 27.45, *B* is the strength of uniform magnetic field perpendicular outwards the plane of triangle. The net *emf* is .... and emf along *ab* is ....

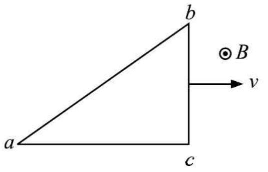


Fig. 27.45

(a) zero, zero
(b) zero, *Bv*(*bc*) with *b* positive, zero
(c) zero, *Bv*(*bc*) with *a* positive
(d) *Bv*(*bc*) with *c* positive, zero
(e) *Bv*(*bc*) with *b* positive, zero

Solution (c) net emf and hence net current in a loop moved with uniform velocity is zero because ϕ = constant and $d\phi/dt = 0$.

22. Two rail tracks are 1 m apart and insulated from each other and insulated from ground. A ***mV*** voltmeter is connected across the rail tracks. When a train travelling at 180 km/h passes through what will be the reading in ***mV*** voltmeter. Given horizontal component of earth's field $\sqrt{3}\times10^{-4}$ T and dip at the place 60°.

(a) 1.5 mV (b) 15 mV

(c) $\dfrac{15}{\sqrt{3}}$ mV (d) none of these

Solution (b) Vertical field will be cut.

$\varepsilon = B_v lv = 3 \times 10^{-4} \times 1 \times 50 = 15\,\text{mV}$

and $B_v = B_H \tan\delta = 3 \times 10^{-4}\text{T}$.

23. A copper wire of length l is bent into a semicircle. It is moved with a velocity v in a region where magnetic field is uniform and perpendicular to the plane of the wire. If the strength of the field is B then emf induced is

(a) Blv (b) $B\dfrac{l}{\pi}v$

(c) $B\dfrac{2l}{\pi}v$ (d) none of these

Solution $\pi r = l$ or $r = \dfrac{l}{\pi}$ $\varepsilon = B(2r)v = B\left(\dfrac{2l}{\pi}\right)v$

24. A small circular ring is kept inside a larger loop as shown. The direction of induced current when the switch is made (i) ON (ii) OFF after it was ON for along time is

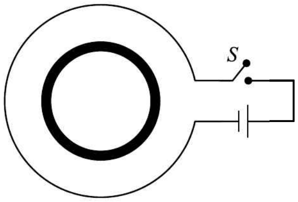


Fig. 27.46

(a) clockwise, anticlockwise
(b) clockwise, clockwise
(c) anti-clockwise, clockwise
(d) anti-clockwise, anti-clockwise

Solution (a) Apply Lenz law.

25. A square loop of Cu of side a enters a magnetic field spread from $-a$ to $+a$ as shown in Fig. 27.47 plot induced emf as a function of x

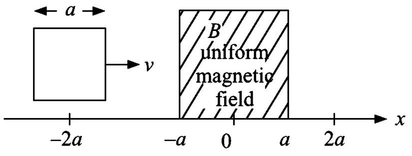
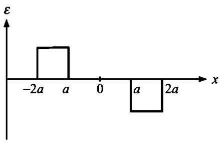
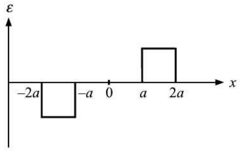
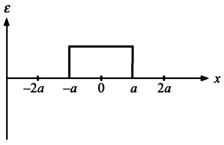


Fig. 27.47

(a)


(b)


(c)


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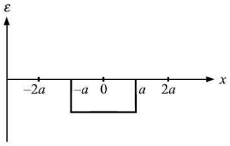
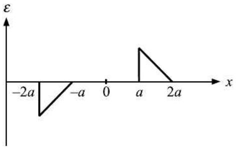


(e)


Solution (b) $\because\ \varepsilon = -\dfrac{d\phi}{dx}$.

26. The armature of a demonstrator generator consists of a flat square coil of side 4 cm and 200 turns. The coil rotates in a magnetic field of 0.75 T. The angular speed so that a max. emf of 1.6 V is generated is ....

(a) $\dfrac{20}{3}$ rad/s (b) $\dfrac{20}{3}$ rotations/s

(c) $\dfrac{20}{3}$ rpm (d) none of these

Solution (a) $\varepsilon_{\max} = NA_0B\omega$

or $$\omega = \frac{\varepsilon_{\max}}{NA_0B} = \frac{1.6}{200\times16\times10^{-4}(0.75)}$$

$$= \frac{20}{3}\ \text{rad/s.}$$

27. The electric flux through a certain area of dielectric is $8.76 \times 10^3 t^4$. The displacement current through the area is 12.9 pA at t = 26.1 ms. Find the dielectric constant of the material.

(a) 2×10^{-8}
(b) 4×10^{-8}
(c) 8×10^{-8}
(d) 2×10^{-7}

Solution (a) $i_D = \varepsilon \dfrac{d\phi_E}{dt}$

or $$\varepsilon = \frac{i_D}{\dfrac{d\phi_E}{dt}}$$

$$= \frac{12.9\times10^{-9}}{4(8.76)\times10^3\times(26.1\times10^{-3})^3}$$

$\simeq 2 \times 10^{-8}$.

Problems for Practice

1. The maximum possible mutual inductance between two coils with coefficients of self induction L_1 and L_2 will be

(a) L_1L_2 (b) $\dfrac{1}{\sqrt{L_1L_2}}$

(c) $\sqrt{\dfrac{L_1}{L_2}}$ (d) $\sqrt{L_1L_2}$

2. The voltage equation of an electric motor is

(a) $E = \dfrac{Eb}{I_aR_a}$
(b) $E = \dfrac{I_aR_a}{E_b}$
(c) $E = E_b - I_aR_a$
(d) $E = E_b + I_aR_a$

3. The expression for induced charge in a coil is

(a) $q = \frac{NR}{(\phi_1 - \phi_2)}$ (b) $q = R(\phi_1 - \phi_2)$

(c) $q = (\phi_1 - \phi_2) - NR$ (d) $q = \frac{N}{R}(\phi_1 - \phi_2)$

4. A magnet is brought towards a coil C as shown in the figure. The direction of induced current in the circuit will be

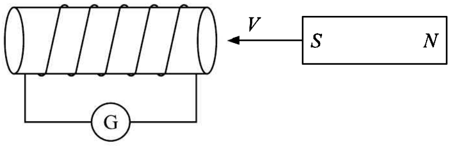


Fig. 27.48

(a) south (b) clockwise
(c) north (d) anti-clockwise

5. The turns ratio of a transformer is 10. It means that

(a) $I_S = 10I_P$ (b) $N_S = \frac{N_P}{10}$

(c) $N_S = 10N_P$ (d) $V_S = \frac{V_P}{10}$

6. The phase difference between the flux linked with a coil rotating in a uniform magnetic field and induced emf produced in it is

(a) zero (b) $\pi/4$ (c) $\pi/2$ (d) π

7. A rod of length l is fixed at P as shown and rotated with ω. If a uniform magnetic field B exists throughout the plane of motion perpendicular to the motion then emf induced between A and B is

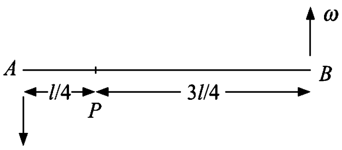


Fig. 27.49

(a) $\frac{Bl^2\omega}{2}$ (b) $\frac{Bl^2\omega}{4}$

(c) $\frac{3Bl^2\omega}{8}$ (d) $\frac{Bl^2\omega}{8}$

8. A current carrying solenoid is carried towards a conducting loop. The direction of induced current in the loop as observed by an observer on the otherside of the loop will be

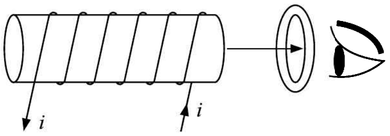


Fig. 27.50

(a) towards west (b) clockwise
(c) towards east (d) anti-clockwise

9. The turns ratio in a step up transformer is 4 : 1. On passing a current of 4A in the primary, the current in the secondary will be

(a) 0.25A (b) 1A (c) 2A (d) 8A

10. The dimension of the quantity $\frac{L}{CVR}$ are equivalent to those of

(a) 1/current (b) current
(c) 1/charge (d) charge

11. The input and output voltage in a step down transformer are 22 kV and 550 V respectively. The ratio of turns in secondary and primary coils will be

(a) 1 : 40 (b) 1 : 20
(c) 40 : 1 (d) 20 : 1

12. When a piece of wire is passed through the space between the pole pieces of a horse shoe magnet in 0.1 s then an emf of 4×10^{-3} V is induced in it. The magnetic flux between the pole pieces will be

(a) 4×10^{-4} Wb (b) 0.1 Wb
(c) 10 Wb (d) 4×10^{2} Wb

13. A 1.2 m wide railway track is parallel to magnetic meridian. The vertical component of earth's magnetic field is 0.25 G. When a train runs on the rails at a speed of 60 kmh^{-1}, then the induced potential difference between the ends of its axle will be

(a) zero (b) 5×10^{-4} V
(c) 10^{-4} V (d) 10^{-3} V

14. The unit of magnetic flux is

(a) Oersted (b) Gauss
(c) Tesla (d) Weber

15. The number of turns in an air core solenoid of length 25 cm and radius 4 cm is 100. Its self inductance will be

(a) $5 \times 4 \times 10^{-3}$ H (b) 5×10^{-4} H
(c) 2.5×10^{-4} H (d) 2.5×10^{-3} H

16. The number of turns in the primary and secondry coils of a transformer are 100 and 300 respectively. If the input power is 60 W the output power will be

(a) 180 W (b) 3×10^{3} W
(c) 60 W (d) 90 W

17. A square of side 22 cm (of copper wire) is converted into a circular loop in 0.4 s. If magnetic field is 1.2 T then emf induced in it is

(a) 0.396 V (b) 0.0396 V
(c) 3.96 V (d) none of these

18. An inductor may store energy in

(a) its coil
(b) its magnetic field
(c) its electric field
(d) both in electric and magnetic field

19. If L and R represent the inductance and resistance respectivley, the dimensional formula of (L/R) is

(a) $[M^0L^0T^{-1}]$ (b) $[M^0L^0T^0]$
(c) $[M^0L^0T]$ (d) $[M^0L^0T^2]$

20. Induced emf produced in a coil rotating in a magnetic field will be maximum when the angle between the axis of coil and direction of magnetic field is

(a) 180° (b) 90° (c) 45° (d) 0°

21. The normal drawn to the surface of a conductor makes an angle θ with the direction of field B. The flux ϕ passing through the area A is given by

(a) $\phi = BA$ (b) $\phi = \vec{B} \cdot \vec{A}$

(c) $\phi = \vec{B} / \vec{A}$ (d) $\phi = \vec{B} \times \vec{A}$

22. A 50 mH coil carries a current of 2A, the energy stored in it in Joules is

(a) 0.05 (b) 0.1
(c) 0.5 (d) 1

23. A magnet falls in the copper ring as shown in the Fig. 27.51 with N pole approaching the ring. Then

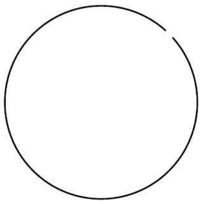


Fig. 27.51

(a) emf is induced without any induced current
(b) both emf and current are induced
(c) as the magnet reaches close to the ring its acceleration falls
(d) it approaches the ring with a constant velocity

24. A coil of copper wire is pulled in a magnetic field with a constant velocity. If its ohmic resistance is increased then
(a) it is easier to pull it
(b) it is difficult to pull it
(c) it cannot be pulled
(d) none of the above

25. If an air plane is moving with 1.5 Mach horizontally the horizontal component of magnetic field at that place is 0.6×10^{-4}T. The emf induced in the wings 10 m apart will be ..... if the dip at the place is 45°.
(a) 0.3 V (b) 3.0 V (c) 3 mV (d) none of these

26. A conducting rod PQ is moving parallel to x-axis in a uniform magnetic field directed in the positive y-direction. The end P of the rod will become

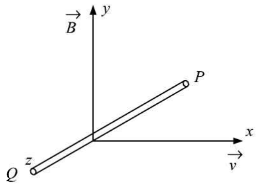


Fig. 27.52

(a) sometime positive and sometime negative
(b) positive
(c) neutral
(d) negative

27. If a bar magnet is dropped vertically into a long metallic tube then its final acceleration a will be
(a) $a = 0$
(b) $a = g$
(c) $a > g$
(d) $a < g$

28. The value of current in the armature of a DC motor is maximum when the motor
(a) starts rotating with mean speed
(b) has gained full speed
(c) is switched off
(d) just starts rotating

29. A square loop of side l is moved in a uniform magnetic field with a velocity v. Then the equivalent circuit is ...... (R is resistance of the loop)

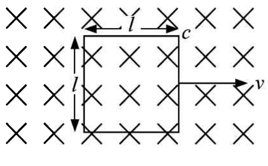
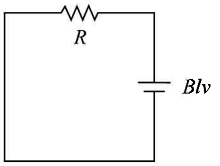
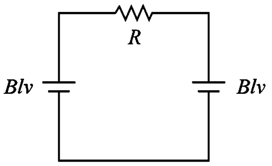
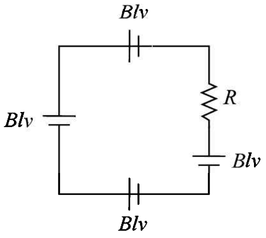


Fig. 27.53

(a)


(b)


(c)


(d) none of these

30. A rod of length l is moved with a speed v in a transverse magnetic field. If the resistance of the rod is R then current in the rod is
(a) Blv/R (b) $Blv/2R$
(c) $2Blv/R$ (d) zero

31. A rectangular loop of area 0.4 m² is lying in a magnetic field of 4×10^{-3} T. If the plane of the loop is at right angles to the magnetic field then the magnetic flux passing through the loop will be
(a) zero (b) 0.6×10^{-3} Wb
(c) 1.6×10^{-3} Wb (d) 4×10^{-3} Wb

32. Two inductance coils, of same self inductance, L are connected in parallel and the distance between them is large. The resultant self inductance of the coil will be
(a) $L/2$ (b) $L/4$ (c) $2L$ (d) L

33. Two coils of self inductances L_1 and L_2 are inclined at 60°. The mutual inductance between them is .... (K is coupling constant)
(a) $K\sqrt{L_1L_2}$ (b) $\frac{\sqrt{3}}{2}K\sqrt{L_1L_2}$
(c) $\frac{K}{2}\sqrt{L_1L_2}$ (d) $\frac{K}{2}(L_1 + L_2)$

34. A coil of area A_0 is lying in a magnetic field whose value changes from B_0 to $4B_0$ in time t(in seconds). The induced emf in the coil will be
(a) $\frac{3B_0}{A_0t}$ (b) $\frac{4A_0B_0}{t}$
(c) $\frac{4B_0}{A_0t}$ (d) $\frac{3A_0B_0}{t}$

35. A rectangular loop of area 0.2 m² is lying in a magnetic field of 5×10^{-2} T at an angle of 60° with the magnetic field. The current induced in this loop will be ...... if the loop turns by 180° in 0.2 s. $R_{\text{loop}} = 5\Omega$
(a) 5×10^{-3} A (b) 5×10^{-4} A
(c) 10^{-2} A (d) zero

36. Magnetic field B_0 exists perpendicular inwards. The resistance of the loop is R. When the switch is made closed the current induced in the circuit is

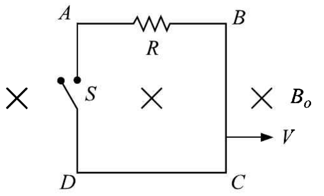


Fig. 27.54

(a) $\frac{Blv}{R}$ (b) $\frac{3Blv}{R}$

(c) $\frac{2Blv}{R}$ (d) zero

37. If the turn ratio of a transformer is 2 and the impedance of primary coil is 250 Ω then the impedance of secondary coil will be

(a) 125Ω (b) 250Ω (c) 500Ω (d) 1000Ω

38. A coil of area 0.01 m^2 is lying in a perpendicular magnetic field of 0.1 T. If a current of 10A is passed in it then the maximum torque acting on the coil will be

(a) 0.001 Nm (b) 0.01 Nm

(c) 0.8 Nm (d) 1.1 Nm

39. The magnetic flux in a coil of 100 turns increases by 12×10^3 Maxwell in 0.2 s due to the motion of a magnet. The emf induced in the coil will be

(a) 0.06 V (b) .6 V (c) 6 V (d) 60 V

40. The length of side of a square coil is 50 cm and number of turns in it is 100. If it is placed at right anglels to such a magnetic field which is changing at the rate of 4 Ts^{-1} then induced emf in the coil will be

(a) 0.1 V (b) 1.0 V (c) 100 V (d) 10 V

41. The quantity in electricity which is equivalent to mass is

(a) I (b) C (c) R (d) L

42. A galvanometer if vibrated fast will

(a) generate emf (b) not generate emf

(c) get damaged (d) shed off needle

43. The current obtained from a generator is

(a) diffusion current

(b) drift current

(c) displacement current

(d) any of these

44. Two coils are joined in series as shown. If their self inductances are L_1 and L_2, the equivalent self inductance is

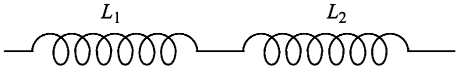


Fig. 27.55

(a) $L_1 + L_2$ (b) $\sqrt{L_1^2 + L_2^2 - L_1L_2}$

(c) $L_1 + L_2 + 2\sqrt{L_1L_2}$ (d) $L_1 - L_2$

45. The emf generated in a loop when it enters a magnetic field with a velocity as shown in Fig. 27.56 is best depicted in

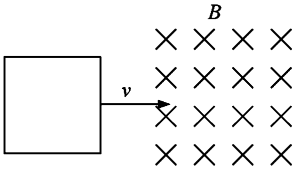
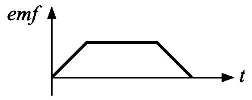
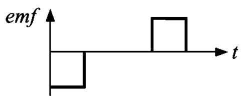
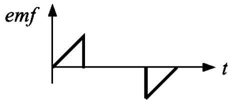
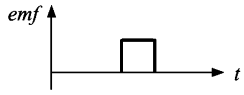


Fig. 27.56

(a)


(b)


(c)


(d)


46. A metallic circular ring is suspended by a string and is kept in a vertical plane. When a magnet is brought towards the ring it will

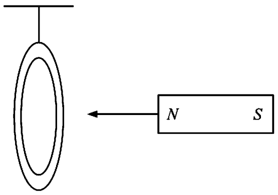


Fig. 27.57

(a) get displaced towards the magnet

(b) remain stationary

(c) get displaced away from the magnet

(d) can't say

47. A conducting rod of length l is falling with velocity v in a uniform horizontal magnetic field B normal to the rod. The induced emf between the ends of the rod will be

(a) zero (b) $Bvl/2$ (c) Blv (d) $2Bvl$

48. The self inductance of two coils wound over one another are L_1 and L_2 then mutual inductance is

(a) $> \sqrt{L_1L_2}$ (b) $< \sqrt{L_1L_2}$

(c) $= \sqrt{L_1L_2}$ (d) $\geq \sqrt{L_1L_2}$

49. If the input power and output power in an electric motor are 540 W and 382 W respectively, its efficiency will be

(a) 100% (b) 91% (c) 71% (d) 50%

50. Two coils P and Q are lying a little distance apart coaxially. If a current I is suddenly set up in the coil P then the direction of current induced in coil Q will be

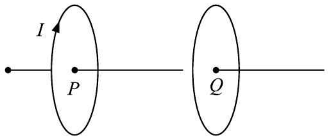


Fig. 27.58

(a) anticlockwise (b) towards north

(c) clockwise (d) towards south

51. The coefficient of self induction of a coil is given by

(a) $L = \frac{dI}{dt}e^2$ (b) $L = \frac{edI}{dt}$

(c) $L = \frac{e}{\left(-\frac{dI}{dt}\right)}$ (d) $L = \frac{dI}{edt}$

52. A capacitor C is connected to a conducting rod of length l moving with a velocity v in a transverse magnetic field B then the charge developed in the capacitor is

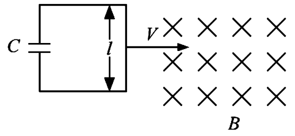


Fig. 27.59

(a) zero (b) $BlvC$

(c) $\frac{BlvC}{2}$ (d) $\frac{BlvC}{3}$

53. The time constant in an L–R circuit is that time in which the value of current in the circuit at make becomes

(a) 37%/I_0 (b) I_0

(c) 63% I_0 (d) $I_0/2$

54. The inductance of a solenoid is 5 Henry and its resistance is 5 Ω. If it is connected to a 10 V battery then the time taken by the current to reach nine tenth of its maximum value is
(a) 1.2 s (b) 1.4 s (c) 2.3 s (d) 4.0 s

55. Two coils are kept mutually perpendicular as shown in the figure. Their mutual inductance is

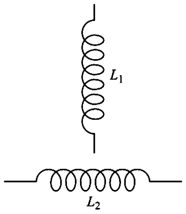


Fig. 27.60

(a) $\sqrt{L_1L_2}$
(b) slightly less than $\sqrt{L_1L_2}$
(c) > $\sqrt{L_1L_2}$
(d) 0

56. The turn ratio (r) for a step up transformer is
(a) $r = 0$ (b) $r > 1$ (c) $r < 1$ (d) $r = 1$

57. Number of turns in a primary coil per unit length is 200 m^{-1} and total number of secondary turns is 80. If the coil is wound on a former of diameter 6cm then mutual inductance at the best could be
(a) 57.6 mH (b) 0.576 μH
(c) 5.76 mH (d) 57.6 μH

58. The expression for the induced emf generated in a coil as a result of change in magnetic flux linked with it is
(a) $e = \frac{d}{dt}(\vec{A}\cdot\vec{B})$ (b) $e = -B \cdot \frac{dA}{dt}$
(c) $e = -\frac{d}{dt}(\vec{A}\times\vec{B})$ (d) $e = -A \cdot \frac{dB}{dt}$

59. The magnetic flux linked with a coil is $\phi \le 8t^2 + 3t + 5$ Wb. The induced emf in the fourth second will be
(a) 145 V (b) 139 V (c) 67 V (d) 16 V

60. A copper disc of radius 0.1 m rotates about its axis in a uniform magnetic field of 0.1 T at 10 rotations per second. The plane of the disc remains normal to the magnetic field. The induced emf along the radius of the disc will be
(a) $2\pi \times 10^{-2}$ V (b) $\pi \times 10^{-2}$ V
(c) $\frac{2\pi}{10}$ V (d) $\frac{\pi}{10}$ V

61. The number of turns in a coil of wire of fixed radius is 600 and its self inductance is 108 mH. The self inductance of a coil of 500 turns will be
(a) 77 mH (b) 76 mH
(c) 75 mH (d) 74 mH

62. An emf of 15 volt is applied in circuit of inductance 5 henry and resistance 10 Ω. The ratio of currents flowing at $t = \infty$ and $t = 1$ second will be
(a) e^{-1} (b) $\frac{e^2}{e^2 - 1}$
(c) $1 - e^{-1}$ (d) $\frac{\sqrt{e}}{\sqrt{e} - 1}$

63. A galvanometer is based on the principle of
(a) force experienced by a current carrying conductor in magnetic field
(b) torque experienced by a current carrying coil in a magnetic field
(c) force and torque experienced by a current carrying coil
(d) Fleming's right hand rule

64. A coil of area 80 cm^2 and number of turns 50 is rotating about an axis perpendicular to a magnetic field of 0.05 T at 2000 rotations per minute. The maximum value of emf induced in it will be
(a) 2/3 V (b) 4π/3 V
(c) 10π/3 V (d) 200 V

65. Hall effect can be used to determine
(a) charge carrier density
(b) eddy current
(c) back emf
(d) mutual inductance

66. The value of coefficient of mutual induction for the arrangement of two coils shown in the figure is

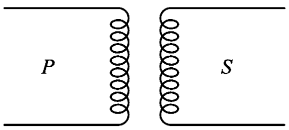


Fig. 27.61

(a) positive (b) maximum
(c) negative (d) zero

67. 220V AC is applied across the terminals A and B of the figure shown. The output obtained through variable terminal x could

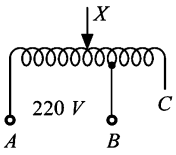


Fig. 27.62

(a) exceed 220 V
(b) not exceeed 220 V
(c) vary between 0 and 220 V
(d) vary between 0 and 150 V

68. A coil and a bulb are connected in series with a DC source. On inserting a soft iron core in the coil its light will
(a) go out (b) become intense
(c) dim (d) remain unchanged

69. The self inductance of a toroid is
(a) $\frac{\mu_0 N^2 r^2 \pi}{2}$ (b) $\frac{\mu_0 N^2 \pi r}{2}$
(c) $\frac{\mu_0 N^2 r}{2}$ (d) $\mu_0 N^2 r \pi$

70. When a conducting ring is moved in a magnetic field, the total charge induced in it depends on the
(a) total change in the magnetic flux
(b) initial magnetic flux
(c) final magnetic flux
(d) the rate of change of magnetic flux

71. When an electrical motor is running, it should generate induced emf. The problem of induced emf is tackled by
(a) using a starter circuit
(b) using a compensator
(c) increasing the rotational frequency
(d) decreasing the rotational frequency

72. Electric current is flowing in same direction in two coaxial coils. On increasing the distance between the two coils the value of current will

(a) not change
(b) increase
(c) decrease
(d) cannot be predicted

73. Number of turns in a generator coil is 10. The area of the coil is $4 \times 10^{-2}\,m^2$. This coil is rotating at the rate of 20 rotations per second about an axis lying in its own plane in a perpendicular magnetic field of 0.3T. The maximum emf induced in the coil will be

(a) zero (b) 13.1V (c) 30.2V (d) 60.4V

74. The rate of change of magnetic flux density through a circular coil of area $10^{-2}\,m^2$ and number of turns 100 is $10^3\,Wbm^{-2}s^{-1}$. The value of emf will be

(a) 10^{-3} V (b) 10^{-2} V
(c) 10V (d) 10^3 V

75. A current is flowing in a wire *C* as shown in the figure. The force on this conducting wire will be towards

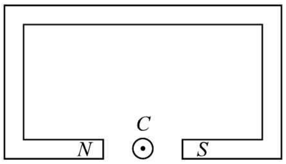


Fig. 27.63

(a) west (b) south
(c) east (d) north

76. The correct relation between the impedance of secondary coil with that of primary coil is

(a) $Z_S = Z_P \left(\frac{N_S}{N_P}\right)^2$ (b) $Z_S = Z_P$

(c) $Z_S = Z_P \left(\frac{N_P}{N_S}\right)^2$ (d) $Z_S = Z_P \frac{N_P}{N_S}$

77. A small piece of wire is passed through a magnetic field between the pole pieces of a horseshoe magnet in 0.1 s. The induced emf generated in it is 4×10^{-3} V. The magnetic flux between the poles is

(a) 4×10^{-6} Wb (b) 4×10^{-4} Wb
(c) 4×10^{-3} Wb (d) 4×10^{-2} Wb

78. The coaxial coils are very close to each other and their mutual inductance is 5mH. If a current 50 sin 500*t* is passed in the primary coil then the maximum emf induced in the secondary coil will be

(a) 125V (b) 150V
(c) 500V (d) 5000V

79. The self inductance of a coil is 2Henry and its resistance is 20Ω. The value of current in it changes from 10A to 9A in 0.01s. The time constant of the coil will be

(a) 1 ms (b) 10 ms (c) 0.1s (d) 2 s

80. The number of turns in the primary of a transformer is 100 and a curent of 8A is flowing in it. If the input power is 1kW then to obtain 500V in the secondary, the number of turns in it will be

(a) 400 (b) 300 (c) 200 (d) 100

81. An ideal coil of inductance 10 Henry is connected in series with a 5Ω resistance and a 5V battery. The value of current in the circuit after 2s will be (in ampere)

(a) $(1 - e^{-1})$ (b) $(1 - e)$
(c) e^{-1} (d) e

ANSWERS

1. (d)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (b)	**10.** (a)
11. (a)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (c)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (b)
21. (b)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (b)	**27.** (a)	**28.** (d)	**29.** (b)	**30.** (d)
31. (c)	**32.** (a)	**33.** (c)	**34.** (d)	**35.** (c)	**36.** (a)	**37.** (d)	**38.** (b)	**39.** (a)	**40.** (c)
41. (d)	**42.** (a)	**43.** (c)	**44.** (c)	**45.** (b)	**46.** (c)	**47.** (c)	**48.** (c)	**49.** (c)	**50.** (c)
51. (c)	**52.** (b)	**53.** (c)	**54.** (c)	**55.** (d)	**56.** (b)	**57.** (d)	**58.** (a)	**59.** (c)	**60.** (b)
61. (c)	**62.** (b)	**63.** (b)	**64.** (b)	**65.** (a)	**66.** (d)	**67.** (a)	**68.** (c)	**69.** (c)	**70.** (a)
71. (a)	**72.** (b)	**73.** (b)	**74.** (d)	**75.** (d)	**76.** (a)	**77.** (b)	**78.** (a)	**79.** (c)	**80.** (a)
81. (a)									

28

AC Current

BRIEF REVIEW OF THE CONCEPTS

AC If the magnitude of current varies with time then such a current is called AC or alternating current. AC is of four types: (a) sinusoidal (b) complex periodic (c) aperiodic (d) random.

Sinusoidal AC If the current or voltage varies in accordance with sine or cosine function or their combination, then such a current or voltage is termed as sinusoidal.

$V = V_P \sin \omega t$ or $V = V_P \sin(\omega t \pm \phi)$

or $I = I_P \cos(\omega t \pm \phi)$ or $I = I_P \sin(\omega t \pm \phi)$

then V is the instantaneous value of voltage, V_P is its peak or maximum voltage and ω is angular frequency

$$\omega = 2\pi f \quad \text{or} \quad \omega = 2\pi/T$$

f is linear frequency $f = 1/T$, T is time period; ϕ is initial phase angle or angle of repose or epoch.

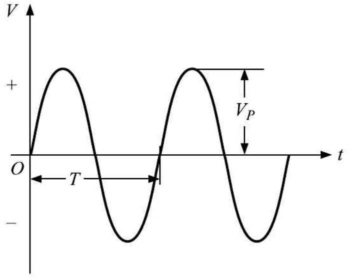


Fig. 28.1

Four values of AC voltage or current are defined as:

(a) peak voltage/peak current (V_P or I_P)
(b) mean or average voltage/average current (V_{av} or I_{av})
(c) RMS voltage/RMS current (V_{rms} or i_{rms})
(d) peak-to-peak voltage/peak-to-peak current (V_{PP} or I_{PP})

Mean or average voltage $(V_{av}) = \dfrac{2}{T}\displaystyle\int_0^{T/2} V_P \sin \omega t\, dt$

Since the mean votlage for a complete cycle is zero, we define it for half the cycle. For sinusoidal voltage V_{avg}

$= 0.63 V_0 = \dfrac{2V_P}{\pi}$ similarly

$i_{\text{avg}} = 0.63 I_P$

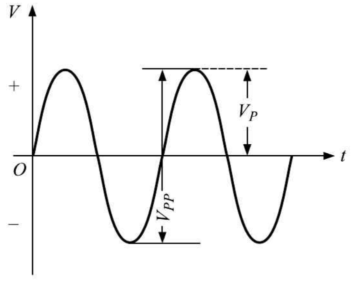


Fig. 28.2

RMS or root mean square voltage It is also called virtual or effective voltage. It is that value of AC voltage which will produce same amount of heat in a given resistance in a given time as is done by DC voltage in the same resistance in the same time.

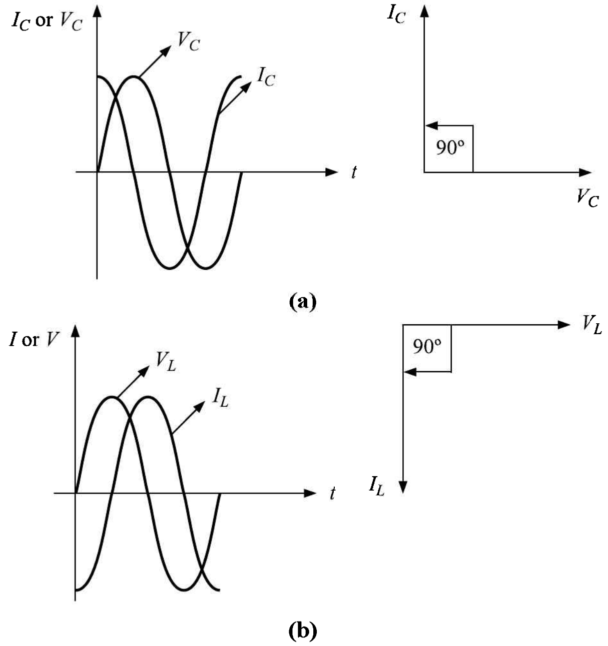
$$V_{\text{rms}} = \frac{1}{T}\int_0^T V^2 dt = \frac{1}{2}\int_0^T V_P^2 \sin^2 \omega t = \frac{V_P^2}{2}$$

or $$V_{\text{rms}} = \frac{V_P}{\sqrt{2}}.$$

Peak-to-peak voltage $V_{PP} = 2V_P$

Reactance The resistance offered by a capacitor or an inductor when AC is applied is called reactance. It also introduces a phase shift of $\dfrac{\pi}{2}$ in voltage and current.

Capacitive reactance $X_C = \dfrac{1}{C\omega}$ where ω is angular frequency. The current leads by $\pi/2$ when AC voltage is applied across a capacitor. Figure 28.3(a) shows phasor diagram in case of capacitor's V and I.



(a)

(b)

Fig. 28.3

Inductive reactance $X_L = L\omega$. Current lags by 90° from the voltage waveform as shown in the phasor diagrams of Fig. 28.3(b). Unit of reactance is ohm.

Note: *C* and *L* act like filters. Capacitance blocks DC and allows AC to pass

$$X_C = \frac{1}{C\omega} \text{ when } \omega \to 0 \; X_C \to \infty$$

$$X_L = L\omega \text{ when } \omega \to 0 \quad X_L \to 0$$

That is, inductor allows DC to pass without attenuation and attenuates AC.

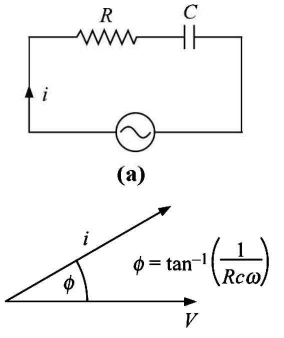
Impedance (*Z*) The net resistance offered in an AC circuit when both AC and DC components are present is called impedance. Unit is ohm.

DC component is resistance which does not offer any phase shift when AC is applied.

AC component offer phase shift between voltage and current when AC is applied across them. *L*, *C* and transformer are common examples.

Admittance (*Y*) Reciprocal of impedance is called admittance

$$Y = 1/Z \text{ unit ohm}^{-1} \text{ or Siemen (S).}$$



(a)

(b)

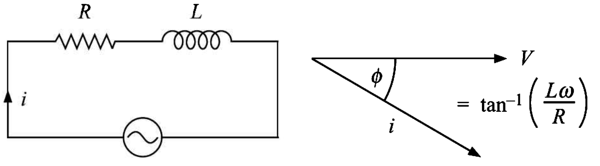
Fig. 28.4

Series *RC* circuit In series *RC* circuit of impedance

$$|Z| = \sqrt{R^2 + \left(\frac{1}{C^2\omega^2}\right)} \qquad \tan\phi = \frac{1}{RC\omega}$$

$$\text{Thus} \quad i = \frac{V_P}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \sin\left(\omega t + \tan^{-1}\frac{1}{RC\omega}\right)$$

Series *RL* Circuit In series *RL* circuit



(a) (b)

Fig. 28.5

$$|Z| = \sqrt{R^2 + L^2\omega^2} \qquad \tan\phi = \frac{L\omega}{R}$$

$$i = \frac{V_P}{|Z|}\sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right)$$

Series *RLC* circuit In series *RLC* circuit [Fig. 28.6] impedance

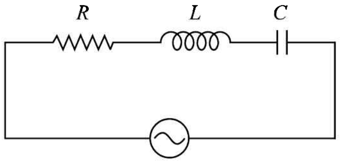


Fig. 28.6

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\text{or} \qquad |Z| = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

$$\tan\phi = -\frac{L\omega - \frac{1}{C\omega}}{R} = \frac{\frac{1}{C\omega} - L\omega}{R}$$

$$i = \frac{V_P}{|Z|}\sin(\omega t + \phi).$$

Three cases arise

(i) when at a particular frequency ω_0

$$X_L = X_C \text{ or } L\omega_0 = 1/C\omega_0$$

$$\text{That is, } \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_0 = \frac{1}{2\pi\sqrt{LC}}$$

This frequency is called resonant frequency.

At resonant frequency $Z = R$, that is, impedance is pure resistance, and, there is no phase shift between voltage and current. Moreover, impedance is minimum at this frequency, so current is maximum at resonance as shown in Fig. 28.7.

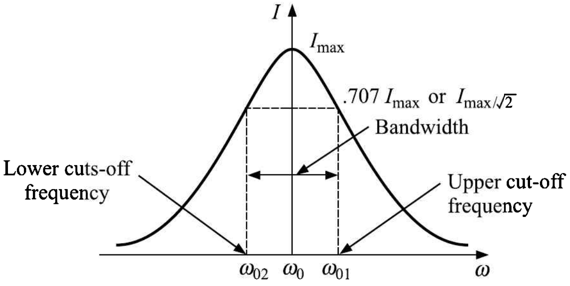


Fig. 28.7

(ii) If $\omega < \omega_0$ or $f < f_0$, the impedance is capacitive as $X_C > X_L$ and hence current leads the voltage.

(iii) If $\omega > \omega_0$ or $f > f_0$, the impedance is inductive as $X_L > X_C$ and hence current lags the voltage waveform.

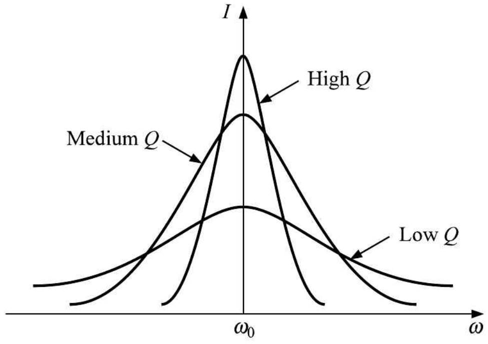


Fig. 28.8

***Q*-factor or quality factor** $\left(Q = \frac{L\omega}{r}\right)$ where *r* is internal resistance of the coil

$$Q = \frac{L\omega}{r} = \frac{\omega_0}{\omega_{01} - \omega_{02}}.$$

If *Q*-factor is large, resonance is sharp and bandwidth is small.

It is clear from Figure 28.8 that smaller the Q-value, poorer the resonance. However, the bandwidth increases.

Bandwidth is the band of allowed frequencies and is defined as the difference between upper and lower cut-off frequencies (or -3dB frequencies). See Fig. 28.9.

Cut-off frequency or -3dB frequency is that frequency at which the power becomes half of the maximum or current falls to $\frac{I_{\max}}{\sqrt{2}}$.

Power (P) $P = V_{\text{rms}}\, i_{\text{rms}} \cos\phi$

$\cos\phi$ is called power factor, ϕ is phase difference between V and I

$$= \frac{V_P}{\sqrt{2}}\frac{I_P}{\sqrt{2}}\cos\phi = \frac{V_P^2}{2|Z|}\cos\phi.$$

when $\phi = 90°$ $P = 0$, that is, when pure inductor or capacitor will be used the power will not be read by energy meters.

There could be two types of power–reactive power and active power. Active power is read by energy meters and is given by $P_{\text{active}} = V_{\text{rms}} I_{\text{rms}} \cos\phi$ and reactive power is not read by energy meters and may be called wattless power

$$P_{\text{reactive}} = V_{\text{rms}} I_{\text{rms}} \sin\phi$$

AC voltmeter and ammeters (either rectifier is used to convert AC to DC and then measure using DC voltmeter or ammeter) or hot wire instruments are used. Whatever, be the method of measuring, they read only rms value.

Transformer An ideal transformer is a lossless element. The principle is mutual induction.

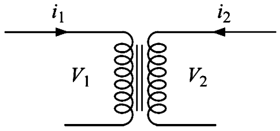


Fig. 28.9

$$P_1 + P_2 = 0$$

or $$V_1 i_1 + V_2 i_2 = 0 \quad \text{or} \quad \frac{V_1}{V_2} = -\frac{i_2}{i_1}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = -\frac{i_2}{i_1}$$

Power transformers are of two types namely step-up and step-down. In step up transformers $V_2 > V_1$ or $N_2 > N_1$. In step down transformers $V_2 < V_1$ or $N_2 < N_1$. However, current in secondary is higher than primary in a step-down transformer.

Step-up transformers are used at the generation end in a power distribution system and step down transformers are used at the distribution end near a locality. This is done in order to prevent power loss due to heating in the transmission line. Power loss $P = I^2R$. If we step up the voltage using step up transformer, $I \to 0$. Therefore, power loss is negligible in transmission line.

$$\text{Efficiency } \eta = \frac{\text{output power}}{\text{input power}} = \frac{V_{\text{out}}}{V_2} = \frac{V_2 - i_2 r}{V_2}$$

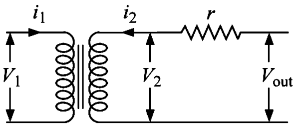


Fig. 28.10

Losses in transformer may be divided in two categories

(a) copper loss (due to resistance of copper winding)

(b) magnetic losses (eddy current loss, flux linkage loss, hysteresis loss)

Eddy current loss is minimised using laminated core in the form of E and I or π square core.

Flux linkage loss is prevented by winding one coil over the other. Hysteresis loss is minimised using soft iron core with four per cent Si.

Generator Generators are of two types, AC generator and DC generators. The basic difference in construction is that in case of AC generators slip rings are used and in DC generators split rings are used.

Split rings are used so that after every half cycle the direction of current reverses.

A coil is moved in the magnetic field and emf is generated.

Emf $V = NBA_0\omega \sin\omega t$

Area at any instant is $A = A_0 \cos\omega t$

where, N = number of turns
B = magnetic induction
A = lb (Area of coil)
ω = angular frequency

SHORT-CUTS AND POINTS TO NOTE

(i) $$i_{\text{avg}} = \frac{\int_0^T I\,dt}{\int_0^T dt} = \frac{1}{T}\int_0^T I\,dt$$

if sinusoidal or complex periodic then integrate for half the cycle.

$i_{av} = 0.636 I_P = \frac{2I_P}{\pi}$ for sinusoidal currents.

$$i_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

rms value is also called apparent or effective or virtual value of AC.

Peak-to-peak voltage $V_{PP} = 2V_P$ where V_P is peak voltage.

(ii) Form factor of AC

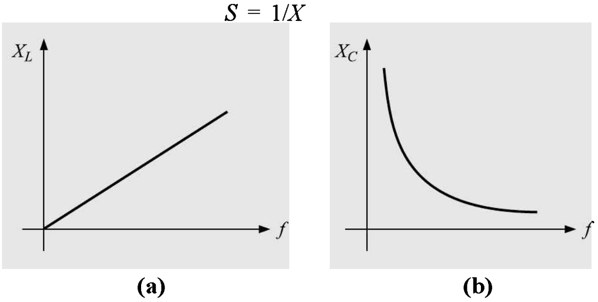
$$F = \frac{i_{\text{rms}}}{i_{av}}$$

$F = \frac{\pi}{2\sqrt{2}}$ for sinusoidal AC.

(iii) Capacitive reactance $X_C = \frac{1}{C\omega}$ Therefore X_C decreases as f increases or ω increases.

$X_L = L\omega$ increases if ω increases. The reciprocal of reactance is called susceptance (S)

$$S = 1/X$$



(a) **(b)**

Fig. 28.11

(iv) Impedance for series RC circuit

$$|Z| = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \frac{1}{C^2\omega^2}}$$

In series RC circuit current leads the voltage waveform by ϕ.

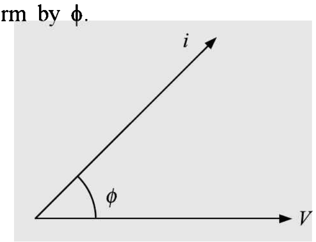


Fig. 28.12

$$\angle\phi = \tan^{-1}\frac{X_C}{R}$$

$$= \tan^{-1}\left(\frac{1}{RC\omega}\right)$$

$$i = \frac{V_P}{|Z|}\sin(\omega t + \phi)$$

(v) For series RL circuit

$$|Z| = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + L^2\omega^2}$$

In series RL circuit current lags the voltage by ϕ

$$\angle\phi = \tan^{-1}\left(\frac{X_L}{R}\right)$$

$$= \tan^{-1}\left(\frac{L\omega}{R}\right)$$

$$i = \frac{V_P}{|Z|}\sin(\omega t - \phi)$$

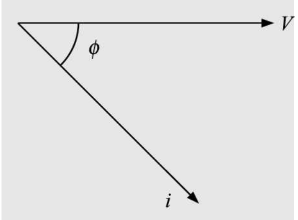


Fig. 28.13

(vi) For series RLC circuit

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

In series RLC circuit phase angle ϕ is given by

$$\angle\phi = \tan^{-1}\left(\frac{\frac{1}{C\omega} - L\omega}{R}\right)$$

Note that phase shift varies with frequency

$$i = \frac{V_P}{|Z|}\sin(\omega t + \phi)$$

if ϕ = positive; impedance is capacitive and $f < f_0$

If $f > f_0$, impedance is inductive and ϕ is negative.

At $f = f_0$ $\phi = 0$ impedance is pure, resistive and minimum. Therefore current is maximum. Fig. 28.14 shows variation of phase shift ϕ with frequency f.

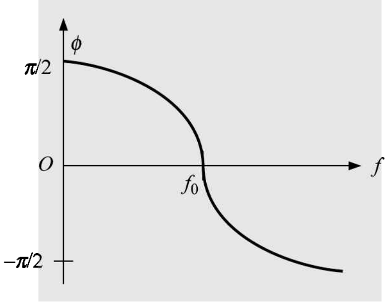
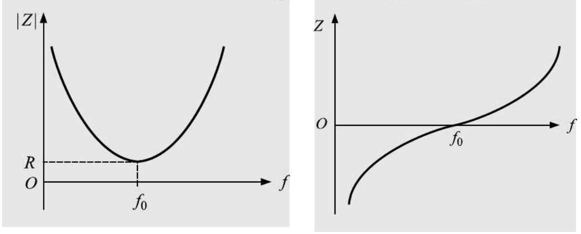


Fig. 28.14

(vii) Impedance $|Z|$ varies with frequency for series RLC circuit as shown in Figure 28.15(a) and (b)



(a) (b)

Fig. 28.15

Note that phase shift between voltage and current is negative of phase angle of impedance. That is,

$$\angle i = -\angle Z$$

(viii) At cut-off frequencies

$$Z = \sqrt{2}R$$

$$R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2 = 2R^2 \text{ or } L\omega - \frac{1}{C\omega} = R$$

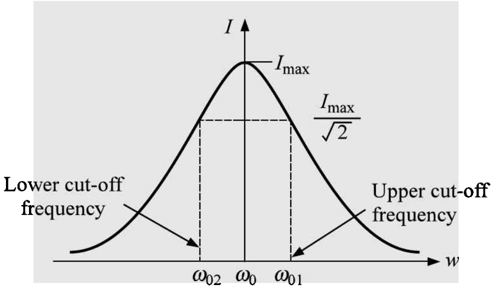


Fig. 28.16

or $$LC\omega^2 - RC\omega - 1 = 0$$

or $$\omega = \frac{RC \pm \sqrt{R^2C^2 + 4LC}}{2LC}$$

represent cut off frequencies.

(ix) Q factor of a coil $Q = \dfrac{L\omega}{r} = \dfrac{\omega_0}{\omega_{01} - \omega_{02}}$

Cut-off frequencies are also called half power frequencies or -3dB frequencies.

(x) Active power $= V_{rms}i_{rms}\cos\phi = \dfrac{V_{rms}^2}{2|Z|}\cos\phi$

(It is read by energy metres)

Reactive power = $V_{rms} i_{rms} \sin\phi$
(also called wattless power).
(not read by energy metres)

$\cos\phi$ is called Power factor

(xi) In series *RC* circuit

$$P = \frac{V^2_{\text{rms}}}{2|Z|}\cos\phi = \frac{V^2_{rms}R}{2|Z|^2} = \frac{V^2_{\text{rms}}R}{2\left(R^2+\frac{1}{C^2\omega^2}\right)}$$

(xii) In series *RL* circuit

$$P = \frac{V^2_{\text{rms}}}{2|Z|}\cos\phi = \frac{V^2_{\text{rms}}R}{2|Z|^2} = \frac{V^2_{\text{rms}}R}{2(R^2+L^2\omega^2)}$$

(xiii) In series *RLC* circuit

$$P = \frac{V^2_{\text{rms}}\cos\phi}{2|Z|} = \frac{V^2_{\text{rms}}R}{2|Z|^2} = \frac{V^2_{\text{rms}}R}{2[R^2+(X_L-X_C)^2]}$$

that is, $\cos\phi = \dfrac{R}{|Z|}$

Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad P_{\text{resonance}} = \frac{V^2_{\text{rms}}}{2R}$$

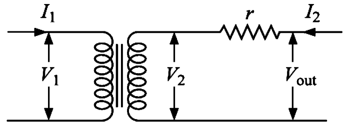
(xiv) Kirchoff's laws can be applied to AC as well if stated as follows.

KCL or Junction law the algebraic sum of all the currents entering a node at any instant is zero.

KVL or Loop law The algebraic sum of all the potential drops in a loop at any instant is zero.

(xv) AC voltage and currents are phasors. so are impedance/ reactances.Therefore treat them analytically as vectors.

(xvi) In transformer $\dfrac{V_1}{V_2} = \dfrac{N_1}{N_2} = -\dfrac{I_2}{I_1}$ negative sign shows phase shift of 180°

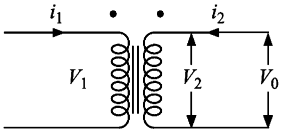


(a)

$$\text{Efficiency } \eta\ \% = \frac{P_{\text{out}}}{P_{\text{in}}}\times 100$$

$$= \frac{V_0}{V_2}\times 100 = \frac{V_2 - I_2 r}{V_2} \text{ x } 100$$

Dot on the transformer shows the winding is in the same direction.



(b)

Fig. 28.17

$$i_1 = \frac{V_1}{\left(\frac{N_1}{N_2}\right)R_{eq}}.$$

(xvii) For *L*, *C* and *R* in parallel circuit or parallel *LCR* circuit is also called anti-resonant circuit

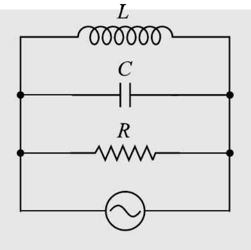


Fig. 28.18

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}$$

$$\frac{1}{|Z|} = \sqrt{\frac{1}{R^2}+\left(\frac{1}{1/C\omega}-\frac{1}{L\omega}\right)^2}$$

$$= \sqrt{\frac{1}{R^2}+\left(C\omega-\frac{1}{L\omega}\right)^2}.$$

(xviii) Fourier analysis can be employed for complex periodic waves.

(xix) In our country AC mains has a frequency 50Hz.

(xx) For maximum power to be transferred in an AC circuit

$$Z_L = Z_S^* \text{ or } R_L + jX_L = R_S - jX_S$$

or $\quad R_L = R_S \quad X_L = -X_S$

that is, if source is inductive then load should be equivalent capacitive or vice versa.

⚠ CAUTION

(i) Considering that impedances/reactances are added like resistors.

- ■ These quantities are phasors, vector algebra should be applied.

(ii) Not remembering trignometric formulae like sin(*A* + *B*), sin(*A* – *B*), cos(*A* + *B*) and cos(*A* – *B*).

- ■ Must remember these formulae.

(iii) Considering power in AC is $V \cdot I$.

- ■ Power in AC is $P = V_{\text{rms}} I_{\text{rms}} \cos\phi$. Also remember about wattless power.

(iv) Considering that Kirchoff's laws can be applied only to resistive networks.

- ■ These can be applied to AC networks provided we consider voltage or current at a particular instant.

(v) Considering that transformer can step up or step down even DC.

- ■ Transformer can only step up or step down AC because mutual inductance can be applied only to AC or time varying currents/voltages. To step up or step down DC, potential divider circuit or rheostat may be used.

(vi) Considering that $V_{\text{rms}} = \dfrac{V_P}{\sqrt{2}} = 0.707\,V_P$

or $V_{av} = 0.633\,V_P$ for all types of ACs.

- $V_{rms} = 0.707 V_P$; $V_{av} = 0.633 V_P$ only for sinusoidal AC.

Apply $V_{rms} = \sqrt{\frac{1}{T}\int_0^T V^2 dtW}$

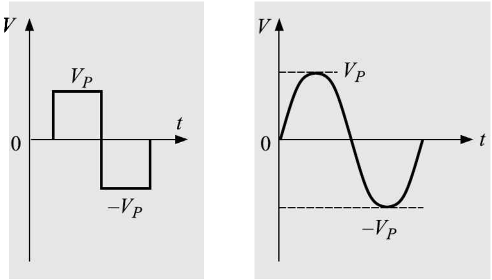


Fig. 28.19

and $V_{av} = \frac{1}{T}\int_0^T V\,dt$ for all other ACs.

$$V_{av} = \frac{2}{T}\int_0^T V\,dt.$$

If voltage goes both positive and negative as illustrated in Fig. 28.19.

(vii) Not remembering the frequency of AC mains.

- Frequency of AC mains in our country is 50 Hz. 50 Hz is minimum frequency required to support TV. In our country a 625 lines PAL – Dsystem is used. Two frames even and odd each of 312 $\frac{1}{2}$ lines appear 25 times a second (Critical flicker frequency) when 50Hz is used.

Solved Problems

1. In a circuit L, C, R connected in series with an alternating voltage source of frequency f, the current leads the voltage by 45°. The value of C is

(a) $\frac{1}{\pi f(2\pi f L - R)}$ (b) $\frac{1}{2\pi f(2\pi f L - R)}$

(c) $\frac{1}{\pi f(2\pi f L + R)}$ (d) $\frac{1}{2\pi f(2\pi f L + R)}$

Solution (d) $\tan 45 = \dfrac{\frac{1}{2\pi f C} - 2\pi L f}{R}$

or $C = \dfrac{1}{2\pi f(2\pi f L + R)}$

2. The self inductance of a motor of an electric fan is 10 H. In order to impart maximum power at 50 Hz, it should be connected to a capacitance of

(a) 4μF (b) 8μF

(c) 1μF (d) 2μF

Solution (c) Maximum power is transferred at resonance

$\therefore$ $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$

or $c = \dfrac{1}{4\pi^2 f_0^2 L} = \dfrac{1}{4\times 10\times (50)^2\times 10}$

$= 10^{-6}\mu F = 1\mu F$

3. A circuit has resistance of 12 Ω and an impedance of 15 Ω. The power factor of the circuit will be

(a) 0.8 (b) 0.4

(c) 1.25 (d) 0.125

Solution (a) $\cos\phi = \dfrac{R}{|Z|} = \dfrac{12}{15} = 0.8.$

4. The phase difference between alternating current and *emf* is π/2. Which of the following cannot be the constituent of the circuit.

(a) C alone (b) R, L

(c) L, C (d) L alone

Solution (b) $0 < \phi < 90°$ for a series RL circuit.

5. The circuit shown in fig. 28.22 acts as a

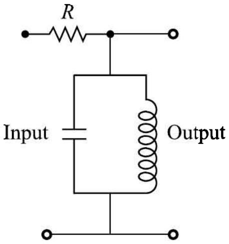


Fig. 28.20

(a) tuned filter (b) low pass filter

(c) high pass filter (d) rectifier

Solution (a)

6. A 50 Hz, 20 V ac source is connected across RC series circuit as shown in fig. 28.21. If the voltage across R is 12 V then voltage across capacitor is

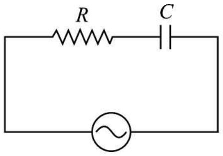


Fig. 28.21

(a) 8 V

(b) 16 V

(c) 10 V

(d) cannot be predicted as values of R and C are not given

Solution (b) $20^2 = 12^2 + V_C^2$ or $V_C = 16$ V

7. In an LCR circuit the capacitance is made 1/4th then what should be the change in inductance that the circuit remains in resonance again.

(a) 8 times (b) 1/4 times
(c) 2 times (d) 4 times

Solution (d) $f_0 = \frac{1}{2\pi\sqrt{LC}}$

To keep frequency unchanged L be made 4 times.

8. Two inductors each equal to L are joined in parallel. The equivalent inductance is
(a) zero (b) $2L$
(c) L (d) $L/2$

Solution (a) Inductances are added like resistors.

$$\frac{1}{L_{er}} = \frac{1}{L_1}+\frac{1}{L_2}$$

$\therefore$ $L_{er} = L/2$.

9. The turn ratio of a transformer is 2 : 3. If the current through primary is 3A, then current through load resistance is
(a) 1A (b) 4.5A
(c) 2A (d) 1.5A

Solution (b) $\frac{I_p}{I_s} = \frac{N_s}{N_p}$ or $\frac{3}{I_s} = \frac{2}{3}$ or $I_s = 4.5$A.

10. The square root of the product of inductance and capacitance has dimensions of
(a) length (b) mass
(c) time (d) dimensionless

Solution (c) $f = \frac{1}{\sqrt{LC}}$ or $\sqrt{LC} = T$.

11. In a series RLC circuit $R = 300\,\Omega$, $L = 60\,\text{mH}$, $C = 0.5\,\mu\text{F}$. $V = 50\,\text{V}$, $\omega = 10^4\,\text{rad/s}$. Find the voltage across capacitor.
(a) 30V (b) 20V
(c) 60V (d) 50V

Solution (b) $I = \frac{V}{|Z|}$

$$= \frac{50}{\sqrt{300^2 + \left(60\times10^{-3}\times10^4 - \frac{1}{0.5\times10^4\times10^{-6}}\right)}}$$

$$= \frac{50}{\sqrt{300^2+400^2}} = \frac{1}{10}\text{A}$$

$$V_C = IX_C = \frac{1}{10}\times200 = 20\,\text{V}$$

12. In a series CR circuit excited by ac mains, $C = 10\,\mu\text{F}$, $R = 300\,\Omega$. Find power factor
(a) $\frac{1}{10}\times200$ (b) $\frac{3}{\sqrt{16}}$
(c) $\frac{3}{\sqrt{18}}$ (d) $\frac{3}{\sqrt{19}}$

Solution (d) $\cos\phi = \frac{R}{|Z|} = \frac{R}{\sqrt{R^2+\frac{1}{C^2\omega^2}}}$

$$= \frac{300}{\sqrt{300^2+\left(\frac{1}{100\pi\times10^{-5}}\right)^2}} = \frac{300}{100\sqrt{9+10}} = \frac{3}{\sqrt{19}}.$$

13. In the given circuit what is the potential drop across resistance

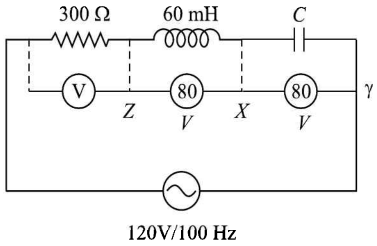


Fig. 28.22

(a) 40V (b) 80V
(c) 120V (d) zero

Solution (c) $\because$ $|V_L| = |V_C|$.

It is possible at resonance.

Therefore $V_R = V_{\text{app}} = 120\,\text{V}$.

14. In the Q13 what is the potential drop across ZY
(a) 160V (b) $80\sqrt{2}$
(c) 80V (d) zero

Solution (d) $V_{ZY} = V_C - V_L = 0$

15. The circuit shown in fig. 28.23 for output voltage V_{AB} acts as

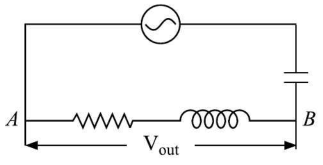


Fig. 28.23

(a) Bandpass filter
(b) Lowpass filter
(c) High pass filter
(d) Band reject filter

Solution (c)

16. For the circuit shown in fig. 28.24, the rms current is

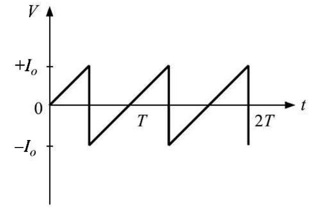


Fig. 28.24

(a) I_0 (b) $I_0/\sqrt{2}$
(c) $I_0/\sqrt{3}$ (d) $I_0/\sqrt{2}$

Solution (c) $I = \frac{2I_0t}{T}$ $0 < t < T/2$

$I = 2I_0(t - T)/T$ $T/2 < t < T$

$$I^2_{rms} = \frac{1}{T}\left(\int_0^T I^2dt + \int_{T/2}^T I^2dt\right)$$

$$= \frac{1}{T}\int_0^{T/2}\left(\frac{2I_0t}{T}\right)^2 dt + \frac{1}{T}\int_{T/2}^{T}\left(\frac{2I_0}{T}(t-T)\right)^2 dt$$

$I_{\text{rms}} = I_0/\sqrt{3}$.

17. The reactance of capacitor is 550 Ω, R = 100 Ω, in a series LCR circuit. Current leads the voltage by 53°. Find X_L

(a) 120 Ω (b) 140 Ω
(c) 210 Ω (d) 110 Ω

Solution (d) $\tan 53 = \dfrac{X_C - X_L}{R}$

$$\Rightarrow \frac{4}{3} = \frac{350 - X_L}{180} \quad \text{or} \quad X_L = 110\,\Omega$$

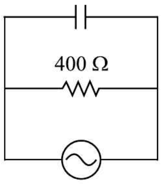
18. A 400 Ω resistor and a 6 μF capacitor are connected in parallel to a source of V_{rms} = 220 V and angular frequency 360 rad/s. Find the current in the resistor and capacitor.

(a) 0.55 A, 0.475 A (b) 0.55 A, 0.726 A
(c) 0.55 A, 0.176 A (d) none of these

Solution (a) $\dfrac{1}{X_C} = C\omega = 6 \times 10^{-6} \times 360 = 2.16 \times 10^{-3}$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + (C\omega)^2}$$

$$= \sqrt{\left(\frac{1}{400}\right)^2 + (2.16 \times 10^{-3})^2}$$



220V/360 rads

Fig. 28.25

$$I_{\text{net}} = 220 \times \frac{1}{Z}$$

$$= 220 \times 10^{-3} \times 3.3 = 0.726;$$

$$I_R = \frac{220}{400} = 0.55\text{A}$$

$$I_C = \sqrt{I_{net}^2 - I_R^2} = \sqrt{(0.726)^2 - (0.55)^2} = 0.475\text{A}$$

Short cut: $I_C = 220 \times \dfrac{1}{X_C}$

$$= 220 \times 2.16 \times 10^{-3} = 0.475\text{A}$$

$$I_R = 220 \times 2.5 \times 10^{-3} = 0.55\text{A}.$$

19. A 120 V ac line transformer is to supply 13000 V for a neon sign. To reduce shock hazard a 8.5 mA fuse is inserted. Find the maximum input power to the transformer

(a) 120 W (b) 121 W
(c) 110 W (d) 104 W

Solution (c) $P_{\text{input}} = P_{\text{output}}$

$$= 13000 \times 8.5 \times 10^{-3} = 110.5\text{W}$$

20. An LC circuit has L = 5 mH and C = 20 μF. $V = 5 \times 10^{-3} \cos\omega t$ is supplied ω is twice the resonant frequency. Find the maximum charge stored in the capacitor.

(a) 66.6 nC (b) 11.3 nC
(c) 23.2 nC (d) 33.3 nC

Solution (d) $\omega = 2\omega_0 = 2 \times \dfrac{1}{\sqrt{LC}} = \dfrac{2 \times 10^4}{\sqrt{10}}$

$$I = \frac{5 \times 10^{-3}}{\sqrt{(X_L - X_C)^2}}$$

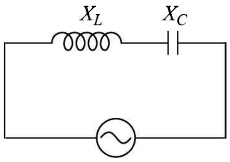


Fig. 28.26

$$= \frac{5 \times 10^{-3}}{\sqrt{(10\sqrt{10} - 2.5\sqrt{10})^2}}$$

$$= \frac{5 \times 10^{-3}}{7.5\sqrt{10}} = 2.1 \times 10^{-4}\text{A}$$

$$V_C = IX_C$$

and $Q = CV_C = I \times \dfrac{1}{C\omega} \times C$

$$= \frac{I}{\omega} = \frac{2.1 \times 10^{-4} \times \sqrt{10}}{2 \times 10^4} = 33.3\,nC$$

21. An AM radio operates 550 kHz to 1650 kHz. If L is fixed and C is varied for tuning then minimum and maximum value of C is

(a) C, 3C (b) C, 6C
(c) C, 9C (d) C, 12C

Solution $\dfrac{f_{\max}}{f_{\min}} = 3$

$$\therefore \quad \frac{\sqrt{LC_{\max}}}{\sqrt{LC_{\min}}} = 3 \quad \text{or} \quad \frac{C_{\max}}{C_{\min}} = 9$$

22. In an ideal transformer turn ratio is 2 : 3. If input voltage is 100 V/60 Hz then output voltage is

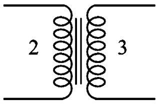


Fig. 28.27

(a) 150 V/90 Hz (b) 150 V/40 Hz
(c) 150 V/60 Hz (d) 66.6 V/60 Hz

Solution (c) $\dfrac{V_1}{V_2} = \dfrac{N_1}{N_2}$

$$\Rightarrow \frac{100}{V_2} = \frac{2}{3}$$ and frequency does not change.

23. A centre tapped transformer is rated 12 – 0 – 12 V. The peak voltage obtained is

(a) 12 V (b) $\dfrac{12}{\sqrt{2}}$ V
(c) $12\sqrt{2}$ V (d) 24 V
(e) $24\sqrt{2}$ V

Solution (c) $V_{\text{rms}} = 12\text{V}$ $\quad V_P = 1/rms\sqrt{2} = 12\sqrt{2}$ V

24. $I = 6\cos\omega t + 8\sin\omega t$ is applied across a 40 Ω resistor. Find the potential difference across the resistor.

(a) 560 V (b) 80 V
(c) 320 V (d) 400 V

Solution (d) $I_{\text{rms}} = \sqrt{6^2 + 8^2} = 10\text{A}$

$$V_R = I_{\text{rms}} \times R = 10 \times 40 = 400\text{V}$$

25. If the output is taken across a capacitor in a series *RLC* circuit then it acts as

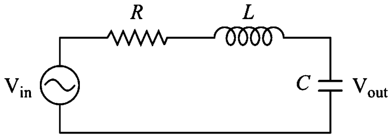


Fig. 28.28

(a) band pass filter
(b) high pass filter
(c) low pass filter
(d) band reject filter

Solution (c)

26. Plot the output voltage V_0 across the capacitor in a series *RC* circuit if square wave is inputted.

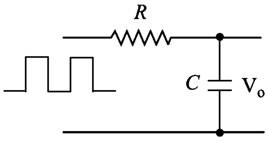
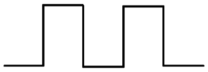
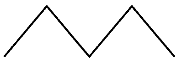
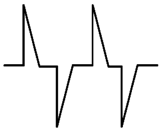
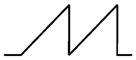


Fig. 28.29

(a)


(b)


(c)


(d)


Solution (b) $V_c = \frac{1}{C}\int i\,dt = \frac{V}{RC}\int dt$

Problems for Practice

1. The inductive reactance of a choke coil of $\frac{1}{4\pi}$ m Henry in an AC circuit of 50 Hz, will be
(a) 0.025 ohm (b) 0.25 ohm
(c) 2.5 ohm (d) 2.5 ohm

2. The impedance of series *LCR* circuit is

(a) $R + \left(\omega L + \frac{1}{\omega C}\right)$

(b) $R + \left(\omega L - \frac{1}{\omega C}\right)$

(c) $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

(d) $\sqrt{R^2 + \left(\omega L + \frac{1}{\omega C}\right)^2}$

3. The resonant frequency in an antiresonant circuit is

(a) $\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$ (b) $\frac{1}{2\pi}\sqrt{LC}$

(c) $\frac{1}{2\pi\sqrt{LC}}$ (d) $\frac{1}{2\pi}\sqrt{\frac{C}{L}}$

4. The alternating emf applied and the current flowing in an AC circuit are represented by $E = E_0 \sin \omega t$ and $I = I_0 \sin(\omega t + \pi/2)$ respectively. The power loss in the circuit will be

(a) $\frac{E_0 I_0}{4}$ (b) $\frac{E_0 I_0}{\sqrt{2}}$ (c) $\frac{E_0 I_0}{2}$ (d) zero

5. If the values of inductance and frequency in an AC circuit are 2 Henry and $10^3/2\pi$ Hz respectively then the value of inductive reactance will be
(a) $2 \times 10^3 \Omega$ (b) $2 \times 10^2 \Omega$
(c) $\frac{2 \times 10^3}{\pi} \Omega$ (d) $10^3 \Omega$

6. In the equation of AC $I = I_0 \sin \omega t$, the current amplitude and frequency will respectively be

(a) I_0, ω (b) $I_{rms}, \frac{\omega}{2\pi}$

(c) $\frac{I_0}{2}, \frac{\omega}{2\pi}$ (d) $I_0, \frac{\omega}{2\pi}$

7. If the value of virtual voltage across *L*, *C* and *R* in an *LCR* circuit are V_L, V_C and V_R respectively then the source voltage will be
(a) $V = V_L - V_C + V_R$ (b) $V = \sqrt{V_R^2 + (V_L^2 - V_C^2)}$
(c) $V = V_L + V_C + V_R$ (d) $V = \sqrt{V_R^2 + (V_L - V_C)^2}$

8. Energy in an inductance coil is stored in the form of
(a) light energy
(b) heat energy
(c) magnetic energy
(d) electrical energy

9. The reading of ammeter and voltmeter in an AC circuit are 10 A and 25 V respectively. The power loss in the circuit will be
(a) less than 150 W (b) 250 W
(c) less than 250 W (d) more than 250 W

10. Alternating current can be produced by a/an
(a) electric motor (b) generator
(c) turbine (d) transformer

11. Which of the following statements is correct for an AC circuit?
(a) the current depends on the components connected in circuit
(b) the current lags the voltage by a phase angle of 90°
(c) the current leads the voltage by a phase angle of 90°
(d) the current and the voltage are in same phase

12. If the current flowing in a choke coil of 2 Henry is decreasing at the rate of 5 amp s^{-1}, then induced emf across the ends of the coil will be
(a) –2.5 V (b) 2.5 V
(c) 10 V (d) 10^3 V

13. The value of admittance at resosnance in an anti-resotant circuit is

(a) $\frac{G^2}{S^2}$ (b) $\sqrt{G^2 + S^2}$

(c) $G^2 + S^2$ (d) $\sqrt{G^2 - S^2}$

14. A coil of self inductance L and resistance R is connected to a cell of emf E V. The value of current flowing in the circuit will be

(a) E/R (b) $\sqrt{E/R^2 + L^2}$

(c) $\sqrt{\frac{EL}{R^2 + L^2}}$ (d) E/L

15. An inductance coil of 1 Henry and a condenser of capacity 1 pF produce resonance. The resonant frequency will be

(a) $\frac{10^6}{2\pi}$ Hz (b) $\frac{10^6}{\pi}$ Hz

(c) $\frac{2\pi}{10^6}$ Hz (d) $2\pi \times 10^6$ Hz

16. In an LCR circuit the values of X_L, X_C and R are 300 Ω, 200 Ω and 100 Ω respectively. The total impedance of the circuit will be

(a) 141Ω (b) 200Ω (c) 310Ω (d) 600Ω

17. The correct curve between admittance (Y) and frequency (f) in an antiresonant circuit will be

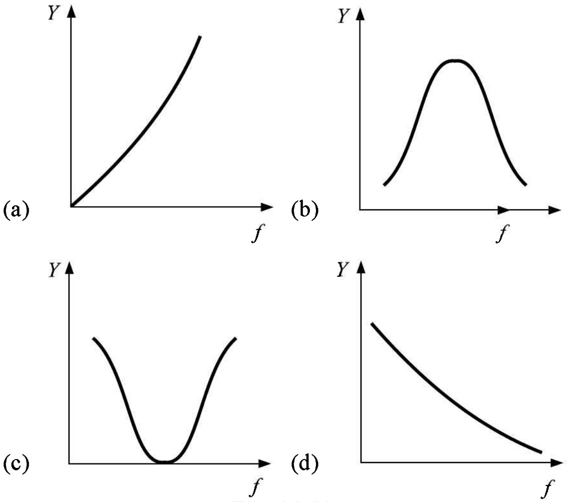


Fig. 28.30

18. The time taken by an AC of the frequency 50 Hz to complete one cycle will be

(a) 0.002 s

(b) 0.02 s

(c) 0.2 s

(d) 2 s

19. If the reactance of a choke coil is X_L and its resistance is R then

(a) $X_L = \infty$ (b) $X_L << R$

(c) $X_L >> R$ (d) $X_L = R$

20. An alternating voltage source is connected in an AC circuit whose maximum value is 170 V. The value of potential at a phase angle of 45° will be

(a) zero (b) 110.12 V

(c) 120.56 V (d) 240 V

21. An RC circuit is as shown in the following diagram. The capacitive reactance and impedance will be

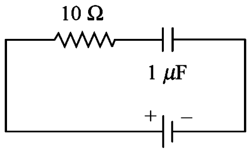


Fig. 28.31

(a) $\omega C, \sqrt{R^2 + C^2\omega^2}$

(b) $1/\omega C, \sqrt{R^2 + \frac{1}{C^2\omega^2}}$

(c) ∞, ∞

(d) zero, R

22. If the phase difference between the emf and the current in an AC circuit is ϕ then the RMS value of wattless current will be

(a) 0 (b) $I_{rms} \sin\phi$

(b) $I_{rms} \tan\phi/2$ (d) zero

23. The values of X_L, X_C and R in a Parallel LCR circuit are 8 Ω, 6 Ω and 10 Ω. The impedance of the circuit is

(a) 0 (b) $10\sqrt{2}$

(c) 9.2 Ω (d) 24.4 Ω

24. Choke coil in AC circuit is used for

(a) increasing voltage

(b) decreasing current

(c) increasing current

(d) decreasing voltage

25. Two bulbs, whose powers are P_1 and P_2 respectively are connected in parallel to a supply line of E V. The total power of the combination will be

(a) $P_1 - P_2$ (b) $P_1^{-1} - P_2^{-1}$

(c) $P_1^{-1} + P_2^{-1}$ (d) $P_1 + P_2$

26. The power factor of wattless current is

(a) 0 (b) 1/2 (c) 1 (d) ∞

27. The self inductance of the motor of an electric fan is 10 Henry. In order to impart maximum power at 50 Hz it should be connected to a capacitance of

(a) 3×10^{-6} Farad

(b) 2×10^{-6} Farad

(c) 10^{-6} Farad

(d) 10^{-4} Farad

28. Energy in a condenser is stored in the form of

(a) magnetic energy (b) heat energy

(c) electrical energy (d) light energy

29. Two heater wires of same length are first connected in series and then in parallel. The ratio of the amount of heat produced in two cases will be

(a) 1 : 2 (b) 1 : 4

(c) 2 : 1 (d) 4 : 1

30. Current in a series RL circuit ...... the voltage waveform

(a) leads by $\pi/2$ rad

(b) lags by $\pi/2$ rad

(c) leads by $< \pi/2$ rad

(d) lags by $< \pi/2$ rad

31. The capacity of a condenser is $2\pi \times 10^{-6}$ farad. It is connected to a voltage source of frequency 50 Hz. The impedance with a resistance of 500 Ω in series is

(a) zero (b) 0.2×10^6 Ω

(c) 0.5×10^3 Ω (d) $500\sqrt{2}$ Ω

32. The value of impedance of a series resonant circuit at cut-off frequencies is

(a) $\sqrt{R^2+\frac{1}{\omega^2C^2}}$ (b) $\sqrt{R^2+\omega^2L^2}$

(c) $\sqrt{2}\,R$ (d) R

33. The sharpness of resonance in a series *LCR* resonant circuit, as the resistance of the circuit is increased,

(a) changes from ∞ to 0
(b) goes on increasing
(c) changes from 0 to ∞
(d) goes on decreasing

34. The equation of alternating voltage and alternating current in an AC circuit are $E = E_0 \sin\omega t$ and $I = I_0 \sin(\omega t - \pi/4)$ ampere respectively. The power loss in the circuit will be

(a) $\frac{E_0I_0}{\sqrt{2}}$ (b) $\frac{E_0I_0}{2}$

(c) zero (d) $\frac{E_0I_0}{2\sqrt{2}}$

35. If in an AC circuit $X_L = X_C$ then the value of power factor will be

(a) 0 (b) 1/2 (c) 1 (d) ∞

36. The electric supply line in houses works at 220 V. The amplitude of emf will be

(a) 110 V (b) 220 V (c) 311 V (d) 440 V

37. Power in a series RLC circuit at off resonant frequency is

(a) $\frac{V_P^2}{2R}$ (b) $\frac{V_P^2R}{2[R^2+(X_L+X_C)^2]}$

(c) $\frac{V_P^2R}{2(R^2+X_C^2)}$ (d) $\frac{V_P^2R}{2[R^2+(X_L-X_C)^2]}$

38. If a tank circuit is used as resonance circuit then

(a) impedance is maximum at resonance
(b) impedance is minimum at resonance
(c) impedance becomes maximum
(d) impedance becomes zero

39. Which of the following phasor diagram best represents a series RC circuit?

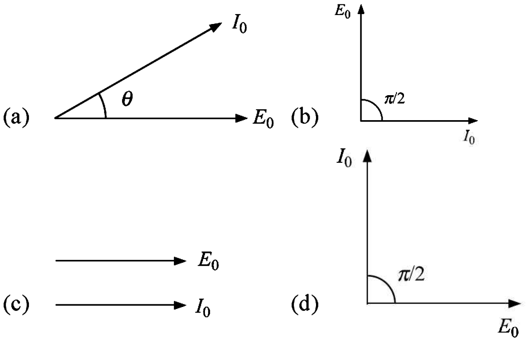


Fig. 28.32

40. The values of resistance and inductive reactance of a choke coil are 6 Ω and 8 Ω respectively. Its power factor will be

(a) 0.2 (b) 0.4 (c) 0.6 (d) 0.8

41. The ratio of apparent power to average power in an AC circuit is equal to

(a) 1/form factor (b) 1/cos θ
(c) cos θ (d) form factor

42. In a purely inductive circuit, on reducing the frequency of applied emf to half, the value of X_L will become

(a) eight times (b) four times
(c) double (d) half

43. If the impedance of an AC circuit is Z = 200 Ω 60°, then the value of resistance will be

(a) 200 Ω 60° (b) 50 Ω 60°
(c) 150 Ω 60° (d) 100 Ω

44. The value of power loss in a pure condenser is

(a) 0 (b) 1/2 (c) 1 (d) ∞

45. The value of *R*/*Z* is equal to

(a) θ (b) sin θ (c) cos θ (d) tan θ

46. The correct phase diagram representing the relation between I_L, I_C and E in an anti-resonant LC circuit is

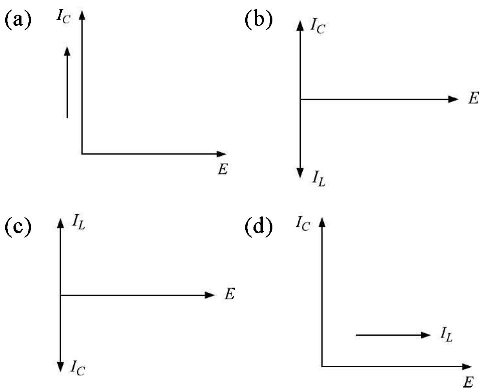


Fig. 28.33

47. The amplitude of effective current in an AC circuit is

(a) $I_{\text{rms}} \cos\theta$ (b) $I_{\text{rms}} \sin\theta$
(c) $I_0 \cos\theta$ (d) infinity

48. The power factor of an *LR* circuit is

(a) between 0 and 1 (b) 0
(c) 1 (d) infinity

49. The time constant of an *RC* circuit shown in Fig. 28.34 is

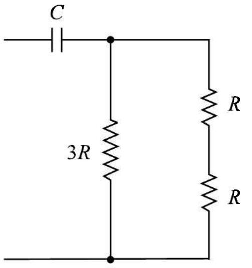


Fig. 28.34

(a) 3*RC* (b) $\frac{2}{3}$ *RC*

(c) $\frac{6RC}{5}$ (d) 2 *RC*

50. In the circuit shown $V_C = -40$ V and $V_L = +40$ V then potential drop across resistance is

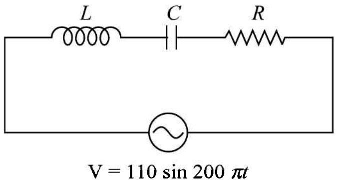


Fig. 28.35

(a) 10 V (b) 40 V
(c) 78 V (d) cannot be predicted

51. If in an *LCR* circuit, *L*, *C* and *R* repesent inductance, capacitance and resistance respectively, then the expressions for dimension of frequency are

(A) $1/RC$ (B) CL
(C) $1/\sqrt{LC}$ (D) R/L
(a) *B*, *C*, *D* (b) *A*, *B*, *C*
(c) *A*, *C*, *D* (d) only *C*

52. On connecting a condenser in parallel to an electric fan connected to an AC circuit, the phase angle

(a) keeps on increasing and decreasing
(b) increases
(c) remains constant
(d) decreases

53. To design a low pass filter the capacitor should be connected in

(a) series
(b) parallel
(c) capacitor cannot be connected
(d) either way

54. An AC voltmeter in an LCR circuit reads 30 V across resistance 80 V across inductance and 40 V across capacitance. The value of applied voltage will be

(a) 25 V (b) 50 V (c) 70 V (d) 150 V

55. In a series *LCR* circuit, on increasing the capacity four times, the value of self inductance in order to keep the resonant frequency unchanged, will be

(a) $L/4$ (b) $L/2$ (c) $2L$ (d) $4L$

56. A toaster and a light bulb are connected in parallel, the more heat is produced in the toaster. The resistance of toaster as compared to that of bulb, will be

(a) equal (b) less (c) more (d) can't say

57. An alternating current of frequency 50 cycles s^{-1} is flowing in a choke coil in an AC circuit. What should be the self inductance of the coil so that its inductive reactance becomes $100\pi\ \Omega$?

(a) $1\,\mu$H (b) 1 mH
(c) 1 H (d) 10 H

58. The peak value of an AC is $2\sqrt{2}$ A. Its RMS value will be

(a) zero (b) $\sqrt{2}$A
(c) 2A (d) $2\sqrt{2}$ A

59. The efficiency of a transformer is 90% $N_P : N_S : : 2 : 1$. The load current is 20A. If input power is 3600W then internal resistance will be

(a) $0.2\,\Omega$ (b) $0.9\,\Omega$
(c) $0.5\,\Omega$ (d) none of these

60. The resistance of three wires are $10\,\Omega$, $20\,\Omega$ and $30\,\Omega$ respectively. These are separately connected to a potential difference *V*. Maximum heat will be generated in the wire of resistance

(a) $30\,\Omega$ (b) $10\,\Omega$
(c) $20\,\Omega$ (d) all of the above

61. The necessary condition of power factor in an *LCR* circuit to be one will be

(a) $L = C$ (b) $X_L = 2 \times C$
(c) $X_L = 1/X_C$ (d) $X_L = X_C$

62. AC voltmeter measures

(a) peak voltage
(b) average voltage
(c) peak inverse voltage
(d) rms voltage

63. The amplitude of voltage in AC circuit is E_0. If a DC voltmeter is connected in this circuit, then its reading will be

(a) $2E_0$ (b) E_{rms} (c) E_0 (d) zero

64. The peak-to-peak value of alternating voltage is

(a) $2E_0$ (b) $E_0/2$ (c) E_{rms} (d) E_0

65. The equation of alternating voltage and alternating current in an AC circuit are $E = 200 \sin 314t$ volt and $I = 100 \sin(314t + \pi/3)$ ampere respectively. The value of power factor will be

(a) -1 (b) 1/4 (c) 1/2 (d) 1

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (b)
11. (a)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (d)
31. (d)	**32.** (c)	**33.** (d)	**34.** (d)	**35.** (c)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (c)
41. (b)	**42.** (d)	**43.** (a)	**44.** (a)	**45.** (c)	**46.** (b)	**47.** (a)	**48.** (a)	**49.** (c)	**50.** (c)
51. (c)	**52.** (d)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (b)	**57.** (c)	**58.** (c)	**59.** (b)	**60.** (b)
61. (d)	**62.** (d)	**63.** (d)	**64.** (a)	**65.** (c)					

29

Ray Optics and Optical Instruments

BRIEF REVIEW OF THE CONCEPTS

Reflection Rebounding of light from a polished surface.

Laws of reflection (a) $i = r$

(b) Incident ray, normal and reflected ray are coplanar.

If the mirror is rotated by an angle θ, reflected ray moves by 2θ

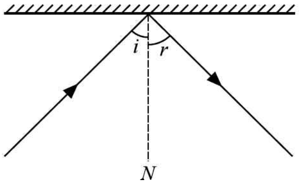
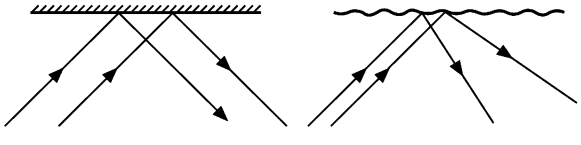


Fig. 29.1

Diffusion Reflection from a rough surface (like wall) is called diffusion. A parallel incident beam will not emerge out parallel because it meets different angles at the reflecting surface. When two mirrors are inclined at an angle θ the number of images are formed for an object placed in front of them.



(a) Reflection (b) Diffusion

Fig. 29.2

(a) Number of images $n = \frac{360}{\theta}$ if $\frac{360}{\theta}$ is odd and object does not lie on angle bisector or placed symmetrically.

$n = \frac{360}{\theta} - 1$ if $\frac{360}{\theta}$ is odd and object placed on angle bisector or placed symmetrically.

(b) Number of images $n = \frac{360}{\theta} - 1$ if $\frac{360}{\theta}$ is even.

Characteristics of image formed with a plane mirror

(a) Erect
(b) Virtual
(c) Size of image = size of object
(d) Image distance = object distance (measured from mirror)
(e) Lateral inversion (left appears right and right appears left)

A, O, M, N ... X Y and so on eleven letters show lateral symmetry. When two mirrors are parallel $\theta = 0$, therefore number of images = ∞.

If the mirror is thick, more than 1 image is formed and the 2nd image (formed due to first reflection from polished surface) is the brightest.

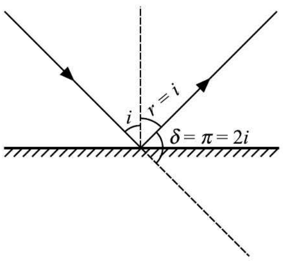


Fig. 29.3

When a ray is reflected from a plane mirror angle of deviation $\delta = \pi - 2i$ radian.

Minimum height of a mirror so that a person can see his full image in the mirror is half the height of the person.

Spherical mirrors are of two types: convex and concave

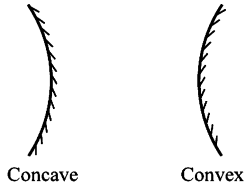


Fig. 29.4

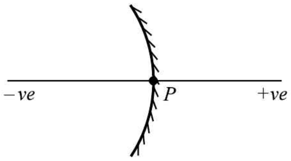


Fig. 29.5

Sign convention Consider pole *P* as origin. All distances to its left are negative and all distances to its right are positive.

Mirror formulae $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

where v = image distance from pole to the mirror
u = object distance from pole to the mirror
f = focal length
R = radius of curvature

Table 29.1

Real image	*Virtual image*
1. Rays actually converge to form image	Rays appear to diverge from
2. Image is inverted	Image is erect
3. Can be obtained on screen	Cannot be obtained on screen
4. Magnification is negative	Magnification is positive

Magnification (lateral)

$$M_{\text{lat}} = -\frac{v}{u} = \frac{I}{O}$$

$$M_{\text{lat}} = \frac{v-f}{f} = \frac{f}{u-f}$$

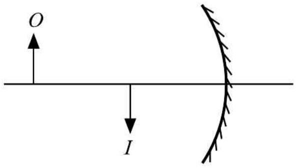


Fig. 29.6

Magnification (axial)

$M_{\text{axial}} = \frac{-v^2}{u^2}$ (valid for small objects)

Lens The part of an isotropic transparent medium bounded by at least one curved surface. Lenses are of two types

(a) convex (b) concave

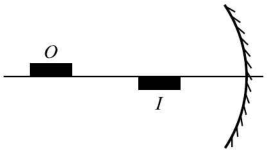


Fig. 29.7

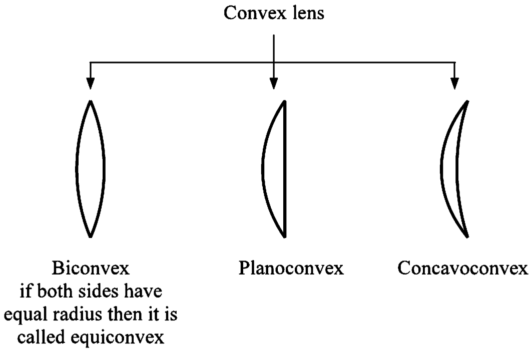


Fig. 29.8

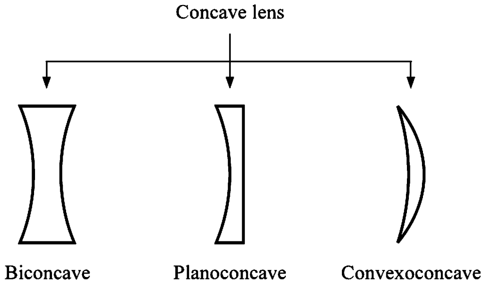


Fig. 29.9

Image formation information for convex lens and concave mirrors

Table 29.2

Position of object	*Position of image and its nature*
At ∞	At focus (Real, inverted and diminished)
Away from $2f$ (or C)	Between f and $2f$ (Real, inverted and diminished)
At $2f$	At $2f$ (Real, inverted and equal in size)
Between f and $2f$	Away from $2f$ (Real, inverted and magnified)
At f	At ∞ (Real, inverted and magnified)
Between Pole and f	Behind the mirror (virtual, erect and magnified) (in front of lens on the side of object)

Lens formulae for thin lenses

Lens makers formula when surrounding medium is air is

$$\frac{1}{f} = (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

Lens makers formula when surrounding medium has refractive index μ_m is

$$\frac{1}{f} = \left(\frac{\mu_L}{\mu_m} - 1\right)\left[\frac{1}{R_1} - \frac{1}{R_2}\right].$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

Lens formula in same medium on two sides

$f = \frac{D^2 - d^2}{4D}$ displacement method and $O = \sqrt{I_1 I_2}$

Lateral magnification $M_{\text{lat}} = \frac{v}{u} = \frac{I}{O}$ for a convex lens

$M_{\text{lat}} = -\frac{v}{u} = \frac{I}{O}$ for a mirror

$M_{\text{lat}} = \frac{f}{u+f}$ and $M_{\text{lat}} = \frac{f-v}{f}$

Axial magnification $M_{\text{axial}} = -\frac{v^2}{u^2}$

If the medium on two sides are different then for focal length use

$$\frac{\mu_3}{f} = \frac{\mu_2 - \mu_1}{R_1} - \frac{\mu_2 - \mu_3}{R_2}.$$

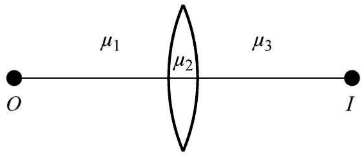


Fig. 29.10

To find v, use $\frac{\mu_3}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R_1} - \frac{\mu_2 - \mu_3}{R_2}$

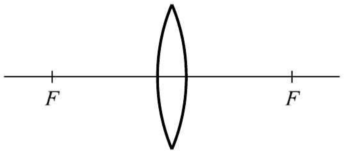


Fig. 29.11

Fig. 29.12

A lens has two principal focii

$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ when two lenses are in contact.

Newton's formula $x_1 x_2 = f^2$

$x_1 x_2 = f_1 f_2$

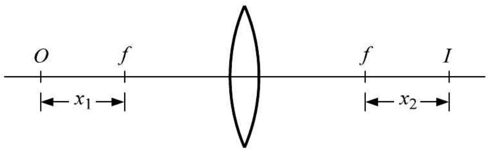


Fig. 29.13

when there is separation d between the lenses

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

Power of the lens $P = \frac{1}{f(\text{m})} = \frac{100}{f(\text{cm})}$. The unit is dioptre (D).

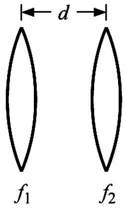


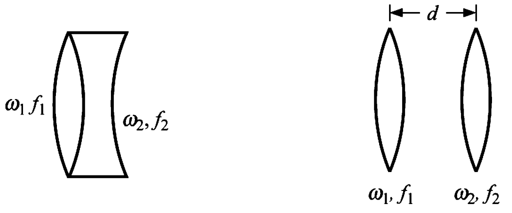
Fig. 29.14

Defects in lenses

(a) **Spherical aberration** or monochromatic aberration is removed using optical stops or cylindrical lens (astigmatism) or aplanatic lens.

Spherical aberration occurs as paraxial and marginal rays do not meet at a point

(b) **Chromatic aberration** A white object appears coloured. It is removed by using achromatic combination. For achromatic combination



(a) **(b)**

Fig. 29.16

Achromatic combination $\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$ where ω_1 and ω_2 are dispersive powers and since it is combination of a convex and a concave lens

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

Chromatic aberration can also be removed using two lenses of same kind separated by a small distance

if $d = \frac{\omega_1 f_2 + \omega_2 f_1}{\omega_1 + \omega_2}$

if $\omega_1 = \omega_2 = \omega$ then $d = \frac{f_1 + f_2}{2}$

If $d = f_1 - f_2$ then spherical aberration is also removed.

Thus if $f_1 = 3f_2$ and $d = 2f_2$ then both the defects are removed.

This method is employed in Huygen's eye piece.

***Refraction** When an **oblique** ray of light enters one medium from another **(optically different or dispersive medium)** then it changes its path. Such a phenomenon is called refraction.

Note: It does not mean that if a ray is incident normal, it, is not refracted.

Laws of refraction There are two laws of refractions

(a) ${}^1\mu_2 = \frac{\sin i}{\sin r}$ (see Fig. 29.17)

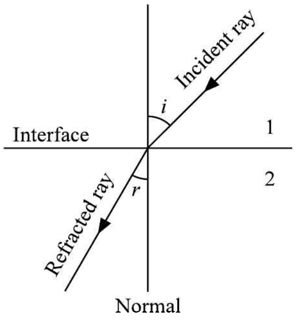


Fig. 29.17

$\mu = \frac{\sin i}{\sin r} = \frac{c}{v}$ or $\frac{v_1}{v_2} = \frac{1}{\sin C}$ where C is critical angle.

$\mu = \frac{\text{real depth}}{\text{apparant depth}}$.

Apply this formula when incidence is normal

$\mu = \frac{\lambda_1}{\lambda_2} = \tan\theta_P$ where θ_P is polarizing angle

$\mu = \frac{\sin\left(\frac{A + D_m}{2}\right)}{\sin A/2}$ in prism, where

$\delta = (\mu - 1)\,\alpha$

If angle of prism α is small and δ is angle of minimum deviation

$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$ is Cauchy's principle.

where A, B and C are constants which depend on the nature of medium.

${}^1\mu_3 = {}^1\mu_2 \times {}^2\mu_3$ ${}^2\mu_1 = \frac{1}{{}^1\mu_2}$

(b) Incident ray, normal and refracted rays are coplanar.

Fermat's principle When a ray of light passes from one point to another by any number of reflections and refractions, the path taken by light ray is the one for which the corres-ponding time taken is the least (or has shortest optical path).

Optical path length is μl if l is the distance travelled in a medium of refractive index μ.

Refraction through a curved surface

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}.$$

In all curved surfaces this formula can be applied with appropriate sign convention and remembering that μ_1 is the refractive index of the medium in which object lies.

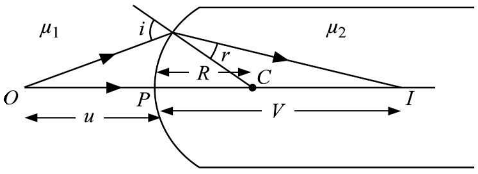


Fig. 29.18

Dispersive power $\omega = \frac{\delta_v - \delta_r}{\delta} = \frac{\mu_v - \mu_r}{\mu - 1}$

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*Most of the authors do not write the exact definition of refraction. This is the correct definition. **Words in bold,** part when added, make the definition correct.

where δ_v and δ_r are minimum deviations for violet and red colours. δ is mean deviation (for yellow colour). μ_v and μ_r are refractive indices for violet and red colours respectively μ is refractive index for yellow or mean colour.

$$\delta = (\delta_v + \delta_r)/2 \text{ (if } \delta \text{ is not given)}$$

$$\text{Similarly } \mu = (\mu_v + \mu_r)/2 \text{ (if } \mu \text{ is not given)}$$

$$\omega\delta = \delta_v - \delta_r \text{ is called angular dispersion.}$$

$$\text{Angular dispersion} \propto \frac{1}{\lambda^3}$$

In a prism $i + e = A + D$

Figure shows graph between D and i,

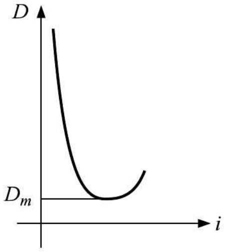


Fig. 29.19

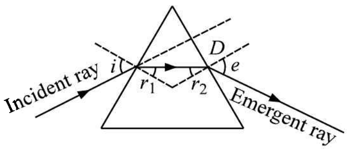


Fig. 29.20

Note that D is minimum for a particular i, called angle of minimum deviation and is denoted by D_m

At minimum deviation $i = e$, $r_1 = r_2$ and ray through prism, is parallel to base of the prism.

Deviation without dispersion (a)

Dispersion without deviation (b)

29.21

Rainbow There are two types of rainbows

(a) **Primary Rainbow** occurs due to one total internal reflection and two refractions from the water drops suspended in air; violet colour on the inner edge and red colour on the outer edge; and 42° violet and 40° red.

(b) **Secondary Rainbow** occurs due to two total internal reflections and two refractions from the water droplets suspended in air; Red colour on inner edge and violet colour on outer edge, angle 51° for violet and 54° for Red.

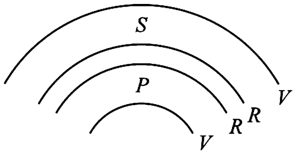


Fig. 29.22

Visual acuity or Resolving power of eye 1/60° or 1 min.

Near point is 15 cm and least distance of distinct vision is (D) = 25 cm.

Eye pieces or occular Commonly used eye pieces are Huygen's and Ramsden. In Huygen's eyepiece both the defects spherical aberration and chromatic aberration are removed

$d = 2f_2$ and $f_1 = 3f_2$ are used so that

$d = (f_1 + f_2)/2$ and $d = f_1 - f_2$.

The drawback is that crosswires cannot be fitted. Therefore, it cannot be used in those applications where measurements are to be taken. In Ramsden's eyepiece both the lens have equal focal lengths.

$$d = 2/3f.$$

It is achromated for two selected colours. Spherical aberration is not removed completely. But crosswires can be connected. It is used where quantitative work is to be undertaken.

Simple microscope or Magnifier Magnification is

$$M = (1 + D/f)$$

Compound microscope Magnification is

$$M = \frac{v_o}{u_o}\left(1 + \frac{D}{f_e}\right); \text{ separation between two lenses}$$

$L = v_o + u_e$.

$$\text{Resolving power of microscope} = \frac{2\mu \sin\theta}{\lambda}$$

Resolving power of microscope is $\frac{\mu \sin\theta}{0.61\lambda}$ for circular aperture

$\mu \sin\theta = 0.004$ and is called numerical aperture.

Telescope (Astronomical) (a) Normal setting

$$M = \frac{f_o}{f_e}, L = f_o + f_e$$

(b) Least distance of distinct vision setting

$$M = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right), L = f_o + u_e$$

Resolving power of a telescope is $\frac{a}{1.22\lambda}$ where a is aperture.

Limit of resolution is reciprocal of resolving power

Telescope (Terrestrial) (a) Normal setting

$$M = \frac{f_o}{f_e}, L = f_o + 4f_{er} + f_e$$

(b) Least distance of distinct vision setting

$$M = -\frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right), L = f_o + 4f_{er} + u_e$$

Rayleigh scattering $\propto \frac{1}{\lambda^4}$. That is why sky appears blue and danger signals are red.

Rayliegh's criterion for just resolution Two light sources close together are said to be just resolved if minima of one falls on the maxima of other

Just resolved Unresolved Widely resolved

Fig. 29.23

SHORT-CUTS AND POINTS TO NOTE

(i) Even number of images can be made if two mirrors are inclined at an angle θ and the object lies on an angle bisector. Otherwise number of images are always odd.

Use geometry to work on problems. It helps a lot. second image is the brightest in a thick plane mirror. Virtual image can be photographed.

(ii) If a lens of focal length f and refractive index μ is immersed completely in a medium of refractive index μ_m then new focal length f_m is given by

$$\frac{f_m}{f} = \frac{(\mu - 1)}{\left(\dfrac{\mu}{\mu_m} - 1\right)}$$

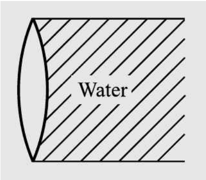


Fig. 29.24

If μ = 1.5 and μ_m = 4/3 (water).

Then $f_m = 4f$.

If μ = 1.5 and the lens is immersed from one side in water μ = 4/3.

Then $f_m = 2f$

where f is focal length of lens in air. If medium is not water then use

$$\frac{\mu_3}{f} = \frac{\mu_2 - \mu_1}{R_1} - \frac{\mu_2 - \mu_3}{R_2}$$

and to find v, use $\dfrac{\mu_3}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R_1} - \dfrac{\mu_2 - \mu_3}{R_2}$

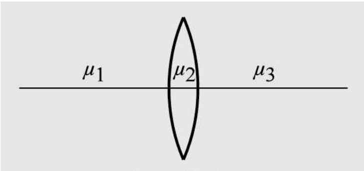


Fig. 29.25

If μ_{lens} = 1.5 then $f = R$ for an equiconvex lens and $f = 2R$ for a planoconvex lens.

(iii) If a lens has refractive index μ_1 and refractive index of surrounding medium is μ_2

(1) If $\mu_1 > \mu_2$ lens behaves normal, that is, a convex lens behaves as a converging lens and a concave lens a diverging lens.

(2) If $\mu_1 < \mu_2$ lens behaves opposite, that is, a convex lens behaves as a diverging lens and vice versa.

(3) If $\mu_1 = \mu_2$ then the system acts as a slab. That is it ceases to be a lens.

(iv) If using $\dfrac{1}{f} = \dfrac{1}{f_1} + \dfrac{1}{f_2} - \dfrac{d}{f_1 f_2}$, use it to find focal length only.

Do not use this focal length to find v or u. Use two separate lenses to find final u or v

(v) If object and screen are fixed, lens is displaced, use

$$f = \frac{D^2 - d^2}{4D}$$

and size of object for images of sizes I_1 and I_2 $O = \sqrt{I_1 I_2}$ where D is distance between object and image and d is displacement between lens positions.

(vi) If the lens is silvered from one side use power to find new focal length

$$P = 2P_L + P_M$$

or $\dfrac{1}{f_{\text{new}}} = \dfrac{2}{f} + \dfrac{2}{R} = \dfrac{2}{f} + \dfrac{1}{f(\mu - 1)}$

or $f_{\text{new}} = \dfrac{f(\mu - 1)}{2\mu - 1}$ for the shown case

$= \dfrac{f}{4}$ if μ = 1.5

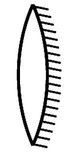


Fig. 29.26

(vii) Due to refraction the sun appears to rise little earlier and set little late (about 3 min difference in the day)

(viii) If angle of prism is small use $\delta = (\mu - 1)\,\alpha$ otherwise use

$$\mu = \frac{\sin(A + D_m)/2}{\sin A/2}.$$

Constant deviation prisms are used in special type of spectrometer.

(ix) While finding position of spot when two or more mediums are placed (Fig. 29.27)

$$\text{Shift } \Delta y = \left(t_1 - \frac{t_1}{\mu}\right) + \left(t_2 - \frac{t_2}{\mu}\right).$$

(x) Lateral shift in a slab $y = \dfrac{t\sin(i - r)}{\cos r}$

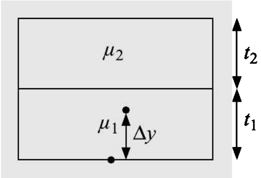


Fig. 29.27

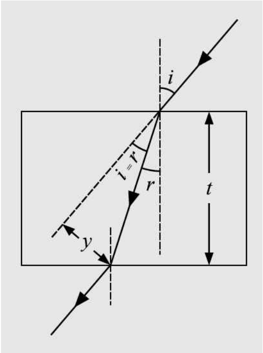


Fig. 29.28

(xi) A body disappears if its refractive index is equal to refractive index of surrounding medium.

During refraction wavelength varies but frequency remains unchanged.

(xii) If a lens is partly covered, intensity or brightness of the image will decrease.

If a lens is cut horizontally its focal length remains unchanged but if cut vertically focal length will change.

(xiii) If t is thickness and α is linear absorption coefficient then for incident light I_0, emergent light

$$I = I_0 e^{-\alpha t}$$

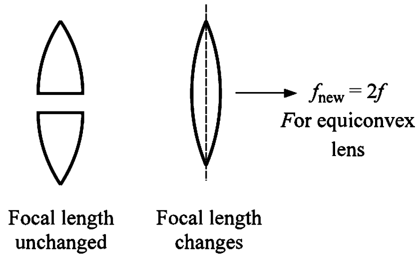


Fig. 29.29

(xiv) Magnifications from spherical surface

$$M_{\text{lat}} = \frac{\mu_1 v}{\mu_2 u},\ M_{\text{axial}} = \frac{\mu_1 v^2}{\mu_2 u^2}$$

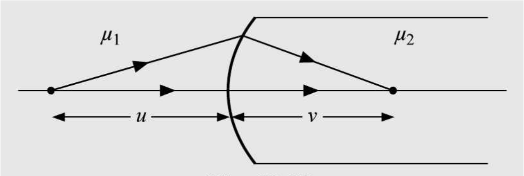


Fig. 29.30

(xv) Note if the image is formed on the object side it is virtual. If the image is formed on the other side it is real.

(xvi) Primary colours are red, blue and green. Complementary colours of primary colours are cyan for red, brown/orange for blue and purple/magenta for green.

(xvii) The minimum distance between object and real image in case of a lens is 4*f*, where *f* is focal length of the lens.

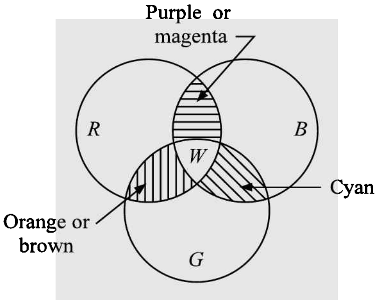


Fig. 29.31

⚠ CAUTION

(i) Considering that real image cannot be formed using plane mirror.

- If the incident beam is converging real image can be made

(ii) Considering that during refractions, rays always bend.

- Rays incident normal do not bend but still refraction occurs. Apply

$$\mu = \frac{\text{real depth}}{\text{apparant depth}} \text{ in such cases.}$$

(iii) Not differentiating between linear (lateral) and axial magnifications.

- Lateral magnification is $M_{\text{lat}} = v/u$ and axial magnification for small objects is v^2/u^2.

(iv) Considering that frequency varies during refraction.

- Frequency of incident radiations never varies during refraction.

(v) Considering that focal length found using

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \text{ can be used to find } v \text{ also.}$$

- Individual lens analysis be employed to find *v* or *u*. Use it to find only focal length of the combination

(vi) Not being sure if colour is determined by wavelength or frequency.

- Colour is determined by wavelength.

(vii) Considering that refraction through a slab produces deviation.

- Only lateral shift is produced. This may be helpful in making images sharp without disturbing object, lens or screen.

(viii) Considering that an achromat can be made using only two lenses convex and concave made of different materials.

- Two similar lenses (both convex) and made of same material (having equal dispersive power) can be used to remove chromatic aberration if $d = \frac{f_1 + f_2}{2}$.

(ix) Considering that refractive index does not depend upon colour or wavelength.

- Refractive index varies with colour according to Cauchy's formula $\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$ that is, blue colour has higher refractive index than red colour. This causes dispersion (in prisms) and chromatic aberration in lenses.

(x) Considering that optical path length is always greater than real path length.

- Optical path length = μx if $\mu > 1$, optical path length $> x$; if $\mu = 1$, optical path length $= x$ and, if $\mu < 1$ optical path length $< x$

(xi) Considering that magnifying/resolving power of a microscope is fixed.

- If we immerse the microscope lens/slide in an oil of refractive index μ resolving power will increase. If *uv* light is selected, it will further increase the resolving power.

(xii) Taking $P = 1/f$ and *f* in centimetres.

- use $P = 100/f$ (*f* in cm).

Solved Problems

1. The light beam travels at a speed $1.94 \times 10^8 \text{ms}^{-1}$ in Quartz. The wavelength found in quartz is 355 nm. What would be the wavelength in air?

(a) 179 nm (b) 549 nm
(c) 355 nm (d) 707 nm

Solution (b) $\mu = c/v = \lambda/\lambda'$

or $\frac{3\times10^8}{1.94\times10^8} = \frac{\lambda}{355}$ or $\lambda = \frac{1065}{1.94} = 549\text{nm}$

2. In 11.5 ns light travels 2.5 m in plastic find its refractive index.

(a) 1.38 (b) 1.48
(c) 1.18 (d) 1.58

Solution (a)

$$\mu = \frac{c}{v} = \frac{3\times10^8}{\frac{2.5}{11.5\times10^{-9}}} = \frac{3\times11.5\times10^{-1}}{2.5} = 1.38$$

3. Two mirrors are inclined at an angle θ. For an object placed in front of them 11 images are noticed. Find the angle between the mirrors.

(a) 30° (b) 32.8° (c) 16.4° (d) 15°

Solution No. of images $= 11 = \frac{360}{\theta} - 1 \Rightarrow \theta = 30°$

4. The rays deviates at 90° after suffering reflection from the mirror. The angle of incidence is

(a) 90° (b) 30°
(c) 60° (d) 45°
(e) none of these

Solution (d) $2\theta = 180 - \delta$ or $2\theta = 180 - 90 \Rightarrow \theta = 45°$

5. To what depth the vessel be filled with water so that it appears half filled

(a) 3/4h (b) 2/3h (c) 5/7h (d) 3/5h

Solution (b) App. depth = Real depth/μ

$$\therefore \text{ Real depth} = \frac{4}{3}\left(\frac{h}{2}\right) = 2/3\text{h}$$

6. A room in 3 m high and 5 m long. A man is standing in front of one of the walls 1 m from the wall. A mirror is to be installed on the wall. Find the minimum height of the mirror so that complete image of the wall behind him is seen

(a) 1.5 m (b) 1 m (c) 2 m (d) 0.5 m

Solution (d) ΔABM and ΔKLM are similar

$$\therefore \frac{KL}{AB} = \frac{KM}{MB} \Rightarrow KL = \frac{3 \times 1}{6} = 1/2\,\text{m}$$

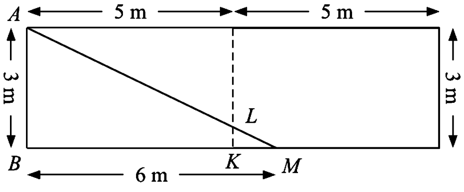


Fig. 29.32

7. A beam is incident parallel on the prism shown in Fig. 29.33. Find the angle between emerging rays.$\mu_{\text{prism}} = 1.66$

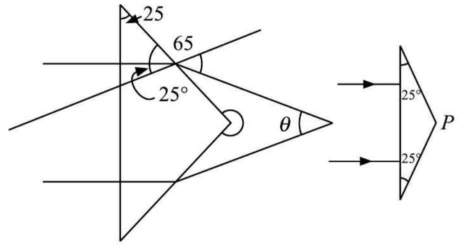


Fig. 29.33

(a) 180° (b) 120° (c) 135° (d) 0°

Solution $\sin C = \frac{1}{1.66}$ or $C = 37°$

$\angle P = 13$

∴ Ray is refracted out.

$$\sin r = \sin 25(1.66) = 0.4226(1.66) = 0.706$$

$$\therefore \quad r = 45°$$

$$\theta = 360 - 230 - 90 = 40°$$

8. A glass slab has $\mu_{\text{violet}} = 1.66$ and $\mu_r = 1.6$. For what value of d the separation will be 1 mm in red and violet rays.

(a) 5/3 cm (b) 10/3 cm
(c) 5 cm (d) 20/3 cm

Solution (b) $\sin r_1 = \frac{\sin 70}{1.66} = \frac{0.9397}{1.66}$ or $r_1 = 34°30'$

$$\sin r_2 = \frac{\sin 70}{1.6} = \frac{0.9397}{1.6} \quad \text{or} \quad r_2 = 30°$$

Using $y = \frac{t \sin(i - r)}{\cos r}$

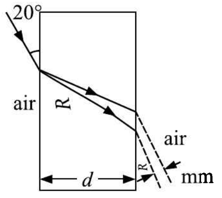


Fig. 29.34

$$y_1 - y_2 = d\left[\frac{\sin(i - r)}{\cos r_1} - \frac{\sin(i - r_2)}{\cos r_2}\right]$$

$$0.1 = d\left[\frac{\sin 35°30'}{\cos 34\ 30'} - \frac{\sin 34°}{\cos 36}\right]$$

or $$0.1 = d\left[\frac{0.5807}{0.8241} - \frac{0.552}{0.8090}\right] = d[0.71 - 0.68]$$

or $$d = \frac{0.1}{0.03} = 10/3\,\text{cm}.$$

9. A glass rod whose one end is hemispherical of radius 2 cm ($\mu_{\text{red}} = 1.52$) is kept in water ($\mu_\omega = 1.33$). The object is kept cm in front of convex surface. Find the magnification

(a) 2.33 (b) 1.33 (c) 2.66 (d) 1.76

Solution (a) $\frac{1.52}{v} - \frac{1.33}{-8} = \frac{1.52 - 1.33}{2}$ or $v = -21.3$ cm;

$$M = -\frac{1.33(-21.3)}{1.52(8)} = 2.33$$

10. Refractive index of water in the situation shown in Fig. 29.35 is μ.. Find the distance seen by the fish F of human eye E

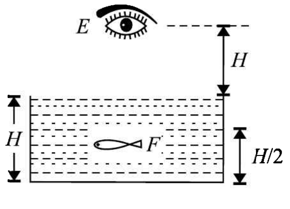


Fig. 29.35

(a) $H + \frac{H}{2\mu}$ (b) $\frac{3H}{2\mu}$

(c) $\frac{H}{2} + H\mu$ (d) $\frac{3\mu H}{2}$

Solution (c) $\frac{H}{2} + \frac{H}{\omega_{\mu_a}} = \frac{H}{2} + \frac{H}{1/\mu}$

11. In the problem 10 what is the distance of fish seen by human eye E.

(a) $H + \frac{H}{2\mu}$ (b) $\frac{H}{\mu} + \frac{H}{2}$

(c) $\frac{3H}{2\mu}$ (d) None of these

Solution (a) $H + \frac{H/2}{a_{\mu_\omega}} = H + \frac{H}{2\mu}$

12. A point object O is placed midway between two concave mirrors distance d apart. What is the value of d for which object and images coincide. Each mirror has focal length F.

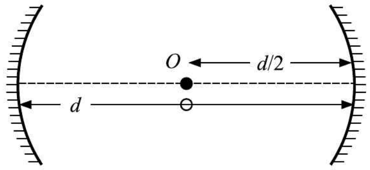
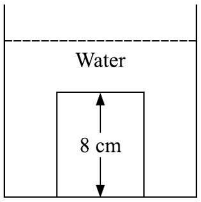


Fig. 29.36

(a) F, $2F$ (b) $2F$, $3F$ (c) F, $4F$ (d) $2F$, $4F$

Solution (d) When $d/2 = F$, the rays from one mirror after reflection will reach parallel to the other mirror. Second mirror will refocus then at O. When object is $2F$, image if formed at $2F$.

13. A cylindrical vessel of diameter 12 cm has $800\pi\,\text{cm}^3$ water. A cylindrical glass piece of glass of diameter 8 cm and height 8 cm is placed in it as shown in Fig. 29.37(a). What is the position of image of bottom of the vessel seen through paraxial rays passing through glass cylinder.



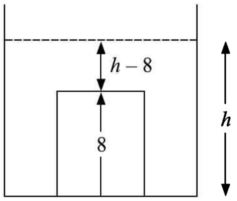
(a)

(a) 6.2 cm above the bottom
(b) 7.1 cm above the bottom
(c) 6.6 cm above the bottom
(d) none of these

Solution (b) $\pi 6^2 h = 800\pi + \pi 4^2(8)$

or $h = \dfrac{928}{36} = 25.8$ cm

$$\Delta y = \left(17.8 - \frac{17.8}{4/3}\right) + \left(8 - \frac{8}{1.5}\right)$$



(b)
Fig. 29.35

= 4.45 + 2.67
= 7.1 cm above the bottom

14. Find angle of minimum deviation of an equilateral prism (μ = 1.732). Also find angle of incidence for this deviation

(a) 60°, 75° (b) 60°, 55°
(c) 60°, 50° (d) 60°, 60°

Solution (d) $\mu = \dfrac{\sin\dfrac{A + D_m}{2}}{\sin A/2}$

or $\dfrac{\sqrt{3}\times 1}{2} = \sin\dfrac{A + D_m}{2} \Rightarrow D_m = 60°$

Using $2i = A + D_m$ we get $i = 60°$

15. An equiconvex lens is 5 cm thick in the middle. Find the position of image for an object far away from the lens.

(a) 10 cm (b) 9.2 cm
(c) 8.72 cm (d) 9.48 cm
(e) 9.68 cm

Solution (b) Image is formed at focus.

$$\frac{1}{f} = (\mu - 1)\left[\frac{2}{R} + \frac{(\mu - 1)t}{\mu R^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{10} - \frac{2.5}{150}\right] = \frac{13}{120}$$

or $f = \dfrac{120}{13} = 9.2.$

16. A slide projector is to project a (35 mm × 23 mm) slide on a 2 m × 2 m screen. Find the focal length of the lens used if screen is 10 m away from the lens

(a) 15.1 cm (b) 17.2 cm
(c) 16.1 cm (d) 18.2 cm

Solution (b) $M = \dfrac{200}{3.5} = \dfrac{10}{4}$ or $u = \dfrac{7}{40}$ m ;

Using $\dfrac{1}{v} - \dfrac{1}{u} = \dfrac{1}{f}$

i.e. $\dfrac{1}{f} = \dfrac{1}{10} + \dfrac{40}{7}$ or $f = 17.2$ cm.

17. Which of the following is a wrong statement:

(a) $D = 1/f$ where f is focal length and D is optical power of a lens.
(b) Power is in diopter, when f is in meters.
(c) Power is in diopter and does not depend upon the system of unit used to measure f
(d) D is positive for convergent lens and D is negative for divergent lens

Solution (c) $\because P = \dfrac{1}{f\,(m)} = \dfrac{100}{f\,(\text{cm})}$

18. Identify the wrong description of the given figures

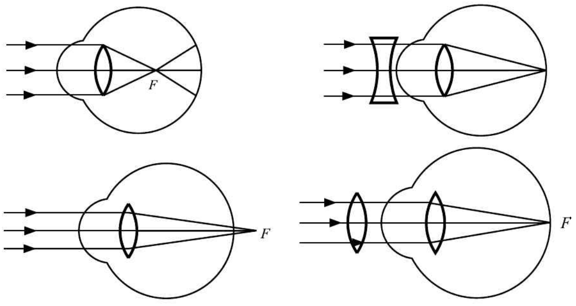


Fig. 29.38

(a) 1 represents far sightedness
(b) 2 is correction for short sightedness
(c) 3 represents far sightedness
(d) 4 represents correction for far sightedness

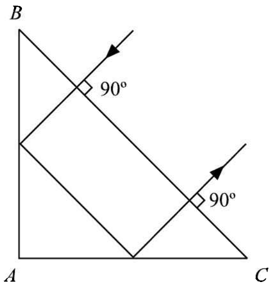
Solution (a)

19. Which mirror be used to obtain a parallel beam of light from a small lamp?

(a) plane mirror (b) convex mirror
(c) concave mirror (d) any of the above

Solution (c)

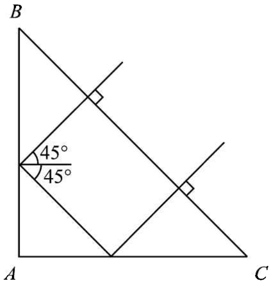
20. As shown in Fig. 29.39(a) $AB = AC$. Find the minimum value of refractive index μ for the given material



(a)

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 1.5 (d) 1.6

Solution (a) $\mu = \dfrac{1}{\sin c} = \dfrac{1}{\sin 45} = \sqrt{2}$



(b)

Fig. 29.39

21. The eyepiece of a refracting telescope $f = 9$ cm. In the normal setting separation between objective and eyepiece is 1.8 m. Find the magnification.

(a) 20 (b) 19 (c) 18 (d) 21

Solution (b) $f_0 = 1.8 - 0.09 = 1.71$ m

$$M = \frac{f_0}{f_e} = \frac{171}{9} = 19$$

22. The focal length of an $f/4$ camera lens is 300 mm. What is the true diameter of the lens.

(a) 75 mm (b) 650 mm
(c) 800 mm (d) 1200 mm

Solution (a) aperture = 300/4 = 75 mm.

23. An object is placed at 15 cm in front of a convex lens of focal length 10 cm. Where shall we place a convex mirror of focal length 13 cm so that object and image coincide

(a) 6 cm from lens
(b) 3 cm from lens
(c) 4 cm from lens
(d) 2 cm from lens

Solution (c) $\dfrac{1}{v} - \dfrac{1}{u} = \dfrac{1}{f}$; $\dfrac{1}{v} = \dfrac{1}{10} - \dfrac{1}{15}$

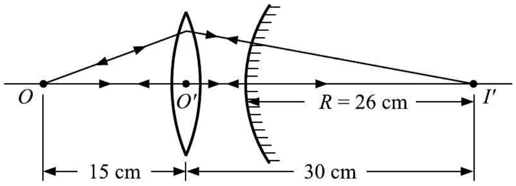


Fig. 29.40

or $v = 30$ cm. In order the ray to retrace the path, ray must be incident normal on the mirror. Hence distance of mirror from I' should be equal to $R = 2f = 26$ cm or 4 cm from lens.

24. In the Fig. 29.41 shown a parallel beam emerges parallel. The relation between μ, μ_1 and μ_2 is

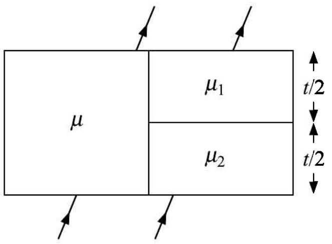


Fig. 29.41

(a) $\mu = \mu_1 + \mu_2$ (b) $\dfrac{1}{\mu} = \dfrac{1}{\mu_1} + \dfrac{1}{\mu_2}$

(c) $\mu = \dfrac{\mu_1 + \mu_2}{2}$ (d) $\dfrac{2}{\mu} = \dfrac{1}{\mu_1} + \dfrac{1}{\mu_2}$

Solution (d) $\dfrac{t/2}{\mu_1} + \dfrac{t/2}{\mu_2} = \dfrac{2}{\mu} = \dfrac{1}{\mu_1} + \dfrac{1}{\mu_2}$.

Problems for Practice

1. Two plane mirrors are inclined to each other at an angle of 60°. If a ray of light incident on the first mirror is parallel to second mirror, it is reflected from the second mirror

(a) parallel to the second mirror
(b) perpendicular to the first mirror
(c) perpendicular to the second mirror
(d) parallel to the first mirror

2. A ray reflected successively from two plane mirrors inclined at a certain angle undergoes a deviation of 300°. Then the number of images observable is

(a) 5 (b) 11
(c) 60 (d) 12

3. Which of the following is not the case with image formed by a plane mirror?

(a) it is virtual
(b) it is at the same distance as the object
(c) it is erect
(d) it is diminished

4. When the ray of light is incident from a denser medium having refractive index 2, what should be the angle of incidence for the ray to go out?

(a) less than 90° (b) less than 30°
(c) less than 40° (d) less than 60°

5. The image formed by convex mirror of focal length 30cm is quarter of the size of the object. Then the distance of the object from the mirror is
(a) 90cm (b) 120cm (c) 60cm (d) 30cm

6. For a convex mirror, the variation of u versus v is given by the graph

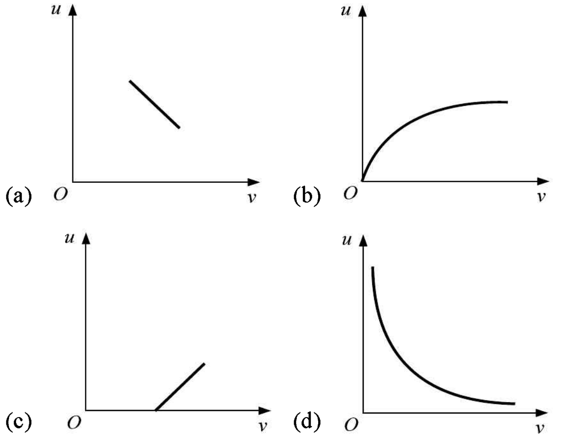


Fig. 29.42

7. A point object is placed at a distance of 30cm from a convex mirror of focal length 30cm. The image will form at
(a) focus
(b) infinity
(c) 15cm behind the mirror
(d) pole

8. The absolute refractive index of any medium is always
(a) zero (b) = 1 (c) > 1 (d) < 1

9. The focal length of a concave mirror is f and the distance of the object to the principal focus is P. Then the ratio of the size of the image of the size of the object is
(a) f/p (b) $\sqrt{Pf}$
(c) $\dfrac{P+f}{f}$ (d) $\dfrac{f+p}{p}$

10. A fish looking from water sees the outside world through a circular horizon. If the fish is $\sqrt{7}$ cm below the surface of water, what will be the radius of circular horizon?
(a) $(3/\sqrt{7})$cm (b) 3 cm
(c) $\sqrt{7}$ cm (d) $3 \times \sqrt{7}$ cm

11. A ray of light enters from a denser medium into rarer medium. The speed of light in the rarer medium is twice that in denser medium. What is the critical angle for total internal reflection to take place?
(a) 45° (b) 30° (c) 60° (d) none of these

12. A luminous object is placed 20cm from the surface of a convex mirror and a plane mirror is set so that the virtual images formed in the two mirrors coincide. If the plane mirror is at a distance of 12cm from the object, then the focal length of the convex mirror is
(a) 30cm (b) 10cm (c) 20cm (d) 40cm

13. Just before setting, the sun may appear to be elliptical. This happens due to
(a) diffraction (b) reflection
(c) dispersion (d) refraction

14. The critical angle for a ray of light suffering total internal reflection will be smallest for the ray travelling from
(a) water to glass (b) water to air
(c) glass to air (d) glass to water

15. A mark at the bottom of a beaker 0.1m deep appears raised by 0.01m when the beaker is filled till brim with a liquid. Refractive index of liquid is
(a) 10 (b) 10/9 (c) 1/10 (d) 9/10

16. A medium shows relation between i and r as shown. If speed of light in the medium is nc then value of n is

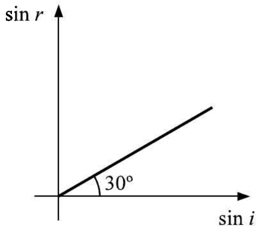


Fig. 29.43

(a) 1.5 (b) 2 (c) $\dfrac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

17. An object is approaching a plane mirror at 5cm per second. A stationary observer sees the image. At what speed will the image approach the stationary observer
(a) 20cm per second (b) 10cm per second
(c) 15cm per second (d) 5cm per second

18. Two plane mirrors are inclined to one another at an angle of 40°. A point object is placed in between them. The number of images formed due to reflection at both mirror is
(a) 9 (b) 6 (c) 8 (d) infinite

19. For a spherical mirror, the graph $1/v$ versus $1/u$ is

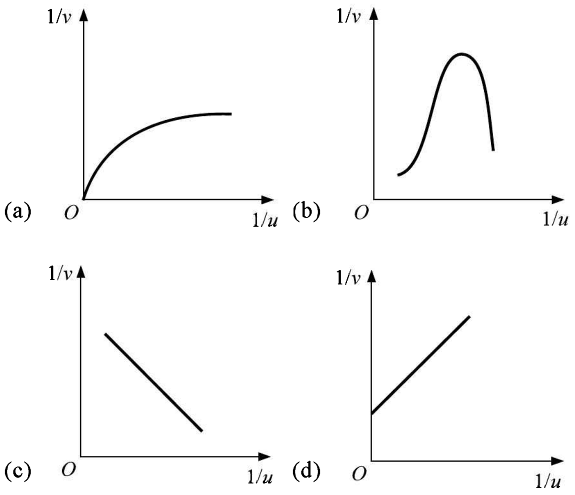


Fig. 29.44

20. A vessel is half filled with a liquid of refractive index μ and the other half with another liquid of refractive index 1.5 μ. The apparent depth is 50% of the actual depth. The value of μ is
(a) 1.4 (b) 1.5 (c) 1.6 (d) 5/3

21. A vessel of depth $2d$cm is half filled with a liquid of refractive index μ_1, and the appear half with a liquid of refractive index μ_2. The apparent depth of the vessel seen perpendicularly is
(a) $2d(1/\mu_1\mu_2)$ (b) $d[(1/\mu_1) + (1/\mu_2)$
(c) $2d[(1/\mu_1) + (1 + \mu_2)]$ (d) $d[\mu_1\mu_2/(\mu_1 + \mu_2)$

22. The wave length of light diminishes μ times (μ = 1.33 for water) in a medium. A diver from inside water looks at an object whose natural colour is green. He sees the object as
(a) red (b) blue (c) green (d) yellow

23. *A* and *B* are two identical beakers. Beaker *A* is filled till brim with liquid of refractive index 1.3 and beaker *B* is filled with some other liquid of refractive index 1.6. Both beakers are viewed from directly above. The ratio of the apparent depth of *A* and *B* is

(a) 1.3×1.6 (b) 1.3/1.6
(c) 1 (d) 1.6/1.3

24. A rectangular block of glass is placed on a printed page lying on a horizontal surface. Then the minimum value of refractive index of glass for which the letters on the page are not visible from any of the vertical faces of the block is

(a) less than $\sqrt{7}$ (b) $> = < \sqrt{7}$
(c) more than $\sqrt{7}$ (d) equal to $\sqrt{7}$

25. It is said that the sun becomes visible before the actual rise and remains visible even after actual setting. This makes the day a bit longer. It happens due to

(a) refraction of light (b) scattering of light
(c) diffraction of light (d) dispersion of light

26. The square root of relative permittivity for a medium is equal to the

(a) speed of light (b) refractive index of medium
(c) intensity of light (d) wavelength of light

27. The speed of light in vacuum is $3 \times 10^8 \text{ms}^{-1}$. Its speed in a medium of refractive index 1.5 will be

(a) 6.5×10^8 (b) $2 \times 10^8 \text{ms}^{-1}$
(c) 4.5×10^8cm (d) 5.5×10^8

28. The value of $\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is

(a) 332ms^{-1} (b) $3 \times 10^6 \text{ms}^{-1}$
(c) $3 \times 10^8 \text{ms}^{-1}$ (d) $3 \times 10^4 \text{ms}^{-1}$

29. A ray of light passes from vacuum into a medium of refractive index μ, the angle of incidence is found to be twice the angle of refraction. Then the angle of incidence is

(a) $2\sin^{-1}(\mu/2)$ (b) $2\sin^{-1}(\mu)$
(c) $2\cos^{-1}(\mu/2)$ (d) $\cos^{-1}(\mu/2)$

30. Critical angle of light passing from glass to air is minimum for

(a) yellow (b) red
(c) green (d) blue

31. An air bubble in a glass slab ($\mu = 1.5$) appears 6cm deep as viewed from one face. It appears 4cm deep when viewed from the other face. The thickness of the glass slab is

(a) 15cm (b) 6.67cm
(c) 10cm (d) none of these

32. A beam of monochromatic blue light of wave length 4200Å in air travels in water (refractive index, $\mu = 4/3$). Its wavelength in water will be

(a) 4000Å (b) 5600Å
(c) 2800Å (d) 3150Å

33. When a ray of light enters a glass slab from air

(a) its wave length decreases
(b) remains unchanged
(c) its frequency increases
(d) its wave length increases

34. The sun has a diameter of 864100 miles and is on an average at a distance of 92900000 miles from the earth. A concave mirror of radius 6m is used to obtain its image. The diameter of the image of the solar disc, is

(a) 28×10^{-1}mm (b) 28m
(c) 28cm (d) 28m m

35. The critical angle for water with respect to air is θ. What is the angular range in which the fish just below the surface of water can see the objects outside

(a) θ (b) 2θ (c) $\theta/2$ (d) none of these

36. How many images of himself, does an observer see if two adjacent walls of a rectangular room are mirrors?

(a) 9 (b) 7 (c) 5 (d) 3

37. On reflection from a rarer medium, a light wave suffers a phase difference of

(a) π (b) $\pi/2$
(c) $\pi/4$ (d) none of these

38. A ray of light is incident on a glass plate at 60°. The reflected and refracted rays are found to be mutually perpendicular. The refractive index of the glass is

(a) 1.15 (b) 1.50
(c) 2 (d) 1.73

39. A light wave of frequency 5×10^{14}Hz passes through a medium of refractive index 2.4. Its wavelength in the medium will be

(a) 2.5×10^{-7}m (b) 1×10^{-7}m
(c) 4×10^{-7} (d) 3.3×10^{-7}

40. Out of the following, which wavelength has minimum momentum of photon?

(a) 100Å (b) 100micro meter
(c) one nanometer (d) one milli micron

41. Monochromatic light passes from denser medium to rarer medium, then its

(a) wavelength decreases
(b) frequency decreases
(c) velocity decreases
(d) velocity increases

42. The absolute refractive indices of water and glass are 1.3 and 1.5 respectively. The refractive index of water with respect to glass is

(a) 1.5/1.3 (b) 1.5×1.3
(c) 1.3/1.5 (d) none of these

43. What is the ratio of luminous intensity of two sources that produce shadows of equal intensities at distances 25cm and 50cm from the photometer screen?

(a) 2 : 1 (b) 1 : 4 (c) 4 : 1 (d) 1 : 2

44. On reflection from a denser medium, the path difference introduced is

(a) λ (b) 2λ (c) $\lambda/2$ (d) zero

45. A person can see the objects lying between 25cm and 10cm from his eye. His vision can be corrected by using lens of power

(a) –0.25D (b) 4D
(c) –0.1D (d) 10D

46. If f_0 and f_e be the lengths of the objective and eye piece of in astronomical telescope, the length of the tube is

(a) $f_o - f_e$ (b) $\sqrt{f_o f_e}$
(c) $f_o f_e/f_o + f_e$ (d) $f_o + f_e$

47. When a parallel beam of light passes through a convex lens red, yellow and blue colours are focused at 19.5, 20.0 and 20.5cm. The dispersive power of the material of the prism is

(a) 0.200 (b) 0.205 (c) 0.195 (d) none of these

48. A myopic person can see things clearly only when they lie between 10cm and 100cm from his eye. Answer the following question.

A lens will enable him to see the moon clearly. His near point shifts by about

(a) –100 cm (b) –10 cm
(c) –110 cm (d) –90 cm

49. To make an achromatic combination a convex lens of focal length 42 cm having dispersive power 0.14 is placed in contact with a concave lens of dispersive power 0.21. The focal length of the concave lens should be

(a) 63 cm (b) 14 cm (c) 21 cm (d) 42 cm

50. Two lenses of focal length + 10 cm and –15 cm are put in contact to form an achromatic combination. The ratio of their dispersive powers is

(a) 2/3 (b) –3/2 (c) 3/2 (d) –2/3

51. Convex lens of power 4D and a concave lens of power 3D are placed in contact. What is the equivalent power of the combination?

(a) 3/4D (b) 7D (c) 4/3D (d) 1D

52. The focal length of lens does not depend on

(a) absolute refractive index of the medium from which the light is incident
(b) radii of curvature of its surfaces
(c) absolute refractive index of its material
(d) direction from which light is incident

53. For which of the following dispersive power is zero?

(a) slab (b) prism (c) lens (d) none of these

54. If we plot a graph between distance of object and that of image formed by a convex lens. What is the nature of the graph?

(a) parabola (b) ellipse
(c) hyperbola (d) straight line

55. A convex lens produces a real image m times the size of the object. What is the distance of the object from the lens

(a) $(m - 1)f$ (b) $(m + 1)f$
(c) $(m + 1)f/m$ (d) $(m - 1)f/m$

56. A convex lens of focal length f is put in contact with a concave lens of the same focal length. The equivalent focal length of the combination is

(a) ∞ (b) $2f$ (c) 0 (d) f

57. We wish to make a planoconvex lens of focal length 16 cm from glass having refractive index 1.5. It is to be used in air. What should be the radius of curvature of the curved surface?

(a) 24 cm (b) 16 cm (c) 8 cm (d) 12 cm

58. A convex lens of focal length 10 cm is placed in contact with a concave lens of focal length 20 cm. What is the nature and focal length of the combination?

(a) convex, 20 cm (b) convex, 10 cm
(c) concave 20 cm (d) concave 10 cm

59. A microscope is focused on an ink mark on the top of a table. If we place a glass slab 3 cm thick on it, how should the microscope be moved to focus the ink spot again? The refractive index of glass is 1.5.

(a) 2 cm downward (b) 1 cm upward
(c) 1 cm downward (d) 2 cm upward

60. What is the minimum and maximum distance between the convex lens and the object for the magnification of a real image to be greater than 1?

(a) f and ∞ (b) f and $2f$
(c) $2f$ and ∞ (d) 0 and f

61. Visual acuity of the eye is

(a) 25 cm (b) 15 cm
(c) 1″ (d) 1′

62. A convex lens of focal length 16 cm forms a real image double the size of the object. The distance of object from the lens is

(a) 32 cm (b) 24 cm
(c) 16 cm (d) 8 cm

63. A liquid placed in a hollow prism of angle 60°. If the angle of minimum deviation is 30°, what is the refractive index of the liquid?

(a) 1.95 (b) 1.41 (c) 1.50 (d) 1.65

64. A ray of light passes through an equilateral prism such that angle of incidence is 45°, what is the angle of minimum deviation?

(a) 45° (b) 60° (c) 50° (d) 30°

65. A bioconvex lens behaves as a diverging lens in water and a converging lens in air. The refractive index of water is 1.33 and that of air is 1. What can we say about the refractive index n of the material of lens?

(a) $n > 1.33$ (b) $0 < n < 1$
(c) $n < 0$ (d) $1 < n < 1.33$

66. An object is placed at the focus of a concave mirror. If the focal length of the mirror be 20 cm, the distance of the image from the pole of mirror is

(a) 40 cm (b) 10 cm
(c) 20 cm (d) ∞

67. A mirror is rotated through angle θ about an axis passing through the point of incidence and in the plane of the mirror. The reflected ray will be rotated through

(a) 2θ (b) θ (c) $4/\theta$ (d) $\theta/2$

68. A ray of light is incident on a plane mirror at an angle of incidence 30°. The ray after reflection is deviated through

(a) 120° (b) 30° (c) 90° (d) 60°

69. Which of the following is NOT the case with the image formed by a convex mirror?

(a) it lies beyond the focus
(b) it is erect
(c) it is virtual
(d) it is diminished

70. A glass prism of refractive index 1.5 is immersed in water as shown in Fig. 29.45. A beam of light incident normally on the face ab is internally reflected from the face ad so as to incident normally on face bd. Given that refractive index of glass is 3/2 and that of water is 4/3. What is the value of $\theta = \angle bad$?

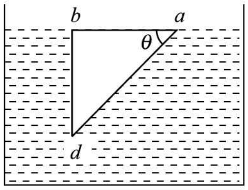


Fig. 29.45

(a) $\sin\theta < (2/3)$
(b) $\sin\theta > (8/9)$
(c) $(2/3) < \sin\theta < (8/9)$
(d) none of these

71. A man runs towards the plane mirror at 2 ms^{-1}. The relative speed of his image with respect to him will be

(a) 10 ms^{-1} (b) 2 ms^{-1}
(c) 4 ms^{-1} (d) 8 ms^{-1}

72. What is the relation between refractive index n and the wavelength of light

(a) $n \propto \lambda^{-1}$ (b) $n \propto \lambda^{-2}$
(c) $n \propto \lambda$ (d) $n \propto \lambda^{2}$

73. A convex mirror forms an image which is $1/n$ times the size of the object. If the focal length of the mirror is f, then the distance of the object is

(a) f/n (b) $(n - 1)f$
(c) nf (d) $(n + 1)f$

74. If the angle of incidence is i and that of refraction is r. Then the speed of light in the medium to which the light is refracted from air is

(a) $v = c\dfrac{\cos i}{\cos r}$ (b) $v = c\dfrac{\sin i}{\sin r}$
(c) $v = c\dfrac{\sin r}{\sin i}$ (d) $v = c\dfrac{\cos r}{\cos i}$

75. Why is light dispersed on passing through a prism?

(a) velocity of light is less in glass than in air
(b) light has seven wavelengths
(c) refractive index is different for different colour
(d) prism is triangular in shape

76. The far point of a normal eye is at

(a) 25 cm (b) 0 cm (c) 15 cm (d) ∞

77. A short-sighted person cannot see clearly beyond 2 m. The focal length of the concave lens to correct his eye to normal vision should be

(a) 0.25 m (b) 2 m
(c) 0.5 m (d) 1 m

78. A myopic person having far point 80 cm uses spectacles of power 1.0 D. How far can he see clearly?

(a) 4 m (b) 1 m (c) 2 m (d) more than 4 m

79. Distance of distinct vision is 25 cm. The focal length of the convex lens is 5 cm. It can act as a magnifier of magnifying power

(a) 6 (b) less than 5
(c) 5 (d) more than 6

80. A beam of light consisting of red, green and blue colours is incident on a right angled prism as shown in Fig. 29.46. The refractive indices of the material of the prism for red, green and blue colours respectively are 1.39, 1.44, 1.47. The prism will

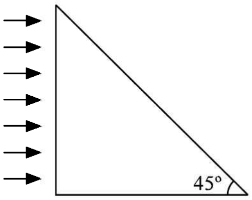


Fig. 29.46

(a) separate part of the red colour from green and blue colours
(b) not separate even partially and colour from the other two colours.
(c) separate all the three colours from one another
(d) separate part of the blue colour from red and green colour.

81. A person cannot clearly see the objects beyond 50 cm. What should be the power of the lens to correct his vision?

(a) –2 D (b) –0.5 D
(c) –1.0 D (d) –1.5 D

82. Ability of the eye to see objects at all distances is called

(a) myopia (b) hypermetropia
(c) accommodation (d) binocular vision

83. A cubical room is shown in the figure ($ABCD$) with the wall CD as a plane mirror. Each side of the room is 3 m. We place a camera at the mid point of the wall AB. At what distance should the camera be focused to photograph an object placed at A

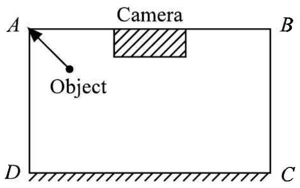


Fig. 29.47

(a) more than 6 m (b) 1.5 m
(c) 3 m (d) 6 m

84. What will be the distance of the object when a concave mirror produces a real image of magnification m? The focal length of the mirror is f.

(a) $(m + 1)f$ (b) $(m - 1)f/m$
(c) $(m - 1)f$ (d) $(m + 1)f/m$

85. Why one cannot see through the fog?

(a) fog refracts light (b) fog scatters light
(c) fog polarises light (d) fog reflects light

86. The final image formed by microscope is

(a) virtual and diminished
(b) real and magnified
(c) virtual and magnified
(d) real and diminished

87. A ray of light is incident from a denser medium on the surface of separation of a rarer medium. The reflected and refracted rays are inclined to each other at 90°. If r be the angle of reflection and r' be the angle of refraction, then what is the critical angle?

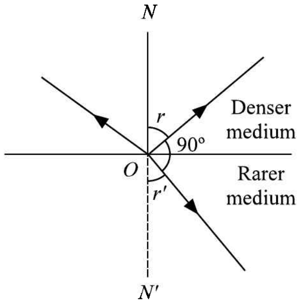


Fig. 29.48

(a) $\tan^{-1}(\sin r)$ (b) $\sin^{-1}(\tan r)$
(c) $\tan^{-1}(\sin r)$ (d) $\sin^{-1}(\tan r')$

88. A symmetrical double convex lens is cut in two equal parts by a plane containing the principal axes. If the power of the original lens was 4D the power of a divided lens will be

(a) 3D (b) 5D
(c) 4D (d) 2D

ANSWERS

1. (d)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (b)
11. (b)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (d)	**18.** (a)	**19.** (c)	**20.** (d)
21. (b)	**22.** (b)	**23.** (d)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (d)	**30.** (d)
31. (a)	**32.** (d)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (d)	**38.** (d)	**39.** (a)	**40.** (b)
41. (d)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (c)	**46.** (d)	**47.** (d)	**48.** (a)	**49.** (a)	**50.** (a)
51. (d)	**52.** (d)	**53.** (d)	**54.** (c)	**55.** (c)	**56.** (a)	**57.** (c)	**58.** (a)	**59.** (b)	**60.** (b)
61. (d)	**62.** (b)	**63.** (b)	**64.** (d)	**65.** (d)	**66.** (d)	**67.** (a)	**68.** (a)	**69.** (a)	**70.** (b)
71. (c)	**72.** (b)	**73.** (b)	**74.** (c)	**75.** (c)	**76.** (d)	**77.** (b)	**78.** (a)	**79.** (a)	**80.** (a)
81. (a)	**82.** (c)	**83.** (a)	**84.** (d)	**85.** (b)	**86.** (c)	**87.** (b)	**88.** (d)		

30

Wave Optics

BRIEF REVIEW OF THE CONCEPTS

Interference When two light waves emitted from two coherent sources superpose then it results in variation of intensity with distance. At certain places intensity is maximum and at other places intensity is minimum.

Coherent sources Two sources/wave trains are said to be coherent if there is a constant or zero phase difference between them. No two different sources could be coherent. They are to be derived from a single source. Their state of polarization remains constant. Laser is highly coherent. If ϕ is phase shift then $\phi \neq f(t)$ and $\frac{d\phi}{dt} = 0$. Coherent sources can be obtained by division of

(a) wavefront
(b) amplitude

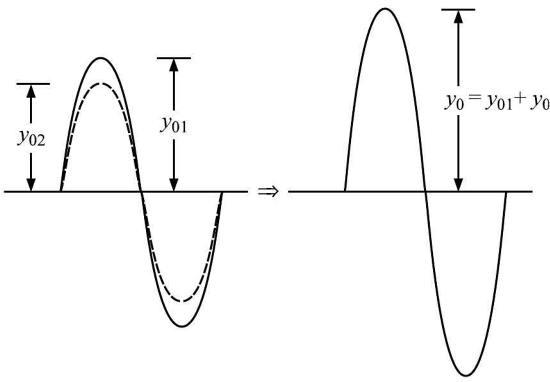
Young's double slit experiment (YDSE), Lloyd's mirror and Fresnel's biprism are examples in which coherent sources are obtained by division of wavefront.

In Newton's rings, thin films and interferometer division of amplitude is used to obtain coherent sources.

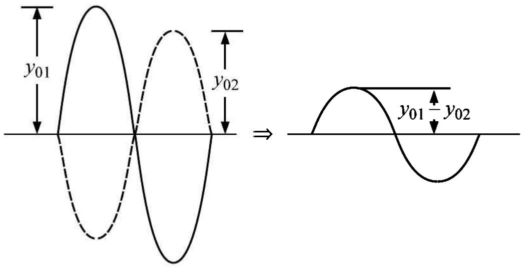
Interference is of two types: constructive interference and destructive interference

Constructive interference occurs when the coherent waves superpose in phase or the path difference is an integral multiple of wavelength or an even multiple of half the wavelength. This type of interference is also called reinforcement.

Destructive interference occurs when the coherent waves superpose out of phase or path difference is an odd integral multiple of half the wavelength.



(a) Constructive Interference



(b) Destructive Interference

Fig. 30.1

$$\frac{I_{\text{bright}}}{I_{\text{dark}}} = \frac{I_{\max}}{I_{\min}} = \left(\frac{y_{01} + y_{02}}{y_{01} - y_{02}}\right)^2 = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2$$

Path difference $\Delta x = n\lambda$ for constructive interference.

Path difference $\Delta x = (2n + 1)\frac{\lambda}{2}$ for destructive interference.

Conditions to obtain sustained interference

Necessary conditions Sources emitting waves must be coherent

Desirable conditions (i) sources should be monochromatic (ii) they shall have same amplitude (iii) they shall emit light continuously (iv) the separation between two sources shall be small.

In YDSE

Fringe width $\beta = \frac{\lambda D}{d}$ (Difference between two successive bright or dark fringe, that is, $x_n - x_{n-1} = \beta$)

$x_n = \frac{n\lambda D}{d}$ for nth bright fringe

$x_n = \frac{(2n-1)\lambda D}{2d}$ for nth dark fringe

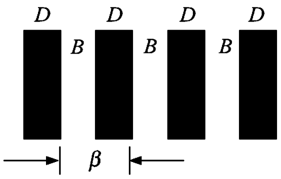


Fig. 30.2 Fringes (dark and bright and fringe width)

Angular fringe width $\theta = \frac{\lambda}{d}\left(=\frac{\beta}{D}\right)$ (in radian)

$$= \frac{\lambda}{d}\times\frac{180}{\pi} \quad \text{(in degrees)}$$

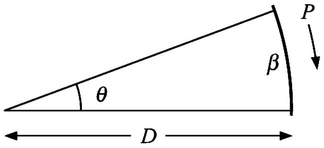


Fig. 30.3

Fringe visibility $= \frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}} = \frac{\sqrt{2I_1I_2}}{I_1+I_2}$

Intensity at any point if amplitude or intensity are equal

$$I = 2a^2(1 + \cos\delta) = 4I'\cos^2(\delta/2)$$

Intensity at any pint when amplitude or intensity are different

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\delta}$$

$$I = I_2 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\delta$$

In Fresnel's biprism

Both sources are virtual

$$D = a + b$$

$$d = 2a\delta = 2a(\mu - 1)\alpha$$

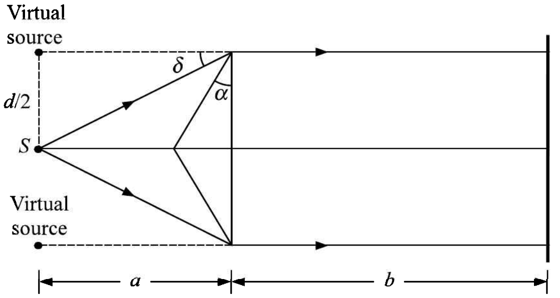


Fig. 30.4

$$\beta = \frac{\lambda D}{d} = \frac{\lambda(a+b)}{2a(\mu-1)\alpha}$$

$$x_n = \frac{n\lambda(a+b)}{2a(\mu-1)\alpha} \quad \text{for } n\text{th bright fringe}$$

$$x_n = \frac{(2n+1)\lambda(a+b)}{4a(\mu-1)\alpha} \quad \text{for } n\text{th dark fringe}$$

where α is angle of prism.

$$d = \sqrt{d_1d_2} \quad \text{if displacement method is used.}$$

In Lloyd's mirror Condition for nth dark and bright fringe obtained in Lloyd's mirror gets reversed to what was obtained in case of YDSE because, of reflection an additional phase shift of π or path difference $\lambda/2$ is achieved.

That is, $x_n = \frac{n\lambda D}{d}$ for nth dark fringe

$$x_n = \frac{(2n-1)\lambda D}{2d} \quad \text{for } n\text{th bright fringe}$$

In Lloyd's mirror one of the source is real and the other is virtual or image source.

If in YDSE we insert a thin slice of thickness t and refractive index μ in front of one of the slits then central fringe shifts to a position where originally was *nth* fringe such that

$$n\lambda = (\mu - 1)t \quad \text{or} \quad \Delta x = \frac{D(\mu-1)t}{d}.$$

If YDSE is immersed in a liquid of refractive index μ then fringes shrink and hence fringe pattern also shrinks.

$$\beta_{\text{new}} = \frac{\beta}{\mu} \quad \text{or} \quad x_{n(\text{new})} = \frac{n\lambda D}{\mu d}$$

If fresnel biprism is immersed in a liquid of refractive index μ' then

$$\beta_{\text{new}} = \frac{\frac{\lambda}{\mu'}(a+b)}{2a\left(\frac{\mu}{\mu'}-1\right)\alpha} = \frac{\lambda(a+b)}{2a(\mu-\mu')\alpha}.$$

For thin films

Path difference $= 2\mu t\cos r = (2n + 1)\frac{\lambda}{2}$ for nth bright fringe in reflected light

$$2\mu t\cos r = \frac{2n\lambda}{2}$$

$= \lambda n$ for nth dark fringe in reflected light

Path difference $2\mu t\cos r = 2n\frac{\lambda}{2} = n\lambda$ for nth bright fringe in refracted light

$= (2n + 1)\frac{\lambda}{2}$ for nth dark fringe in refracted light.

Wedge shaped film

Fringe width $\beta = \frac{\lambda}{2\theta}$, $\beta = \frac{\lambda x_n}{2t}$, $\theta = \frac{t}{x_n}$

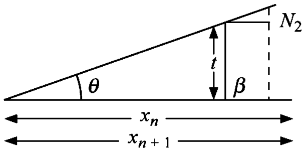


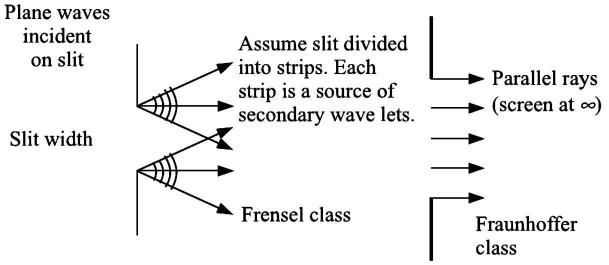
Fig. 30.5

If plates are put in a liquid of refractive index μ

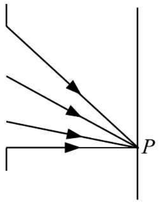
$$\beta = \frac{\lambda}{2\mu\theta} = \frac{\lambda x_n}{2\mu t}, \qquad 2\mu t = n\lambda, \qquad t_{\min} = \frac{\lambda}{2}.$$

Time of coherence (t_c) is the time during which electric field vector is in the sinusoidal form. Its value is 10^{-10} s. Coherence length $L_C = ct_c$. If path difference $> L_C$ coherence nature is lost. Therefore we cannot keep distance > 3 cm between two sources or slits.

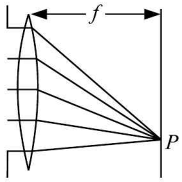
Diffraction The bending of wave from obstacles of size of the order of wavelength. For diffraction planar wavefront is required. Diffraction is of two types (a) Fresnel's class of diffraction (b) Fraunhoffer class of diffraction.



Fresnel class diffraction is near field diffraction. It may be assumed that rays from small imaginary strips to point P are not parallel.



Fresnel Class



Fraunhoffer Class

Fraunhoffer class diffraction is far field diffraction. The rays are parallel as the screen is at a very large distance. A converging lens will bring them at sharp focus on screen placed at the focus of lens.

Table 29.1

Interference	*Diffraction*
1. Fringes are formed due to superposition of wave trains emitted from two coherent sources	Fringes are formed due to super position of bent waves or due to super-position of secondary wavelets
2. Intensity of each fringe is equal	Intensity falls as the fringe order increases.
3. Number of fringes are very large	Number of fringes are finite and small
4. Fringe width is equal for each fringe	Fringe width of primary and secondary maxima are different.

Huygen's principle

(a) Each point on the primary wavefront is a source of secondary wavelets
(b) Secondary wavelets move only in forward direction.
(c) Secondary wavelets can superpose to produce disturbances
(d) Secondary wavelets move with c (speed of light) as well as primary wavefronts

Diffraction fro

Path difference $BC = d\sin\theta$

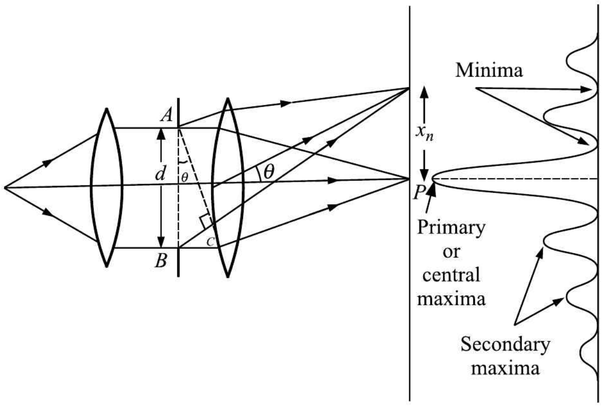


Fig. 30.6

For minima (Dark fringe)

$$d\sin\theta = n\lambda,$$

$$\sin\theta = \tan\theta = \frac{x_n}{D} \text{ or } \frac{dx_n}{D} = n\lambda, \quad x_n = \frac{n\lambda D}{d}$$

Normally $D \simeq f$ (focal length of focusing lens).

$$\text{Fringe width } \beta_{\text{primary}} = \frac{2\lambda D}{d} \quad \text{fringe width } \beta_{\text{sec.}} = \frac{\lambda D}{d}$$

$$\text{Angular fringe width } \beta_{\text{primary}} = \frac{2\lambda}{d}$$

$$\text{Angular fringe width } \beta_{\text{secondary}} = \frac{\lambda}{d}$$

$$\text{If } \beta = \frac{\pi d\sin\theta}{\lambda} \text{ then } I = \frac{I_0'\sin^2\beta}{\beta^2}$$

If aperture is circular $\sin\theta = \dfrac{1.22\lambda}{r}$, r is radius of aperture.

$$\text{Radius of first dark ring } R = \frac{1.22\lambda D}{r}, \frac{1.22\lambda f}{r}.$$

Fresnel diztance $z_f = \dfrac{a^2}{\lambda}$ where a is size of the aperture. Fresnel distance is the distance beyond which spreading due to diffraction becomes greater than the size 'a' of the aperture.

Polarisation If plane of vibration of the light is fixed then light will travel only in one direction. Such a state is called plane polarized light.

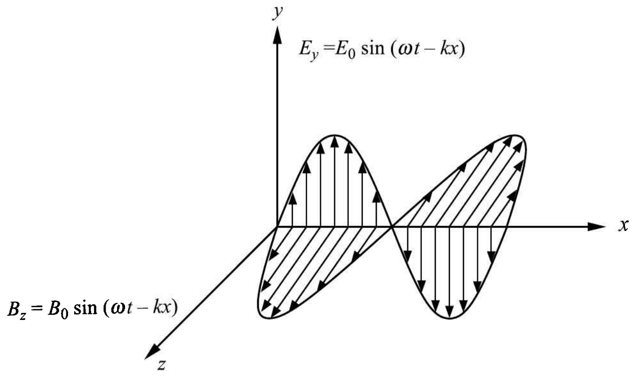


Fig. 30.7

In the Fig. 30.7, electric field is along y-axis and magnetic field along z-axis. Wave travels along x-axis, that is, plane of polarisation is y-z.

$E_y = E_0\sin(\omega t - kx)$ and magnetic field is along z-axis

$B_z = B_0\sin(\omega t - kx)$ then the wave travels along x-direction.

Only transverse waves can be polarised. Longitudinal waves cannot be polarised. Plane polarized light can be achieved using

(a) reflection
(b) refraction
(c) scattering
(d) nicol prism
(e) birefracting crystals

Brewster's law If light is incident on the interface of two media such that angle between reflected and refracted rays is 90° then reflected rays are completely polarized. Angle of incidence is called angle of polarization (θ_P)

Then $\mu = \tan\theta_P$

Malus's law When the plane of polarization is rotated by an angle θ then intensity of emergent light is given by $I = I_0\cos^2\theta$. I_0 is intensity of incident polarised light.

In birefracting crystals there are two rays—ordinary and extraordinary. The extraordinary ray does not follow law of refraction. If the velocity of extra ordinary ray is greater than that of the ordinary ray such crystals are called negative crystals examples are Iceland spar, tourmaline, saphire, ruby, emerald and apatite. If the ordinary ray has higher velocity than the extraordinary ray then such crystals are called positive crystals. Examples are Quartz, Iron oxide.

If the amplitudes of two waves are unequal and angle between the two is $\frac{\pi}{2}$ or path difference is $\lambda/4$ then an ellipticalliy polarized wave front is formed. On the other hand if amplitudes are equal then circularly polarized wavefront results. It could be elliptically polarized if amplitudes are equal but angle between the two is $0 < \theta < \pi/2$

SHORT CUTS AND POINTS TO NOTE

(i) Coherent sources are those in which wave trains have constant or zero phase difference. The coherent sources cannot be two separate sources. They must be derived from a single source either by division of wavefront or by division of amplitude. Only two different lasers could be coherent.

(ii) If two slits have unequal sizes (they correspond to intensity). The intensity of the resultant is

$$I = \left(\sqrt{I_1}\right)^2 + \left(\sqrt{I_2}\right)^2 + 2\sqrt{I_1 I_2}\cos\phi$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

$$= k\,(S_1 + S_2 + 2\sqrt{S_1 S_2}\cos\phi)$$

where S_1 and S_2 are size of the slits.

Coherence length $l_{\text{coh}} = \frac{\lambda^2}{\Delta\lambda}$,

Coherence radius $\rho_{\text{coh}} = \frac{\lambda}{\phi}$ $\quad \beta = \frac{\phi}{2}$

(iii) In YDSE $E\theta = E_m \cos\beta = 2E_0 \cos\beta$

$I\theta = I_m \cos^2\beta = 4I_0 \cos^2\beta$

Interference from narrow slits, that is, slit width $<< \lambda$

Phase difference $\beta = \frac{\pi d}{\lambda}\sin\theta$

Maximum intensity occurs at $d\sin\theta = n\lambda$

minimum intensity occurs at $d\sin\theta = (2n+1)\frac{\lambda}{2}$

When slits are not so narrow, position of nth bright fringe

$$x_n = \frac{n\lambda D}{d} = \frac{\lambda D}{d}$$

fringe width $\beta = \frac{\lambda D}{d}$

angular fringe width $= \frac{\lambda}{d}$ (rad) $= \frac{\lambda}{d} \times \frac{180}{\pi}$ (degrees)

$x_n = \frac{(2n+1)\lambda D}{2d}$ for nth dark fringe

(iv) If the light reaching P is direct (not reflected) from two sources then P will be a bright fringe if the path difference $= n\lambda$

(v) If the light reaching P after reflection forms a bright fringe (at P) then path difference $= (2n+1)\frac{\lambda}{2}$

because reflection causes an additional path difference of $\lambda/2$ (or phase difference π rad).

If the interference occurs due to reflected light central fringe (or ring in Newton's rings) will be dark.

If the interference occurs due to transmitted light central fringe (or ring in Newton's rings) will be bright.

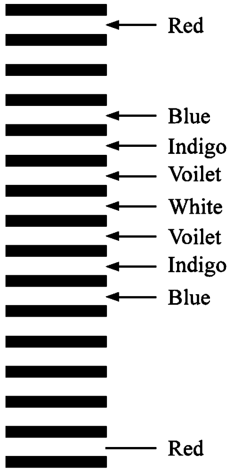


Fig. 30.8

(vi) If white light is used in YDSE central fringe is white surrounded by coloured fringes in VIBGYOR order.

(vii) Each fringe in YDSE has equal intensity while in diffraction the intensity falls as the fringe order increases.

(viii) To locate the central fringe in YDSE, illuminate it with white light. The central fringe is white.

(ix) Fringes can be displaced by introducing a thin slice in front of one of the slits. If t is thickness of the slice and μ its refractive index then $(\mu - 1)t = n\lambda$ describes the shift. Central fringe is now shifted at the place of nth fringe previously occupied OR $\Delta x = \frac{D}{d}(\mu - 1)t$

(x) The fringes shrink by $1/\mu$ if YDSE is immersed in a liquid of refractive index μ (>1).

The separation between two sources can neither be very large nor very small. The maximum separation between the slits could be equal to coherence length $\simeq$ 3 cm.

Fresnel distance $= \frac{a^2}{\lambda}$ where a is slit width

(Note $\beta = \frac{\lambda D}{d}$ if $d \to 0$ $\beta \to \infty$ and if $d \to \infty$, $\beta \to 0$).

In Fresnel biprism $d = 2a(\mu - 1)\,\alpha$ and $D = a + b$

So $x_n = \frac{n\lambda(a+b)}{2a(\mu-1)\alpha}$ for nth bright fringe.

If displacement method is employed to find d, then $d = \sqrt{d_1 d_2}$ where d_1 and d_2 are distance between images of virtual sources in magnified and diminished cases.

(xi) In Newton's ring radius of nth ring is given by $r_n = \sqrt{n\lambda R}$ R is radius of curvature of plano- convex lens.

For nth dark fringe in thin films $2\mu t\cos r = n\lambda$ (reflected light) and

$2\mu t\cos r = (2n+1)\frac{\lambda}{2}$ for nth dark fringe (transmitted or refracted light)

In wedge shaped films $2\mu t = n\lambda$ (if immersed in a liquid of refractive index μ).

In air $\mu = 1$, $t = \frac{n\lambda}{2}$ and $t_{\min} = \lambda/2$

[Maximum number of fringes = 1,50,000 called Haidenger fringes]

In a zone plate the focal length is given by

$$f_n = \frac{r_n^2}{(2p+1)n\lambda}$$

(xii) Diffraction occurs due to planar wavefront. If Fresnel diffraction is to be studied, then source should be at ∞. If Fraunhoffer diffraction is to be studied develop planar wavefront (parallel rays) with the help of a collimating lens. That is, optical aid is required.

If aperture is circular then radius of first dark ring $R = \frac{1.22\lambda D}{d} = \frac{1.22\lambda f}{d}$ where f is focal length of the focussing lens.

In single slit diffraction experiment $\beta_{\text{primary}} = \frac{2\lambda D}{d}$, $\beta_{\text{sec}} = \frac{\lambda D}{d}$ $d \sin\theta = n\lambda$ for nth minima.

(xiii) For diffraction to occur, the size of obstacle should be of the order of wavelength. As every obstacle has dimensions of the order of wavelength of sound, therefore, diffraction is more pronounced in sound than in light.

(xiv) Bragg's law is diffraction of X- rays from crystals. $2d \sin\theta = n\lambda$

(xv) In diffraction grating if there are N slits/lines per inch then grating element $(a + b) = \frac{2.54}{\text{N}}$

and $(a + b) \sin\theta = n\lambda$ where n is order of the spectrum.

Resolving power of grating is $\frac{\lambda}{d\lambda} = n\text{N}$

If white light is used central maxima will be white followed by coloured fringes in VIBGYOR order even in Diffraction

(xvi) Resolving power of a prism $= \frac{td\mu}{d\lambda}$ where t is length of the base

(xvii) Only transverse waves can be polarised. Sound waves being longitudinal cannot be polarised.

(xviii) The crystals in which ordinary ray travels faster than extraordinary ray or $\mu_{\text{extraordinary}} > \mu_{\text{ordinary}}$ are called positive crystals.

The crystals in which ordinary ray travels slower than extraordinary ray are known as negative crystals. In negative crystals $\mu_{\text{ordinary}} > \mu_{\text{extraordinary}}$.

(xix) The substances which rotate the plane of polarization are called optically active. The substances which rotate the plane of polarisation to its **left** or anticlockwise are called **Leveo rotatory** and the substances which rotate the plane of polarisation to its **right** or **clockwise** are called **dextrorotatory.**

(xx) According to Brewster's law $\mu = \tan\theta_P$ where θ_P is polarizing angle. The angle between reflected and refracted rays should be 90°.

(xxi) The intensity of plane polarized light is $I_0/2$ if the incident unpolarized light has intensity I_0.

If plane of polarization rotates by θ, polarized light varies according to Malus law, $I = I_0\cos^2\theta$

(xxii) The light sources like lasers are highly monochromatic and coherent.

Though sodium lamp gives a doublet D_1 & D_2 lines of wavelength 5890 Å and 5896 Å, it may be considered monochromatic for most of the experiments.

(xxiii) When white light is incident on a YDSE central fringe is white

since $\lambda_V = 390\,\text{nm}$

and $\lambda_R = 780\,\text{nm}$

and $x_n = \frac{n\lambda D}{d}$

reveals that 2nd voilet maxima and 1st red maxima superpose. That is there will be two colour bands of voilet to red on both sides of the white fringe, and, then general illumination will be observed as illustrated in Fig. 30.9

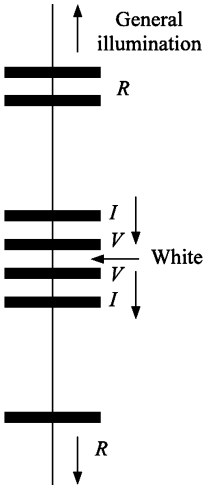


Fig. 30.9

(xxiv) For point sources or spherical wavefronts $I \propto r^{-2}$

For cylindrical sources amplitude $A \propto r^{-1/2}$, r being distance from the source

⚠ CAUTION

(i) Considering path difference = $n\lambda$ for bright fringes in all cases.

- Path difference = $n\lambda$ for bright fringes only for transmitted or refracted light. If interference occurs due to reflected light then path difference = $n\lambda$ for dark fringes or destructive interference.

(ii) Considering slit width as amplitude of the wave.

- Slit width acts like intensity. Therefore to find resultant intensity use

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{S_1} + \sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}}\right)^2$$

(iii) Applying same formula $x_n = n\lambda D/d$ even when sources are placed horizontally

- Note that path difference is $d\cos\theta$ where d is separation between the sources $d\cos\theta = n\lambda$ for nth bright fringe as shown in Fig. 30.10.

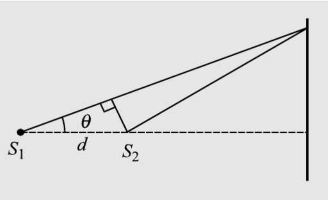


Fig. 30.10

(iv) Considering that Malus's law can be applied even to incident unpolarized light.

- ■ Malus's law can be applied only to polarised light

(v) Not knowing the relation between polarised light and unpolarised light intensity.

- ■ Polarised light intensity is half the incident unpolarised light (total) intensity as illustrated in Fig. 30.11.

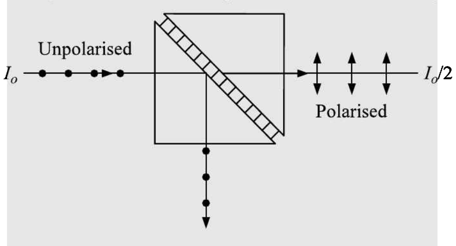


Fig. 30.11

(vi) Considering any wavefront if meets an obstacle will cause diffraction.

- ■ Planar wavefront when meets an obstacle of the size of the order of wavelength only then diffraction occurs.

(vii) Considering interference and diffraction are alike.

- ■ For interference one needs coherent sources which can be derived from a single source by division of wavefront or by division of amplitude. In diffraction, bent rays or secondary wavelets superpose to form fringes.

(viii) Considering that equal amplitude of superposing waves is necessary for interference.

- ■ It may be a desirable condition. If amplitudes are unequal, interference does occur

$$\frac{I_{\text{max}}}{I_{\text{min}}} = \frac{I_{\text{bright}}}{I_{\text{dark}}} = \left(\frac{y_{01}+y_{02}}{y_{01}-y_{02}}\right)^2 .$$

(ix) Considering only monochromatic light is necessary for interference or diffraction to occur.

- ■ Interference and diffraction do occur with white light.

  In YDSE central fringe will be white surrounded by coloured fringes in VIBGYOR order. Similarly in a single slit experiment or in diffraction grating the central fringe is white surrounded by coloured fringes in VIBGYOR order.

(x) Not considering the effect of refractive index μ' of the medium

- ■ Fringes shrink in YDSE by a factor of μ' and in Fresnel's biprism $\beta_{\text{new}} = \dfrac{\lambda(a+b)}{2a(\mu-\mu')}$.

Solved Problems

1. In *YDSE*, an electron beam is used to obtain interference pattern. If speed of electrons is increased.
   (a) no interference pattern will be observed
   (b) distance between two consecutive fringes will increase
   (c) distance between two consecutive fringes will decrease
   (d) distance between two consecutive fringe remains same.

Solution (c) $\lambda = h/mv$

If v increases λ decreases.

Therefore $\beta = \lambda D/d$ will decrease.

2. In *YDSE* the angular position of a point on the central maxima whose intensity is 1/4th of the maximum intensity is
   (a) $\sin^{-1}\left(\frac{\lambda}{d}\right)$ (b) $\sin^{-1}\left(\frac{\lambda}{2d}\right)$
   (c) $\sin^{-1}\left(\frac{\lambda}{3d}\right)$ (d) $\sin^{-1}\left(\frac{\lambda}{4d}\right)$

Solution (c) $I = I_{\text{max}} \cos^2(\phi/2)$

or $\cos\phi/2 = +1/2$

or $\phi = 2\pi/3$

$$\frac{2\pi}{\lambda} d \sin\theta = \frac{2\pi}{3} \text{ or } \theta = \sin^{-1}\left(\frac{\lambda}{3d}\right)$$

3. A *YDSE* uses a monochromatic source. The shape of the fringes formed on the screen is
   (a) hyperbola (b) circle
   (c) straight line (d) portion

Solution (c)

4. When an unpolarized light of intensity I_0 is incident on a polarizing sheet, the intensity of the light which does not get transmitted is
   (a) $I_0/2$ (b) $I_0/4$
   (c) zero (d) I_0

Solution (a)

5. The intensity due to a source at a point is I_0. What will be the intensity at the same point when slit width is doubled.
   (a) $2I_0$ (b) $4I_0$
   (c) I_0 (d) $I_0/2$

Solution (a)

6. Two waves of intensity I undergo interference. The maximum intensity obtained is
   (a) $I/2$ (b) $2I$ (c) I (d) $4I$

Solution (d) $I_{\text{max}} = I + I + 2\sqrt{I}\sqrt{I}\cos\theta = 4I.$

7. The wave theory in its original form was first postulated by
(a) Issac Newton (b) Thomas Young
(c) Christian Huygens (d) Augustan Dean Fresnel

Solution (c)

8. Two coherent light beams of intensity I and $4I$ are superposed. The minimum and maximum possible intensities in the resulting beam are
(a) $9I$ and I (b) $9I$ and $3I$
(c) $5I$ and I (d) $5I$ and $3I$

Solution (a) $\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{4I}+\sqrt{I}}{\sqrt{4I}-\sqrt{I}}\right)^2 = \frac{9}{1}$

9. A single slit of width a is illuminated by violet light of wavelength 400 nm and width of the diffraction pattern is measured as y. If the half of the slit is covered and illuminated with 600 nm. The width of the diffraction pattern will be
(a) $y/3$
(b) pattern vanishes and width is zero
(c) $3y$
(d) none of these

Solution (c) $\beta = \frac{2\lambda D}{d}$ $\frac{y}{y'} = \frac{\frac{2\times 400\,D}{d}}{\frac{2\times 600\,D}{d/2}}$ or $y' = 3y$

10. When unpolarized light beam is incident in air into glass (n = 1.5) at polarizing angle
(a) Reflected beam is completely polarized
(b) Reflected and refracted beams are partially polarized
(c) the reason for (a) is that almost all the light is reflected
(d) all the above

Solution (a)

11. Select the right option
(a) Christian Huygens, a contemporary of Newton established the wave theory of light by assuming that light waves are transverse
(b) Maxwell provided the compelling theoretical evidence that light is transverse in nature.
(c) Thomas Young experimentally proved the wave behaviour of light and Huygens assumption
(d) All the statements given above correctly answer the question what is light

Solution (b)

12. On placing a thin sheet of mica of thickness 12×10^{-5} cm in the path of one of the interfering beams in *YDSE*, the central fringe shift is equal to a fringe width. Find the refractive index of mica. Given λ = 600 nm.
(a) 1.5 (b) 1.48 (c) 1.61 (d) 1.56

Solution (a) $\frac{\lambda D}{d} = (\mu - 1)t\frac{D}{d}$ or $\mu = \frac{\lambda}{t} + 1$ = 1.5

13. The waves emitted by a radio transmitter are
(a) linearly polarized (b) unpolarized
(c) monochromatic (d) elliptically polarized

Solution (a)

14. Dichroism amounts to
(a) Selective absorption of unpolarized light
(b) Selective absorption of dispersed light
(c) Selective absorption of scattered light
(d) Selective absorption of one of the polarized component

Solution (d)

15. Two Nicol prisms are kept perpendicular One of them is illuminated with a light intensity (natural) I_0. Two more Nicol prisms are introduced in between symmetrically. Find the light intensity emitted from the last Nicol prism

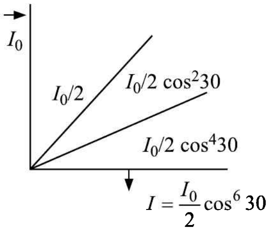


Fig. 30.12

(a) $\frac{27I_0}{64}$ (b) $\frac{27I_0}{128}$
(c) $\frac{9I_0}{32}$ (d) $\frac{9I_0}{64}$

Solution (b) $I = \frac{I_0}{2}\cos^6 30 = \frac{I_0}{2}\left(\frac{\sqrt{3}}{2}\right)^6 = \frac{27I_0}{128}$.

16. The angle between refracted and reflected rays is 90° at the water air interface. The angle of incidence in water.
(a) 60° (b) 53°
(c) 30° (d) 37°

Solution (d) $\tan\theta = 3/4$
$\theta = 37°$

17. In a birefracting crystal ordinary ray travels faster than extraordinary ray. The crystal is called
(a) positive crystal
(b) negative crystal
(c) no such demarcation exists
(d) dextro rotatory
(e) levorotatory

Solution (a)

18. If in a birefracting crystal the magnitude of E_x and E_y are equal and phase angle between the two is 60° then the waves are
(a) linearly polarised
(b) plane polarised
(c) circularly polarized
(d) elliptically polarized

Solution (d)

19. Antinodal curves correspond to ______ interference
(a) constructive
(b) destructive
(c) where intensity is less than maximum but not completely zero
(d) none of these

Solution (a)

20. A radio station operating at a frequency 1500 kHz has two vertical dipole antennas spaced 400 m apart oscillating in phase. In which directions is the intensity greatest
(a) 0, ±30°, ±90° (b) 0, ±30°, ±60°
(c) 0, ±45, ±90° (d) ±30°, ±60°, ±90°

Solution (a) $\lambda = \frac{C}{f} = \frac{3\times 10^8}{1.5\times 10^6}$ = 200 m

$\sin\theta = \frac{n\lambda}{d} = \frac{n(200)}{400} = \frac{n}{2}$, i.e., θ = 0, ±30°, ±90°.

21. In the above question where will minimum intensities be found

(a) ±14.5°, ±48.6° (b) ±30°, ±45°
(c) 14.5°, ±68.5° (d) ±14.5°, ±79.6°

Solution (a) $\sin\theta = \dfrac{(2n+1)}{4}$

$\theta = \sin^{-1}1/4$ or $\theta = \pm 14.5°$
$\theta = \sin^{-1}3/4 = \pm 48.6°$

22. When exposed to sunlight, thin films of oil on water often exhibit brilliant colours due to the phenomenon of

(a) dispersion (b) interference
(c) diffraction (d) angular acceleration

Solution (b)

23. Two glass plates are 10 cm long. At one end a piece of paper 0.02 mm thick is placed to make a wedge as shown in Fig. 30.13. Find the separation between two fringes. Assume $\lambda = 500$ nm

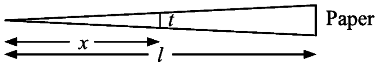


Fig. 30.13

(a) 1.25 mm (b) 1.5 mm
(c) 2.5 mm (d) none of these

Solution (a) $2t = n\lambda$; $t/x = h/l$

$$\frac{2x_0 h}{l} = n\lambda;$$

$$x_n - x_{n-1} = \frac{\lambda l}{2h} = \frac{500\times10^{-9}\times0.1}{2\times2\times10^{-5}}$$

$= 1.25 \times 10^{-3}$m

24. A commonly used lens coating material is MgF_2 with $n = 1.38$. Find the thickness of non-reflective coating for 550 nm light if it is applied to glass of $n = 1.52$.

(a) 400 nm (b) 200 nm
(c) 300 nm (d) 100 nm

Solution (d) 550 nm wavelength in MgF_2 will be

$$\lambda = \frac{\lambda_{air}}{n} = \frac{550}{1.38} = 400\text{nm}$$

$t = \lambda/4 = 100$ nm.

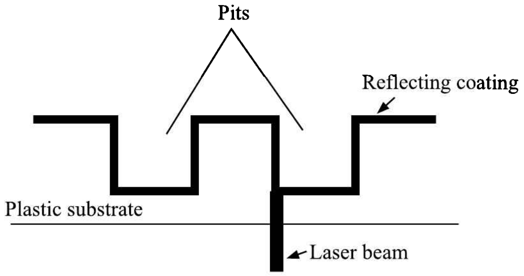


Fig. 30.14

25. A *CD* (compact disc) is read from the bottom by a semiconductor laser with wavelength 790 nm passing through a plastic substrate of refractive index 1.8. When the beam encounters 0 pit, part of the beam is reflected from the pit and part from the flat region. These two beams interfere with each other. What must be the minimum depth of the pit so that part of the beam reflected from the pit and part reflected from the flat surface cancel out. (This cancellation allows the player to recognize beginning and end of a bit).

(a) 0.197 μ m (b) 0.395 μ m
(c) 0.22 μ m (d) 0.11 μ m

Solution (d) $\lambda = \dfrac{\lambda_{air}}{\mu}$ and $t = \dfrac{\lambda}{4}$

$$= \frac{\lambda_{air}}{4\mu} = \frac{790}{4\times1.8} = 110\text{nm} = 0.11\,\mu\text{m}$$

Problems for Practice

1. The fringe width for red colour as compared to that for violet colour is approximately

(a) four times (b) eight times
(c) three times (d) double

2. When a thin film of thickness t is placed in the path of light wave emerging out of S_1 then increase in the length of optical path will be

(a) $(\mu - 1)t$ (b) μt
(c) μ/t (d) $(\mu + 1)t$

3. Which of following formula is correct in YDSE?

(a) $d = d_1 d_2$ (b) $d = \dfrac{\omega_a}{D}$
(c) $d = \sqrt{d_1 d_2}$ (d) $d = 2a(\mu - 1)d$

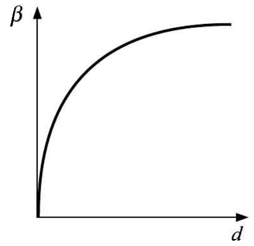
4. If the ratio of maximum and minimum intensities in an inference pattern is 36 : 1 then the ratio of amplitudes of two interfering waves will be

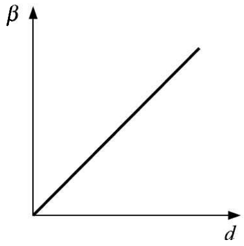
(a) 4 : 7 (b) 7 : 4
(c) 5 : 7 (d) 7 : 4

5. The colour of a bright fringe nearest the central achromatic fringe in the interference pattern with white light will be

(a) yellow (b) red
(c) green (d) violet

6. The correct curve between fringe width β and distance (d) between the slits is

(a)


(b)


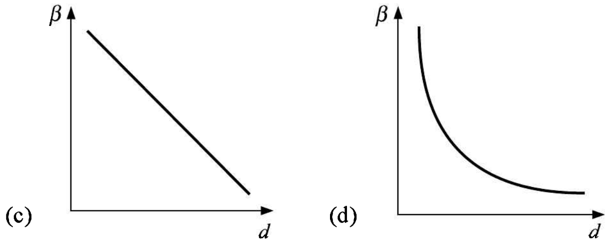


Fig. 30.15

7. If in a Young's double slit experiment, the distance between the slits is halved and the distance between slit and screen is doubled, then the fringe width will become
(a) four times (b) unchanged
(c) half (d) double

8. In Young's double slit experiment if the width of third fringe is 10^{-2}cm then the width of 5th fringe will be
(a) 2×10^{-2}cm (b) 10^{-2}cm
(c) 4×10^{-2}cm (d) 5×10^{-2}cm

9. If the path difference between the interfering waves is $n\lambda$ then the fringes obtained on the screen will be
(a) coloured (b) white
(c) bright (d) dark

10. In Fresnel's biprism experiment the distance between two coherent sources is 0.9 mm. The interference fringes are observed at a distance of 100 cm from the coherent sources. If the second dark fringe is at a distance of 1 mm from the central fringe then the wavelength of light will be
(a) 3×10^{-5}cm (b) 6×10^{-5}cm
(c) 6×10^{-5}m (d) 6×10^{-5}mm

11. Light of wavelength 5880 Å is incident on a thin glass plate (μ = 1.5) such that the angle of refraction in the plate is 60°. The minimum thickness of the plate, so that it appears dark in the reflected light will be
(a) 6312 Å (b) 4372 Å
(c) 3920 Å (d) 5840 Å

12. Two coherent sources of wavelength 6.2×10^{-7}m produce interference. The path difference corresponding to 10th order maximum will be
(a) 12.4×10^{-6}m (b) 6.2×10^{-6}m
(c) 3.1×10^{-6}m (d) 1.5×10^{-6}m

13. The device which produces highly coherent sources is
(a) Lloyd's mirror (b) LASER
(c) Fresnel's Biprism (d) Young's double slit

14. The intensity of central fringe in the interferences pattern produced by two identical slits is I. When one of the slits is closed then the intensity at the same point is I_0. The relation between I and I_0 is
(a) $I = I_0/2$ (b) $I = I_0$
(c) $I = 2I_0$ (d) $I = 4I_0$

15. The oil layer on the surface of water appears coloured due to interference. For this effect to be visible the thickness of oil layer will be
(a) 0.65 μm (b) 3.25 μm
(c) 0.35 μm (d) 1.63 μm

16. The relation between fringe width (β) and angular fringe width (ω_a) is
(a) $\beta = \omega_a - D$ (b) $\beta = D\omega_a$
(c) $\beta = \frac{\omega_a}{D}$ (d) $\beta = \omega_a - D$

17. The nature of light which is verified by the interference event is
(a) quantum nature (b) particle nature
(c) wave nature (d) dual nature

18. In Young's double slit experiment if the maximum intensity of light is $I_{\max}$ then the intensity at path difference $\lambda/2$ will be
(a) $\frac{I_{\max}}{4}$ (b) $\frac{I_{\max}}{2}$ (c) $I_{\max}$ (d) zero

19. A thin sheet of mica is placed in the path of S_2. The fringes will get shifted towards
(a) S_2
(b) S_1
(b) both sides
(d) first towards S_2 and then towards S_1

20. Intensity of light depends on
(a) wavelength (b) amplitude
(c) velocity (d) frequency

21. In Fresnel's biprism experiment, when light of wavelength 6000 Å is used then 16th bright fringe is obtained at point P. If light of wavelength 4800 Å is used then the order of fringe obtained at point P will be
(a) 16 (b) 20 (c) 22 (d) 18

22. In Young's double slit experiment two light beams of wavelength λ_1 = 6000 Å and λ_2 = 4800 Å are used. The distance between two slits is 2.5 mm. The distance between the central maxima obtained with two beams will be
(a) 2.872 mm (b) 2.652 mm
(c) 1.872 mm (d) zero

23. If the whole biprism experiment is immersed in water then the fringe width becomes ——— time if the refractive indices of biprism material and water are 1.5 and 1.33 respectively.
(a) 1/3 times (b) 4/3 times
(c) 3 times (d) 3/4 times

24. The equation of two interfering waves are $y_1 = b \cos \omega t$ and $y_2 = b \cos (\omega t + \phi)$ respectively. Destructive interference will take place at the point of observation for the following value of ϕ.
(a) 720° (b) 180° (c) 360° (d) 0°

25. Light of wavelength 7500 Å is incident on a thin glass plate (μ = 1.5) so that the angle of refraction obtained is 30°. If the plate appears dark then the minimum thickness of plate will be
(a) $5000/\sqrt{3}$ Å (b) $8000/\sqrt{3}$ Å
(c) $1000\sqrt{3}$ Å (d) $4000\sqrt{3}$ Å

26. The path of difference between two interfering waves at a point on the screen is $\lambda/8$. The ratio of intensity at this point and that at the central fringe will be
(a) 853 (b) 85.3 (c) 8.53 (d) 0.853

27. White light is normally incident on a soap film. The thickness of the film is 5×10^{-7}m and its refractive index is 1.33. Which wavelength will be reflected maximum in the visible region?
(a) 8860 Å (b) 3800 Å
(c) 5320 Å (d) 26600 Åe

28. In double slit experiment the distance between two slits is 0.6 mm and these are illuminated with light of wavelength 4800 Å. The angular width of dark fringe on the screen distant 120 cm from slits will be
(a) 16×10^{-4}radian
(b) 8×10^{-4}radian
(c) 6×10^{-4}radian
(d) 4×10^{-4}radian

29. The parallel rays of white light are made incident normally on an air film of uniform thickness. 250 fringes are seen in the transmitted light between 4000 Å and 6500 Å. Thickness of air film is
(a) 0.11 mm (b) 0.15 mm
(c) 0.17 mm (d) 0.13 mm

30. The ratio of slit width's in a Young's double slit experiment is 4 : 9. The ratio of maximum and minimum intensities will be
(a) 169 : 25 (b) 13 : 5
(c) 25 : 1 (d) 81 : 16

31. For sustained interference we need two sources which emit radiation
(a) having constant phase difference
(b) with same intensity
(c) with same amplitude
(d) in a narrow beam

32. The amplitude of the light waves emerging from the two slits in young's experiment is in the ratio of 2 : 3. The intensity of the minimum to that of the consecutive maximum will be in the ratio of
(a) 4 : 9 (b) 1 : 9 (c) 2 : 3 (d) none of these

33. How will the diffraction pattern of single slit change when yellow light is replaced by blue light?
(a) fainter (b) wider
(c) narrower (d) brighter

34. What happens to the fringe pattern when the Young's double slit experiment is performed in water instead of air?
(a) enlarged (b) shrinks
(c) disappears (d) unchanged

35. The fringe width β of the diffraction pattern and the slit width d are related as
(a) β is independent of d
(b) $\beta \propto 1/d$
(c) $\beta \propto d$
(d) $\beta \propto \sqrt{d}$

36. S is the size of the slit, d is the separation between the slits and D is the distance where Young's double slit interference pattern is being observed. If λ be the wavelength of light, then for sharp fringes, the essential condition is
(a) $SD > \lambda d$ (b) $\frac{S}{D} < \frac{\lambda}{d}$
(c) $\frac{S}{D} > \frac{\lambda}{d}$ (d) $S\lambda < dD$

37. In Young's double slit experiment, a minimum is obtained when the phase difference of the superimposing wave is (given that $n = 1, 2, 3, 4$ ....)
(a) $(2n + 1)\pi$ (b) zero
(c) $n\pi$ (d) $(n + 1)\pi$

38. During interference of light, light energy is
(a) redistributed
(b) created at the maxima
(c) destroyed at the minima
(d) non conserved

39. Diffraction pattern cannot be observed with
(a) one narrow slit
(b) two narrow slits
(c) large number of narrow slits
(d) one wide slit

40. Which of the following cannot produce colours with white light?
(a) diffraction (b) interference
(c) polarisation (d) dispersion

41. How many colours comprise white light?
(a) 3 (b) 7 (c) 14 (d) ∞

42. In Young's double slit experiment, the intensity of light at a point on the screen where the path difference is λ is K unit λ being the wavelength of light used the intensity at a point where the path difference is $\lambda/4$ will be
(a) K/4 (b) K/2 (c) K (d) zero

43. Two coherent monochromatic light beams of intensities I and $4I$ are superposed. The maximum and minimum possible intensities in the resulting beam are
(a) $9I$ and $3I$ (b) $5I$ and I
(c) $5I$ and $3I$ (d) $9I$ and I

44. Two coherent sources emitting light of wavelength λ are a distance $\lambda/4$ apart. I_0 is the intensity due to either of the two sources. The intensity at a point in a direction making an angle θ as shown in Fig. 30.16.

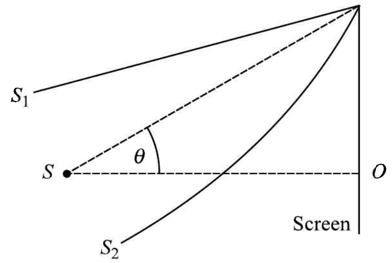


Fig. 30.16

(a) $4I_0\cos^2(\pi/2 \sin\theta)$ (b) $4I_0\cos^2\theta$
(c) $4I_0\cos^2(\pi/4 \sin\theta)$ (d) $4I_0\cos^2\theta/2$

45. Which of the following phenomenon is not common to sound and light waves?
(a) polarization (b) interference
(c) reflection (d) diffraction

46. In Young's double slit experiment the 7th maximum with wavelength λ_1 is at a distance d_1 and that with wavelength λ_2 is at distance d_2. Then d_1/d_2 is
(a) λ_1^2/λ_2^2 (b) λ_2/λ_1
(c) λ_2^2/λ_1^2 (d) λ_1/λ_2

47. The figure given below shows the path of white light rays which leave in phase from two small sources S_1 and S_2 and travel to a point Y on the screen. The path difference $S_2Y - S_1Y = 11.4 \times 10^{-7}$ m. What wavelength of light gives the brightest illumination at Y?

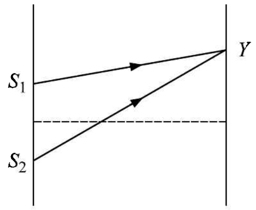


Fig. 30.17

(a) 7.6×10^{-7}m (b) 5.2×10^{-7}m
(c) 3.8×10^{-7}m (d) 6.4×10^{-7}m

48. The maximum intensity produced by two coherent waves of intensity I_1 and I_2 will be
(a) $I_1 + I_2$ (b) $I_1 + I_2 + 2\sqrt{I_1 I_2}$
(c) $I_1^2 + I_2^2$ (d) 0

49. The amplitude of two interfering waves are a and $2a$ respectively. The resultant amplitude in the condition of constructive interference will be
(a) $5a$ (b) a
(c) $3a$ (d) $2a$

50. In a biprism experiment fifth dark fringe is obtained at a point. If a thin transparent film is placed in the path of one of waves, then seventh bright fringe is obtained at the same point. The thickness of the film in terms of wavelength λ and refractive index μ will be
(a) $2.5\,(\mu - 1)\lambda$ (b) $1.5\lambda(\mu - 1)$
(c) $2.5\lambda(\mu - 1)$ (d) $1.5\,(\mu - 1)\lambda$

51. What will be the radius of first HPZ at a distance of 0.5 m from a plane wavefront, if the wavelength of light used in 5000 Å?
(a) 0.05 km (b) 0.05 mm
(c) 0.05 m (d) 0.05 cm

52. The correct formula for number of HPZ blocked by the obstacle or exposed through the aperture is
(a) $n = r^2 b\lambda$ (b) $n = r^2/\lambda$
(c) $n = r^2/b\lambda$ (d) $n = b\lambda/r^2$

53. What should be the size of the aperture of the objective of a telescope that can just resolve the two stars of angular width of 10^{-3} degree by light of wavelength 5000 Å?
(a) 3.5 km (b) 3.5 mm
(c) 3.5 cm (d) 3.5 m

54. The correct formula for the limit of resolution of a telescope is
(a) $d\theta = a\lambda/1.22$ (b) $d\theta = a/1.22\lambda$
(c) $d\theta = \dfrac{1.22\lambda}{a}$ (d) $d\theta = 1.22\lambda a$

55. In Fresnel's diffraction event the radii of fifth and sixth HPZ for wavelengths λ_5 and λ_6 at a distance of 2 m from the HPZ are same. The value of λ_5 and λ_6 will be
(a) $\sqrt{3} : \sqrt{4}$ (b) 6 : 5
(c) 5 : 6 (d) 3 : 4

56. In a YDSE $\lambda = 500$ nm, $d = 1$ mm, $D = 1$ m. The minimum distance from the central maximum for which the intensity is half of maximum intensity is
(a) 2.5×10^{-4}m (b) 5×10^{-4}m
(c) 1.25×10^{-4}m (d) 1.67×10^{-4}m

57. Light of wavelength 6000 Å is incident normally on slit of width 0.2 mm. The angular width of central maximum on a screen distant 9 m will be
(a) 0.09° (b) 0.18°
(c) 0.36° (d) 0.82°

58. The wave representing the rectilinear propagation is
(a) infrared rays (b) X-rays
(c) radio waves (d) telephone waves

59. The radius of first circle of a zone plate is 0.13 mm. Its principal focal length for wavelength 7200 Å will be
(a) 0.062 m (b) 1.25 m
(c) 0.125 m (d) 0.25 m

60. The value f_2/f_5 for a zone plate will be
(a) 5/2 (b) 3 (c) 3/9 (d) 2/5

61. The principal focal length of a zone plate for wavelength of 5000 Å is 21 cm. Its value for wavelength of 7000 Å will be
(a) 0.28 mm (b) 0.15 m
(c) 0.34 cm (d) 0.028 mm

62. Plane microwaves are incident on a long slit having a width 5 cm The wavelength of the microwave if first minimum is obtained at 30° is
(a) 2.5 cm (b) 5 cm
(c) 1.25 cm (d) 1.4 cm

63. A convex lens of diameter 8 cm focusses a parallel beam of wavelength 620 nm. If the light is focussed at 20 cm from the lens then the radius of central bright ring is
(a) 3.8 μm (b) 3.8 mm
(c) 1.9 μm (d) 1.9 mm

64. Which colour should be used to increase the resolving power of a microscope?
(a) yellow (b) red
(c) green (d) violet

65. The ratio of intensities of consecutive maxima in the diffraction pattern due to a single slit is
(a) $1 : 4/\pi^2 : 9/\pi^2$ (b) $1 : 4 : 9$
(c) $1 : 4/9\pi^2 : 4/25\pi^2$ (d) $1 : 2 : 3$

66. The condition for obtaining maxima in the diffraction pattern due to a single slit is
(a) $a \sin\theta = (2n + 1)\lambda$ (b) $a \sin\theta = (2n + 1)\lambda/2$
(c) $a \sin\theta = n\lambda$ (d) $a \sin\theta = n\lambda/2$

67. Which criterion describes limit of resolution that maxima of one should fall on the minima of the other, then two sources are just resolved?
(a) Rayleigh's (b) Fresnel's
(c) Huygen's (d) Fraunhoffer's

68. A plane wave of wavelength 6250 Å is incident normally on a slit of width 2×10^{-2}cm. The width of the principal maximum on a screen distant 50 cm will be
(a) 312.5×10^{-3}cm (b) 312.5×10^{-3}m
(c) 312 cm (d) 312.5×10^{-3}mm

69. To achieve Fraunhoffer class of diffraction, wavefront required is
(a) cylindrical (b) spherical
(c) planar (d) any of the (a), (b), (c)

70. The wave nature can be best judged using
(a) interference (b) diffraction
(c) polarisation (d) scattering

ANSWERS

1. (d)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (b)
11. (c)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (b)	**22.** (d)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (d)	**27.** (c)	**28.** (b)	**29.** (d)	**30.** (c)
31. (a)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (a)	**38.** (a)	**39.** (d)	**40.** (c)
41. (b)	**42.** (b)	**43.** (d)	**44.** (c)	**45.** (a)	**46.** (d)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (c)
51. (d)	**52.** (c)	**53.** (c)	**54.** (c)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (b)	**59.** (c)	**60.** (d)
61. (b)	**62.** (a)	**63.** (a)	**64.** (d)	**65.** (c)	**66.** (b)	**67.** (a)	**68.** (a)	**69.** (c)	**70.** (b)

Photoelectric Effect and Daul Nature of Matter

31

BRIEF REVIEW OF THE CONCEPTS

Photons are packets of energy emitted by a source of radiation. They travel in a straight line with speed of light.

According to Planck's quantum theory, a source of radiation emits energy in the form of photons which travel in straight line. That is, the particles of light are called **Photons.**

Energy of photon $E = h\nu = \dfrac{hc}{\lambda}$

$h = 6.626 \times 10^{-34}\,\text{Js} = 4.136 \times 10^{-15}\,\text{eVs}$

All photons travel with speed of light $c = 3 \times 10^8\,\text{ms}^{-1}$ in vacuum.

Their velocity changes in the medium due to change in wavelength.

Rest mass of photon is zero, that is, photons cannot exist at rest.

Each photon has a definite energy and a definite momentum.

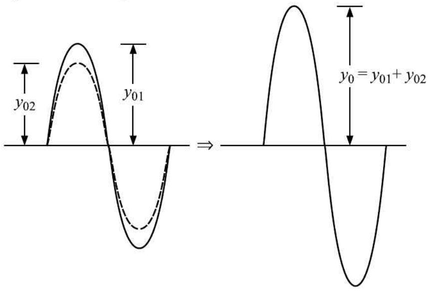
Energy $E = h\nu$ momentum $p = \dfrac{h}{\lambda} = mc = \dfrac{E}{c}$.

A photon may collide with a material particle. The total energy and total momentum remain conserved in such a collision. The photon may get absorbed and/or a new photon may be emitted. Thus the number of photons may not be conserved.

The increase in intensity of light means increase in number of photons crossing a given area in a given time. The energy of each photon remains the same.

Photoelectric effect When light of sufficiently small wavelengths is incident on a metal surface, electrons are ejected from its surface. This phenomenon is called **Photoelectric effect.** Electrons so emitted are called **photoelectrons**. Experimental setup is shown in Fig. 31.1(a). When photons are incident on a photocathode, electrons are ejected. The anode at positive potential attracts these electrons and current is seen.

It is found experimentally that

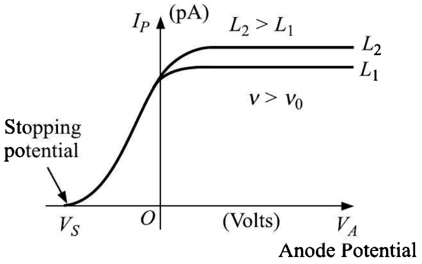


(a)

(a) No emission occurs until the incident radiations have a frequency greater than a certain minimum called threshold frequency ν_0 irrespective of time of exposure

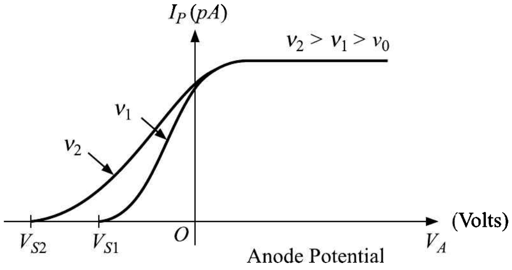
(b) No emission occurs until the incident radiations have a frequency greater than a certain minimum called threshold frequency ν_0 irrespective of intensity of incident radiations

(c) If frequency of incident radiations is greater than the threshold frequency ν_0 then photocurrent increases on increasing the intensity as illustrated in Fig. 31.1(b)



(b)

(d) If frequency of incident radiations is greater than the threshold frequency ν_0 then stopping potential increases if frequency of incident radiation is increased as shown in Fig. 31.1(c). Wave theory cannot explain these effects.



(c)

Fig. 31.1

Einstein explained this theory in 1905. He was awarded Noble Prize in 1921 for his contribution to explain Photoelectric effect.

$$(\text{KE})_{\max} = h\nu - h\nu_0 = h\nu - \phi$$

max. KE of photo electrons = energy of incident radiation (photon) – work function.

Work function Minimum energy given to electrons to bring it out of the metal is called work function. It is equivalent to ionization energy or it is the minimum energy given to an electron in its ground state of upper most filled level to transmit it to the vacuum level. It is denoted by ϕ

$$\phi = h\nu_0$$

Stopping potential is that negative voltage given to the anode at which photocurrent stops (becomes zero).

$$eV_S = (KE)_{max}$$

Thus $eV_S = h\nu - h\nu_0$ or $V_S = \frac{h\nu}{e} - \frac{h\nu_0}{e} = \frac{h\nu}{e} - \frac{\phi}{e}$

The curve between stopping potential and frequency ν (or 1/λ) is a straight line as shown in Fig. 31.2.

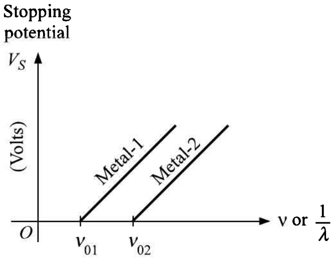


Fig. 31.2

The slope is h/e if curve is between V_S and ν.

The slope is hc/e if curve is between V_S and 1/λ.

Matter waves The wavelength of matter waves is given by de- Broglie relation and confirms the dual nature of metal.

$$\lambda = h/p$$

Experiments like Davisson's and Germer's experiment confirm wave nature of electrons. de Broglie was awarded the Noble Prize in 1929.

Classical mechanics works well for particles of linear size > 10^{-4} cm. For smaller particles quantum mechanics is to be used. Quantum mechanics take into account dual nature (wave nature and particle nature) of electrons, protons and other subatomic particles.

SHORT-CUTS AND POINT TO NOTE

(i) Photoelectric effect can be explained with Einstein's equation

$(KE)_{max} = h\nu - h\nu_0 = h\nu - \phi$

and $(KE)_{max} = eV_S$ where V_S is stopping potential.

If the incident light has a wavelength greater than threshold wavelength no electron emission takes place.

(ii) Stopping potential does not depend upon intensity of incident radiations. (or power rating of the source). That is, maximum KE of ejected electrons is independent of intensity of light incident.

(iii) Cesium has least work function. Work function can be further decreased by oxide coating a substance.

(iv) Stopping potential depends on the wavelength/frequency of the incident radiations. Smaller the wavelength (or greater the frequency) more is the stopping potential.

(v) Stopping potential/$(KE)_{max}$ of photo electrons depends upon work function. More the work function less is the stopping potential or KE_{max}

(vi) Stopping potential in volts = $(KE)_{max}$ in eV (That is, remove only e)

(vii) $\lambda(\text{nm}) = \frac{1242}{E(\text{eV})}$ for photons

$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2Km}}$ for particles like electrons, protons and so on where K is kinetic energy of the particle and m its mass.

(viii) The slope of stopping potential versus frequency is h/e in photoelectric effect and slope of stopping potential versus 1/λ is hc/e.

(ix) $\lambda = \frac{12.27}{\sqrt{V}}$ Å for electron

$\lambda = \frac{0.286}{\sqrt{V}}$ Å for proton and neutron

$\lambda = \frac{0.101}{\sqrt{V}}$ Å for α-particles

$\lambda = \frac{h}{mC_{rms}} = \frac{h}{\sqrt{3mkT}}$ for gas particles

(x) Number of photons incident per second

$$= \frac{I_P}{e} = \frac{\text{Photocurrent}}{\text{charge on an electron}}$$

assuming each photon emits an electron.

(xi) Force exerted by photons $F = \frac{dp}{dt} = \frac{P}{c}$ if absorbing surface where c = speed of light $F = 2P/c$ if perfectly reflecting surface where P is power.

(xii) Momentum of photons $p = E/c = h/\lambda$

(xiii) If no light is incident even then a small current flows due to γ-rays etc on entering the photocell. Such a current is called dark current.

(xiv) Photo cells are of three types:

(1) Photo emissive

(2) Photo conductive or light dependent resistors (LDR).

(3) Photo voltaic or solar cells.

Photoelectric effect was studied initially using photo emissive cells.

(xv) Compton shift $\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi)$ ϕ is the angle at which photon is scattered.

CAUTION

(i) Considering that same formula may be applied for photons and material particles like electrons, protons.

- $\lambda(\text{nm}) = \frac{1242}{E(\text{eV})}$ is valid only for photons (mass less particles)

(ii) Considering that all photoelectrons emitted have same KE or velocity.

- Only few photoelectrons have maximum KE.

(iii) Considering that if the source has more power rating than photons emitted from it have more energy.

- More power rating means more intensity, that is, more number of photons emitted per second. Therefore photo current will increase provided the frequency of the photons is greater than threshold frequency.

(iv) Considering that when a photon enters a material it is always absorbed.

- All photons incident do not eject photoelectrons. Efficiency is around 1.0%.

(v) Considering that as energy is conserved hence number of photons remain conserved.

- Photons may be absorbed or even can cause other photons. Hence number of photons is not conserved.

(vi) Considering that absorption process of photon is only photoelectric effect.

- Photon can be absorbed in four ways
  (1) adsorption (heating the material)
  (2) Pair production
  (3) photoelectric effect
  (4) compton scattering

(vii) Considering that Compton wavelength is Compton shift

- Compton wavelength $\lambda_C = \frac{h}{m_0c} = 0.024\text{A}^\circ$

  $\Delta\lambda = \frac{h}{m_0c}(1 - \cos\phi) = 0.024\,(1 - \cos\phi)$.

(viii) Considering that even a moving photon has zero mass

- Mass of moving photon is $\frac{h\nu}{c^2}$ or $\frac{h}{c\lambda}$ or $\frac{E}{c^2}$.

(ix) Considering that photoelectrons are emitted after some time of incidence.

- There is no time lag between the incidence of photons and emission of photoelectrons.

(x) Considering that the distance of source from photocathode has no bearing.

- Intensity ∝ $1/r^2$. Therefore, Photocurrent ∝ $1/r^2$.

Solved Problems

1. A radio station emits 10 kW power at 90.8 MHz. Find the number of photons emitted per second.

(a) $10^{28} \times 1.6$ (b) 1.6×10^{29}
(c) 1.6×10^{30} (d) 1.6×10^{32}

Solution (b) number of photons emitted per sec

$$= \frac{10^4}{6.626\times10^{-34}\times90.8\times10^6}$$

$$= \frac{10^{32}}{621.64} = 1.6\times10^{29}\text{photon/s}$$

2. A photosensitive metallic surface has work function $h\nu_0$. If photons of energy $2h\nu_0$ fall on this surface, the electrons come out with a maximum velocity $4 \times 10^6\,\text{ms}^{-1}$. When the photon energy is increased to $5h\nu_0$ then maximum velocity is

(a) $2 \times 10^7\,\text{ms}^{-1}$ (b) $2 \times 10^6\,\text{ms}^{-1}$
(c) $8 \times 10^6\,\text{ms}^{-1}$ (c) $8 \times 10^5\,\text{ms}^{-1}$

Solution $$\left(\frac{\frac{1}{2}mv_1^2}{\frac{1}{2}mv_2^2}\right)_{\max} = \frac{h\nu_1 - \phi}{h\nu_2 - \phi}$$

or $$\frac{v_{1\max}}{v_{2\max}} = \sqrt{\frac{2h\nu_0 - h\nu_0}{5h\nu_0 - h\nu_0}} = \frac{1}{2}$$

3. (*A*) The energy *E* and momentum *p* of a photon are related as $E/C = p$

(*R*) The photon behaves as a particle

(a) *A* and *R* are both correct and *R* is correct explanation of *A*
(b) *A* and *R* are correct but *R* is not correct explanation of *A*
(c) *A* is correct but *R* is false
(d) both *A* and *R* are false

Solution (a)

4. The light rays having photons of energy 1.8 eV are falling on a metal surface having a work function 1.2 eV. What is the stopping potential to be applied to stop the emitting electrons?

(a) 3 V (b) 1.2 V
(c) 0.6 V (d) 1.4 V

Solution (c) $eV_s = hf - \phi$ or $V_s = 1.8 - 1.2 = 0.6\text{V}$

5. A photon of energy 10.2 eV collides in elastically with *H*-atom in ground state. After a certain time interval of few μs another photon of energy 15 eV collides in elastically with the same H atom, the observation made by a suitable detector is

(a) 1 photon with energy 10.2 eV and an electron with 1.4 eV
(b) Two photons with 10.2 eV
(c) 2 photon with 1.4 eV
(d) one photon with 3.4 eV and 1 electron with 1.4 eV

Solution (a) 10.2 eV photon will excite it to 2nd orbit which on de-excitation will emit 10.2 eV photon and 15 – 13.6 = 1.4 eV.

∴ electron emitted will have energy 1.4 eV.

6. From the fig. 31.3 describing photoelectric effect we may infer correctly that

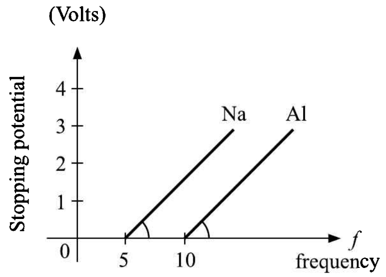


Fig. 31.3

(a) Na and Al both have same threshold frequency
(b) Maximum KE for both the metals depend linearly on the frequency

(c) the stopping potentials change differently for equal change in frequency
(d) Al is better photosensitive material than Na

Solution (b) $(KE)_{\max} = hf - \phi$

7. A photocell is illuminated by a small bright source placed 1 m away. When the same source of light is placed 1/2 m away, the number of electrons emitted by photocathode would
(a) decrease by a factor of 4
(b) increase by a factor of 4
(c) decrease by a factor of 2
(d) increase by a factor of 2

Solution (b) ∵ $I_p \propto 1/r^2$
∴ Current increases by a factor of 4.

8. If the kE of a free electron doubles then its de Broglie wavelength changes by a factor
(a) 1/2 (b) $1/\sqrt{2}$
(c) 2 (d) $\sqrt{2}$

Solution (b) $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2(kE)m}}$

9. A laser used to weld detatched ratina emits light with a wavelength 652 nm in pulses that are of 20 ms duration. The average power during each pulse is 0.6 W. Find the energy in each pulse in eV and in a single photon
(a) 7.5×10^{15} eV, 1.9 eV
(b) 7.5×10^{15} eV, 1.19 eV
(c) 7.5×10^{16} eV, 0.19 eV
(d) 7.5×10^{16} eV, 1.9 eV

Solution (d) $E(eV) = \frac{1240}{652}$
= 1.9 eV (in a single photon)
Energy in 20 ms = $0.6 \times 20 \times 10^{-3} = 1.2 \times 10^{-2}$ J
$= \frac{1.2\times10^{-2}}{1.6\times10^{-19}}$ eV $= 7.5 \times 10^{16}$ eV

10. Threshold wavelength for tungsten is 272 nm. Light of frequency. 1.45×10^{15} Hz is incident. Find the stopping potential
(a) 1.37 V (b) 1.47 V
(c) 1.57 V (d) 1.51 V

Solution (b) $\phi = \frac{1240}{272} = 4.56$ eV;
$hf = \frac{6.625\times10^{-34}\times1.45\times10^{15}}{1.6\times10^{-14}} = 6.03$ eV
$eV_s = hf - \phi$
or $V_s = 6.03 - 4.56 = 1.47$V.

11. What should be the minimum work function of a metal so that visible light is able to cause emission?
(visible light = 400 – 700 nm)
(a) 1.77 eV (b) 1.87 eV
(c) 1.97 eV (d) none of these

Solution (a) $E(\text{eV}) = \frac{1240}{700} = 1.77$ eV

12. A photon has momentum 9×10^{-28} kg ms^{-1}. What will be the stopping potential if photocathode has work function 1.3 eV?
(a) 0.44 V (b) 1.4 V
(c) 0.59 V (d) 0.39 V

Solution (d) $E = pC = \frac{2.7\times10^{-19}\times3\times10^{8}}{1.6\times10^{-19}} = 1.69$ eV
$V_s = 1.69 - 1.3 = 0.39$ V

13. When the incident frequency is f_0, k is the $(KE)_{\max}$ of the electrons emitted and ϕ is work function of the surface. If incident frequency is doubled new $(KE)_{\max}$ will be
(a) $2k$ (b) $2k - \phi$
(c) $2k + \phi$ (d) $2k + 2\phi$
(e) $2k - 2\phi$

Solution (c) $k = hf - \phi$ or $2k = h(2f) - 2\phi$
or $2k + \phi = 2hf - \phi$.

14. A node voltage is at +3 V. Incident radiation has frequency 1.4×10^{15} Hz work function of the photocathode is 2.8 eV. The minimum and maximum kE of photoelectrons in eV is
(a) 3, 6 (b) 0, 3
(c) 0, 6 (d) 2.8, 5.8

Solution (a) $hf = \frac{6.625\times10^{-34}\times1.4\times10^{15}}{1.6\times10^{-19}} = 5.8$ eV;
$(kE)_{\max} = hf - \phi$
$3 = 5.8 - 2.8$.
Since anode voltage is 3 V, the electrons emitted with zero KE will acquire an energy = 3 eV and the electrons emitted with 3 eV will acquire 3 + 3 = 6 eV
∴ min KE = 3 eV and max KE = 6 eV.

15. A surface has work function 3.3 eV. Which of the following will cause emission?
(a) 100 W incandescent lamp
(b) 40 W fluorescent lamp
(c) 20 W sodium lamp
(d) 20 W Hg lamp

Solution (e) Min. wavelength of visible region is 400 nm
$E(ev) = 1242/400 = 3.1$ eV
∴ No visible light can cause emission. We require UV light. Only Hg lamp gives UV light.

16. A man wants current ~mA. He should use
(a) photo multiplier tube
(b) photocell and amplifier
(c) photo multiplier tube and amplifier
(d) photo cell and two stage amplifier

Solution (c)

17. When a photo multiplier tube was used, the photo current recorded is 60 μA. The actual photo current is
(a) >60 μA
(b) = 60 μA
(c) <60 μA
(d) none of these

Solution (c) ∵ The dark current will also be added in it. Actual current is 60 μA - dark current.

18. Tungsten has work function 4.8 eV. We wish to use tungsten as photocathode with a 600 nm wavelength. What shall we do?
(a) coat tungsten with Cs
(b) oxide coat tungsten
(c) Cu_2O_2 be coated on tungsten
(d) none of these

Solution (b)

19. Find the wavelength of 100 eV electron
(a) 1.23 Å (b) 1.72 Å
(c) 1.24 nm (d) 12.4 nm

Solution (a) $\lambda = \frac{12.27}{\sqrt{v}}$ Å $= \frac{12.27}{\sqrt{100}} = 1.227$ Å

20. Find the wavelength of 10 MeV α-particles

(a) 3 Å (b) 3 pm

(c) 3 fm (d) 30 fm

Solution (c) $\lambda(\text{Å}) = \dfrac{0.101}{\sqrt{10^7}} = 0.03 \times 10^{-3}\,\text{Å}$

$= 3 \times 10^{-15}\,\text{m} = 3\,\text{fm}$

21. The wavelength associated with 1 MeV proton is

(a) 28.6 pm (b) 2.86 pm

(c) 2.86 fm (d) 28.6 fm

Solution (d) $\lambda(\text{Å}) = \dfrac{0.286}{\sqrt{V}} = \dfrac{0.286}{\sqrt{10^6}}$

$= 2.86 \times 10^{-14}\,\text{m} = 28.6\,\text{fm}.$

Problems for Practice

1. An electron is accelerated through a potential difference of 100 V. Its velocity will be

(a) $3.78 \times 10^7\,\text{ms}^{-1}$

(b) $0.95 \times 10^7\,\text{ms}^{-1}$

(c) $5.67 \times 10^7\,\text{ms}^{-1}$

(d) $0.6 \times 10^7\,\text{ms}^{-1}$

2. The photoelectric currents at distances r_1 and r_2 of light source from photoelectric cell are I_1 and I_2 respectively. The value of I_1/I_2 will

(a) r_2/r_1 (b) $(r_1/r_2)^2$

(c) r_1/r_2 (d) $(r_2/r_1)^2$

3. The correct curve between the stopping potential (V) and intensity of incident light (I) is

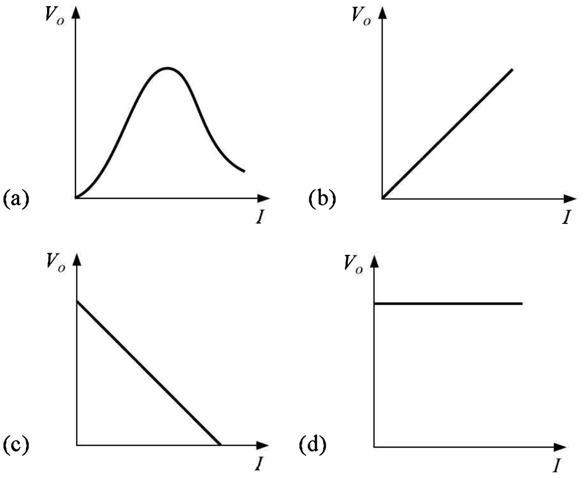


Fig. 31.4

4. A radio transmitter is working at frequency 880 KHz and power 10 KW. The number of photons emitted per second

(a) 0.075×10^{-33} (b) 1.71×10^{33}

(c) 13.27×10^{34} (d) 1327×10^{34}

5. The work function of a metal is X eV. When light of energy $2X$ is made incident on it then the maximum kinetic energy of emitted photoelectron will be

(a) XeV (b) $3X$eV (c) $2X$eV (d) 2 eV

6. If the energy of incident photon and work function of metal are E eV and ϕ eV respectively, then the maximum velocity of emitted photoelectron will be

(a) $\sqrt{2(E-\phi)/\text{m}}$ (b) $2\text{m}\sqrt{(E-\phi_0)}$

(c) $2/\text{m}[E - \phi_0]$ (d) $\text{m}/2[E - \phi_0]$

7. The momentum of a photon of energy 1 MeV, in Kgms^{-1}, will be

(a) 5×10^{-22} (b) 0.33×10^6

(c) 7×10^{-24} (d) 10^{-22}

8. The slope of $V_s - v$ as shown curve is equal to

(a) e (b) h

(c) ϕ_0 (d) h/e

9. The work function of photosensitive surface is 3.3 eV. The threshold frequency for it is

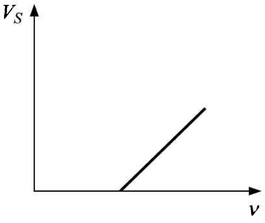


Fig. 31.5

(a) 4×10^{11} Hz (b) 5×10^{56} Hz

(c) 8×10^{10} Hz (d) 8×10^{14} Hz

10. The work functions of tungsten and sodium are 5.06 eV and 2.53 eV respectively. If the threshold wavelength of sodium is 5896 Å, then the threshold wavelength of tungsten will be

(a) 11792 Å (b) 4321 Å

(c) 2948 Å (d) 5896 Å

11. The photoelectric equation is

(a) $hv = hv_0$ (b) $hv = hv_0 + \text{E}_\text{K}$

(c) $hv = hv_0 - \text{E}_\text{K}$ (d) $hv = hv_0 + \text{E}_\text{K}/\text{V}$

12. The work function of aluminium is 4.2 eV. Light of wavelength 2000 Å is incident on it. The threshold frequency will be

(a) 10^{19} Hz (b) 10^{18} Hz

(c) 10^{15} Hz (d) 10^{13} Hz

13. The value of threshold wavelength for photoelectric effect is 7000 Å which of the following radiations will not produce photoelectric effect?

(a) infrared (b) ultraviolet

(c) yellow (d) violet

14. In a photoelectric cell, the cathode with work function W_2 is replaced by W_1 ($\text{W}_2 > \text{W}_1$). If the current before this change is I_1 and that after the change is I_2 and other circumstances remain same and if $hv > \text{W}_2$, then

(a) $I_1 < I_2 < 2I_1$ (b) $I_1 > I_2$

(c) $I_1 < I_2$ (d) $I_1 = I_2$

15. The work function of a metal is 3.3 eV. The minimum frequency for photoelectric emission by it will be

(a) $0.5 \times 10^{-5}\,s^{-1}$ (b) $0.8 \times 10^{15}\,s^{-1}$

(c) $8 \times 10^{15}\,s^{-1}$ (d) $0.8 \times 10^{15}\,s^{-1}$

16. There are two light sources *A* and *B*. The intensity of source *A* is more than that of source *B*. The frequency of light emitted by source *B* is higher than that emitted by source *A*. The photoelectric current obtained will be more from source.

(a) *A* (b) *B*

(c) same from *A* and *B* (d) none of these

17. The momentum of photon of electromagnetic radiation is 3.3×10^{-29} kg ms^{-1}. The frequency of waves associated with it is

(a) 7.5×10^{12} Hz (b) 1.5×10^{13} Hz

(c) 6×10^{2} Hz (d) 3×10^{3} Hz

18. The process of photoelectric emission depends on

(a) wavelength of incident light (λ)

(b) work function of surface

(c) nature of surface

(d) all of the above

19. The curve between current (i) and potential difference (V) for a photo cell will be

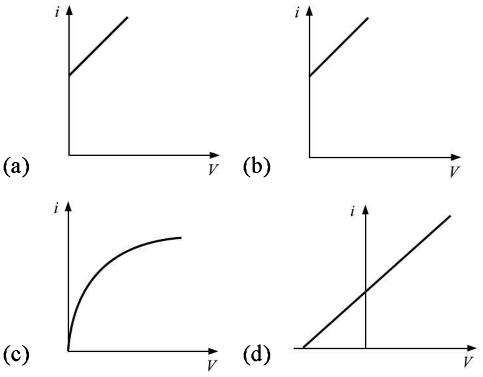


Fig. 31.6

20. The maximum value of stopping potential in the following diagram is

(a) –4V (b) –1V

(c) –3V (d) –2V

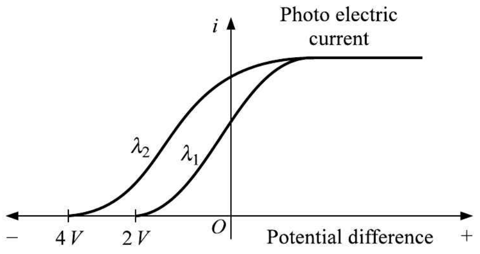


Fig. 31.7

21. If *h* is Planck's constant, the momentum of a photon of wavelength 0.01 Å is

(a) $10^{2}h$ (b) h

(c) $10^{12}h$ (d) $10^{-2}h$

22. If the distance of 100 W lamp is increased from a photo cell, the saturation current 'i' in the photocell varies with distance 'd' as

(a) $i \propto 1/d$ (b) $i \propto d^2$

(c) $i \propto 1/d^2$ (d) $i \propto d$

23. Threshold wavelength for a metal having work function ω_0 is λ. What is the threshold wavelength for the metal having work function $2\omega_0$?

(a) λ/2 (b) 4λ (c) λ/4 (d) 2λ

24. A photosensitive plate is irradiated from a distance of 1 m. The photoelectric current is 0.1 A. If the source of radiations is brought nearer to a distance of 50 cm, the photoelectric current will be

(a) 0.8 A (b) 0.1A (c) 0.2A (d) 0.4A

25. 5% of the energy supplied to a lamp is radiated as a visible light. How many quanta of light are emitted per second by 100 watt lamp? Assume the average wavelength of visible light as 555 nm.

(a) 2.1×10^{19} (b) 0.7×10^{19}

(c) 1.4×10^{19} (d) 2.8×10^{19}

26. The photoelectric effect supports a theory of light proposed by

(a) Einstein (b) Newton

(c) Huygen (d) none of these

27. The energy of a photon corresponding the visible light of maximum wavelength is nearly

(a) 1.0 eV (b) 7.0 eV (c) 1.6 eV (d) 3.2 eV

28. The screen is placed at a distance of 4 m from a source of luminous intensity 32 cd. The illumination is found to be 2 lux. What is the angle between the normal to the screen and the ray of light?

(a) 60° (b) 45° (c) 30° (d) 0°

29. Moving with the same velocity, which of the following has large wavelength of the matter waves?

(a) proton (b) α particle

(c) β particle (d) neutron

30. The stopping potential as a function of incident radiation frequency is ploted for two different photoelectric surfaces *A* and *B*. The graphs show that the work function of *A* is

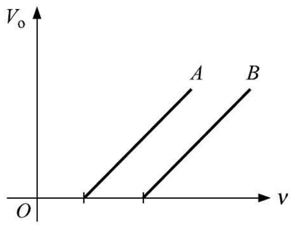


Fig. 31.8

(a) same as that of *B*

(b) greater than that of *B*

(c) no comparison can be done from the given graph

(d) smaller than that of *B*

31. The best metal to be used for photoemission is

(a) lithium (b) cesium

(c) potassium (d) sodium

32. Light of wavelength *L* strike a photoelectric surface and electrons are ejected with an energy K. If K is to be increased to exactly twice its original value, the wavelength must be changed to *L* where

(a) $L' > L/2$ (b) $L > L' > L/2$

(c) $L' = L/2$ (d) $L' < L/2$

33. An AIR radio station is broadcasting the waves of wavelength 300 m. If the radiating power of the transmitter is 10 kW, then the number of photons radiated per second is

(a) 1.5×10^{31} (b) 1.5×10^{35}
(c) 1.5×10^{29} (d) 1.5×10^{33}

34. The rest mass of a photon of wavelength λ is

(a) h/λ (b) zero (c) hc/λ (d) $h/c\lambda$

35. Photoelectric effect was discovered by

(a) Lenard (b) Einstein
(c) Hertz (d) Hallwach

36. A stopping potential of 0.82 V is required to stop the photoelectrons emitted from a metallic surface by light of wavelength 4000 Å. The stopping potential for wavelength 3000 Å will be

(a) 2.03 V (b) 1.1 V (c) 2.5 V (d) 1.85 V

37. Which conservation law is obeyed in Einstein's photoelectric equation?

(a) momentum (b) charge
(c) mass (d) energy

38. If the energy of photon is 10 eV and work function is 5 eV, then the value of stopping potential will be

(a) 50 V (b) 2 V (c) 5 V (d) 15 V

39. When a photon of energy 7 eV is made incident on a metal then the emitted electron is stopped by a stopping potential of –5.5 V. The work function of metal will be

(a) 12.5 eV (b) –1.5 eV
(c) 37.5 eV (d) 1.5 eV

40. In the following diagram if $V_2 > V_1$ then

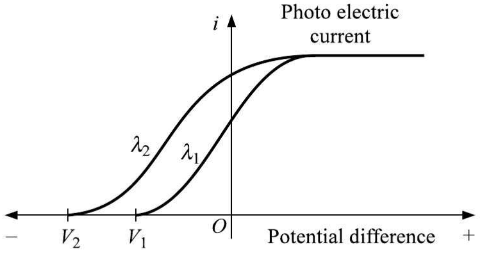


Fig. 31.9

(a) $\lambda_1 > \lambda_2$ (b) $\lambda_1 = \sqrt{\lambda_2}$
(c) $\lambda_1 = \lambda_2$ (d) $\lambda_1 < \lambda_2$

41. The work function of aluminium is 4.2 eV. The threshold wavelength for it for photoelectric emission will be

(a) 4200 Å (b) 1100 Å
(c) 3000 Å (d) 2955 Å

42. What does in the photoelectric effect support quantum nature of light?

(a) distribution of kinetic energy among the electrons
(b) threshold frequency
(c) quantisation of electric charge on the electrons
(d) instantaneous emission of electrons

43. How many kinds of lasers are there?

(a) solid only
(b) liquid only
(c) gaseous only
(d) all of the above

44. Photons of wavelength 660 nm are emitted from a 60 W lamp. What is the nnumber of photons emitted per second? Take $h = 6.6 \times 10^{-34}$ Js

(a) 2×10^{20} (b) 2×10^{22}
(c) 2×10^{16} (d) 2×10^{18}

45. When a photon collides with an electron which of the following characteristics of the photon increase?

(a) wavelength (b) energy
(c) frequency (d) none of these

46. Which of the following makes use of photoelectric effect?

(a) Cathode Ray oscillograph
(b) Radar
(c) Television receiver
(d) Television camera

47. Which of the following does not change, when a photon enters glass from air?

(a) wavelength (b) energy
(c) velocity (d) momentum

48. The threshold frequency of potassium is 3×10^{14} Hz. The work function is

(a) 0.5×10^{-19} J (b) 2.0×10^{-19} J
(c) 1.0×10^{-19} J (d) 4.0×10^{-19} J

49. Light of two different frequencies whose photons have energies 1 eV and 2.5 eV respectively successively illuminate a metal of work function 0.5 eV. The ratio of maximum speeds of the emitted electrons will be

(a) 1 : 1 (b) 1 : 5 (c) 1 : 4 (d) 1 : 2

50. Which of the following achieve conversion of electromagnetic wave energy into electrical energy?

(a) photocell (b) coolidge tube
(c) cathode ray tube (d) vacuum tube

ANSWERS

1. (d)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (d)	**9.** (d)	**10.** (c)
11. (b)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (a)
21. (c)	**22.** (c)	**23.** (a)	**24.** (d)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (d)
31. (b)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (d)	**37.** (d)	**38.** (c)	**39.** (d)	**40.** (a)
41. (d)	**42.** (b), (d)	**43.** (d)	**44.** (a)	**45.** (a)	**46.** (d)	**47.** (b)	**48.** (b)	**49.** (d)	**50.** (a)

32

Bohr's Theory and Atomic Physics

BRIEF REVIEW OF THE CONCEPTS

Thomson's model (1898)

(a) Atom as a whole is neutral, that is, positive charges and negative charges are equal.

(b) The positive charge and the whole mass is uniformly distributed like a cake and electrons embedded appear as cherries in the cake. Therefore, it is also called plum pudding model.

(c) It cannot explain α-particle scattering and spectrum of an atom. Leonard in 1903 suggested that atom is made up of tiny particles called electrons and similar tiny particles carrying positive charge. He could not explain why the heating of metals does not eject positively charged particles.

Rutherford's model

(a) The whole positive charge is concentrated in a small region called nucleus. The size of the nucleus is of the order of 10^{-15} m or 1 fm.

(b) The electrons revolve around the nucleus in circular orbits. The size of an atom is 10^{-10} m. There exists a large empty space around the nucleus.

(c) Atoms are electrically neutral.

Distance of closest approach $r = \dfrac{2Ze^2}{4\pi\varepsilon_0(KE)}$

Impact parameter $b = \dfrac{Ze^2\cot\theta/2}{4\pi\varepsilon_0(KE)}$

It could not explain why the electrons revolving around the nucleus do not fall into the nucleus following a spiral path, that is, it could not explain stability of the atom.

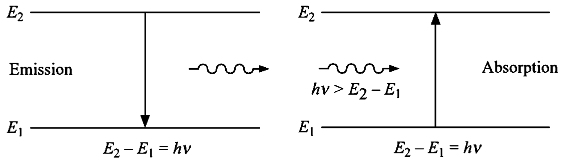
The number of particles scattered through an angle θ is given by $N(\theta) \propto \dfrac{Z^2}{\sin^4(\theta/2)\,(KE)^2}$

Bohr's model

(a) The electrons move around the nucleus in circular orbits.

(b) The orbits are stable called stationary orbits. They have special values of radii such that the angular momentum is quantised, that is, $mvr = n\hbar$ where, $\hbar = h/2\pi$.

(c) The energy is emitted when electrons make a transition from higher to lower orbit and energy is absorbed when electrons jump from lower to higher orbit.



(a) **(b)**

Fig. 32.1

The centripetal force is equal to the electrostatic force.

Radius of nth orbit $r_n = \dfrac{n^2\varepsilon_0 h^2}{\pi m z e^2}$

Binding energy of nth orbit $E_n = -\dfrac{mZ^2e^4}{8\varepsilon_0^2h^2n^2}$.

de-Broglie theory The electrons revolve around the nucleus in stable circular orbits in the form of stationary waves.

Only those circular orbits are possible whose circumference is integral multiple of de-Broglie wavelength associated with the electron, that is, $2\pi r = n\lambda$

velocity of electron in the nth orbit

$$v_n = \frac{2\pi Ze^2}{4\pi\varepsilon_0 nh} = \frac{c}{137}\frac{z}{n} = 2.2 \times \frac{10^6 z}{n}.$$

$$\alpha = \frac{2\pi e^2}{4\pi\varepsilon_0 ch} \text{ is called fine structure constant } = \frac{1}{137}.$$

Angular frequency of electron

$$= \frac{8\pi^2 Z^2e^4m}{(4\pi\varepsilon_0)^2n^3h^3} = \frac{4.159\times10^6 Z^2}{n^3}\text{ rads}^{-1}.$$

Electric current due to electron motion in nth orbit

$$I_n = \frac{4\pi Z^2e^5m}{n^3h^3(4\pi\varepsilon_0)^2} = \frac{1.06Z^2}{n^3}\text{ mA}.$$

Magnetic induction produced in the nth orbit

$$B_n = \frac{\mu_0 I_n}{2r_n} = \frac{8\pi^4Z^3e^7m^2}{n^5h^5(4\pi\varepsilon_0)^3} = \frac{12.58Z^3}{n^5}\text{ Tesla}.$$

Magnetic moment produced in the nth orbit

$$M_n = \frac{e\hbar n}{2m} = \frac{ehn}{4\pi m} = 9.26 \times 10^{-24}n\text{Am}^2$$

$$= n \text{ Bohr Magneton}.$$

KE of electron $= \dfrac{e^2Z^2}{8\pi\varepsilon_0 r_n} = \dfrac{13.6Z^2}{n^2}eV$.

PE of electron $= -2\text{KE} = -\dfrac{e^2Z^2}{4\pi\varepsilon_0 r_n} = \dfrac{-27.2Z^2}{n^2}eV$

Binding energy of electron $= \text{KE} + \text{PE} = E_n$

$$= -\frac{e^2Z^2}{8\pi\varepsilon_0 r_n} = -\frac{13.6Z^2}{n^2}\ \text{eV.}$$

Ionisation Potential $= \dfrac{E_n}{e} = \dfrac{13.6Z^2}{n^2}\text{V}$

Rydberg constant $R = \dfrac{me^4}{8\varepsilon_0^2 ch^3} = 1.09737 \times 10^7 \text{m}^{-1}$

In Balmer series $\lambda(nm) = \dfrac{364.56\,n^2}{n^2-4}$ where, $n = 3, 4, 5, \ldots$

Representation of waves associated with orbital electrons in an atom

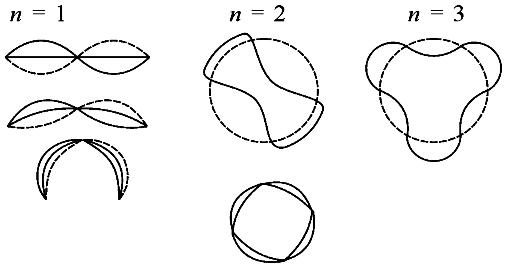


Fig. 32.2

The number of waves N in distance d is $N = d/\lambda$.

Hydrogen spectrum

$$\frac{1}{\lambda} = R\left[\frac{1}{n^2} - \frac{1}{m^2}\right]$$

wave number $\bar{v} = 1/\lambda$

Lyman series

$$\frac{1}{\lambda} = R\left[\frac{1}{1^2} - \frac{1}{n^2}\right] \quad n = 2, 3, \ldots$$

$\lambda_{\max} = 1216\text{A}°$; $\lambda_{\min} = 912\text{A}°$

This series lies in UV region

It shows both emission and absorption spectrum.

Balmer series

$$\frac{1}{\lambda} = R\left[\frac{1}{2^2} - \frac{1}{n^2}\right] \quad n = 3, 4, 5, \ldots$$

$\lambda_{\max} = 656.3\,\text{nm}$, $\lambda_{\min} = 364.6\,\text{nm}$

This series lies in visible region. It shows only emission spectrum.

Paschen series

$$\frac{1}{\lambda} = R\left[\frac{1}{3^2} - \frac{1}{n^2}\right] \quad n = 4, 5, 6\ldots$$

$\lambda_{\max} = 1875.1\,\text{nm}$ $\lambda_{\min} = 810.7\,\text{nm}$

It lies in IR region. This shows only emission spectrum.

Brackett series

$$\frac{1}{\lambda} = R\left[\frac{1}{4^2} - \frac{1}{n^2}\right]$$

$\lambda_{\max} = 4047.7\,\text{nm}$, $\lambda_{\min} = 1457.2\,\text{nm}$

It lies in IR region. It shows only emission spectrum.

Pfund series

$$\frac{1}{\lambda} = R\left[\frac{1}{5^2} - \frac{1}{n^2}\right]$$

$\lambda_{\max} = 7451.5\,\text{nm}$, $\lambda_{\min} = 2276.8\,\text{nm}$

It lies in deep IR region. It shows only emission spectrum.

The number of spectral lines emitted $N = \dfrac{n(n-1)}{2}$ if the electron is in the nth orbit.

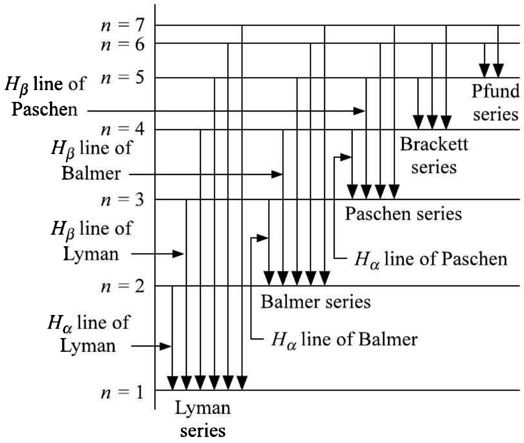


Fig. 32.3

Schrodinger equation

$$\frac{-h^2}{8\pi^2 m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right] - \frac{Ze^2\psi}{4\pi\varepsilon_0 r} = E\psi$$

Probability of finding an electron $P(r) = |\psi(r,t)|^2$

$P(r) = \dfrac{4}{r_B^3} r^2 e^{\frac{-2r}{r_B}}$, where, r_B = Bohr radius

Quantum numbers

Principal quantum number (n) (1, 2, ...) describes major shell

Orbital angular quantum number l (0 to $n - 1$) and represents number of subshells

$l = 0 \rightarrow \underset{\text{(sharp)}}{s}$, $l = 1 \rightarrow \underset{\text{(principal)}}{p}$

$l = 2 \rightarrow \underset{\text{(diffused)}}{d}$, $l = 3 \rightarrow \underset{\text{(fundamental)}}{f}$

Magnetic quantum number (m) Its value varies from $-l$ to $+l$. The total number of states $= 2l + 1$. It represents orientation of orbit of electrons.

For example, if $l = 1$ then $m = -1, 0, 1$

It gives clue that the p-orbitals have 3-orientations and so on.

Spin quantum number (s) It has two possible values.

Total angular momentum quantum number (J) J $= l \pm s$ (orbital + spin)

Total magnetic quantum number varies from –J to +J

Spin quantum number is $\pm 1/2\hbar$

Fermions The particles which obey Pauli's exclusion principle are called Fermions. Electrons, neutrons, protons are Fermions. They have spin equal to $(2n + 1)\hbar/2$. They also follow Fermi Dirac statistics. In other words, the spin half particles or spin multiple of half odd integral particles are called Fermions.

Bosons The particles which follow Bose-Einstein statistics are called Bosons. Bosons have integral multiple of $\hbar$ ($n\hbar$). Photons, Gravitons, Phonon, Exciton, cooper pair and so on are Bosons.

SHORT-CUTS AND POINTS TO NOTE

(i) Bohr model could not explain atoms or ions having more than one electron. Even 656.3 nm line was found to split in 5 lines when seen through high resolving power microscope. No explanation exists for such a splitting in Bohr model. Bohr arbitrarily assumed orbits to be stationary. No explanation was given as to why the moving electrons do not lose energy. It could not explain zeeman or stark effect.

(ii) Radius for *nth* orbit $r_n = 0.53n^2$ Å for hydrogen

$r_n = \frac{0.53n^2}{z}$ for other atoms.

(iii) Velocity in *nth* orbit $v_n = \frac{2.2\times10^6}{n}$ ms^{-1} for hydrogen

$V_n = \frac{2.2\times10^6 Z}{n}$ ms^{-1} for other atoms.

(iv) Angular frequency

$\omega_n = \frac{4.159\times10^6}{n^3}$ rads^{-1} for hydrogen

$\omega_n = \frac{4.159\times10^6 Z^2}{n^3}$ rads^{-1} for other atoms.

(v) Linear frequency $f_n = \frac{6.62\times10^5 Z^2}{n^3} s^{-1}$.

(vi) Period of revolution of electron in n^{th} orbit

$$T_n = \frac{1.5\times10^{-16} n^3}{z^2}.$$

(vii) Electric current due to an electron motion in the *nth* orbit $= \frac{1.06Z^2}{n^3}$ mA.

(viii) Magnetic induction $B_n = \frac{12.58Z^3}{n^5} T$ for electron revolving in n^{th} orbit.

(ix) Magnetic moment $M = \frac{e\hbar}{2m} = \frac{eh}{4\pi m}$

$= 9.26 \times 10^{-24}$Am^2.

It is for the first orbit of hydrogen and is called Bohr magneton $M_n = nM = n \times 9.26 \times 10^{-24}$Am^2 $= n$ Bohr magenton.

(x) PE $\text{PE}_n = -2\text{KE}$

$\text{PE} = \frac{-27.2\, Z^2}{n^2}$ eV

$\text{KE} = \frac{13.6\, z^2}{n^2}$ eV.

Binding Energy

$\text{BE} = \frac{-13.6\, Z^2}{n^2}$ eV.

(xi) Ionisation Potential $\frac{13.6\, Z^2}{n^2}$ V

Rydberg constant $R = 1.0973 \times 10^7$m^{-1};
1 Rydberg $= -13.6$eV in energy form $= Rhc$

(xii) Excitation Potential $13.6z^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$.

(xiii) Note that only Lyman series is observed in emission and absorption spectrum. All others are observed in only emission spectrum.

(xiv) *n*th excited state means $(n + 1)$th orbit.

(xv) Number of spectral lines emitted when the electron is in *n*th state $n(n - 1)/2$.

(xvi) The energy level difference goes on decreasing as n increases. The total number of elements for a given quantum number n is
Total number of elements $= 2[n^2 + (n - 1)^2 + \ldots 1^2]$
For example, if $n = 3$, then
total number of elements $= 2[3^2 + 2^2 + 1^2] = 28$.

(xvii) The distance of closest approach $r = \frac{2ze^2}{4\pi\varepsilon_0(KE)}$

The impact parameter $b = \frac{ze^2\cot(\theta/2)}{4\pi\varepsilon_0(KE)}$

(xviii) Hydrogen like elements are He$^+$, Li^{+2} and so on.
Bohr radius $r_b = 0.53$Å.

(xix) $\lambda(\text{nm}) = \frac{1242}{E(\text{eV})} = \frac{1242}{E_2 - E_1}$

for photons or spectral lines.

(xx) Radius of the orbit if the particle is other than an electron. For example, for a muon it is

$$r_n^* = \frac{r_n m_e}{m^*}$$

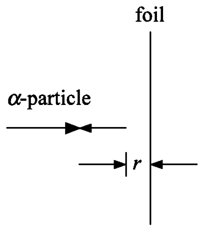
and energy $E_n^* = m^* E_n$ where, m^* is reduced mass

$$m^* = \frac{m_1 m_2}{m_1 + m_2}.$$

CAUTION

(i) Thinking that impact parameter and closest distance of approach are same.

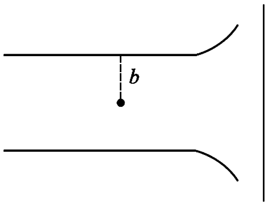
- Closest distance of approach is the minimum separation between α-particles and foil. It depends upon KE of the incident α-particle.



(a)

$$r = \frac{2ze^2}{4\pi\varepsilon_0(KE)}$$

Impact parameter is the radius of the incident cylindrical beam of α-particles as illustrated in Fig. below α-particles scattered at different angles lie on different radius of the cylindrical beam.



(b)

Fig. 32.4

(ii) Considering that Bohr's Model can be applied to any atom.

- It can be applied only to hydrogen like atoms/ions, even it does not satisfy hydrogen spectrum completely.

  Models based on Quantum mechanics are used which include quantum numbers n, l, m and s and use Pauli's exclusion principle.

(iii) Confusion betweeen binding energy, ionisation energy and excitation energy

- Binding energy is the sum of KE + PE of the orbital electron. Ionisation energy is the minimum energy required to remove an electron from its ground state to vacuum level (or ∞)

  Assuming electron is present in nth state.

  $E_{\text{ionization}} = E_{\infty} - E_n$

  Excitation Energy is the amount of energy required to transit electron from lower level n_1 to higher level n_2

  $E_{\text{excitation}} = E_{n2} - E_{n1}$

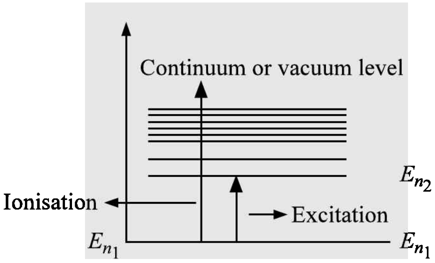


Fig. 32.5

(iv) Not recognising H_α, H_β lines of a series

- H_α line is $E_3 \to E_2$ and H_β line is $E_4 \to E_2$ in Balmrer series

that is, $E_{n+1} \to E_n$ is H_α

$E_{n+2} \to E_n$ is H_β line of the series

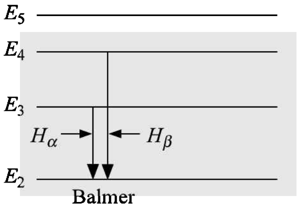


Fig. 32.6

(v) Not remembering which series has its absorption spectrum (in hydrogen) and presuming that every series has absorption spectrum also.

- Only Lyman series has both emission and absorption spectrum. All others show only emission spectrum.

(vi) Confusion in the meaning of excited state

- nth excited state means $(n + 1)$th state

(vii) Confusing that Fermions are the only electrons

- All particle having spin odd half multiple of $\hbar$ are Fermions. Thus protons, neutrons and electrons are Fermions. They follow Pauli's exclusion principle.

(viii) Considering that ionisation potential/ionisation energy; excitation potential/excitation energy are equivalent terms.

- The minimum energy to ionise the atom is called ionisation energy. The potential difference through which an electron be accelerated to acquire this much energy is called ionisation potential. Ionization energy of hydrogen atom in ground state is 13.6 eV and ionisation potential is 13.6 V. That is, energy needed to take atom from ground state to excited state is called excitation energy. The potential through which an electron should be accelerated to acquire this much energy is called the excitation potential.

(ix) Considering that spectral lines are just single lines.

- They consist of fine lines. For example, 656.3 nm line seen in hydrogen spectrum, when seen under high resolution microscope was found to split to 5 lines. Even zeeman effect showed that each spectral line splits into two, when magnetic field is applied.

(x) Not remembering the relation between eV and J.

- 1 eV = 1.6×10^{-19} J

Solved Problems

1. The total energy of an electron in the first excited state of hydrogen is about −3.4 eV. Its KE in this state is

(a) 3.4 eV
(b) 6.8 eV
(c) −3.4 eV
(d) −6.8 eV

Solution (a) KE = −B.E.

2. Energy levels A, B and C of a certain atom correspond to increasing values of energy, i.e., $E_A < E_B < E_C$. If λ_1, λ_2 and λ_3 are wavelengths corresponding to transitions C to B, B to A and C to A respectively then

(a) $\lambda_3 = \lambda_1 + \lambda_2$ (b) $\lambda_3 = \dfrac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}$

(c) $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (d) $\lambda_1^2 + \lambda_2^2$

Solution (b) $E_C - E_A = E_C - E_B + E_B - E_A$ (From Fig.)

$$\frac{hc}{\lambda_3} = \frac{hc}{\lambda_1} + \frac{hc}{\lambda_2} \quad \text{or} \quad \frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

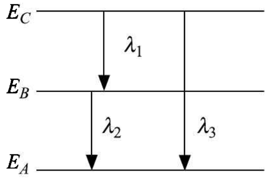


Fig. 32.7

3. The ground state energy of Hydrogen atom is –13.6 eV. What is the potential energy of the electron in this state.
   (a) 0 eV (b) –27.2 eV
   (c) 1 eV (d) 2 eV

Solution (b) $PE = 2BE$

4. Radius of first Bohr orbit is r. The radius of 2nd Bohr orbit is
   (a) $8r$ (b) $2r$
   (c) $4r$ (d) $2\sqrt{2}\,r$

Solution (c) $r_n = n^2 r$ $\therefore$ $r_2 = 4r$

5. The electron in a hydrogen atom makes a transition from n_1 to n_2 state. The time period of the electron in n_1 is 8 times that in n_2. The possible values of n_1 and n_2 are
   (a) $n_1 = 8, n_2 = 1$ (b) $n_1 = 4, n_2 = 2$
   (c) $n_1 = 2, n_2 = 4$ (d) $n_1 = 1, n_2 = 8$

Solution (b) $T_n = 1.5 \times 10^{-6} n^3/z^2$

$\therefore$ $n_1 = 4$ and $n_2 = 2$

6. Bohr's atom model assumes
   (a) the nucleus is of infinite mass and is at rest
   (b) electron in a quantized orbit will not radiate energy
   (c) mass of the electron remains constant
   (d) all of these

Solution (d)

7. Identify the wrong statement in the following: Coulomb's law correctly describes the electric force that
   (a) binds the electrons of an atom to its nucleus
   (b) binds protons and neutrons in the nucleus of an atom
   (c) binds atoms together to form molecules
   (d) binds atoms and molecules to form solids

Solution (b) Nuclear force binds protons and neutrons.

8. The energy that should be added to an electron to reduce its de Broglie wavelength from 1 mm to 0.5 mm is
   (a) four times the initial energy
   (b) equal to initial energy
   (c) twice the initial energy
   (d) thrice the initial energy

Solution (d) $A = \dfrac{h}{p} = \dfrac{h}{\sqrt{2m(KE)}}$

$\therefore$ New energy should be 4 times and hence energy to be added is thrice the initial energy.

9. A photon excites a hydrogen atom. The de-excitation emits a max. wavelength photon in lyman series. The wavelength of latter photon will be
   (a) 102 nm (b) 112 nm
   (c) 122 nm (d) none of these

Solution (c) $\lambda = \dfrac{1240}{E\,(\text{eV})}(\text{nm}) = \dfrac{1240}{10.2} = 122\,\text{nm}$

10. A positive ion having only one electron ejects it if a photon of $\lambda \le 228$ Å is absorbed by it. The ion is
    (a) He^+ (b) Li^{++}
    (c) Be^{+++} (d) B^{+4}

Solution (a) $E(eV) = \dfrac{1240}{22.8} = 54.38$

Using $E = \dfrac{13.6\,z^2}{1^2} \Rightarrow z = 2$ $\therefore$ ion is He^+.

11. Find the ratio of magnetic dipole moment to angular momentum in a hydrogen like atom
    (a) $\dfrac{e}{m}$ (b) $\dfrac{e}{2m}$ (c) $\dfrac{e}{3m}$ (d) $\dfrac{2e}{m}$ (e) $\dfrac{3e}{m}$

Solution (b) $\dfrac{M}{L} = \dfrac{iA}{mvr} = \dfrac{\dfrac{ev}{2\pi r} \times \pi r^2}{mvr} = \dfrac{e}{2m}$.

12. The average KE of molecules in a gas at temperature T is $\dfrac{3}{2}\text{kT}$. Find the temperature at which the average KE of molecules equals to binding energy of its atoms.
    (a) 1.05×10^5 K (b) 1.05×10^4 K
    (c) 1.05×10^3 K (d) none of these

Solution (a) $\dfrac{3}{2}\text{kT} = 13.6\,\text{eV}$ or $\dfrac{3}{2} \times \dfrac{1.38 \times 10^{-23}}{1.6 \times 10^{-19}} \times T = 13.6$

or $T = 1.05 \times 10^5$ K

13. Hydrogen atom in ground state takes up a photon of $\lambda = 50$ nm (UV light). Find the kE with which it is emitted.
    (a) 14.4 eV (b) 12.2 eV
    (c) 13.6 eV (d) 11.2 eV

Solution (d) $KE = 1240/\lambda - BE$

$$\Rightarrow \left(\frac{1240}{50} - 13.6\right) \text{eV} = 11.2\,\text{eV}$$

14. A filter transmits only the radiation of wavelength ≥440 nm. Radiation from a hydrogen discharge tube goes through such a filter and is incident on a metal of work function 2.0 eV. Find the stopping potential which can stop emission.
    (a) 1.4 V (b) 1.3 V
    (c) 0.85 V (d) 0.55 V

Solution (d) H_B line has wavelength

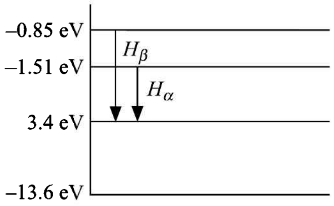


Fig. 32.8

$$\lambda = \left(\frac{1240}{50} - 13.6\right) = 486.1\,\text{nm}$$

Apply $eV_s = h\nu - \phi = 2.55 - 2.0 = 0.55\,\text{eV}$

$V_s = 0.55\,\text{V}$

15. Find the maximum work function a metal can have so that light from Balmer series can cause emission.

(a) 3.4 eV (b) 2.55 eV

(c) 1.89 eV (d) none of these

Solution (a) The maximum energy of photon emitted in Balmer series is 3.4 eV.

16. The electron is present in the -1.51 eV energy state. Find the angular momentum of the electron.

(a) $2\hbar$ (b) $\hbar$

(c) $3\hbar$ (d) $4\hbar$

(e) none of these

Solution (c) Angular momentum $L = n\hbar \Rightarrow 3\hbar$

$\because$ -1.51 eV corresponds to $n = 3$

17. Which of the following products in a hydrogen atom is independent of principal quantum number n?

(a) vr (b) vn

(c) Er^2 (d) En

Solution (b)

18. The radius of shortest orbit in one electron system is 18 pm. It may be

(a) ${}_1^1H$ (b) ${}_1^2H$

(c) He^+ (d) Li^{++}

Solution (d) $r_n = \frac{n^2 r_B}{z} \Rightarrow \frac{0.53\text{ Å}}{z} = 0.18\text{Å} \Rightarrow z = 3$

19. An atom initially at an energy level $E = -6.52$ eV, absorbs a photon of wavelength 860 nm. What is the internal energy of atom after absorbing photon?

(a) 5.08 eV (b) 1.44 eV

(c) -1.44 eV (d) -5.08 eV

Solution (d) $E(ev) = \frac{1240}{860} = 1.44$ eV

$E_{net} = -6.52 + 1.44 = -5.08$ eV

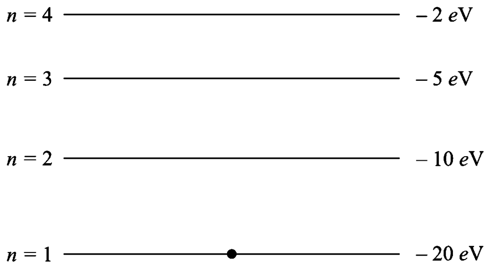


Fig. 32.9

20. What is minimum frequency of the photon required to carry out transition $n = 2$ to $n = 3m$ fig. 32.9?

(a) 1.21×10^{15} Hz

(b) 1.61×10^{15} Hz

(c) 1.21×10^{14} Hz

(d) 1.61×10^{14} Hz

Solution (a) $\lambda = \frac{1240}{5} = 248$ nm; $f = \frac{3\times10^8}{248\times10^{-9}}$

21. If A_n is the area enclosed in the nth orbit in a hydrogen atom then the graph log (A_n/A_1) against log$n$

(a) will have slope 2 (straight line)

(b) will have slope 4 (straight line)

(c) will be a monotonically increasing non linear curve

(d) will be a circle

Solution (b) $r_n = n^2 r_1$

$\therefore$ $\log\left(\frac{\pi r_n^2}{\pi r_1^2}\right)$ against. Log (n) has slope = 4

22. The hydrogen atom emits a photon of 656.3 nm line. Find the momentum of the photon associated with it.

(a) 10^{-27} kg ms^{-1} (b) 10^{-23} kg ms^{-1}

(c) 10^{-25} kg ms^{-1} (d) none of these

Solution (a) $p = \frac{h}{\lambda} = \frac{6.625\times10^{-34}}{656.3\times10^{-9}}$

$= 1.01 \times 10^{-27}$ kg ms^{-1}

23. A hydrogen atom emits UV radiation of wavelength 102.5 nm. Find the quantum numbers of the state involved.

(a) 3, 1 (b) 2, 1

(c) 4, 1 (d) 4, 2

Solution $E(eV) = \frac{1240}{102.5} = 12.1;$

$E_n = -13.6 + 12.1 = -1.5$ eV

That is, transition is from $n = 3$ to $n = 1$.

24. An electron with kE 6 eV is incident on a hydrogen atom in its ground state. The collision

(a) must be elastic

(b) may be partially elastic

(c) must be completely inelastic

(d) must be partially inelastic

Solution (a) Unless energy ≥ 10.2 eV electron from ground state of hydrogen cannot be excited.

Problems for Practice

1. The wavelength of radiation required to excite an electron from first to third Bohr orbit in a doubly ionised lithium atom will be

(a) 113.74 Å (b) 113.74 cm

(c) 113.74 mm (d) 113.74 m

2. An electron revolving in an orbit of radius 0.5 Å in a hydrogen atom executes 10^{16} revolutions per second. The magnetic moment of an electron due to its orbital motion will be

(a) 256×10^{-26} amp – m^2

(b) 1256×10^{-26} amp – m^2

(c) 653×10^{-26} amp – m^2

(d) zero

3. The energy necessary to remove the electron from $n = 10$ the state in hydrogen atom will be

(a) 0.136 eV (b) 0.0136 eV

(c) 1.36 eV (d) 13.6 eV

4. How many revolutions does an electron complete in one second in the first orbit of an hydrogen atom?

(a) 6.56×10^{15} (b) 6.56×10^{14}

(c) 6.56×10^{12} (d) 6.56

5. The time taken by a particle moving with velocity c/10 in crossing a nucleus will approximately be

(a) 10^{-21} s (b) 10^{-17} s

(c) 10^{-12} s (d) 10^{-8} s

6. According to Bohr's atom model, the diameter of first orbit of hydrogen atom will be

(a) 0.529 Å (b) 0.725 Å

(c) 2.25 Å (d) 1 Å

7. An hydrogen atom is excited from $n = 1$ to $n = 3$ state. The amount of energy absorbed by the atom will be

(a) 13.6 eV (b) 25 eV

(c) –13.6 eV (d) 12.1 eV

8. If E_m and J_m are the magnitude of total energy and angular momentum of an electron in the mth Bohr orbit respectively then

(a) $E_m \propto 1/J_m^2$ (b) $E_m \propto 1/J_m$

(c) $E_m \propto J_m^2$ (d) $E_m \propto J_m$

9. The quantum number corresponding to the orbit of diameter 0.0001 mm in hydrogen atom will be approximately

(a) 9 (b) 31

(c) 29 (d) 49

10. According to Bohr's hypothesis, which of the following quantities is discrete?

(a) angular velocity

(b) potential energy

(c) angular momentum

(d) momentum

11. According to classical theory, the path of an electron in Rutherford atom will be

(a) parabolic (b) circular

(c) straight line (d) spiral

12. The energy emitted by a source is in the form of

(a) electrons (b) neutrons

(c) photons (d) protons

13. The energy required to excite an electron from $n = 2$ to $n = 3$ energy state is 47.2 eV. The charge number of the nucleus, around which the electron is revolving, will be

(a) 15 (b) 20 (c) 5 (d) 10

14. The radius of first Bohr orbit in hydrogen atom is r_0, then the radius of first orbit in helium atom will be

(a) $2r_0$ (b) $r_0/2$ (c) $4r_0$ (d) r_0

15. In the Bohr model of hydrogen atom, the ratio of the kinetic energy and total energy of electron in the nth quantum state will be

(a) 1 (b) –1

(c) –12 (d) 2

16. The wavelength of first line of Balmer series is 6563 Å. The wavelength of first line of Lyman series will be

(a) 600 Å (b) 1215.4 Å

(c) 2500 Å (d) 7500 Å

17. In the elements with principal quantum number, $n > 3$ does not exist in nature, the number of possible elements will be

(a) 14 (b) 28 (c) 60 (d) 105

18. The value of principal quantum number for an ionised atom is

(a) ∞ (b) 4 (c) 1 (d) 0

19. The energy of an electron in the ground state of hydrogen atom is –13.6 eV. The energy required for the transition from $n = 2$ to $n = 3$ will be

(a) 1.89 eV (b) 2 eV

(c) 4 eV (d) 2.89 eV

20. In which of the following atoms, the wavelength of spectral lines, corresponding to transition from$n = 2$ to $n = 1$, will be minimum?

(a) hydrogen

(b) deuterium

(c) singly ionised lithium

(d) doubly ionised lithium

21. The value of Rydberg's constant is

(a) 1.1×10^7 per metre (b) 11×10^7 per cm

(c) 9.1×10^7 per metre (d) 1.1×10^8 per metre

22. Suppose the elements with principal quantum number $n > 4$ were not allowed in nature, what would have been the total number of elements in nature?

(a) 60 (b) 110 (c) 10 (d) 28

23. The kinetic and potential energies of the orbital electrons in the hydrogen atom without taking their sign in to account are E_k and E_p. Which of the following gives the value of total energy with proper sign?

(a) $(E_k - E_p)/2$ (b) $E_p - E_k$

(c) $(E_p - E_k)/2$ (d) $E_k - E_p$

24. The meaning of negative energy is that the system is

(a) bound (b) unstable

(c) in free state (d) none of these

25. The angular momentum of electrons in an atom produces

(a) zeeman effect (b) magnetic moment

(c) nuclear fission (d) light

26. For given value of n, the maximum number of electrons in an orbit can be

(a) $2n$ (b) n^2 (c) $2n^2$ (d) n

27. The orbital velocity of the electrons in the ground state of hydrogen atom is v. What will be its orbital velocity when excited to the energy state –1.51 eV?

(a) $v/2$ (b) $v/3$ (c) $v/4$ (d) $v/9$

28. The proof of quantisation of energy states in an atom is obtained by the experiment performed by

(a) Millikan (b) Rutherford

(c) Frank and Hertz (d) Thomson

29. If the series limit wavelength of the Lyman series for the hydrogen atom is 912 Å, then the series limit wavelength for the Balmer series of the hydrogen atom is

(a) (912/2) Å (b) 912 Å

(c) 912 × 2 Å (d) 912 × 4 Å

30. The following figure below indicates the energy levels of a certain atom. When the system moves from $2E$ level to E, a photon of wavelength λ is emitted. The wavelength of photon produced during its transition from level $4E/3$ to level E is

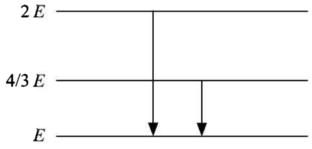


Fig. 32.10

(a) 3λ (b) 4λ/3 (c) λ/3 (d) 3λ/4

31. In an atom, two electrons move round the nucleus in circular orbits of radii R and $4R$. The ratio of the times taken by them to complete one revolution is
(a) 1/8 (b) 1/4 (c) 4 (d) 8

32. The radius of Bohr's first orbit is r_0. The radius of the electron in first orbit of singly ionised helium atom is
(a) $r_0/2$ (b) $2r_0$ (c) $4r_0$ (d) $r_0/4$

33. Ionising energy for hydrogen atom in the ground state is E. What is the ionisation energy of Li^{++} atom in the second excited state?
(a) $9E$ (b) $6E$ (c) $3E$ (d) E

34. The Band spectrum is obtained whenever the incandescent vapours of the excited substance are in their
(a) molecular state (b) nuclear state
(c) bulk state (d) atomic state

35. The energy difference between the states $n = 2$ and $n = 3$ is ε eV, in an hydrogen atom. The ionisation potential of the atom in volts is
(a) 13.2ε (b) 7.2ε (c) 3.2ε (d) 5.6ε

36. At some place in universe, an atom consists of a positron revolving round an antiproton. The ratio of the wavelength of corresponding spectral lines from this atom and ordinary hydrogen is
(a) > 1 (b) ∞ (c) < 1 (d) 1

37. The ratio of the radius of the orbit for an electron orbiting the hydrogen nucleus to that of an electron orbiting a deuterium nucleus is
(a) 2 : 1 (b) 1 : 3 (c) 1 : 1 (d) 1 : 2

38. In Rutherford's α-particle scattering experiment, the ratio of number of α-particles scattered through an angle of 60° and 120° is
(a) 9 : 1 (b) 1 : 2
(c) $\sqrt{3}$: 1 (d) 3 : 1

39. The wavelength of first line of Balmer series of an hydrogen atom is λ Å. The wavelength of the line of a doubly ionised lithium atom ($Z = 3$) is
(a) $\lambda/27$ (b) $\lambda/3$ (c) $\lambda/9$ (d) $\lambda/8$

40. An hydrogen atom emits green light when it transits from $n = 4$ energy level to the $n = 2$ level. Which colour of light would the atom emit when it transits from $n = 3$ level to the $n = 2$ level?
(a) green (b) violet (c) red (d) yellow

41. The observation of a faint line very near to blue line of the hydrogen spectrum led to the discovery of
(a) parity (b) deuterium
(c) ghost lines (d) positron

42. The energy of an hydrogen atom in its ground state is -13.6 eV. The energy of the level corresponding to $n = 5$ is
(a) -0.85 eV (b) -5.40 eV
(c) -2.72 eV (d) -0.54 eV

43. In hydrogen atom H_α line arise due to the transition of electron from third orbit to second orbit. In the spectrum of singly ionized helium there is a line having the same wavelength as that of H_α line, this is due to the transition of electron between the orbits
(a) 5 to 3 (b) 6 to 4
(c) 3 to 2 (d) 1 to 2

44. The radius of first permitted orbit of an electron having $z = 15$ is r_0. What is the radius of the fifth orbit of the electron of same element?
(a) $3r_0$ (b) $5r_0$ (c) $25r_0$ (d) $9r_0$

45. When the hydrogen atom gets to its lowest excited level, its radius is --- Bohr radius
(a) four times (b) half
(c) same (d) twice

46. The elliptical orbit of the electrons in an atom were proposed by
(a) Sommerfield (b) de Broglie
(c) J.J. Thomson (d) Bohr

47. If the potential energy of the electron in the first allowed orbit in hydrogen atom is PE. Then
(a) total energy is $PE/2$
(b) ionization potential is $-PE/2$ expressed in volts
(c) kinetic energy is $-PE/2$
(d) all of these are true

48. If the mass of mu-meson in 207 times the mass of an electron, then the ratio of the radius of a mu-meson in ground state orbit to that of corresponding electron is about
(a) $(207)^2$ (b) $(1/207)^2$
(c) (1/207) (d) 207

49. The difference in angular momentum associated with the electron in the two successive orbits of hydrogen atom is
(a) $h/2$ (b) $(n - 1)\,(h/2\pi)$
(c) $h/2\pi$ (d) h/π

50. Band spectrum is produced by
(a) He (b) Na (c) H_2 (d) H

51. In terms of the Ryberg's constant (R), what is the minimum value of the wavelength in the Lyman series?
(a) $1/8R$ (b) $1/R$
(c) $1/4R$ (d) $1/16R$

52. Suppose the mass of an electron decreases by 25%. How will it affect the Rydberg constant?
(a) become one fourth
(b) reduced to 75% of its original value
(c) it is doubled
(d) remains unchanged

53. The ratio of the angular momentum of the orbital electron in the first orbit to that in the second orbit is
(a) 0.5 (b) 2
(c) 1 (d) none of these

54. According to the classical theory, the Rutherford's atom is
(a) partially stable (b) unstable
(c) stable (d) positive

55. The spin of a neutron is
(a) zero (b) $\hbar$ (c) $\hbar/2$ (d) none of these

56. The value of n_2 corresponding to minimum and maximum wavelength in Balmer series respectively are
(a) 0 and 0 (b) ∞ and 3
(c) 0 and 3 (d) 3 and 0

57. For the given value of n and l, the possible number of electrons will be
(a) 2 (b) $2n(2l + 1)$
(c) $2\,(2l + 1)$ (d) $(2l + l)$

58. The relation between wavelength and wave number is
(a) $\vec{v} = 1/\lambda$ (b) $\vec{v} = 2\lambda$
(c) $\vec{v} = \lambda$ (d) none of these

59. The velocity of an electron in an atom according to Bohr's theory is
(a) $nh/2\pi mr$ (b) $2\pi mr$
(c) $1/2\pi mr$ (d) none of these

60. A photon of energy 16 eV ionises an hydrogen atom in the lowest energy state. The kinetic energy of an electron coming out of the atom will be
(a) 16 eV (b) 29.6 eV
(c) 2.4 eV (d) 13.6 eV

61. The ratio of the speed of an electron in the ground state of hydrogen atom to that of light is
(a) 2/237 (b) 1/237
(c) 1/137 (d) 1/2

62. When an electron jumps from n_1 orbit to n_2 orbit then the formula for energy radiated out is
(a) $h\nu = E_1/E_2$ (b) $h\nu = E_1 - E_2$
(c) $E_2 - E_1 = h\nu$ (d) $E_1 + E_2 = h\nu$

63. If the transition of an electron takes place from $n = 4$ state, then the maximum number of spectral lines obtained from transition to ground state will be
(a) 24 (b) 18
(c) 12 (d) 6

64. Which of the following subshells is represented by the quantum numbers $n = 4$ and $l = 1$?
(a) 4*d* (b) 4*p* (c) 4*s* (d) 4*f*

65. The energy required to remove an electron from $n = 2$ state in hydrogen will be
(a) 3.4 eV (b) 6.8 eV
(c) 13.6 eV (d) 27.2 eV

66. In basic metals the valence electron is
(a) *d*-electron (b) *p*-electron
(c) *s*-electron (d) *f*-electron

67. According to Bohr's theory the radius of electron in an orbit described by principal quantum number *n* and atomic number *Z* is proportional to
(a) Z^2n^2 (b) (n^2/Z)
(c) (Z^2/n) (d) Z^2/n^2

68. Which of the following pair is correct?
(a) Chadwick - neutron
(b) Roentgen - electron
(c) J-J Thomson - photon
(d) Rutherford - X-rays

69. The maximum wavelength of Lyman series is
(a) $1/Rc$ (b) $1/R^2$
(c) $4/(3R)$ (d) c/R

70. The ratio of ionisation potentials for second energy state of hydrogen and fifth energy state of lithium will be
(a) 25 : 36 (b) 16 : 25
(c) 19 : 16 (d) 36 : 25

71. A photon of energy 13.4 eV is incident on hydrogen atom in the ground state. The electron of the atom will go to the energy state
(a) 4 (b) 3 (d) 6 (d) 8

72. The ratio of excitation potentials of second energy states of helium and hydrogen will
(a) 1 : 2 (b) 4 : 1 (c) 2 : 1 (d) 9 : 4

73. The ratio of the area of orbit of first excited state of electron to the area of orbit of ground level, for hydrogen atom will be
(a) 16 : 1(b) 8 : 1 (c) 4 : 1 (d) 2 : 1

74. The maximum wavelength of Brackett series of hydrogen atom will be
(a) 3589 Å (b) 62160 Å
(c) 40400 Å (d) 14440 Å

75. The angular momentum of an electron in hydrogen atom is proportional to
(a) $1/r$ (b) $1/\sqrt{r}$ (c) $\sqrt{r}$ (d) r^2

76. The excitation potential of second orbit of helium will be
(a) 7.55 V (b) 10.21 V
(c) 15.33 V (d) 21.7 V

77. The minimum wavelength of Paschen series of hydrogen atom will be
(a) 970 Å (b) 1022 Å
(c) 8107 Å (d) 18700 Å

78. The frequency (f_n) of an electron in circular orbit is
(a) $\propto n^2$ (b) $\propto 1/n^2$
(c) $\propto 1/n^3$ (d) $\propto n$

79. The excitation energy of H from $n = 1$ to $n = 2$ is
(a) 10.2 eV (b) 3.4 eV
(c) 1.51 eV (d) 1.89 eV

80. The value of Rydberg's constant is
(a) 1.1×10^{-7} per metre
(b) 11×10^{-7} per metre
(c) 0.1 to 10^{-7} per metre
(d) 9.1×10^{-7} per metre

ANSWERS

1. (a)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (c)
11. (d)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (d)
21. (a)	**22.** (a)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (b)	**28.** (c)	**29.** (d)	**30.** (a)
31. (a)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (b)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (c)
41. (b)	**42.** (d)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (a)	**47.** (d)	**48.** (c)	**49.** (c)	**50.** (c)
51. (b)	**52.** (b)	**53.** (a)	**54.** (b)	**55.** (c)	**56.** (b)	**57.** (c)	**58.** (a)	**59.** (a)	**60.** (c)
61. (c)	**62.** (c)	**63.** (d)	**64.** (b)	**65.** (a)	**66.** (c)	**67.** (b)	**68.** (a)	**69.** (c)	**70.** (a
71. (d)	**72.** (b)	**73.** (a)	**74.** (c)	**75.** (c)	**76.** (a)	**77.** (c)	**78.** (c)	**79.** (c)	**80.** (a)

Nuclear Physics 33

BRIEF REVIEW OF THE CONCEPTS

Mass of proton, $m_p = 1.673 \times 10^{-27}$ kg $= 938.2723\,\text{MeV/c}^2$ $(= 938.27\,\text{MeV/c}^2)$
Mass of neutron, $m_n = 1.675 \times 10^{-27}$ kg $= 939.5656\,\text{MeV/c}^2$ $(= 939.57\,\text{MeV/c}^2)$
Mass of electron, $m_e = 9.1093 \times 10^{-31}$ kg $= 510.99\,\text{keV/c}^2$ $(= 0.511\,\text{MeV/c}^2)$
$\left(\frac{1}{12}\text{ of carbon}\right) = 1$ amu or $1u = 1.6605 \times 10^{-27}$ kg $= 931.478\,\text{MeV/c}^2$ $(= 931.5\,\text{MeV/c}^2)$
Number of neutrons $= N$; Number of protons $= Z$ (atomic number); Total mass number $A = N + Z$

Isotopes elements having same atomic number Z, but different A mass number, for example, ${}^{12}_{6}\text{C}$, ${}^{14}_{6}\text{C}$ and ${}^{1}_{1}\text{H}$, ${}^{2}_{1}\text{H}$, ${}^{3}_{1}\text{H}$ are isotopes of carbon and hydrogen.

Isotones elements having same number of neutrons (N) but different atomic number, for example, (Z) ${}^{4}_{2}\text{He}$, ${}^{3}_{1}\text{H}$.

Isobars elements having same mass number (A) but different atomic number (Z) are called isobars, for example, ${}^{14}_{6}\text{C}$, ${}^{14}_{7}\text{N}$ are isobars. Protons and neutrons together are called nucleons.

Stability criterion A survey of periodic table carefully reveals that those elements in which $N/Z = 1$ or 1.6 are stable. Amongst these, the elements having even N and even Z are the most stable and are termed as magic numbers.

The heaviest stable nuclide is ${}^{209}{}_{83}\text{Bi}$. Lead (${}^{208}{}_{82}\text{Pb}$) is the most stable heaviest element. All transuranic elements finally disintegrate into lead (Pb). The elements or nuclides which decay with time are termed as **radioactive nuclides.**

The elements/nuclide, whose binding energy per nucleon is less than 7.5 MeV, are (mostly) unstable.

Nuclear radius $R = R_0 A^{1/3}$ where $R_0 = 1.1 \times 10^{-15}\,\text{m}$ or 1.1 fm.

Note that the density within a nucleus is independent of mass number.

Nuclear spin All nucleons (protons or neutrons) are spin particles. They have spin odd half multiple of $\hbar\left(=\dfrac{h}{2\pi}\right)$. They follow Fermi Dirac statistics or Pauli's exclusion principle and are called Fermions.

Nuclear force is a short range force extending upto 10 fm. It is fifty-sixty times stronger than electromagnetic force. Nuclear force is independent of charge. Nuclear force between two protons is same as nuclear force between two neutrons or nuclear force between a proton and a neutron. It is not a central force. It cannot be solely determined by distance. It depends upon the spins of the nucleons as well.

Heisenberg in 1932 proposed exchange force theory. Yukawa extended this theory and calculated mass of π-mesons. According to this theory, proton does not remain proton forever and similarly neutron does not remain as neutron forever. They go on changing. For instance,

$${}^{1}_{0}n \rightleftarrows {}^{1}_{0}n + \pi^{0};\ {}^{1}_{1}p \rightleftarrows {}^{1}_{0}n + \pi^{+};$$

$${}^{1}_{0}n \rightleftarrows {}^{1}_{1}p + \pi^{-};\ {}^{1}_{1}p \rightleftarrows {}^{1}_{1}p + \pi^{\circ}$$

where π^0, π^+ and π^- are π-mesons having mass around $200\,m_e$. Later on π-mesons were confirmed in cosmic rays. The heavy nuclides require more neutrons so that coulomb repulsion between protons could be balanced by nuclear force.

Binding energy $B = (Zm_p + Nm_n - M)c^2$ where M is mass of the nucleus. The term in the bracket is called mass defect. Binding energy per nucleon

$$\frac{B}{A} = \left(\frac{Zm_p}{A} + \frac{Nm_n}{A} - \frac{M}{A}\right)c^2$$

Mass excess Let A be the mass number of a nucleus. Let *mu* (atomic mass units) be the mass of neutral atom and *au* is mass of the nuclide in amu then excess mass

$$= (mu - au)$$

$$= (m - a)\ \frac{931.5}{c^2}\ \text{MeV} \times c^2$$

$$= (m - a)\ 931.5\,\text{MeV}.$$

Packing fraction $P = (m - A)/A$

Fig. 33.1 shows binding energy per nucleon *vs* mass number. The nuclides showing binding energy per nucleon greater than 7.5 MeV/nucleon are stable.

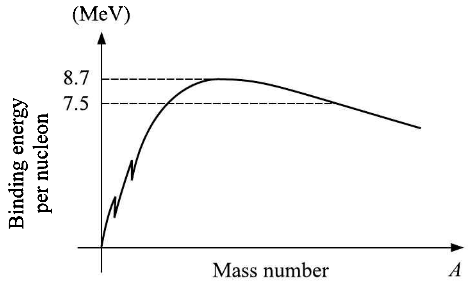


Fig. 33.1

Binding energy is the algebraic sum of volume energy, surface energy and coulomb energy

$$B = a_1A + a_2A^{2/3} - \frac{a_3Z(Z-1)}{R}$$

$$= a_1A + a_2A^{2/3} - \frac{a_3Z(Z-1)}{A^{1/3}}$$

or Binding energy per nucleon

$$\frac{B}{A} = a_1 + \frac{a_2}{A^{1/3}} - \frac{a_3(Z-1)Z}{A^{4/3}}$$

Radioactive decay Stable nuclides have definite atomic number and number of neutrons. Unstable nuclides decay by alpha emission or β-emission. When the residual nucleus gets de-excited γ-rays are also produced.

Q-value of the reaction

$$Q = u_i - u_f = (M_R - M_P)c^2 \text{ where,}$$

$M_R \rightarrow$ mass of reactants

$M_P \rightarrow$ mass of products

For the α-decay,

$$Q = \left[m\left({}^A_ZX\right) - m\left({}^{A-4}_{Z-2}Y\right) - m\left({}^4_2\text{He}\right)\right]c^2$$

A stream of α-particles coming out from a bulk is called α-rays.

Alpha decay

$$\underbrace{{}^A_ZX}_{\text{Parent nucleus}} \rightarrow \underbrace{{}^{A-4}_{Z-2}Y}_{\text{Daughter nucleus}} + {}^4_2\text{He}$$

In alpha decay, proton number decreases by 2 and mass number decreases by 4. The residual nucleus is thus different and is termed as daughter nucleus.

Conditions for α-decay Mass number $A > 210$ and $N/Z > 1.6$.

Three types of β-decay (a) β^-(or electron emission), β^+(positron emission) and electron capture.

β decay kept scientists puzzled for about 20years. We consider radioactivity as a collision process. Momentum could not be conserved as emitted β-particles have different energies as shown in Fig. 33.2. It was then suggested — consider β-emission as a two particle emission. The second particle was soon detected as a nutrino. Nutrino has rest mass zero. It has a spin quantum number 1/2.

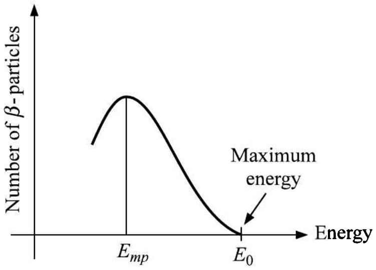


Fig. 33.2

To understand β-emission, we must have an idea of conservation rules.

Conservation rules

(a) momentum is conserved
(b) mass number is conserved
(c) charge number is conserved
(d) particle number is conserved
(e) parity is conserved

β^- *decay*

${}^A_ZX \rightarrow {}_{Z+1}^{\;\;A}Y + {}^{\;0}_{-1}\beta + \bar{\nu}$ antinutrino, (antiparticle of nutrino) is assumed to be emitted to conserve particle number

$$A = A + 1 - 1$$

The daughter nucleus has atomic number larger by 1 while the mass number remains the same. It is assumed that neutron is converted to a proton, that is,

$${}^1_0n \rightarrow {}^1_1p + {}^{\;0}_{-1}e + \bar{\nu}$$

Note that the electron emitted from nucleus are called Beta (β^-) particles.

Condition of β^- decay $N/Z > 1.6$ for heavy nuclide and $N/Z > 1$ for light nuclides.

Positron is an antiparticle of electron. Particle number conservation demands emission of nutrino (particle).

$${}^A_ZX \rightarrow {}_{Z-1}^{\;\;A}Y + \underbrace{{}^{\;0}_{+1}\beta}_{\text{Positron}} + \underbrace{\nu}_{\text{Nutrino}}$$

The daughter nucleus has proton number or atomic number 1 less than the parent while the mass number remains same. It is assumed that a proton is converted to neutron to achieve $N/Z = 1.6$ or 1.

$${}^1_1p \rightarrow {}^1_0n + \underbrace{{}^{\;0}_{+1}e}_{\text{Positron}} + \underbrace{\nu}_{\text{Nutrino}}$$

Condition for β^+ decay $N/Z < 1.6$ or 1.

Electron capture electrons from the K-shell may be captured by nucleus. In this process, X-ray is emitted rather than γ-ray and makes it a distinguished process. The process otherwise matches with the positrion emission.

$${}^A_ZX + {}^{\;0}_{-1}e \rightarrow {}_{Z-1}^{\;\;A}Y + \nu \quad (+\text{ X-ray})$$

Gamma (γ) emission The daughter nucleus after α-decay or β-decay comes to the excited state. It de-excites by γ-emission.

Artificial radioactive samples can only decay by γ emission. In naturally occuring radioactive substances, γ emission follows α or β emission. α, β and γ decay is collectively called radioactive decay.

Particles and antiparticles Particles live in positive energy sea and antiparticles live in negative energy sea. They are separated by $2m_0c^2$ (Fig. 33.3) when particle and its antiparticle unite energy is produced. For example,

$$e^- + e^+ \rightarrow \gamma\text{-ray } (E_\gamma = 1.02\,\text{MeV})$$

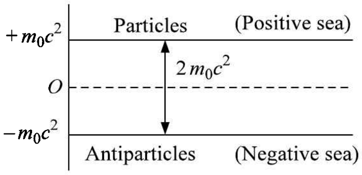


Fig. 33.3

Particle is assigned a number +1 and an antiparticle is assigned a number–1.

Law of radioactive decay $\frac{dN}{dt} = -\lambda N$

where λ is decay constant or disintegration constant or $$\int_{N_0}^{N}\frac{dN}{N}=\int_{0}^{t}-\lambda\,dt \quad \text{or} \quad N = N_0 e^{-\lambda t}$$

Fig. 33.4 shows activity time graph

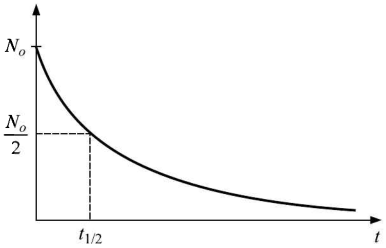


Fig. 33.4

The quantity $-\frac{dN}{dt}$ gives the number of decay per second and is called activity.

Thus $-\frac{dN}{dt} = \lambda N = A$, (activity)

or $A = A_0 e^{-\lambda t}$

The SI unit of activity is Bq. 1Bq = 1dps (disintegrations per second) The practical units of activity are curie and rutherford where.

1 Curie (Ci) = 3.7×10^{10}dps;

1 Rutherford (R) = 10^6dps.

Activity per unit mass is called specific activity.

Half life $t_{1/2} = 0.693/\lambda$

Average life $t_{av} = \frac{1}{\lambda} = \frac{t_{1/2}}{0.693} = 1.44 t_{1/2}$.

Properties of α-rays

(a) It is a stream of He nuclides
(b) Since they have two unit positive charge, thus they can be deflected by electric as well as magnetic fields.
(c) Their ionising power is very high (maximum amongst α, β and γ)
(d) There penetrating power is minimum. They can travel few cm in air.
(e) They produce scintillation on striking fluorescent material like barium platinocynide.
(f) They affect photographic plate.

Properties of β^-particles

(a) It is a stream of electrons.
(b) They are deflected by electric as well as magnetic fields.
(c) Their penetrating power is greater than α particles but less than that of γ-rays.
(d) Their ionising power is less than that of α particles but greater than that of γ-rays.
(e) They produce scintillation on striking a fluorescent material.
(f) They affect photographic plate.

Note: β^+ **rays** They possess same properties as β^- rays except that they are positively charged.

Properties of γ rays

(a) They are electromagnetic radiation (no charge, rest mass zero).
(b) They are not deflected by electric or magnetic fields.
(c) γ-ray travels with speed of light in vacuum.
(d) Their penetrating power is maximum amongst α, β and γ.
(e) Their ionising power is minimum amongst α, β and γ.
(f) They affect photographic plate.

Nuclear fission Occurs when a heavy nucleus splits (usually $A > 230$) into two lighter nuclei of nearly equal mass

$${}^{235}_{92}\text{U} + {}^{1}_{0}\text{n} \rightarrow {}^{236}_{92}\text{U}$$

$${}^{236}_{92}\text{U} \rightarrow {}^{137}_{53}\text{I} + {}^{97}_{39}\text{Y} + 2\text{n}$$

$${}^{97}_{39}\text{Y} \xrightarrow{\beta^-} {}^{97}_{40}\text{Zr} \xrightarrow{\beta^-} {}^{97}_{41}\text{Nb} \xrightarrow{\beta^-} {}^{97}_{42}\text{Mo}$$

$${}^{137}_{53}\text{I} \xrightarrow{\beta^-} {}^{137}_{54}\text{Xe} \xrightarrow[\text{delayed neutron}]{\text{n}} {}^{136}_{54}\text{Xe}$$

$${}^{236}_{92}\text{U} \rightarrow {}^{140}_{56}\text{Ba} + {}^{94}_{36}\text{Kr} + 2\text{n}$$

Large number of fission reactions are possible. Out of which the above two are most probable. Fig. 33.5 shows percentage yield *vs* mass number of fission products of ${}^{236}_{92}$U. Note that one reaction gives two neutrons per reaction and the other gives three neutrons per reaction. One neutron is delayed neutron. Thus, on an average 2.47 neutrons per reaction are emitted. Delayed neutrons play an important role in controlling fission rate.

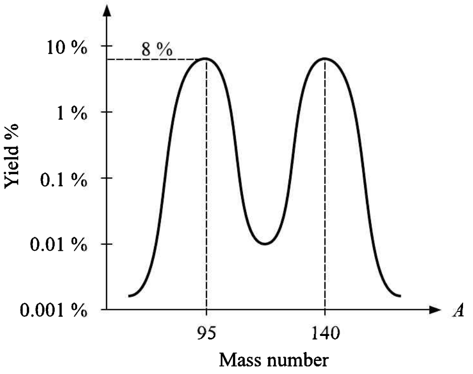


Fig. 33.5

About 200MeV/reaction energy is released in each fission. Neutrons take away about 5MeV energy in each reaction. As the fragments decay, an additional 15-20MeV energy is released.

Nuclear fission may be explained with liquid drop model as illustrated in Fig. 33.6.

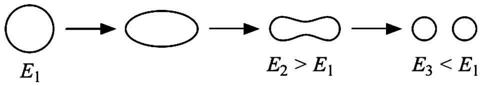


Fig. 33.6

Initially assume that the nuclide is in state E_1. It gains energy by itself, for a short time, according to Hiesenberg's uncertainty principle $\Delta E.\Delta t \approx \hbar$ and reaches a higher energy state E_2. The shape gets distorted like a dumb bell and finally breaks up into two nuclides at state E_3 releasing energy $(E_1 - E_3)$ as illustrated in Fig. 33.6 and Fig. 33.7.

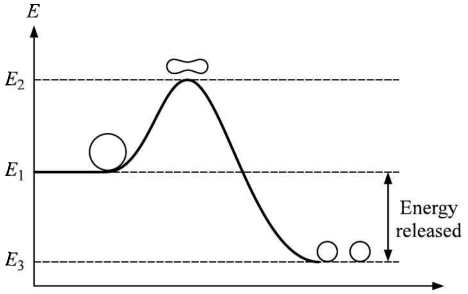


Fig. 33.7

Table 33.1 shows fission probabilities of various substances. Note ^{240}Pu is 1.5 times efficient than ^{236}U. This is why it is the most desirable fissionable material.

Table 1.2 Fission Probability

Nuclide	***Fission Probability Relative to*** $\mathbf{{}^{236}_{92}U}$
^{236}U	1 (assumed arbitrarily)
^{238}U	$< 10^{-3}$
^{240}Pu	1.5
^{244}Am	$< 2 \times 10^{-4}$

Critical mass is the minimum amount of fissionable material required to carry out fission reaction. It is 10 kg for ^{236}U.

Critical reaction One neutron per reaction used to carry out further chain fission reactions while the other neutrons are absorbed.

Moderator Slows down the neutrons.

Thermal neutrons/Slow neutrons Neutrons having energy of the order of room temperature ~ 0.02 eV are termed as slow or thermal neutrons. The normal nuclear reactors use ^{236}U or ^{235}U while breeder reactors use ^{238}U and produce nuclear fuel which is much efficient than that consumed.

$$ {}^{238}_{92}U + {}^{1}_{0}n \rightarrow {}^{239}_{92}U \xrightarrow{\beta^-} {}^{239}_{93}Np \xrightarrow{\beta^-} {}^{239}_{94}Pu $$
239

Pu is about 1.5 times more efficient than ^{235}U. Thus a breeder reactor converts a non-fissionable material to fissionable material which is rather efficient.

Nuclear fusion occurs when two light nuclie unite or fuse together to form a heavy nucleus.

To carry out nuclear fusion, the temperature should be of the order of 10^7K.

$$\left.\begin{array}{l} {}^{1}_{1}H + {}^{1}_{1}H \rightarrow {}^{2}_{1}H + e^{+} + \nu \\ {}^{2}_{1}H + {}^{1}_{1}H \rightarrow {}^{3}_{1}H + e^{+} + \nu \text{ or } {}^{3}_{2}He \\ {}^{3}_{1}H + {}^{1}_{1}H \rightarrow {}^{4}_{2}He \\ {}^{3}_{2}He + {}^{1}_{1}H \rightarrow {}^{4}_{2}He + e^{+} + \nu \end{array}\right] \begin{array}{l} \text{Overall energy} \\ \text{released is of the order of 27 MeV} \end{array}$$

OR $4\ {}^{1}_{1}H \rightarrow {}^{4}_{2}He + 2e^{+} + 2\nu$

Such a reaction is called thermal nuclear fusion reaction.

In stars, where the temperature is 10^8K, another cycle known as proton carbon cycle takes place

$$ {}^{4}_{2}He + {}^{4}_{2}He + {}^{4}_{2}He \rightarrow {}^{12}_{6}C $$
$$ {}^{1}_{1}H + {}^{12}_{6}C \rightarrow {}^{13}_{7}N + \gamma $$
$$ {}^{13}_{7}N \rightarrow {}^{13}_{6}C + {}^{0}_{-1}e + \bar{\nu} $$

The process continues until $A = 56$ (Iron). The elements heavier than iron can be produced by neutron absorption and subsequent β decay.

$$ {}^{1}_{1}H + {}^{13}_{6}C \rightarrow {}^{14}_{7}N + \gamma $$
$$ {}^{1}_{1}H + {}^{14}_{7}N \rightarrow {}^{15}_{8}O + \gamma $$
$$ {}^{15}_{8}O \rightarrow {}^{15}_{7}N + {}^{0}_{+1}e + \nu $$
$$ {}^{1}_{1}H + {}^{15}_{7}N \rightarrow {}^{12}_{6}C + {}^{4}_{2}He $$

Nuclear fusion in laboratory Lawson criterion $n\tau \geq 10^{14}$ s-cm^{-3} where n is the density of interacting particles and τ is time of confinement. The quantity $n\tau$ is called Lawson number. Lawson showed that in order to achieve energy output > energy input $n\tau \geq 10^{14}$ s-cm^{-3}.

Nuclear holocaust It is the name given to large scale destruction and devastation that would be caused by the use of nuclear weapon. The radioactive waste will hang like a cloud in the earth's atmosphere. It will absorb the solar radiations and there may be a long nuclear winter.

Radiation hazards human beings flora and founa are affected.
(a) pathologically (b) genetically

The safe limit of radiation dose is $50\,\mu$ severt (micro severt) per annum including X-rays and UV rays for human being.

SHORT-CUTS AND POINTS TO NOTE

(i) The particles inside the nucleus are called nucleons.

(ii) The radius of the nucleus is given by $R = R_0A^{1/3}$, where $R_0 = 1.1 \times 10^{-15}$ m. The nucleus density is independent of mass number A.

(iii) For nuclides to be stable, $N/Z = 1$ for light nuclides and $N/Z = 1.6$ for heavy nuclides.

(iv) Binding energy $B = [Zm_p + Nm_n - M]c^2$

Binding energy per nucleon

$$\frac{B}{A} = \left[\frac{Zm_p + Nm_n - M}{A}\right]c^2$$

The term in brackets is called mass defect.

(v) Packing fraction $P = (M - A)/A$ where M is atomic mass and A is mass number. The more negative value of P means the nuclide is more stable.

Fig. 33.8 shows variation of packing fraction with A.

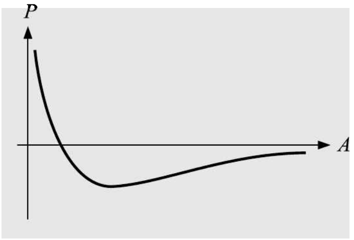


Fig. 33.8

(vi) The nuclear force keeps the neutrons and protons inside the nucleus. It is independent of charge. This is non-central and of short range. It depends upon distance and spin. It is of exchange nature.

According to Yukawa theory

$$ {}^{1}_{1}p \longleftrightarrow {}^{1}_{0}n + \pi^{+}, \quad {}^{1}_{1}p \longleftrightarrow {}^{1}_{1}p + \pi^{0} $$
$$ {}^{1}_{0}n \longleftrightarrow {}^{1}_{1}p + \pi^{-}, \quad {}^{1}_{1}n \longleftrightarrow {}^{1}_{1}n + \pi^{0} $$

At distances less than nuclear range these forces become repulsive.

Yukawa Potential $u(r) = u_0\,(r_0/r)\ e^{-r/r0}$

(vii) Nuclear fission is modelled on liquid drop model.

Nuclear fission of ${}^{235}_{92}U$ generates 200 MeV per reaction. 5 MeV energy is taken by neutrons. On an average 2.47 neutrons are emitted per reaction. To carry out controlled chain reaction, 1 neutron per reaction is used.

Critical mass of the fuel is 10 kg for uranium. Uncontrolled chain reaction is used in bombs or weapons.

In breeder reactor, more efficient fuel is generated than consumed. These reactors convert a non-fissionable material ${}^{238}_{92}U$ into a fissionable material ${}^{239}_{94}Pu$ or ${}^{240}_{94}Pu$ or even ^{236}U.

Thermal neutrons are used to carry out chain reactions.

Enriched uranium is converting 0.7% ${}^{238}_{92}\text{U}$to 2.3% of ${}^{238}_{92}\text{U}$ for reactor grade and 5% – 7% for weapon grade.

(viii) Neutron reproduction factor

$$= \frac{\text{rate of production of neutron}}{\text{rate of loss of neutrons}}$$

$$= \frac{\text{number of neutrons produced}}{\text{neutrons absorbed + neutron leakage}}$$

Heavy water is used as moderator.

(ix) A fusion reaction $4{}^1_1\text{H} \rightarrow {}^4_2\text{He} + 2e^+ + 2\nu$, is feasible if the temperature is 10^7K. Such reactions are also called thermonuclear reactions. If temperature is 10^8K carbon cycle is also feasible

(x) $\frac{-dN}{dt} = \lambda N$ or $N = N_0 e^{-\lambda t}$; $\lambda N = A = -\frac{dN}{dt}$ is activity.

(xi) $t_{1/2}$(half life) = $0.693/\lambda$; $t_{av} = 1/\lambda = 1.4\ t_{1/2}$.

(xii) Law of successive transformation

$$dN_2/dt = \lambda_1 N_1 - \lambda_2 N_2$$

(xiii) Number of nuclei left after n half lives $N = N_0/2^n$

Mass left after n half lives $m = m_0/2^n$.

(xiv) α decay is explained by Gamow's theory of tunnelling. β emission is explained using Nutrino hypothesis

$$t_{1/2} = 0.693/\lambda \qquad t_{av} = 1/\lambda = 1.4 t_{1/2}$$

(xv) Relation between range and energy of α-particle

$$R = 0.318E^{3/2}$$

Geiger Nuttal law $\log\lambda = A + B\log_e R$.

Range of α particles is 2.6 cm to 8.6 cm in air.

(xvi) α rays are detected using scintillation counter. β and γ rays are detected by *GM* (Geiger-Muller) counter.

(xvii) Radioactivity is a nuclear process as it is not associated with atomic electrons.

(xviii) Lighter particles are called Leptons. For example $\bar{e}, e^+, \mu^-, \mu^+, \nu, \bar{\nu}$.

(xix) Heavy particles are called Baryons. For example p^+, p^-, n, $\bar{n}$, hyperons (lambda, sigma, ksi, omega).

(xx) Bosons are integral spin particles for example. Photons, gravitons, magnons, pion, kaon, η-meson.

(xxi) If $E\gamma \geq 1.02$ MeV then $\gamma \rightarrow e^- + e^+$. This process is called pair production.

$e^- + e^+ \rightarrow \gamma$ (1.02 MeV). This is called pair annihilation.

⚠ CAUTION

(i) Difficult to remember which types of neutrons cause fission

- ■ Only thermal or slow neutrons having energy ~ 0.02 eV cause fission.

(ii) Confusion between fermions and bosons

- ■ Fermions are half spin particles. They follow Pauli's exclusion principle and Fermi Dirac statistics.

  Bosons are integral spin particles. They follow Bose-Einstein's statistics.

(iii) Confusion between half-life and average life.

- ■ $t_{1/2} = \frac{0.693}{\lambda}$ in which particles reduce to half the amount

  $t_{av} = \frac{1}{\lambda}$.

(iv) Confusion between curie and rutherford units

- ■ 1 curie (Ci)= 3.7×10^{10} dps

  1 rutherford (R) = 10^6 dps.

  1 Bq (Becqueral) = 1 dps. 1mci = 37R

(iv) Not able to remember the formula for (radioactive) particles left after n half-lives

- ■ $N = \frac{N_0}{2^n}$ are the number of radioactive nuclides left after n half-lives.

(vi) Non-clarity on concepts of radioactive equilibrium

- ■ When rate of decay of daughter element is equal to the rate of decay of parent element then radioactive equilibrium is achieved.

  that is, $\lambda_1 N_1 = \lambda_2 N_2$.

(vii) Assuming that all β particles emitted from a source have constant energy

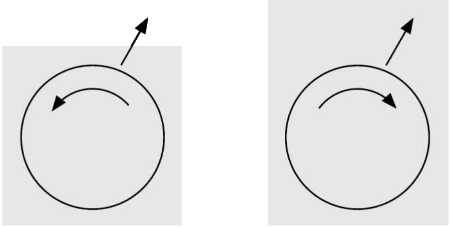
- ■ Their energies are different. It is explained by nutrino hypothesis or two particle emission.

(viii) Considering that nuclear force is a central force

- ■ Nuclear force is not a central force. It is a short range attractive force. It does not depend upon charge. It is an exchange force.

(ix) Considering that nutrinos have mass

- ■ Rest mass of nutrinos is zero. They travel with speed of light and they have spin $\frac{\hbar}{2}$. They follow Pauli's exclusion principle. The linear momentum vectors and spin vectors are mutually opposite for antinutrino as shown in Fig. 33.9.



Nutrino Antinutrino

Fig. 33.9

(x) Non-clarity on the process of absorption of γ rays

- ■ By photoelectric effect, Compton scattering and by pair production. The absorption coefficient $\mu \propto \lambda^3$ and $\mu \propto Z^4$

  $I = I_0 e^{-\mu x}$ is the law of absorption.

(xi) Considering that nuclear density depends upon mass number

- ■ Nuclear density does not depend upon mass number as $\frac{A}{\frac{4}{3}\pi R^3} = \frac{A}{\frac{4}{3}\pi R_0^3 A}$.

(xii) Considering that a neutron cannot be a fermion as it has no charge

- ■ Chargeless particles like neutrons and nutrino are also fermions. They are spin half particles.

(xiii) Considering that thermal neutrons destabilises the nucleus and then the fission occurs.

- The energy to change its state comes from within. According to the uncertainy, principle, that is, $\Delta E.\Delta t \approx \hbar$, energy can increase by itself.

(xiv) Thinking that electrons live inside the nucleus because during β^- emission electrons are emitted

- Electrons cannot live inside the nucleus, otherwise their speed will become more than the speed of light.

(xv) Considering that a radioactive element can release any particle electron, proton, neutron or alpha.

- Naturally occurring radioactive elements release only α, β and γ. Elements having atomic number $Z > 83$ are radioactive. Artificially prepared radioactive samples can decay by γ, proton and neutron.

Solved Problems

1. In the reaction ${}_1^2\text{H} + {}_1^3\text{H} \rightarrow {}_2^4\text{He} + {}_0^1\text{n}$, if the binding energies of ${}_1^2\text{H}, {}_1^3\text{H}$ and ${}_2^4\text{He}$ are respectively a, b and c (in MeV) then the energy (in MeV) released is

(a) $a + b + c$ (b) $a + b - c$
(c) $c - a - b$ (d) $c + a - b$

Solution (b) $Q = (MR - MP)c^2 = a + b - c$

2. Fission of nuclei is possible because the binding energy per nucleon in them

(a) increases with mass number at low mass number
(b) decreases with mass number of low mass number
(c) increases with mass number at high mass number
(d) decreases with number of high mass number

Solution (d)

3. A star converts He to oxygen. Find the energy released in each reaction of oxygen. mass of He = 4.0026 amu and mass of O = 15.9994 amu

(a) 7.26 MeV (b) 7 MeV
(c) 10.24 MeV (d) 5.12 MeV

Solution (c) $E = \Delta mc^2 = (4 \times 4.0026 - 15.9994)931.5$
$= 10.24\,\text{MeV}$

4. The intensity of gamma radiation from a given source is I. On passing through 36 mm of lead it is reduced to $I/8$. The thickness of lead which reduces the intensity to $I/2$ is

(a) 6 mm (b) 9 mm
(c) 18 mm (d) 12 mm

Solution (d) $\dfrac{I/2}{I/8} = \dfrac{e^{-\mu x}}{e^{-\mu 36}}$

or $\dfrac{8}{2} = \dfrac{e^{36\mu}}{e^{\mu x}}$ or $\dfrac{2^3}{2} = \dfrac{(e^{\mu x})^3}{e^{\mu x}}$

$(e^{\mu x})^3 = e^{36\mu}$

$\therefore \quad x = 12.$

5. Starting with a sample of pure ^{66}Cu, 7/8 of it decays into Zn in 15 minutes the corresponding half life is

(a) 10 min (b) 15 min
(c) 5 min (d) 7.5 min

Solution (c) Sample left = 1/8 = $1/2^n$
$x = 3$, that is, 3 half lives have passed $3t_{1/2} = 15$
or $t_{1/2} = 5\,\text{min}$.

6. The radius of is 3.6 fermi find the radius of ${}_{13}^{27}\text{Al}$ nucleus

(a) 6 Fermi (b) 8 Fermi
(c) 4 Fermi (d) 5 Fermi

Solution (a) $\dfrac{R_1}{R_2} = \dfrac{A_1^{1/3}}{A_2^{1/3}}$

$$R_2 = \left(\frac{125}{27}\right)^{1/3} \times 3.6 = 6 \text{ Fermi.}$$

7. (A) It is not possible to use ^{35}Cl as the fuel for fusion energy
(R) The binding energy of ^{35}Cl is too small

(a) Both A and R the correct and R is correct explanation of A
(b) A and R are correct but R is not correct explanation of A
(c) A is true but R is false
(d) both A and R are false

Solution (c)

8. ${}_{86}^{222}\text{A} \rightarrow {}_{84}^{210}\text{B}$. In this reaction how many α and β emissions have occurred

(a) 6α, 3β (b) 3α, 4β
(c) 4α, 3β (d) 3α, 6β

Solution (b) Since the mass number has decreased by 12. Therefore, 3α emissions have occurred. The charge number will decrease by 6 with 3α emission, 4β emission will make charge 84 units.

9. The phenomenon of radioactivity is

(a) exothermic change which increases or decreases with temperature
(b) increases on applied pressure
(c) nuclear process does not depend upon external factors
(d) none of these

Solution (c)

10. Mean life of a radioactive sample is 100s. Find its half life in minutes

(a) 0.693 (b) 1
(c) 10^{-4} (d) 1.155

Solution (d) $t_{av} = 1/\lambda = 100\,\text{s}$
$t_{1/2} = 0.693/\lambda = 69.3\,s$.

11. Consider two nuclei of same radioactive nuclide. One of the nuclei was created in a supernova explosion 5 billion years ago. The other was created in a nuclear reactor 5 minutes ago. The probability of decay during the next time is

(a) different for each nuclei
(b) nuclei created in explosion decays first
(c) nuclei created in the reactor decays first
(d) independent of time of creation

Solution (d) It depends only on the number of nuclei present at that time.

12. Protons are placed in a magnetic field in the z-direction (magnitude = 2.3 T). The energy difference between a state with *z* component of proton spin angular momentum parallel to the field and antiparallel to the field.

(a) 4.05×10^{7} eV (b) 4.05×10^{-7} eV
(c) 2.025×10^{7} eV (d) 2.025×10^{-7} eV

Solution (b) $U_1 = |s_z|B$

$= -2.7928 \times (2.3\text{T}) \times 3.152 \times 10^{-8}$ (eV/T)

$= -2.025 \times 10^{-7}$ eV

(when B and $|S_z|$ are parallel)

$U_2 = +2.025 \times 10^{-7}$ eV

(when B and $|S_z|$ are antiparallel)

$\therefore \quad \Delta U = U_2 - U_1 = 4.05 \times 10^{-7}$ eV

13. The hyperfine lines in the spectrum is related to

(a) Zeenan effect (b) Stark effect
(c) Lande's splitting (d) nuclear magnetic spin

Solution (a)

14. Find the binding energy of ${}^{62}_{28}\text{Ni}$ given $m_H = 1.008$ u, $M_n = 1.0087$ u

(a) 545.3 MeV (b) 595.3 MeV
(c) 645.3 MeV (d) 695.3 MeV ${}^{61}_{28}\text{M} = 61.9283$ u

Solution (a) $E_B = (28m_H + 34m_n - 61.9283)931.5$
$= 545.3$ MeV

15. ${}^{57}_{27}\text{Co}$ will emit ______ radiation.

(a) β^- (b) β^+
(c) α (d) electron capture

Solution (d) ${}^{57}_{27}\text{Co}$ decays to ${}^{57}_{26}\text{Fe}$. The mass of ${}^{57}_{26}\text{Fe}$ is 0.000897 u less than ${}^{57}_{27}\text{Co}$ makes it suitable for electron capture.

16. ^{57}Co decays by electron capture. Its half life is 272 days. Find the activity left after a year if present activity is 2 μCi

(a) 0.788 μCi (b) 0.431 μCi
(c) 0.39 μCi (d) none of these

Solution (a) $\lambda = \dfrac{0.693}{t_{1/2}} = 2.95 \times 10^{-8}\,\text{s}^{-1}$

$$N_0 = \frac{-dN/dt}{\lambda} = \frac{7.4 \times 10^4}{2.95 \times 10^{-8}}$$

$= 2.51 \times 10^{12}$ nuclei

$N(t) = N_0 e^{-\lambda t}$

$= 2.51 \times 10^{12}\, e^{-2.95 \times 10^{-8} \times 3.516 \times 10^{7}}$

$= 0.394(2.51 \times 10^{12})$

Activity $= \lambda N(t)$

$= 0.394(2.51 \times 10^{12}) \times 2.95 \times 10^{-8} = 0.788$ μCi

Alternate method

$$\frac{dN(t)}{dt} = \frac{dN(0)e^{-\lambda t}}{dt}$$

$= (2\,\mu\text{Ci})(e^{-2.95 \times 10^{-8} \times 3.516 \times 10^{7}})$

$= 2(0.394) = 0.788$ μCi

17. During a diagnostic x-ray examination a 1.2 kg portion of the broken leg receives an equivalent dose of 0.4 mSv. Find the absorbed dose in mGy and number of x-ray photons received if energy of *x*-ray is 50 keV.

(a) 0.4 mGy, 3×10^{15} (b) 0.32 mGy, 6×10^{10}
(c) 0.4 mGy, 6×10^{10} (d) 0.32 mGy, 3×10^{15}

Solution (c) for x-ray *RBE* = 1

$\therefore$ absorbed dose $= \dfrac{0.4\text{ mSv}}{1\text{ Sv/Gy}} = 0.4$ mGy

Total energy absorbed $= 0.4 \times 10^{-3} \times 1.2$
$= 4.8 \times 10^{-4}\,\text{J} = 3 \times 10^{15}$ eV

Number of photons of x-ray absorbed $= \dfrac{3 \times 10^{15}}{50 \times 10^{3}}$

$= 6 \times 10^{10}$

18. When ${}^{7}_{3}\text{Li}$ ($M_{Li} = 7.016004$ u) is bombarded by a proton 2α-particles result. ($M_{He} = 4.002603$ u). Find the reaction energy.

(a) 13.35 MeV (b) 14.85 MeV
(c) 16.05 MeV (d) 17.35 MeV

Solution (d)

$Q = [7.016004 + 1.007825 - 2(4.002603)] \times 931.5$

$= 0.018623 \times 931.5 = 17.35$ MeV

19. What mass of ${}^{235}_{92}\text{U}$ has to undergo fission each day to provide 3000 MW of power each day.

(a) 3.2 g (b) 320 g
(c) 3.2 kg (d) 32 kg

Solution (c) 1 fission or 235 u gives 200 MeV

Mass of uranium $= \dfrac{m \times 200 \times 1.6 \times 10^{-13}}{2.35 \times 1.66 \times 10^{-27}}$

$= 10^{6} \times 3000 \times 86400$ or $m = 3.2$ kg

20. A bone fragment found in a cave contains 0.21 times as much ${}^{14}_{6}\text{C}$ as an equal amount of carbon in air when the organism containing bone died. Find the approximate age of fragment. $t_{1/2}$ of ^{14}C = 5730 years

(a) 1.15×10^{4} *y* (b) 1.3×10^{4} y
(c) 1.24×10^{4} *y* (d) 1.4×10^{4} y

Solution (b) $\lambda = \dfrac{0.693}{t_{1/2}} = \dfrac{0.693}{5730} = 1.209 \times 10^{-4}\,y^{-1}$

$$\frac{N}{N_0} = e^{-\lambda t} = 0.21 \quad \text{or} \quad t = \frac{2.303\log(1/0.21)}{\lambda}$$

$$t = \frac{2.303(0.6794)}{1.209 \times 10^{-4}} = \frac{1.564 \times 10^{4}}{1.2} = 1.3 \times 10^{4}\,y$$

21. The net nuclear force between the particles is F_1 if both are neutrons, F_2 if both are protons, F_3 if one neutron and one proton then

(a) $F_1 = F_2 > F_3$ (b) $F_1 = F_2 = F_3$
(c) $F_1 < F_2 < F_3$ (d) $F_1 > F_2 > F_3$

Solution (b) Nuclear force is independent of charge.

22. In which of the following decays atomic number increases

(a) α (b) β^+ (c) β^- (d) γ

Solution (c) ${}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + {}^{0}_{-1}e + \bar{v}$

23. As the mass number *A* varies which of the quantity related to nucleus does not change

(a) mass (b) volume
(c) binding energy (d) density

Solution (d)

Problems for Practice

1. When N^{14} is bombarded by α particle, then a proton is emitted. The new element formed is
(a) $_8O^{17}$ (b) $_9Ne^{18}$ (c) $_6C^{12}$ (d) $_7N^{14}$

2. The average binding energy per nucleon is approximately
(a) 89 eV (b) 18.9 MeV
(c) 81 KeV (d) 8 MeV

3. If the mass defect in a fusion process is 0.3%, then the energy released in the fusion of 1 kg of material will be
(a) 2.7×10^4 eV (b) 2.7×10^{14} J
(c) 2.7×10^{-14} J (d) 2.7×10^{14} erg

4. The mass density of nucleus varies with its mass number as
(a) A (b) A^2
(c) $1/A$ (d) remains constant

5. The maximum energy obtained during atomic explosion is in the
(a) X-ray region (b) infrared region
(c) ultraviolet region (d) visible region

6. The temperature necessary for fusion reaction is
(a) 3×10^6 K (b) 3×10^4 K
(c) 3×10^3 K (d) 3×10^2 K

7. The fission of U^{238} is possible by
(a) fast protons
(b) only slow neutrons
(c) only fast neutrons
(d) fast as well as slow neutrons

8. The correct relation between the packing fraction P and mass number A is

(a) $P = \dfrac{A}{M - A}$ (b) $P = \dfrac{M - A}{A}$

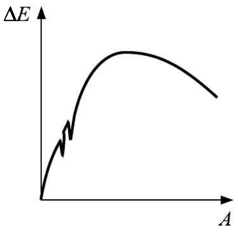
(c) $P = \dfrac{A}{M + A}$ (d) $P = \dfrac{M + A}{A}$

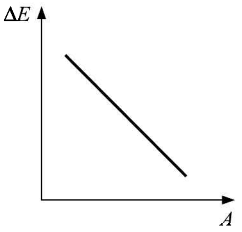
9. In the fusion process there are
(a) isotopes (b) isotopes of hydrogen
(c) isotopes of carbon (d) isotopes of helium

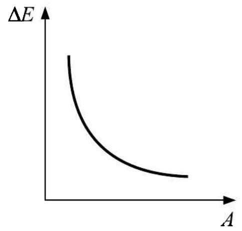
10. If 200 MeV energy is obtained per fission of $_{92}U^{235}$, then the number of fission per second to produce 1 kW power will be
(a) 3.125×10^{13} (b) 0.125×10^{13}
(c) 1.25×10^{18} (d) 3.2×10^{-8}

11. If all the atoms of 1 kg deuterium undergo nuclear fusion then the amount of energy released will be
(a) 8×10^{23} MeV (b) 6×10^{27} calorie
(c) 56.9×10^{13} J (d) 2×10^7 kwh

12. The curve between binding energy per nucleon (E) and mass number A is

(a)


(b)


(c)


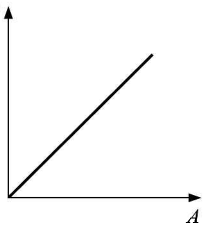
(d)


Fig. 33.10

13. Atomic reactor is based on
(a) nuclear fission
(b) controlled chain reaction
(c) nuclear fusion
(d) uncontrolled chain reaction

14. The energy of thermal neutrons is nearly
(a) 0.025 Joule (b) 0.25 eV
(c) 0.025 eV (d) 200 MeV

15. For making atom bomb, what else is needed except U^{235}?
(a) meson (b) neutron
(c) proton (d) electron

16. The energy emitted per second by the sun is approximately
(a) 3.8×10^{-26} J (b) 3.8×10^{26} J
(c) 3.8×10^{14} J (d) 3.8×10^{12} J

17. The mass defect for helium nucleus is 0.0304 amu. The binding energy per nucleon of helium nucleus is
(a) 28.3 MeV
(b) 200 × 10 MeV
(c) 9.31 MeV
(d) 7.075 MeV

18. The first atomic reactor was made by
(a) Bohr (b) Fermi
(c) Rutherford (d) Taylor

19. For maintaining sustained chain reaction, the following is required
(a) neutrons (b) electrons
(c) positrons (d) protons

20. When beryllium is bombarded by α particles, then ——— is obtained.
(a) neutron (b) electron
(c) proton (d) positron

21. The fissionable material used in the bomb dropped over the city of Nagasaki in Japan in 1945 was
(a) Np (b) U (c) Pu (d) Th

22. The sun maintains its shining because of
(a) burning of carbon
(b) the fission of helium
(c) fusion of hydrogen nuclei
(d) chemical reaction

23. The ratio of the volume of atom to the volume of nucleus is
(a) 10^{15} (b) 10^{10} (c) 10^5 (d) 10

24. In a nuclear reactor, which of the following quantities is conserved?
(a) momentum (b) energy
(c) mass (d) all of these

25. The mass of U^{235} consumed per hour by a nuclear reactor of capacity 100 kW will be ———, if the energy obtained per fission is 200 MeV.
(a) 0.45×10^{-5} kg (b) 4.5×10^{-5} kg
(c) 4.5×10^{-5} kg (d) 45×10^{5} kg

26. The nucleus with maximum binding energy per nucleon is
(a) ${}_{26}Fe^{56}$ (b) ${}_{2}He^{4}$
(c) ${}_{92}U^{238}$ (d) ${}_{8}O^{16}$

27. When two nuclei of masses x and y respectively fuse to form a nucleus of mass m and some energy is released, then
(a) $X + Y < m$ (b) $X + Y > m$
(c) $X - Y = m$ (d) $X + Y = m$

28. The energy of neutrons obtained during fission is approximately
(a) 1 MeV (b) 2 KeV
(c) 4 GeV (d) zero

29. Out of both atom bomb and hydrogen bomb, which one is more harmful?
(a) sometimes atom bomb, some-times hydrogen bomb
(b) atom bomb
(c) hydrogen bomb
(d) cannot be said

30. The critical mass of the fissionable material is
(a) 1 kg (b) 10 kg (c) 20 kg (d) 75 kg

31. An electric field can deflect
(a) gamma rays (b) α particles
(c) X-rays (d) neutrons

32. Radium was isolated by
(a) Becquerel
(b) Pierre curie and Madam curie
(c) Irene curie and Joliot
(d) Rutherford

33. Plutonium decays with half-life of 24000 years. If plutonium is stored for 72000 years, the fraction of it that remains is
(a) 1/2 (b) 1/3
(c) 1/4 (d) 1/8

34. The half-life of radioactive radon is 3.8 days. The time at the end of which, $(1/20)^{th}$ of the radon sample will remain undecayed is nearly
(a) 1.76 days (b) 3.8 × 20 days
(c) 20 days (d) 16.4 days

35. If α, β and γ rays of same energy are arranged in the ascending order of their ranges in air. The order will be
(a) γ, β, α (b) α, β, γ
(c) γ, α, β (d) β, α, γ

36. The relation between half-life T of a radioactive sample and its mean life τ is
(a) $\tau = T$ (b) $\tau = 2.718T$
(c) $\tau = 0.693T$ (d) $T = 0.693\tau$

37. When Boron (${}^{10}_{5}B$) is bombarded by a neutron, alpha particles are emitted. The resulting nucleus has the mass number
(a) 15 (b) 11 (c) 7 (d) 6

38. Of the following particles the one which is unstable in free space is
(a) alpha particle (b) electron
(c) proton (d) neutron

39. After 2 hours (1/16)th of the initial amount of a certain radioactive isotope remains undecayed. The half-life of the isotope is
(a) 60 min (b) 15 min
(c) 45 min (d) 30 min

40. A positron is emitted by a radioactive nucleus of atomic number 90. The product nucleus will have atomic number
(a) 88 (b) 89
(c) 90 (d) 9

41. If parent decays to daughter nucleus with a rate r and daughter nucleus has average life τ and number of nuclie of daughter nucleus at any instant is N then for radioactive equilibrium to be achieved, we have
(a) $r = N\tau$ (b) $r = \tau N$
(c) $r\tau = N$ (d) $\tau = r$

42. Radioactive irradiations are harmful to the living organs. To avoid harm during storage, radioactive samples are stored in lead boxes. Lead is used because it is a
(a) good absorber (b) heavy
(c) bad conductor (d) strong

43. In radioactivity, the mass number and the atomic number of a radioactive nucleus does not change when it emits
(a) a gamma ray (b) an alpha particle
(c) a beta particle (d) all of the above

44. Micro-curie of radioactivity is equal to
(a) 3.7×10^{4} Bq (b) 1 μBq
(c) 3.7×10^{10} Bq (d) 3.7×10^{2} Bq

45. A radioactive substance has a half-life of four months. 3/4 of the substance will decay in
(a) 12 months (b) 8 months
(c) 4 months (d) 3 months

46. When a radioactive isotope ${}_{88}Ra^{228}$ decays in series by the emission of three α particles and a β particle, the isotope finally formed is
(a) ${}_{83}X^{216}$ (b) ${}_{84}X^{220}$
(c) ${}_{86}X^{222}$ (d) ${}_{88}X^{215}$

47. The activity of a radioactive element decreases to one-third of the original activity I_0 in a period of nine years. After a further lapse of nine years its activity will be
(a) $I_0/6$ (b) I_0
(c) $(2/3)I_0$ (d) $I_0/9$

48. Atoms of an element differing in mass, though possessing the same chemical properties are called
(a) isomers (b) isotones
(c) isotopes (d) isobars

49. In a nuclear reaction involving a deutron and ${}_{3}Li^{7}$, one finds ${}_{3}Li^{8}$. The other product must be
(a) gamma ray (b) neutron
(c) proton (d) nothing is formed

50. After a time equal to four half-lives, the amount of radioactive material remaining undecayed is
(a) 50% (b) 25% (c) 12.5% (d) 6.25%

51. The wavelength of gamma rays is of the order of
(a) 10^{-8} m (b) 10^{-12} m
(c) 10^{-7} m (d) 10^{-10} m

52. Neutron was discovered by
(a) Thomson (b) Rutherford
(c) Chadwick (d) Anderson

53. A freshly prepared radioactive source of half-life 2 hours emits radiation of intensity which is 64 times the permissible safe level. The minimum time after which it would be possible to work safely with this source is
(a) 128 hours (b) 6 hours
(c) 12 hours (d) 24 hours

54. Half-period of lead is
(a) 1590 days (b) infinite
(c) 1590 years (d) zero

55. A radioactive element $_ZX^A$ emits an α particle and changes to
(a) $_{Z+2}^{\;\;4}Y$ (b) $_{Z-2}^{\;\;A}Y$
(c) $^{A-4}{}_{Z}Y$ (d) $_{Z-2}^{A-4}Y$

56. The nucleus $_6C^{12}$ absorbs an energetic neutron and emits beta particle β. The resulting nucleus is
(a) $_6C^{13}$ (b) $_7N^{14}$
(c) $_5B^{13}$ (d) $_7N^{13}$

57. B^{210} has a half-life of 5 days. The time taken for 7/8th of the sample to decay is
(a) 20 days (b) 15 days
(c) 10 days (d) 3.4 days

58. Energy transferred to a person through gamma rays is measured in units of
(a) roentgen (b) curie
(c) rutherford (d) none of the above

59. Weak nuclear force is exhibited during
(a) α emission (b) β emission
(c) γ emission (d) fission

60. Which one of the following is not a mode of radioactive decay?
(a) fusion (b) gamma emission
(c) alpha decay (d) electron emission

61. A nuclear reaction along with the masses of the particles taking part in it is as follows

$$A \;+\; B \;\rightarrow\; C \;+\; D \;+\; Q\ \text{MeV}$$

1.002u 1.004u 1.001u 1.003u

The energy Q liberated in the reaction is
(a) 1.862 MeV (b) 1.234 MeV
(c) 0.931 MeV (d) 0.465 MeV

62. If F_N and F_e represent nuclear and electrostatic forces between proton and proton. Then which of the following is correct if separation between two protons is 1Å?
(a) $F_N > F_e$ (b) $F_N = F_e$
(c) $F_N < F_e$ (d) $F_N \geq F_e$

63. When the mass equal to 1 amu is converted completely into energy, the energy produced is
(a) 1.5×10^{-18} J (b) 1.5×10^{-10} J
(c) 1.5×10^{-14} J (d) 1.5×10^{-12} J

64. The energy released when 1 amu of mass is completely converted into energy is
(a) 10^6 MeV (b) 931 MeV
(c) 100 MeV (d) 1 MeV

65. How much energy will approximately be released if all the atoms of 1 kg of deuterium could undergo fusion?
(a) 2×10^7 kwh (b) 8×10^{23} MeV
(c) 9×10^{13} J (d) 6×10^{27} calories

66. A radioactive substance decays by α β emission and a capacitor of capacitance C decays through R such that the activity to charge ratio remains constant. The half-life of the substance is
(a) $0.693RC$ (b) RC
(c) $\dfrac{RC}{0.693}$ (d) none of these

67. In the nuclear process
$_6C^{11} \rightarrow {}_5B^{11} + e^+ + X$
X stands for
(a) neutrino (b) photon
(c) neutron (d) antineutrino

68. The mother and daughter elements, with the emission of alpha particles are
(a) isomers (b) isotopes
(c) isobars (d) isodiopheres

69. ^{208}Ra has half-life 120 days. Find the amount of ^{208}Ra if the activity level is one mci.
(a) 0.184 μg (b) 0.184 mg
(c) 0.134 μg (d) 0.314 mg

70. The particle used to transmute boron to nitrogen is
(a) protons (b) α particles
(c) neutrons (d) β particles

71. Enriched uranium means increasing to percentage of ^{235}U from 0.7% to
(a) 7% (b) 11.1%
(c) 6.2% (d) 2.3%

72. The average number of neutrons emitted per fission reaction of ^{235}U is
(a) 3.0 (b) 2.72 (c) 2.47 (d) 2.16

73. On what theory, fission is explained?
(a) Fermi theory (b) einstein's theory
(c) O'ham's theory (d) liquid drop model

74. In a controlled reaction, the number of neutrons used per reaction to carry out chain reaction is
(a) 1 (b) 2 (c) 1.5 (d) 1.2

75. The equation $4\ {}_1^1H^+ \rightarrow {}_2^4He + 2e^+ + 26$ MeV represents
(a) fission (b) β decay
(c) γ decay (d) fusion

76. According to Lawson criterion, the fusion can take place only if $n\tau \geq$ --- where n is density of fusing elements and τ is time of captivity.
(a) 10^{17}s-cm^{-3} (b) 10^{14}s-cm^{-3}
(c) 10^{20}s-cm^{-3} (d) 10^{9}s-cm^{-3}

77. In stable nuclei, the number of neutrons (N) is related to the atomic number Z in a neutral atom in general as
(a) $N < Z$ (b) $N > Z$
(c) $N \geq Z$ (d) $N = Z$

78. If binding energy per nucleon is less than --- MeV/nucleon then the nuclide may be unstable.
(a) 7.5 (b) 3.4 (c) 5.2 (d) 8.9

79. The heaviest stable nuclide occurring in nature is
(a) $_{83}^{209}Bi$ (b) $_{82}^{208}Pb$
(c) $^{238}{}_{92}U$ (d) none of these

80. The nucleon is the name used for
(a) only neutron
(b) all light nuclei
(c) proton and neutron both
(d) only hydrogen nuclei

ANSWERS

1. (a)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (d)	**18.** (d)	**19.** (a)	**20.** (a)
21. (c)	**22.** (c)	**23.** (a)	**24.** (d)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (b)
31. (b)	**32.** (b)	**33.** (d)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (c)	**38.** (d)	**39.** (d)	**40.** (b)
41. (c)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (b)	**46.** (a)	**47.** (d)	**48.** (c)	**49.** (c)	**50.** (d)
51. (b)	**52.** (c)	**53.** (c)	**54.** (b)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (a)	**59.** (b)	**60.** (a)
61. (a)	**62.** (c)	**63.** (b)	**64.** (b)	**65.** (c)	**66.** (a)	**67.** (a)	**68.** (d)	**69.** (a)	**70.** (b)
71. (d)	**72.** (c)	**73.** (d)	**74.** (a)	**75.** (d)	**76.** (b)	**77.** (c)	**78.** (a)	**79.** (a)	**80.** (c)

34

Semiconductors

BRIEF REVIEW OF THE CONCEPTS

Semiconductors have a unique property that their conductivity increases with rise in temperature. Fig. 34.1 illustrates how resistivity falls with rise in temperature. This phenomenon can be explained only on the basis of band theory.

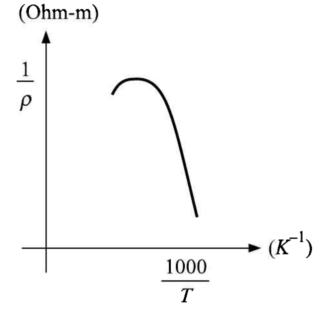


Fig. 34.1

The energy bands which are completely filled at 0K are called valence bands. The bands with higher energies are called conduction bands. We will refer to valence band as the top most filled band and conduction band as, lowest conduction band, that is, just above the valence band.

E_V is the top most energy of the valence band and E_C is the bottom most energy of the conduction band then $E_g = E_C - E_V$ represents forbidden energy gap.

In metals, $E_g \to 0$, that is, valence band and conduction band overlap so that a large number of electrons lie in the conduction band.

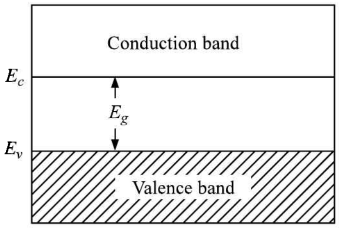


Fig. 34.2

In semiconductors, $E_g \sim 1\,\text{eV}$. At room temperature it is about 1 – 2 per cent filled. At 0K semiconductor is a perfect insulator. Fermi level is an imaginary level which lies between valence band and conduction band such that the probability of finding an electron is 50 per cent or 1/2. At 0K Fermi level is the highest filled level. Fermi level is used as reference level.

E_g(for Ge) = 0.71 eV and E_g(for Si) = 1.12 eV.

In insulators, $E_g \sim 6\,\text{eV}$, for example, Diamond has $E_g = 6.3\,\text{eV}$.

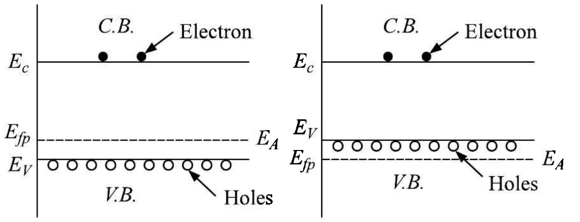
Semiconductors are of two types (a) intrinsic (b) extrinsic or doped

In intrinsic semiconductor no impurity from 3rd or 5th group of the periodic table has been added. So that the density of electrons in conduction band is equal to density of holes in valence band, that is, $n_i = h_i$ where n is electron density and h is hole density. (Subscript i stands for intrinsic.)

Extrinsic semiconductor is of two types p and n types. These are also called doped semiconductors. In p-type majority carriers are holes. Thus, $h_p > n_p$. Third group impurity (B, Al, Ga, In) is added to make p-type semiconductor.

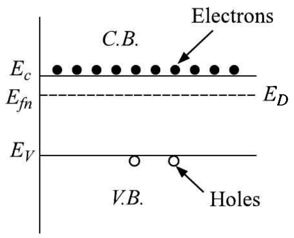
In n-type majority carriers are electrons. It is made by doping donor impurity, that is, impurity from the 5th group of periodic table like P, As, Bi, Sb. Thus, $n_n > h_n$.

In thermal equilibrium condition $n_e . h_e = n_i^2$. (subscript e denotes extrinsic.) Fig. 34.3(a) and Fig. 34.3(b) show energy-band diagram of p and n-type semiconductors. In heavily doped p or n-type acceptor or donor impurity level lie in VB and CB respectively.



***p*-type semiconductor** **heavily doped *p*-type**

(a)



(b)

Fig. 34.3 *n*-type semiconductor

Conductivity in semiconductors is due to holes and electrons both. Moreover, total conduction is due to diffusion and drift currents.

In an intrinsic semiconductor conductivity

$$\sigma = ne\mu_n + he\mu_h$$

where μ_n and μ_h are mobility of electrons and holes respectively.

For *n*-type semiconductor

Assuming $n \cong N_D \quad \sigma_n = ne\mu_n \simeq N_D\, e\mu_n$.

For *p*-type semiconductor

Assuming $h \cong N_A \quad \sigma_p = he\mu_n \simeq N_A e\mu_h$.

***pn* junction** When *p*-and *n*-type semiconductors of same material either both Si or both Ge are joined to form a homojunction, such a junction is called *pn* junction.

Depletion layer Close to the junction, a layer exists devoid of carriers due to the migration of carriers to the opposite sides. The electrons migrated from *n*-side are taken up by acceptor impurity atoms near the junction and become negatively ionised. Similarly, donor impurity atoms close to the *pn* junction on *n*-side get positively ionised and hence create a potential barrier or fictitious battery.

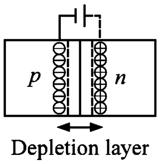


Fig. 34.4

Forward biasing If positive terminal of an external battery is connected to *p*-type and negative terminal to *n*-type. This reduces potential barrier and hence depletion layer width decreases. The current is therefore due to majority carriers. See Fig. 34.5 and 34.7.

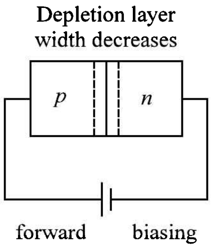


Fig. 34.5

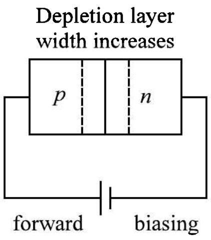


Fig. 34.6

Reverse biasing If positive terminal of the external battery is connected to *n*-type and negative terminal to *p*-type. The potential barrier increases. The depletion layer width increases and current is due to minority carriers. See Fig. 34.6 and 34.7

Note that when forward biased by a voltage greater than, barrier potential, a large current flows and when reverse biased a very small current flows. Hence *pn* junction acts very closely like a valve.

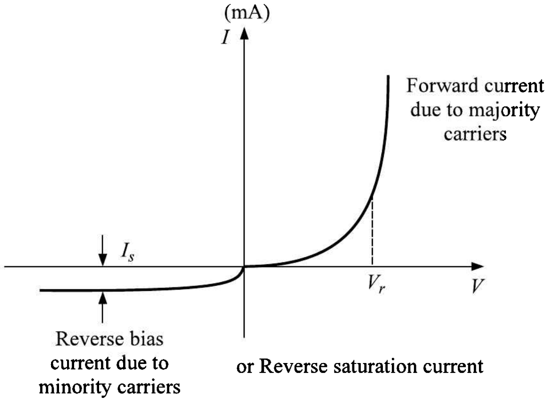


Fig. 34.7

Equation of current in *pn* junction

$$I = I_S\,[e^{V/VT} - 1]$$

where, I_S = reverse saturation current, V = applied potential, V_T = thermal voltage

$$V_T = \frac{kT}{e} = 0.026\text{V at } 300\text{K}.$$

where *k* is Boltzmann's const *T* is temperature of an electron and *e* is the charge.

Dynamic or incremental resistance

$$r = \frac{\Delta V}{\Delta I} = \frac{dV}{dI}$$

is called dynamic resistance as illustrated in Fig. 34.8 below. Its value is different at different points.

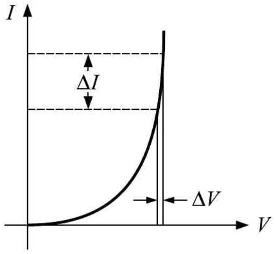
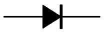
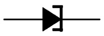
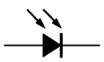
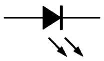


Fig. 34.8

Table 34.1

Type of Diode	*Circuit Symbol*	*Applications*
1. General purpose diode		Demodulator, voltage multiplier, clamping, rectifier, clipping, peak detector
2. Avalanche or Zener diode or breakdown diode		Load regulator
3. Tunnel diode		Oscillator, stable/monostable multivibrator
4. Varactor or Varicap		Frequency modulation, VFC (Voltage to frequency converter)
5. Photodiode		Burglar alarm, fire alarm, automatic switching of light, nuclear detector, optical fiber communication (detector).
6. Light emitting diode (LED)		Indicator, remote control, seven segment and fourteen-segment alphanumeric displays. Optical fiber communication

From the equation of the diode

$$\frac{dI}{dV} = \frac{I_S}{V_T} e^{V/V_T} = \frac{1}{V_T}$$

In case diode is forward biased

or $r = \frac{dV}{dI} = \frac{V_T}{I}$ if $I = 1\,\text{mA}$ then $r_f = 26\,\Omega$ (low).

In case diode is reverse biased $I \to 0 \quad \therefore \; r_r \to \infty$ In an actual diode $r_r \geq 10^4\,\Omega$

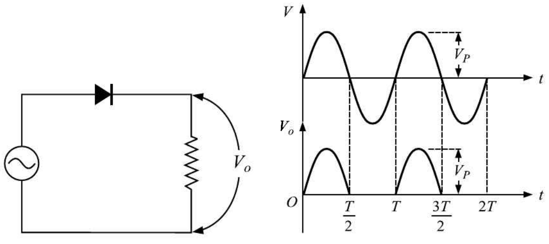
Ideal diode Ideal diode is like a voltage controlled switch. When forward biased it acts like an ON switch (zero resistance) and when reverse biased it acts like an OFF switch (infinite resistance)

Rectifier It is a circuit which converts AC to unidirectional pulsating output. In other words, it converts AC to DC.

Rectifiers are of two types:

(a) Half wave rectifier and Fullwave rectifier.

(b) *Half-wave rectifier* Fig. 34.9 (a) illustrates half wave rectifier



(a) (b)

Fig 34.9 Half-wave rectifier

$V_o = V_i = V_P \sin\omega t \quad (0 < t < T/2)$

$V_o = 0 \quad (T/2 < t < T)$

$$V_{\text{out}}(\text{DC}) = \frac{V_P}{\pi}; \quad V_{\text{out}}(\text{rms}) = \frac{V_P}{5}$$

$$\text{Ripple factor} = \frac{V_{\text{AC}}}{V_{\text{DC}}} = \sqrt{\left(\frac{V_{\text{rms}}}{V_{\text{DC}}}\right)^2 - 1} = 1.21$$

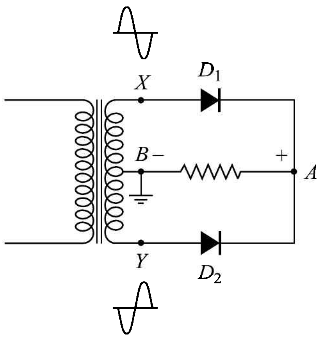
$$\text{Rectification efficiency } \eta = \frac{P_{\text{DC}}}{P_{\text{rms}}} \times 100 = 40.6\%.$$

Frequency of output signal = frequency of input signal.

Full-wave rectifier gives output in both the half cycles as shown in Fig. 34.10. The circuit using centre tapped transformer or bridge rectifier is developed.

$V_o = V_i = V_P \sin\omega t \quad 0 < t < T/2$

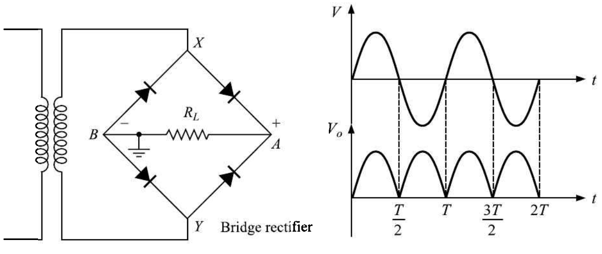
$= -V_i = -V_P \sin\omega t \quad 0 < t < T/2$



(a)

$$V_{\text{out}}(\text{DC}) = \frac{2V_P}{\pi} \quad V_{\text{out}}(\text{rms}) = \frac{V_P}{\sqrt{2}}$$

Ripple factor = 0.48



(b) (c)

Fig. 34.10 Full-wave rectifier

$$\text{Rectification efficiency } \eta = \frac{P_{\text{DC}}}{P_{\text{rms}}} \times 100 = 81.2\%.$$

Frequency of output signal = twice the frequency of input signal.

These characteristics speak themselves that we shall prefer full-wave rectifier.

Negative resistance

In region *AB* of Fig. 34.11. $I \propto 1/V$. This region is termed as negative resistance.

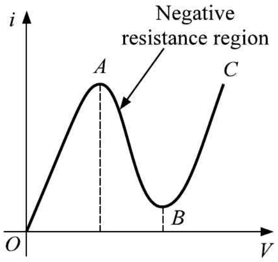


Fig. 34.11

The devices having negative resistance are

(a) tunnel diode (b) tetrode (vacuum tube)

(c) thyristors

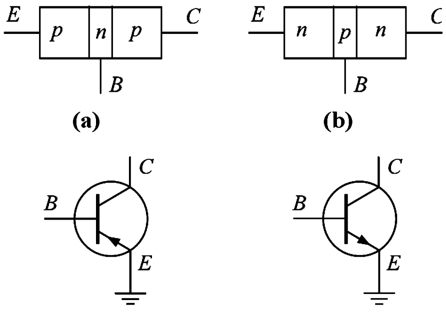
Drawback of diode (*pn* junction) It cannot be used as an amplifier.

Transistor Transistor is made from the words TRANSfer + resISTOR (TRANS from transfer and ISTOR from resistor).

Thus, transistor is a device which gives transfer of resistor. This device is designed to make amplifier.

Transistor is basically of three-types (a) UJT (b) BJT (c) FET. UJT (unjunction transistor): BJT (bipolar junction transistor). BJT is of two types *pnp* and *npn*. FET (field effect transistor) is of three types. JFET (junction field effect transistor), MOSFET (metal oxide semiconductor field effect transistor) and IGFET (insulated gate field effect transistor).

In a BJT, emitter is heavily doped and base region should be as thin as possible.



(a) (b)

(c) (d)

Fig. 34.12

$$I_E = I_B + I_C$$
$$I_C = \beta I_B + (\beta + 1)\, I_{CBO}$$
$$I_C = \alpha I_E + I_{CBO}$$

Here, I_{CBO} is collector base junction current when emitter is open. It is minority carrier current.

$$\beta = \frac{I_C}{I_B}\ (> 1); \qquad \alpha = \frac{I_C}{I_E}\ (< 1)$$

$$\beta = \frac{\alpha}{1-\alpha} \qquad \alpha = \frac{\beta}{1+\beta}.$$

Transistor can operate in three regions. In saturation region, transistor acts like an ON switch (dynamic resistance is 8 Ω). In cut off region, transistor behaves as an OFF switch (resistance ≥ 10^4 Ω). In active region transistor acts as an amplifier.

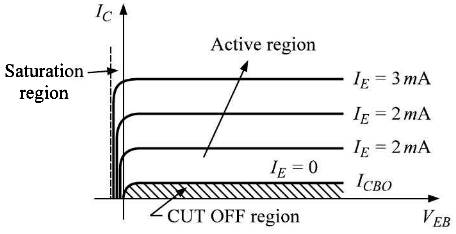


Fig. 34.13

CUT OFF region is achieved when both collector base junction and emitter base junction are reverse biased. Saturation region is achieved when both collector base (*CB*) junction and *EB* (emitter-base) junction are forward biased.

In active region, *EB* junction is forward biased and *CB* junction is reverse biased. Cut off and saturation regions are used in logic gates.

Transconductance or Transfer conductance or mutual conductance

$$g_m = \frac{\partial I_C}{\partial V_{BE}}$$

It is clear from the above table that

Common base amplifier is a voltage amplifier as it amplifies only voltage.

Common collector is a current amplifier as it amplifies only current.

Common emitter amplifies both current and voltage. It gives a phase shift of 180° or π-radian between input and output signal.

Common collector amplifier is also known as emitter follower or a buffer amplifier.

Amplifiers may be of four types

(a) Class A
(b) Class B
(c) Class AB
(d) Class C

Class A amplifier Complete signal 0 – 360° is amplified. Therefore, it is used when the signal is small, that is, at the input stage or first stage.

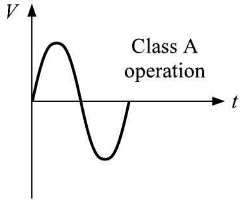


Fig. 34.14

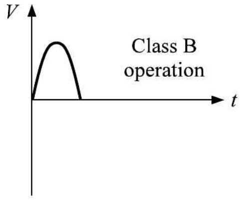


Fig. 34.15

Class B amplifier Class B amplifiers amplify half the signal 0 – 180° or 180° to 360°.

Normally output stage of an audio amplifier is class B. Rather, class-B push-pull amplifier.

Class AB amplifier Class AB amplifiers amplify more than 180° but less than 360° of the signal.

Class C amplifier Amplify only pulses when the signal is large. Class C amplifiers are used in transmitters.

Amplifier is a circuit which gives power gain.

Table 34.2

Property	*Common base amplifier*	*Common emitter amplifier*	*Common collector amplifier*
Input impedance	low	medium high	medium high
Output impedance	high	high	low
Current gain (A_i)	$A_i = \alpha < 1$	$A_i = \beta > 1$	$A_i = (\beta + 1) > 1$
Voltage gain (A_V)	$A_V = \alpha \frac{R_L}{R_i} > 1$	$A_V = \frac{\beta R_L}{R_i} > 1$	$A_V = (\beta + 1) \frac{R_L}{R_i} < 1$
	$= A_i \frac{R_L}{r_e} > 1$	$= A_i \frac{R_L}{r_b} > 1$	$= A_i \frac{R_L}{r_b} < 1$
Power gain A_P	$A_P = A_V . A_i = \frac{\alpha^2 R_L}{r_e} > 1$	$A_P = \beta^2 \frac{R_L}{r_b} > 1$	$A_P = (\beta + 1)^2 \frac{R_L}{r_b} > 1$
Phase shift between input and output	nil	180° or π-rad	nil

Oscillator *LC* tank circuit is a basic oscillator. Due to the dissipative element (internal resistance of the inductance) the oscillations are damped. Therefore, positive feedback or negative resistance is required.

Barkhausen criterion should be satisfied for an oscillator to generate sustained oscillations

$$\beta A_V \geq 1$$

where β is feedback factor and A_V is the voltage gain. The criterion lists two points

(a) there should be a positive feedback

(b) $\beta = 1/A_v$.

Therefore, frequency selective feedback network should be used so that at a particular frequency called the frequency of oscillation, $\beta A_v \geq 1$.

Oscillators may be of two types

(a) audio frequency oscillator (AFO)

(b) radio frequency oscillator (RFO)

To design an AF oscillator, one requires an RC circuit. RC phase shift oscillator and Wein's bridge oscillator are popular AF oscillators. AF oscillators have frequency $f \leq 20\,\text{kHz}$.

Radio frequency oscillators are LC oscillators. Hartley, colptitts, clapp's, crystal oscillators are some of the popular RF oscillators, RF oscillators operate at high frequencies (> 100 kHz). These are used to generate carrier wave and as local oscillator in a radio receiver.

In another categorisation oscillators may be of two types (a) sine wave generator (b) relaxation oscillator. Relaxation oscillators generate any wave other than sine or cosine, that is, square, rectangular, sawtooth, triangular or any such kind.

Logic gates logic is of two types (a) positive logic (b) negative logic.

In positive logic, high state (+5V) is assigned '1' and low state (0V) is assigned '0'. In negative logic, low state is assigned '1' and high state '0'.

AND gate A positive logic AND gate assumes high state if and only if all the inputs are high. Circuit symbol of two input AND gate is

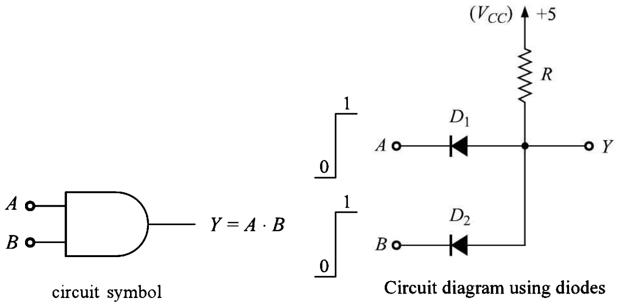


Fig. 34.16

Truth table of AND gate

A	B	$Y = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

OR gate A positive logic OR gate assumes a high state if any of the input is high.

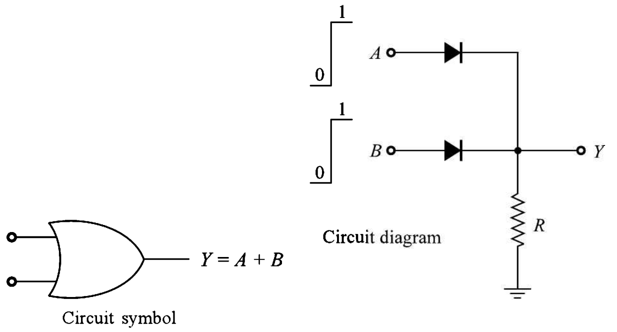


Fig. 34.17

Truth table of OR gate

A	B	$Y = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

NOT gate or Inverter

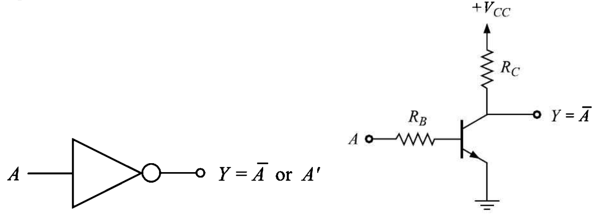


Fig. 34.18

Truth table of NOT gate

A	$Y = \overline{A}$
0	1
1	0 **Fig. 34.**

NAND (Negated AND)

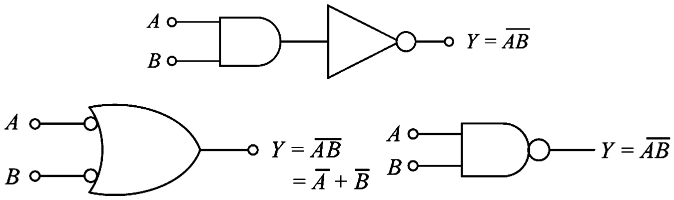
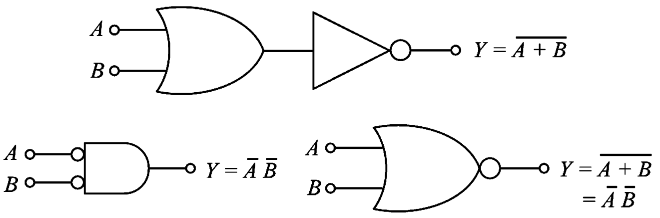


Fig. 34.19

Truth table of NAND

A	B	$Y = \overline{AB}$
0	0	1
1	0	1
0	1	1
1	1	0

NOR (Negated OR)



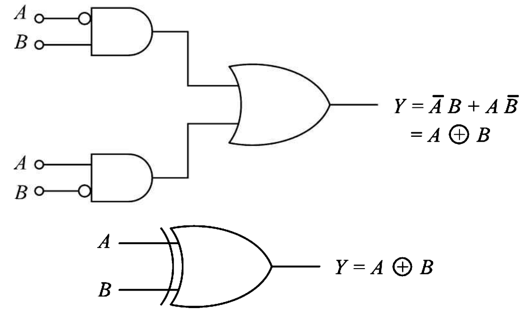
Circuit symbols for NOR gate

Fig. 34.20

Truth table of NOR gate

A	B	$Y = \overline{A} + \overline{B}$
0	0	1
0	1	0
1	0	0
1	1	0

XOR (Exclusive OR)



Circuit symbol of XOR

Fig. 34.21

Truth table of XOR gate

A	B	$Y = A + B$
0	0	0
0	1	1
1	0	1
1	1	0

NAND and NOR gates are called Universal gates as

(a) Any gate/logic can be developed using a single gate or combination of similar gates

(b) They follow associative and distributive laws

(c) They can be manufactured economically

Conversion of gates

Short both the inputs

NOT from NAND

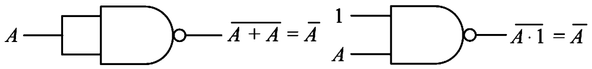


Fig. 34.22

NOT from NOR

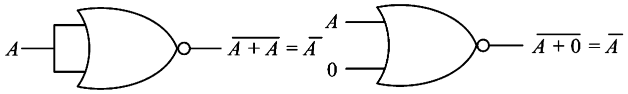


Fig. 34.23

AND from NAND

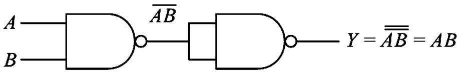


Fig. 34.24

AND from NOR

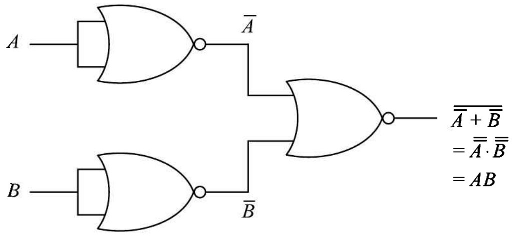


Fig. 34.25

OR from NAND

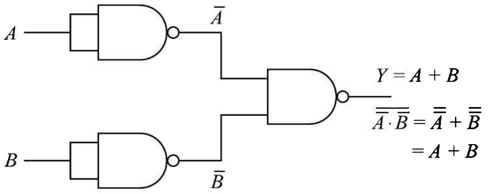


Fig. 34.26

OR from NOR

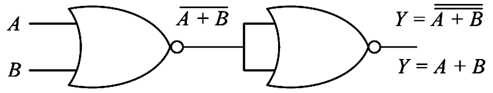


Fig. 34.27

De-Morgan Laws

$\overline{A + B} = \bar{A}\,\bar{B}$

$\overline{AB} = \bar{A} + \bar{B}$

When positive logic is changed to negative logic or vice-versa AND changes to OR ; OR to AND ; NAND changes to NOR and NOR changes to NAND. It is called Duality principle.

Phantom OR or wired OR

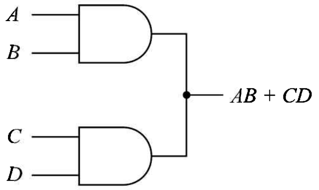


Fig. 34.28

Half adder adds two bits, namely, augend and addend

Note from the half adder table that sum bit S can be implemented using XOR gate and carry bit C is implemented using AND gate.

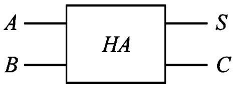


Fig. 34.29

SHORT-CUTS AND POINTS TO NOTE

(i) Energy bands are formed due to the degenerated energy levels in a crystal structure or bulk of the material.

(ii) The property of the semiconductors, that the resistivity falls with rise in temperature can be explained only through band theory.

(iii) A semiconductor is perfect insulator at 0K.

(iv) In a semiconductor, conduction is due to drift current as well as due to diffusion current.

(v) The mobility of electrons is two-three times the mobility of holes. Therefore, *npn* devices are fast and hence preferred.

(vi) In intrinsic (or pure) semiconductor, $n_i = h_i$

(vii) In extrinsic semiconductor, if n_e is the electron density and h_e is the hole density. For $n_e > h_e > n_i$ then the material is *n*-type and if $h_e > n_e > n_i$ then material is *p*-type. Moreover, in thermal equilibrium the condition is $n_e h_e = n_i^2$

(viii) If $E_g \simeq 0\,\text{eV}$, the material is good conductor or metal and if $E_g \simeq 1\,\text{eV}$, the material is a semiconductor. If $E_g \simeq 6\,\text{eV}$ then the material is an insulator or a bad conductor.

(ix) In *n*-type semiconductor $\sigma_n = ne\mu_e \cong N_D\,\mu_e e$

In *p*-type semiconductor $\sigma_p = he\mu_h \cong N_A\,\mu_h e$

(x) A *pn* junction or diode acts like a valve or voltage controlled switch. When forward biased, it acts like ON switch (allows current to pass without attenuation, that is, $r = 0\Omega$, when reverse biased, it acts like an OFF switch, that is, current is zero or $r \to \infty$.

(xi) On forward biasing the diode, $r \to 0$, depletion layer width decreases.

On reverse biasing the diode $r \to \infty$ depletion layer width increases.

(xii) Diode cannot be used as an amplifier since it is a two terminal device.

(xiii) Transistor amplifies by converting power of DC source into AC (of the signal applied). It uses the principle $P = I^2R$ if R_{in} is much less than R_{out} and current at input and output is equal then power gain is obtained.

(xiv) *CB* (common base) amplifier gives only voltage gain
current gain $A_i = \alpha < 1$,

voltage gain $A_V = \alpha \dfrac{R_L}{R_i} = \alpha \dfrac{R_L}{r_e} > 1$,

power gain $A_P = \alpha^2 \dfrac{R_L}{r_e} > 1$,

phase shift-nil

(xv) CE (common emitter) amplifier
current gain $A_i = \beta > 1$;

voltage gain $A_V = \beta \dfrac{R_L}{r_b} > 1$

power gain $A_P = \beta^2 \dfrac{R_L}{r_b} > 1$

phase shift = 180° or π-rad.

(xvi) *CC* (common collector) amplifier is called power amplifier or current booster or emitter follower

current gain $A_i = (\beta + 1) > 1$

voltage gain $A_V = (\beta + 1)\dfrac{R_L}{r_b} < 1$,

Power gain $A_P = (\beta + 1)^2 \dfrac{R_L}{r_b} > 1$

Phase shift = nil

(xvii) Class–*A* amplifier amplifies complete signal (0°–360°), Class–*B* amplifies half the signal (0°–180° or 180°–360°). Class *AB* amplifies more than half but not complete, that is, > 180° but < 360°. Class *C* amplifies pulses. It is used only in communication systems.

(xviii) $I_E = I_C + I_B$; $\alpha = \dfrac{I_C}{I_E}$; $\beta = \dfrac{I_C}{I_B}$, $\beta = \dfrac{\alpha}{1-\alpha}$ and $\alpha = \dfrac{\beta}{1+\beta}$.

(xix) Oscillator can be designed with any of the two techniques. Either use negative resistance device or use positive feedback. When in the frequency selective network, Barkhausen criterion is satisfied then the oscillations are generated. *LC* oscillators are high frequency or radio oscillators while *RC* oscillators are low frequency or audio oscillators.

(xx) Use the relations in order to simplify logical expressions.

$A + A = A$;

$A + \bar{A} = 1, A + 1 = 1, \bar{\bar{A}} = A$

$A + AB = A$; $A(A + B) = A$; $A + \bar{A}B = A + B$;

$A(\bar{A} + B) = AB$; $1 + 1 = 1, 0 + 1 = 1, 0.1 = 0, 1.1 = 1$

$\bar{1} = 0, \bar{0} = 1$ or $\bar{\bar{1}} = 1, \bar{\bar{0}} = 0$

Also remember Demorgan laws, $\overline{A+B} = \bar{A}\,\bar{B}$ and $\overline{AB} = \bar{A} + \bar{B}$

(xxi) NAND and NOR are universal gates. Not gate is unipolar while all other gates are bipolar.

(xxii) Devices like tunnel diode, tetrode and thyristors have negative resistance.

(xxiii) Transistor may operate in three-regions (a) cutoff (like an OFF switch) (b) active or linear region (amplifier is made in this region) (c) saturation region (like ON switch).

Saturation and cutoff regions are used in logic gates. Logic gates are also called switching circuits.

Switching circuits may be of two types, combinational or combinatorial and sequential. Sequential circuit possess memory.

If the logic is changed from positive to negative or vice versa then AND changes to OR, OR to AND, NAND to NOR and NOR to NAND. This is called duality principle. We may also say that if 0 and 1 are interchanged then, '+' and '.' are interchanged.

XOR gate acts as sum bit of the half adder. Carry bit of the half adder is designed using AND gate. Two half adders may be used to make a full adder.

If Amplifiers are connected in series (tandem) having gains A_1, A_2, A_3, ... then the overall gain is $A = A_1A_2A_3$ ....

CAUTION

(i) Forgetting that *pn* junction gives conduction only in one direction.

- ■ In *pn* junction current flows only from *p* to *n* side (arrow mark shows the direction of current).

(ii) Considering that output of the rectifier is DC, therefore, its frequency is zero.

- ■ Frequency of output of a half-wave rectifier is same as that of input signal and frequency of output of a full-wave rectifier is twice the frequency of input signal.

(iii) Considering that rectification efficiency of half-wave rectifier is 50% and that of full-wave rectifier is 100%, as in half wave rectifier half signal and in full-wave rectifier full signal is obtained.

- ■ Rectification efficiency is 40.6% for half-wave and 81.2% for full-wave rectifier.

(iv) Considering that an amplifier means it should increase the amplitude of voltage or current.

- ■ Amplitude should be increased along with increase in power.

(v) Not able to recall correctly the current gain in *CE* and *CB* configurations.

- ■ $\dfrac{I_C}{I_E} = \alpha$ and $\alpha < 1$. It is current gain in *CB* configuration.

$\beta = \frac{I_C}{I_B}$ and $\beta > 1$. It is current gain in *CE* configuration.

(vi) Considering that the transistor cannot be used as a rectifier.

- If only collector base or emitter base junction is considered then the rectifier can be designed.

(vii) Considering that Kirchoff laws cannot be employed in electronic circuits.

- Kirchoff laws can be used even in circuits containing transistors or *pn* junction. A transistor can be considered as a junction.

(viii) Not remembering the formulae for voltage gain and power gain.

- Voltage gain in *CE* amplifiers is $A_V = \beta\frac{R_L}{r_b}$ and in *CB* amplifier voltage gain is $A_V = \alpha\frac{R_L}{r_e}$. Remember that power gain $A_p = A_V A_i$ is a ratio, therefore, it has no dimensions.

(ix) Assuming that the oscillator can be developed only with *L* and *C*.

- Low frequency or audio frequency oscillators are made with *R* and *C*. You have to remember that the basic requirement to make an oscillator is to fulfil Barkhausen criterion, that is, $\beta A_V \geq 1$

  However high frequency oscillators (used to generate carrier wave) are *LC* or crystal oscillators.

(x) Not able to remember logic identities to simplify Boolean expression

- Must always remember logic identities

  $A + A = A$;

  $A + \bar{A} = 1$; $\bar{\bar{A}} = A$

  $A\bar{A} = 0$, $A.A = A$, $A + AB = A$,

  $A + \bar{A}B = A + B$

  $A(A + B) = A$; $A(\bar{A} + B) = AB$

(xi) Assuming that 1 + 1 = 2 even in binary addition

- In binary addition 1 + 1 = 10 (2 written in binary form).

Solved Problems

1. Choose the only false statement from the following

(a) In conductors valence and conduction band may overlap

(b) Substances with energy gap of the order of 10 eV are insulators

(c) The resistivity of semiconductor increases with rise in temperature

(d) The conductivity of semiconductor increases with rise in temperature

Solution (c) is false

2. Zener diode is used for

(a) Amplification

(b) Rectification

(c) Stabilization

(d) Producing oscillations in oscillator

Solution (c)

3. Application of forward bias to a *pn* junction.

(a) widens the depletion zone

(b) increases the potential difference across the depletion zone

(c) increases the number of donors in the *n* side

(d) increases the electric field in the depletion zone

Solution (c)

4. Carbon silicon and Germanium atoms have 4 valence electrons each. Their valence and conduction bands are separated by energy band gaps represented by $(Eg)_C$, $(Eg)_{Si}$, $(Eg)_{Ge}$ respectively which one of the following relationship is true in their case?

(a) $(E_g)_C > (E_g)_{Si}$

(b) $(E_g)_C < (E_g)_{Si}$

(c) $(E_g)_C = (E_g)_{Si}$

(d) $(E_g)_C < (E_g)_{Ge}$

Solution (a) The alletrope of C (diamond) is insulator

$\therefore \quad (E_g)_C > (E_g)_{Si}$

5. The electrical conductivity of semiconductor increases when electromagnetic radiation of wavelength shorter than 2480 nm is incident on it. The band gap in eV for semiconductor is:

(a) 1.1 eV (b) 2.5 eV

(c) 0.5 eV (d) 0.7 eV

Solution (c) $E_g(eV) = \frac{1240}{\lambda(nm)} = \frac{1240}{2480} = \frac{1}{2} = 0.5\,\text{eV}$

6. In a common base amplifier, the phase difference between the input signal voltage and output signal voltage is

(a) $\pi/4$ (b) π

(c) 0 (d) $\pi/2$

Solution (c) In CE amplifier only a phase shift of $\pi/2$ exists.

7. In a full wave rectifier circuit operating from 50 Hz mains, the funamental frequency in the ripple would be

(a) 50 Hz (b) 100 Hz

(c) 22 Hz (d) 70.7 Hz

Solution (b) $f_{(\text{out})} = 2\,f_{\text{input}}$.

8. Which of the followign gates is a universal gate

(a) OR (b) AND

(c) NOT (d) NAND

Solution (d)

9. Consider an *npn* transistor amplifier in CE configuration. The current gain in the transistor is 100. If the collector current changes by 1 mA what will be the change in emitter current.

(a) 1.1 mA (b) 1.01 mA
(c) 0.01 mA (d) 10 mA

Solution (b) $\beta = 100$ and $\alpha = \frac{\beta}{1+\beta} = \frac{100}{101}$;

Given $\frac{\Delta I_C}{\Delta I_E} = \frac{100}{101} = \frac{1\,\text{mA}}{\Delta I_E}$

$\therefore \quad \Delta I_E = 1.01\,\text{mA}.$

10. In a semiconducting material the mobilities of electron and hole are μ_e and μ_h respectively. Which of the following is true?

(a) $\mu_e > \mu_h$ (b) $\mu_e = \mu_h$
(c) $\mu_e < \mu_h$ (d) $\mu_e < 0;\ \mu_h > 0$

Solution (a) Mobility of electrons is 2 to 3 times larger than that of holes.

11. The voltage gain of the amplifier shown is

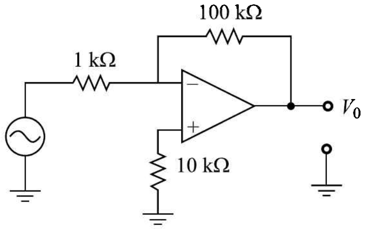


Fig. 34.30

(a) 10 (b) 100
(c) 9.9 (d) 1000

Solution (b) $A_V = \frac{R_f}{R_i} = \frac{100\,\text{k}\Omega}{1\,\text{k}\Omega} = 100$

12. (*A*) The logic gate NOT can be built using diode.

(*R*) The output and input voltage of the diode have 180° phase shift

(a) *A* and *R* are true and *R* is correct expansion of *H*
(b) *A* and *k* are true but *R* is not correct expansion of *A*
(c) *A* is true but *R* is false
(d) both *A* and *R* are false

Solution (d) NOT gate is made using CE transistor configuration.

13. (*A*) The number of electrons in a *p*-type semiconductor is less than number of electrons in a pure silicon semiconductor at room temperature.

(*R*) It is due to law of mass action.

(a) *A* and *R* are true and *R* is correct explanation of *A*
(b) *A* and *R* are true but *R* is not correct explanation of *A*.
(c) *A* is true but *R* is false
(d) *A* and *R* both are false

Solution (a) According to law of mass action

$$h_e n_e = n_i^2$$

14. (*A*) In a CE transistor amplifier the input current is much less than output current.

(*R*) The common emitter transistor amplifier has very high input impedance.

(a) *A* and *R* are true and *R* is correct explanation of *A*
(b) *A* and *R* are true but *R* is not correct explanation of *A*
(c) *A* is true but *R* is false
(d) *A* and *R* both are false

Solution (c) See table 34.2. The input impedance is moderately high.

15. The amplification factor of a triode valve is 15. If the grid voltage is changed by 0.3 V. The change in plate voltage in order to keep the current constant (in volt) is

(a) 0.02 (b) 0.002
(c) 4.5 (d) 5.0

Solution (c) $\mu = \frac{\Delta V_p}{\Delta V_g} \Rightarrow 15 = \frac{\Delta V_p}{0.3}$ or $\Delta V_p = 4.5\,\text{V}$

16. In a full wave rectifier, input AC current has a frequency *v*, the output frequency of the current is

(a) $2v$ (b) $v/2$
(c) v (d) none

Solution (a)

17. The forward voltage of the diode is increased, the width of depletion layer

(a) increases (b) decreases
(c) fluctuates (d) no change

Solution (b)

18. Identify the property which is not a characteristic for a semiconductor

(a) At a very low temperature it behaves as an insulator
(b) At higher temperatures two types of charge carriers will cause conductivity
(c) The charge carriers are electrons and holes in the valence band at higher temperature
(d) The semiconductor is electrically neutral

Solution (c) Electrons exist in conduction band and holes exist in valence band.

19. The type of transition shown in Fig. 34.33 takes place in

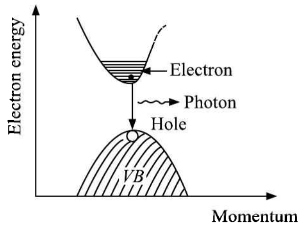


Fig. 34.31

(a) Si (b) Ge
(c) C (d) GaAs

Solution (d) This is an example of direct band gap used in LED and lasers.

20. If $V = V_m \sin\omega t$ is input then find the voltage at *P*

(a) V_m (b) $\frac{3}{2}V_m$
(c) $\frac{V_m}{2}$ (d) $2V_m$

Solution (d) this is the circuit of voltage doubler

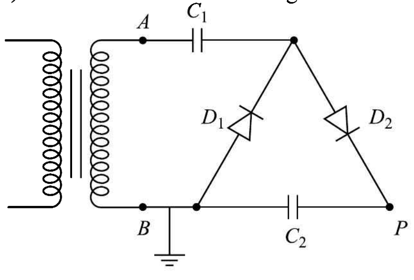


Fig. 34.32

21. The current through diode D_1 is

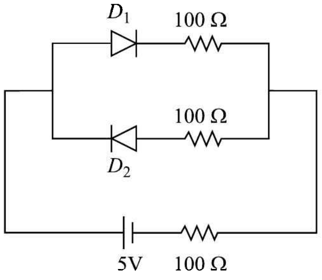


Fig. 34.33

(a) 2.5 mA (b) 25 mA
(c) zero (d) 33 mA

Solution (b) $\frac{5}{200}$ = 25 mA

22. The capacitance shown by reverse biased *pn* junction is termed as

(a) fixed capacitance (b) drift capacitance
(c) diffusion capacitance (d) plate capacity

Solution (c)

23. A silicon diode has knee or cut in voltage equal to ________ volts.

(a) 0.2 V (b) 0.3 V
(c) 0.6 V (d) 0.8 V
(e) none

Solution (c)

24. Tunnel diode is another name for

(a) Power diode (b) varactor diode
(c) Photodiode (d) Esakidiode
(e) Zener diode

Solution (d)

25. To make a full wave rectifier, the AC input be applied across

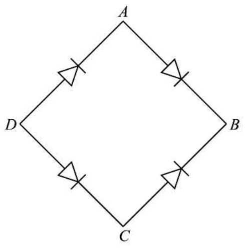


Fig. 34.34

(a) *BD* (b) *BC*
(c) *AD* (d) *AC*
(e) *AB*

Solution (d) Alternating current is applied across AC and output is taken across *BD*.

26. Ebers Moll model describes the working of a

(a) *pn* junction diode (b) BJT
(c) MOSFET
(d) UJT (unijunction transistor)

Solution (b)

27. A transistor has h_{FE} = 95, find h_{FB}.

(a) 1.9 (b) 0.20
(c) 0.94 (d) 0.99

Solution (d) $h_{FB} = \alpha$; $h_{FE} = \beta$

$$\alpha = \frac{\beta}{\beta + 1} = \frac{95}{96}$$

28. Name of gate and write its output

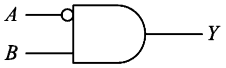


Fig. 34.35

(a) NAND (b) XAND
(c) INHIBIT (d) XNOR

29. If temperature rises by 10°C which of the following current doubles

(a) I_C (b) I_B
(c) I_{CBO} (d) I_E
(e) none

Solution (c)

30. An amplifier has low output impedance and high input impedance. It is a

(a) CB amplifier
(b) CE amplifier
(c) CC amplifier
(d) Push-pull class *B* amplifier

Solution (b)

31. In which of the following circuits capacitor is discharged if it was charged initially

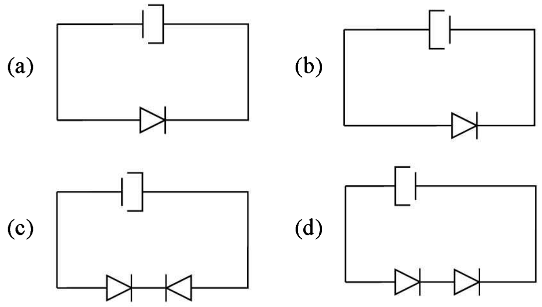


Fig. 34.36

Solution (a)

32. From the truth table find the gate it represents

(a) XOR
(b) Associated NAND
(c) XNOR
(d) NOR

Solution (c) $Y = \bar{A}\bar{B} + AB$ or $Y = \overline{\bar{A}B + A\bar{B}}$

A	*B*	*Y*
0	0	1
0	1	0
1	0	0
1	1	1

33. A circuit used in a local oscillator in radio receivers is

(a) op-amp (b) AF oscillator
(c) RF oscillator (d) phase locked loop

Solution (c)

34. For an input sinusoidal wave train, the output waveform is

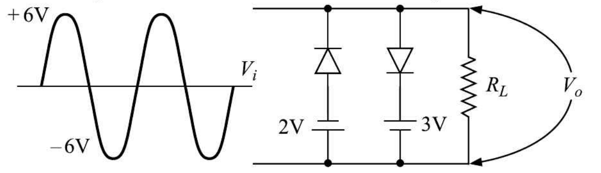
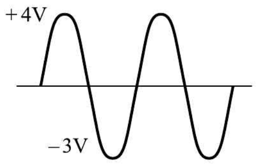
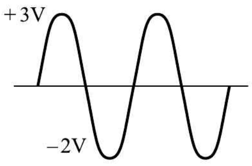


Fig. 34.37

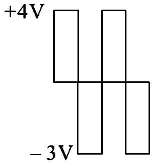
(a)



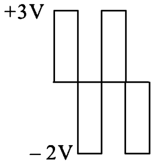
(b)



(c)



(d)



Solution (d) Shunt clipping circuit operation.

35. The conductivity of a pure semiconductor is roughly ∝ $T^{3/2}e^{-\Delta E/2kT}$ where ΔE is band gap. The band gap for Ge is 0.74 eV at 4 K and 0.67 eV at 300 K. By what factor does the conductivity of pure Ge increase as the temperature rises from 4 K to 300 K?

(a) 10^{40} (b) 10^{46}
(c) 10^{201} (d) 10^{401}

Solution (a)

36. Find binary equivalent of $(75.75)_{10}$

(a) 1001011.11
(b) 1001011.01
(c) 1010011.11
(d) 1100101.01

Solution (a) 1001011.11

Problems for Practice

1. In a p type semiconductor the acceptor level is situated 57 meV above the valence band. The maximum wavelength of light required to produce a hole will be

(a) 57×10^{-3} Å (b) 11.61×10^{-33} watt
(c) 57 Å (d) 217100 Å

2. The band of maximum energy in which electrons are present is called

(a) valence band (b) forbidden band
(c) conduction band (d) fermi band

3. On doping germanium with donor atoms of density 10^{17} cm^{-3}, its conductivity in mhocm^{-1} will be ——— if $\mu = 3800$ cm^2/V-s and $n_i = 2.5 \times 10^{13}$ cm^{-3}.

(a) 240 (b) 180
(c) 120 (d) 60.8

4. The three axes of a crystal lattice are mutually perpendicular and two of the lattice parameters are equal. The crystal system is

(a) orthorhombic
(b) hexagonal
(c) cubic
(d) tetragonal

5. Which of the following is not true for crystalline substances?

(a) flat surface
(b) sharp melting point
(c) short range order
(d) isotropic electric conductivity

6. The lattice of Na and Al are bcc and fcc respectively. Presuming them to be closed packed, their packing factors are respectively,

(a) 0.52 and 0.52
(b) 0.68 and 0.34
(c) 0.68 and 0.74
(d) 0.34 and 0.34

7. The safe limits of temperature for germanium and silicon are

(a) 8×10^{2}°C and 2×10^{2}°C
(b) 80°C and 2×10^{2}°C
(c) 2×10^{3}°C and 80°C
(d) 2×10^{2}°C and 800°C

8. The number of densities of electrons and holes in pure silicon at 27°C are equal and its value is 1.5×10^{16} m^{-3}. On doping with indium, the hole density increases to 4.5×10^{22} m^{-3}. The electron density in doped silicon will be

(a) 10^{8} m^{-3} (b) 10^{7} m^{-3}
(c) 50×10^{9} m^{-3} (d) 5×10^{9} m^{-3}

9. The relation between the number of free electrons in semiconductors (n) and its temperature (T) is

(a) $n \propto T$ (b) $n \propto \sqrt{T}$
(c) $n \propto T^{3/2}$ (d) $n \propto T^{2}$

10. There are 10^{18} m^{-3} free electrons in a semiconductor. It is doped with pentavalent impurity atoms of density 10^{24} m^{-3}. The free electron density increases by a factor of

(a) 10^{6} (b) 10^{5} (c) 10^{4} (d) 10^{3}

11. A solid having uppermost energy band partially filled with electrons is called

(a) insulator (b) semiconductor
(c) conductor (d) none of these

12. In a full-wave rectifier circuit operating from 50 Hz mains frequency, the fundamental frequency in the ripple would be

(a) 70.7 Hz (b) 100 Hz
(c) 450 Hz (d) 25 Hz

13. In an amplitude modulated wave for audio frequency of 500 Hz, the appropriate carrier frequency will be

(a) 5×10^{5} c/s^{-1} (b) 50 c/s^{-1}
(c) 500 c/s^{-1} (d) 10^{4} c/s^{-1}

14. What is the order of the forbidden gap in the energy bands of silicon?

(a) 2.1eV (b) 1.1eV (c) 3.1eV (d) 0.1eV

15. Which of the following is NOT a parameter defined for a transistor amplifier?

(a) power gain (b) energy gain
(c) current gain (d) resistance gain

16. For a npn transistor the p type crystal only acts as a

(a) collector only
(b) emitter only
(c) base only
(d) either emitter or collector

17. Which of the following represents the digital signal?

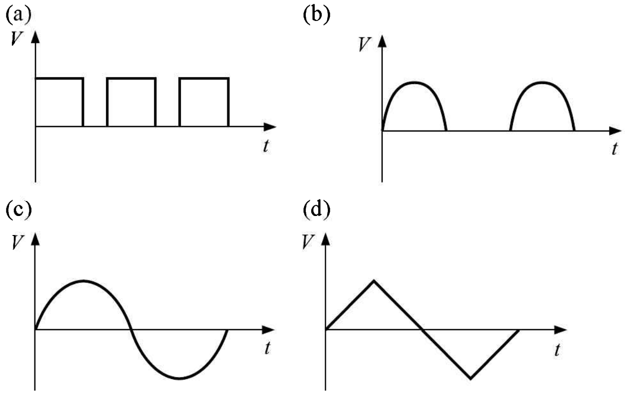


Fig. 34.38

18. The electron mobility in an *n* type germanium is 3900 cm^2/V-s and its conductivity is 6.24 mhocm^{-1}, then the impurity concentration will be

(a) 10^{16}/cm^3 (b) 10^{15}/cm^3
(c) 10^{13}/cm (d) 10^{12}/cm^3

19. The potential barrier of silicon at 30°C is 0.7 eV, then its value at 13° will be

(a) 0.9 eV (b) 0.6 eV
(c) 0.67 eV (d) 0.7 eV

20. Polymers are the substances which are

(a) amorphous in nature
(b) semi-solids
(c) crystalline in nature
(d) none of these

21. A silicon specimen is made into a *p* type semiconductor by doping, on an average one indium atom per 5 × 10^7 silicon atoms. If the number density of atom in the silicon specimen is 5 × 10^{28} atoms/m^3, then the number of acceptor atoms in silicon per cm^3 will be

(a) 1.0 × 10^{15} (b) 2.5 × 10^{36}
(c) 2.5 × 10^{30} (d) 1.0 × 10^3

22. A solid which is not transparent to visible light and whose electrical conductivity decreases with temperature is formed by

(a) ionic bonding (b) van der waal bonding
(c) metallic bonding (d) covalent bonding

23. The number of atoms per unit cell in gold is

(a) 1 (b) 2 (c) 3 (d) 4

24. Radar is a device

(a) for detecting distant enemy planes
(b) similar to telephone
(c) for measuring radioactivity
(d) for destroying enemy planes from a distance

25. What is the process of superimposing signal frequency on the carrier frequency called?

(a) detection (b) reception
(c) modulation (d) transmission

26. In which case is the junction diode forward biased

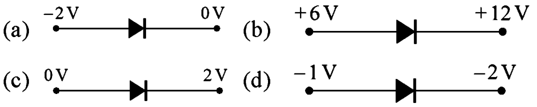


Fig. 34.39

27. A transistor having β equal to 80 has a change in base current of 250 μ A, then the change in collector current is

(a) (250 + 8) μA (b) 250/80 μA
(c) (250 − 80) μA (d) 80 × 250 μA

28. In the arrangement shows in a Fig. 34.42 below, the current through diode is

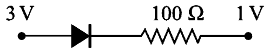


Fig. 38.40

(a) 1 mA (b) 10 mA
(c) 20 mA (d) zero

29. Choose the pair of impurities that may be added to obtain a *p* type crystal.

(a) indium and gallium
(b) nitrogen and phosphorous
(c) antimony and bismuth
(d) boron and aluminium

30. Which of the following bonds produces a solid that reflects light in visible region and whose electrical conductivity decreases with temperature and has melting point of the order of 1000°C?

(a) ionic bonding (b) metallic bonding
(c) covalent bonding (d) van der waal bonding

31. Which of the following solid is partly ionic and partly covalent crystal?

(a) sulphur (b) diamond
(c) zinc (d) none of these

32. Two identical *pn* junctions may be connected in series with a battery in three ways as shown in Fig. 34.43. The potential drop across the *p-n* junction are equal in

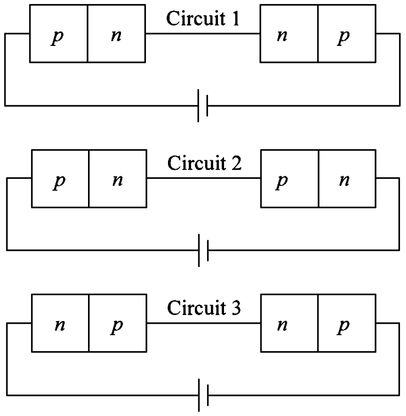


Fig. 34.41

(a) circuit 3 and circuit 1
(b) circuit 1 and circuit 2
(c) circuit 2 and 3
(d) only circuit 1

33. In a common-base configuration I_E = 1 mA and I_C = 0.95 mA. The value of base current is

(a) 1.95 mA (b) 0.95 mA
(c) 1.05 mA (d) 0.05 mA

34. Which one of the arrangements in Fig. 34.44 shows the correct biasing of the collector in a transistor?

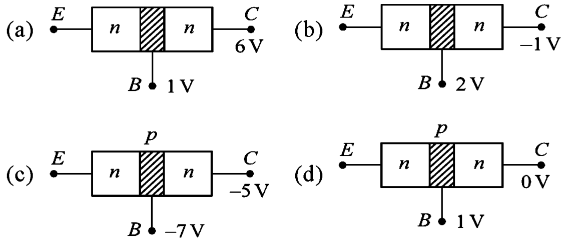


Fig. 34.42

35. The distance between two atoms in contact is known as
(a) lattice parameter
(b) atomic radius
(c) number of constituent particles
(d) packing factor

36. For the given circuit shown in Fig. 34.45, to act as full-wave rectifier, the AC input should be connected across .... and the DC output would appear across ....

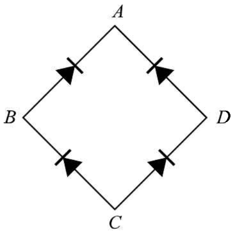


Fig. 34.43

(a) $A-B, C-D$ (b) $A-C, B-D$
(c) $B-C, A-D$ (d) $B-D, A-C$

37. The value of α
(a) is always infinity
(b) is always less than 1
(b) lies between 50 and 150
(d) is always greater than 150

38. Acceptor type or *p* type semiconductor is formed by adding impurity of valency
(a) 6 (b) 5
(c) 4 (d) 3

39. *npn* transistors are preferred to *pnp* transistors because they are
(a) capable of handling large power
(b) electrons which have high mobility than holes and hence make fast devices
(c) are full of electron gas
(d) have no valence band

40. There is no hole current in good conductors because they
(a) have large forbidden energy gap
(b) have overlapping valance and conduction bands
(c) are full of electron gas
(d) have no valence band

41. A hole in a *p* type semiconductor is
(a) a missing atom (b) a donor level
(c) an excess electron (d) a missing electron

42. A transistor has $\alpha = 0.95$. It has a change in emitter current of 100 milliampere. Then the change in collector current is
(a) 95 mA (b) 100 mA
(c) 99.05 mA (d) 100.95 mA

43. At absolute zero temperature a crystal of pure germanium
(a) contains no electron
(b) behaves as a perfect conductor
(c) behaves as a perfect insulator
(d) none of these

44. The relation between α and β is given by
(a) $\beta = \frac{\alpha}{1-\alpha}$ (b) $\alpha = \beta$
(c) $\beta = \frac{\alpha}{1+\alpha}$ (d) $\alpha = \frac{1}{\beta}$

45. In a *pnp* transistor circuit, the collector current is 10 mA. If 90% of the holes emitted reach the collector, then the
(a) the base current is –1 mA
(b) the emitter current will be 11 mA
(c) the base current is 1 mA
(d) the emitter current will be 9 mA

46. In *npn* transistor, the emitter current is
(a) slightly less than the collector current
(b) equal to the base current
(c) slightly more than the collector current
(d) equal to the collector current

47. A typical example of a semiconductor is
(a) germanium (b) quartz
(c) mica (d) platinum

48. When two semiconductors of *p* and *n* type are brought in contact, they form a *pn* junction which acts like an
(a) oscillator (b) amplifier
(c) rectifier (d) conductor

49. If I_E, I_C and I_B represent the emitter current,, collector current and base current respectively in a transistor then
(a) I_B is much smaller than I_E
(b) I_C is slightly smaller than I_E
(c) I_B is much greater than I_E
(d) I_C is slightly greater than I_E

50. When an impurity is doped into an intrinsic semiconductor, the conductivity of the semiconductor
(a) remains the same (b) increases
(c) decrease (d) becomes equal to zero

51. A hole in a *p* type semiconductor is
(a) conduction band (b) forbidden energy gap
(c) valence band (d) none of these

52. The impurity atoms with which pure silicon should be doped to make a *p*-type semiconductor are those of
(a) antimony (b) boron
(c) phosphorus (d) aluminium

53. With rise in temperature, the electrical conductivity of intrinsic semiconductor
(a) first increases and then decrease
(b) increases
(c) first decreases and then increase
(d) decreases

54. PIV (peak inverse voltage) of diode in full-wave rectifier made with centre tapped transformer should be
(a) $= 2V_P$ (b) $> 2V_P$
(c) $< 2V_P$ (d) $= V_P$

55. The hardest material out of the following is
(a) diamond (b) aluminium
(c) glass (d) steel

56. The impurity element to be doped in pure germanium in order to make *p* type semiconductor is
(a) Bi (b) Sb
(c) B (d) P

57.  is the symbol of

(a) a capacitor (b) tunnel diode
(c) varactor diode (d) photodiode

58. Which of the following properties can be different along different directions in a crystalline solids?
(a) refractive index
(b) mechanical strength
(c) electrical conductivity
(d) all of these

59. The ripple factor in a full-wave rectifier is
(a) 0.48 (b) 1.21 (c) 0.62 (d) 1.0

60. The concentration of acceptor atom in a *p* type germanium crystal is $4 \times 10^3/\text{cm}^3$. If $\mu_p = 1900\,\text{cm}^2/V\text{-}s$ at 300 K, then its conductivity will be

(a) 1.126×10^{-10} mhocm^{-1}

(b) 12.16 mhocm^{-1}

(c) 1.1216 milli mhocm^{-1}

(d) zero

61. The circuit shown in Fig. 34.46 is used to represent a transistor. Such a model is called

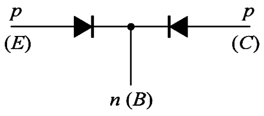


Fig. 34.44

(a) eber's moll model

(b) shockley's model

(c) bell model

(d) quantum model

62. In which of the transistor amplifiers is the voltage gain highest?

(a) common emitter

(b) common collector

(c) common base

(d) same for all types of the amplifiers

63. The number of junctions in a semiconductor diode

(a) no junction (b) one

(c) two (d) three

64. To get an output $Y = 1$ from circuit in Fig. 34.47 the input must be

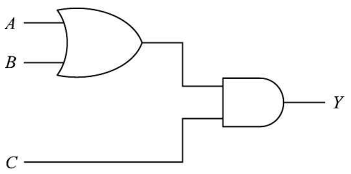


Fig. 34.45

	A	B	C
(a)	0	1	0
(b)	1	0	0
(c)	1	0	1
(d)	1	1	0

65. Which of the following in not the function of NOT gate?

(a) invert the input signal

(b) change the logic in a digital circuit

(c) stop a signal

(d) complement a signal

66. What is the output *Y* of the gate circuit shown in the Fig. 34.48.

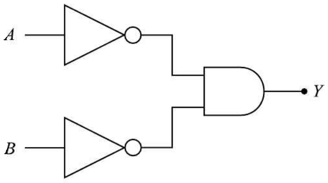


Fig. 34.46

(a) $\bar{A}.\bar{B}$ (b) $\overline{\overline{A.B}}$

(c) $\overline{A.B}$ (d) none of these

67. In the Boolean algebra, which of the following is not equal to *A*?

(a) $\overline{\bar{A}+\bar{A}}$ (b) $A.A$

(c) $A + A$ (d) $\bar{A}.A$

68. Which of the following is not true in Boolean algebra?

(a) $[\bar{1} + 0].1 = 0$ (b) $[\bar{1} + \bar{1}].1 = 0$

(c) $[1 + \bar{1}].1 = 0$ (d) $[\bar{1} + 0].1 = 0$

69. For a common base amplifier, the value of resistance gain and voltage gain are 300 and 280 respectively. The current gain will be

(a) 0.63 (b) 0.73 (c) 0.83 (d) 0.93

70. The value of α a for transistor is 0.9. What would be the change in the collector current corresponding to a change of 4 mA in the base current in a common emitter arrangement?

(a) 18 mA (b) 36 mA

(b) 72 mA (d) none of these

71. The diode used to generate FM is

(a) tunnel diode (b) photodiode

(c) varactor (d) zener diode

72. Emitter of a transistor cannot be interchanged with the collector because

(a) emitter is heavily doped

(b) emitter is moderately doped

(c) emitter is slightly doped

(d) collector is heavily doped

73. To avoid noise to enter the amplifier, it is

(a) DC coupled

(b) *RC* coupled

(c) transformer coupled

(d) none of these

74. The base region of a transistor should be

(a) thick (b) moderately thin

(c) very thin (d) very thick

75. If a transistor has $\alpha = 0.99$, $r_i = 2\,\text{k}\Omega$ and $R_L = 10\,\text{k}\Omega$. It is used in *CE* configuration then the voltage gain is

(a) 99 (b) 495 (c) 195 (d) 590

76. Two amplifiers of gain A_1 and A_2 are *RC* coupled then the net gain is

(a) $A_1 + A_2$ (b) $A_1 - A_2$

(c) $(A_1 + A_2)/2$ (d) $A_1 \times A_2$

77. A transistor in ——— configuration gives maximum current gain

(a) common emitter (b) common base

(c) common collector (d) any of these

78. The transconductance in *CB* amplifier is

(a) $\partial V_{EB}/\partial I_C$ (b) $\partial I_C/\partial V_{EB}$

(c) $\partial I_B/\partial V_{EB}$ (d) $\partial I_C/\partial V_{CB}$

79. The cut in voltage or barrier potential for a Si diode is

(a) 0.3V (b) 0.5V (c) 0.7V (d) 0.9V

80. The demodulating circuit in AM is

(a) rectifier (b) filter

(c) peak detector (d) voltage multiplier

81. The best oscillator from the frequency stability point is

(a) wein's bridge (b) colpitt's oscillator

(c) clapp's oscillator (d) crystal oscillator

82. The solar cell is a *pn* junction operating in

(a) reverse bias (b) forward bias

(c) unbiased (d) none of these

83. Zener diode is used in

(a) load regulator (b) demodulator

(c) current amplifier (d) none of these

84. *pn* junction cannot be used as
(a) an amplifier (b) rectifier
(c) voltage multiplier (d) demodulator

85. In semiconductors the forbidden energy gap between *VB* and *CB* is of the order of
(a) 1 MeV (b) 1 KeV
(c) 5 eV (d) 1 eV

86. AF (audio frequency) amplifier in the last stage uses
(a) wein's bridge (b) push pull class *B* amplifier
(c) class *A* amplifier (d) class *C* amplifier

87. The correct relation between n_i and p_i for an intrinsic semiconductor of $E_g = 1.2\,\text{eV}$ will be
(a) $n_i < p_i$ (b) $n_i = p_i = 0$
(c) $n_i > p_i$ (d) $n_i = p_i$

88. The majority current carriers in *p* type semiconductors are
(a) only free electrons
(b) only holes
(c) both electrons and holes
(d) none of these

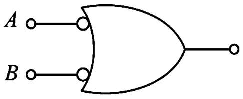
89. The binary equivalent of $(62)_{10}$ is
(a) 111101 (b) 111100
(c) 111110 (d) 111111

90. The drift current in a *pn* junction is obtained when it is
(a) forward biased
(b) reverse biased
(c) forward or reverse biased
(d) none of these

91. The temperature coefficient of resistance of a semiconductor is
(a) positive (b) negative
(c) imaginary (d) zero

92. Negative resistance is shown by ——— diode
(a) tunnel diode
(b) varactor diode
(c) zener diode
(d) none of these

93.



Y represents ——— gate

Fig. 34.57

(a) NAND (b) NOR
(c) XOR (d) OR

ANSWERS

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (c), (d)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (d)
21. (a)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (c)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (a), (d)	**30.** (a)
31. (d)	**32.** (c)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (d)	**37.** (b)	**38.** (d)	**39.** (b)	**40.** (b)
41. (d)	**42.** (a)	**43.** (c)	**44.** (a)	**45.** (b), (c)	**46.** (c)	**47.** (a)	**48.** (c)	**49.** (a), (b)	**50.** (b)
51. (c)	**52.** (b)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (c)	**57.** (c)	**58.** (d)	**59.** (a)	**60.** (a)
61. (a)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (c)	**66.** (a)	**67.** (d)	**68.** (c)	**69.** (d)	**70.** (b)
71. (c)	**72.** (a)	**73.** (b), (c)	**74.** (c)	**75.** (b)	**76.** (d)	**77.** (c)	**78.** (b)	**79.** (c)	**80.** (c)
81. (d)	**82.** (b)	**83.** (a)	**84.** (a)	**85.** (d)	**86.** (b)	**87.** (d)	**88.** (b)	**89.** (c)	**90.** (b)
91. (b)	**92.** (a)	**93.** (a)							

Principles of Communication

35

BRIEF REVIEW OF THE CONCEPTS

Communication may broadly be defined as the transfer of information from one point to another. When the information is to be conveyed over a distance, a communication system is required. Within a communication system the information transfer is achieved usually by superimposing or modulating the information on to an electromagnetic wave (or carrier). Modulated carrier is then transmitted to the required destination via transmission lines. At the destination it is received and original information is retrieved by means of demodulation or detection. Sophisticated techniques have been developed for this process, using electromagnetic carrier waves operating at radio frequencies as well as microwave and millimeter wave frequencies or even infrared rays infrared rays in optical communications.

Transducer It is a device which converts a physical quantity varying with time into electrical signal or vice versa. It is of two types. Input transducer and output tranducer. Input transducer converts a physical quantity varying with time into electrical signal. For example microphone converts sound into electrical signal. Output transducer converts electrical signal back to a physical quantity varying with time. For example, speaker converts electrical signal back to sound.

Channel is used to refer to the frequency range allocated to a particular transmission. For example, a television channel.

Noise introduction of unwanted signal or some distortion in the process of transmission and reception. The signal gets deteriorated. Since the noise will be received along with the signal and if noise is several times the signal, it may mask the signal making it unintelligible.

Modulation is of two types analog and digital. In analog modulation some characteristic of a high frequency sine wave (called carrier) is varied in accordance with the instantaneous value of the modulating signal. If amplitude of the carrier wave is varied in accordance with the modulating signal then Amplitude modulation (AM) results. If the frequency of the carrier is varied in accordance with the instantaneous value of the modulating signal then Frequency modulation (FM) results. If phase of the carrier wave is varied in accordance to the instantaneous value of the modulating signal then Phase modulation (PM) results. FM and PM may be categorized as angle modulation.

Modulation is required for long distance communications. If we transmit sound waves directly say at 20 kHz (largest frequency of sound) then the length of the antenna required = $\frac{\lambda}{2}$ = 7.5 km $\left(\lambda = \frac{c}{f} = \frac{3\times10^8}{20\times10^3} = 15\,\text{km}\right)$ which is impractical in present day technology. The second reason for modulation is that if all the radio stations transmit at 20 kHz (or same frequency) their signal will mix up and nothing would be audible.

Amplitude modulation Let $e_c = E_c \sin \omega_c t$ be the carrier wave and $e_m = E_m \sin\omega_m t$ be the modulating signal. Then modulated signal e is given by

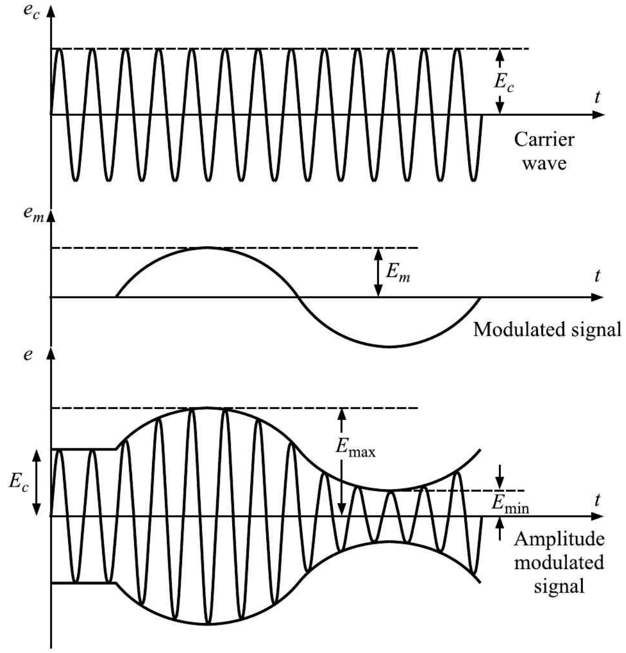


Fig. 35.1

$$e = E_C\left(1 + \frac{E_m}{E_C}\sin\omega_m t\right)\sin\omega_C t$$

$$= E_C\,(1 + m_a \sin\omega_m t)\,\sin\omega_C t$$

where $m_a = \frac{E_m}{E_C}$ is modulation index. It is normally expressed in % and should be less than 100%

$$m_a = \frac{E_{\max} - E_C}{E_C} = \frac{E_C - E_{\min}}{E_C} = \frac{E_{\max} - E_{\min}}{2E_C} = \frac{E_{\max} - E_{\min}}{E_{\max} + E_{\min}}$$

$$e = E_C\,(1 + m_a \sin\omega_m t)\,\sin\omega_C t$$

$$= E_C \sin\omega_C t + \frac{m_a}{2} E_C \cos(\omega_C - \omega_m)t$$

$$+ \frac{m_a E_C}{2} \cos(\omega_C + \omega_m)\,t$$

The spectrum of AM will consist of carrier wave, lower side band (LSB) or [component of $(\omega_C - \omega_m)$] and upper side band (USB) or component of $(\omega_C + \omega_m)$]

$$P_{\text{tot}} = P_{\text{carrier}} + P_{\text{LSB}} + P_{\text{USB}}$$

$$= P_{\text{carrier}}\left(1 + \frac{m_a^2}{2}\right)$$

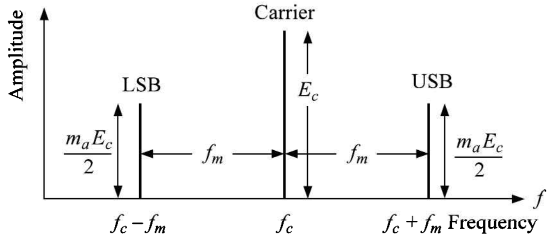


Fig. 35.2

$$\text{or } \frac{P_{\text{tot}}}{P_{\text{carrier}}} = 1 + \frac{m_a^2}{2}$$

$$\frac{I_{\text{tot}}}{I_{\text{carrier}}} = \sqrt{1 + \frac{m_a^2}{2}}$$

If several modulating signals are present then

$$m_{\text{tot}} = \sqrt{m_1^2 + m_2^2 + m_3^2 + \ldots}$$

Moreover, total modulation index should not exceed unity.

Types of AM

A3 → double side band full carrier

A3A → single side band reduced carrier

A3H → single side band full carrier

A3J → single side band suppressed carrier called SSB (Single side band) transmission.

A3B → two independent side bands with suppressed carrier

A5C → vestigial side band transmission (used for video transmission in TV)

Frequency modulation Mathematically, frequency of the frequency modulated signal f is given by

$f = f_C\,(1 + kE_m \cos\omega_m t)$

where k is conversion factor which converts voltage to frequency since $\cos\omega_m t$ will lie between ±1, therefore, frequency of modulated signal will lie between $f = f_c\,(1 \pm kE_m)$

Maximum deviation $\delta = k\,f_C\,E_m$

We may also write $\omega = \omega_C(1 + kE_m \cos\omega_m t)$

$$\text{and} \quad \theta = \int \omega\, dt$$

$$= \int \omega_C\,(1 + kE_m \cos\omega_m t)\, dt$$

$$\text{or} \quad \theta = \omega_C t + \frac{kE_m\omega_C \sin\omega_m t}{\omega_m}$$

$$= \omega_C t + \frac{kE_m f_C}{f_m} \sin\omega_m t$$

$$= \omega_C t + \frac{\delta}{f_m} \sin\omega_m t$$

And the instantaneous voltage of FM signal is given by

$$e = A \sin\left(\omega_C t + \frac{\delta}{f_m} \sin\omega_m t\right)$$

$$= A \sin(\omega_C t + m_f \sin\omega_m t)$$

$$\text{Frequency modulation index } m_f = \frac{\delta}{f_m} = \frac{kf_C E_m}{f_m}$$

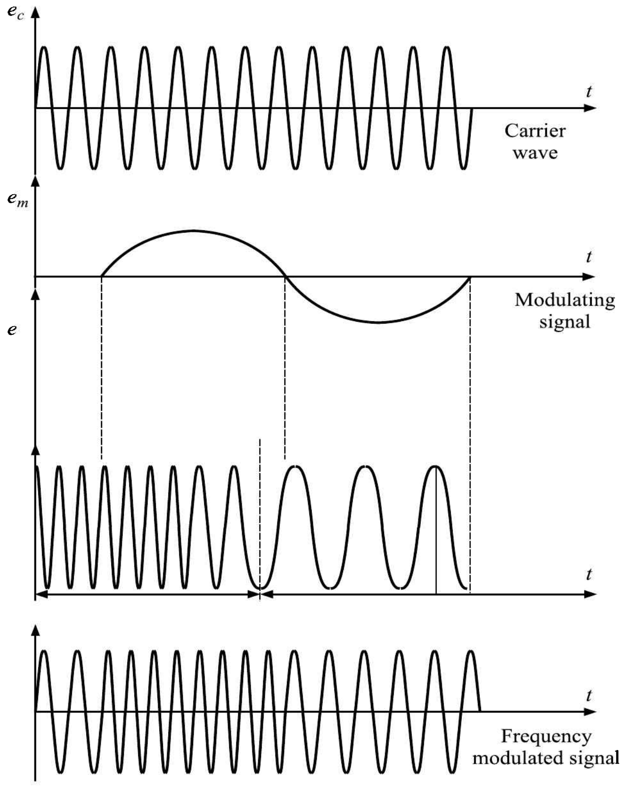


Fig. 35.3

Band width $\Delta = 2(f_m + \delta)$ if $m_f > 6$. For $m_f = 2.4$, 5.5, 8.6 and 11.8 and so on, the carrier component completely disappears. Hence these values of m_f be used.

Bessels functions are used to solve the FM equations.

Phase modulation If the phase in the equation $e_C = E_C \sin(\omega_C t + \phi)$ is varied in accordance to instantaneous value of the modulating signal, the resulting wave is phase modulated.

The expression for a PM wave is

$e = A \sin(\omega_C t + \phi_m \sin\omega_m t)$

where ϕ_m is maximum value of phase shift.

We can write $m_p = \phi_m$ as modulation index equation of PM appears identical to FM. The difference being that the modulation index in PM is proportional to amplitude of the modulating voltage while in FM it is inversely proportional to modulating frequency. Thus, if frequency of modulating signal is changed, modulation index in PM will remain constant while in FM it will vary.

However, if FM is received on a PM receiver, bass frequencies will have more deviation of phase than a PM transmitter would have given them, that is, the signal would be unduly bass boosted. If PM is received on an FM receiver it will lack bass and may be corrected by applying a bass booster

Ground or surface waves These waves progress along the surface of the earth. They are vertically polarised to prevent shortcircuiting the electric component. A wave induces current in the ground over which it passes and thus loses some energy by absorption. However, this is made up by energy diffracted downwards from a wavefront and act like a leakage capacitor.

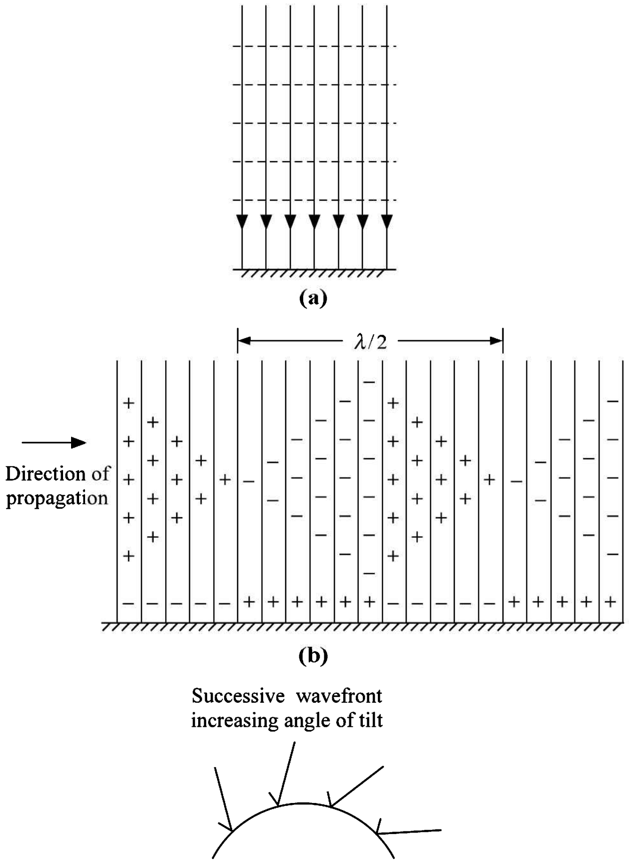


Fig. 35.4

Attenuation also occurs due to diffraction as angle of tilt of successive wavefronts increases as shown in Fig. 35.4(c)

Electric field at a distance due to ground waves is given by $E = \frac{120\pi h_t I}{\lambda d}$ and the signal received by the receiving antenna of height h_r is given by

$$V(\text{volts}) = \frac{120\pi h_t h_r I}{\lambda d}$$

where $120\pi = 377\Omega$ is characteristic impedance, I is antenna current, h_t = effective height of the transmitting antenna, distance from transmitting antenna d, and λ is wavelength

VLF propagation When propagation is over a good conductor like sea water at frequencies below 100 kHz then attenuation is small. Ship communication use frequency ranging from 10 to 110 kHz.

Sky wave propagation–the ionosphere Ionosphere is the upper portion of the atmosphere, which continually absorbs large quantities of radiant energy from the sun, thus, becoming heated and ionised. Temperature, density, composition and the type of radiations received stratify the ionosphere. The most important ionising agents are UV, γ, α, β and λ rays from the sun as well as cosmic rays from the meteors. The overall result shown in Fig. 35.5 is a range of four main layers D, E, F_1 and F_2 in ascending order. The last two combine at night to form a single layer.

The D layer is the lowest existing at an average height of 70 km with an average thickness of 10 km. The degree of its ionisation depends on the altitude of the sun. It disappears at night. It reflects LF and VLF rays and absorbs MF and HF waves to certain extent.

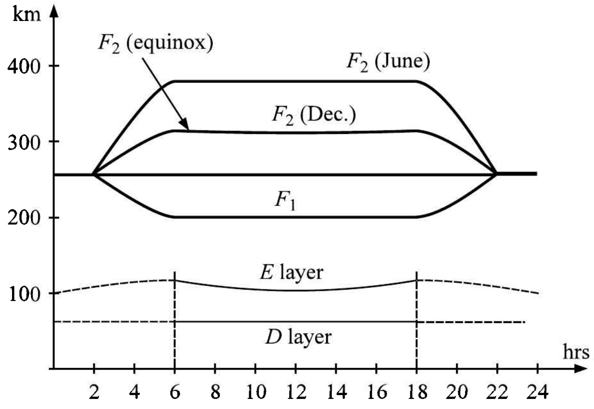


Fig. 35.5

The E layer, is a thin layer of very high density. Like the D layer it also disappears in the night. It reflects MF and surface waves and some HF waves at daytime.

The F_1 layer exists at a height of 180 – 200 km and combines with the F_2 layer during night. F layer absorbs HF waves. F_2 layer is most important reflecting layer for HF. Its height is 250 to 400 km with an average thickness of 200 km and average height of about 300 km.

Reflection mechanism As the ionisation density increases, the refractive index of the layer decreases. The incident ray is gradually bent and suffers total internal reflection. The reflected wave is same except that it is attenuated. Critical frequency f_c (kHz) = $9\sqrt{N}$ where N is electron density or ionisation density. Maximum usable frequency (*muf*) = $f_c \sec\theta$ refractive index $\mu = \sqrt{k}$ where k is dielectric constant

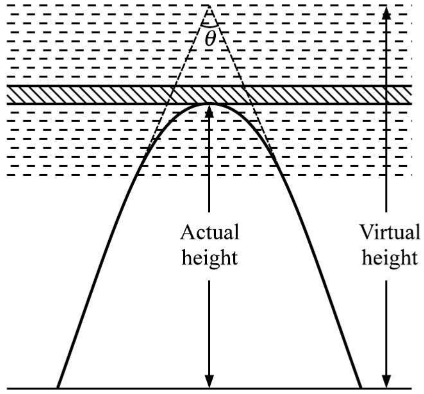


Fig. 35.6

$$\mu = \sqrt{k} = \sqrt{1 - \frac{81N}{f^2}}$$

Attenuation constant $\alpha = K\sqrt{\frac{f_c}{f}}$ where,

$K \rightarrow$ constant

$f_c \rightarrow$ critical frequency of E layers

f is wave frequency

Skip distance is the shortest distance from a transmitter, measured along the surface of the earth at which a sky wave of fixed frequency ($> f_c$) will be returned to earth. It is frequency specific and depends upon the angle of incidence.

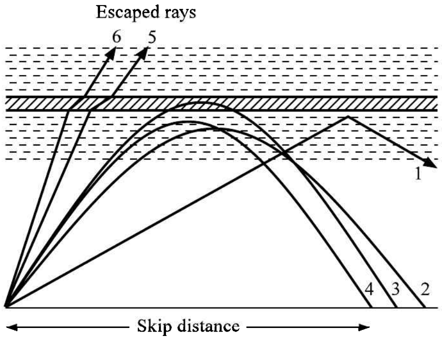


Fig. 35.7

Space waves travel in straight lines and depend on line of sight condition. They are limited by the curvature of the earth.

Radio horizon = $\frac{4}{3}$ optical horizon. The empirical formula is $d_t = 4\sqrt{h_t}$ where d_t is the distance from transmitting antenna, h_t is height of transmitting antenna. The same relation will be true for receiving antenna and the total distance will, therefore be,

$$d = d_t + d_r = 4\left(\sqrt{h_t} + \sqrt{h_r}\right)$$

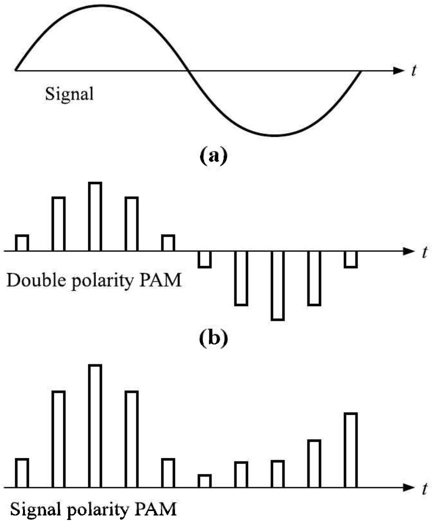
If h_t = 225 m then the radio horizon = $4\sqrt{h_t}$ = 60 km.

Duct or super refraction for microwaves occurs due to temperature inversion zones present in the space.

For UHF rays tropospheric scattering called **troposcatter** or **forward scatter propagation** occurs. Frequency range 900 to 5000 MHz.

Satellite and Probe tracking The requirement for tracking and communicating with satellite in close orbits involve the use of fast-rotating circularly polarized antennas together with fairly low-noise and medium-power transmitter and receivers.

Pulse modulation Pulse modulation may be subdivided broadly into two categories – **analog** and **digital**. In analog form, the sample amplitude may be infinitely variable, while in the digital form a code is sent which indicates the sample amplitude to the nearest predetermined level. Pulse amplitude and pulse time modulation are both analog while pulse code modulation and delta modulation are both digital.



(a)

(b)

(c)

Fig. 35.8

Pulse amplitude modulation (PAM): This is the simplest form of pulse modulation, as illustrated in Fig. 35.8. It is of two types, single polarity and double polarity. Double polarity is self illustrated. Each pulse is made proportional to amplitude of the signal at the instant of the sampling. The pulse are then transmitted by wire, cable or modulated using carrier. In single polarity, the amplitude of the pulse is made so large that it remains positive even when the input is negative. PAM is hardly used but whenever used, pulses are made to frequency modulate the carrier and termed as PAMFM.

Pulse-time modulation (PTM): In PTM the signal is sampled as before, but the pulses indicating instantaneous sample amplitudes have a constant amplitude. However, one of their timing characteristic is varied and is made proportional to the amplitude of the signal (at that instant) being sampled. The variable characteristic may be width, position or frequency. Thus, PTM is of three types, namely, pulse width modulation (PWM) or pulse duration modulation (PDM), pulse position modulation (PPM) and pulse frequency modulation. The last one, that is, pulse frequency modulation has no practical significance.

Sampling theorem According to this theorem if the sampling rate in any pulse modulation system exceeds twice the maximum signal frequency then the original signal frequency can be reconstructed in the receiver with negligibly small distortion. This is called Nyquist theorem.

PWM or PDM As illustrated in Fig. 35.9 below the pulse width at a particular instant is proportional to the instantaneous value of the signal at that very instant. Zero amplitude for instance gives 1 μs as the average width of the pulse. If the recurrence rate of the pulse is 8000 pps, the time between occurrence of next pulse is $\frac{10^6}{8000} = 125\,\mu$s. It is quite sufficient to accommodate the varying widths but also to permit time division multiplexing. This leads to a drawback of varying power due to varying duration.

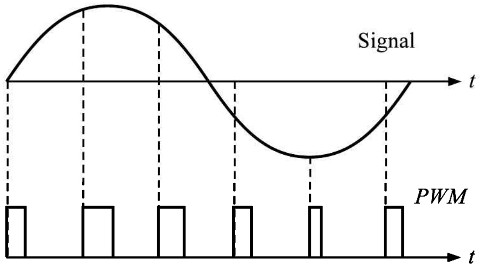


Fig. 35.9

PPM It is generated from PWM as illustrated in Fig. 35.10. Each pulse has a leading anda trailing edge. The repetition rate of leading edge is fixed but not of trailing edge. Their positions depend on pulse width, which is determined by the signal amplitude at that instant. Thus, trailing edges of PWM pulses are infact pulse position modulated. If the pulse train obtained from PWM is differentiated, leading and trailing edges are separated. If the position of trailing edge of unmodulated pulse is counted as zero displacement, then the other trailing edge will arrive earlier or later. They will therefore have a time displacement other than zero, that is, displacement is proportional to the instantaneous value of the signal voltage. The leading edges of pulses are removed with a diode clipper or rectifier and remaining pulses are pulse position modulated.

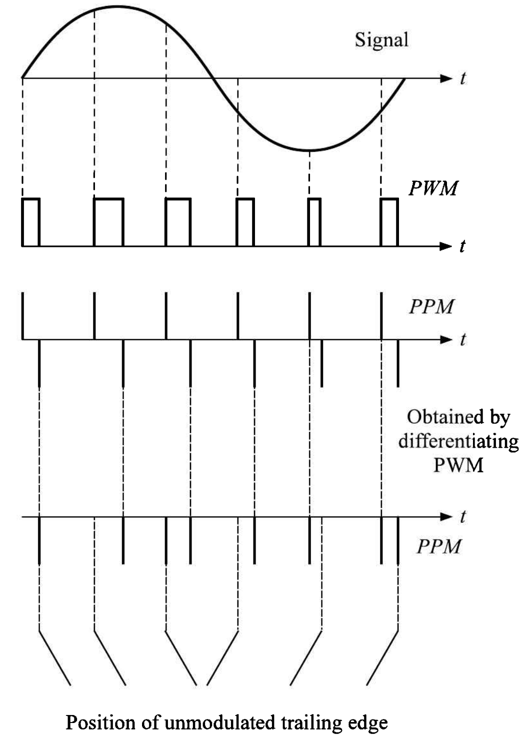


Fig. 35.10

Pulse coded modulation (PCM) It is a digital process. It also uses sampling technique which generates code. The Fig. 35.11 explains how a signal can be quantised or coded. For example, when V < –5.25V code is 000 when –5.25V < V < –3.5V code is 001, and when –3.5V < V < –1.75V code is 010 and so on.

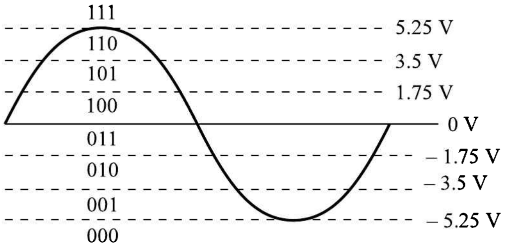


Fig. 35.11

PCM requires complex encoding and quantising circuit. It also requires larger bandwidth as compared to analog systems. Its main advantage is that it can be made computer compatible using ASCII codes.

Delta modulation The simplest form of delta modulation is differential PCM. Just one bit per sample is sent to indicate whether the signal is longer or smaller than the previous sample. It uses a simple coding and decoding procedures, and quantising process is also quite simple. In delta modulation the transmission rate must be close to 100k bit/s to give the same performance as for a telephone channel as PCM gives with 64k bit/s (8000 samples/s × 8 bit per sample)

Detection systems The simplest detector uses a peak detector (rectifier + capacitor filter) to detect AM wave. The diode will rectify and give either positive half or negative half cycle and capacitor filter gives only peak value, thereby detecting information from the carrier. However, the information is slightly distorted as we cannot retrieve exactly the input as capacitor charges nearly linearly for short intervals. But to a large extent it is replica of input. Normally, we use negative part of the input for detection (in the Figure positive part is shown), as it helps in achieving automatic gain control or AGC.

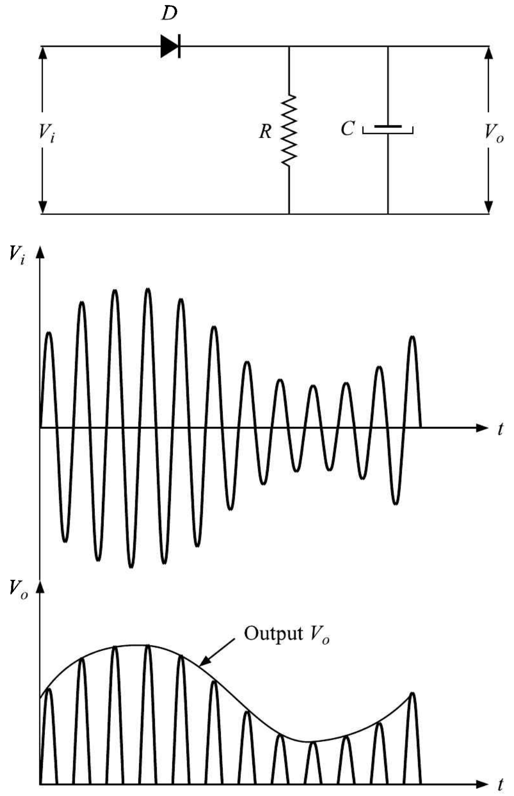


Fig. 35.12

FM detection The simplest FM detection is achieved using an *LC* circuit operated at OFF resonant frequency as shown in Fig. 35.13. Then the frequency variation will be converted to current and hence voltage variation (if taken across a resistance).

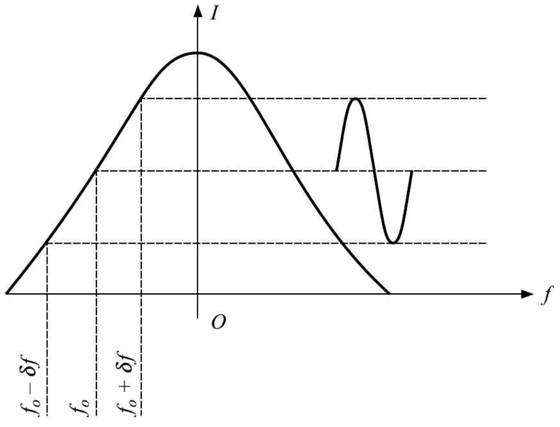


Fig. 35.13

ASK (amplitude shift keying) Binary ASK also called OOK (ON-OFF keying) as is clear from Fig. 35.14(a) when a 1 is transmitted, wave is present when a 0 is transmitted wave is absent.

FSK (frequency shift keying) During transmission of a '1', frequency increases (say doubles in the Fig. 35.14(b) shown) and it remains uninterrupted during 0 transmission.

PSK (phase shift keying) As illustrated in Fig. 35.14(c) a phase shift of 180° is observed whenever there is a transition 0 → 1 or 1 → 0 data transmission. Fig. 35.14(a), (b) and (c) depict ASK, FSK and PSK respectively.

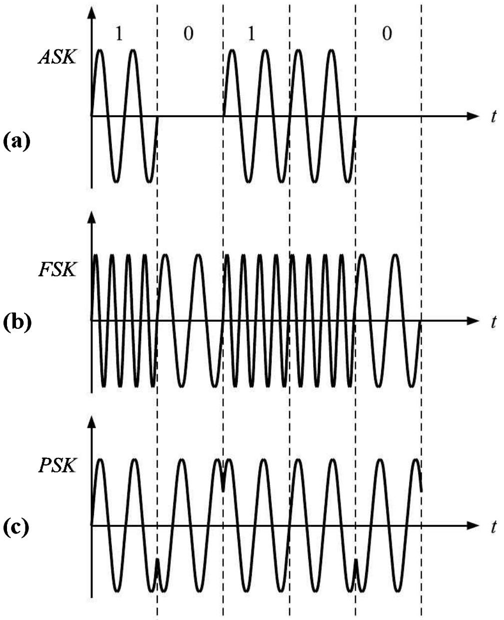
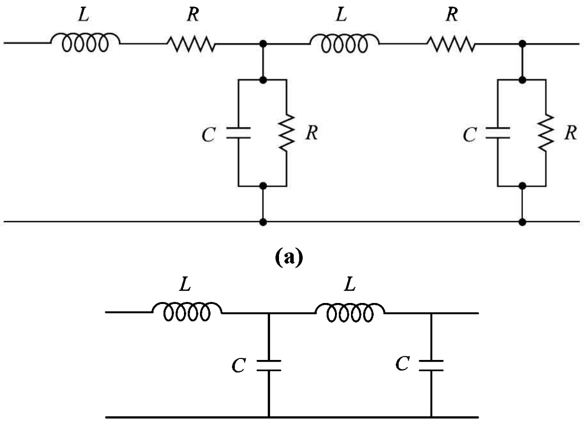


Fig. 35.14

These techniques are used to transmit binary data by analog means.

Transmission lines Transmission line is a mode of carrying the information. The equivalent circuit of a transmission line is shown in Fig. 35.15 (a) and (b)



(a)

(b)

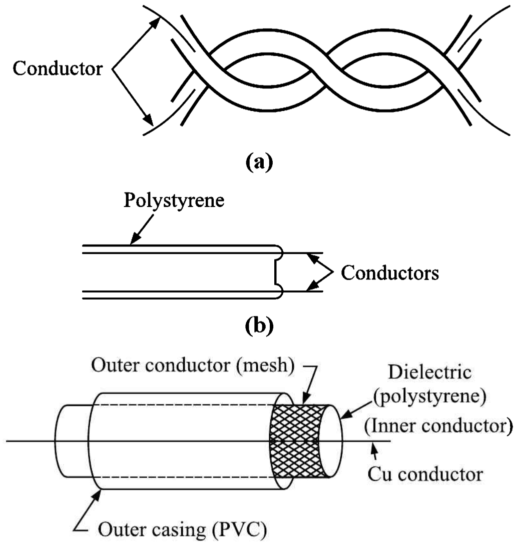
Fig. 35.15

The common transmission lines are twisted pair, twin parallel line and coaxial cable. For long distance communication satellite links and microwave links (space wave communication) are used. Optical fibres are also showing promising deal. Each type has its limitations and properties listed in following table. Parallel twin wire is suitable for balanced properties, for example, connected a folded dipole antenna to a TV receiver.

Coaxial cable can be used utpo 18 GHz

Table 35.1

Cable	*Bandwidth*
Twisted pair	2-3 MHz
Twin wire	30-60 MHz
Coaxial cable	upto 5 GHz



(a)

(b)

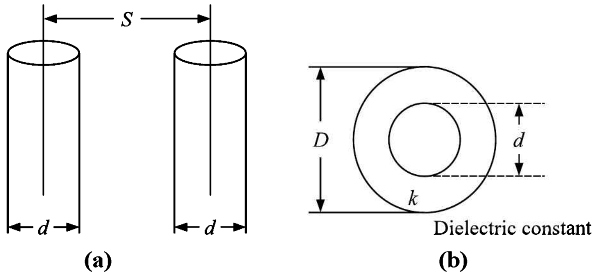
(c)

Fig. 35.16

Characteristic impedance $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{R + jL\omega}{G + jC\omega}}$

For co-axial cables $Z_0 = \frac{138}{\sqrt{k}} \log \frac{D}{d}$

There could be three types of losses in transmission lines namely, radiation, conductor heating and dielectric heating.



(a) (b)

Fig. 35.17

Standing wave ratio (SWR) The ratio of maximum current to minimum current along a transmission line is called standing wave ratio.

$$\text{SWR} = \frac{I_{\max}}{I_{\min}} = \frac{V_{\max}}{V_{\min}} = \frac{Z_0}{R_L} \text{ or } \frac{R_L}{Z_0} \text{ whichever is larger}$$

wavelengths of interest in optical fiber range = 0.8μm to 1.7μm (IR region)

Acceptance Angle A ray which is incident at the core/cladding interface at angle $\geq \theta_C$ will be transmitted. The ray enters the fiber core at an angle θ_a to the fiber axis and is refracted at the air/core interface before transmission to the core/cladding interface at the critical angle. Hence θ_a is maximum angle to the axis at which light may enter the fiber in order to be propagated and is called acceptance angle.

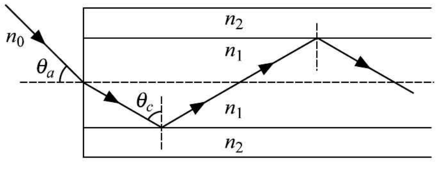


Fig. 35.18

$$n_0 \sin\theta_a = n_1\sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}.$$

Factors which cause losses in fibers are intrinsic absorption, linear scattering, ray light scattering, mie scattering, simulated brillouin scattering, raman scattering and dispersion.

$$\text{Attenuation (dB)} = 10 \log \frac{P_i}{P_0}$$

If α_{dB} is attenuation (in dB) per unit length

$$\alpha_{\text{dB}} l = 10 \log \frac{P_i}{P_0}$$

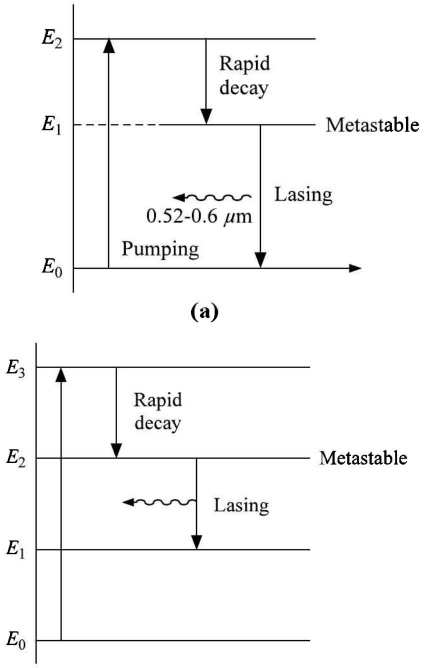
LASER (Light Amplification by stimulated emission of radiation): Light emission is of two types (a) spontaneous emission (b) stimulated emission. In spontaneous emission atom returns to the lower energy state in an entirely random manner. For example, in an incandescent bulb.

In stimulated emission a photon having energy equal to energy difference between two states ($E_2 - E_1$) interacts with the atom in the higher energy state E_2 causing it to return to lower energy state E_1 with the creation of second photon. Both the photons happen to be in phase and have the same polarisation and hence amplitudes are added up or energy increased which is called light amplification. In a normal case according to Einstein's condition

$$\frac{\text{stimulated emission rate}}{\text{spontaneous emission rate}} = \frac{1}{\exp \frac{hf}{kT} - 1}$$

That is, spontaneous emission is a much more dominant mechanism in thermal equilibrium. Therefore, to form laser, suitable conditions are to be created so that stimulated emission dominates. Notable among these are population inversion and resonating cavity.

Population inversion Under normal conditions (thermal equilibrium) lower levels contain more electrons than higher levels. It is necessary to create non equilibrium condition such that number of electrons in higher levels are more. This is termed as population inversion and is achieved by means of pumping. Optical pumping and electrical pumping are two popular methods of excitation. Fig. 35.19(a) illustrates 3 level lasing system as in ruby laser and Fig. 35.19(b) illustrates 4-level transition. The disadvantage of 3 level system is that transition from E_0 to E_2 should be very fast, that is, pumping rate should be quite large. In 4-level system, E_2 (metastable) to E_1 gives lasing action.



(a)

(b)

Fig. 35.19

Febry-perot resonator is required to achieve stimulating radiation or optical feedback and hence laser oscillations are generated. The resonator consists of two mirrors M_1 and M_2 at the ends as shown in Fig. 35.20. M_2 is partially reflecting. Therefore, output beam comes out from this side (M_2). Some of the rays are reflected back from M_2 and after reflection from M_1 and acts as stimulating radiations.

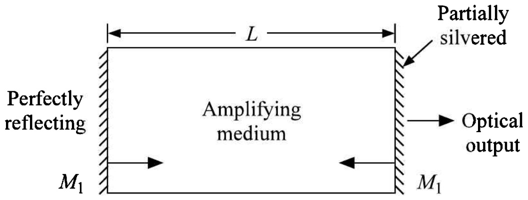


Fig. 35.20

Lasers are of three types Solid, liquid and gas lasers. Ruby laser, glass laser, semiconductor lasers are examples of solid lasers. Normally solid lasers are pulsed lasers. Some organic liquids are used to form liquid laser. He-Ne and CO_2 laser are examples of gas lasers. CO_2 laser is used in surgery.

Lasers used in communication system are semiconductor lasers, Nd : YAG (a glass laser), glass fiber laser, diode laser and ND : YAG laser are used in cosmetics hair removal triple treatment and so on.

Total Laser efficiency

$$(\eta_T) = \frac{\text{Total number of output photons}}{\text{Total number of injected electrons}}$$

$$\text{External power efficiency } \eta_{EP} = \eta_T \left(\frac{E_g}{V}\right)$$

High power lasers can be used to cut thick metal sheets/bars or may be used for welding. Lasers are used for blood-less operation and to see the inner view. Lasers are also used for eye operation/ sight correctness and so on. Laser printers are also available. Laser is used to store data in the CD (compact disc) and so on.

Semiconductors Lasers are formed with GaAs, AlGaAs, AlGaAs, PALGaP, AlGaP and so on. Coupled cavity or distributed feedback lasers are used in communication systems.

LEDs are also used in fiber-optics. Edge emitted, truncated edge emitting In GaAsP, V-grooved edge emitting and mesa structure edge emitting LEDs are used. All are made with GaAs or other III-V semiconductors. Super luminescent AlGaAs LEDs are also in use.

Reliability Reliability is defined in two ways, electrical reliability and optical reliability.

Electrical Reliability R_E(dB)

$$= 10 \log_{10} \frac{\text{electric power output (at delector)}}{\text{electric power input (at source)}}$$

$$= 10 \log \frac{I_{\text{out}}^2}{I_{\text{in}}^2}$$

The –3dB point occurs when $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{\sqrt{2}}$

Optical reliability R_O(dB)

$$= 10 \log_{10} \frac{\text{optical power out}}{\text{optical power in}}$$

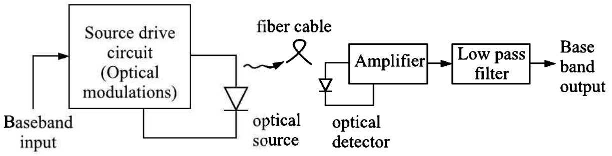
$$= 10 \log_{10} \frac{I_{\text{out}}}{I_{\text{in}}}$$

The –3dB point occurs when $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{2}$

Avalanche photo diodes, p-i-n photodiodes, photo transistors and photo-conductive detectors are used as detectors in optical fiber communication.

Direct Intensity Modulation (DIM) Fig. (a) Analog optical fiber communication system using direct modulation (b) optical carrier modulation with base band.

Fig. (a) shown block diagram for an analog optical fiber system which uses direct modulation of the optical source intensity with the base band signal. No electrical modulation or demodulation is required in this technique. Hence it is inexpensive and easy to implement.



(a)

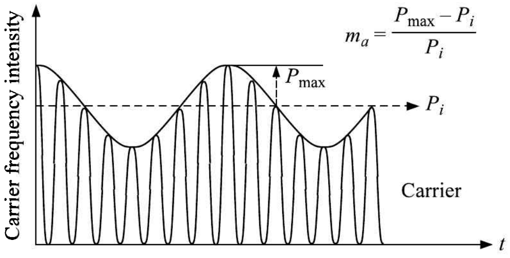
The transmitted optical power waveform is a function of time give by

$P_{opt}(t) = Pi\,(1 + m(t))$

where Pi is average input power transmitted without modulation and $m(t)$ is the intensity of modulating signal

$m(t) = ma \cos \omega_m t$

where m_a is modulation index $ma = \frac{P_{\max} - P_i}{P_i}$. Thus, $P_{opt}(t) = P_i\,(1 + m_a \cos \omega_m t)$.



(b)

Modem (Modulator – demodulator): used in two way communication system. If digital system is used, Modem converts analog signal to digital form before it is transmitted. It receives a digital signal and converts it into analog form. Modems are therefore attached with telephone lines so that two way communication is feasible. Fax machine is also a form of modem. Fax (facsimilie) is used to transmit data written on a paper or text written on a paper.

Remote sensing Remote sensing utilizes electromagnetic energy to detect and quantify information about an object that is not in contact with the sensing device. IRS-1A and IRS-1B are Indian remote sensing satellites. These are placed in sun synchronous orbits around the earth. The important applications of remote sensing satellites are (a) spying work (b) ground survey, water survey, forest survey underground location of coal, oil and ores, fishing zones in the sea and so on (c) preparation of waste land maps (d) draught estimation (e) estimation of crop yield (f) detection of crop diseases.

SHORT-CUTS AND POINTS TO NOTE

(i) Modulation is required for long distance communication. Modulation is of two types, analog and digital. Analog modulation may be further of two types, amplitude modulation (AM) and angle modulation. Angle modulation is of two types frequency modulation and phase modulation. Frequency modulation is noise immune and amplitude modulation is noise prone. AM (Radio) is carried from 550 kHz-1650 kHz (medium wave band) and 3 MHz to 10 MHz (short wave band).

(ii) Modulation index (AM)

$$= m_a = \frac{E_m}{E_C} = \frac{E_{\max} - E_C}{E_C}$$

$$= \frac{E_C - E_{\min}}{E_C} = \frac{E_{\max} - E_{\min}}{2E_C}$$

$$= \frac{E_{\max} - E_{\min}}{E_{\max} + E_{\min}}$$

$$P_{\text{total}} = P_{\text{carrier}} \left(1 + \frac{m_a^2}{2}\right) \qquad \frac{I_{\text{tot}}}{I_{\text{carrier}}} = \sqrt{1 + \frac{m_a^2}{2}}$$

If several modulating signals are present then $m_{tot} = \sqrt{m_1^2 + m_2^2 + m_3^2 + \ldots}$ $m_{\text{tot}} \le 1$.

(iii) Maximum deviation in FM $\delta = kf_C\, E_m$;

$$\theta = \omega_C t + \frac{\delta}{f_m} \sin \omega_m t$$

Frequency modulation index $m_f = \frac{\delta}{f_m} = \frac{kf_C E_m}{f_m}$

FM (Radio) is carried out at 80-120 MHz.

(iv) Ground waves propagate along the surface of the earth. These are vertically polarised to prevent short circuiting of the electric component. Electric field at a distance d is given by $E\ \frac{120\pi\, h_t I}{\lambda d}$ and signal received by an antenna of height h_r is given by V (volts) $\frac{120\pi\, h_t h_r I}{\lambda d}$

(v) Sky waves and space waves are reflected by ionosphere if frequency of the signal < 30 MHz. The Critical frequency is given by $f_C(\text{kHz}) = 9\sqrt{N}$ where N is electron density or ionisation density. Maximum usable frequency (*muf*) $= f_C \sec\theta$

Refractive index $\mu = \sqrt{k} = \sqrt{1 + \frac{81N}{f^2}}$

Attenuation constant $\alpha = K \sqrt{\frac{f_C}{f}}$.

(vi) Radio horizon $= \frac{4}{3}$ optical horizon. The emperical formula is

$d = d_t + d_r$

$d_t(\text{km}\sqrt{h_t} = 4$ and $d(\text{km}\left(\sqrt{h_t} + \sqrt{h_r}\right) = 4$.

(vii) Duct super refraction of microwaves occurs due to temperature inversion zones present in the space. For UHF rays troposcatter or forward scatter propagation occurs. Frequency range is 900 to 5000 MHz for UHF

(viii) In satellite communication uplink and downlink frequencies differ. Downlink frequency being slightly higher.

(ix) Pulse modulation is of two types, analog and digital. PAM and PTM (are example of analog modulation while PCM is an example of digital modulation.

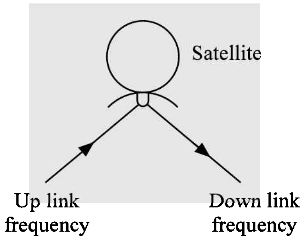


Fig. 35.21

PTM may be of two types, PPM and PWM. Delta modulation is also a kind of PCM or digital modulation.

(x) Receivers may be of two types, tuned radio frequency (TRF) receivers and superheterodyne receivers. Super heterodyne receivers use local oscillators and intermediate frequency amplifiers before the signal is detected. In this way the reception becomes free of signal frequency but depends only on intermediate frequency which is fixed. For example, for a radio it is 455 kHz. (AM) and 10.7 MHz) FM for TV operating at VHF range is 26 to 46 MHz and for UHF range it is 36 to 46 MHz while those operating at microwave have IF 60 – 70 MHz.

(xi) A rectifier with peak detection is used in the AM wave detection and FM detection is achieved by an *LC* circuit tuned at off resonant frequency.

(xii) Amplitude shift keying is also called ON OFF keying or (OOK) ASK, FSK and PSK are used for binary data transmission by analog means.

(xiii) Transmission lines have equivalent circuit as *LCR* or *LC* circuit. Twisted pair, twin line and co-axial cable and so on are transmission lines in the electrical regime and optical fiber in the light wave (electromagnetic wave regime of IR range). Microwave links, space waves satellites and so on, are used for microwave, space wave transmission.

(xiv) Standing wave ratio SWR $= \frac{I_{\max}}{I_{\min}} = \frac{V_{\max}}{V_{\min}} = \frac{Z_0}{R_L}$ or $\frac{R_L}{Z_0}$.

(xv) Acceptance angle of an optical fiber $n_0 \sin\theta_a$ or $\sin\theta_a = n_1\sqrt{1-\left(\frac{n_2}{n_1}\right)^2}$

Attenuation is defined in (dB) as 10 log $\frac{P_i}{P_0}$ α dB is attenuation per unit length.

(xvi) Lasing material must have metastable state. The energy difference between metastable state and just upper or lower excited state must lie in the range of interest.

(xvii) Population inversion means increasing the density of electrons in excited state.

(xviii) Febry perot cavity or resonator is required to generate feed back oscillations in the laser to act as stimulating radiations

(xix) APDs (Avalanche photodiodes) are best suited for detection in fiber optic communication

(xx) External power efficiency $\eta_{EP} = \eta_T \left(\frac{E_g}{\gamma}\right)$ where η_T is total efficiency given by

$$n_T = \frac{\text{total number of output protons}}{\text{total number of injected electrons}}$$

(xxi) Reliability

Electrical reliability R_E(dB) $= 10 \log_{10} \frac{I_{\text{out}}}{I_{\text{in}}}$

and –3dB point occurs When $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{2}$

Optical reliability R_O(dB) $= 10 \log_{10} \frac{I_{\text{out}}}{I_{\text{in}}} =$

and –3dB point occur When $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{2}$.

(xxii) Modem (Modulator-demodulator) are used for two way communications. In digital systems modems convert analog to digital or vice versa. Telephone, Fax are simple modems.

(xxiii) Transducers are of two types self excited and those which require excitation source. Self excited do not require battery while others require battery.

(xxiv) A communication satellite covers 42% area of the globe. Therefore, at least three communication satellites are required to cover the whole globe. Maximum number of geostationary satellites which can be launched is 180 at a spacing of 2° each.

Table 35.2

Frequency band	*Applications*
Low frequency 30 kHz – 300 kHz	Long wave radio and communication over large distances
Medium frequency (300 kHz – 3 MHz)	Medium wave, local and distant radio
High frequency (3 MHz –30 MHz)	Short wave radio and communication amateur and CB radio
Very high frequency (30 MHz – 300 MHz)	FM radio, police, meteorology
UHF (300 MHz – 3 GHz)	TV (bands 4 and 5) aircraft landing system
Microwaves > 3 GHz	Radar, communication satellites, mobile telephones and TV links.

(xxv) Lasers are monochromatic, coherent and collimated

(xxvi) Optical fibers are single mode type and multimode type. Single mode type have smaller cross-section of the core while multimode have larger cross-section of the core. Further, they are step index type and graded index type. Parabolic index variation is most preferred in present day technology.

(xxvii) MASER is microwave amplification by stimulated emission of radiation. It is used as a microwave amplifier or oscillator. The principle of MASER is identical to that of LASER. Only frequency range is ≤ 10^{11} Hz in masers.

(xxviii) Light intensity is modulated in optical fiber communication. All the methods like ASK, FSK, PSK or PCM and so on can be employed even in light intensity modulation.

(xxix) If h is height of the antenna then the range to which it can serve is $r = \sqrt{2Rh}$ where R is radius of the earth.

CAUTION

(i) Considering all pulse modulations as a form of digital modulation.

- Only PCM, delta and differential PCM are digital. PPM, PDM or PWM, PAM are analog modulation techniques.

(ii) Considering that there is no difference between frequency and phase modulation.

- In frequency modulation m_f (frequency modulation index) is inversely proportional to modulating frequency f_m. While in PM it does not vary with modulating frequency. Moreover, FM is more noise immune.

(iii) Assuming AM with both the sidebands is advantageus.

- AM with single side band suppressed carrier is better as it contains maximum modulating power.

(iv) Considering that with several waves modulation, indices are added algebraically.

- Modulation indices are added according to root mean square rule

$$M_{\text{tot}} = \sqrt{m_1^2 + m_2^2 + \ldots}$$

(v) Considering all antennas to be isotropic.

- Mostly antennas are directional. For example, antennas like dish, butterfly, Yagi and so on are directional. Using isotropic antenna, range covere is $\sqrt{2R_e h}$.

(vi) Considering that ionosphere reflects all the waves.

- Ionosphere reflects waves of VHF order. UHF rays and microwaves are transparent to ionosphere. The critical frequency is given by f_C(kHz) = $9\sqrt{N}$.

(vii) Considering that any number of communication satellites can be installed in space.

- Minimum three for global communication and maximum 180 geostationary satellites can be launched (at a gap of 2° each)

(viii) Considering mobile phones operate at VHF

- Mobile phones operate at microwaves.

(ix) Considering modems are used in one way communication like radio or TV.

- Modems are used in two way communications. They convert digital signal to analog and vice-versa.

(x) Assuming transducers only convert time varying physical quantities into electrical signal.

- They also convert electrical signal back to physical quantities. The former type are called input transducers and latter type the output transducer

(xi) Considering that light (laser) in fiber optics can be only intensity or amplitude modulated.

- All possible modulation techniques may also be employed in optical communication including AM, FM, PM and PCM and so on. Even ASK, FSK, and PSK can be need.

(xii) Considering that electrical and optical reliability in fiber optics is same

‰ –3dB point in electrical regime is $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{\sqrt{2}}$

and –3dB point in optical regime is $\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{1}{2}$.

Solved Problems

1. A 500 W carrier is modulated to a depth 75%. Find the power of modulated wave.

(a) 640.5 W (b) 500 W
(c) 600 W (d) 359.5 W

Solution (a) $P_{\text{tot}} = P_{\text{carrier}}\left(1 + \frac{m_a^2}{2}\right)$

$$= 500\left[1 + \frac{(0.75)^2}{2}\right] = 640.5\,\text{W}$$

2. A transmitter radiates 10 kW at 80% modulation index. The power in carrier is nearly

(a) 5.58 kW (b) 6.58 kW
(c) 7.58 kW (d) 10 kW

Solution (c) $P_{\text{carrier}} = \frac{P_{\text{tot}}}{\left(1 + \frac{m_a^2}{2}\right)} = \frac{10}{1 + \frac{0.8^2}{2}} = 7.58\,\text{kW}.$

3. Find the modulation index if an AM transmitter shows an antenna current of 8 A when only carrier is transmitted and 9 A when current is modulated using sine wave. Also find antenna current if modulation index is 60%

(a) 72.9%, 8.55 A (b) 65.6%, 8.55 A
(c) 82.9%, 8.55 A (d) none of these

Solution (a) $m_a = \sqrt{2\left[\left(\frac{I_{\text{tot}}}{I_{\text{carrier}}}\right)^2 - 1\right]}$

$$= \sqrt{2\left[\left(\frac{9}{8}\right)^2 - 1\right]} = 72.9\%\ \text{(approx)}$$

$$I_{\text{tot}} = I_{\text{carrier}}\sqrt{1 + \frac{m_a^2}{2}} = 8\sqrt{1 + \frac{(.6)^2}{2}}$$

= 8.55 A (approx).

4. If $E_C = 20 \sin 10^6 \pi t$ and $E_m = 12 \sin 500 \pi t$ are carrier and modulating signal then the modulation index is

(a) 60% (b) 40%
(c) 50% (d) 56%

Solution (a) $m_a = \frac{E_m}{E_C} \times 100 = 60\%$.

5. In an AM wave maximum voltage was found to be 16 V and minimum voltage was found to be 4 V then the percentage of modulation is

(a) 40% (b) 25%
(c) 33% (d) 60%

Solution (d) $m_a = \frac{E_{\max} - E_{\min}}{E_{\max} + E_{\min}} \times 100$

$= \frac{12}{20} \times 100 = 60\%$.

6. If antenna current for carrier wave is 8 A and total antenna current is 10 A then the modulating power of the AM wave for total power to be 10 kW is

(a) 8 kW (b) 6.4 kW
(c) 2 kW (d) 3.6 kW

Solution (d) $\frac{P_{\text{tot}}}{P_{\text{carrier}}} = \frac{I^2_{\text{tot}}}{I^2_{\text{carrier}}} = \frac{(10)^2}{8^2}$

or $P_{\text{carrier}} = 10 \times \frac{64}{100} = 6.4$ kW

Modulating power $= P_{\text{tot}} - P_{\text{carrier}} = 10 - 6.4 = 3.6$ kW.

7. Single side band transmission in AM is called

(a) A3H (b) A3J
(c) A3B (d) A5C

Solution (b) A3J is SSB transmission.

8. A 400 Hz modulating voltage fed into an FM generator produces a frequency deviation of 2.4 kHz. The modulation index is

(a) 5.6 (b) 6.0
(c) 6.2 (d) 8.3

Solution (b) $m_f = \frac{\delta}{f_m} = \frac{2.4 \times 10^3}{400} = 6$.

9. When the modulating frequency in an FM system is 500 Hz and the modulating voltage is 2.5 V, the modulation index is 50. Find the maximum deviation. Also find the modulation index if modulating signal is $e_m = 4.5 \sin 400\pi t$.

(a) 25 kHz, 225 (b) 25 kHz, 50
(c) 25 kHz, 45 (d) 25 kHz, 125

Solution (a) $\delta = m_f \times f_m = 50 \times 500 = 25$ kHz

$kf_C = \frac{\delta}{E_m} = \frac{25}{2.5} = 10$ kHz/V

case (ii) $\delta_{\text{new}} = kf_C\,(E_{m2}) = 10 \times 4.5 = 45$ kHz

and $m_f = \frac{45 \times 10^3}{200} = 225$.

10. The equation of an FM signal is $e = 6 \sin(10^6 t + 4 \sin 10^3 t)$ then the modulating frequency is

(a) 1000 Hz (b) 500 Hz
(c) 300 Hz (d) 159 Hz

Solution (d) $f_m = \frac{10^3}{2\pi} = 159$ Hz.

11. In the above question number 10 modulation index is

(a) 4 (b) $\frac{4}{2\pi}$
(c) $\frac{4}{\pi}$ (d) none of these

Solution (a) $m_f = 4$.

12. In question number 10, the power dissipated in 50 Ω resistor is

(a) 0.18 W (b) 0.8 W
(c) 0.36 W (d) 1.8 W

Solution (c) $P = \frac{E^2_{\text{rms}}}{R} = \frac{\left(6/\sqrt{2}\right)^2}{50} = \frac{18}{50} = 0.36$ W.

13. A piece of *RC* – 59A/U coaxial cable has a 75 Ω characteristic impedance and a nominal capacitance of 64 pF/m. The inductance per meter is

(a) 0.36 μH (b) 0.56 μH
(c) 0.16 μH (d) none of these

Solution (a) $Z_0 = \sqrt{\frac{L}{C}}$ or $L = Z_0^2 C$

$= (75)^2 \times 64 \times 10^{-12} = 0.36\,\mu$H.

14. A coaxial cable has 75 Ω characteristic impedance and diameter of the inner conductor is 0.584 mm and dielectric constant of the insulation is 2.23 then the outer diameter is

(a) 5.77 mm (b) 4.77 mm
(c) 3.77 mm (d) 2.77 mm

Solution (c) $Z_0 = \frac{138}{\sqrt{k}} \log \frac{D}{d}$

or $\log \frac{138}{\sqrt{k}} \log \frac{D}{d} = \frac{75\sqrt{2.23}}{138} = 0.81$

or $D = 3.77$ mm.

15. Find the characteristic impedance of an air-dielectric parallel-wire line.

(a) 83 Ω (b) 103 Ω
(c) 288 Ω (d) 377 Ω

Solution (a) $Z_{0\min} = 276 \log 2 = 276\,(.3010) = 83\,\Omega$.

16. If the core and cladding have a refractive index 1.5 and 1.45 respectively then find the acceptance angle.

(a) 13° 16′ (b) 23° 16′
(c) 33° 16′ (d) 29° 33′

Solution (b) $n_0 \sin\theta_a = n_1 \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}$

that is, $\sin\theta_a = 1.5 \sqrt{1 - \left(\frac{1.45}{1.50}\right)^2} = 0.395$

or $\theta_a = 23° \; 16'$.

17. Mean optical power launched into an 8 km fiber is $120\,\mu$W and mean output power is $4\,\mu$W then the find overall attenuation.

(a) 14.77 dB (b) 16.77 dB
(c) 3.01 dB (d) none of these

Solution (a) $10 \log \frac{120}{4} = 10 \log 30$

$= 10(1.4771) = 14.77$ dB.

18. Find attenuation per km in dB in the above question.

(a) 1.623 dB/km
(b) 1.846 dB/km
(c) 1.686 dB/KM
(D) 1.886 dB/KM

Solution (b) $\frac{14.77}{8} = 1.846$ dB/km.

Problems for Practice

1. In a communication system, noise is most likely to affect the signal
(a) at the transmitter
(b) in the channel or in the transmission line
(c) in the information source
(d) at the receiver

2. Which of the following is false? Modulation is used to
(a) separate different transmission
(b) reduce the bandwidth
(c) allow the use of practicable antenna
(d) ensure transmission to long distances

3. For efficient transmission and reception the height of antenna should at least be
(a) $\lambda/2$ (b) λ (c) $\lambda/3$ (d) $\lambda/4$

4. The channel band width allotted to FM radio is
(a) 100 kHz (b) 150 kHz
(c) 200 kHz (d) 300 kHz

5. The increase in modulation index increases the band width in case of
(a) FM
(b) AM
(c) vestigial sideband transmission
(d) none of these

6. Quantisation is done in which of the following cases
(a) PCM (b) PAM
(c) PPM (d) PWM
(e) all of these

7. The optimum frequency for a cable is
(a) 2.5 GHz (b) 3 GHz
(c) 10 GHz (d) 300 GHz

8. The digital transmission is mostly achieved via
(a) telephone links (b) satellite links
(c) computer links (d) none of these

9. Modems are
(a) A/D converters (b) D/A converters
(c) both (a) and (b) (d) none of these

10. The bandwidth in PDM is determined from
(a) minimum pulse width
(b) maximum pulse width
(c) position of pulses
(d) none of these above

11. If n_1 and n_2 are refractive index of core and cladding then
(a) $n_1 = n_2$ (b) $n_1 > n_2$
(c) $n_1 < n_2$ (d) no confirmed relation

12. The waves used in telecommunication are
(a) IR (b) UV
(c) microwave (d) cosmic rays

13. In AM the complete information can be transmitted using
(a) carrier and both the sidebands
(b) carrier and one side band
(c) only carrier
(d) only one sideband

14. The amplitude modulated current is given by $i = 125\,[1 + 0.6 \sin 2900t] \sin 10^6 t$. The depth of modulation is
(a) 60% (b) 6%
(c) 36% (d) none of these

15. The audio signal used to modulate $60 \sin(2\pi \times 10^6 t)$ is $15 \sin 300\pi t$. The depth of modulation is
(a) 50% (b) 40%
(c) 25% (d) 15%

16. In an FM system a 7 kHz signal modulates 108 MHz carrier so that frequency deviation is 50 kHz. The carrier swing is
(a) 7.143 (b) 8
(c) 0.71 (d) 350

17. The frequency which will pass through a D-region having refractive index 0.5 and electron density 400cm^{-2} is
(a) 400.2 kHz (b) 303.23 kHz
(c) 207.3 kHz (d) none of these

18. Super solar cycle repeats after every
(a) 11 years (b) 50 years
(c) 100 years (d) 1 year

19. If the probability of occurrence of message is $p = \frac{1}{8}$ then the amount of information conveyed is
(a) 4 bits (b) 8 bits
(c) 3 bits (d) 1 bit

20. The Shannon-Hartley theorem states that
(a) redundancy is essential
(b) only binary codes may be used
(c) the maximum rate of information transmission depends on the depth of modulation
(d) the maximum rate of information transmission depends upon the channel bandwidth

21. The number of AM broadcast stations that can be accommodated in a 300 kHz band width for the highest modulating frequency 15 kHz will be
(a) 10 (b) 12 (c) 15 (d) 8

22. The band width required by an AM signal is equal to
(a) f_C (b) f_m (c) $2f_m$ (d) $f_m/2$

23. A carrier is modulated simultaneously by 3 sine waves of modulation indices 0.3, 0.4 and 0.45 respectively, the net modulation index is
(a) 1.15 (b) 0.67 (c) 0.57 (d) none of these

24. The total power in AM transmission is
(a) $P_{\text{carrier}}\,(1 + m_a^2)$ (b) $P_{\text{carrier}}\left(1+\frac{m_a^2}{2}\right)$
(c) $P_{\text{carrier}}\left(1+\frac{m_a^2}{4}\right)$ (d) $P_{\text{carrier}}\,(1 + 2m_a^2)$

25. Vestigial sideband transmission is used in
(a) radio transmission (AM)
(b) TV transmission (AM)
(c) radio transmission (digital)
(d) TV transmission (digital)

26. An FM signal with a deviation δ is passed through a mixer and has its frequency reduced 5 fold. The deviation in the output of the mixer is
(a) δ (b) $\delta/5$ (c) 2.5δ (d) 5δ

27. An FM signal with a modulation index m_f is passed through a frequency tripler. The wave in output of the tripler will have a modulation index of
(a) $3m_f$ (b) m_f (c) $m_f/3$ (d) $9m_f$

28. In PM modulation index is
(a) directly proportional to modulating amplitude
(b) inversely proportional to modulating frequency
(c) directly proportional to modulating frequency
(d) inversely proportional to modulating amplitude

29. The saving in power in case of SSB suppressed carrier over DSB full carrier with 100% modulation is
(a) 50% (b) 33.33%
(c) 66.67% (d) 83.3%

30. Superheterodyne receiver replaced tuned frequency receiver because the latter suffered from
(a) inadequate selectivity at high frequencies
(b) gain variation over frequency
(c) instability
(d) insufficient gain and sensitivity

31. A superheterodyne receiver with an IF = 455 kHz is tuned to 1250 kHz. The image frequency is
(a) 795 kHz (b) 1705 kHz
(c) 1910 kHz (d) 2160 kHz

32. The ratio of maximum to minimum frequency for a medium frequency broadcast receiver is
(a) 2 : 1 (b) 2.4 : 1
(c) 2.8 : 1 (d) 3.4 : 1

33. A receiver has poor IF selectivity and, therefore, it will also have
(a) poor sensitivity
(b) double spotting
(c) the variable selectivity
(d) poor padder capacitor

34. In an AM broadcasting system, modulation index is very unlikely to cross
(a) 100% (b) 70% (c) 80% (d) 50%

35. In a short circuited line, the first current node lies at a distance of
(a) $\lambda/4$ from the short circuit point
(b) $\lambda/2$ from the short circuit point
(c) λ from the short circuit point
(d) $3\lambda/4$ from the short circuit point

36. Characteristic impedance of free space is given as
(a) $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ (b) $\sqrt{\mu_0\varepsilon_0}$ (c) $\sqrt{\frac{\varepsilon_0}{\mu_0}}$ (d) $\frac{\mu_0}{\varepsilon_0}$

37. The main component of atmosphere responsible for absorption of electromagnetic waves is
(a) N_2
(b) water vapours
(c) N_2 and water vapour
(d) O_2 and water vapour

38. During night the ionosphere consists of
(a) *D* and *E* layers (b) *E* and *F* layers
(c) *D*, *E* and *F* layer (d) only *F* layer

39. Critical frequency for E_2 layer is
(a) 1 – 10 MHz (b) 3 – 30 MHz
(c) 10 – 30 MHz (d) 5 – 12 MHz

40. Which of the following frequencies cannot be used for reliable communication beyond horizon without use of repeaters?
(a) 15 MHz (b) 900 MHz
(c) 5 GHz (d) 25 kHz

41. When microwave signals propagate along the curvature of the earth, this effect is called
(a) ducting
(b) tropospheric scatter
(c) ionospheric reflection
(d) faraday effect

42. A PIN diode is used as a
(a) microwave detector
(b) microwave switch
(c) mixer
(d) a metal semiconductor point contact diode

43. Out of the following, which has the maximum band width?
(a) twin line pair (b) twisted pair
(c) coaxial cable (d) none of these

44. If the pulse repetition frequency is 300, the maximum unambiguous range will be
(a) 500 km (b) 500 nmi
(c) 5000 m (d) 500 radar mile

45. The height of a TV antenna is 200 m. The population density is 4000 per km^2. Find the population benefitted.
(a) 3.2×10^5 (b) 3.2×10^6
(c) 3.2×10^7 (d) 3.2×10^8

46. Which of the following system is analog modulation system?
(a) PCM (b) differential PCM
(c) delta (d) PWM

47. Which of the following is not a PTM?
(a) PDM (b) PWM (c) PPM (d) PAM

48. Compounding is used
(a) to protect small signals in PCM from quantising distortion
(b) in PCM transmitters to allow amplitude limiting in the receiver
(c) in PCM receivers, to overcome impulse noise
(d) to overcome quantising noise in PCM

49. An on-line, real time data transmission system is most likely to require a circuit that is
(a) simplex (b) semi duplex
(c) duplex (d) time-shared

50. The main disadvantage of PCM is
(a) its incompatibility with TDM
(b) the complex circuitry required
(c) the large bandwidth required
(d) the high error rate during quantisation

51. The most common modulation system used for telegraphy is
(a) PWM (b) FSK
(c) PCM (d) delta modulation

52. Satellite earth-stations possessing only receive capabilities are used for
(a) telephony and telegraphy
(b) only meteorology
(c) only television reception
(d) only telephony

53. In India each field in TV has number of lines
(a) $312\frac{1}{2}$ (b) 625 (c) 825 (d) 525

54. The frequency of each field in India is
(a) 25 Hz (b) 30 Hz
(c) 50 Hz (d) 60 Hz

55. To design high power transmitters for radio, we use ——— at the last stage amplifier
(a) tunnel diode (b) transistor
(c) power transistors (d) vacuum tubes

56. The modulation index of an AM wave is changed from 0 to 1 the transmitted power is
(a) unchanged (b) halved
(c) doubled (d) increases by 50%

57. When the modulation index of an AM wave is doubled the antenna current is also doubled. The AM system being used is
(a) A3H (b) A5C (c) A3J (d) A3

58. Which of the following frequencies cannot be used for reliable beyond the horizon terrestrial communications without repeaters?
(a) 20 kHz (b) 15 MHz
(c) 900 MHz (d) 12 GHz

59. UHF range frequencies propagate by means of
(a) sky waves (b) ground waves
(c) surface waves (d) space waves

60. In electromagnetic waves polarisation is
(a) caused by reflection
(b) due to transverse nature of the waves
(c) due to longitudinal nature of the waves
(d) is always vertical in an isotropic medium

61. Which of the following is suitable for low power microwave oscillators only?
(a) tunnel (b) LSA
(c) gunn (d) IMPATT

62. The output from a laser is monochromatic, this means that it is
(a) IR (b) polarised
(c) narrow beam (d) single frequency

63. Communication lasers are used with optical fibers, rather than in open links, to
(a) ensure that beam does not spread
(b) prevent atmospheric interference
(c) prevent interference due to other lasers
(d) ensure that people are not blinded by them

64. Ruby laser differs from Ruby maser in that the former
(a) does not require pumping
(b) needs no resonator
(c) produces much lower powers
(d) is an oscillator

65. Identical telephone numbers in different parts of a country are distinguished by their
(a) language digits (b) access digits
(c) area codes (d) central office codes

66. To permit selection of 1 out of 16 equiprobable events, the number of bits required is
(a) 2 (b) $\log_{10}16$
(c) 4 (d) 8

67. Quantisation error occurs in
(a) time division multiplex
(b) frequency division multiplex
(c) PCM
(d) PWM

68. Which of the following represents carrier wave amplitude?
(a) $E_{\max} - E_{\min}$ (b) $\frac{E_{\max} + E_{\min}}{2}$
(c) $E_{\max} + E_{\min}$ (d) $\frac{E_{\max} - E_{\min}}{2}$

69. Which of the following represents modulating wave amplitude?

(a) $\frac{E_{\max}+E_{\min}}{2}$ (b) $\frac{E_{\max}-E_{\min}}{2}$

(c) $E_{\max}+E_{\min}$ (d) $E_{\max}-E_{\min}$

70. The antenna current of an AM transmitter is 10A when only carrier wave is transmitted and increases to 11A when carrier is sinusoidally modulated. The percentage modulation is

(a) 42%

(b) 21%

(c) 63%

(d) none of these

71. $e = 20\ [1 + .5 \sin 10^3 \times 2\pi t] \sin 10^6 \times 2\pi t$ is an AM signal then the LSB has frequency

(a) $(10^6 - 10^3)$Hz

(b) $(10^6 + 10^3)$Hz

(c) 10^3Hz

(d) $(10^6 - 10^3)$Hz to $(10^6 + 10^2)$Hz

72. $e = [24 + 8 \sin 10^3\pi t] \sin 10^6\pi t$ is an AM signal. The percentage modulation is

(a) 33.3% (b) 80.1%

(c) 12.8% (d) 28.3%

73. 500 Watt carrier is modulated to a depth of 60%. The total power is

(a) 590 W (b) 690 W

(c) 410 W (d) none of these

74. The antenna current of an AM broadcast transmitter modulated by 50% is 11A. Find the carrier current.

(a) 10.35A (b) 9.25A

(c) 10A (d) 5.5A

75. A transmitter radiates 9 kW power with only carrier and 10.125 W with modulated carrier. The depth of modulation is

(a) 25% (b) 50%

(c) 12.5% (d) none of these

76. A transistor class-C amplifier has a maximum permissible collector dissipation 20 W and collector efficiency is 75%. Find the maximum modulated carrier power.

(a) 42.7 W (b) 28.1 W

(c) 21.1 W (d) none of these

ANSWERS

1. (b)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (a)
11. (b)	**12.** (c)	**13.** (d)	**14.** (a)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (b)	**20.** (d)
21. (a)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (a)	**28.** (a)	**29.** (d)	**30.** (d)
31. (d)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (a)	**37.** (d)	**38.** (b)	**39.** (d)	**40.** (c)
41. (a)	**42.** (a), (b)	**43.** (c)	**44.** (a)	**45.** (c)	**46.** (d)	**47.** (d)	**48.** (a)	**49.** (c)	**50.** (c)
51. (b)	**52.** (c)	**53.** (a)	**54.** (a)	**55.** (d)	**56.** (d)	**57.** (c)	**58.** (d)	**59.** (d)	**60.** (b)
61. (a)	**62.** (d)	**63.** (c)	**64.** (d)	**65.** (c)	**66.** (c)	**67.** (c)	**68.** (b)	**69.** (b)	**70.** (a)
71. (a)	**72.** (a)	**73.** (a)	**74.** (a)	**75.** (b)	**76.** (a)				

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Experiments

Experiment 1

Vernier Callipers

It is a device used to measure the internal and external diameter and depth of a calorimeter or beaker or any hollow object. It was invented by a French mathematician Pierre Vernier and so is called Vernier Callipers.

Least Count (L C) or Vernier Constant (V C)

It is the minimum distance that can be measured accurately with the vernier scale. It is calculated as the difference between the value of one Main Scale Division (M S D) and one Vernier Scale Division (V S D)

$$1\text{ M S D} = 1\text{ mm}$$

(10 V S D Coincide with 9 M S D)

$$10\text{ V S D} = 9\text{ M S D}$$

$$1\text{ V S D} = \frac{9}{10}\text{ M S D}$$

$$\text{L C} = 1\text{ M S D} - 1\text{ V S D}$$

$$= 1\text{ M S D} - \frac{9}{10}\text{ M S D} = \frac{1}{10}\text{ M S D}$$

$$\frac{1}{10} \times 1\text{ mm} = \frac{1}{10}\text{ mm} = 0.1\text{ mm} = 0.01\text{ cm}$$

In general LC can be calculated:

If n V S D coincide with (n–1) M S D.

Then

$$n\text{ V S D} = (n-1)\text{M S D}$$

$$1\text{ V S D} = \frac{n-1}{n}\text{ M S D}$$

$$\text{L C/V C} = 1\text{ M S D} - 1\text{ V S D} = 1\text{ M S D} - \left(\frac{n-1}{n}\right)\text{ M S D.}$$

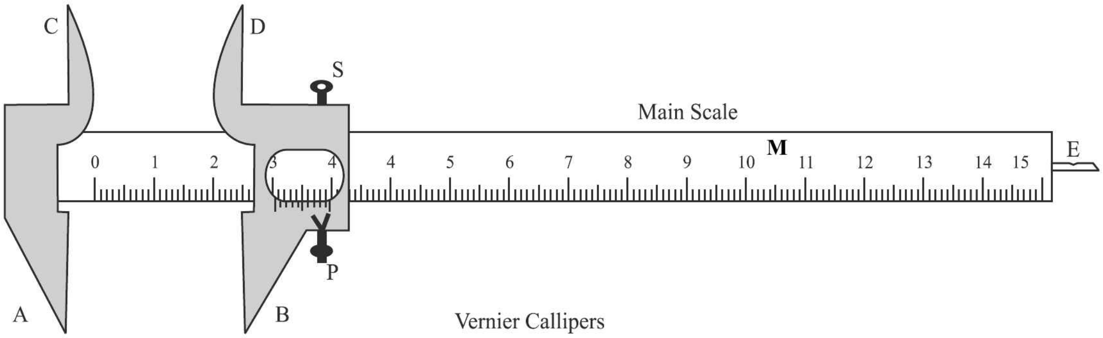
$$= \frac{1}{n}\text{ M S D} = \frac{\text{Smallest division of main}}{\text{No. of division on VS}}$$

Construction

It consists of mainly three parts (1) Main scale (2) Vernier scale (3) A thin metallic strip

(1) **Main scale:** It is made up of a steel metallic strip M, graduated in cm and mm at one edge and in inches, and tenth of an inch on the other edge on the same side. It has two fixed jaws A and C which one fixed at right angle to the main scale.

(2) **Vernier scale:** It slides on the metallic strip M. It can be fixed at any point by screw S. It is also graduated on both the sides, the side which slide over the main scale has 10 divisions over a length of 9 mm. It contains two moveable jaws B and D fixed at right angle to the main scale. The object whose length or external diameter is to be calculated is held between the lower jaws A and B, while the internal diameter of a hollow object like calorimeter beaker etc., can be measured by using the upper jaws C and D.



Vernier Callipers

(3) **A thin metallic strip:** A thin metallic strip E is connected to the vernier scale from the back side of the main scale which moves along with vernier scale. It is used to measure the depth of the hollow object.

Zero error:

On bringing the jaws of vernier callipers in contact with each other. If zero of vernier scale does not coincide with the zero of main scale then instrument is said to possess a zero error. It is of two types:

(1) Positive zero error

(2) Negative zero error

Positive zero error: If zero of the vernier scale lies to right of the zero of main scale then error is called positive zero error. It can be calculated by noting the main scale reading N on left of zero of vernier scale and also the vernier division x coincide with any M S division then zero error = + (N + x × V C) and zero correction = – (N + x × V C).

Negative zero error: If the zero of the vernier scale lies to the left of this zero of the main scale then error is called Negative zero error. It is added to the observed reading. It can be calculated by nothing down the division of V S coincide with any of the main scale. If x division of vernier scale coincide with main scale the zero error = – (10 – x × V C). Suppose x = 8 Then zero error = – 2 × V C and zero correction = + 2 × V C.

Total Reading of Vernier Callipers: If with the body between the jaws, the zero of vernier scale lies ahead of Nth division of main scale then main scale reading (M S R) = N

If nth division of vernier scale coincide with any division main scale then vernier scale reading (V S R) = n × (LC)

Total Reading = M S R + V S R

= N + [n × (V C)]

In the total reading, zero error can be added or subtracted depending upon its nature, we get the correct final reading.

Procedure

Used to measure internal diameter:

(1) Put the hallow body in lies upper jaws C and D without any undue pressure as shown in figure below.

(2) Note down the reading as explained above

***Used to measure the external diameter*:**

(1) Put the solid body between the lower jaws A and B and shown below in figure

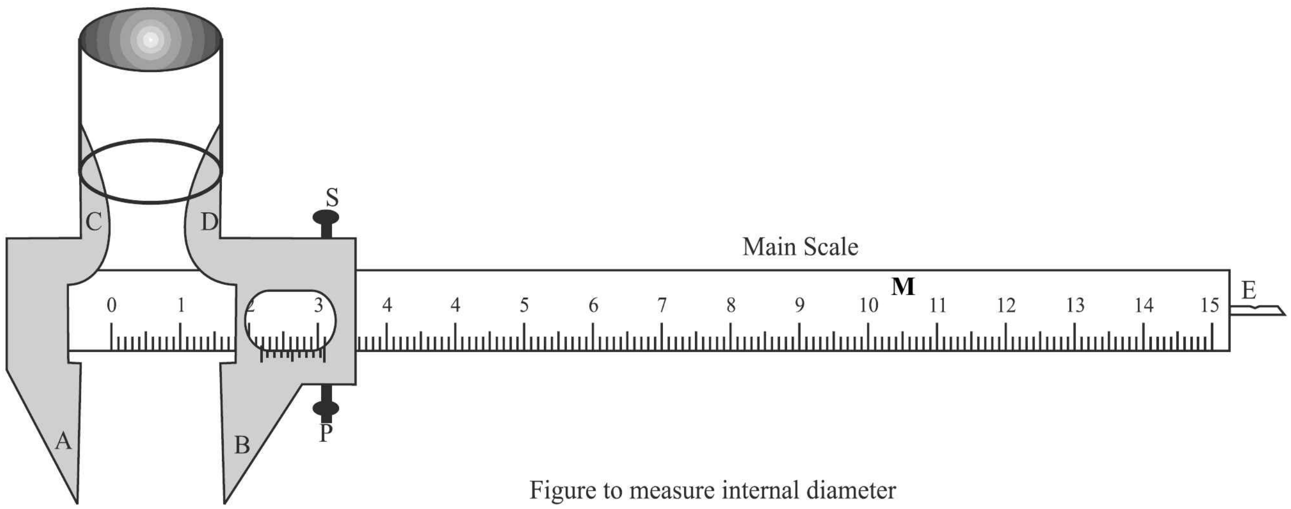


Figure to measure internal diameter

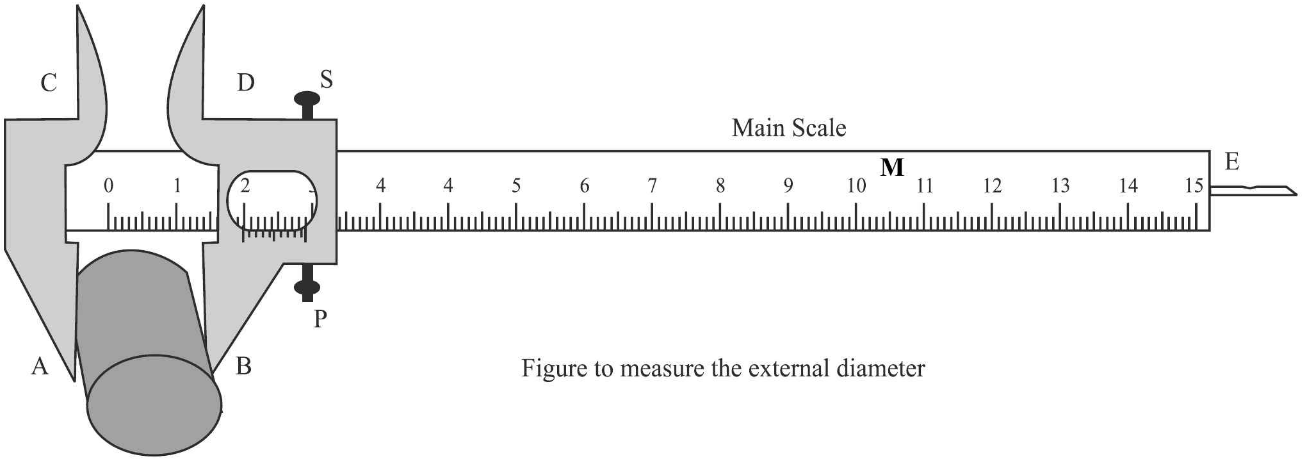


Figure to measure the external diameter

(2) Note down the reading as explained

Used to measure the depth:

(1) Keep the right edge of the main scale strip M on the Upper edge of the beaker

(2) Press the jaws BD down ward till the metallic strip touch the bottom of beaker as shown below in the figure.

(3) Note down the reading as explained.

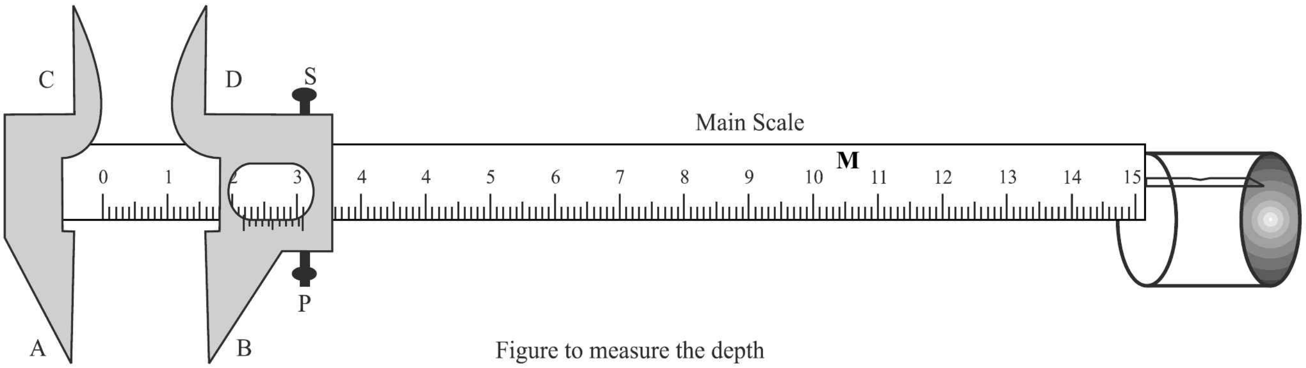


Figure to measure the depth

Problems for Practice

1. Find the wrong statement.
- (a) The positive zero error is added to the observed reading and negative zero error is subtracted from the zero error.
- (b) The vernier constant is the least count of the instrument.
- (c) Vernier constant is defined as the difference between the value of smallest main scale division and the value of one vernier division.
- (d) The vernier scale is always moveable.

2. A vernier calliper has 20 divisions on the vernier scale which coincide with 19 on the main scale. The least count of the instrument is 0.1 mm. The main scale divisions are of
- (a) 0.5 mm
- (b) 1 mm
- (c) 2 mm
- (d) $\frac{1}{4}$ mm

3. Negative zero error is ______ total reading.
- (a) Added in
- (b) Subtracted in
- (c) Multiplied in
- (d) Divided by

4. Positive zero error is ______ total reading.
- (a) Added in
- (b) Subtracted in
- (c) Multiplied in
- (d) Divided by

5. Vernier callipers is named so because it was invented by
- (a) Max Vernier
- (b) Jack Vernier
- (c) Pierre Vernier
- (d) R. Vernier

6. The side of a cube is measured by vernier calliper in which 10 division of VS coincide with 9 division of MS and 1 MS is 1 mm. The main scale read 10 mm and first division of vernier scale coincide with MS. Mass of cube is 2.736 gm. The density of cube is
- (a) 2.66 gm/cm^3
- (b) 2.6 gm/cm^3
- (c) 2.7 gm/cm^3
- (d) 2.70 gm/cm^3

7. In a vernier calliper the smallest division on the main scale is 0.5 mm, while the vernier scale has 50 divisions. Its vernier constant is
- (a) 0.001 mm
- (b) 0.01 mm
- (c) 0.1 mm
- (d) 1.00 mm

8. Find the correct statement:
- (a) The least count of vernier callipers is the length of the smallest unit on the main scale.
- (b) The least count of vernier callipers are length of the smallest unit on the vernier scale.
- (c) The upper jaws of vernier calliper are used to measures the internal diameter of the cylinder.
- (d) Vernier calliper can not measure the depth of a cylinder.

9. In a vernier calliper, smallest division on the main scale is 1 mm, while the vernier scale have 20 divisions. When fixed jaw to touches with moveable jaw, zero of vernier scale lies to the right of zero of the main scale and 15th division of vernier scale. The type of zero error and its value is
- (a) Negative, 0.75 mm
- (b) Negative 0.15 mm
- (c) Positive, 0.75 mm
- (d) Positive 0.15 mm

10. The specifications of two vernier callipers are given as

(1) 9 M S D = 10 V S D and 1 M S D = 0.1 cm
(2) 19 M S D = 20 V S D and 1 M S D = 0.05 cm
(3) 49 M S D = 50 V S D and 1 M S D = 0.05 cm

Which of the above is more accurate?
- (a) 1
- (b) 2
- (c) 3
- (d) All the above

11. What part of vernier callipers is called vernier scale?
- (a) The one that is fixed
- (b) The one that is moveable
- (c) A thin metallic strip
- (d) None of above

12. Zero correction is
- (a) Same as zero error
- (b) Opposite in sign to the zero error
- (c) Sometime same sometime opposite
- (d) None of above

13. If nth division of main scale coincides with (n + 1)th division of vernier scale given one main scale division is equal to x unit. The least count of the callipers is

(a) $\frac{x}{x+1}$ (b) $\frac{x+1}{n}$

(c) $\frac{x}{n}$ (d) $\frac{x}{x-1}$

14. The smallest division on the main scale of a vernier callipers is 1 mm and 10 V S D coincide with 9 M S D while measuring the diameter of sphere. The zero of V S lies between 2 and 2.1 cm and 6th division of V S coincide with any of main scale. The diameter of the sphere is

(a) 1.26 cm

(b) 2.26 cm

(c) 22.6 cm

(d) 2.26 mm

15. The main scale of a vernier callipers reads in millimeter and its V S is divided into 10 equal parts which coincide with a divisions of the M S. When two jaws touch each other the 6th division of the vernier scale coincide with a M S and zero of the vernier lies to the right of zero of main scale. When a body is placed between its two jaws the zero of the vernier scale lies slightly to the left of 2.8 cm and 8th V D coincide with a M S. The measured length of the body is

(a) 2.72 cm (b) 2.84 cm

(c) 2.78 cm (d) 2.8 cm

16. The least count of vernier callipers is 0.1 mm. The main scale reading which coincides with zero of vernier scale is 10 and zero the division of vernier scale coincide with main scale. If each main scale division is 1 mm, the measured value is

(a) 10 cm (b) 1 cm

(c) 0.1 cm (d) 2 cm

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (a)	**7.** (b)	**8.** (c)	**9.** (c)	**10.** (b)
11. (b)	**12.** (b)	**13.** (a)	**14.** (b)						

Experiment 2

Screw gauge: It is used to determine thickness/diameter of thin sheet/ wire. It is called a screw gauge because it is based upon the principle of a screw and is used to measure the gauges of thin sheets. If a screw is related in a nut having evenly spaced threads than in addition to the circular motion of the screw there is linear motion of the screw head in the forward or backward direction along the axis of the screw. The linear distance moved by the screw divided by the number of rotations given to the screw is called pitch of the screw gauge. It is also given by the distance between two consecutive threads. A circular scale is fixed on the one end of the screw which has 100 equal divisions. When we divide pitch by 100 we get the least count i.e., the minimum value that can be accurately measured by screw gauge.

$$\text{Pitch} = \frac{\text{Distance moved on linear scale}}{\text{No. of rotation given to screw}}$$

$$\text{LC/VC} = \frac{\text{Pitch}}{\text{No. of divisions on the circular scale}}$$

e.g., if pitch = 1 mm

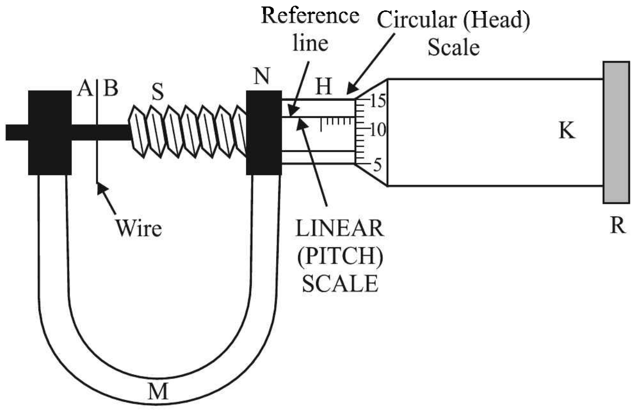
No. of divisions on circular scale = 100

$$\text{LC / VC} = \frac{1\text{mm}}{100} = 0.01 \text{ mm} = 0.001 \text{ cm}$$

Diagram:

Screw gauge is made up of a U-Shaped metal piece M. At one end of this, there is a fixed metal piece A, called stud. The other end of this piece N carries a cylindrical hub H. On this hub along its axis, a line is drawn which is called as reference line. On this line graduations are in millimeter or half millimeter are done depending upon the pitch of screw. This is called linear scale or main scale or pitch scale.

Figure



A hollow cylindrical cap marked K is fixed with the end of the screw which is capable of rotating over the hub when screw is rotated. A scale is marked on this cap, which is called circular scale or vernier scale. The object whose diameter/thickness is to be calculated is placed between the stud and screw of the screw gauge.

Zero error and zero correction

(1) If zero of circular scale exactly coincides with the reference line on main scale, the zero error is nil.

(2) If zero of circular scale lies below the reference line of main scale then error is +ve error and zero correction is –ve. e.g., if 2 division of circular scale coincide with reference line the zero error = 2 × L C mm and zero correction = – 2 × L C mm.

(3) If zero of circular scale lies below the reference line of main scale then error is –ve error and zero correction is +ve, e.g., if 97 division of circular scale coincides with reference line then zero error = – (100–99) × L C mm = – 3 × L C mm and zero correction = + 3 × L C mm

Back lash error

Due to loose fitting or mishandling or wear and tear there is some gap between the nut and screw. In such situations if a screw is advanced in one direction then it is rotated back the screw may not move along the axis. This type of error is called back lash error. To avoid this error the screw should be rotated in only one particular direction.

Total Reading

Let the body is placed between faces A and B, the edge of the callipers ahead of Nth division of linear scale or main scale. The main scale reading is (MSR) = N × pitch. If nth division of circualr scale coincides with reference line then vernier scale reading is (VSR) = n × L C mm

Total Reading = MSR + VSR = N × pitch + n × L C

Zero error and zero correction must be added or subtracted as explained.

Procedure

Thickness of thin sheet

(1) Take a plane paper. Make it many folds so that we get 16 pages.

(2) Place this folded paper over face A and move the face B toward A using the racket head R, stop when R slips without moving the screw.

(3) Note down the uncovered reading on the main scale. This reading is called Main Scale Reading (MSR) as N.

(4) Note down the number of divisions on the vernier scale (circular scale) coinciding with reference line. It gives Vernier Scale Reading (VSR).

(5) Repeat the above process to take 5-6 readings at different positions of paper.

(6) Find the mean of all readings.

(7) Dividing the means by 16 we get the thickness of one paper sheet.

Diameter of wire

(1) Place the wire over the face A and move the face B such that the racket head R slips.

(2) Note down no of division on the main scale. It gives MSR

(3) Note down the vernier scale coinciding with reference line. It gives Vernier Scale Reading (VSR)

(4) Find the total reading by applyingv zero error and zero correction

(5) Take mean of different values of diameter

Problems for Practice

1. The screw advances by 1 mm on 4 complete rotations and number of division on the circular scale is 250. The least count of screw gauge is

(a) 0.001 mm (b) 0.0001 cm
(c) 1 mm crometer (d) All of the above

2. If a screw gauge advances by 2 mm on 4 complete rotation given to it. Pitch of the screw gauge is

(a) 1 mm (b) 0.5 mm
(c) 2 mm (d) 1.5 mm

3. In a screw gauge, the value of one division on the linear scale is 1mm, while the circular scale has 100 divisions. While the screw touches the stud without any object, the zero on circular scale advances 17 divisions beyond the reference line. The type and amount of zero error is

(a) Positive, 0.17 mm
(b) Negative 0.17 mm
(c) Positive, 0.017 mm
(d) Negative 0.017 mm

4. In a screw gauge, the value of one division on the main scale is 0.5 mm, while the circular scale have 100 divisions. The least count is

(a) 0.05 mm (b) 0.005 mm
(c) 0.0005 mm (d) 0.00005 mm

5. Which of the following is most accurate instrument for measuring length?

(a) A vernier calliper having 20 divisions on the vernier scale which coincide with 19 division one the main mm scale.
(b) A screw gauge having pitch 1 mm and 50 divisions one the circular scale.
(c) A vernier scale of least count 0.01 mm.
(d) A screw gauge of least count 0.001 mm.

6. A screw gauge having 100 equal divisions and pitch of length 1 mm is used to measure the diameter of a wire of length 5.6 cm. The main scale reading is 1mm and 47th circular division coincide with the main scale. The surface area of the wire in cm^2 is

(a) 2.59 cm^2 (b) 2.5 cm^2
(c) 2.6 cm^2 (d) 2.60 cm^2

7. Positive error in a screw gauge is ________ total reading.

(a) added in
(b) subtracted in
(c) multiplied in
(d) divided by

8. Negative error in a screw gauge is -------- total reading.

(a) added in (b) subtracted in
(c) multiplied in (d) divided by

9. Find the correct statement.

(a) Least count of screw gauge is equal to its pitch.

(b) With the help of ratchet an object can be lightly held between the studs of screw.

(c) Zero correction is same as that of zero error.

(d) The pitch of the screw gauge depends on the number of divisions on the circular scale.

10. Find the correct statement.

(a) Least count of screw gauge depends only on the number of circular scale divisions.

(b) The screw gauge with a pitch 0.5 mm is more precise than the instrument with a pitch a pitch of 1 mm. If both have the same number of circular division.

(c) In a positive error the zero of main scale coincide with zero of vernier scale.

(d) In a positive zero error the zero of circular scale lies below the reference line.

11. Find the incorrect statement.

(a) The screw gauge is also called micrometer.

(b) The backlash error can be avoided by giving forward and backward movements repeatedly.

(c) The screw gauge with a pitch of 0.5 mm is more precise than the instrument with a pitch of 1 mm if both have the same number divisions.

(d) The pitch of screw gauge depends on the number of divisions on the circular scale.

12. The pitch of a screw gauge is 1 mm and there are 50 divisions on its cap. When noting is put in between the stud, 44th division of the circular scale coincides with the reference line when a glass plate is placed between the studs, the main scale reads three divisions and circular scale reads 26 divisions. The thickness of plate is

(a) 3.64 cm (b) 2.64 mm

(c) 2.64 cm (d) 3.64 mm

ANSWERS

1. (d) **2.** (b) **3.** (c) **4.** (b) **5.** (d) **6.** (c) **7.** (b) **8.** (a) **9.** (b) **10.** (b)

11. (b) **12.** (b)

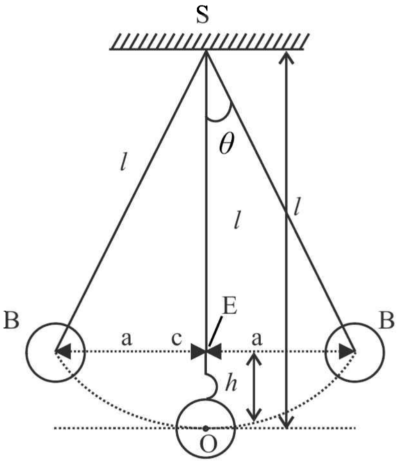
Experiment 3

Simple pendulum: Dissipation of energy by plotting a graph between square of amplitude and time

Simple pendulum

An ideal simple pendulum is a heavy point mass (a metallic bob) tied to one end of weightless, flexible and perfectly inextensible string about which it can vibrate. The point from where it is suspended is called point of suspension and point about which it oscillate is called point of oscillation. The distance between the point of oscillation and point of suspension is called effective length (lenght of thread + length of hook + radius of bob) of the pendulum.

Figure



Time taken by the bob of simple pendulum to complete on vibration is called time period. It is denoted by T and given by $T = 2\pi\sqrt{\frac{l}{g}}$

l = effective length of pendulum, g = acceleration to gravity

Seconds pendulum: The simple pendulum whose time period is 2 sec is called 2nd pendulum. It length is 99.7 cm.

Energy of simple pendulum: The energy of oscillation of simple pendulum is the change in the energy of bob from it lowest position to extreme position. If a simple pendulum of mass m and length l swing in a small angular amplitude θ, then bob of simple pendulum rises by h height from the mean position.

Form fig from Δ ESA

$$\frac{ES}{SA} = \cos\theta$$

$$\frac{l-h}{l} = \cos\theta$$

$$l - h = l\cos\theta$$

$$h = l - l\cos\theta$$

$$h = l(1-\cos\theta) = 2l\sin^2 \theta/2 = 2l\frac{\theta^2}{4}$$

$$h = l\frac{\theta^2}{2}$$

As a is the linear amplitude of the oscillation

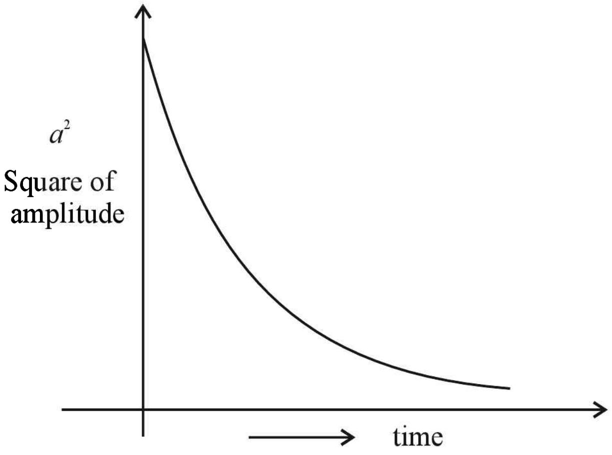
$$\theta = \frac{a}{l}$$

$$\therefore \quad h = l\frac{a^2}{2l^2} = \frac{a^2}{2l}$$

The energy of oscillation of simple pendulum is given by

$$E = mgh = mg\frac{a^2}{2l} = \frac{mg}{2l}a^2$$

Figure



The energy of oscillation of simple pendulum is directly proportional to the square of amplitude of vibration. As amplitude goes on decreasing, energy also goes on decreasing. The graph between square of amplitude and time is as shown below:

Procedure

(1) Find the least count of stop watch.
(2) Tie one end of a long thread about 150 cm to the hook attached with the bob.
(3) Pass the thread through two halves of the cork cut along its length. Hold the cord firmly in a clamp stand such that its slit is perpendicular to the edge of the table.
(4) Draw two mutually perpendicular lines on the floor with a piece of paper at point O and mark points on either side at a gap of 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm, etc.
(5) Adjust the position of bob by sliding the stand so that the bob lies at point O centre of line .
(6) Set the pendulum into oscillation beyond 7 cm.
(7) When the amplitude of vibration decreases to 7 cm, start the stop watch.
(8) Note the time for which amplitude decrease to 6 cm, 5 cm, 4 cm, 3 cm, 2 cm, 1 cm etc.
(9) Plot the graph between square of amplitude and time period and study the dissipation of energy.

Problems for Practice

1. If a girl is swinging in a swing and another sits with her, the time period will
(a) decrease (b) increase
(c) reason same (d) None of above

2. A simple pendulum is vibrating in a evacuated chamber, it will oscillate with
(a) increasing amplitude
(b) constant amplitude
(c) decreasing amplitude
(d) first (c) then (a)

3. The length of seconds pendulum is
(a) 98 cm (b) 99.7 cm
(c) 100 cm (d) None of above

4. What effect occurs on the frequency of the pendulum, if is taken from the earth susface to deep into a mine?
(a) increases
(b) decreases
(c) first increases then decreases
(d) None of above

5. To show that a simple pendulum execute simple harmonic motion, it is necessary to assume that
(a) length of pendulum is small
(b) amplitude of oscillation is small
(c) acceleration due to gravity is small
(d) mass of pendulum is small

6. The property of a string used as suspension of simple pendulum is
(a) it should be weightless
(b) it should be inextensible
(c) it should be flexible
(d) all the above

7. Time period of simple pendulum is 2 sec if its length increased 4 times, then its time period will become
(a) 12 sec (b) 8 sec
(c) 16 sec (d) 4 sec

8. A simple pendulum is executing simple harmonic motion with a time period T. If the length of the pendulum is increased by 21% the percentage increase in the time period of the pendulum is
(a) 21% (b) 10%
(c) 30% (d) 50%

9. In a simple pendulum the period of oscillation T is related to the length of the pendulum l as
(a) $\frac{l}{T}$=constant (b) $\frac{l^2}{T}$ = constant
(c) $\frac{l}{T^2}$ = constant (d) $\frac{l^2}{T^2}$ = constant

10. A simple pendulum of length l has a brass bob attached to its lower end and its time period is T. If a steel bob of same size having density x times that of brass, replaces the brass bob and length is changed so that time period becomes $2T$, then new length is

(a) $2l$ (b) $4l$

(c) $4lx$ (d) $4\frac{l}{x}$

11. If the length of simple pendulum is increased by 300% then the time period will be increased by

(a) 100%

(b) 200%

(c) 300%

(d) 400%

12. If the metal bob of a simple pendulum is replaced by a wooden bob, then its time period will

(a) Increase

(b) Decrease

(c) Remain the same

(d) First increase then decrease

13. A simple pendulum executing Simple Harmonic Motion is falling freely along with the support, then

(a) its periodic time decreases

(b) its periodic time increases

(c) it does not oscillate at all

(d) None of these

14. If the length of the seconds pendulum on the surface of earth is 1 m. The length of seconds pendulum on the surface of moon where g is $\frac{1}{6}$ th the value of g on the surface of earth is

(a) 6 m (b) $\frac{1}{6}$ m

(c) $\frac{1}{36}$ m (d) 36 m

15. Damping effect the amplitude

(a) Linearly

(b) Exponentially

(c) Inversally

(d) Square of amplitude

16. Slope of curve between amplitude and time provides

(a) acceleration

(b) speed

(c) velocity

(d) total energy

17. The relation between time period and length of pendulum is

(a) $T = 2\pi\frac{l}{g}$ (b) $T = 2\pi\sqrt{\frac{l}{g}}$

(c) $T = 2\pi\frac{l^2}{g}$ (d) $T = 2\pi\frac{l^2}{g^2}$

18. Plot of total energy with displacement of an dissipation loss simple pendulum must be

(a) Straight line parallel to displacement axis

(b) Parabola with its foci on energy axis

(c) Straight ling parallel to energy axis

(d) Parabola with its foci on displacement axis

19. Total energy of a metallic bob during its motion is

(a) $\frac{1}{2}mu + \frac{mga^2}{2l}$ (b) $\frac{1}{2}mu^2 + \frac{mga^2}{2l}$

(c) $\frac{1}{2}mu^2 + \frac{mga}{2l}$ (d) $\frac{1}{2}mu + \frac{mga}{2l}$

ANSWERS

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (b)
11. (a)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (b)	

Experiment 4

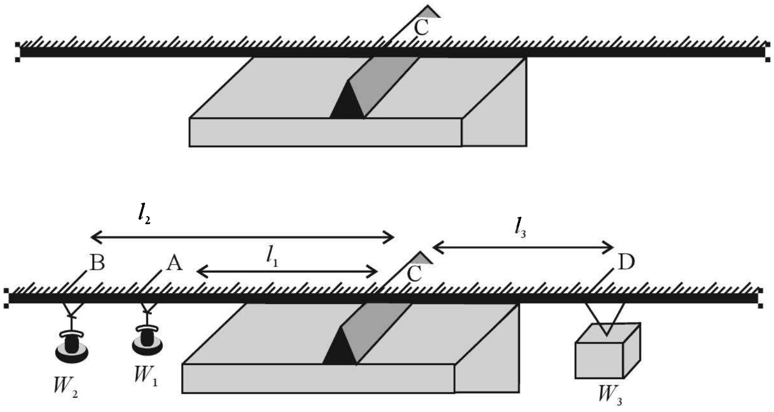
Metre scale: Mass of given object by principle of moments.

Principle of moments

A lever is a rigid bar (metre scale), straight on bend, able to oscillate or rotate about a fixed point. This fixed point is called fulcrum. The lever works on the principle of moment's i.e., when the lever is in equilibrium, the algebraic sum of the moments of all the forces acting on it, about the fulcrum is zero.

When the metre rod is supported at its centre of gravity C on kinfe edge, then moment of weight over the metre rod is zero and meter rod well be in equilibrium. If the weights W_1, and W_2 are suspended one the left side of C (the CG of rod) at a distances l_1 and l_2, then these weights will produce anti-clockwise moment and the rod will rotate in anticlock wise direction. If another weight W_3 is suspended on the right side of as shown in figure.

Figure



then it will produce clock wise moment. By adjusting the distance l_3 of weight W_3 from C, bring the system again in equilibrium. At equilibrium, sum of clockwise moments is equal to the sum of anticlockwise moments

i.e., $W_3 l_3 = W_1 l_1 + W_2 l_2$

$$W_3 = \frac{W_1 l_1 + W_2 l_2}{l_3}$$

Knowing W_1 W_2 l_1 l_2 l_3 W_3 (weight) of unknown body can be calculated.

Procedure

(1) Balance the meter rod on the sharp wedge and find the position C of the centre of gravity.

(2) Suspend two known weight W_1 and W_2 at A and B on left side of CG and unknown weight W_3 at D on the right side of CG.

(3) Adjust line position of W_3 so that meter scale again get balanced on the knife edge.

(4) Note line positions of W_1 W_2 and W_3 on the metre rod and find the weight W_3 by above relation.

Problems for Practice

1. **Let m_1 and m_2 be the mass of body and mass of weight used and l_1 and l_2 be the distance of their loops form wedge then according to principle of moments**

(a) $m_1 g l_1 = m_2 g l_2$ (b) $m_1 g l_2 = m_2 g l_1$

(c) $m_1 = \frac{m_2 l_2}{g l_1}$ (d) $m_1 = \frac{m_2 g l_2}{l_1}$

2. A physical balance works on the principle of

(a) Inertia

(b) Moments

(c) Energy conservation

(d) Mass conservation

3. The principle of moment is also known as

(a) Bernoullis theorem (b) Hamiltons theorem

(c) Stocke's theorem (d) Varignons theorem

4. If the origin of coordinate system lies at the centre of mass, the sum of moments of the masses of the system about the centre of mass

(a) may be greater then zero

(b) may be less then zero

(c) may be equal to zero

(d) is always zero

5. Find the incorrect statements.

(a) The mass of an object comes into action only when body is in the state of motion.

(b) The gravitational mass of a body in independent of its motion.

(c) The mass of an object is measured by comparing its mass with a standard mass.

(d) According to Newton, gravitational and inertial masses are identical.

ANSWERS

1. (a) **2.** (b) **3.** (d) **4.** (d) **5.** (a)

Experiment 5

Youngs Modulus of elasticity of the material of a metallic wire

Youngs Modulus of elasticity: It is defined as the ratio of normal stress to longitudinal strain. It is denoted by Y.

$$\therefore Y = \frac{\text{Normal stress}}{\text{Longitudinal strain}}$$

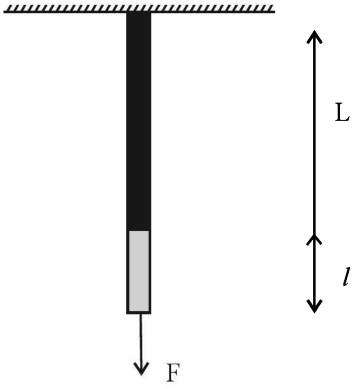
Consider a wire or a rod of length L and area of cross section a is fixed at one end as in figure. Suppose a normal force F is applied to its free end of the rod and its length increases by l, then normal stress $= \frac{F}{a}$ and longitudinal strain $= \frac{l}{L}$

$$\therefore Y = \frac{FL}{al} = \frac{FL}{\pi r^2 l} = \frac{mgL}{\pi r^2 l}$$

Units of Young's Modulus are N/m^2 in SI units and dyne/cm^2 in CGS system.

Hence, to find the Young's Modulus of a material wire, some weight is suspended on it to produce extension which is measured and Y is calculated.

Figure



Procedure

(1) Remove the kinks in the two wires if any by pulling the wires length wise between two wooden pieces.

(2) Find the least count and zero correction of screw gauge.

(3) Measure the diameter of wire at four/five places in two mutually perpendicular direction at every position.

(4) Find the pitch and least count of micrometer attached to the Searl's apparatus as shown in figure

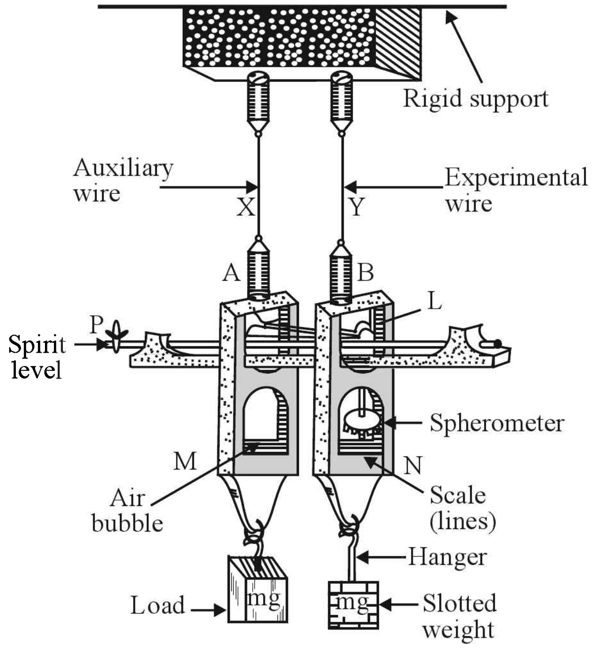
(5) Find the breaking load of the wire which can be calculated by

Breaking load = Breaking stress × area of cross section [**Breaking stress** Brass = 34 – 38 kg/mm^2,
Copper = 30– 35 kg/mm^2, Iron 36 – 40 kg/mm^2
Steel 78 – 82 kg/mm^2]

(6) Add a fixed load on side where micrometer is not attached and hanger on other side. Bring the bubble of spirit level in the centre by moving the micrometer screw. Note the micrometer reading.

(7) Add a weight of half kilogram in a hanger and wait for some time and bring the air bubble at the centre by moving micrometer reading.

Figure



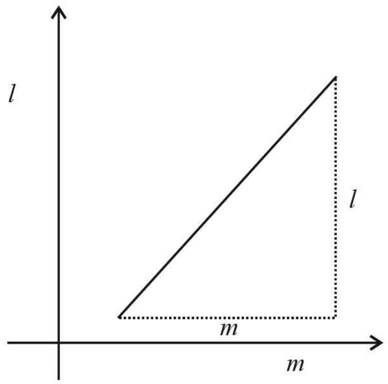
(8) Take at least six observation by increasing the load in 500 gm and noting the micrometer reading

(9) Now reduce the load by removing 500 gm weight every time and note the reading of micrometer in each observation. Move the micrometer screw always in the same direction to avoid backlash error.

(10) Measure the length of the wire with a meter rod from the point of suspension to the point of attachment with frame B

(11) Plot a graph between mass (m) along x-axis and extension (l) along y-axis. The graph obtained is as:

Figure



(12) Calculate $Y = \frac{gL}{\pi r^2}\left(\frac{m}{l}\right) = \frac{gl}{\pi r^2}\left[\frac{1}{\text{slope of graph}}\right]^l$

(13) Compare the calculated value with actual value from table. Find error and percentage error.

Problems for Practice

1. Which has more Young's Modulus, a thin steel wire or thick steel wire?

(a) thin (b) thick
(c) both have same (d) None of above

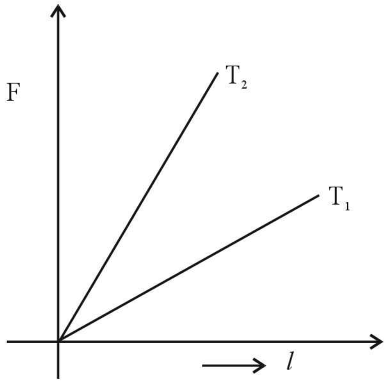
2. SI units of Young's modulus is

(a) dyne/cm^2 (b) N/m
(c) N/m^2 (d) N.m.

3. An Indian rubber cord of L metre long and area of cross-section A m^2 is suspended vertically. Density of rubber is ϑ kg/m^3 and Young's Modulus of rubber is Y N/m^2. If the cord extend by l metre under its own weight, then extension l is

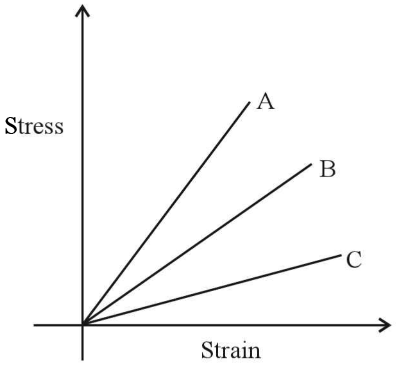
(a) $L^2 \vartheta g/2y$ (b) $L^2 \vartheta g/y$
(c) $L^2\vartheta g/4y$ (d) $y/L^2 \vartheta g$

4. The diagram show the change in length (l) of a thin uniform wire caused by the application of stress F at two different temperature T_1 and T_2. The variation show that



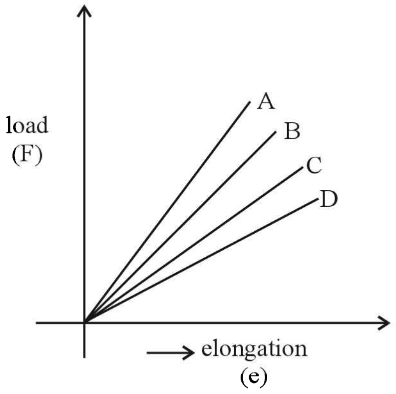
(a) $T_1 > T_2$ (b) $T_1 < T_2$
(c) $T_1 = T_2$ (d) None of above

5. The strain stress curves of three wires of different material as shown in fig are A B C are the elastic limits of the wire. The figure shows that



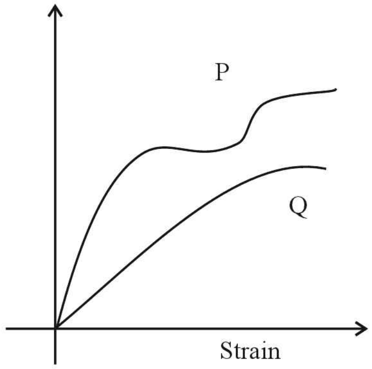
(a) Elasticity of wire A is maximum
(b) Elasticity of wire B is maximum
(c) Elasticity of wire C is maximum
(d) None of above is true

6. The load verses elongation graph for four wires of the same material is as shown in figure, which of the wire is thickest



(a) A (b) B
(c) C (d) D

7. The figure shows stress verses→ elongation strain curve for two materials P and Q, from the curve we know that



(a) P is brittle but Q is ductile
(b) P is ductile but Q is brittle
(c) both P and Q are ductile
(d) both P and Q are brittle

8. The Young's Modulus of a perfect plastic material is

(a) infinite (b) zero
(c) finite (d) one

9. The Young's Modulus of a perfect rigid material is

(a) zero (b) one
(c) infinite (d) None of these

10. The breaking force of a wire of radius 1mm is 10N. The breaking force for a wire of radius 2 mm is

(a) 80 N (b) 40 N
(c) $\frac{10}{3}$ (d) 30

11. In an experiment of determination of Young's Modulus of the material of a wire, the length as well as the radius of the wire is double. The Young's Modulus of wire

(a) becomes double
(b) becomes four times
(c) remain unchanged
(d) becomes sixteen times

12. Within elastic limit the ratio of normal stress and longitudinal strain is known as

(a) modulus of elasticity (b) youngs modulus

(c) modulus of rigidity (d) bulk modulus of elasticity

13. Two wires one of copper and other of steel are of same length and diameter. All reined and same force is applied at their one end. The combined length is increased by 0.5 cm. The wire will have

(a) different strain and same stress

(b) same stress and same strain

(c) different stress, and different strain

(d) different stress and same strain

14. An increase is 0.2% in the length of a wire of cress-section 10^{-6} m^2 cause a tension of 2000 N in the wire. The Youngs Modulus of the material of the wire is

(a) 10^{11} N/m^2 (b) 10^{12} N/m^2

(c) 10^{10} N/m^2 (d) 10^{9} N/m^2

15. The length of a wire is increase by 2 mm under a given load. The increase in length of wire of same material but of length and radius twice that of the first under the same load will be

(a) 0.25 m (b) 1 mm

(c) 8 mm (d) 4 mm

ANSWERS

1. (c) **2.** (c) **3.** (b) **4.** (b) **5.** (a) **6.** (a) **7.** (b) **8.** (b) **9.** (c) **10.** (b)
11. (c) **12.** (b) **13.** (a) **14.** (b) **15.** (b)

Experiment 6

Surface tension of water by capillary rise and effect of detergents.

Surface Tension

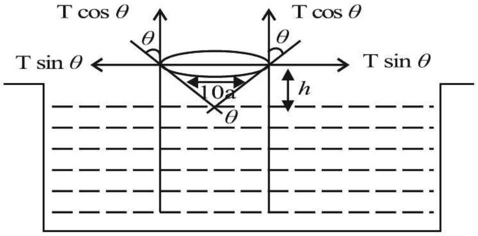
Surface tension is the property of a liquid by virtue of which it behaves like an elastic stretched membrane with a tendency to contract so as to occupy a minimum surface area. It is measured as the force per unit length on an imaginary line drawn on the liquid surface acting perpendicular to it on either side at every point and tangentially to the liquid surface

$$\text{Surfaec Tension} = \frac{\text{Total force on either side of imaginary line}}{\text{length of the line}}$$

Units: SI N/m CGS dyne/cm

When a capillary tube (a glass tube of very fine bore) is dipped into the liquid which wets the tube, that liquid rises in it, where as the liquid which does not wet the tube fall into it. The phenomenon of rise or fall of liquid in the capillary tube is called capillary action or capillarity. This is due to the surface tension of liquid. It surface tension is T, then for a tube of radius r the upward force on total liquid molecules becomes $2\pi r$ T cos θ, θ is the angle of contact. Due to this upward force the liquid starts rising in the tube till the weight of the liquid raised become equal to this force.

Figure

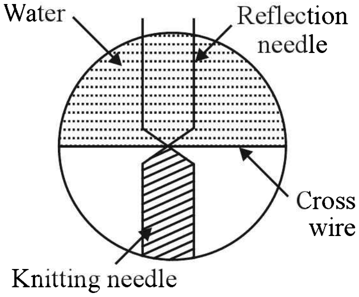


Total weight of liquid raised upward

$$\pi r^2 h\rho g + \frac{1}{3}\pi r^3 hg$$

$$= \pi r^2\left(h + \frac{r}{3}\right)\rho g \qquad \text{.... (1)}$$

Figure



Total upward force due to surface tension

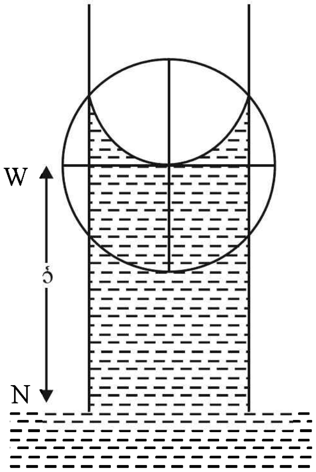
$$= 2\pi r\, T\cos\theta \qquad \text{.... (2)}$$

When water stops rising these two forces are equal

$$2\pi r T\cos\theta = \pi r^2\left(h + \frac{r}{3}\right)\rho g$$

$$T = \frac{\left(h + \frac{r}{3}\right) r\rho g}{2\cos\theta}$$

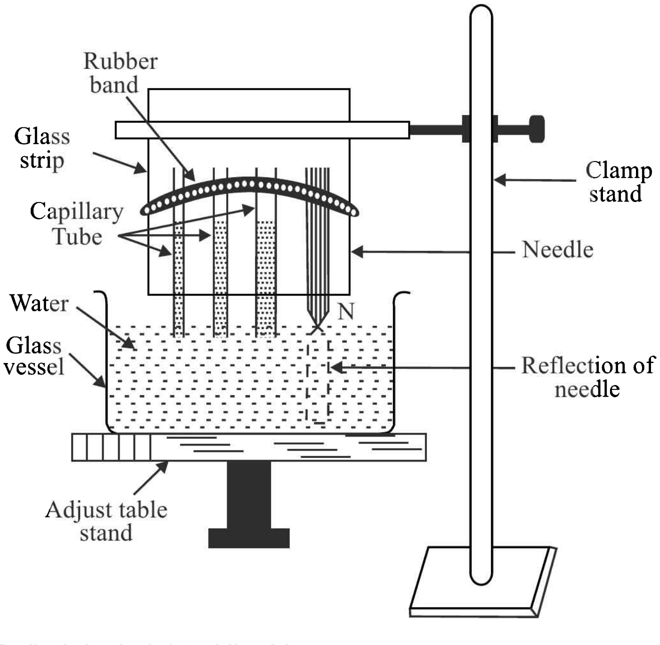
Figure



Knowing the values of all parameters the surface tension can be calculated easily

Procedure

Figure



To find the height of liquid:

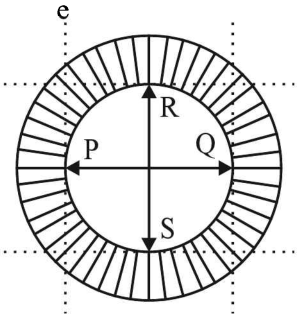
(1) Fill the glass dish with water and place it on the adjustable stand.

(2) Mount the capillary tube and needle on capillary tube holder and fix it in clamp stand keeping the tube vertical.

(3) Put the capillary tube in water and adjust the needle such that it just touches the surface of water in the dish by moving adjustable stand.

(4) Now focus the microscope on the tip of the needle and note the reading on the vertical scale of the travelling microscope. Now move the vertical scale upward such that horizontal cross-wire of the microscope lies tangentially the water meniscus in the tube and note down the reading again. The difference between the two give the height. Repeat for other tubes.

To find the diameter of capillary tube:



(1) Place line capillary tube on the adjustable stand in horizontal position. Focus the microscope on the hole of one of the capillary tubes. Adjust the microscope by moving horizontally so that vertical cross-wire touch the point P. Note the reading of microscope.

(2) Now move the microscope in horizontal direction to let it touch the same vertical cross wire at Q of the tube and note down the reading.

(3) Similarly read the value on the vertical scale at point R and S.

(4) Difference of two give the diameter.

(5) Repeat the process for others tubes also.

Effect of detergents

It is clear from the relation that the height to which the liquid rises in a capillary tube is directly proportional to the surface tension of liquid. The surface tension of pure water is high and on adding the detergent in water, the surface tension of water decreases. Therefore, the height to which liquid rises in the capillary tube decreases. Further, if concentration of detergent is more in liquid then it will risen to less.

Problems for Practice

1. The surface tension of pure water is
   (a) 32 dyne/cm (b) 72 dyne/cm
   (c) 60 dyne/cm (d) None of above

2. How does the surface tension vary with temperature?
   (a) It decreases, with rise in temperature
   (b) It increases, with rise in temperature
   (c) Temperature has no effect on it
   (d) None of these

3. Why edges of glass become round on heating?
   (a) due to viscosity (b) due to surface tension
   (c) both a and b (d) None of these

4. Surface tension is
   (a) gravitational effect
   (b) a proton effect
   (c) an atomic phenomenon
   (d) a molecular phenomenon

5. If a capillary tube is dipped into the liquid inside and outside are same than angle of constant is

(a) 0° (b) 90°
(c) 45° (d) 30°

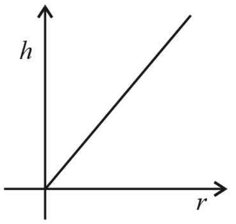
6. Find the correct statement.

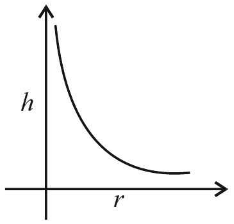
(a) Due to surface tension all the liquids rise in a capillary tube.
(b) Surface tension increases with temperature.
(c) Surface tension acts only at the free surface of the liquid and air.
(d) Only liquids possess surface tension.

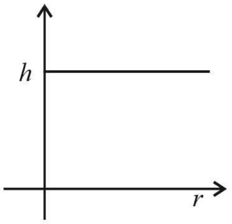
7. Find the incorrect statement.

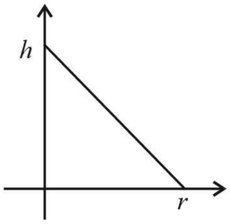
(a) If the capillary tube is not properly cleaned then the height of rise of liquid is less.
(b) The surface tension of liquid decreases with impurity contents.
(c) Where water risen in a capillary tube the radius of the meniscus is equal to the radius of the capillary tube.
(d) The rise of liquid in the narrow tube is greater and the depression of liquid in the wider tube is greater.

8. The correct curve between the height or depression *h* of liquid in a capillary tube and its radius is

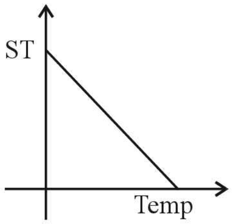
(a)


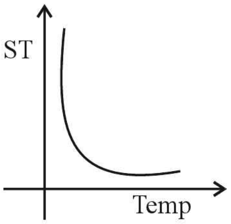
(b)


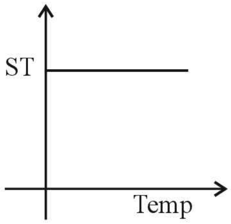
(c)


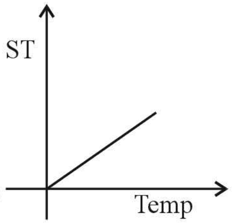
(d)


9. Which graph represents the variation of surface tension over small temperature ranges for water?

(a)


(b)


(c)


(d)


10. Find the incorrect statement.

(a) The molecules on the free surface of a liquid always possess a greater potential energy as compared to the molecules in the body of the liquid.
(b) It the glass surface is lightly oiled then the angle of contact between water and glass become obtuse.
(c) Only liquids possess surface tension.
(d) Surface tension acts only at the free surface of liquid and air.

11. When a capillary tube is dipped in a liquid, water rises by h_1 when inner surface is coated with wax, rise is h_2 than

(a) $h_2 = h_1$ (b) $h_2 > h_1$
(c) $h_2 < h_1$ (d) None of above

12. In a surface tension experiment with a capillary tube water rises upto 0.1 m. If the same experiment is repeated in an artificial satellite, which is revolving around the earth; water will rise in the capillary tube upto a height of

(a) 0.2 m (b) 0.1 m
(c) Full length of tube (d) 0.98 m

ANSWERS

1. (b) **2.** (a) **3.** (b) **4.** (d) **5.** (b) **6.** (d) **7.** (d) **8.** (a) **9.** (b) **10.** (d)
11. (c) **12.** (c)

Experiment 7

Coefficient of viscosity of a given viscous liquid by measuring terminal velocity of a given spherical body.

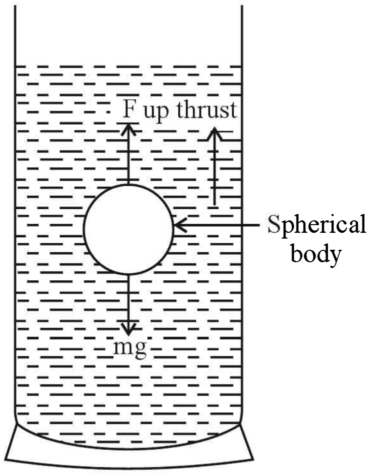
Viscosity

It is the property of liquids or (gases) due to which a back ward dragging force, a viscous drag, acts tangentially on the layers of the liquid in motion and it tries to stop the motion. When liquid flows over a solid surface, liquid layer in contact with the fixed surface does not move. Upper and upper layers move forward with increasing velocity. Due to relative motion of different layers a backward dragging force F acts tangentially to every layer which depends upon the area of two layers A and velocity gradient $\frac{dv}{dx}$. Then the viscous force is given by

$$F = -\eta A \frac{dv}{dx}$$

Negative sign is due to reason that force acts opposite to the velocity of liquid, η is constant called coefficient of viscosity when a spherical body of radius r falling through a large column of a viscous fluid of density ρ and coefficient of viscosity η. Let ρ be the density of material of body then following three forces act on the body:

Figure



(1) Weight of the body acting vertically downward $mg = \frac{4}{3}\pi r^3 \rho g$

(2) Upward thrust equal to the weight of the fluid displaced which is given by $U = \frac{4}{3}\pi r^3 \sigma g$

(3) Viscous force acting in upward direction (opposite to motion of body) $F = 6\pi \eta r \upsilon$.

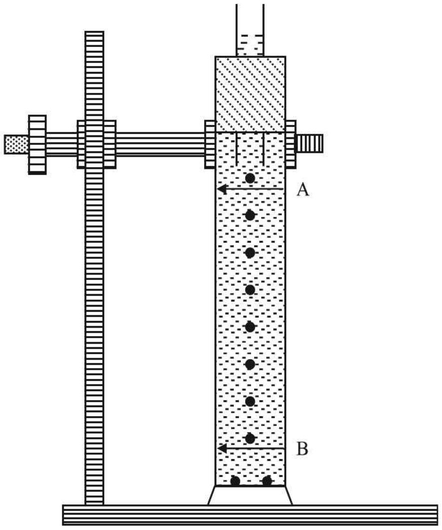
In equilibrium total upward force = Total downward from

$$\frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r\upsilon = \frac{4}{3} r^3 \rho g$$

$$\vartheta = \frac{2r^2(\rho - \sigma)g}{9\eta}$$

This equation give the terminal velocity of the body i.e., the constant velocity with which the body moves in equilibrium.

Figure



Procedure

(1) Find the least count and zero correction of screw gauge.

(2) Fix the rubber stopper in the open mouth of 1 m long glass cylinder and pass the glass tube in the hole of the stopper.

(3) Fill the glass cylinder with glycerine with the help of funnel and ensure that there is no air bubble in it. Record its temperature.

(4) Mark two points about 10 cm from above and second 10 cm from below.

(5) Take some lead shots or steel balls. Take one ball. Find its diameter (radius) and drop it into the glass tube.

(6) After falling some distance in the glycerine the ball acquires the terminal velocity. When the ball reaches the upper mark start stop watch and stop when it reaches the lower time. Note down the time.

(7) Repeat the step 5 and 6 with different balls.

(8) Note the temperature of glycerine.

(9) Plot a graph between r^2 and υ slope of graph give $\left(\upsilon/_{r^2}\right)$. Hence calculate η.

Problems for Practice

1. In which liquid, the terminal velocity of a body will have lesser value—benzene or honey?

(a) honey (b) benzene
(c) same in both (d) None of these

2. SI units of Coefficient of viscocity is

(a) poise (b) decapoise
(c) centipoise (d) None of these

3. The relation between decapoise and poise

(a) 1 poise = 10 decapoise
(b) 1 decapoise =10 poise
(c) poise = decapoise
(d) both a and b

4. How does the viscosity of the liquid change with temperature?

(a) decrease with increase in temperature
(b) increase with increase in temperature
(c) no effect of temperature
(d) sometime decreases sometime increases

5. If radius of spherical body is doubled what will be effect on terminal velocity?

(a) remain same
(b) become double
(c) become 4 time
(d) become half

6. Name the historical experiment that made use of Stokes law and positive concept of terminal velocity.

(a) J.J. Thomson's experiment

(b) X. Scelleury experiment

(c) Millikan's Oil drop experiment

(d) All the above

7. The nature of graph between radius of spherical body (*r*) and its terminal velocity (*v*) is

(a) Straight line

(b) Parabola

(c) Ellipse

(d) Hyperbola

8. Why liquid taken must be transparent?

(a) ball must visible

(b) white liquids are good conducts

(c) both a and B

(d) None of above

9. The terminal velocity of a sphere moving through a viscous medium is

(a) inversely proportional to the radius of the sphere

(b) directly proportional to the radius of the sphere

(c) directly proportional to the square of the radius of sphere

(d) inversely proportional to the square of the radius of sphere

10. Viscosity of gases is

(a) About ten hundred times less than those of liquids

(b) About five hundred times less than those of liquids

(c) About twenty times less than those of liquids

(d) about hundred times less than those of liquids

11. If a small sphere is allowed to fall vertically in a large quantity of still liquid of density smaller than that of the material of the sphere

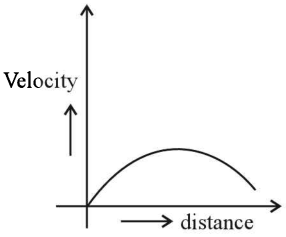
(a) At first its velocity increases but soon approaches a constant value

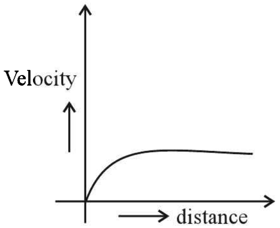
(b) It falls with a constant velocity all along from the very beginning

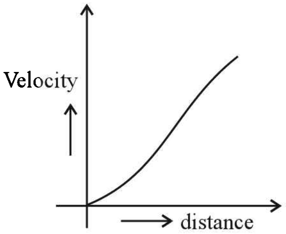
(c) At first it falls with a constant velocity, which after some time goes on decreasing

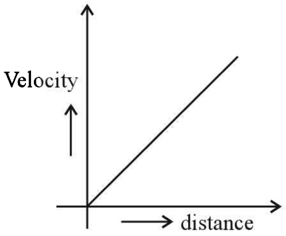
(d) Nothing can be said about the motion

12. A small spherical solid ball is dropped in a viscous liquid. The variation of its velocity with distance travelled in the liquid is best represented by

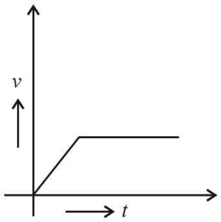
(a)


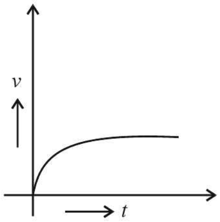
(b)


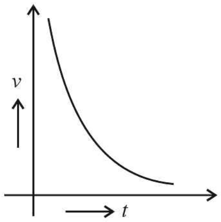
(c)


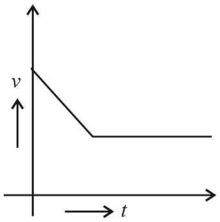
(d)


13. A small spherical solid ball is dropped from a great height in a viscous liquid. Its v-t graph in the liquid is best described by

(a)


(b)


(c)


(d)


ANSWERS

1. (a) **2.** (b) **3.** (b) **4.** (a) **5.** (c) **6.** (c) **7.** (b) **8.** (a) **9.** (c) **10.** (d)
11. (a) **12.** (b) **13.** (a)

Experiment 8

Plotting a cooling curve for the relationship between the temperature of the hot body and time.

Newton's law of cooling:

According to Newton's law of cooling the rate of loss of heat of a body is directly proportional to the temperature differences between the body and surroundings, provided this temperature difference is very small and other conditions remain same.

Consider a body at a temperature 'T' place in a surrounding at temperature 'To'. If $d\theta$ is the amount of heat lost by body in time 'dt' then

$$-\frac{d\theta}{dt}\times(T-To) \text{ or } \frac{d\theta}{dt}=-k(T-To)$$

Negative sign shows that a decrease in heat with an increase in time.

If dT is the change in temperature at a result of thin loss of heat.

$$d\theta = m\,s\,dT$$

$$\frac{d}{dt}(m\,s\,T) = -k(T-To)$$

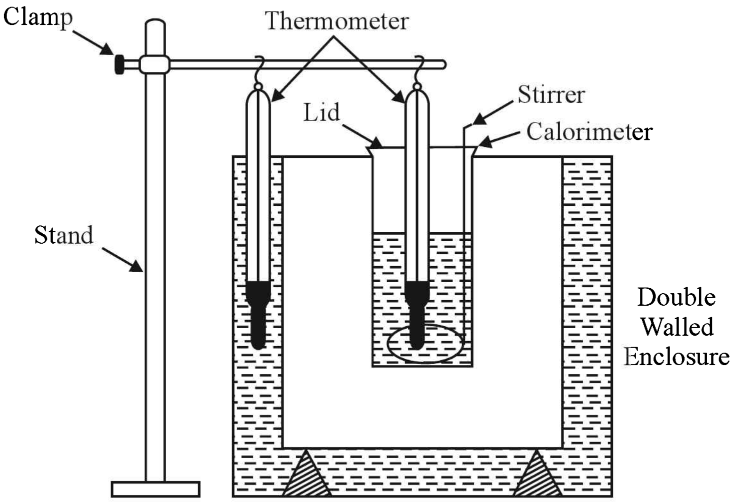
$$ms\frac{dT}{dt} = -k(T-To)$$

$$\frac{dT}{dt} = -\frac{k}{ms}(T-To) = -k(T-To)$$

Integrating we get

$\log(T - To) = -kt + c$. c is constant of integration graph between log $(T - To)$ and t is a straight line with –ve slope.

Figure



Procedure

(1) Find the least count of two thermometers.

(2) Note their readings corresponding to room temperature and add correction if any.

(3) Fill the space between its two walls with water which acts as surroundings at constant temperature.

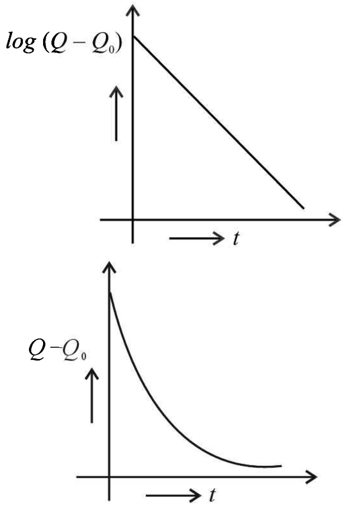
(4) Heat the water upto 70-80°C and put it into calorimeter so that 80% of calorimeter is filled.

(5) Place correct thermometer in the calorimeter and a stirrer, bulb of thermometer should be dipped into liquid and should not touch with the walls of calorimeter.

(6) Note the temperature of hot water first after 2 min then 3 min then 4 min because rate of cooling is fast earlier.

(7) Plot graph between $(Q - Qo)$ and t and In $(Q - Qo)$ and t. The graph obtained will be

Figure



Problems for Practice

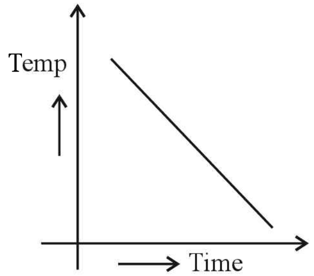
1. Newton's law of cooling is applicable if temperature difference between hot body and surrounding is
(a) less than 40°C
(b) greater than 40°C
(c) equal to 40°C
(d) None of these

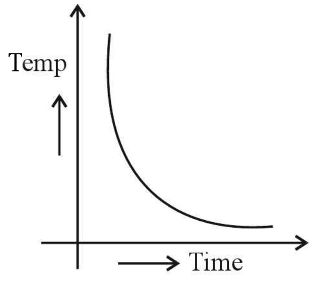
2. On what factor does the rate of loss of heat of body depend?
(a) Nature of surface of body
(b) Surface area of body
(c) Temperature of body and surroundings
(d) All of the above

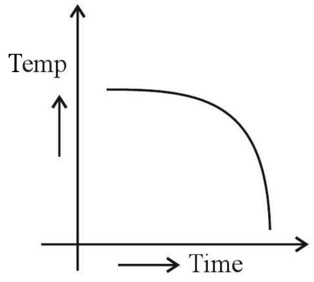
3. Why is the outer surface of calorimeter blackened?
(a) To increase the radiation loss
(b) To decrease the radiation loss
(c) To keep the temp of the hot water uniform
(d) None of above

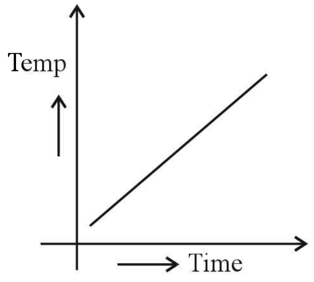
4. The mode of transfer of heat is
(a) Radiation
(b) Conduction
(c) Convection
(d) All of the above

5. A steel block heated to 100°C is left in a room to cool. The curve showing the correct cooling behaviour is

(a)


(b)


(c)


(d)


6. The colour of a star is due to its
(a) Size (b) Distance
(c) Temperature (d) Weight

7. As compared to a person with white skin another person with dark skin will experience.
(a) more heat and more cold
(b) less heat and less cold
(c) more heat and less cold
(d) less heat and more cold

8. A vessel contains hot water is left to cool in air. It cools from 90°C to 80°C in t_1 min from 80°C to 70°C in t_2 min and 70°C to 60°C in t_3 min, then
(a) $t_1 > t_2 > t_3$ (b) $t_1 < t_2 < t_3$
(c) $t_1 = t_2 = t_{3\backslash}$ (d) $t_1 > t_2 = t_3$

9. Which of the following statement is wrong
(a) A dull surface cools at a faster rate as compared to shining surface
(b) If a body is broken into small pieces that rate of cooling increases
(c) In Newton's law of cooling, the proportionality constant is a uniform constant
(d) Newton's law of cooling is an experimentally obtained law

10. Which of the following statement is correct?
(a) A body of low heat capacity cools slowly.
(b) In Newton's law of cooling, the proportionality constant in a universal constant.
(c) In Newton's law of cooling, the rate of heat loss involves all the three modes of heat conduction i.e., conduction, convection and radiation.
(d) The rate of cooling decreases as a substance approaches its meeting point.

11. Newton's law of cooling is a special case of
(a) Stefan's law (b) Boliz mann's law
(c) Wien's law (d) Planck's constant

12. Newton's law of cooling is used in lab for determination of
(a) specific heat of solids
(b) latent heat of solids
(c) specific heat of liquids
(d) latent heat of liquids

ANSWERS

1. (a) **2.** (d) **3.** (b) **4.** (a) **5.** (b) **6.** (c) **7.** (a) **8.** (b) **9.** (c) **10.** (d)
11. (a) **12.** (c)

Experiment 9

Speed of sound in air at room temperature using a resonance tube.

Theory: If l_1 and l_2 are the lengths of air column for first and second resonance length respectively and l is the end correction

$$l_1 + l = \frac{\lambda}{4};$$

$$l_2 + l = \frac{3\lambda}{4}$$

$$l_2 - l_1 = \frac{3\lambda}{4} - \frac{\lambda}{4} = \frac{\lambda}{2} \quad \lambda = 2(l_2 - l_1)$$

Velocity of sound is given by

$$V = v\ \lambda = 2\upsilon(l_2 - l_1)$$

Knowing $l_1 l_2$ and υ speed of sound can be calculated

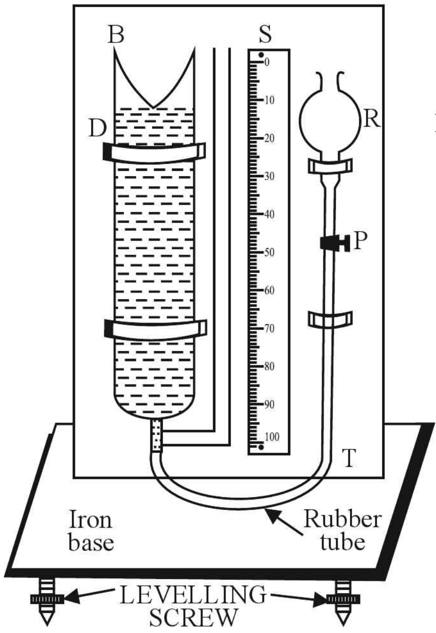
As $V \times \sqrt{T}$

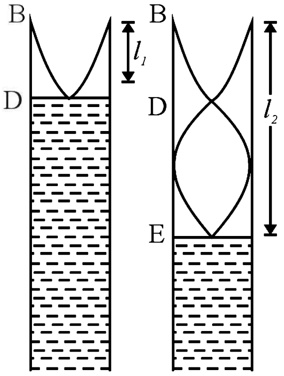
$$\frac{V_o}{V_t} = \sqrt{\frac{273}{273+t}}$$

$$V_o = V_t\sqrt{\frac{273}{273+t}}$$

Velocity calculated can be compared by measuring the room temperature.

Procedure





(1) Note down the room temperature.

(2) Set the resonance apparatus vertical with the help of plumb line by adjusting the levelling screws provided at the base of the apparatus.

(3) Fill a portion of the resonance tube and the reservoir with water. Check the working of the pinch cork so that there is no leakage of water.

(4) Open the pinch cork and raise the reservoir R.

(5) Add more water till water reaches the highest point in the tube. Close the pinch cork and than lower the reservoir keeping the pinch cork tight.

(6) Take a tuning fork of frequency V_1. Strike it against the rubber pad gently and place it just above the open end of the tube with care so that prongs of the tuning fork should not touch the tube.

(7) Open the pinch cork and allow the water level to fall gradually till the intensity of sound heard is maximum. At this stage close the pinch cork at once and read the water level on the scale. Record this reading of water level as first resonance when falling.

(8) Now open the pinch cork and allow the water level to fall 3 cm less than three times l_1. Close the pinch cork. Bring the vibrating tuning fork near the mouth of the tube and open the pinch cork, allow the water level to fall gradually till the intensity of sound heard is maximum and then close the pinch cork at once. Read the water level on the scale. Record this reading of water level as second resonance when falling.

(9) Now open the pinch cork and allow the water to fall by 3 cm and then close the pinch cork. Raise the reservoir to maximum height. Open the pinch cork so that water level in resonance tube starts increasing gradually. Note the water level at second resonance and first resonance when water level is rising.

(10) Repeat the above process three times.

(11) Take a set of three readings with another turning fork.

Problems for Practice

1. A turning fork produces
(a) Transverse wave
(b) Longitudinal wave
(c) Both (a) and (b)
(d) sometimes transverse sometimes longitudinal

2. In a resonance tube experiment strong resonance are observed at two successive lengths 0.40 m and 0.74 m above the surface of water. If the velocity of sound is 340 m/sec, then frequency of the turning fork is
(a) 128 Hz
(b) 500Hz
(c) 384 Hz
(d) 256 Hz

3. If an empty vessel is filled with water, its frequency
(a) decreases
(b) increases
(c) remains the same
(d) none of these

4. In a long cylindrical tube, the water level is adjusted and air column above it is made to vibrate in unison with a vibrating turning fork kept at the open end. Maximum sound is heard when air column length are equal to
(a) $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$
(b) $\frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$
(c) $\frac{\lambda}{4}, \frac{\lambda}{2}, \frac{3\lambda}{4}$
(d) $\frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}$

5. A resonance column of air contains
(a) Stationary longitudinal wave
(b) Stationary transverse waves
(c) Transverse progressive waves
(d) Longitudinal progressive waves

6. In an experiment for determination of velocity of sound by resonance tube methods using a tuning fork of frequency 512 Hz, first resonance was observed at 30.7 cm and record was obtained by 63.2 cm. The maximum possible error in velocity of sound (actual velocity of sound in air is 332 m/s is
(a) 80 cm
(b) 110 cm
(c) 58 cm
(d) 204 cm

7. A student determines the velocity of sound with the help of resonance tube apparatus. If the observed length for fundamental frequency is 24.70 cm the length of third harmonic will be
(a) 73.1cm
(b) 75.4 cm
(c) 74.1 cm
(d) 72.7 cm

8. If the temperature increase, then what happens to the resonance length
(a) decreases
(b) increases
(c) remains the same
(d) not definite

9. In a resonance column experiment, the first resonance is obtained when the level of water in the tube is at 25 cm from the open end. Resonance will also be obtained when the water level is at a distance of
(a) 40 cm form open end
(b) 75 cm from open end
(c) 60 cm from open end
(d) 90 cm from open end

10. In a resonance pipe the first and second resonances are obtained at depths 22.7 cm and 70.2 cm respectively. The end correction will be
(a) 10.5 cm
(b) 1.05 cm
(c) 2.05 cm
(d) 110.5 cm

11. The end correction of a resonance column is 1.0 cm. It the shortest length resonating with a tuning fork is 15.0 cm, the next resonating length is
(a) 47 cm
(b) 45 cm
(c) 30 cm
(d) 48 cm

12. Which of the following statement is true?
(a) Some frequencies of open and close pipes are common if they have the same length
(b) In case of open pipe ***nth*** overtone is always the (n+1)th harmonics
(c) Closed pipe contains only odd number of overtones while open pipe contains all overtones
(d) The fundamental frequency of closed pipe is greater than the fundamental frequency of open pipe if both give the same length

13. An empty vessel is partially filled with water then the frequency of vibration of air column in the vessel
(a) remains same
(b) decreases
(c) increases
(d) None of these

14. Which of the following statement is wrong?
(a) End correction in organ pipes is always positive
(b) The total end correction in a pipe is e = 0.6 D, where D is the diameter of the pipe
(c) A narrow pipe has less end correction as compared to a wider pipe
(d) The length of air column in the second resonance is always greater than three times the length of air column in the first resonance

ANSWERS

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (a) **6.** (c) **7.** (a) **8.** (b) **9.** (b) **10.** (b)
11. (a) **12.** (b) **13.** (c) **14.** (d)

Experiment 10

Specific heat capacity of a given (1) solid and (2) liquid by method of mixtures

Las of mixtures (Principle of Calorimeter)

The law of mixtures states that when two substances at different temperature are mixed, then the heat is exchange between them, the substance at higher temperature loses heat are that at lower temperature gain heat. Exchange of heat energy continues till both the substances attain a common temperature called equilibrium temperature.

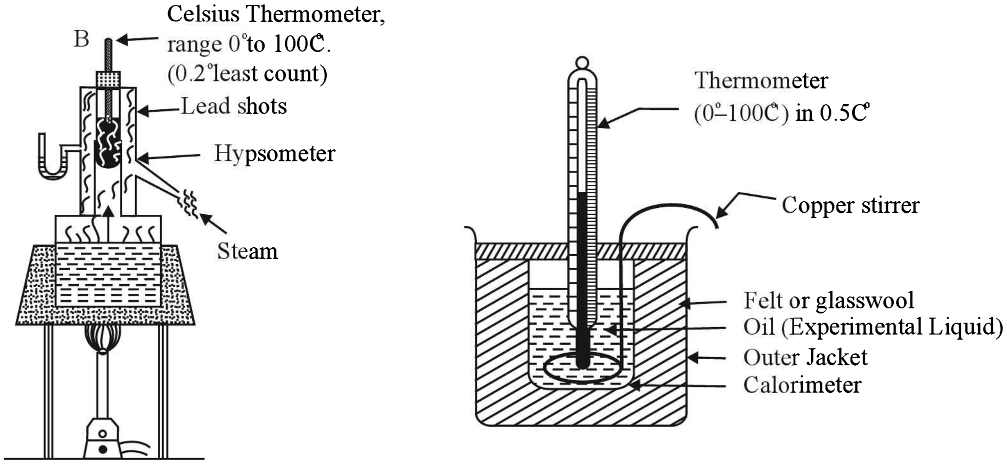
The total amount of heat lost by hot body is equal to the total amount of heat gained by cooler body provided (1) no heat is lost to the surrounding and (2) the substances do not react chemically to produce or absorb heat.

i.e., Total heat lost by hot body = Total heat gained by cold body

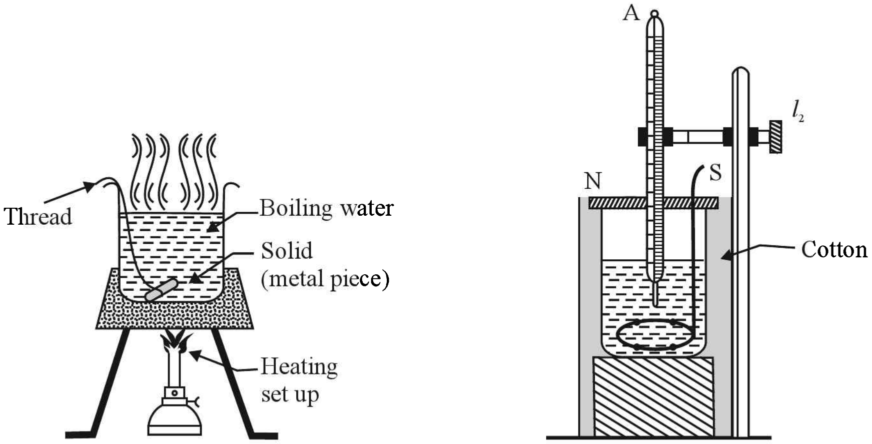
If a body of mass *m* and specific heat *S* its temp rises or falls then amount of heat lost or gained is given by

Q = ms dT (dT = change in temperature)

Figures



Specific heat of solid



Specific heat of liquid

Procedure

(1) Mark the thermometer used for measuring temperature of water in calorimeter at room temperature as A and other used in hypsometer as B.

(2) Take about 100 gm of lead shots (or solid) in the tube of hypsometer and add sufficient quantity of water in the hypsometer.

(3) Insert the thermometer marked B in the tube such that bulb is surrounded by lead shots and fix the tube inside the mouth of hypsometer.

(4) Place hypsometer on the wire gauge placed on the tripod stand and start heating it using the burner.

(5) Find the weight of calorimeter + stirrer and lid using physical balance.

(6) Fill about $\frac{2}{3}$rd of calorimeter with cold water and find the weight of calorimeter + stirrer + lid + cold water.

(7) Place the calorimeter in the wooden jacket. Insert thermometer labelled as A through the lid of the jacket. Note down its temperature.

(8) Observe the temperature of solid in the hypsometer when the temperature become steady for 5 minutes. Note its temperature.

(9) Take out the tube containing solid and transfer the hot solid quickly to the cold water in the calorimeter and cover it with lid. Stir the mixture in the calorimeter till its temperature become constant. Note its temperature.

(10) Take out the calorimeter and find its weight

(11) Calculate the mass of calorimeter, mass of cold water and mass of solids. Knowing their temperature calculate the specific heat of solid by using principle of calorimeter.

(12) Instead of water in the calorimeter use liquid (kerosene) whose specific heat is to be calculated. By applying same principle the specific heat of liquid can be calculated .

Problems for Practice

1. SI units of specific heat is
(a) J/kg (b) J/kg K
(c) J kg (d) J K

2. Which of the following represent the correct statement of specific heat?
(a) Amount of heat required to raise the temp of whole of substance by 1°C
(b) Amount of heat required to raise the temp of 1gm of substance 1°C
(c) Amount of heat required to convert 1gm solid into liquid
(d) Amount of heat required to convert 1gm of liquid in gas

3. One kg of ice at 0°C is mixed with 1kg of water at 10°C. The resultant temperature will be
(a) equal to 0°C (b) lesser than 0°C
(c) greater than 10°C (d) between 0°C and 10°C

4. Steam is passed into 54 gm of water at 30°C till the temperature of mixture become 90°C. If the latent heat of steam is 536 cal/gm, the mass of the mixture will be
(a) 70 gm (b) 60 gm
(c) 80 gm (d) 90 gm

5. Two spheres P and Q have diameter in the ratio of 1 : 2, densities in the ratio 2 : 1 and specific heats in the ratio 1 : 3. The ratio of their thermal capacities is
(a) 12 : 1 (b) 1 : 12
(c) 2 : 9 (d) 1 : 2

6. Assuming that all the energy of a water fall 2000 m is utilized in heating water, the rise in temperature is
(a) 5°C (b) 10°C
(c) 15°C (d) 1.5°C

7. A calorimeter is placed in the wooden box
(a) because it is difficult to hold hot calorimeter
(b) to protect it from getting oxidized due to atmospheric air
(c) to avoid radiation loss
(d) None of the above

8. A calorimeter is made up of copper because it
(a) is cheap (b) looks good
(c) is a good conductor (d) does not get used

9. Quantity of heat required to raise the temperature of 5 kg of water through 20°C is
(a) 100 cal (b) 4 cal
(c) 10^5 cal (d) 10^9cal

10. Specific heat of water is
(a) 1 cal/kg (b) 1000 cal/kg
(c) 10^5 cal/kg (d) None of the above

11. Water equivalent of a body is measured in
(a) degree celcius (b) calorie
(c) gram (d) dyne

12. One degree Celcius is the temperature through which
(a) One cal of heat, heats 1gm of substance
(b) One cal of heat, heats 1gm of water
(c) One K cal of heat, heats 1gm of water
(d) None of the above

13. Specific heat of a body depends upon
(a) mass of body
(b) amount of heat supplied
(c) rise of temperature
(d) None of these

14. Substance having the highest specific heat among copper, kerosene oil and water is
(a) copper (b) kerosene
(c) water (d) None of the above

15. Some amount of heat is supplied to same massed of two samples A and B. Specific heat of A is double than that of B. If the rise in temp of A is θ, that of B will be
(a) 2θ (b) θ
(c) $\frac{\theta}{2}$ (d) None of these

ANSWERS

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (a) **6.** (c) **7.** (a) **8.** (b) **9.** (b) **10.** (b)
11. (a) **12.** (b) **13.** (c) **14.** (d)

Experiment 11

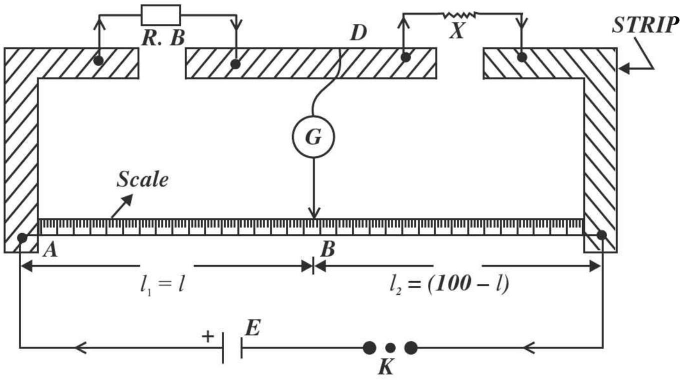
Resistivity of the material of a given wire using metre bridge.

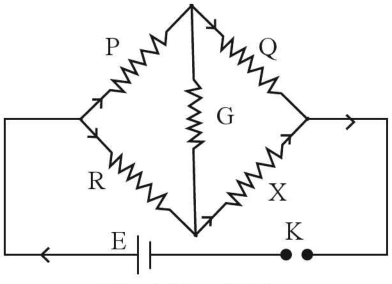
Resistivity of conductor

The property of a conductor to oppose the flow of current through it is called resistance. It is denoted by R. It depends upon the two factors. It is directly proportional to the length (*l*) of the conductor and inversely proportional to the area of cross section of the conductor i.e., $R = j\,l/A$, where j is constant of proportionality and is called resistivity or specific resistance of a conductor. Its units are ohm m and it is resistance of a conductor of unit cube.

The find the resistivity of a conductor we need to measure the resistance of the conductor using meter bridge as

Figures





Wheat Stone Bridge

It is based opon the principle of Wheat Stone Bridge i.e., in a balance bridge no current flow through galvanometer

and $\frac{P}{Q} = \frac{R}{X}$ $X = \frac{RQ}{p} = R\left(\frac{100-l}{l}\right)$

Knowing R Q and P, X can be calculated

Procedure

(1) Take out a suitable resistance R from the Resistance Box. Press jockey first at A and then a/c. Observe the deflection in the galvanometer. If the deflection is opposite, the connection are correct.

(2) Adjust the value of resistance R in such a way that null point is obtained between 40 to 60 cm on the wire.

(3) Take four five observations by taking out different resistance form resistance box. Find the mean value of resistance.

(4) Cut the wire just outside screws and measure its length with a metre scale and diameter and, hence, radius by screw gauge.

(5) Find the resistively using relation $\rho = \frac{XA}{l} = \frac{X\pi r^2}{l}$.

(6) Compare the calculated value with the actual value and find error and percentage error.

Problems for Practice

1. When is the sensitivity of Wheatstone bridge maximum?

(a) When all four resistances are of different order
(b) When all four resistances are of same order
(c) Both (a) and (b)
(d) None of the above

2. After everything remaining same, in which case will the balancing length be small for a series combination or a parallel combination of two resistances?

(a) paralled combination
(b) series combination
(c) mixed combination
(d) same in all above

3. With a certain resistance in the left gap and the right gap of the meter bridge null point is obtained at 40 cm. If resistances in the two gap are inter changed. The null point now will be at

(a) 40 cm (b) 60 cm
(c) 80 cm (d) 10 cm

4. For a given R in the left gap, if the resistance wire in the right gap is replaced by another wire of small resistance, in which direction the null point will shift?

(a) towards left
(b) towards right
(c) does not shift at all
(d) All of the above

5. Why are alloys used for making standard resistance coil?

(a) low temperature coefficient of resistance
(b) high resistivity
(c) resistance does not change large with change in temperature
(d) all the above

6. Why all the copper strips filled on the metre bridge are made thick?

(a) To reduce resistance
(b) To increase resistance
(c) To send more current
(d) To change the resistivity

7. Which of the following statement is wrong?

(a) The zero defection of a galvanometer is marked at the middle.
(b) A galvanometer is marked with positive and negative terminals.
(c) A galvanometer can measure current.
(d) In a moving coil galvanometer the deflection of the coil is directly proportioned to current.

8. Which of the following statement is wrong?

(a) The Wheatstone bridge condition does not change even if we interchange the positions of the galvanometer and cell.
(b) Resistance can be accurately determined by measuring the current and voltage across it.
(c) The wire of the metre bridge is made of a material of high resistivity and high temperature coefficient.
(d) The balance point on the meter wire is obtained by sliding the jockey on the wire.

9. The resistance of a conductor does not depend on its

(a) Shape (b) Temperature
(c) Length (d) Material

10. The specific resistance of manganin is 50×10^{-8} ohm m. The resistance of a cube of manganin of length 50 cm will be

(a) 2.5×10^{-5} ohm (b) 10^{-8} ohm
(c) 10^{-6} ohm (d) 5×10^{-4} ohm

11. The resistivity of Iron is 1.08×10^{-7} ohm m. The resistance of the given wire of a particular thickness and length is 1 ohm. If the diameter and length of the wire, both are doubled, the resistivity will be

(a) 2.08×10^{-7} ohm m (b) 1.08×10^{-7} ohm m
(c) 4×10^{-7} ohm m (d) 3.24×10^{-7} ohm m

ANSWERS

1. (b) **2.** (a) **3.** (b) **4.** (b) **5.** (d) **6.** (a) **7.** (b) **8.** (c) **9.** (a) **10.** (c)
11. (b)

PART B MATHEMATICS

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Coordinates and Straight Lines

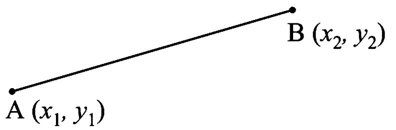
1

BRIEF REVIEW OF THE CONCEPTS

DISTANCE FORMULA

The distance between two points

A(x_1, y_1) and B(x_2, y_2) is given by



$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notes 1. If the distance between two points is given, then use ± sign.

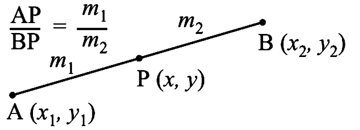
2. The distance of any point A(x, y) from origin is $\sqrt{(x_2 - x_1)}$.

SECTION FORMULA

1. The coordinates of the point P(x, y) dividing the line segment joining the two points A(x_1, y_1) and B(x_2, y_2) internally in the ratio $m_1 : m_2$, are given by

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2},\ y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

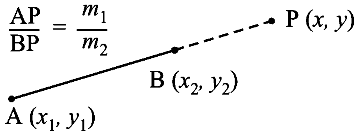
$$\frac{AP}{BP} = \frac{m_1}{m_2}$$



2. The coordinates of the point P(x, y), dividing the line segment joining the two points A(x_1, y_1) and B(x_2, y_2) externally in the ratio $m_1 : m_2$, are given by

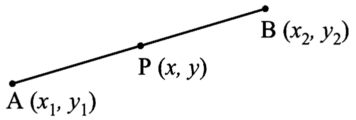
$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2},\ y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

$$\frac{AP}{BP} = \frac{m_1}{m_2}$$



3. The coordinates of the mid point of the line segment joining the two points A(x_1, y_1) and B(x_2, y_2) are given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



Point to Note: The coordinates of any point on a line joining the two points A and B are given by $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$. Such a point divides the given line in the ratio λ : 1. If λ is positive, then the point divides internally and if λ is negative, then the point divides externally.

Key Points to Remember

In order to prove that a given figure is a

(i) **Square:** Prove that the four sides are equal and the diagonals are also equal.

(ii) **Rhombus** (but not a square)**:** Prove that the four sides are equal but the diagonals are not equal.

(iii) **Rectangle:** Prove that the opposite sides are equal and the diagonals are also equal.

(iv) **Parallelogram** (but not a ractangle)**:** Prove that the opposite sides are equal but diagonals are not equal.

Note that in each of these cases diagonals bisect each other.

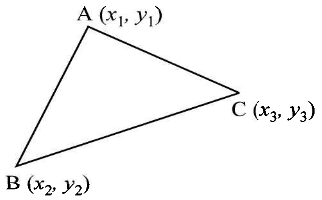
AREA OF A TRIANGLE

The area of ΔABC with vertices A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) is given by:

$$\Delta = \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\,[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$



Notes:

(i) Area of a triangle is always taken as positive.

(ii) If area of a triangle is given, then use ± sign.

(iii) If the three points A, B, C are collinear, then area of ΔABC is zero.

(iv) The area of a quadrilateral, whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, is

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}\right]$$

If the area of a quadrilateral is zero, then its four vertices lie on a stright line i.e., points are collinear.

(v) The area of a polygon of n sides with vertices $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, ..., $A_n(x_n, y_n)$ is

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots. + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right].$$

(vi) If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the equations of the sides of a triangle, then the area of the triangle is

$$= \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2,$$

where C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant i.e., $C_1 = a_2b_3 - a_3b_2$, $C_2 = a_3b_1 - a_1b_3$ and $C_3 = a_1b_2 - a_2b_1$.

LOCUS

The locus of a moving point is the path traced by it under certain geometrical condition or conditions.

For example, if a point moves in a plane under the geometrical condition that its distance from a fixed point O in the plane is always equal to a constant quantity a, then the curve traced by the moving point will be a circle with centre O and radius a. Thus, locus of the point is a circle with centre O and radius a.

Working rule to find the locus of a point

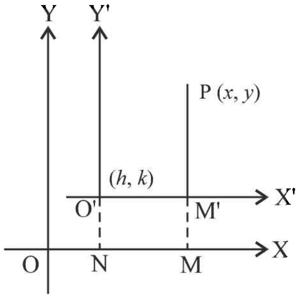
(a) Let the coordinates of the moving point P be (h, k).

(b) Using the given geometrical conditions, find the relation between h and k. This relation must contain only h, k and known quantities.

(c) Express the given relation in h and k in the simplest form and then put x for h and y for k. The relation, thus obtained, will be the required equation of the locus of (h, k).

TRANSLATION OF AXES

Sometimes a problem with a given set of axes can be solved more easily by translation of axes. The translation of axes involves the shifting of the origin to a new point, the new axes remaining parallel to the original axes.



Let OX, OY be the original axes and O' be the new origin. Let coodinates of O' referred to original axes, i.e., OX, OY be (h, k).

Let O'X' and O'Y' be drawn parallel to and in the same direction as OX and OY respectively. Let P be any point in the plane having coordinates (x, y) referred to old axes and (X, Y) referred to new axes. Then,

$$x = OM = ON + NM = ON + O'M'$$
$$= h + X = X + h \quad \text{or} \quad X = x - h$$

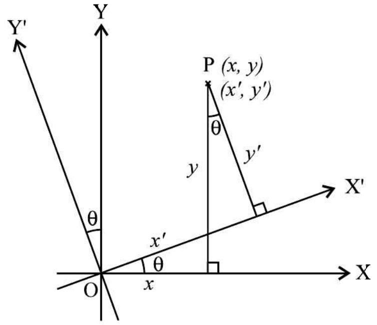
and

$$y = MP = MM' + M'P = NO' + M'P$$
$$= k + Y = Y + k. \quad \text{or} \quad Y = y - k$$

Thus, the point whose coordinates were (x, y) has now the coordinates (x – h, y – k).

ROTATION OF AXES

Rotation of Axes without Changing the Origin Let OX, OY be the original axes and OX', OY' be the new axes obtained by rotating OX and OY through an angle θ in the anticlockwise sense. Let P be any point in the plane having coordinates (x, y) w.r.t. axes OX and OY and (x', y') w.r.t. axes OX' and OY'. Then,



$$x = x'\cos\theta - y'\sin\theta; \quad y = x'\sin\theta + y'\cos\theta$$

and

$$x' = x\cos\theta + y\sin\theta; \quad y' = -x\sin\theta + y\cos\theta.$$

Notes:

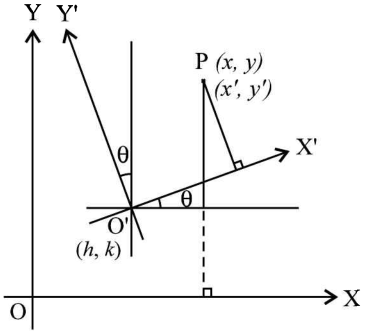
1. The above transformation of coordinates may be displayed by a table

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

2. If f(x, y) = 0 is the equation of a curve then its transformed equation is
$$f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0.$$
3. If f(x', y') = 0 is the transformed equation, then the equation w.r.t. original axes is
$$f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = 0.$$
4. The angle of rotation of coordinate axes in order to remove the xy term in the equation
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ is}$$
$$\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right).$$

5. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is translated to the form in which first degree terms are missing, then the coordinates of the new origin are $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$, where $ab \neq h^2$.

Change of Origin and Rotation of Axes If origin is changed to O' (h, k) and axes are rotated about the new origin O' by angle θ in the anticlockwise sense such that the new coordinates of P (x, y) become (x', y'), then the equations of transformation will be



$$x = h + x' \cos\theta - y' \sin\theta$$

and $$y = \beta + x' \sin\theta + y' \cos\theta.$$

GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form: $ax + by + c = 0$, where a, b, c are any real numbers not all zero, always represents a straight line.

Equation of a straight line is always of first degree in x and y.

Slope of a Line If a line makes an angle θ $\left(\theta \neq \frac{\pi}{2}\right)$ with the positive direction of x-axis, the slope or gradient of that line is usually denoted by m, i.e., $\tan\theta = m$.

Notes:

(a) The slope of a line parallel to x-axis = 0 and perpendicular to x-axis is undefined.

(b) If three points A, B, C are collinear, then slope of AB = slope of BC = slope of AC.

(c) If a line is equally inclined to the axes, then it will make an angle of 45° or 135° with x-axis (i.e., positive direction of x-axis) and hence its slope will be $\tan 45°$ or $\tan 135°$, $= \pm 1$.

(d) Slope of the line joining two points (x_1, y_1) and (x_2, y_2) is given as

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}.$$

Intercept of a Line on the Axes

(i) **Intercept of a line on x-axis** If a line cuts x-axis at a point (a, 0), then a is called the intercept of the line on x-axis. $|a|$ is called the length of the intercept of the line on x-axis. Intercept of a line on x-axis may be positive or negative.

(ii) **Intercept of a line on y-axis** If a line cuts y-axis at a point (0, b), then b is called the intercept of the line on y-axis and $|b|$ is called the length of the intercept of the line on y-axis. Intercept of a line on y-axis may be positive or negative.

Equations of Lines Parallel to Axes

Equation of *x*-axis The equation of x-axis is $y = 0$.

Equation of *y*-axis The equation of y-axis is $x = 0$.

Equation of a line parallel to y-axis The equation of the straight line parallel to y-axis at a distance 'a' from it on the positive side of x-axis is $x = a$.

If a line is parallel to y-axis, at a distance a from it and is on the negative side of x-axis, then its equation is $x = -a$.

Equation of a line parallel to x-axis The equation of the straight line parallel to x-axis at a distance b from it on the positive side of y-axis is $y = b$.

If a line is parallel to x-axis, at a distance b from it and is on the negative side of y-axis, then its equation is $y = -b$.

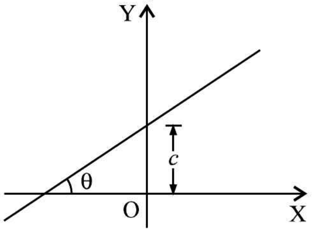
EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

Slope-Intercept Form

The equation of a straight line whose slope is m and which cuts an intercept c on the y-axis is given by

$$y = mx + c.$$

If the line passes through the origin, then $c = 0$ and hence the equation of the line will become $y = mx$.



Point-Slope Form

The equation of a straight line passing through the point (x_1, y_1) and having slope m is given by

$$(y - y_1) = m(x - x_1).$$

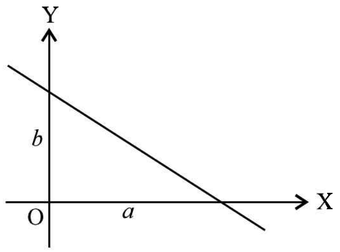
Two-Point Form

The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Intercept Form

The equation of a straight line which cuts off intercepts a and b on x-axis and y-axis respectively is given by

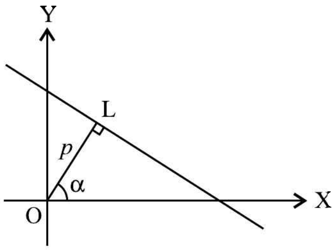


$$\frac{x}{a} + \frac{y}{b} = 1.$$

Normal Form (or Perpendicular Form)

The equation of a straight line upon which the length of the perpendicular from the origin is p and the perpendicular makes an angle α with the positive direction of x-axis is given by

$$x\cos\alpha + y\sin\alpha = p.$$



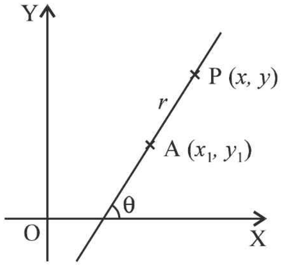
In normal form of equation of a straight line p is always taken as positive and α is measured from positive direction of x-axis in anti-clockwise direction between 0 and 2π.

Parametric Form (or Symmetric Form)

The equation of a straight line passing through the point (x_1, y_1) and making an angle θ with the positive direction of x-axis is given by

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

where r is the distance of the point (x, y) from the point (x_1, y_1).



The coordinates (x, y) of any point P on the line at a distance r from the point $A(x_1, y_1)$ can be taken as

$(x_1 + r\cos\theta, y_1 + r + \sin\theta)$
or $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

REDUCTION OF THE GENERAL EQUATION TO DIFFERENT STANDARD FORMS

Slope-Intercept Form: The general form,

Ax + By + C = 0, of the straight line can be reduced to the form y = mx + c by expressing y as:

$$y = -\frac{A}{B}x - \frac{C}{B} = mx + c$$

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

Thus, slope of the line Ax + By + C = 0 is

$$m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{A}{B}.$$

Intercept Form: The equation

Ax + By + C = 0 can be reduced to the form $\frac{x}{a}+\frac{y}{b} = 1$ by expressing it as:

$$Ax + By = -C$$

or $-\frac{A}{C}x - \frac{B}{C}y = 1$, where $C \neq 0$

or $\frac{x}{-C/A} + \frac{y}{-C/B} = 1$, which is of the form $\frac{x}{a}+\frac{y}{b} = 1$,

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$ are intercepts on x-axis and y-axis, respectively.

Intercept of a straight line on x-axis can be found by putting y = 0 in the equation of the line and then finding the value of x. Similarly intercept on y-axis can be found by putting x = 0 in the equation of the line and then finding the value of y.

Normal Form

To reduce the equation Ax + By + C = 0 to the form x cos α + y sin α = p, first express it as:

$$Ax + By = -C \quad ...(1)$$

CASE 1. If C < 0 or – C > 0, then divide both sides of eqn. (1) by $\sqrt{A^2+B^2}$, we get

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}}$$

which is of the form x cos α + y sin α = p,

where, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$

and $p = -\frac{C}{\sqrt{A^2+B^2}}$

CASE 2. If C > 0 or – C < 0, then divide Eqn. (1) by $-\sqrt{A^2+B^2}$, we get

$$\frac{-A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

which is of the form x cos α + y sin α = p,

where $\cos\alpha = -\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = -\frac{B}{\sqrt{A^2+B^2}}$

and $p = \frac{C}{\sqrt{A^2+B^2}}$.

ANGLE BETWEEN TWO INTERSECTING LINES

The angle θ between two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2},$$

provided no line is perpendicular to x-axis and the acute angle θ is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

Notes:

(a) If both the lines are perpendicular to x-axis, then the angle between them is 0°.

(b) If slope of one line is not defined (one of the lines is perpendicular to x-axis and other makes an angle θ with the positive direction of x-axis), then angle between them = π – θ.

(c) The two lines are parallel if and only if $m_1 = m_2$.

(d) The two lines are perpendicular if and only if $m_1 \times m_2 = -1$.

CONDITION FOR TWO LINES TO BE COINCIDENT, PARALLEL, PERPENDICULAR OR INTERSECTING

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) **Coincident,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;

(ii) **Parallel,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$;

(iii) **Perpendicular,** if $a_1a_2 + b_1b_2 = 0$;

(iv) **Intersecting,** if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ i.e., if they are neither coincident nor parallel.

EQUATION OF A LINE PARALLEL TO A GIVEN LINE

The equation of a line parallel to a given line ax + by + c = 0 is ax + by + k = 0, where k is a constant.

Thus to write the equation of any line parallel to a given line, do not change the coefficient of x and y and change the constant term only.

EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE

The equation of a line perpendicular to a given line ax + by + c = 0 is bx – ay + k = 0, where k is a constant.

POINT OF INTERSECTION OF TWO GIVEN LINES

Let the two given lines be

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Solving these two equations, the point of intersection of the given two lines is given by

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right).$$

CONCURRENT LINES

The three given lines are concurrent if they meet in a point.

Short-cut Method : Following three methods can be used to prove that the three lines are concurrent

1. Find the point of intersection of any two lines by solving them simultaneously. If this point satisfies the third equation also, then the given lines are concurrent.
2. The three lines
   $P \equiv a_1x + b_1y + c_1 = 0$, $Q \equiv a_2x + b_2y + c_2 = 0$,
   $R \equiv a_3x + b_3y + c_3 = 0$ are concurrent if
   $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$
3. The three lines P = 0, Q = 0 and R = 0 are concurrent if there exist constants l, m and n, not all zero at the same time, such that
   $$lP + mQ + nR = 0.$$
   This method is particularly useful in theoretical results.

POSITION OF TWO POINTS RELATIVE TO A LINE

Two points (x_1, y_1) and (x_2, y_2) are on the same side or on opposite sides of the line ax + by + c = 0 according as the expressions: $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite signs.

LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of the perpendicular from the point (x_1, y_1) to the line ax + by + c = 0 is given by

$$p = \frac{|a x_1 + b x_2 + c|}{\sqrt{a^2 + b^2}}.$$

The length of perpendicular from origin (0, 0) to the given line is $\sqrt{a^2 + b^2}$.

DISTANCE BETWEEN TWO PARALLEL LINES

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

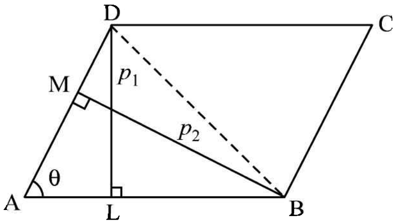
$$d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.$$

(a) The distance between two parallel lines can also be obtained by taking a suitable point (take y = 0 and find x or take x = 0 and find y) on one straight line and then finding the length of the perpendicular from this point to the second line.

(b) Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$\text{Area} = \frac{p_1p_2}{\sin\theta},$$

where p_1 = DL = distance between lines AB and CD,
p_2 = BM = distance between lines AD and BC,
θ = angle between adjacent sides AB and AD.



In the case of a rhombus, $p_1 = p_2$. Thus,

$$\text{Area of rhombus} = \frac{p_1^2}{\sin\theta}.$$

$$\text{Also, area of rhombus} = \frac{1}{2} d_1d_2$$

where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.

EQUATIONS OF STRAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

The equations of the striaght lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line y = mx + c are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

EQUATIONS OF THE BISECTORS OF THE ANGLES BETWEEN TWO LINES

The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

Any point on a bisector is equidistant from the given lines.

Equation of the Bisector of the Acute and Obtuse Angle between Two Lines Let the equations of the two lines be

$$a_1x + b_1y + c_1 = 0 \qquad ...(1)$$

and $$a_2x + b_2y + c_2 = 0 \qquad ...(2)$$

where $c_1 > 0$ and $c_2 > 0$.

Then the equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the acute or obtuse angle between the lines (1) and (2) according as $a_1a_2 + b_1b_2 < 0$ or > 0.

Similarly, the equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the acute or obtuse angle between the lines (1) and (2) according as $a_1a_2 + b_1b_2 > 0$ or < 0.

Note: If $a_1a_2 + b_1b_2 > 0$, then the origin lies in obtuse angle and if $a_1a_2 + b_1b_2 < 0$, then the origin lies in acute angle.

EQUATIONS OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO GIVEN LINES

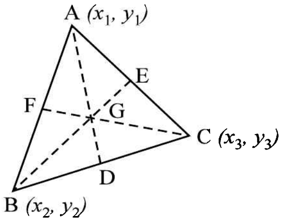
The equation of any line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$(a_1x + b_1y + c_1) + k\,(a_2x + b_2y + c_2) = 0,$$

where k is a parameter. The value of k can be obtained by using one more conditions which the required line satisfies.

STANDARD POINTS OF A TRIANGLE

Centroid of a Triangle The point of intersection of the medians of the triangle is called the centroid of the triangle. The centroid divides the medians in the ratio 2 : 1 (2 from the vertex and 1 from the opposite side).

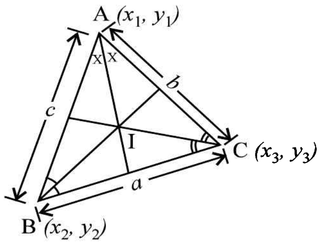


The coordinates of the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Incentre of a Triangle

The point of intersection of the internal bisectors of the angles of a triangle is called the incentre of the triangle.



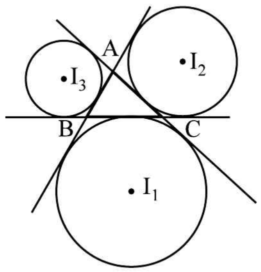
The coordinates of the incentre of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right).$$

Note: The incentre of the triangle formed by (0, 0), (a, 0) and (0, b) is

$$\left(\frac{ab}{a + b + \sqrt{a^2 + b^2}}, \frac{ab}{a + b + \sqrt{a^2 + b^2}}\right).$$

Ex-Centres of a Triangle A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.



Let ABC be a triangle then there are three excircles, with three excentres I_1, I_2, I_3 opposite to vertices A, B and C respectively. If the vertices of triangle are A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) then

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right).$$

Circumcentre The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is same and this distance is known as the circum-radius of the triangle.

Short-cut Method

1. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the ΔABC and let circumcentre be $P(x, y)$. Then (x, y) can be found by solving

$$(OA)^2 = (OB)^2 = (OC)^2$$

or $$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2 = (x - x_3)^2 + (y - y_3)^2$$

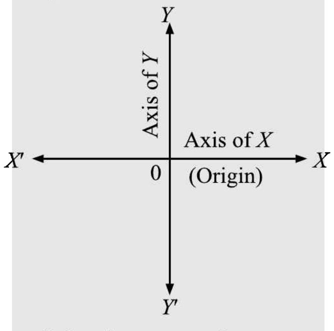
2. Let D, E and F be the mid points of the sides BC, CA and AB of the ΔABC respectively.

   Then, OD ⊥ BC, OE ⊥ AC, OF ⊥ AB.

   slope of OD × slope of BC = – 1

   slope of OE × slope of AC = – 1

   slope of OF × slope of AB = – 1



   Solving any two of the above equations, we get the circumcentre (x, y).

3. (a) If the equations of the three sides of the triangle are given, first of all find the coordinates of the vertices of the triangle by solving the equations of the sides of the triangle taken two at a time.

   (b) Find the coordinates of the middle points of two sides of the triangle.

   (c) Find the equations of the perpendicular bisectors of these two sides and solve them. This will give the coordinates of the circumcentre of the triangle.

4. If angles A, B, C and vertices A(x_1, y_1), B(x_2, y_2) and C (x_3, y_3) of a ΔABC are given, then its circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Notes:

1. The circumcentre of a right angled triangle is the mid point of its hypotenuse.
2. The circumcentre of the triangle formed by (0, 0), (x_1, y_1) and (x_2, y_2) is

$$\left\{\frac{y_2(x_1^2 + y_1^2) - y_1(x_2^2 + y_2^2)}{2(x_1y_2 - x_2y_1)}, \frac{x_2(x_1^2 + y_1^2) - x_1(x_2^2 + y_2^2)}{2(x_2y_1 - x_1y_2)}\right\}.$$

ORTHOCENTRE

The orthocentre of a triangle is the point of intersection of altitudes.

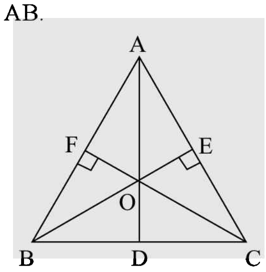
Key Points to Remember

1. Let O be the orthocentre. Since AD ⊥ BC, BE ⊥ CA and CF ⊥ AB, then

   OA ⊥ BC

   OB ⊥ CA

   and OC ⊥ AB.



Solving any two of these, we can get coordinates of O.

2. (a) Write down the equations of any two sides of the triangle.

   (b) Find the equations of the lines perpendicular to these two sides and passing through the opposite vertices.

   (c) Solve these equations to get the coordinates of the orthocentre.

3. If angles A, B and C and vertices A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) of a ΔABC are given, then orthocentre of ΔABC is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Notes:

1. If any two lines out of three lines, i.e., AB, BC and CA are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.
2. The orthocentre of the triangle with vertices (0, 0), (x_1, y_1) and (x_2, y_2) is

$$\left\{(y_1 - y_2)\left[\frac{x_1x_2 - y_1y_2}{x_2y_1 - x_1y_2}\right], (x_1 - x_2)\left[\frac{x_1x_2 + y_1y_2}{x_1y_2 - x_2y_1}\right]\right\}.$$

Solved Problems

1. The ends of a rod of length l move on two mutually perpendicular lines. The locus of the point on the rod which divides it in the ratio 1 : 2 is

(a) $\frac{x^2}{1}+\frac{y^2}{4}=\frac{l^2}{9}$ (b) $\frac{x^2}{4}+\frac{y^2}{1}=\frac{l^2}{9}$

(c) $\frac{x^2}{1}-\frac{y^2}{4}=\frac{l^2}{9}$ (d) $\frac{x^2}{4}-\frac{y^2}{1}=\frac{l^2}{9}$

Solution (a) Let the equation of the rod be = 1 where $a^2 + b^2 = l^2$ ...(1)

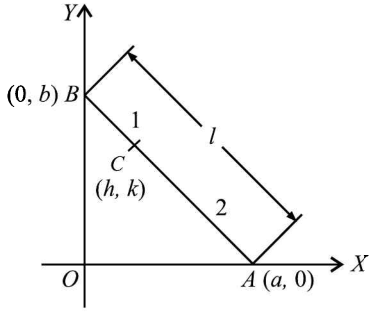
The point A (a, 0) is on x-axis and B (0, b) is on y-axis. Let C (h, k) divide AB in the ratio 2 : 1. So, by section formula,

$$h = \frac{2\cdot 0+1\cdot a}{2+1} = \frac{a}{3} \therefore a = 3h$$

$$k = \frac{2\cdot b+1\cdot 0}{2+1} = \frac{2b}{3} \therefore b = \frac{3k}{2}$$

Putting for a and b in (1), we get

$$9h^2 + 9\,\frac{k^2}{4} = l^2.$$



So, the required locus is $\frac{\mathbf{x^2}}{\mathbf{1}}+\frac{\mathbf{y^2}}{\mathbf{4}}=\frac{\mathbf{l^2}}{\mathbf{9}}$.

2. If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then the centroid of the triangle having vertices $\left(\alpha, \frac{1}{\alpha}\right)$, $\left(\beta, \frac{1}{\beta}\right)$ and $\left(\gamma, \frac{1}{\gamma}\right)$ are

(a) (p, q) (b) (p, – q)

(c) (– p, q) (d) (– p, – q)

Solution (a) Since α, β, γ are the roots of the equation

$$x^3 - 3px^2 + 3qx - 1 = 0$$

$\therefore$ $\alpha + \beta + \gamma = 3p$, $\alpha\beta + \alpha\gamma + \beta\gamma = 3q$, $\alpha\beta\gamma = 1$.

Let G (x, y) be the centroid of the given triangle, then

$$x = \frac{\alpha+\beta+\gamma}{3} = p$$

and $$y = \frac{1/\alpha+1/\beta+1/\gamma}{3} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{3\alpha\beta\gamma}$$

$$= \frac{3q}{3} = q.$$

$\therefore$ Coordinates of the centroid of the given triangle are **(p, q).**

3. The locus of the point whose distance from x-axis is twice that from y-axis, is

(a) y = x (b) y = 2x

(c) x = y (d) x = 2y

Solution (b) Let the coordinates of the point P be (α, β). Then, distance of P from x-axis is β and from y-axis is α.

We have, β = 2α (given).

$\therefore$ The locus of P is **y = 2x.**

4. The coordiantes of the orthocentre of the triangle, having vertices (0, 0), (2, – 1) and (– 1, 3), are

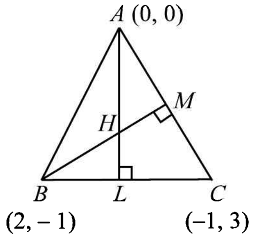
(a) (4, –3) (b) (–4, 3)

(c) (4, 3) (d) (– 4, –3)

Solution (d) Let A ≡ (0, 0), B ≡ (2, – 1) and C ≡ (– 1, 3).

Since AL ⊥ BC, therefore equation of AL is

$$y - 0 = -\frac{2+1}{-1-3}(x - 0)$$



$$\left(\because \text{ slope of } BC = \frac{-1-3}{2+1}\right)$$

$\Rightarrow$ $3x - 4y = 0$ ...(1)

Since BM ⊥ AC, therefore equation of BM is

$$y + 1 = -\frac{0+1}{0-3}(x - 2)$$

$$\left(\because \text{ slope of AC} = \frac{0-3}{0+1}\right)$$

$\Rightarrow$ $x - 3y = 5$ ...(2)

Solving equations (1) and (2), we get

x = – 4, y = –3.

Thus, the coordinates of the orthocentre are

H (– 4, – 3).

5. The coordiantes of the orthocentre of the triangle, formed by lines xy = 0 and x + y = 1, are

(a) (0, 0) (b) (2, – 1)

(c) (– 2, 1) (d) none of these

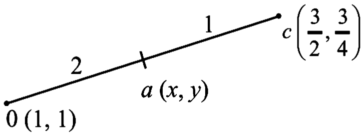
Solution (a) The three sides of the triangle are x = 0, y = 0 and x + y = 1. So, the three vertices of the triangle are O (0, 0), A (1, 0) and B (0, 1). The ΔOAB is a right angled triangle, right angled at O. Therefore, O(0, 0) is the orthocentre of the triangle.

6. If a triangle has its orthocentre at (1, 1) and circumcentre at $\left(\frac{3}{2}, \frac{3}{4}\right)$, then the coordinates of the centroid of the triangle are

(a) $\left(\frac{4}{3}, -\frac{5}{6}\right)$ (b) $\left(\frac{4}{3}, \frac{5}{6}\right)$

(c) $\left(-\frac{4}{3}, \frac{5}{6}\right)$ (d) $\left(-\frac{4}{3}, -\frac{5}{6}\right)$

Solution (b) Since the centroid divides the line joining the orthocentre and circumcentre in the ratio 2 : 1 internally, therefore, if the centroid is (x, y), then



$$x = \frac{2\cdot\frac{3}{2}+1\cdot 1}{2+1} = \frac{4}{3} \quad\text{and}\quad y = \frac{2\cdot\frac{3}{4}+1\cdot 1}{2+1} = \frac{5}{6}.$$

∴ Coordinates of centroid are $\left(\mathbf{\frac{4}{3}, \frac{5}{6}}\right)$.

7. A ladder of length 'a' rests against the floor and a wall of a room. If the ladder begins to slide on the floor, then the locus of its middle point is

(a) $x^2 + y^2 = a^2$ (b) $2(x^2 + y^2) = a^2$

(c) $x^2 + y^2 = 2a^2$ (d) $4(x^2 + y^2) = a^2$

Solution (d) Let AB be the ladder. Let the cross section of the floor and wall be taken as the coordinates axes. Let P (x, y) be the mid-point of AB whose locus is required. Then the coordinates of A and B are (2x, 0) and (0, 2y) respectively.

Given, AB = a

$\Rightarrow \sqrt{(2x-0)^2+(0-2y)^2} = a \Rightarrow 4x^2 + 4y^2 = a^2$

or $\mathbf{4(x^2 + y^2) = a^2}$,

which is the required locus..

8. If the point A is symmetric to the point B (4, – 1) with respect to the bisector of the first quadrant, then the length of AB is

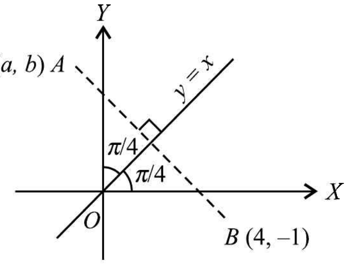
(a) 5 (b) $5\sqrt{2}$

(c) $3\sqrt{2}$ (d) 3

Solution (b) Let A ≡ (a, b).

The coordinates of the mid point C of AB are $\left(\frac{a+4}{2}, \frac{b-1}{2}\right)$.

Since it lies on the line y = x (the bisector of first quadrant)



$$\therefore \quad \frac{b-1}{2} = \frac{a+4}{2}$$

$\Rightarrow \quad a - b = -5 \qquad ...(1)$

Since AB is ⊥ to the line y = x,

∴ product of slope of AB and slope of the line y = x is equal to – 1

$\Rightarrow \quad \frac{b+1}{a-4} \times 1 = -1 \Rightarrow b + 1 = -a + 4$

$\Rightarrow \quad a + b = 3 \qquad ...(2)$

Solving (1) and (2), we get a = – 1, b = 4.

∴ Coordinates of the point A are (– 1, 4).

∴ $AB = \sqrt{(-1-4)^2+(4+1)^2} = \mathbf{5\sqrt{2}}$.

9. A line through the point A (2, 0), which makes an angle of 30° with the positive direction of x-axis is rotated about A in clockwise direction through an angle 15°. The equation of the straight line in the new position is

(a) $(2-\sqrt{3})x - y - 4 + 2\sqrt{3} = 0$

(b) $(2-\sqrt{3})x + y - 4 + 2\sqrt{3} = 0$

(c) $(2-\sqrt{3})x - y + 4 + 2\sqrt{3} = 0$

(d) none of these

Solution (a) Let AB be the initial position of the line and AC be its new position.

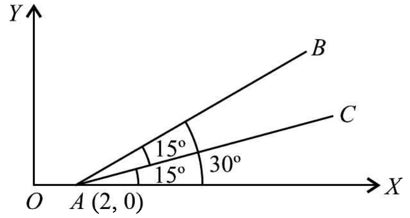
Slope of the line

AC = tan 15° = $(2-\sqrt{3})$ (See Q. No. 37)

∴ Equation of the line AC is

$(y - 0) = (2-\sqrt{3})(x - 2)$

or $y = (2-\sqrt{3})x - 4 + 2\sqrt{3}$



or $\mathbf{(2-\sqrt{3})x - y - 4 + 2\sqrt{3} = 0}$.

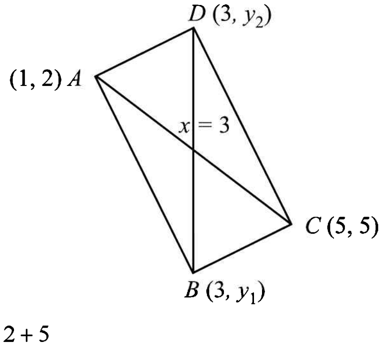
10. A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line x = 3, then the coordinates of the other vertices are

(a) (3, – 1), (3, – 6) (b) (3, 1), (3, 5)

(c) (3, 2), (3, 6) (d) (3, 1), (3, 6)

Solution (d) Let A ≡ (1, 2) and C ≡ (5, 5). Since the vertices B and D lie on the line x = 3, therefore, let B ≡ $(3, y_1)$ and D ≡ $(3, y_2)$.

Since AC and BD bisect each other, so they have same middle point



i.e., $\frac{y_1+y_2}{2} = \frac{2+5}{2}$

or $y_1 + y_2 = 7 \qquad ...(1)$

Also, $BD^2 = AC^2$

$\Rightarrow (y_1 - y_2)^2 = (1 - 5)^2 + (2 - 5)^2 = 25$

or $y_1 - y_2 = \pm 5 \qquad ...(2)$

Solving (1) and (2), we get $y_1 = 6, y_2 = 1$

or $y_1 = 1, y_2 = \mathbf{6}$.

Thus, the other vertices of the rectangle are **(3, 1)** and **(3, 6).**

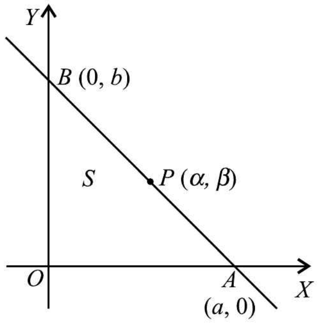
11. Through the point P(α, β), where αβ > 0 the straight line $\frac{x}{a}+\frac{y}{b}$ = 1 is drawn so as to form with coordinate axes a triangle of area S. If ab > 0, then the least value of S is

(a) αβ (b) 2αβ

(c) 4αβ (d) none of these

Solution (b) The equation of the given line is

$$\frac{x}{a}+\frac{y}{b} = 1 \qquad ...(1)$$



This line cuts x-axis and y-axis at A (a, 0) and B(0, b) respectively.

Since area of ΔOAB = S (Given)

∴ $\left|\frac{1}{2}ab\right|$ = S or ab = 2S (∵ ab > 0) ...(2)

Since the line (1) passes through the point P (α, β)

∴ $\frac{\alpha}{a}+\frac{\beta}{b} = 1$ or $\frac{\alpha}{a}+\frac{a\beta}{2S} = 1$ [Using (2)]

or $a^2\beta - 2aS + 2\alpha S = 0$.

Since a is real, ∴ $4S^2 - 8\alpha\beta S \ge 0$

or $4S^2 \ge 8\alpha\beta S$ or $S \ge 2\alpha\beta$ $\left(\because S = \frac{1}{2}ab > 0 \text{ as } ab > 0\right)$

Hence the least value of **S** = **2αβ.**

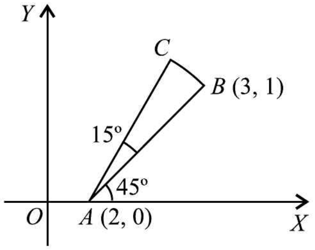
12. A line joining two points A (2, 0) and B (3, 1) is rotated about A in anticlockwise direction through an angle 15°. If B goes to C in the new position, then the coordinates of C are

(a) $\left(2, \sqrt{\frac{3}{2}}\right)$ (b) $\left(2, -\sqrt{\frac{3}{2}}\right)$

(c) $\left(2+\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ (d) none of these

Solution (c) Slope of line

AB = $\frac{0-1}{2-3}$ = 1



= tan 45°

∴ ∠BAX = 45°.

Given ∠CAB = 15°.

∴ ∠CAX = 60°.

∴ Slope of line AC = tan 60° = $\sqrt{3}$.

Now, line AC makes an angle of 60° with positive direction of x-axis and

$$AC = AB = \sqrt{(3-2)^2+(1-0)^2} = \sqrt{2}.$$

∴ Coordinates of C are (2 + $\sqrt{2}$ cos 60°, 0 + $\sqrt{2}$ sin 60°)

i.e $$\left(2+\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right).$$

13. The image of the point (– 8, 12) with respect to the line mirror 4x + 7y + 13 = 0 is

(a) (16, – 2) (b) (– 16, 2)

(c) (16, 2) (d) (– 16, – 2)

Solution (d) Equation of the given line is

$$4x + 7y + 13 = 0 \qquad ...(1)$$

Let Q (α, β) be the image of the point P (– 8, 12) w.r.t. line (1).

Then, PQ ⊥ line (1) and PC = CQ.

Equation of the line PC is

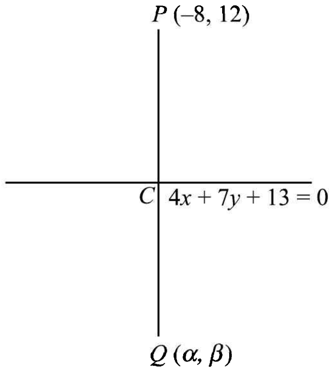
$$(y - 12) = \frac{7}{4}(x + 8)$$

[PC is ⊥ to the line (1) and passes through (– 8, 12)]

or $$7x - 4y + 104 = 0 \qquad ...(2)$$

Solving (1) and (2), we get

x = – 12 and y = 5. ∴ C ≡ (– 12, 5).



Since C is mid point of PQ,

∴ $-12 = \frac{\alpha-8}{2}$ and $5 = \frac{\beta+12}{2}$

⇒ α = – 16 and β = –2.

∴ **Q ≡ (– 16, – 2).**

14. The image of the point (3, – 8) under the transformation (x, y) → (2x + y, 3x – y) is

(a) (– 2, 17) (b) (2, 17)

(c) (– 2, – 17) (d) (2, – 17)

Solution (a) Let (x_1, y_1) be the image of the point (x, y) under the given transformation.

Then, $x_1 = 2x + y = 2(3) - 8 = -2$

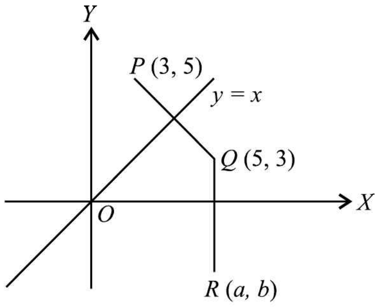
and $y_1 = 3x - y = 3(3) - (-8) = 17$.

Hence, the image is **(–2, 17).**

15. The image of the point P (3, 5) with respect to the line y = x is the point Q and the image of Q with respect to the line y = 0 is the point R (a, b), then (a, b) =

(a) (5, 3) (b) (5, – 3)
(c) (– 5, 3) (d) (– 5, – 3)

Solution (b) Let (x_1, y_1) be the image of the point P (3, 5) with respect to the line y = x. Then, $x_1 = 5$, $y_1 = 3$.



∴ Q = (5, 3).

Since the image of the point Q (5, 3) w.r.t. the line y = 0 is (a, b).

∴ a = 5 and b = – 3.

∴ (a, b) = **(5, – 3).**

16. Without changing the direction of coordinates axes, origin is transferred to (α, β) so that the linear terms in the equation $x^2 + y^2 + 2x - 4y + 6 = 0$ are eliminated. The point (α, β) is

(a) (– 1, 2) (b) (1, – 2)
(c) (1, 2) (d) (– 1, – 2)

Solution (a) The given equation is

$$x^2 + y^2 + 2x - 4y + 6 = 0 \quad \text{...(1)}$$

Putting $x = x' + \alpha$, $y = y' + \beta$ in (1), we get

$$x'^2 + y'^2 + x'(2\alpha + 2) + y'(2\beta - 4) + (\alpha^2 + \beta^2 + 2\alpha - 4\beta + 6) = 0$$

To eliminate linear terms, we should have

$2\alpha + 2 = 0$ and $2\beta - 4 = 0$

⇒ $\alpha = -1$ and $\beta = 2$

∴ $(\alpha, \beta) \equiv$ **(– 1, 2).**

17. The point $(2t^2 + 2t + 4, t^2 + t + 1)$ lies on the line x + 2y = 1 for

(a) all real values of t (b) some real values of t
(c) $t = \frac{-4 \pm \sqrt{7}}{8}$ (d) none of these

Solution (d) The point $(2t^2 + 2t + 4, t^2 + t + 1)$ lies on the line

$x + 2y = 1$ if $(2t^2 + 2t + 4) + 2(t^2 + t + 1) = 1$

i.e., $4t^2 + 4t + 5 = 0$.

Here, discriminent = $16 - 4 \cdot 4 \cdot 5 = -64 < 0$.

∴ No real value of t is possible.

Hence, the given point cannot lie on the line.

18. P is a point on either of the two lines $y - \sqrt{3}|x| = 2$ at a distance of 5 units from their point of intersection. The coordinates of the foot of the perpendicular from P on the bisector of the angle between them are

(a) $\left[0, \frac{1}{2}(4+5\sqrt{3})\right]$ or $\left[0, \frac{1}{2}(4-5\sqrt{3})\right]$ depending on which line the point P is taken

(b) $\left[0, \frac{1}{2}(4+5\sqrt{3})\right]$

(c) $\left[0, \frac{1}{2}(4-5\sqrt{3})\right]$

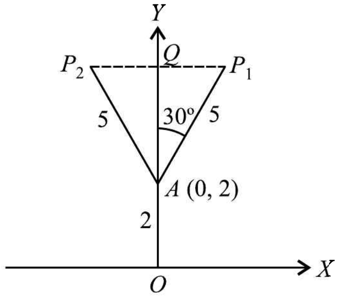
(d) $\left[\frac{5}{2}, \frac{5\sqrt{3}}{2}\right]$

Solution (b) Equation of two lines are

$y = \sqrt{3}x + 2$, if $x \ge 0$

and $y = -\sqrt{3}x + 2$, if $x \le 0$.

Clearly $y \ge 2$.



Also, y-axis is the bisector of the angle between the two lines. P_1, P_2 are two points on these lines, at a distance 5 units from A. Q is the foot of the ⊥ from P_1 and P_2 on the bisector (y-axis).

Then, the coordinates of Q are (0, 2 + 5 cos 30°)

$$= \left(0,\ 2 + \frac{5\sqrt{3}}{2}\right) = \left(\mathbf{0,\ \frac{1}{2}(4 + 5\sqrt{3})}\right).$$

19. Let P be the image of the point (– 3, 2) with respect to x-axis. Keeping the origin as same, the coordinate axes are rotated through an angle 60° in the clockwise sense. The coordinates of point P with respect to the new axes are

(a) $\left(\frac{2\sqrt{3}-3}{2}, -\frac{(3\sqrt{3}+2)}{2}\right)$

(b) $\left(\frac{2\sqrt{3}-3}{2}, \frac{3\sqrt{3}+2}{2}\right)$

(c) $\left(-\frac{(2\sqrt{3}-3)}{2}, \frac{3\sqrt{3}+2}{2}\right)$

(d) none of these

Solution (a) Since P is the image of the point (– 3, 2) with respect to x-axis, therefore, the coordinates of P are (– 3, – 2).

Let (x', y') be the coordinates of P with respect to new axes. Then,

$$x' = x\cos\alpha + y\sin\alpha = -3\cos(-60^\circ) - 2\sin(-60^\circ)$$

$$= -\frac{3}{2} + \sqrt{3} = \frac{2\sqrt{3}-3}{2}.$$

$$y' = -x\sin\alpha + y\cos\alpha = 3\sin(-60^\circ) - 2\cos(-60^\circ)$$

$$= -\left(\frac{3\sqrt{3}+2}{2}\right)$$

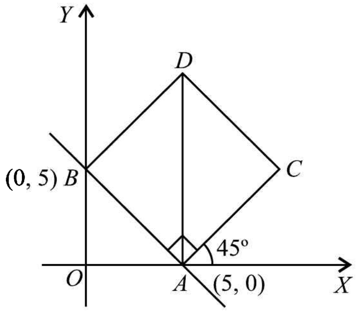
∴ Coordinates of P are $\left(\mathbf{\frac{2\sqrt{3}-3}{2}, -\left(\frac{3\sqrt{3}+2}{2}\right)}\right)$.

20. A square is constructed on the portion of the line $x + y = 5$ which is intercepted between the axes, on the side of the line away from origin. The equations to the diagonals of the square are

(a) $x = 5, y = -5$ (b) $x = 5, y = 5$

(c) $x = -5, y = 5$ (d) $x - y = 5, x - y = -5$

Solution (b) Clearly, the equations of the two diagonals are $x = \mathbf{5}$ and $y = \mathbf{5}$.



21. The area of the region enclosed by $4|x| + 5|y| \le 20$ is

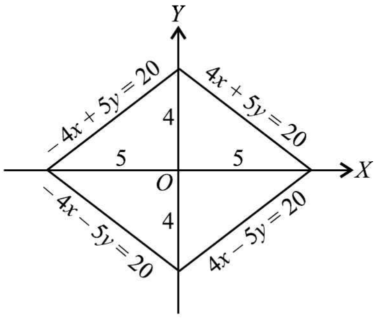
(a) 10 (b) 20

(c) 40 (d) none of these

Solution (c) The four lines enclosing the given region are

$4x + 5y = 20$, $4x - 5y = 20$, $-4x + 5y = 20$ and $-4x - 5y = 20$.

Clearly, the four lines form a rhombus having diagonals of length 10 and 8.



$$\therefore \quad \text{Required area} = \frac{1}{2} \times 10 \times 8 = \mathbf{40}.$$

22. If a, b, c are in H.P. then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c} = 0$ always passes through a fixed point, that point is

(a) $(-1, -2)$ (b) $(-1, 2)$

(c) $(1, -2)$ (d) $\left(1, -\frac{1}{2}\right)$

Solution (c) Since a, b, c are in H.P., $\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{b}$ are in A.P.

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b} \qquad \text{...(1)}$$

The given line is $\frac{x}{a}+\frac{y}{b}+\frac{1}{c} = 0$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b} + \left(\frac{2}{b}-\frac{1}{a}\right) = 0 \qquad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{1}{a}(x - 1) + \frac{1}{b}(y + 2) = 0$$

⇒ The given line passes through the point of intersection of $x - 1 = 0$ and $y + 2 = 0$ i.e., $\mathbf{(1, -2)}$ which is a fixed point.

23. The figure which is is designed by the line $ax \pm by \pm c = 0$ is:

(a) rectangle (b) square

(c) rhombus (d) none of these

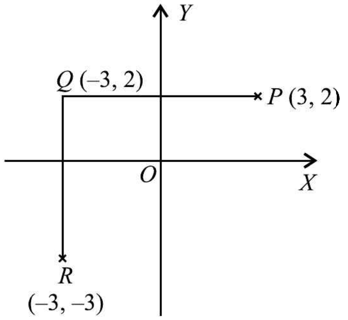
Solution (c)

24. The point (3, 2) is reflected in the y-axis and then moved a distance 5 units towards the negative side of y-axis. The coordinates of the point thus obtained are

(a) $(3, -3)$ (b) $(-3, 3)$

(c) $(3, 3)$ (d) $(-3, -3)$

Solution (d) Reflection in the y-axis gives the new position as $(-3, 2)$.



When it moves towards the negative side of y-axis through 5 units, then the new position is $(-3, 2 - 5)$ i.e., $\mathbf{(-3, -3)}$.

25. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

Solution (d) Let $A \equiv (1, \sqrt{3})$, $B \equiv (0, 0)$ and $C \equiv (2, 0)$.

Then, $AB = \sqrt{1+3} = 2$, $BC = 2$ and

$$CA = \sqrt{(1-2)^2 + (\sqrt{3}-0)^2} = 2.$$

Thus, $AB = BC = CA$.

As ΔABC is an equilateral triangle, the incentre coincides with the centroid of the triangle which is

$$I\left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) \quad \text{i.e., } \mathbf{I}\left(\mathbf{1}, \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\right).$$

26. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

(a) square (b) circle

(c) straight line (d) two intersecting lines

Solution (a) If α and β are the lengths of perpendiculars, then $|\alpha| + |\beta| = \mathbf{1}$ (given), whose graph is a square.

27. The condition to be imposed on β so that $(0, \beta)$ lies on or inside the triangle having sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$ is

(a) $0 < \beta < \frac{5}{3}$ (b) $0 < \beta < \frac{7}{2}$

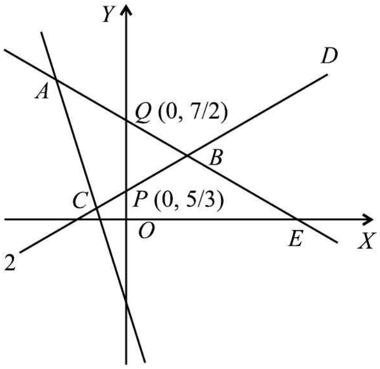
(c) $\frac{5}{3} \le \beta \le \frac{7}{2}$ (d) none of these

Solution (c) Clearly point $(0, \beta)$ lies on y-axis.

Drawing the graph of the three straight lines, we see that

$$Q \equiv \left(0, \frac{7}{2}\right) \text{ and } P \equiv \left(0, \frac{5}{3}\right).$$

Therefore, the point $(0, \beta)$ lies on or inside ΔABC, when



$$\frac{5}{3} \le \beta \le \frac{7}{2}.$$

28. The image (or reflection) of the point (4, – 13) in the line $5x + y + 6 = 0$ is

(a) (1, 2)	(b) (– 4, 13)
(c) (– 1, – 14)	(d) (3, 4)

Solution (c) Let Q (α, β) be the image of the point P(4, – 13).

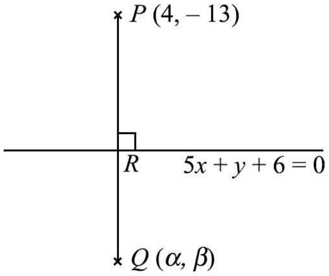
Then, the coordinates of the mid point R are $\left(\frac{\alpha+4}{2}, \frac{\beta-13}{2}\right)$.

Since the point R lies on the line $5x + y + 6 = 0$

$$\therefore \quad 5\left(\frac{\alpha+4}{2}\right)+\left(\frac{\beta-13}{2}\right) + 6 = 0$$

$\Rightarrow 5\alpha + \beta + 19 = 0$ ...(1)

Also, slope of PQ is



$$= \frac{\beta+13}{\alpha-4}$$

and slope of given line = – 5.

Since PQ is $\perp$ to given line

$$\therefore \quad \left(\frac{\beta+13}{\alpha-4}\right)(-5) = -1$$

$\Rightarrow \quad 5\beta + 65 = \alpha - 4$ or $\alpha - 5\beta - 69 = 0$ ...(2)

Solving (1) and (2), we get $\alpha = -1$ and $\beta = -14$

$\therefore$ The image point is **(–1, –14).**

29. The new coordinates of a point (4, 5) when the origin is shifted to the point (1, – 2) are

(a) (5, 3)	(b) (3, 5)
(c) (3, 7)	(d) none of these

Solution (c) According to the question,

$x + 1 = 4 \Rightarrow x = 3$

and $y - 2 = 5 \Rightarrow y = 7$

So, the point is **(3, 7).**

30. If two vertices of an equilateral triangle have integral coordinates then the third vertex will have
(a) coordinates which are irrational
(b) atleast one coordinate which is irrational
(c) coordinates which are rational
(d) coordinates which are integers.

Solution (b) Let the vertices of the equilateral triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3). If none of x_i and y_i (i = 1, 2, 3) are irrational, then

$$\text{area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \text{rational.}$$

But the area of an equilateral triangle

$$= \frac{\sqrt{3}}{4}(\text{side})^2 = \text{irrational.}$$

Thus, the two statements are contradictory. Therefore, both the coordinates of the third vertex cannot be rational.

31. Let P (2, – 4) and Q (3, 1) be two given points. Let R(x, y) be a point such that $(x - 2)(x - 3) + (y - 1)(y + 4) = 0$. If area of ΔPQR is $\frac{13}{2}$, then the number of possible positions of R are

(a) 2	(b) 3
(c) 4	(d) none of these

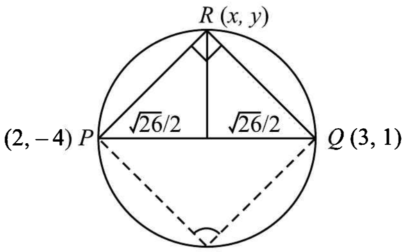
Solution (a) We have

$$(x - 2)(x - 3) + (y - 1)(y + 4) = 0$$

$$\Rightarrow \quad \left(\frac{y+4}{x-2}\right)\times\left(\frac{y-1}{x-3}\right) = -1$$

$$\Rightarrow \quad RP \perp RQ \text{ or } \angle PRQ = \frac{\pi}{2}.$$

$\therefore$ The point R lies on the circle whose diameter is PQ.



Now, area of $\Delta PQR = \frac{13}{2}$

$$\Rightarrow \quad \frac{1}{2} \times \sqrt{26} \times (\text{altitude}) = \frac{13}{2}$$

$$\Rightarrow \quad \text{altitude} = \frac{\sqrt{26}}{2} = \text{radius}$$

$\Rightarrow$ there are **two** possible positions of R.

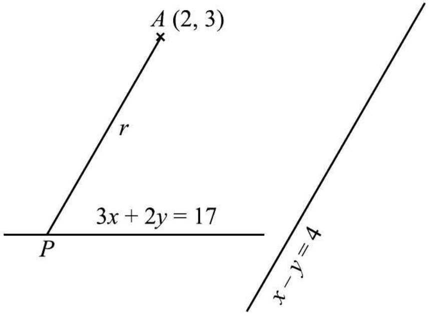
32. The distance of the point (2, 3) from the line $3x + 2y = 17$, measured parallel to the line $x - y = 4$ is

(a) $4\sqrt{2}$ (b) $5\sqrt{2}$

(c) $\sqrt{2}$ (d) none of these

Solution (c) Coordinates of any point on the line through (2, 3) and parallel to the line x – y = 4, at a distance r, are

$$(2 + r\cos\theta,\ 3 + r\sin\theta),\ \text{where } \tan\theta = 1.$$



If this point lies on the line 3x + 2y = 17,

then $3(2 + r\cos\theta) + 2(3 + r\sin\theta) = 17$

$$\Rightarrow 6 + 3r\cdot\frac{1}{\sqrt{2}} + 6 + 2r\cdot\frac{1}{\sqrt{2}} = 17 \Rightarrow \frac{5}{\sqrt{2}}r = 5$$

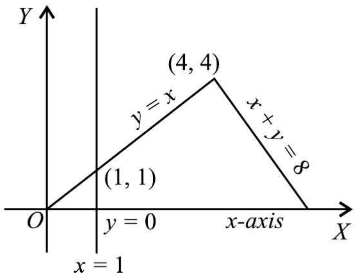
$$\Rightarrow\quad r = \sqrt{2}.$$

33. The point (1, β) lies on or inside the triangle formed by the lines y = x, x-axis and x + y = 8, if

(a) $0 < \beta < 1$ (b) $0 \le \beta \le 1$

(c) $0 < \beta < 8$ (d) none of these

Solution (b) The point (1, β) lies on the line x = 1, for all real β. Clearly, from the figure, it will lie on or inside the triangle formed by the given lines if $\mathbf{0 \le \beta \le 1}$.

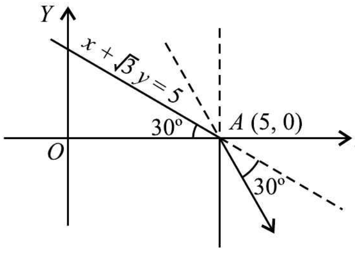


34. A ray of light travelling along the line $x + \sqrt{3}y = 5$ is incident on the x-axis and after refraction it enters the other side of the x-axis by turning $\frac{\pi}{6}$ away from the x-axis. The equation of the line along which the refracted ray travels is

(a) $x + \sqrt{3}y - 5\sqrt{3} = 0$ (b) $x - \sqrt{3}y - 5\sqrt{3} = 0$

(c) $\sqrt{3}x + y - 5\sqrt{3} = 0$ (d) $\sqrt{3}x - y - 5\sqrt{3} = 0$

Solution (c) The refracted ray passes through the point (5, 0) and makes an angle 120° with positive direction of x-axis



∴ The equation of the refracted ray is

$$(y - 0) = \tan 120°\,(x - 5)$$

$$\Rightarrow\quad y = -\sqrt{3}\,(x - 5) \quad\text{or}\quad \mathbf{\sqrt{3}x + y - 5\sqrt{3} = 0.}$$

35. Through the point (1, 1), a straight line is drawn so as to form with coordinate axes a triangle of area S. The intercepts made by the line on the coordinate axes are the roots of the equation

(a) $x^2 - |S|\,x + 2\,|S| = 0$

(b) $x^2 + |S|\,x + 2\,|S| = 0$

(c) $x^2 - 2\,|S|\,x + 2\,|S| = 0$

(d) none of these

Solution (c) If a, b are the intercepts made by the line, then the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$.

Since it passes through (1, 1), ∴ $\frac{1}{a} + \frac{1}{b} = 1$

$$\Rightarrow\quad \frac{a+b}{ab} = 1. \qquad \text{...(1)}$$

Also, area of the triangle made by the straight line on the coordinate axes is S

$$\therefore\quad \frac{1}{2}ab = |S| \text{ i.e., } ab = 2\,|S| \qquad \text{...(2)}$$

So, by (1), $a + b = 2\,|S|$ ...(3)

From (2) and (3), the intercepts a and b are the roots of the equation $\mathbf{x^2 - 2\,|S|\,x + 2\,|S| = 0.}$

36. A ray of light is sent along the line which passes through the point (2, 3). The ray is reflected from the point P on x-axis. If the reflected ray passes through the point (6, 4), then the coordinates of A are

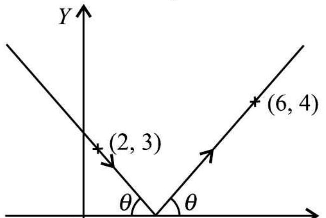
(a) $\left(\frac{26}{7}, 0\right)$ (b) $\left(0, \frac{26}{7}\right)$

(c) $\left(-\frac{26}{7}, 0\right)$ (d) none of these

Solution (a) Let P ≡ (α, 0).

Let the reflected ray makes an angle θ with +ve direction of x-axis, then the incident ray makes angle (π – θ) with positive direction of x-axis.

The slope of the incident ray is



$$= \frac{0-3}{\alpha-2} = \tan(\pi - \theta) \text{ i.e., } \tan\theta = \frac{3}{\alpha-2} \qquad \text{...(1)}$$

The slope of the reflected ray is

$$= \frac{4-0}{6-\alpha} = \tan\theta \text{ i.e., } \tan\theta = \frac{4}{6-\alpha} \qquad \text{...(2)}$$

From (1) and (2), we get

$$\frac{3}{\alpha-2} = \frac{4}{6-\alpha} \Rightarrow 18 - 3\alpha = 4\alpha - 8$$

$$\Rightarrow 7\alpha = 26 \text{ or } \alpha = \frac{26}{7}.$$

$\therefore$ The coordiantes of A are $\left(\mathbf{\frac{26}{7}, 0}\right)$.

37. A line passing through the point P (4, 2), meets the x-axis and y-axis at A and B respectively. If O is the origin, then locus of the centre of the circum circle of ΔOAB is

(a) $x^{-1} + y^{-1} = 2$ (b) $2x^{-1} + y^{-1} = 1$

(b) $x^{-1} + 2y^{-1} = 1$ (d) $2x^{-1} + 2y^{-1} = 1$

Solution (b) (Let the coordinates of A and B be (a, 0) and (0, b) respectively.

Then, equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Since, it passes through the point P (4, 2)

$$\therefore \quad \frac{4}{a} + \frac{2}{b} = 1. \qquad \text{...(1)}$$

Now, centre of the circumcircle of ΔOAB = $\left(\frac{a}{2}, \frac{b}{2}\right)$.

So, equation (1) can be written in the form

$$\frac{2}{a/2} + \frac{1}{b/2} = 1$$

$\therefore$ locus of circumcentre is

$$\frac{2}{x} + \frac{1}{y} = 1 \text{ or } \mathbf{2x^{-1} + y^{-1} = 1.}$$

38. If the point $(2\cos\theta, 2\sin\theta)$ does not fall in that angle between the lines $y = |x - 2|$ in which the origin lies then θ belongs to

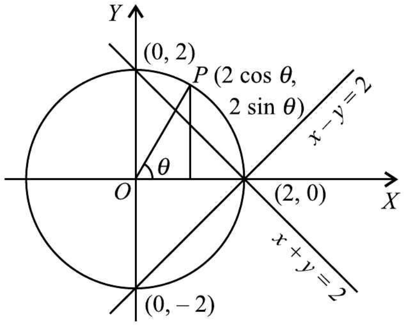
(a) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) none of these

Solution (b) Clearly the point $(2\cos\theta, 2\sin\theta)$ lie on the circle

$$x^2 + y^2 = 4.$$

The two lines represented by the equation $y = |x - 2|$ are $y = x - 2$ and $y = 2 - x$.



From the figure, θ can be vary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

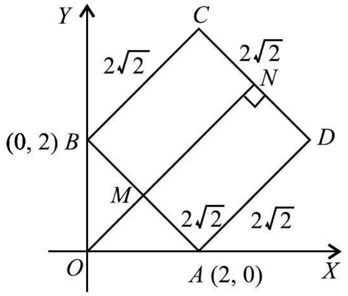
39. On the portion of the straight line x + y = 2 which is intercepted between the axes, a square is constructed away from the origin, with this portion as one of its side. If p denotes the perpendicular distance of a side of this square from the origin, then the maximum value of p is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $3\sqrt{2}$ (d) $4\sqrt{2}$

Solution (c) p = ON = OM + MN

= $\perp$ distance from



O to the line AB + DA

$$= \frac{2}{\sqrt{2}} + 2\sqrt{2} = \sqrt{2} + 2\sqrt{2} = \mathbf{3\sqrt{2}}.$$

40. Let S_1, S_2, ... be squares such that for each n $\geq$ 1, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

(a) 7 (b) 8

(c) 9 (d) 10

Solution (b), (c) (d) Let a be the side of the square, then diagonal d = a$\sqrt{2}$.

Given : $a_n = \sqrt{2}\ a_{n+1}$

$$\Rightarrow a_{n+1} = \frac{a_n}{\sqrt{2}} = \frac{a_{n-1}}{(\sqrt{2})^2} = \frac{a_{n-2}}{(\sqrt{2})^3} = \ldots = \frac{a_1}{(\sqrt{2})^n}$$

$$\therefore \quad a_{n+1} = \frac{a_1}{(\sqrt{2})^n} \Rightarrow a_n = \frac{a_1}{(\sqrt{2})^{n-1}} = \frac{10}{2^{\frac{n-1}{2}}}.$$

Now, area of $S_n < 1 \Rightarrow a_n^2 < 1 \Rightarrow \frac{100}{2^{n-1}} < 1$

$$\Rightarrow 2^n > 200 > 2^7 \Rightarrow n > 7$$

$\therefore$ n = **8, 9, 10.**

41. If a, c, b are three terms of a G.P., then the line

ax + by + c = 0

(a) has a fixed direction
(b) always passes through a fixed point
(c) forms a triangle with the axes whose area is constant
(d) always cuts intercepts on the axes such that their sum is zero.

Solution (c) Since a, c, b are in G.P.,

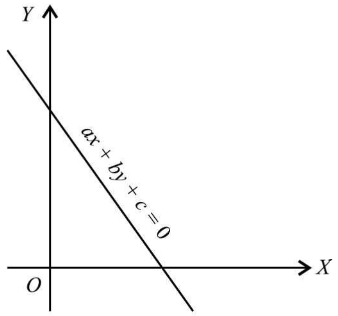
$$\therefore \quad c^2 = ab \qquad \text{...(1)}$$

The area of the triangle

$$= \frac{1}{2} \times \left(-\frac{c}{a}\right) \times \left(-\frac{c}{a}\right)$$

$$= \frac{1}{2} \times \frac{c^2}{ab} = \frac{1}{2} \qquad \text{[Using (1)]}$$

= **constant.**



42. If the points $P_i\left(\frac{a_i^3}{a_i-1}, \frac{a_i^2-3}{a_i-1}\right)$, i = 1, 2, 3 are collinear and a_1, a_2 and a_3 are distinct real numbers, then

(a) $a_1a_2a_3 - \Sigma a_1a_2 - 3\Sigma a_1 = 0$
(b) $a_1a_2a_3 + \Sigma a_1a_2 - 3\Sigma a_1 = 0$
(c) $a_1a_2a_3 - \Sigma a_1a_2 + 3\Sigma a_1 = 0$
(d) $-a_1a_2a_3 + \Sigma a_1a_2 - 2\Sigma a_1 = 0$

Solution (c) Let the given points lie on the line lx + my + n = 0

$$\therefore \quad l\left(\frac{a_i^3}{a_i-1}\right) + m\left(\frac{a_i^2-3}{a_i-1}\right) + n = 0$$

$$\Rightarrow \quad la_i^3 + ma_i^2 + na_i - 3m - n = 0$$

$$\therefore \quad \sum a_1 = -\frac{m}{l}, \sum a_1a_2 = \frac{n}{l} \text{ and } a_1a_2a_3 = \frac{3m+n}{l}$$

$$\therefore \quad a_1a_2a_3 + 3\sum a_1 - \sum a_1a_2 = \mathbf{0}.$$

43. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P., with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(a) lie on a straight line
(b) lie on an ellipse
(c) lie on a circle
(d) are vertices of a triangle

Solution (a) If x_1, x_2, x_3 and y_1, y_2, y_3 are in G.P., then let

$x_2 = rx_1$, $\quad x_3 = r^2x_1$

and $\quad y_2 = ry_1$, $\quad y_3 = r^2y_1$

with common ratio r, then the points are (x_1, y_1), (rx_1, ry_1) and (r^2x_1, r^2y_1)

$$\text{Now, } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

$$= x_1y_1 \begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix}$$

$$= x_1 y_1 (0) = 0$$

(Since two columns are identical)

Thus, these points lie on a straight line.

44. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2) x + (b_1 - b_2) y + c = 0$, then the value of 'c' is

(a) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
(b) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
(c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$
(d) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$

Solution (a) (a_1, b_1) and (a_2, b_2) satisfy the equation, so that

$$a_1 (a_1 - a_2) + b_1 (b_1 - b_2) + c = 0$$

and $$a_2 (a_1 - a_2) + b_2 (b_1 - b_2) + c = 0$$

Adding $(a_1 + a_2)(a_1 - a_2) + (b_1 + b_2)(b_1 - b_2) + 2c = 0$

$$\Rightarrow 2c = -[a_1^2 - a_2^2 + b_1^2 - b_2^2]$$

or $$c = \frac{1}{2}[a_2^2 + b_2^2 - a_1^2 - b_1^2]$$

45. Locus of centroid of the triangle whose vertices are (a cos t, a sin t), (b sin t, – b cos t) and (1, 0), where t is a parameter, is

(a) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(b) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(d) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$

Solution (b) Centroid of the triangle is

$$x = \frac{a\cos t + b\sin t + 1}{3}$$

$$\Rightarrow 3x - 1 = a \cos t + b \sin t \quad \text{...(1)}$$

and $$y = \frac{a\sin t - b\cos t + 0}{3}$$

$$\Rightarrow 3y = a \sin t - b \cos t \quad \text{...(2)}$$

Spuaring and adding (1) and (2), we get

$$(3x - 1)^2 + (3y)^2 = (a \cos t + b \sin t)^2 + (a \sin t - b \cos t)^2$$

$$(3x - 1)^2 + (3y)^2 = a^2 + b^2$$

46. Let A(2, –3) and B(–2, 1) be vertices of a triangle ABC. If the centroid of this triangle moves on the line 2x + 3y = 1, then the locus of the vertex C is the line

(a) 3x + 2y = 5 (b) 2x – 3y = 7
(c) 2x + 3y = 9 (d) 3x – 2y = 3

Solution (c) Let C ≡ (x, y) and G ≡ (α, β) then 2α + 3β = 1

$$\Rightarrow 2\left(\frac{x+2-2}{3}\right) + 3\left(\frac{y-3+1}{3}\right) = 1$$

$$\Rightarrow 2x + 3y - 6 = 3 \Rightarrow 2x + 3y = 9.$$

47. Let P (2, 0) and Q (0, 2) be two points and O be the origin. If A (x, y) is a point such that xy > 0 and x + y < 2, then

(a) A cannot be inside the ΔOPQ
(b) A lies outside the ΔOPQ
(c) A lies either inside ΔOPQ or in the third quadrant
(d) none of these

Solution (c)

48. If the equations of the sides of a triangle are x + y = 2, y = x and $\sqrt{3}$ y + x = 0, then which of the following is an exterior point of the triangle?

(a) orthocentre (b) incentre
(c) centroid (d) none of these

Solution (a)

49. The line (p + 2q) x + (p – 3q) y = p – q for different values of p and q passes through the fixed point

(a) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (b) $\left(\frac{2}{5}, \frac{2}{5}\right)$

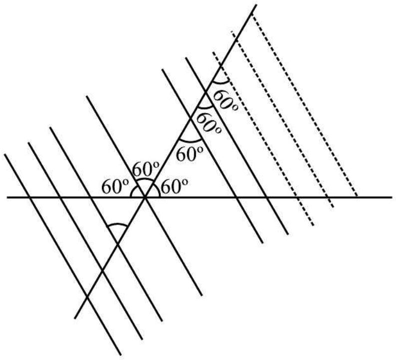
(c) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{3}{5}\right)$

Solution (d)

50. Number of equilateral triangles with $y = \sqrt{3}(x - 1) + 2$ and $y = \sqrt{3}x$ as two of its sides, is
(a) 0 (b) 1
(c) 2 (d) none of these

Solution (d) The sides are,

$$y = \sqrt{3}(x-1)+2 \text{ and } y = -\sqrt{3}x$$



The two lines are at an angle of 60° to each other. Now any line parallel to obtuse angle bisector will make equilateral triangle with these lines as its two sides.

Problems for Practice

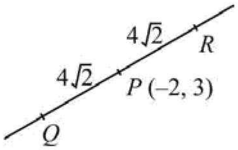
Level of Difficulty I

1. The length of the median through A of a triangle whose vertices are A (– 1, 3), B (1, – 1) and C (5, 1), is
(a) 5 (b) 4 (c) 1 (d) none of these

2. If (1, 4) be the C.G. of a triangle and the coordiantes of its any two vertices be (4, – 8) and (– 9, 7), then the area of the triangle is-
(a) 84 (b) 132 (c) $\frac{333}{2}$ (d) none of these

3. The coordinates of the points at a distance $4\sqrt{2}$ units from the point (–2, 3) in the direction making an angle of 45° with the positive direction of x-axis, are
(a) (2, 7), (–6, –1) (b) (2, 7), (6, –1)
(c) (2, –7), (–6, –1) (d) none of these

4. The coordinates of the foot of the perpendicular drawn from the point (2, 3) to the line y = 3x + 4 are



(a) $\left(-\frac{1}{10}, \frac{37}{10}\right)$ (b) $\left(\frac{1}{10}, -\frac{37}{10}\right)$

(c) $\left(-\frac{1}{10}, -\frac{37}{10}\right)$ (d) none of these

5. The equation of the straight line that can be drawn through the point (4, – 5) so that its distance from the point (– 2, 3) is equal to 12, is
(a) x – 2y = 3(b) 2x + y = 3
(c) 2x – y = 3(d) no such line is possible

6. The equation of the bisector of the acute angle between the lines 3x + 4y – 11 = 0 and 12x – 5y – 2 = 0 is
(a) 11x + 3y + 17 = 0 (b) 11x + 3y – 17 = 0
(c) 3x + 11y + 17 = 0 (d) none of these

7. If P (1, 0), Q (– 1, 0), R (2, 0) are three given points, then the locus of point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is
(a) a straight line || to x-axis
(b) a straight line || to y-axis
(c) circle through the origin
(d) circle with centre at the origin.

8. If the centroid and circumcentre of a triangle are (3, 3) and (6, 2) respectively, then the orthocentre is
(a) (– 3, 5) (b) (– 3, 1)
(c) (3, – 1) (d) (9, 5)

9. The equations ax + by + c = 0 and dx + ey + f = 0 represent the same straight line if and only if
(a) $\frac{a}{d} = \frac{b}{e}$ (b) c = f
(c) $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$ (d) a = d, b = e, c = f

10. If the lines ax + 12y + 1 = 0, bx + 13y + 1 = 0 and cx + 14y + 1 = 0 are concurrent, then a, b, c are in
(a) H.P. (b) G.P. (c) A.P. (d) none of these

11. The vertices of a ΔOBC are O (0, 0), B (– 3, –1) and C (– 1, – 3). The equation of a line parallel to BC and intersecting sides OB and OC whose distance from the origin is $\frac{1}{2}$, is
(a) $x + y + \frac{1}{\sqrt{2}} = 0$ (b) $x + y - \frac{1}{\sqrt{2}} = 0$
(c) $x + y - \frac{1}{2} = 0$ (d) $x + y + \frac{1}{2} = 0$

12. The medians AD and BE of a triangle with vertices A (0, b), B (0, 0) and C (a, 0) are perpendicular to each other if
(a) $a = \frac{b}{2}$ (b) $b = \frac{a}{2}$ (c) ab = 1 (d) $a = \pm\sqrt{2}\,b$

13. The point which divides the join of (1, 2) and (3, 4) externally in the ratio 1 : 1
(a) lies in the IIIrd quadrant
(b) lies in the IInd quadrant
(c) lies in the Ist quadrant
(d) cannot be found

14. The point (– 4, 5) is the vertex of a square and one of its diagonals is 7x – y + 8 = 0. The equation of the other diagonal is

(a) $7x - y + 23 = 0$ (b) $x + 7y = 31$
(c) $x - 7y = 31$ (d) none of these

15. A point equidistant from the lines $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is
(a) $(1, -1)$ (b) $(1, 1)$
(c) $(0, 0)$ (d) $(0, 1)$

16. The base of a triangle lies along the line $x = a$ and is of length a. The area of the triangle is a^2, if the vertex lies on the line
(a) $x = 0$ (b) $x = -a$
(c) $x = 3a$ (d) $x = -3a$

17. If P and Q are two points on the line $4x + 3y + 30 = 0$ such that $OP = OQ = 10$, where O is the origin, then the area of the ΔOPQ is
(a) 48 (b) 16 (c) 32 (d) none of these

18. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is
(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

19. If G be the centroid and I be the incentre of the triangle with vertices A $(-36, 7)$, B $(20, 7)$ and C$(0, -8)$ and $GI = \frac{25}{3}\sqrt{205}\ \lambda$, then the value of λ is
(a) 25 (b) $\frac{1}{25}$ (c) $\frac{4}{25}$ (d) none of these

20. The range of values of θ in the interval $(0, \pi)$ such that the points $(3, 5)$ and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y - 1 = 0$, is
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$
(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) None of these

Level of Difficulty II

21. A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB so that the triangle OAB is equilateral is
(a) $x - 2 = 0$ (b) $y - 2 = 0$
(c) $x + y - 4 = 0$ (d) none of these

22. Three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$
(a) form a triangle
(b) are only concurrent
(c) are concurrent with one line bisecting the angle between the other two
(d) none of these

23. If a, b, c are three terms of an A.P., then the line $ax + by + c = 0$
(a) has a fixed direction
(b) always passes through a fixed point
(c) always cuts intercepts on the axes such that their sum is zero
(d) forms a triangle with the axes whose area is constant.

24. The coordinates of a point on the line $x + y = 4$ that lies at a unit distance from the line $4x + 3y - 10 = 0$ are
(a) $(3, 1)$ (b) $(-7, 11)$
(c) $(3, -1)$ (d) $(7, -11)$

25. If the straight line drawn through the point P$(\sqrt{3}, 2)$ and making an angle $\frac{\pi}{6}$ with the x-axis meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q, then the length of PQ is
(a) 4 (b) 5 (c) 6 (d) none of these

26. If a and b are the intercepts of a straight line on the x-axis and y-axis respectively and p be its perpendicular distance from the origin, then
(a) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ (b) $\frac{1}{p^2} = \frac{1}{a^2} - \frac{1}{b^2}$
(c) $p^2 = a^2 + b^2$ (d) none of these

27. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, then the value of $4p_1^2 + p_2^2$ is
(a) $4a^2$ (b) $2a^2$ (c) a^2 (d) none of these

28. A line L has intercepts a and b on the coordinate axes. When the axes are rotated through an angle, keeping the origin fixed, the same line L has intercepts p and q. Then,
(a) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
(b) $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2} - \frac{1}{q^2}$
(c) $\frac{1}{a^2} + \frac{1}{b^2} = 2\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$
(d) none of these

29. If a straight line cuts intercepts from the axes of coordinates the sum of the reciprocals of which is a constant k, then the line passes through the fixed point
(a) (k, k) (b) $\left(\frac{1}{k}, \frac{1}{k}\right)$
(c) $(k, -k)$ (d) $(-k, k)$

30. A and B are two fixed points. The locus of a point P such that $\angle APB$ is a right angle, is
(a) $x^2 + y^2 = a^2$ (b) $x^2 - y^2 = a^2$
(c) $2x^2 + y^2 = a^2$ (d) none of these

31. The sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$, is
(a) 3 (b) -3 (c) 4 (d) -4.

32. If the family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through the point for all values of a and b, then the coordinates of the point are
(a) $(2, 1)$ (b) $(2, -1)$
(c) $(-2, 1)$ (d) none of these

33. The lines $x\cos\alpha + y\sin\alpha = p_1$ and $x\cos\beta + y\sin\beta = p_2$ will be perpendicular if
(a) $\alpha \pm \beta = \frac{\pi}{2}$ (b) $\alpha = \frac{\pi}{2}$
(c) $|\alpha - \beta| = \frac{\pi}{2}$ (d) $\alpha = \beta$

34. If the vertices P, Q, R of a ΔPQR are rational points, which of the following points of the ΔPQR is (are) always rational point(s)?
(a) centroid (b) incentre
(c) circumcentre (d) orthocentre
(A rational point is a point both of whose coordinates are rational numbers)

35. If P $(1, 2)$, Q $(4, 6)$, R $(5, 7)$ and S (a, b) are the vertices of a parallelogram PQRS, then

(a) a = 2, b = 4 (b) a = 3, b = 4
(c) a = 2, b = 3 (d) a = 3, b = 5

36. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)
(a) lie on a straight line
(b) lie on an ellipse
(c) lie on a circle
(d) are vertices of a triangle

37. If a, b, c are in A.P., then the straight line ax + by + c=0 will always pass through a fixed point whose coordinates are
(a) (– 1, – 2) (b) (1, 2)
(c) (– 1, 2) (d) (1, – 2)

38. If $\alpha + \beta + \gamma = 0$, the line $3\alpha x + \beta y + 2\gamma = 0$ passes through the fixed point
(a) $\left(2, \frac{2}{3}\right)$ (b) $\left(\frac{2}{3}, 2\right)$
(c) $\left(-2, \frac{2}{3}\right)$ (d) none of these

39. The number of integer values of m, for which the x-coordinate of the point of intersection of the lines 3x + 4y = 9 and y = mx + 1 is also an integer, is
(a) –2 (b) 0 (c) 4 (d) 1

40. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
(a) square (b) circle
(c) straight line (d) two intersecting lines

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (c)
11. (a)	**12.** (d)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (b), (c)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (a)
21. (b)	**22.** (c)	**23.** (b)	**24.** (a), (b)	**25.** (c)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (a)
31. (d)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (d)	**38.** (b)	**39.** (a)	**40.** (a)

2

Pair of Straight Lines

BRIEF REVIEW OF THE CONCEPTS

PAIR OF STRAIGHT LINES PASSING THROUGH ORIGIN

An equation of the form $ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree. It represents a pair of straight lines both passing through the origin.

(a) The lines are real and distinct if $h^2 > ab$.
(b) The lines are real and coincident if $h^2 = ab$.
(c) The lines are imaginary if $h^2 < ab$.
(d) If the two lines represented by
$ax^2 + 2hxy + by^2 = 0$ are $y = m_1x$ and $y = m_2x$,

then $$(y - m_1x)(y - m_2x) = y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$$

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}.$$

ANGLE BETWEEN THE PAIR OF STRAIGHT LINES

The angle θ between the pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

and the acute angle is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \frac{2\sqrt{h^2 - ab}}{|a + b|}.$$

Key Points to Remember

1. The two lines are coincident if and only if $h^2 - ab = 0$
2. The two lines are perpendicular if and only if $a + b = 0$, i.e., coefficient of x^2 + coefficient of $y^2 = 0$.
3. The equation to the pair of straight lines passing through the origin and perpendicular to the pair of lines is given by $bx^2 - 2hxy + ay^2 = 0$.
   Thus, to obtain the equation to the pair of lines passing through the origin and perpendicular to the pair of lines
   $$ax^2 + 2hxy + by^2 = 0$$
   We should interchange the coefficients of x^2 and y^2 and change the sign of the term containing xy.

Note:

Angle θ between the pair of lines $ax^2 + 2hxy + by^2 = 0$ can also be obtained from

$$\cos\theta = \frac{a + b}{\sqrt{(a - b)^2 + 4h^2}}$$

EQUATION OF THE BISECTORS OF THE ANGLES BETWEEN THE PAIR OF LINES

The equations of the bisectors of the angles between the pair of lines represented by

$$ax^2 + 2hxy + by^2 = 0 \text{ is } \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \qquad \text{...(1)}$$

Key Points to Remember

1. If $a = b$, the bisectors are
   $x^2 - y^2 = 0$ i.e., $x - y = 0$, $x + y = 0$.
2. If $h = 0$, the bisectors are $xy = 0$, i.e., $x = 0$, $y = 0$.
3. Since in eqn. (1), coefficient of x^2 + coefficient of $y^2 = 0$, the two bisectors are always perpendicular to each other.

GENERAL EQUATION OF PAIR OF STRAIGHT LINES

The general equation of second in degree x and y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(1)}$$

will represent a pair of straight lines if and only if

(i) $$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and
(ii) $h^2 - ab \geq 0$.

Key Points to Remember

1. **Angle between the lines:** The angle θ between the lines represented by eqn. (1) is given by
   $$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|.$$

Note that the angle between the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is same as the angle between the lines $ax^2 + 2hxy + by^2 = 0$.

2. The two lines represented by eqn. (1) will be parallel if
$$h^2 = ab \text{ and } bg^2 = af^2.$$

3. **Point of intersection of the lines:** The point of intersection of the two lines represented by eqn. (1) is
$$\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$$
The point of intersection can also be determined with the help of partial differentiation as follows.
Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Then
$$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g$$
[Differentiating ϕ w.r.t. x, treating y as constant]
and $$\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f$$
[Differentiating ϕ w.r.t.y, treating x as constant]
For point of intersection, $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$. Thus, we have
$$ax + hy + g = 0 \text{ and } hx + by + f = 0$$
Solving the two equations, we get
$$\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$$
$$\therefore \quad (x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$$

4. **Bisectors:** The equations of the bisectors of the angles between the lines represented by eqn. (1) are given by
$$\frac{(x - x')^2 - (y - y')^2}{a - b} = \frac{(x - x')(y - y')}{h},$$
where (x', y') is the point of intersection of the lines represented by (1).

5. **Distance between the parallel lines:** If the two lines represented by eqn. (1) are parallel, then the distance between the two parallel lines is given by
$$d = 2\sqrt{\frac{g^2 - ac}{a(a + b)}}.$$

6. The equation to the pair of straight lines through the origin and parallel to the pair of lines represented by eqn. (1) is
$$ax^2 + 2hxy + by^2 = 0.$$

7. The equation to the pair of lines through (α, β) and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $b(x - \alpha)^2 - 2h(x - \alpha)(y - \beta) + a(y - \beta)^2 = 0$.

8. The area of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by $\left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|$.

9. The product of the perpendiculars from (α, β) to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a - b)^2 + 4h^2}}$.

10. The product of the perpendiculars from (α, β) to the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is
$$\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c}{\sqrt{(a - b)^2 + 4h^2}}.$$

11. The equation to the pair of lines through the origin and forming an equilateral triangle with the line $ax + by + c = 0$ is given by $(ax + by)^2 - 3(ay - bx)^2 = 0$. Also, the area of the equilateral triangle is $\frac{c^2}{\sqrt{3}\,(a^2 + b^2)}$.

12. The two pairs of straight lines $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ and $ax^2 + by^2 + 2hxy = 0$ form a
(a) square if $(a - b)fg + h(f^2 - g^2) = 0$, $a + b = 0$
(b) rectangle if $(a - b)fg + h(f^2 - g^2) \neq 0$, $a + b = 0$
(c) rhombus if $(a - b)fg + h(f^2 - g^2) = 0$, $a + b \neq 0$
(d) parallelogram if $(a - b)fg + h(f^2 - g^2) \neq 0$, $a + b \neq 0$.

13. The lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ form an isosceles triangle if $\frac{l^2 - m^2}{lm} = \frac{a - b}{h}$.

14. The centroid of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is
$$\left(\frac{2(bl - hm)}{3(bl^2 - 2hlm + am^2)}, \frac{2(am - hl)}{3(bl^2 - 2hlm + am^2)}\right).$$

15. The orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is
$$\left(\frac{-ln(a + b)}{am^2 - 2hlm + bl^2}, \frac{-mn(a + b)}{am^2 - 2hlm + bl^2}\right).$$

Method to find the Separate Equations of the Lines when their Joint Equation is Given

Short-cut Methods

Method I:
(a) Write down the equation of pair of lines (1) as a quadratic equation in x (or y).
(b) Write down the value of x in terms of y (or value of y in terms of x).
(c) Do cross multiplication and bring all the terms on LHS making RHS equal to zero. The two equations thus obtained will be the equations of the two lines.

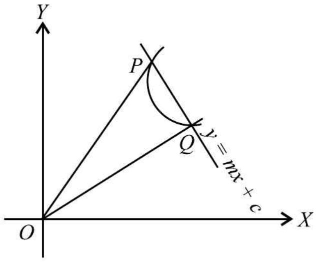
Method II:
(a) Factorize the homogeneous part $ax^2 + 2hxy + by^2$ into two linear factors, say $a_1x + b_1y$ and $a_2x + b_2y$.
(b) Add constants c_1 and c_2 in the linear factors obtained in (a) to get $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$.
(c) Write $ax^2 + by^2 + 2hxy + 2gx + 2fy + c$
$$= (a_1x + b_1y + c_1) \times (a_2x + b_2y + c_2).$$
Comparing coefficients of x and y on both sides, we get
$$a_1c_2 + a_2c_1 = 2g \text{ and } b_1c_2 + b_2c_1 = 2f.$$
Solving the above two equations, we obtain the values of c_1 and c_2.
(d) Substitute the values of c_1 and c_2 in $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, we get the required equations.

Note: A pair of lines is equally inclined to another pair of lines if the joint equation of bisectors of angles between both pair of lines is same.

Equations of the lines joining the origin to the points of intersection of a given line and a given curve

Let the straight line be $y = mx + c$ or $\frac{y - mx}{c} = 1$ and the curve be $ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0$

Let the line cuts the curve at the points P and Q, then the joint equation of OP and OQ is



$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{y - mx}{c}\right) + k\left(\frac{y - mx}{c}\right)^2 = 0.$$

This is done by making the equation of the curve homogeneous with the help of the equation of the line.

Solved Problems

1. The gradient of one of the lines $x^2 + hxy + 2y^2 = 0$ is twice that of the other, then h =

(a) ± 3 (b) $\pm \frac{3}{2}$

(c) ± 2 (d) ± 17.

Solution (a) If m_1, m_2 are gradients of lines represented by $ax^2 + 2hxy + by^2 = 0$, then

$$m_1 + m_2 = -\frac{2h}{b},\ m_1m_2 = \frac{a}{b}.$$

If $m_1 = m$, then $m_2 = 2m$ (Given)

$$\therefore \quad m + 2m = -\frac{h}{2} \text{ or } 9m^2 = \frac{h^2}{4} \quad ...(1)$$

and $$m \cdot 2m = \frac{1}{2} \text{ or } m^2 = \frac{1}{4} \quad ...(2)$$

Eliminating m^2 from (1) and (2), we get

$h^2 = 9$ or $h = \pm$**3.**

2. The coordinates of the centroid of the triangle whose sides are $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$ are

(a) $\left(\frac{8}{3}, \frac{-8}{3}\right)$ (b) $\left(-\frac{8}{3}, \frac{8}{3}\right)$

(c) $\left(\frac{8}{3}, \frac{8}{3}\right)$ (d) $\left(-\frac{8}{3}, -\frac{8}{3}\right)$

Solution (c) We have, $12x^2 - 20xy + 7y^2 = 0$

$\Rightarrow \quad (2x - y)(6x - 7y) = 0$

So, the equations of three sides of the triangle are $2x - y = 0$, $6x - 7y = 0$, $2x - 3y + 4 = 0$.

The coordinates of vertices of the triangle are (0, 0), (7, 6) and (1, 2).

[Solving the above three equations, pairwise]

$\therefore$ Centroid of the triangle is

$$\left(\frac{0+7+1}{3}, \frac{0+6+2}{3}\right) \text{ i.e., } \left(\mathbf{\frac{8}{3}, \frac{8}{3}}\right).$$

3. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then

(a) $pq = 1$ (b) $pq = -1$

(c) $pq = 2$ (d) none of these

Solution (b) The equation of the bisectors of the angles between the lines $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p} \quad \text{or} \quad \frac{x^2 - y^2}{2} = -\frac{xy}{p}$$

i.e., $$x^2 + \frac{2}{p}xy - y^2 = 0 \quad ...(1)$$

Also, $x^2 - 2qxy - y^2 = 0$ ...(2)

is the equation of the bisectors of the angles between the same lines (given).

From (1) and (2), by comparing coefficients, we get

$$\frac{1}{1} = \frac{2/p}{-2q} = \frac{-1}{-1} \text{ i.e., } 1 = -\frac{1}{pq} \text{ or } \mathbf{pq = -1.}$$

4. If the pair of straight lines $ax^2 + 2hxy - ay^2 = 0$ and $bx^2 + 2gxy - by^2 = 0$ be such that each bisects the angle between the other, then

(a) $hg = -ab$ (b) $hg = ab$

(c) $hb = ag$ (d) $hb = -ag$

Solution (a) The equation of the bisectors of the angles between the lines $ax^2 + 2hxy - ay^2 = 0$

is $$\frac{x^2 - y^2}{a - (-a)} = \frac{xy}{h}$$

or $$hx^2 - 2axy - hy^2 = 0 \quad ...(1)$$

Also, $bx^2 + 2gxy - by^2 = 0$ ...(2)

is the equation of the bisectors of the angles between the same lines (given).

From (1) and (2), by comparing coefficients

$$\frac{h}{b} = \frac{-2a}{2g} = \frac{-h}{-b} \text{ i.e., } \mathbf{hg = -ab.}$$

5. The lines bisecting the angle between the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ are given by

(a) $(a - b)(x^2 - y^2) - 4hxy = 0$

(b) $(a - b)(x^2 + y^2) + 4hxy = 0$

(c) $(a - b)(x^2 - y^2) + 4hxy = 0$

(d) none of these

Solution (c) The equation of the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

i.e., $$x^2 - y^2 = \frac{a - b}{h}\, xy$$

or $$x^2 - \frac{a - b}{h}\, xy - y^2 = 0 \qquad \text{... (1)}$$

The equation of the bisectors of the angles between the lines (1) is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-\frac{a - b}{2h}} \qquad \left[\text{Here } a = 1, b = 1, h = -\frac{a - b}{2h}\right]$$

or $$x^2 - y^2 = 2xy\left(-\frac{2h}{a - b}\right)$$

or $$\boldsymbol{(a - b)\,(x^2 - y^2) + 4hxy = 0.}$$

6. The equations $a^2x^2 + 2h\,(a + b)xy + b^2y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ represent
   (a) two pairs of perpendicular straight lines
   (b) two pairs of parallel straight lines
   (c) two pairs of straight lines which are equally inclined to each other
   (d) none of these

Solution (c) The given equations are

$$a^2x^2 + 2h(a + b)xy + b^2y^2 = 0 \qquad \text{... (1)}$$

and $$ax^2 + 2hxy + by^2 = 0 \qquad \text{... (2)}$$

The equation of the bisectors of the angles between the line pair (1) is

$$\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a + b)} \quad \text{or} \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

which is same as equation of the bisectors of angles between the line pair (2). Thus, **the two line pairs are equally inclined to each other.**

7. The lines $ax^2 + 2hxy + by^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ for
   (a) $\lambda = 1$ only (b) $\lambda = 2$ only
   (c) for any value of λ (d) none of these

Solution (c) Equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is

$$\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h} \quad \text{or} \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

which is same as the equation of the bisectors of angles between the lines $ax^2 + 2hxy + by^2 = 0$. Thus, the two line pairs are equally inclined to each other for **any value of** $\boldsymbol{\lambda}$**.**

8. The line $y = mx$ bisects the angle between the lines $ax^2 + 2hxy + by^2 = 0$ if
   (a) $h(m^2 - 1) + m\,(b - a) = 0$
   (b) $h(m^2 - 1) + m\,(a - b) = 0$
   (c) $h(m^2 + 1) + m\,(a - b) = 0$
   (d) none of these

Solution (b) Equation of bisectors of angles between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \text{ or } h(x^2 - y^2) = (a - b)xy \text{...(1)}$$

But one of the lines given by (1) is $y = mx$...(2)

$\therefore$ (2) satisfies (1)

$\Rightarrow \quad h(x^2 - m^2x^2) = (a - b)x \cdot mx$

or $\quad \boldsymbol{h(m^2 - 1) + m(a - b) = 0.}$

9. All chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin pass through the point
   (a) (1, 2) (b) (– 1, 2)
   (c) (1, – 2) (d) (– 1, – 2)

Solution (c) Let $lx + my = 1$ ...(1)

be any chord of the curve

$$3x^2 - y^2 - 2x + 4y = 0, \qquad \text{...(2)}$$

which subtends a right angle at the origin.

Making (2) homogeneous with the help of (1), we obtain

$$3x^2 - y^2 - 2x\,(lx + my) + 4y\,(lx + my) = 0$$

or $$(3 - 2l)x^2 + (4m - 1)y^2 + 2(2l - m)xy = 0.$$

The angle subtended is right angle,

$$(3 - 2l) + (4m - 1) = 0 \text{ or } l - 2m = 1,$$

which shows that (1) passes through the fixed point **(1, –2).**

10. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive direction of the axes, then a, b and h satisfy the relation
   (a) $a + b = 2|h|$ (b) $a + b = -2h$
   (c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$

Solution (b) Bisector of the angle between the positive directions of the axes is $y = x$.

Since it is one of the lines of the given pair of lines

$ax^2 + 2hxy + by^2 = 0$, we have

$x^2\,(a + 2h + b) = 0$ or $\boldsymbol{a + b = -2h.}$

11. The equation $x^2y^2 - 9y^2 + 6x^2y + 54y = 0$ represents
   (a) a pair of straight lines and a circle
   (b) a pair of straight lines and a parabola
   (c) a set of four straight lines forming a square
   (d) none of these

Solution (c) We have, $x^2y^2 - 9y^2 - 6x^2y + 54y = 0$

$\Rightarrow \quad y^2(x^2 - 9) - 6y(x^2 - 9) = 0$

$\Rightarrow \quad y(y - 6)\,(x - 3)\,(x + 3) = 0$

$\Rightarrow \quad y = 0, y = 6, x = 3, x = \boldsymbol{-3.}$

So, the given equation represents **four straight lines which form a square.**

12. If the ratio of gradients of the lines, represented by $ax^2 + 2hxy + by^2 = 0$ is 1 : 3, then the value of the ratio $h^2 : ab$ is
   (a) $\frac{1}{3}$ (b) $\frac{3}{4}$
   (c) $\frac{4}{3}$ (d) 1

Solution (c) Let m, $3m$ be the gradients of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$$\therefore \quad m + 3m = 4m = -\frac{2h}{b} \qquad \text{...(1)}$$

and $$m \cdot 3m = 3m^2 = \frac{a}{b} \qquad \text{...(2)}$$

Substituting the value of m from (1) in (2), we get

$$3\left(-\frac{h}{2b}\right)^2 = \frac{a}{b} \Rightarrow \frac{h^2}{ab} = \mathbf{\frac{4}{3}}.$$

13. The equation $2x^2 + kxy + 2y^2 = 0$ represents a pair of real and distinct lines if

(a) $k \in (-4, 4)$
(b) $k \in R$
(c) $k \in (-\infty, -4) \cup (4, \infty)$
(d) none of these

Solution (c) The given equation will represent a pair of real and distinct lines if $h^2 > ab$

i.e., $\left(\frac{k}{2}\right)^2 > (2)(2)$ or $k^2 > 16$ or $(k-4)(k+4) > 0$

i.e., $\mathbf{k \in (-\infty, -4) \cup (4, \infty)}$.

14. The equation of the image of the lines $y = |x|$ by the line $x = 2$ is

(a) $y = |x - 4|$ (b) $|y| = x + 4$
(c) $|y| + 4 = x$ (d) none of these

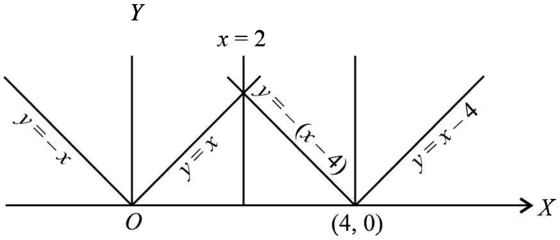
Solution (a) The image of the lines

$y = x$ and $y = -x$ by the line $x = 2$

are given by the lines

$y = x - 4$ and $y = -(x - 4)$.

Their combined equation is $\mathbf{y = |x - 4|}$.



15. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value

(a) 2 (b) –1
(c) 1 (d) –3

Solution (a) Given pair of lines is $x^2 - 2cxy - 7y^2 = 0$

Given: $m_1 + m_2 = 4m_1m_2 \Rightarrow -\frac{2c}{7} = \frac{-4}{7} \Rightarrow c = 2$

$$\left[\because m_1 + m_2 = -\frac{2h}{b} = -\frac{2c}{7}; m_1m_2 = \frac{a}{b} = -\frac{1}{7}\right]$$

16. The straight lines joining the origin to the points of intersection of the two curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ are at right angles, if

(a) $g(a' + b') = g'(a + b)$
(b) $g(a + b) = g'(a' + b')$
(c) $g(a' - b') = g'(a - b)$
(d) none of these

Solution (a) The given curves are

$$ax^2 + 2hxy + by^2 + 2gx = 0, \quad \text{...(1)}$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0. \quad \text{...(2)}$$

From (1), we obtain

$$2x = -\frac{1}{g}(ax^2 + 2hxy + by^2) \quad \text{...(3)}$$

Making (2) homogeneous with the help of (3), we obtain

$$a'x^2 + 2h'xy + b'y^2 - \frac{g'}{g}(ax^2 + 2hxy + by^2) = 0$$

or, $g(a'x^2 + 2h'xy + b'y^2) - g'(ax^2 + 2hxy + by^2) = 0$ ...(4)

(4) represents two straight lines joining the origin to the points of intersection of (1) and (2). The lines will be ⊥ if, coefficient of x^2 + coefficient of $y^2 = 0$

i.e., $(ga' - g'a) + (gb' - g'b) = 0$

Hence, $\mathbf{g'(a + b) = g(a' + b')}$.

17. If the equation $hxy + gx + fy + c = 0$, $(h \neq 0)$ represents two straight lines, then

(a) $2fgh = c^2$ (b) $2fg = ch$
(c) $fgh = c^2$ (d) $fg = ch$

Solution (d) Comparing the given equation with the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 0$, $b = 0$, $h = h/2$, $g = g/2$, $f = f/2$, $c = c$.

Since the given equation represents a pair of straight lines, therefore

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 0 + 2\left(\frac{f}{2}\right)\left(\frac{g}{2}\right)\left(\frac{h}{2}\right) - 0 - 0 - c\left(\frac{h}{2}\right)^2 = 0$$

$$\Rightarrow \frac{fgh}{4} - \frac{ch^2}{4} = 0 \text{ or } \mathbf{fg = ch}.$$

18. The pair of lines $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ are rotated about the origin by $\frac{\pi}{6}$ in the anti-clockwise sense. The equation of the pair in the new position is

(a) $x^2 - \sqrt{3}xy = 0$ (b) $xy - \sqrt{3}y^2 = 0$
(c) $\sqrt{3}x^2 - xy = 0$ (d) none of these

Solution (c) The given equation of pair of straight lines can be rewritten as

$$(\sqrt{3}x - y)(x - \sqrt{3}y) = 0$$

Their separate equations are $y = \sqrt{3}x$ and $y = \frac{1}{\sqrt{3}}x$

or $y = \tan 60° \, x$ and $y = \tan 30° \, x$.

After rotation, the separate equations are

$y = \tan 90° \, x$ and $y = \tan 60° \, x$

or $x = 0$ and $y = \sqrt{3}x$.

∴ The combined equation in the new position is

$x(\sqrt{3}x - y) = 0$ or $\mathbf{\sqrt{3}x^2 - xy = 0}$.

19. If the lines represented by the equation $3y^2 - x^2 + 2\sqrt{3}x - 3 = 0$ are rotated about the point $(\sqrt{3}, 0)$ through an angle 15°, one clockwise direction and other in anticlockwise direction, then the equation of the pair of lines in the new position is

(a) $y^2 - x^2 + 2\sqrt{3}x + 3 = 0$
(b) $y^2 - x^2 + 2\sqrt{3}x - 3 = 0$
(c) $y^2 - x^2 - 2\sqrt{3}x + 3 = 0$
(d) none of these

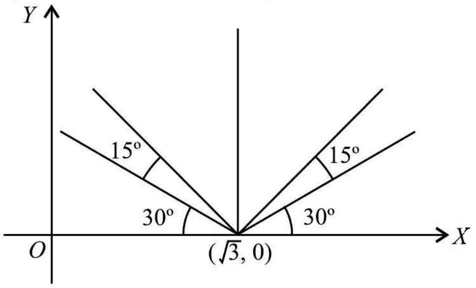
Solution (b) The given equation of pair of straight lines can be rewritten as $(\sqrt{3}y - x + \sqrt{3})(\sqrt{3}y + x - \sqrt{3}) = 0$

Their separate equations are

$$\sqrt{3}y - x + \sqrt{3} = 0 \text{ and } \sqrt{3}y + x - \sqrt{3} = 0$$

or $y = \frac{1}{\sqrt{3}}x - 1$ and $y = -\frac{1}{\sqrt{3}}x + 1$.

or $y = (\tan 30°)x - 1$ and $y = (\tan 150°)x + 1$.



After rotation through an angle of 15°, the lines are

$(y - 0) = \tan 45° (x - \sqrt{3})$

and $(y - 0) = \tan 135° (x - \sqrt{3})$

or $y = x - \sqrt{3}$ and $y = -x + \sqrt{3}$.

Their combined equation is

$(y - x + \sqrt{3})(y + x - \sqrt{3}) = 0$

or $\mathbf{y^2 - x^2 + 2\sqrt{3}\,x - 3 = 0.}$

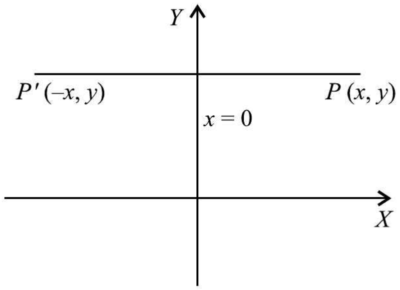
20. The image of the pair of lines represented by $3x^2 + 4xy + 5y^2 = 0$ by the line mirror $x = 0$ is

(a) $3x^2 - 4xy + 5y^2 = 0$
(b) $3x^2 - 4xy - 5y^2 = 0$
(c) $5y^2 - 4xy - 3x^2 = 0$
(d) none of these

Solution (a) Replacing x by $-x$, the image of the pair of lines

$3x^2 + 4xy + 5y^2 = 0$ is

$3x^2 + 4y(-x) + 5y^2 = 0$ or $\mathbf{3x^2 - 4xy + 5y^2 = 0.}$



21. The equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$, where λ is a real number, represents a pair of straight lines. If θ is the angle between the lines, then $\operatorname{cosec}^2\theta$ =

(a) 3 (b) 9
(c) 10 (d) 100

Solution (c)

22. The equation $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents

(a) pair of lines (b) a parabola
(c) a hyperbola (d) none of these

Solution (d)

23. The new equation of the curve
$12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$,
after removing the first degree terms is

(a) $12X^2 - 7XY - 12Y^2 = 0$
(b) $12X^2 + 7XY - 12Y^2 = 0$
(c) $12X^2 + 7XY + 12Y^2 = 0$
(d) none of these

Solution (b)

24. Mixed term xy is to be removed from the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, one should rotate the axes through an angle θ, given by $\tan 2\theta$ equal to

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$
(c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

Solution (d)

25. Given the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$, the angle through which the axes be rotated so that the term in xy required from the transformed equation, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{2\pi}{3}$ (d) none of these

Solution (a), (c)

Problems for Practice

Level of Difficulty I

1. The angle between the lines whose joint equation is $2x^2 - 3xy + y^2 = 0$, is

(a) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\tan^{-1}(\sqrt{3})$
(c) $\tan^{-1}\left(\frac{1}{3}\right)$ (d) none of these

2. If the equation $3x^2 - 8xy + \lambda y^2 = 0$ repersents two perpendicular lines, then the value of λ is

(a) 3 (b) −3 (c) 2 (d) none of these

3. If the lines represented by $2x^2 + 8xy + ky^2 = 0$ are coincident, then the values of k is

(a) 8 (b) −8 (c) 4 (d) none of these

4. The angle between the lines represented by
$x^2 + 2xy\sec\theta + y^2 = 0$ is

(a) 4θ (b) 2θ (c) θ (d) none of these

5. If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then the value of λ is

(a) 4 (b) 3 (c) 2 (d) 1

6. The distance between the two lines represented by the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is

(a) $\frac{8}{5}$ (b) $\frac{6}{5}$ (c) $\frac{11}{5}$ (d) none of these

7. The angle between the lines joining the origin to the points of intersection of the line $y = 3x + 2$ and the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is

(a) $\tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ (b) $\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

(c) $\tan^{-1}\left(\frac{3\sqrt{2}}{3}\right)$ (d) none of these

8. If the curve $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts on the line $lx + my = 1$, a length which subtends a right angle at the origin, then

(a) $c(l^2 + m^2) + 2(gl + fm + 1) = 0$

(b) $c(l^2 + m^2) - 2(gl + fm + 1) = 0$

(c) $c(l^2 + m^2) + 2(gl + fm - 1) = 0$

(d) none of these

9. The pair of lines which join the origin to the points of intersection of the line $y = mx + c$ with the curve $x^2 + y^2 = a^2$ are at right angles, if

(a) $c^2 = a^2(1 + m^2)$ (b) $2c^2 = a^2(1 + m^2)$

(c) $2c^2 = a^2(1 - m^2)$ (d) none of these

10. The angle between the lines joining the origin to the points of intersection of the line $x\sqrt{3} + y = 2$ and the curve $x^2 + y^2 = 4$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

11. The separate equations of the straight lines whose joint equatiion is $x^2 - 5xy + 6y^2 = 0$, are

(a) $x + 2y = 0, x - 3y = 0$

(b) $x - 2y = 0, x - 3y = 0$

(c) $x - 2y = 0, x + 3y = 0$

(d) $x - 2y = 0, x - 3y = 0$

12. The value of λ for which the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represents two straight lines is

(a) 1 (b) 2 (c) -1 (d) -2

13. The angle between the straight lines joining the origin to the points of intersection of $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y = 1$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

14. If the lines joining the origin to the points of intersection of $y = mx + 1$ with $x^2 + y^2 = 1$ are $\perp$, then m is equal to

(a) 2 (b) 1 (c) -1 (d) -2

15. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1}m$, then $m =$

(a) $\frac{1}{5}$ (b) 1 (c) $\frac{7}{5}$ (d) 7

Level of Difficulty II

16. Which of the following pair of straight lines intersect at right angles?

(a) $2x^2 = y(x + 2y)$ (b) $(x + y)^2 = x(y + 3x)$

(c) $2y(x + y) = xy$ (d) $y = \mp 2x$

17. If the equation $3x^2 + 6xy + my^2 = 0$ represents a pair of straight lines inclined at an angle π, then m is equal to

(a) 3 (b) 6 (c) 9 (d) any real number

18. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals

(a) 3 (b) -1 (c) 1 (d) -3

19. If the gradient of one of the lines $x^2 + hxy + 2y^2 = 0$ is twice that of the other, then $h =$

(a) ± 2 (b) ± 3 (c) ± 1 (d) $\pm\frac{3}{2}$

20. If the lines $px^2 - qxy - y^2 = 0$ make the angles α and β with x-axis, then the value of $\tan(\alpha + \beta)$ is

(a) $\frac{-q}{1+p}$ (b) $\frac{q}{1+p}$ (c) $\frac{p}{1+q}$ (d) $\frac{-p}{1+q}$

21. The point of intersection of the pair of straight lines given by the equation $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$, is

(a) $(1, 1)$ (b) $(1, -1)$

(c) $(-1, 1)$ (d) $(-1, -1)$

22. The equation of the bisectors of the angles between the lines joining the origin to the points of intersection of the straight line $x - y = 2$ with the curve
$5x^2 + 11xy - 8y^2 + 8x - 4y + 12 = 0$ is

(a) $x^2 + 30xy + y^2 = 0$ (b) $x^2 - 30xy + y^2 = 0$

(c) $x^2 + 30xy - y^2 = 0$ (d) none of these

23. The straight lines represented by the equation $135x^2 - 136xy + 33y^2 = 0$ are equally inclined to the line

(a) $x - 2y = 7$(b) $x + 2y = 7$

(c) $x - 2y = 4$(d) $3x + 2y = 4$

24. If the distance of a given point (α, β) from each of two straight lines through the origin is d, then

(a) $(\alpha x - \beta y)^2 = d^2(x^2 + y^2)$

(b) $(\alpha y - \beta x)^2 = d^2(x^2 + y^2)$

(c) $(\alpha x + \beta y)^2 = d^2(x^2 + y^2)$

(d) none of these

25. The three lines whose combined equation is $(3x^2 + 2xy - 3y^2)(x - y + 2) = 0$ form a triangle which is

(a) equilateral (b) right angled

(c) obtuse angled (d) none of these

26. Two pairs of straight lines have the equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. One line will be common among them if

(a) $a + 8h - 16b = 0$ (b) $a - 8h + 16b = 0$

(c) $a - 6h + 9b = 0$ (d) $a + 6h + 9b = 0$

27. The equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines

(a) which are equally inclined to one another

(b) two of which are at right angles

(c) two of which are coincident

(d) which pass through origin

28. The condition that the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ should be λ times the slope of the other is

(a) $ab(1 + \lambda)^2 = h^2\lambda$ (b) $ab(1 + \lambda)^2 = 2h^2\lambda$

(c) $ab(1 + \lambda)^2 = 3h^2\lambda$ (d) $ab(1 + \lambda)^2 = 4h^2\lambda$

29. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents a pair of perpendicular straight lines, then

(a) $p = 12, q = 1$ (b) $p = 1, q = 12$

(c) $p = -1, q = 12$ (d) $p = 1, q = -12$

30. The circumcentre of the triangle formed by the lines $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is

(a) (0, 0) (b) (– 2, – 2)

(c) (– 1, – 1) (d) (– 1, – 2)

ANSWERS

1. (c)	**2.** (b)	**3.** (a)	**4.** (c)	**5.** (c)	**6.** (a)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (c)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b), (c)	**15.** (a)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (a)
21. (c)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (b), (d)	**27.** (a), (d)	**28.** (d)	**29.** (a)	**30.** (c)

3

Circles

BRIEF REVIEW OF THE CONCEPTS

CIRCLE

A circle is the locus of a point which moves in a plane such that its distance from a fixed point always remains constant. The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

STANDARD EQUATION OF A CIRCLE

The equation of a circle with the centre at (h, k) and radius a, is

$$(x - h)^2 + (y - k)^2 = a^2$$

If the centre of the circle is at the origin and radius is a, then the equation of circle is $x^2 + y^2 = a^2$.

GENERAL EQUATION OF A CIRCLE

The general equation of a circle is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad ...(1)$$

where g, f and c are constants.

The coordinates of its centre are $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

CONDITIONS FOR AN EQUATION TO REPRESENT A CIRCLE

A general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in x, y represents a circle if

(i) coefficient of x^2 = coefficient of y^2, i.e., $a = b$,

(ii) coefficient of xy is zero, i.e., $h = 0$.

To Find the Centre and Radius of a Circle whose Equation is GIven

Short-cut Methods

1. Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
2. The coordinates of centre will be (h, k), where

$$h = -\frac{1}{2} \text{ (coefficient of } x)$$

and $$k = -\frac{1}{2} \text{ (coefficient of } y)$$

3. Radius = $\sqrt{h^2 + k^2 - \text{constant term}}$

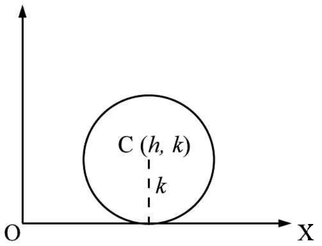
Nature of the Circle

1. If $g^2 + f^2 - c > 0$, then the general eqn. (1) represents real circle with centre $(-g, -f)$.
2. If $g^2 + f^2 - c = 0$, then the general eqn. (1) represents a circle whose centre is $(-g, -f)$ and radius is zero i.e., the circle coincides with the centre represented by a point $(-g, -f)$. It is, therefore called a *point circle*.
3. If $g^2 + f^2 - c < 0$, the radius of the circle is imaginary but the centre is real. Such a circle is called a *virtual circle or imaginary circle* as it is not possible to draw such a circle.

Different Forms of the Equation of a Circle

1. Circle with centre at the point (h, k) and which touches the axis of x

Since the circle touches the x-axis, the radius of the circle $= k$.



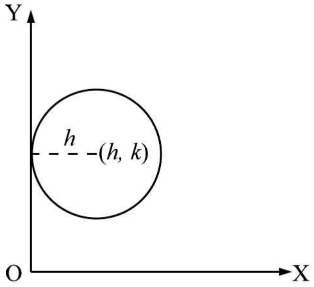
Therefore, the equation of the circle is:

$$(x - h)^2 + (y - x)^2 = k^2$$

or $$x^2 + y^2 - 2hx - 2ky + h^2 = 0.$$

2. Circle with centre at the point (h, k) and which touches the axis of y

Since the circle touches the y-axis, the radius of the circle $= h$.



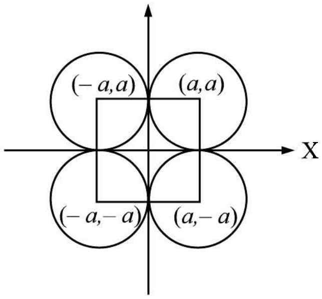
Therefore, the equation of the circle is

$$(x-h)^2-(y-k)^2=h^2$$

or $$x^2+y^2-2hx-2ky-k^2=0.$$

3. Circle with radius *a* and which touches both the coordinate axes

Since the centre of the circle may be in any of the four quadrants, therefore it will be any one of the four points $(\pm a, \pm a)$. Thus, there are four circles of radius a touching both the coordinate axes, and their equations are

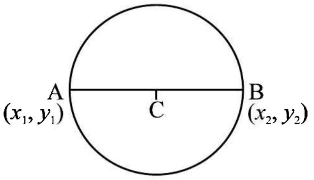


$$(x\pm a)^2+(y\pm a)^2=a^2$$

or $$x^2+y^2\pm 2ax\pm 2ay+a^2=0.$$

4. Equation of the Circle on the Line Joining two Points as as Diameter

The equation of the circle drawn on the line segment joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is



$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Its Centre $=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and

Radius $=\sqrt{\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}$.

Intercepts made by a Circle on the axes

(a) The Length of the intercept made by the circle $x^2+y^2+2gx+2fc+c=0$ on

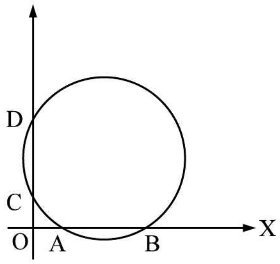
x-axis $= AB = 2\sqrt{g^2-c}$

y-axis $= CD = 2\sqrt{f^2-c}$

(b) Intercepts are always positive.

(c) If the circle touches x-axis then $|AB| = 0$

Thus, $c = g^2$.



(d) If the circle touches y-axis, then $|CD| = 0$

Thus, $c = f^2$.

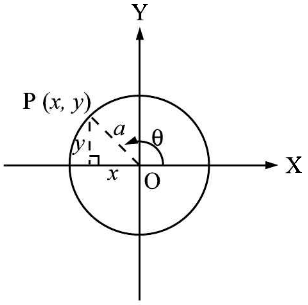
(e) If the circle touches both the axes, then

$|AB| = 0 = |CD|$

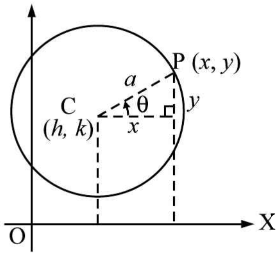
Thus, $c = g^2 = f^2$.

Parametric Equations of a Circle

(a) The parametric equations of a circle $x^2+y^2=a^2$ are $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta < 2\pi$. θ is called parameter and the point $P(a\cos\theta, a\sin\theta)$ is called the point 'θ' on the circle $x^2+y^2=a^2$. Thus, the coordinates of any point on the circle $x^2+y^2=a^2$ may be taken as $(a\cos\theta, a\sin\theta)$.

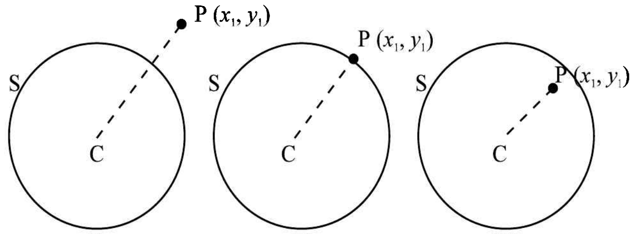


(b) The parametric equations of a circle $(x-h)^2+(y-k)^2=a^2$ are $x = h + a\cos\theta, y = k + a\sin\theta, 0 \le \theta < 2\pi$. is called the point '$\theta$' on this circle. Thus the coordinates of any point on this circle may be taken as $(h + a\cos\theta, k + a\sin\theta)$.



POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let $S \equiv x^2+y^2+2gx+2fy+c=0$, be a circle and $P(x_1, y_1)$ be a point in the plane of S, then $S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c$. The point $P(x_1, y_1)$ lies outside, on or inside the circle S according as $S_1 >, =$ or < 0, respectively.



(i) (ii) (iii)

Note: Let S be a circle and $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane of S, then they lie

(i) on the same side of S iff S_1 and S_2 have same sign,

(ii) on the opposite sides of S iff S_1 and S_2 have opposite signs,

CIRCLE THROUGH THREE POINTS

The equation of the circle through three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

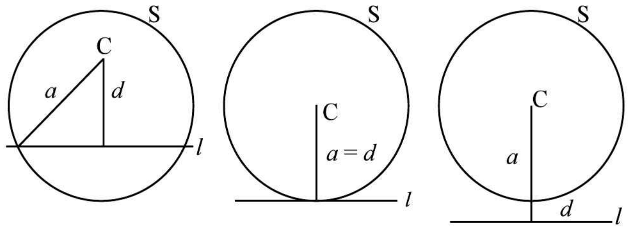
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Notes: 1. If the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct points, then those points are concyclic if $a_1a_2 = b_1b_2$.

Also, the equation of the circle passing through those concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)xy = 0$.

2. The equation of the circumcircle of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is $ab(x^2 + y^2) + c(bx + ay) = 0$.

INTERSECTION OF A LINE AND A CIRCLE



(*a*) (*b*) (*c*)

Let S be a circle with centre C and radius *a*. Let *l* be any line in the plane of the circle and *d* be the perpendicular distance from *C* to the line *l*, then

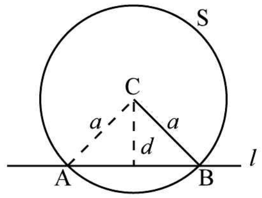
(a) *l* intersects *S* in two distinct points iff $d < a$

(b) *l* intersects *S* in one and only point iff $d = a$, i.e., *the line l touches the circle iff perpendicular distance from the centre to the line l must be equal to radius of the circle.*

(c) *l* does not intersect S iff $d > a$.

LENGTH OF INTERCEPT MADE BY A CIRCLE ON A LINE

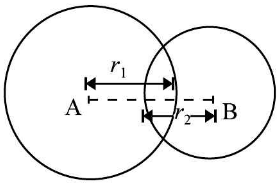
If a line *l* meets a circle S, with centre *C* and radius *a*, in two distinct points and if *d* is the perpendicular distance of centre *C* from the line *l*, then the length of the intercept made by the circle on the line $|AB| = 2\sqrt{a^2 - d^2}$.



Point to Note: If the points of intersection of a line *l* and a circle S are known, then the distance between these points is the required length of intercept and there is no need of using the above formula.

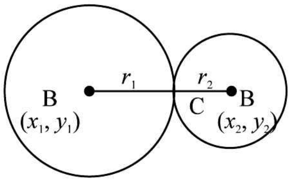
CONTACT OF TWO CIRCLES

The two circles having centres at $A(x_1, y_1)$ and $B(x_2, y_2)$ and radii r_1 and r_2 respectively will



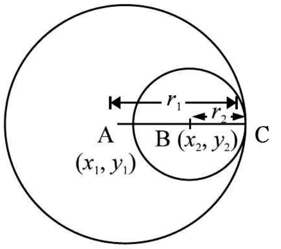
(a) intersect in two real distinct points if and only if $|r_1 - r_2| < AB < r_1 + r_2$,

(b) touch each other externally if and only if $AB = r_1 + r_2$ and their point of contact *C* is given by



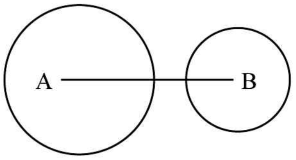
$$C \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$

(c) touch each other internally if and only if $AB = |r_1 - r_2|$, and their point of contact C is given by

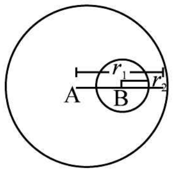


$$C \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right).$$

(d) One circle lies outside the other if $AB > r_1 + r_2$.



(e) One circle is contained in the other if $AB < |r_1 - r_2|$.



Notations

If $S \equiv x^2 + y^2 - a^2$, then $S_1 \equiv x_1^2 + y_1^2 - a^2$ and $T \equiv xx_1 + yy_1 - a^2$.

If $S \equiv x^2 + y^2 + 2gx + 2fy + c$, then

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

S_1 is obtained from *S* by putting x_1 and y_1 in place of *x* and *y* respectively and *T* is obtained by putting xx_1 in place of x^2, yy_1, in place of y^2, $\frac{x + x_1}{2}$ in place of *x* and $\frac{y + y_1}{2}$ in place of *y*.

TANGENT TO A CIRCLE AT A GIVEN POINT

1. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) on it is: $xx_1 + yy_1 = a^2$.
2. Equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

at the point (x_1, y_1) on it is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

3. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ on it is:

$$x\cos\theta + y\sin\theta = a$$

[Parametric form of equation of tangent]

Point to Note: The equation of the tangent at the point (x_1, y_1) on the circle $S = 0$ is $T = 0$.

Equation of the Tangent in Slope Form

The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1+m^2}$.

The coordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right).$$

Condition of Tangency

The straight line $y = mx + c$ will be a tangent to the circle

$$x^2 + y^2 = a^2 \text{ if } c = \pm a\sqrt{1+m^2}.$$

Note: A line will touch a circle if and only if the length of the perpendicular from the centre of the circle to the line is equal to the radius of the circle.

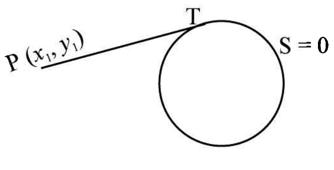
Tangents From a Point Outside the Circle

Short-cut Method

1. Let the point be (x_1, y_1).
   Write the equation of a straight line passing through the point (x_1, y_1) and having slope m i.e.,
   $$(y - y_1) = m(x - x_1) \quad ...(1)$$
2. Find the length of the perpendicular from the centre of the circle to the line (1) and equate it to the radius of the circle. Call this equation as (2).
3. Obtain the value of m from the eqn. (2).
4. Substitute this value of m in equation (1) to obtain the required equation of tangent.

Length of the Tangent From a Point to a Circle

The length of the tangent that can be drawn from the point $P(x_1, y_1)$ to the circle $S = 0$ is $\sqrt{S_1}$, where the coefficients of x^2 and y^2 in the equation of the circle are unity. Thus, the length of the tangent from the point P (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by



$$PT = \sqrt{x_1^2 + y_1^2 + 2gx, +2fy_1 + c}$$

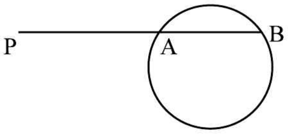
Power of a Point with Respect to a Circle

The power of a point $P(x_1, y_1)$ with respect to a given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is defined as:

$$\text{Power} = PA \times PB$$

where PAB is any line through P intersecting the circle in A and B.

If PT is the length of the tangent from a point P to a given circle, the PT^2 is also the power of the point P with respect to the given circle. Thus, power of the point $P(x_1, y_1)$ w.r.t. the circle $S = 0$ is S_1.

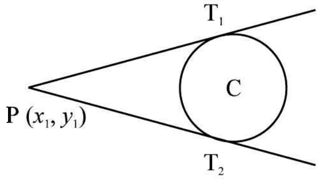


Notes:

1. The power of the point outside the circle is positive.
2. The power of the point on the circle is zero.
3. The power of the point inside the circle is negative.

Pair of Tangents

The equation of the pair of tangents drawn from the point $P(x_1, y_1)$ to the circle



$S = 0$ is $SS_1 = T^2$, where

$S : x^2 + y^2 + 2gx + 2fy + c,$

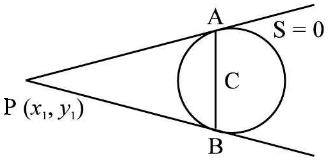
$S_1 : x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

and $T : xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$

Note: The pair of tangents from (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angles if $g^2 + f^2 = 2c$.

CHORD OF CONTACT OF TANGENTS

Let P be a point outside a circle. If two tangents PA and PB be drawn to the circle, then the chord AB is called the **chord of contact** of tangents drawn from the point P to the circle.



The equation of the chord of contact of tangents drawn from the point $P(x_1, y_1)$ is given by

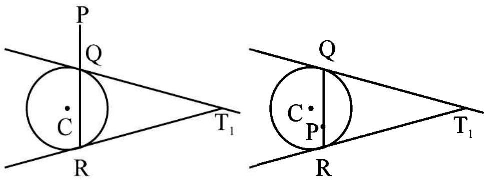
$T = 0$, i.e., $xx_1 + yy_1 = a^2$ for the circle $x^2 + y^2 = a^2$

and $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Pole and Polar

If a straight line is drawn from a point P to meet the circle at Q and R and the tangents to the circle at Q and R meet at a point T_1, then the locus of T_1 is called the polar of P with respect to the circle.



The point P is called the *pole of its polar.*

The polar of the point $P(x_1, y_1)$ w.r.t. the circle $S = 0$ is given by $T = 0$, i.e.,

$$xx_1 + yy_1 + = a^2, \text{ for the circle } x^2 + y^2 = a^2 \text{ and}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Notes:

1. If the point P lies outside the circle, then the polar and the chord of contact of this point P are same straight line.
2. If the point P lies on the circle, then the polar and the tangent to the circle at P are same straight line.
3. The coordinates of the pole of the line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$ are $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.
4. If (x_1, y_1) is the pole of the line $lx + my + n = 0$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then $\frac{x_1 + g}{l} = \frac{y_1 + f}{m} = \frac{r^2}{lg + mf - n}$, where r is the radius of the circle.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal to a circle, at any point on the circle is a straight line which is perpendicular to the tangent to the circle at that point and always passes through the centre of the circle.

1. Equation of the normal to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) on it is: $\frac{x}{x_1} = \frac{y}{y_1}$.
2. Equation of the normal to the circle
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
at the point (x_1, y_1) on it is
$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$

Alternate Method Equation of the line joining (x_1, y_1) and centre of the circle gives the equation of normal to the circle.

CONJUGATE POINTS AND LINES

Two points are said to be *conjugate points* with respect to a circle if the polar of either passes through the other.

Two straight lines are said to be *conjugate lines* if the pole of either lies on the other.

Notes:

1. $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ if $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$
2. If P and Q are conjugate points w.r.t. a circle with centre at O and radius r, then $PQ^2 = OP^2 + OQ^2 - 2r^2$.

CHORD WITH A GIVEN MIDDLE POINT

The equation of the chord of the circle $S = 0$, whose middle point is (x_1, y_1), is given by $T = S_1$, i.e.,

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and

$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$ for the circle $x^2 + y^2 = a^2$.

Notes:

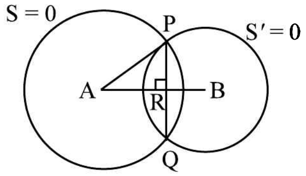
1. If (x_1, y_1) is the mid point of the chord PQ of the circle $S = 0$, then the length $PQ = 2\sqrt{-S_1}$, where $S_1 < 0$.
2. The coordinates of middle point of the chord $lx + my + n = 0$ of the circle $x^2 + y^2 = a^2$ are $\left(\frac{-ln}{l^2 + m^2}, \frac{-mn}{l^2 + m^2}\right)$.
3. Let (x_1, y_1) be the mid point of the chord PQ of the circle $S = 0$. If the tangents at P and Q meet at R, then the area of $\Delta PQR = \frac{(-S_1)^{3/2}}{\sqrt{S_1 + a^2}}$, where a is the radius of the circle.

COMMON CHORD OF TWO CIRCLES

The chord joining the point of intersection of two circles is called their common chord.

If $S = 0$ and $S' = 0$ be two interesting circles, then the equation of their common chord is

$$S - S' = 0.$$



Note: In order to find the equation of the common chord of two circles, first of all make the coefficients of x^2 and y^2 in the equation of the two circles equal to unity.

Length of the Common Chord The length of the common chord PQ is given by

$$PQ = 2PR = 2\sqrt{(AP)^2 - (AR)^2},$$

where AP is the radius of the circle $S = 0$ and AR is the length of perpendicular from A to the common chord PQ.

Notes:

1. The length of the common chord of two circles becomes maximum when it is a diameter of the smaller one between them.
2. If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

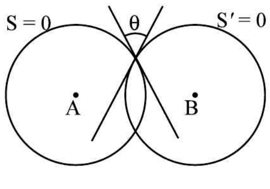
ANGLE OF INTERSECTION OF TWO CIRCLES

The angle between the two circles is the angle between their tangents at their point of intersection.

The angle of intersection θ of two circles

$$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$



is given by $\cos\theta = \pm\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}.\sqrt{g_2^2 + f_2^2 - c_2}}$

Orthogonal Intersection of two Circles

Two circles are said to intersect orthogonally when they intersect at right angles.

The condition for the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to intersect orthogonally is given by

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2.$$

Note: If P and Q are conjugate points w.r.t. the circle $S = 0$, then the circle on PQ as diameter is orthogonal to $S = 0$.

RADICAL AXIS

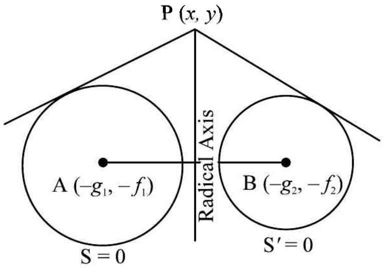
The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from it to the two circles are equal. The equation of radical axis of two circles

$$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

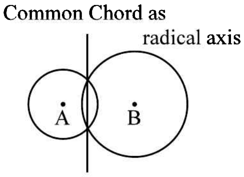
is $$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

or $$S - S' = 0.$$

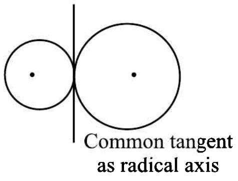


Notes:

1. If the two circles cut each other at two different points, then their radical axis will be their common chord.



2. If the two circles touch each other (internally or externally), then their radical axis will be the common tangent at the point of contact.



PROPERTIES OF RADICAL AXIS

1. The radical axis of two circles is perpendicular to the line joining the centres.
2. **Radical Centre** The radical axis of three circles taken in pairs meet at a point, called the radical centre of the circles. Coordinates of radical centre can be found by solving the equations $S_1 = S_2 = S_3 = 0$.
3. The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.
4. If two circles cut a third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle.

EQUATION OF A CIRCLE THROUGH THE INTERSECTION OF A LINE AND A CIRCLE

Equation of the family of circles passing through the points of intersection of the circle S = 0 and the line $L = 0$ is

$$S + kL = 0, \text{ where } k \text{ is a parameter, } k \neq -1.$$

EQUATION OF A CIRCLE THROUGH THE INTERSECTION OF TWO CIRCLES

Equation of the family of circles passing thought the points of intersection of circles $S = 0$ and $S' = 0$ is $S + kS' = 0$, where k is a parameter, $k \neq -1$.

COAXAL SYSTEM OF CIRCLES

A system of circles is said to be coaxal if each pair of circles of the system has the same radical axis.

Equation of a System of Coaxal Circles in the Simplest Form

The equation of a system of coaxal circles is

$$x^2 + y^2 + 2gx + c = 0,$$

where g is a constant and c is a parameter and the line of centre of the circles is taken as x-axis and the radical axis as the y-axis.

Notes:

1. The equation of a system of coaxal circles, when the equation of the radical axis and one circle of the system are

   $P \equiv lx + my + n = 0$

   and $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

   respectively, is $S + kP = 0$ (k is an arbitrary constant)

2. The equation of a coaxal system of circles when the equation of any two circles of the system are

   $$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

   and $$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0,$$

   is $$S + k(S - S') = 0$$

   or $$S' + k(S - S') = 0,$$

   i.e., $$S + kS' = 0\ (k \neq -1).$$

LIMITING POINTS OF A COAXAL SYSTEM

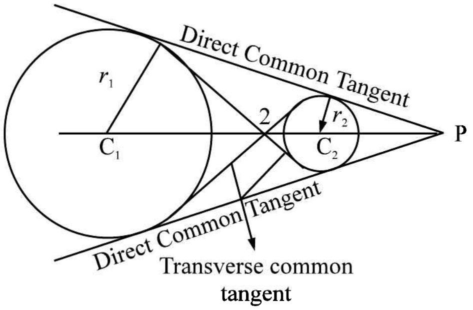
Limiting points of coaxal system of circles, are point circles of the system i.e., circles of radius zero. The limiting points of the system of circles

$$x^2 + y^2 + 2gx + c = 0 \text{ are } (\pm\sqrt{c}, 0).$$

COMMON TANGENTS TO TWO CIRCLES

Direct Common Tangents

The direct Common tangents to the two circles meet at a point (say P) which lies on the line joining the centres C_1 and C_2 of the two circles and divide C_1C_2 externally in the ratio of their radii say (r_1 and r_2)



Short-cut Method

1. Find the coordinates of centres C_1, C_2 and radii r_1, r_2 of two given circles.
2. Find the coordinates of the point P dividing C_1C_2 in the ratio $r_1 : r_2$ externally. Let $P \equiv (h, k)$.
3. Write the equation of any line through the point $P(h, k)$, i.e.,
$$(y - k) = m\,(x - h) \qquad ...(1)$$
4. Find the two values of m, using the fact that the length of perpendicular on line (1) from the centre C_1 of one circle is equal to its radius r_1.
5. Substituting these values of m in eqn. (1), the equations of two direct common tangents are obtained.

Transverse Common Tangents

The transverse common tangents to the two circles intersect at a point (say θ) which lies on the line joining the centres C_1 and C_2 of the two circles and divide C_1C_2 internally in the ratio of their radii r_1 and r_2.

Short-cut Method

1. Find the coordinates of centres C_1, C_2 and radii r_1, r_2 of two given circles.
2. Find the coordinates of the point θ dividing C_1C_2, in the ratio $r_1 : r_2$ internally. Let $P \equiv (h, k)$.
3. Write the equation of any line through the point $P\,(h, k)$
$$(y - k) = m\,(x - h) \qquad ...(1)$$
4. Find the two values of m, using the fact that the length of perpendicular on (1) from the centre C_1 of one circle is equal to its radius r_1.
5. Substituting these values of m in eqn. (1), the equations of two transverse common tangents are obtained.

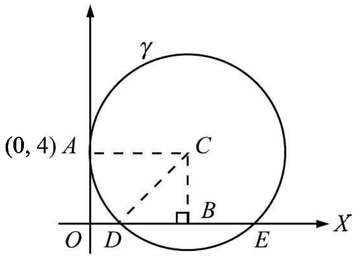
Notes:

1. If two circles do not intersect ($c_1c_2 > r_1 + r_2$), then they have two transverse and two direct common tangents.
2. If two circles intersect ($c_1c_2 < r_1 + r_2$), then they have two direct tangents only.
3. If two circles touch externally ($c_1c_2 = r_1 + r_2$), then they have one transverse and two direct common tangents.
4. If two circles touch internally ($c_1c_2 = r_1 - r_2$), then they have only one common tangent.

Solved Problems

1. The equation of the circle which touches the axis of y at a distance + 4 from the origin and cuts off an intercept 6 from the axis of x is
(a) $x^2 + y^2 - 10x - 8y + 16 = 0$
(b) $x^2 + y^2 + 10x - 8y + 16 = 0$
(c) $x^2 + y^2 - 10x + 8y + 16 = 0$
(d) none of these

Solution (a) Let the centre of the circle be C. Let the circle touches y-axis at A and cuts off intercept DE from x-axis. Let B be the mid point of DE.



Now, $OA = 4 = BC$ and $BD = 3$.

$\therefore$ Radius of the circle $= \sqrt{BD^2 + BC^2}$

$= \sqrt{9 + 16} = 5.$

Also, $OB = AC =$ radius $= 5$.

$\therefore$ Centre of the circle is (5, 4).

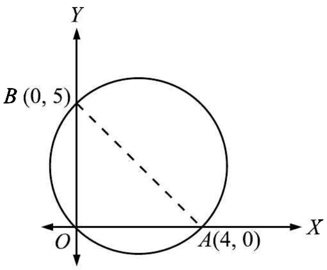
Thus, equation of the required circle is
$$(x - 5)^2 + (y - 4)^2 = (5)^2$$
or $\quad \mathbf{x^2 + y^2 - 10x - 8y + 16 = 0.}$

2. The equation of a circle passing through the origin and making intercepts 4, 5 on the coordinate axes is
(a) $x^2 + y^2 - 4x + 5y = 0$
(b) $x^2 + y^2 - 4x - 5y = 0$
(c) $x^2 + y^2 + 4x + 5y = 0$
(d) none of these

Solution (b) Let the intercepts on the coordinate axes be OA and OB.

Then $OA = 4$ and $OB = 5$.

$\therefore$ the coordinates of A and B are (4, 0) and (0, 5) respectively



Since $\angle AOB = \frac{\pi}{2}$, AB is a diameter.

$\therefore$ The required equation of the circle is

$$(x-4)(x-0)+(y-0)(y-5)=0$$

or $\quad \mathbf{x^2+y^2-4x-5y=0.}$

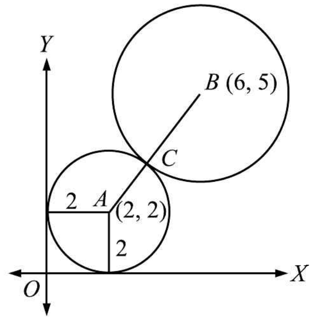
3. A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. The equation of the circle with centre at (6, 5) and touching the above circle externally is
(a) $x^2+y^2+12x-10y+52=0$
(b) $x^2+y^2-12x+10y+52=0$
(c) $x^2+y^2-12x-10y+52=0$
(d) none of these

Solution (c) Given $AC = 2$.

$\therefore \quad A \equiv (2, 2)$. Let $B \equiv (6, 5)$.

$\therefore \quad AB = \sqrt{(2-6)^2+(2-5)^2} = 5.$

$\therefore \quad BC = AB - AC = 5 - 2 = 3.$



[$\because$ the two circles touch each other externally]

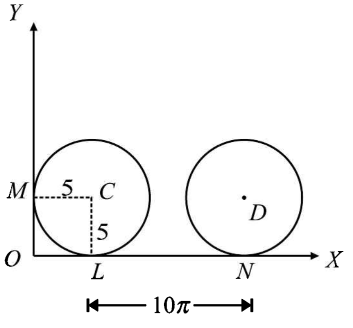
$\therefore$ The equation of the required circle is

$$(x-6)^2+(y-5)^2=3^2$$

or $\quad \mathbf{x^2+y^2-12x-10y+52=0.}$

4. A circle of radius 5 touches the coordinate axes in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, then its equation in the new position is
(a) $x^2+y^2-10(2\pi+1)x-10y+100\pi^2+100\pi+25=0$
(b) $x^2+y^2+10(2\pi+1)x-10y+100\pi^2+100\pi+25=0$
(c) $x^2+y^2-10(2\pi+1)x+10y+100\pi^2+100\pi+25=0$
(d) none of these

Solution (a) Let C be the centre of the circle in its initial position and D be its centre in the new position.



Then $C \equiv (5, 5)$ and $D \equiv (5 + 10\pi, 5)$

Now centre of the circle in the new position is $(5 + 10\pi, 5)$ and its radius is 5, therefore its equation will be

$$(x-5-10\pi)^2+(y-5)^2=5^2$$

or $\quad x^2+25+100\pi^2-10x-20\pi x+100\pi+y^2+25-10y=25$

or $\quad \mathbf{x^2+y^2-10(2\pi+1)x-10y+100\pi^2+100\pi+25=0.}$

5. The line joining (5, 0) and $(10\cos\theta, 10\sin\theta)$ is divided internally in the ratio 2 : 3 at P. If θ varies, then the locus of P is
(a) a pair of straight lines
(b) a circle
(c) a straight line
(d) none of these

Solution (b) Let $P \equiv (\alpha, \beta)$, $A \equiv (5, 0)$ and $B \equiv (10\cos\theta, 10\sin\theta)$.

Since the point P divides AB internally in the ratio 2 : 3, therefore

$$\alpha = \frac{20\cos\theta+15}{5} = 4\cos\theta + 3$$

and $$\beta = \frac{20\sin\theta+0}{5} = 4\sin\theta.$$

$\Rightarrow \quad \alpha - 3 = 4\cos\theta$ and $\beta = 4\sin\theta$.

Squaring and adding, we get

$$(\alpha-3)^2+\beta^2=16.$$

$\therefore$ Locus of $P(\alpha, \beta)$ is $(x-3)^2+y^2=16$, which represents a **circle.**

6. The abscissae of two points A and B are the roots of the equation $x^2+2ax-b^2=0$ and their ordinates are the roots of the equation $x^2+2px-q^2=0$. The equation of the circle with AB as diameter is
(a) $x^2+y^2+2ax+2py+b^2+q^2=0$
(b) $x^2+y^2-2ax-2py-b^2-q^2=0$
(c) $x^2+y^2+2ax+2py-b^2-q^2=0$
(d) none of these

Solution (c) Given equations are

$$x^2+2ax-b^2=0 \qquad \text{...(1)}$$

and $$x^2+2px-q^2=0 \qquad \text{...(2)}$$

Let the roots of (1) be α and β and that of (2) be γ and δ.

Then, $\alpha+\beta=-2a$ and $\alpha\beta=-b^2$.

and $\quad \gamma+\delta=-2p$, $\gamma\delta=-q^2$.

Let $A \equiv (\alpha, \gamma)$ and $B = (\beta, \delta)$.

The equation of the circle with AB as diameter is

$$(x-\alpha)(x-\beta)+(y-\gamma)(y-\delta)=0$$

$\Rightarrow \quad x^2+y^2-(\alpha+\beta)x-(\gamma+\delta)y+\alpha\beta+\gamma\delta=0$

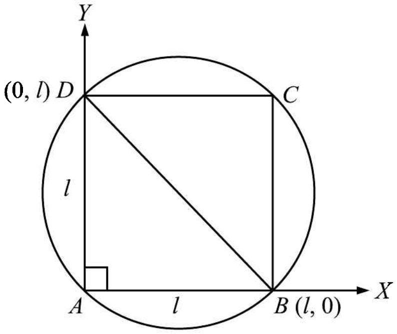
$\Rightarrow \quad \mathbf{x^2+y^2+2ax+2py-b^2-q^2=0.}$

7. $ABCD$ is a square with side whose length is l, if AB and AD be taken as axes, then the circle circumscribing the square will have the equation
(a) $x^2+y^2=l(x+y)$ (b) $x^2+y^2=l(x-y)$
(c) $x^2+y^2=l(x+2y)$ (d) $x^2+y^2=l(2x+y)$

Solution (a) Each side of the square is l and AB, AD are the coordinate axes.

$\therefore$ Coordinates of B and D are $(l, 0)$ and $(0, l)$ respectively.

Since $\angle BAD = 90°$, BD is a diameter of the circumcircle of square $ABCD$.



$\therefore$ Equation of circumcircle is

$$(x - l)(x - 0) + (y - 0)(y - l) = 0$$

or $$\mathbf{x^2 + y^2 = l\,(x + y).}$$

8. Two rods of lengths a and b slide along the axes which are rectangular is such a manner that their ends are concyclic. The locus of the centre of the circle passing through these points is

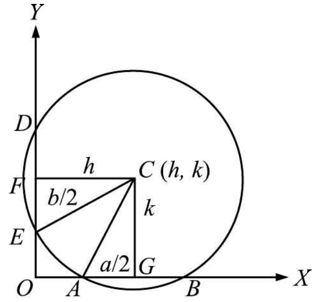
(a) $4(x^2 + y^2) = a^2 + b^2$ (b) $x^2 - y^2 = a^2 - b^2$

(c) $4(x^2 - y^2) = a^2 - b^2$ (d) $x^2 + y^2 = a^2 + b^2$

Solution (c) Let AB and DE be the two rods of lengths a and b respectively so that the points A, B, D and E are concyclic.

Let the centre of the circle be $C\,(h, k)$

Then $k^2 + \dfrac{a^2}{4} = h^2 + \dfrac{b^2}{4} = (\text{radius})^2$



$\Rightarrow$ $4(h^2 - k^2) = a^2 - b^2$.

$\therefore$ Locus of (h, k) is $\mathbf{4\,(x^2 - y^2) = a^2 - b^2}$.

9. The circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other in two distinct points if

(a) $r < 2$ (b) $r > 8$

(c) $2 < r < 8$ (d) $2 \le r \le 8$

Solution (c) Centres of the given circles are

$C_1 \equiv (5, 0)$ and $C_2 \equiv (0, 0)$.

Also, their radii are $r_1 = 3$ and $r_2 = r$.

Since the two circles cut each other in two distinct points, therefore

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$\Rightarrow$ $r - 3 < 5 < r + 3 \Rightarrow \mathbf{2 < r < 8}$.

10. The two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if

(a) $\dfrac{1}{a^2} - \dfrac{1}{b^2} = \dfrac{1}{c}$ (b) $\dfrac{1}{a^2} - \dfrac{1}{b^2} = \dfrac{1}{c^2}$

(c) $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}$ (d) $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$

Solution (c) Clearly the two circles will touch if the distance between their centres = sum of their radii. Now, the centre and radius of $x^2 + y^2 + 2ax + c = 0$ are $(-a, 0)$ and $\sqrt{a^2 - c}$ respectively.

The centre and radius of $x^2 + y^2 + 2bx + c = 0$ are $(0, -b)$ and $\sqrt{b^2 - c}$ respectively.

Hence, the two circles will touch if

$$\sqrt{(-a - 0)^2 + (0 + b)^2} = \sqrt{a^2 - c} + \sqrt{b^2 - c}$$

Squaring both sides, we get

$$a^2 + b^2 = a^2 - c + b^2 - c + 2\sqrt{(a^2 - c)(b^2 - c)}$$

or $$c = \sqrt{(a^2 - c)(b^2 - c)}$$

Squaring again, we get

$$c^2 = (a^2 - c)(b^2 - c) = a^2b^2 - c\,(a^2 + b^2) + c^2$$

or $$c\,(a^2 + b^2) = a^2b^2$$

Dividing by a^2b^2c, we obtain $\mathbf{\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}}$.

11. The equation of the tangent, from the point (0, 1) to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$, is

(a) $y - 1 = 0$ (b) $4x + 3y + 3 = 0$

(c) $4x + 3y - 3 = 0$ (d) $y + 1 = 0$

Solution (a), (c) Equation of circle is

$$x^2 + y^2 - 2x - 6y + 6 = 0. \quad \text{...(1)}$$

Its centre is (1, 3) and radius $= \sqrt{1 + 9 - 6} = 2$.

Equation of any line through (0, 1) is

$$y - 1 = m\,(x - 0)$$

or $$mx - y + 1 = 0 \quad \text{...(2)}$$

If it touches the circle (1), then the length of ⊥ from centre (1, 3) on (2) is numerically equal to radius 2 i.e.,

$$\frac{m - 3 + 1}{\sqrt{m^2 + 1}} = \pm 2$$

or $$(m - 2)^2 = 4(m^2 + 1).$$

$\therefore$ $\mathbf{m = 0, -\dfrac{4}{3}}$.

Substituting these values of m in (2), the required tangents are $y - 1 = 0$ and $4x + 3y - 3 = 0$.

12. The equation of the circle which touches both the axes and the straight line $4x + 3y = 6$ in the first quadrant and lies below it is

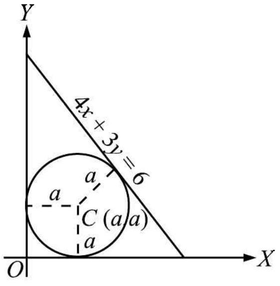
(a) $4x^2 + 4y^2 - 4x - 4y + 1 = 0$

(b) $x^2 + y^2 - 6x - 6y + 9 = 0$

(c) $x^2 + y^2 - 6x - y + 9 = 0$

(d) $4(x^2 + y^2 - x - 6y) + 1 = 0$

Solution (a) Let a be the radius of the circle which touches both the axes and lies in the first quadrant, then its centre is $C\,(a, a)$.



Since the circle also touches the line $4x + 3y = 6$,

$\therefore$ length of ⊥ from centre (a, a) on the line = radius of the circle.

i.e., $$\frac{4a + 3a - 6}{\sqrt{16 + 9}} = \pm a \Rightarrow a = \frac{1}{2}, 3.$$

Since the circle lies below the given line, $\therefore$ $a = \dfrac{1}{2}$.

$\therefore$ Equation of the required circle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

or $$\mathbf{4x^2 + 4y^2 - 4x - 4y + 1 = 0.}$$

13. If the equation of a circle is $(4a - 3)x^2 + ay^2 + 6x - 2y + 2 = 0$, then its centre is

(a) (3, –1) (b) (3, 1)
(c) (– 3, 1) (d) none of these

Solution (c) Since the given equation represents a circle, therefore, $4a - 3 = a$ i.e., $a = 1$

($\because$ coefficients of x^2 and y^2 must be equal).

So, the circle becomes $x^2 + y^2 + 6x - 2y + 2 = 0$.

$\therefore$ The coordinates of centre are **(–3, 1).**

14. If the lines $12x + 5y + 16 = 0$ and $12x + 5y - 10 = 0$ are tangents to the same circle, then the radius of this circle is

(a) 1 (b) 2
(c) 4 (d) none of these

Solution (a) Since the given two lines are parallel and distance between two parallel tangents is equal to the diameter of the circle.

$$\therefore \text{ Radius} = \frac{1}{2}\cdot\frac{|16-(-10)|}{\sqrt{12^2+5^2}} = \frac{13}{13} = \mathbf{1.}$$

15. If the coordinates of two consecutive vertices of a regular hexagon which lies completely above the x-axis, are (–2, 0) and (2, 0), then the equation of the circle, circumscribing the hexagon, is

(a) $x^2 + y^2 - 4\sqrt{3}\,y - 4 = 0$

(b) $x^2 + y^2 + 4\sqrt{3}\,y - 4 = 0$

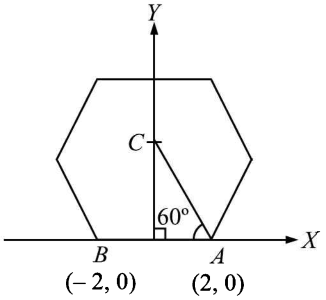
(c) $x^2 + y^2 - 4\sqrt{3}\,x - 4 = 0$

(d) $x^2 + y^2 + 4\sqrt{3}\,x - 4 = 0$

Solution (a) It is clear from the figure that coordinates of the circumcentre are (0, 2 tan 60°) or $(0, 2\sqrt{3})$

Also, circumradius $= \sqrt{(0-2)^2 + (2\sqrt{3}-0)^2}$

$= \sqrt{4+12} = 4.$



$\therefore$ Equation of the circumcircle is

$(x - 0)^2 + (y - 2\sqrt{3})^2 = (4)^2$

i.e., $\mathbf{x^2 + y^2 - 4\sqrt{3}\,y - 4 = 0.}$

16. The two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$

(a) touch externally (b) touch internally
(c) intersect (d) do not touch

Solution (b) Given circles are

$$x^2 + y^2 - 2x - 4y = 0 \quad ...(1)$$

and $$x^2 + y^2 - 8y - 4 = 0 \quad ...(2)$$

Let A and B be the centres and r_1 and r_2 the radii of circles (1) and (2) respectively, then

$A \equiv (1, 2)$, $B \equiv (0, 4)$, $r_1 = \sqrt{5}$, $r_2 = 2\sqrt{5}$.

Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$.

and $r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = |\sqrt{5} - 2\sqrt{5}| = \sqrt{5}$.

Thus $AB = |r_1 - r_2|$, hence the two circles **touch each other internally.**

17. If the circle $x^2 + y^2 - 4x - 6y + k = 0$ does not touch or intersect the axes and the point (2, 2) lies inside the circle, then

(a) $4 < k < 9$ (b) $4 < k < 12$
(c) $9 < k < 12$ (d) none of these

Solution (c) The centre of the given circle is (2, 3) and the radius $= \sqrt{4+9-k}$ i.e., $\sqrt{13-k}$.

Since the given circle does not touch or intersect the coordinate axes and the point (2, 2) lies inside the circle

$\therefore$ x-coordinate of centre > radius i.e., $2 > \sqrt{13-k}$,

y-coordinate of centre > radius i.e., $3 > \sqrt{13-k}$

and $4 + 4 - 8 - 12 + k < 0$

$\Rightarrow$ $4 > 13 - k$, $9 > 13 - k$ and $-12 + k < 0$

$\Rightarrow$ $k > 9$, $k > 4$ and $k < 12$

$\Rightarrow$ $\mathbf{9 < k < 12}$

18. If the tangents PA and PB are drawn from the point $P(-1, 2)$ to the circle $x^2 + y^2 + x - 2y - 3 = 0$ and C is the centre of the circle, then the area of the quadrilateral $PACB$ is

(a) 4 (b) 16
(c) does not exist (d) none of these

Solution (c) The given circle is $S : x^2 + y^2 + x - 2y - 3 = 0$.

Since $S]_{P(-1,\,2)} = 1 + 4 - 1 - 4 - 3 = -3 < 0$, the point $P(-1, 2)$ lies inside the circle. Consequently, the tangents from the point $P(-1, 2)$ to the circle do not exist. Thus, the quadrilateral $PACB$ cannot be formed.

19. The number of common tangents to the circles
$x^2 + y^2 - 6x - 2y + 9 = 0$ and
$x^2 + y^2 - 14x - 8y + 61 = 0$ is

(a) 2 (b) 3
(c) 1 (d) 4

Solution (d) The equations of the circles are

$$x^2 + y^2 - 6x - 2y + 9 = 0 \quad ...(1)$$

and $$x^2 + y^2 - 14x - 8y + 61 = 0 \quad ...(2)$$

Centre of (1) is $C_1 \equiv (3, 1)$ and radius $r_1 = 1$

Centre of (2) is $C_2 \equiv (7, 4)$ and radius $r_2 = 2$

d = distance between centres $= C_1C_2 = \sqrt{16+9} = 5$.

So, $r_1 + r_2 < d$, $\therefore$ the two circles do not cut each other and hence the number of common tangents is **4**.

20. If the line $(y - 2) = m(x + 1)$ intersects the circle $x^2 + y^2 + 2x - 4y - 3 = 0$ at two real distinct points, then the number of possible values of m is

(a) 2 (b) 1
(c) any real value of m (d) none of these

Solution (c) The given line passes through the point (–1, 2).

Given circle is $S \equiv x^2 + y^2 + 2x - 4y - 3 = 0$.

Since $S]_{(-1,\,2)} = 1 + 4 - 2 - 8 - 3 < 0$,

$\therefore$ (– 1, 2) is an interior point of the circle. Thus, **m can have any real value.**

21. If the line $3x + ay - 20 = 0$ cuts the circle $x^2 + y^2 = 25$ at real, distinct or coincident points, then a belongs to the interval

(a) $[-\sqrt{7}, \sqrt{7}]$

(b) $(-\sqrt{7}, \sqrt{7})$

(c) $(-\infty - \sqrt{7}] \cup [\sqrt{7}, \infty)$

(d) none of these

Solution (c) The length of the $\perp$ from the centre (0, 0) of the given circle to the line $3x + ay - 20 = 0$ is

$$= \frac{|3(0)+a(0)-20|}{\sqrt{9+a^2}} = \frac{20}{\sqrt{9+a^2}}$$

Radius of the given circle = 5.

Since the line cuts the circle at real, distinct or coincident points,

$$\therefore \quad \frac{20}{\sqrt{9+a^2}} \le 5 \quad \Rightarrow a^2 + 9 \ge 16 \Rightarrow a^2 - 7 \ge 0$$

$$\Rightarrow (a - \sqrt{7})\,(a - \sqrt{7}) \ge 0$$

$$\Rightarrow \boldsymbol{a \in (-\infty, -\sqrt{7}\,] \cup [\,\sqrt{7}, \infty)}.$$

22. If (α, β) is a point on the chord PQ of the circle $x^2 + y^2 = 19$, where the coordinates of P and Q are $(3, -4)$ and $(4, 3)$ respectively, then

(a) $\alpha \in [3, 4]$, $\beta \in [-4, 3]$

(b) $\alpha \in [-4, 3]$, $\beta \in [3, 4]$

(c) $\alpha \in [3, 3]$, $\beta \in [-4, 4]$

(d) none of these

Solution (a) Clearly, the point (α, β) is either an internal point or one of the end points of the line segment joining $P(3, -4)$ and $Q(4, 3)$.

$\therefore \quad \boldsymbol{3 \le \alpha \le 4}$ and $\boldsymbol{-4 \le \beta \le 3}$.

23. From any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = a^2 \sin^2 \alpha$. The angle between them is

(a) $\alpha/2$ (b) α

(c) 2α (d) none of these

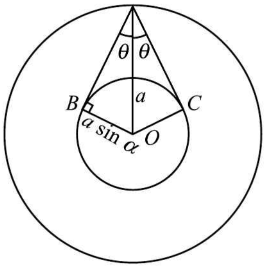
Solution (c) Let the angle between the tangents be 2θ.

From the figure,

$$\sin\theta = \frac{a\sin\alpha}{a} = \sin\alpha$$

$\Rightarrow \quad \theta = \alpha$.

Thus, the required angle $= 2\theta = \boldsymbol{2\alpha}$.



24. If the point $(k + 1, k)$ lies inside the region bounded by the curve and y-axis, then k belongs to the interval

$$x = \sqrt{25 - y^2}\,.$$

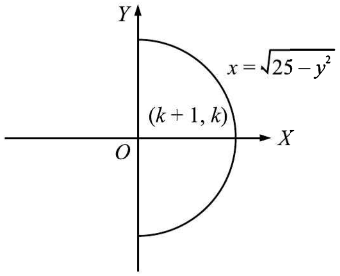
(a) $(-1, 3)$ (b) $(-4, 3)$

(c) $(-\infty, -4) \cup (3, \infty)$ (d) none of these

Solution (a) Since the point $(k + 1, k)$ lies inside the region bounded by $x = \sqrt{25 - y^2}$ and y-axis,

$\therefore \quad (k + 1)^2 + k^2 - 25 < 0$

and $\quad k + 1 > 0$



$\Rightarrow \quad 2k^2 + 2k - 24 < 0$ and $k > -1$

$\Rightarrow \quad k^2 + k - 12 < 0$ and $k > -1$

$\Rightarrow \quad (k + 4)(k - 3) < 0$ and $k > -1$

$\Rightarrow \quad -4 < k < 3$ and $k > -1$

$\Rightarrow \quad \boldsymbol{-1 < k < 3}$.

25. If the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 24x - 10y + a^2 = 0$, $a \in I$, have exactly two common tangents, then the number of possible values of a is

(a) 11 (b) 13

(c) 0 (d) 2

Solution (b) The equations of the circles are

$x^2 + y^2 = 4$ ...(1)

and $\quad x^2 + y^2 - 24x - 10y + a^2 = 0$ ...(2)

Centre of (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$

Centre of (2) is $C_2 \equiv (12, 5)$ and radius $r_2 = \sqrt{169 - a^2}$

d = distance between centres $= C_1C_2$

$= \sqrt{144 + 25} = 13$.

If the two circles have exactly two common tangents, then

$169 - a^2 > 0$ and $r_1 + r_2 > d$

$\Rightarrow \quad (a - 13)(a + 13) < 0$ and $2 + \sqrt{169 - a^2} > 13$

$\Rightarrow \quad -13 < a < 13$ and $169 - a^2 > 121$

$\Rightarrow \quad -13 < a < 13$ and $a^2 - 48 < 0$

$\Rightarrow \quad -13 < a < 13$ and $-\sqrt{48} < a < \sqrt{48}$

$\Rightarrow \quad -\sqrt{48} < a < \sqrt{48}$

Since a is an integer,

$\therefore \quad a = -6, -5, -4, \ldots, 4, 5, 6$.

$\therefore$ The number of possible values of a is **13.**

26. The equation of a circle of radius 2 touching the circles $x^2 + y^2 - 4\,|x| = 0$ is

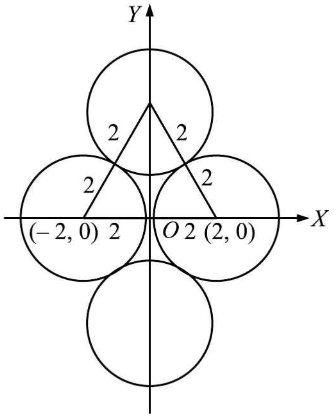
(a) $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$

(b) $x^2 + y^2 + 4\sqrt{3}y + 8 = 0$

(c) $x^2 + y^2 - 4\sqrt{3}y + 8 = 0$

(d) none of these

Solution (b), (c) The given circles are



$x^2 + y^2 - 4x = 0,\ x > 0$

i.e., $\quad (x - 2)^2 + y^2 = 2^2,\ x > 0$.

and $\quad x^2 + y^2 + 4x = 0,\ x < 0$

i.e., $\quad (x + 2)^2 + y^2 = 2^2,\ x < 0$.

Clearly, from the figure, the centres of the required circles are at $(0, \sqrt{12})$ and $(0, -\sqrt{12})$.

$\therefore$ Equations of the required circles are

$$(x - 0)^2 + (y \mp \sqrt{12})^2 = 2^2$$

i.e., $\quad \boldsymbol{x^2 + y^2 + 2\sqrt{12}\,y + 8 = 0}$

and $\quad \boldsymbol{x^2 + y^2 - 2\sqrt{12}\,y + 8 = 0}$.

27. The coordinates of a point on the line $y = 2$ from which the tangents drawn to the circle $x^2 + y^2 = 25$ are perpendicular, are

(a) $(\sqrt{46}, 2)$ (b) $(-\sqrt{46}, 2)$

(c) $(\sqrt{37}, 2)$ (d) $(-\sqrt{37}, 2)$

Solution (a), (b) Let the point on the given line be $(x_1, 2)$.

Since the tangents drawn from $(x_1, 2)$ to the given circle are at right angles, so the point $(x_1, 2)$ must also lie on the director circle whose equation is

$$x^2 + y^2 = 2.25 \quad \text{i.e.,} \quad x^2 + y^2 = 50.$$

$$\therefore \quad x_1^2 + 4 = 50 \Rightarrow x_1 = \sqrt{46}.$$

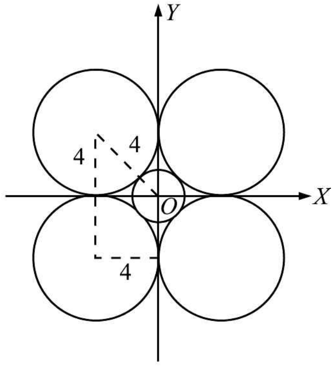
So, the points are **$(\sqrt{46}, 2)$ and $(-\sqrt{46}, 2)$.**

28. If the equations of four circles are $(x \pm 4)^2 + (y \pm 4)^2 = 4^2$, then the radius of the smallest circle touching all the four circles is

(a) $4(\sqrt{2}+1)$ (b) $4(\sqrt{2}-1)$

(c) $2(\sqrt{2}-1)$ (d) none of these

Solution (b) Clearly, from the figure, the radius of the smallest circle touching the given circles is



$$= \sqrt{4^2 + 4^2} - 4 \quad \text{i.e.,} \quad \mathbf{4\sqrt{2} - 4}.$$

29. Locus of the centre of a circle of radius 4 which touches the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ externally is

(a) $x^2 + y^2 - 4x + 2y - 44 = 0$

(b) $x^2 + y^2 + 4x + 2y - 44 = 0$

(c) $x^2 + y^2 - 4x - 2y - 44 = 0$

(d) none of these

Solution (a) Let the centre of the circle S_1 be $C_1(x_1, y_1)$

Its radius $= r_1 = 4$.

Given circle is $S_2 \equiv x^2 + y^2 - 4x + 2y - 4 = 0$

Its centre is $C_2(2, -1)$ and radius

$$= r_2 = \sqrt{4+1+4} = 3.$$

Also, d = distance between the centres.

$$= \sqrt{(x_1 - 2)^2 + (y_1 + 1)^2}.$$

Since the two circles touch each other externally,

$$\therefore \quad d = r_1 + r_2$$

$$\Rightarrow \quad \sqrt{(x_1 - 2)^2 + (y_1 + 1)^2} = 4 + 3$$

$$\Rightarrow \quad x_1^2 + y_1^2 - 4x_1 + 2y_1 + 5 = 49$$

$\therefore$ locus of (x_1, y_1) is

$$\mathbf{x^2 + y^2 - 4x + 2y - 44 = 0.}$$

30. A variable circle passes through the point $P(1, 2)$ and touches the x-axis. The locus of the other end of the diameter through P is

(a) $(x - 1)^2 = 8y$ (b) $(x + 1)^2 = 8y$

(c) $(y - 1)^2 = 8x$ (d) none of these

Solution (a) The equation of any circle touching x-axis is of the form

$$(x - h)^2 + (y - k)^2 = k^2.$$

Let the coordinates of the other end of the diameter through P be (α, β)

Then, $\dfrac{\alpha+1}{2} = h$ and $\dfrac{\beta+2}{2} = k$

i.e., $\alpha = 2h - 1$ and $\beta = 2k - 2$ ...(1)

Also, $(h - 1)^2 + (k - 2)^2 = (\text{radius})^2 = k^2$

$$\Rightarrow \quad \left(\frac{\alpha+1}{2} - 1\right)^2 + \left(\frac{\beta+2}{2} - 2\right)^2 = \left(\frac{\beta+2}{2}\right)^2$$

$$\Rightarrow \quad (\alpha - 1)^2 + (\beta - 2)^2 = (\beta + 2)^2.$$

$$\Rightarrow \quad (\alpha - 1)^2 = 8\beta.$$

$\therefore$ Locus of (α, β) is $\mathbf{(x - 1)^2 = 8y}$.

31. A point moves such that the sum of the squares of its distances from the sides of a square of side unity is equal to 9. The locus of the point is a circle such that

(a) Centre of the circle coincides with that of square.

(b) Centre of the circle is $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) radius of the circle is 2

(d) all the above are true

Solution (d) Let the sides of the square be $y = 0$, $y = 1$, $x = 0$ and $x = 1$.

Let the moving point be (x, y).

Then, $y^2 + (y - 1)^2 + x^2 + (x - 1)^2 = 9$ is the equation of the locus.

$$\Rightarrow \quad 2x^2 + 2y^2 - 2x - 2y - 7 = 0,$$

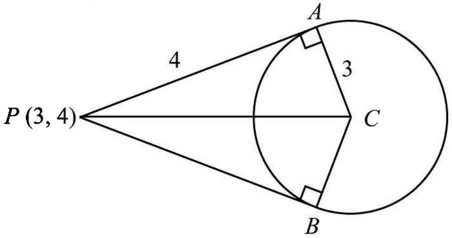
which represents a circle having centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ (which is also the centre of the square) and radius 2.

32. The area of the triangle formed by the tangents from the point $(4, 3)$ to the circle $x^2 + y^2 = 9$ and the line joining their point of contact is

(a) 12 (b) 6

(c) 4 (d) none of these

Solution (a) Length of the tangents from the point $P(3, 4)$ to the circle $x^2 + y^2 = 9$ is



$$PA = PB = \sqrt{(3)^2 + (4)^2 - 9} = 4$$

$\therefore$ Area of ΔPAB = 2 (area of ΔPAC)

$$= 2 \times \frac{1}{2} \times 3 \times 4 = \mathbf{12}.$$

33. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2px - 2qy + q^2 = 0$ are prependicular if

(a) $p = q/2$ (b) $q = p/2$

(c) $p^2 = q^2$ (d) $p^2 + q^2 = 1$

Solution (c) The equations of pair of tangents drawn from the origin to the given circle, are

$SS_1 = T^2$

i.e., $(x^2 + y^2 - 2px - 2qy + q^2)(0 + 0 - 0 - 0 + q^2)$

$= (x\cdot 0 + y\cdot 0 - p(x + 0) - q(y + 0) + q^2)^2$

⇒ $q^2(x^2 + y^2 - 2px - 2qy + q^2) - (-px - qy + q^2)^2 = 0$

The two tangents are ⊥ if

$q^2 + q^2 - p^2 - q^2 = 0$

(sum of coefficients of x^2 and $y^2 = 0$)

⇒ $\mathbf{q^2 = p^2}$.

34. Given the circles $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 6x - 2y + 6 = 0$. Let P be a point (α, β) such that the tangents from P to both the circles are equal. Then

(a) $10\alpha + 2\beta + 11 = 0$ (b) $10\alpha - 2\beta + 11 = 0$

(c) $2\alpha - 10\beta + 11 = 0$ (d) $2\alpha + 10\beta + 11 = 0$.

Solution (b) Lengths of tangents from the point $P(\alpha, \beta)$ to the given circles are equal

∴ $\sqrt{\alpha^2 + \beta^2 - 4\alpha - 5} = \sqrt{\alpha^2 + \beta^2 + 6\alpha - 2\beta + 6}$

⇒ $\mathbf{10\alpha - 2\beta + 11 = 0.}$

35. The tangent to the circle $x^2 + y^2 = 9$, which is parallel to y-axis and does not lie in third quadrant, touches the circle at the point

(a) $(-3, 0)$ (b) $(3, 0)$

(c) $(0, 3)$ (d) $(0, -3)$

Solution (b) Any line parallel to y-axis is $x = k$.

If it touches the circle $x^2 + y^2 = 9$, then ⊥ distance from the centre (0, 0) of the circle to the line $x = k$, must be equal to radius 3.

i.e., $\dfrac{|0 - k|}{1} = 3 \Rightarrow k = \pm 3$

∴ $k = 3$.

(∵ line does not lie in the IIIrd quadrant)

∴ The equation of the tangent line is $x = 3$.

This meets the circle when $9 + y^2 = 9 \Rightarrow y = 0$.

∴ Point of contact is **(3, 0).**

36. If the circle $x^2 + y^2 + 6x + 8y + a = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - b = 0$, then $a + b$ is equal to

(a) 38 (b) − 38

(c) 42 (d) none of these

Solution (b) Given circles are

$S_1 : x^2 + y^2 + 6x + 8y + a = 0$ ...(1)

and $S_2 : x^2 + y^2 + 2x - 6y - b = 0$ ...(2)

The equation of common chord of the two circles is

$S_1 - S_2 = 0$ i.e., $4x + 14y + (a + b) = 0$ ...(3)

Since the circle S_1 bisects the circumference of circle S_2, therefore, (1) passes through the centre of circle S_2 i.e., $(-1, 3)$

∴ $4(-1) + 14(3) + a + b = 0 \Rightarrow \mathbf{a + b = -38.}$

37. If a chord of the circle $x^2 + y^2 = 32$ makes equal intercepts of length l on the coordinate axes, then

(a) $|l| < 8$ (b) $|l| < 16$

(c) $|l| > 8$ (d) none of these

Solution (a) Since the chord makes equal intercepts of length l on the coordinate axes, so its equation can be written in the form $x \pm y = \pm l$.

Since the chord intersects the given circle at two distinct points, therefore, the length of the ⊥ from the centre (0, 0) of the given circle to the chord must be less than the radius

i.e., $\left|\dfrac{\pm l}{\sqrt{2}}\right| < \sqrt{32} \Rightarrow l^2 < 64 \Rightarrow \mathbf{|l| < 8.}$

38. The greatest distance of the point P (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

(a) 10 (b) 15

(c) 5 (d) none of these

Solution (b) Since $S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$

so, P lies outside the circle. Join P with the centre C (2, 1) of the given circle. Suppose PC cuts the circle at A and B then, PB is the greatest distance of P from the circle

$PC = \sqrt{(10-1)^2 + (7-1)^2} = 10$

$BC = \sqrt{4 + 1 + 20} = 5$

$PB = PC + CB = 10 + 5 = 15.$

39. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which makes equal intercepts on the positive coordinate axes is

(a) $x + y = 2$ (b) $x + y = 2\sqrt{2}$

(c) $x + y = 4$ (d) $x + y = 8$

Solution (b) Let the equation of the required tangent be $x + y = a$ then, length of the perpendicular from the centre upon the tangent = radius

i.e., $\left|\dfrac{-2 + 2 - a}{\sqrt{2}}\right| = 2 \Rightarrow a = 2\sqrt{2}$

Hence, the equation of tangent is

$x + y = 2\sqrt{2}$

40. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is

(a) $2ax - 2by + (a^2 + b^2 + 4) = 0$

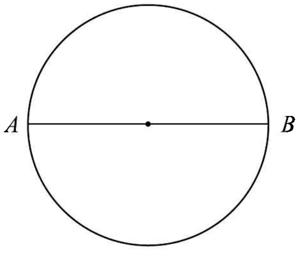
(b) $2ax + 2by - (a^2 + b^2 + 4) = 0$

(c) $2ax + 2by + (a^2 + b^2 + 4) = 0$

(d) $2ax - 2ab - (a^2 + b^2 + 4) = 0$

Solution (b) Let the variable circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)



circle (1) cuts the circle $x^2 + y^2 - 4 = 0$ orthogonally

∴ $2g\,.\,0 + 2f.\,0 = c - 4 \Rightarrow c = 4$.

Since circle (1) passes through (a, b),

∴ $a^2 + b^2 + 2ga + 2fb + c = 0$

∴ Locus of centre $(-g, -f)$ is

$2ax + 2by - (a^2 + b^2 + 4) = 0$.

41. A variable circle passes through the fixed point $A(p, q)$ and touches x-axis. The locus of the other end of the diameter through A is

(a) $(y - p)^2 = 4qx$ (b) $(x - q)^2 = 4py$

(c) $(x - p)^2 = 4qy$ (d) $(y - q)^2 = 4px$.

Solution (c) Equation of the circle having AB as diameter is $(x-p)(x-\alpha)+(y-q)(y-\beta)=0$

or $x^2+y^2-(p+\alpha)x-(q+\beta)y+p\alpha+q\beta=0$ ...(1)

putting $y=0$, we get $x^2-(p+\alpha)x+p\alpha+q\beta=0$ ...(2)

Since circle (1) touches x-axis

$\therefore$ discriminant of equation (2) = 0

$\Rightarrow$ $(p+\alpha)^2=4(p\alpha+q\beta)\Rightarrow(p-\alpha)^2=4q\beta$

$\therefore$ Locus of $B(\alpha,\beta)$ is $(p-x)^2=4qy$

or $(x-p)^2=4qy.$

42. If the lines $2x+3y+1=0$ and $3x-y-4=0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is

(a) $x^2+y^2+2x+2y-23=0$

(b) $x^2+y^2-2x-2y-23=0$

(c) $x^2+y^2-2x+2y-23=0$

(d) $x^2+y2+2x-2y-23=0$

Solution (c) According to question two diameters of the circle are $2x+3y+1=0$ and $3x-y-4=0$

Solving, we get $x=1, y=-1$

$\therefore$ Centre of the circle is $(1,-1)$

Given $2\pi r=10\pi\Rightarrow r=5$

$\therefore$ required circle is $(x-1)^2+(y+1)^2=5^2$

or $x^2+y^2-2x+2y-23=0.$

43. The locus of the centres of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-4x+6y+4=0$ orthogonally is

(a) $8x-12y+5=0$ (b) $8x+12y-5=0$

(c) $12x-8y+5=0$ (d) none of these

Solution (a)

44. If $(2,-1)$ lies on the circle $x^2+y^2+2gx+2fy+c=0$ which is concentric with the circle $x^2+y^2+4x-6y+3=0$, then c is

(a) 19 (b) -19

(c) 21 (d) none of these

Solution (b)

45. A square is inscribed in the circle $x^2+y^2-2x+4y+3=0$. Its sides are parallel to the coordinate axes. Then one vertex of the square is

(a) $(1+\sqrt{2},-2)$ (b) $(1-\sqrt{2},-2)$

(c) $(1,-2+\sqrt{2})$ (d) none of these

Solution (d)

46. Two distinct chords drawn from the point (p,q) on the circle $x^2+y^2=px+qy$, where $pq\neq 0$, are bisected by the x-axis. Then

(a) $|p|=|q|$ (b) $p^2=8q^2$

(c) $p^2<8q^2$ (d) $p^2>8q^2$

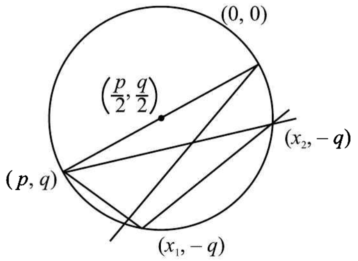
Solution (d) Given circle is $x^2+y^2=px+qy$.

Since the centre of the circle is $\left(\frac{p}{2},\frac{q}{2}\right)$, so (p,q) and $(0,0)$ are the end points of a diameter. As the two chords are bisected by x-axis, the chords will cut the circle at the points $(x_1,-q)$ and $(x_2,-q)$, where x_1, x_2 are real.

The equation of the line joining these points is $y=-q$.

Solving $y=-q$ and $x^2+y^2=px+qy$, we get

$x^2-px+2q^2=0.$



The roots of this equation are x_1 and x_2. Since the roots are real and distinct, $\therefore$ discriminent > 0

i.e., $p^2-8q^2>0$ or $\mathbf{p^2>8q^2}$.

47. If the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ cut the coordinate axes in concyclic points, then

(a) $a_1a_2=b_1b_2$ (b) $a_1b_1=a_2b_2$

(c) $a_1b_2=a_2b_1$ (d) none of these

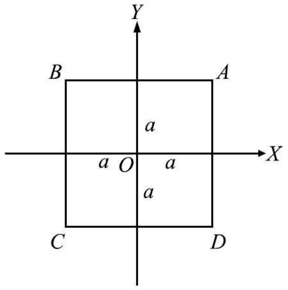
Solution (a)

48. A point moves so that the sum of the squares of its distances from the four sides of a square is constant. The locus of the point is

(a) a circle (b) an ellipse

(c) a hyperbola (d) none of these

Solution (a) Take the centre of the square as origin and axes parallel to its sides.



Let side of square be $2a$.

The equations of sides are

$AD: x=a,\ BC: x=-a$

$AB: y=a,\ CD: y=-a$

Let $P(x,y)$ be any point on locus.

Distances of P from the sides of square are

$x-a, x+a, y-a$ and $y+a$

By the given condition,

$(x-a)^2+(x+a)^2+(y-a)^2+(y+a)^2=$ constant

$=4c^2$ (say)

$\therefore 2x^2+2y^2=4c^2-4a^2$

or $x^2+y^2=2(c^2-a^2)$, which is a **circle.**

49. The locus of the centres of circles passing through the origin and cutting the circle $x^2+y^2+6x-4y+2=0$ orthogonally is

(a) $2x-3y+1=0$ (b) $2x+3y+1=0$

(c) $3x-2y+1=0$ (d) none of these

Solution (c)

50. For the two circles $x^2+y^2=16$ and $x^2+y^2-2y=0$, there is/are

(a) one pair of common tangents

(b) two pairs of common tangents

(c) three common tangents

(d) no common tangent

Solution (d)

Solved Problems

Level of Difficulty I

1. The equation of the circle whose centre is (1, 2) and which passes through the point (4, 6) is
   (a) $x^2 + y^2 + 2x - 4y - 20 = 0$
   (b) $x^2 + y^2 + 2x + 4y - 20 = 0$
   (c) $x^2 + y^2 - 2x - 4y - 20 = 0$
   (d) none of these

2. The equation of the circle which passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is the point (2, –3), is
   (a) $(x - 2)^2 + (y + 3)^2 = 109$
   (b) $(x - 2)^2 + (y + 3)^2 = (111)^2$
   (c) $(x - 2)^2 + (y + 3)^2 = (97)^2$
   (d) none of these

3. The equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point (–2, –7), is
   (a) $x^2 + y^2 + 8x + 6y - 27 = 0$
   (b) $x^2 + y^2 - 8x + 6y - 27 = 0$
   (c) $x^2 + y^2 + 8x + 6y + 27 = 0$
   (d) none of these

4. If the equations of the two diameters of a circle are $x + y = 6$ and $x + 2y = 4$ and the radius of the circle is 10, then the equation of the circle is
   (a) $x^2 + y^2 + 16x + 4y - 32 = 0$
   (b) $x^2 + y^2 - 16x + 4y + 32 = 0$
   (c) $x^2 + y^2 - 16x + 4y - 32 = 0$
   (d) none of these

5. The equation of the circle which touches the line $5x + 12y = 1$ and which has its centre at (3, 4) is
   (a) $(x - 3)^2 + (y - 4)^2 = \left(\frac{62}{11}\right)^2$
   (b) $(x - 3)^2 + (y - 4)^2 = \left(\frac{62}{17}\right)^2$
   (c) $(x - 3)^2 + (y - 4)^2 = \left(\frac{62}{13}\right)^2$
   (d) none of these

6. The sides of a square are $x = 2, x = 3, y = 1$ and $y = 2$. The equation of the circle drawn on the diagonals of the square as its diameter, is
   (a) $x^2 + y^2 - 5x - 3y + 8 = 0$
   (b) $x^2 + y^2 + 5x - 3y + 8 = 0$
   (c) $x^2 + y^2 + 5x + 3y - 8 = 0$
   (d) none of these

7. The length of the intercept made by the circle whose diameter is the line joining the end points (–4, 3) and (12, –1) on y-axis, is
   (a) $2\sqrt{13}$ (b) $4\sqrt{13}$
   (c) $8\sqrt{13}$ (d) none of these

8. The equation of the circle, if its centre is (4, 5) and its circumference passes through the center of the circle $x^2 + y^2 + 4x - 6y = 12$, is
   (a) $x^2 + y^2 - 8x - 10y + 1 = 0$
   (b) $x^2 + y^2 + 8x - 10y + 1 = 0$
   (c) $x^2 + y^2 - 8x + 10y + 1 = 0$
   (d) none of these

9. The equation of that diameter of the circle $x^2 + y^2 - 6x + 2y - 8 = 0$, which passes through the origin, is
   (a) $x - 3y = 0$
   (b) $x + 3y = 0$
   (c) $3x - y = 0$
   (d) none of these

10. Equation of tangent to the circle $x^2 + y^2 = 10$, at the point whose abscissa is 1, is
    (a) $x + 3y = 10$ (b) $3x + y = 10$
    (c) $x - 3y = 10$ (d) $3x - y = 10$

11. The value of k for which the line $4x + 3y + k = 0$ touches the circle $2x^2 + 2y^2 = 5x$ is
    (a) $\frac{5}{4}$ (b) $\frac{-5}{4}$ (c) $\frac{45}{4}$ (d) $\frac{-45}{4}$

12. The equation of the tangent to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$, which is parallel to the line $3x - 4y - 1 = 0$, is
    (a) $3x - 4y + 20 = 0$ (b) $3x + 4y + 20 = 0$
    (c) $3x - 4y - 10 = 0$ (d) $3x + 4y - 10 = 0$

13. The equation of the tangent to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$, which is perpendicular to the line $4x + 3y = 7$, is
    (a) $3x - 4y + 31 = 0$ (b) $3x - 4y - 19 = 0$
    (c) $3x + 4y + 31 = 0$ (d) $3x + 4y - 19 = 0$

14. If the equation of a circle is $3x^2 + 3y^2 + kxy + 9x + (k - 6)y + 3 = 0$, then its radius is
    (a) $\frac{3}{2}$ (b) $\frac{\sqrt{17}}{2}$
    (c) $\frac{2}{3}$ (d) none of these

15. The equation $x^2 + y^2 - 8x + 6y + 25 = 0$ represents
    (a) a circle
    (b) a pair of straight lines
    (c) a point
    (d) none of these

16. The slope of the tangent at the point (h, h) of the circles $x^2 + y^2 = a^2$ is
    (a) 0 (b) 1
    (c) –1 (d) depends on h

17. The equation of the circle on the join of the points A and B as diameter, where the ordinates of A, B are the roots of the equation $y^2 - 7y + 12 = 0$ and the abscissae are the roots of the equation $x^2 - 3x + 2 = 0$, is
    (a) $x^2 + y^2 - 3x + 7y + 14 = 0$
    (b) $x^2 + y^2 + 3x - 7y + 14 = 0$
    (c) $x^2 + y^2 + 3x + 7y + 14 = 0$
    (d) $x^2 + y^2 - 3x - 7y + 14 = 0$

18. The number of points on the circle $x^2 + y^2 - 4x - 10y + 13 = 0$ which are at a distance 1 from the point (–3, 2) is

(a) 1 (b) 2
(c) 3 (d) none of these

19. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$ is
(a) 1 (b) 2 (c) 3 (d) 4

20. If the point $(2, k)$ lies outside the circles $x^2 + y^2 + x - 2y - 14 = 0$ and $x^2 + y^2 = 13$, then
(a) $k \in (-3, -2) \cup (3, 4)$
(b) $k \in (-3, 4)$
(c) $k \in (-\infty, -3) \cup (4, \infty)$
(d) $k \in (-\infty, -2) \cup (3, \infty)$

Level of Difficulty II

21. The circles $x^2 + y^2 + 4x + 6y + 3 = 0$ and $2(x^2 + y^2) + 6x + 4y + c = 0$ will cut orthogonally if c equals
(a) 18 (b) 16 (c) 12 (d) 4

22. If $2x - 3y = 0$ is the equation of the common chord of the circles $x^2 + y^2 + 4x = 0$ and $x^2 + y^2 + 2\lambda y = 0$, then the value of λ is
(a) 0 (b) 1 (c) 2 (d) 3

23. If the line $2x - y + k = 0$ is a diameter of the circle $x^2 + y^2 + 6x - 6y + 5 = 0$, then k is equal to
(a) 12 (b) 9 (c) 6 (d) 3

24. If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then
(a) $2 < r < 8$ (b) $r < 2$
(c) $r = 2$ (d) $r > 2$

25. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq. units. Then the equation of the circle is
(a) $x^2 + y^2 + 2x - 2y = 62$
(b) $x^2 + y^2 + 2x - 2y = 47$
(c) $x^2 + y^2 - 2x + 2y = 47$
(d) $x^2 + y^2 - 2x + 2y = 62$

26. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is
(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 - x - y = 0$
(d) $x^2 + y^2 + x - y = 0$.

27. If the distances from the origin of the centres of the three circles $x^2 + y^2 + 2\alpha_i x = a^2$ $(i = 1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circles $x^2 + y^2 = a^2$ are in
(a) A. P. (b) G. P.
(c) H. P. (d) none of these

28. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is
(a) 2 (b) 1
(c) -1 (d) -2

29. The locus of the mid point of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$ which makes an angle of 120° at the centre is
(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 + x + y - 1 = 0$
(c) $x^2 + y^2 - 2x - 2y - 1 = 0$
(d) none of these

30. Extremities of a diagonal of a rectangle are (0, 0) and (4, 3). The equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal are
(a) $16x + 8y \pm 25 = 0$
(b) $6x - 8y \pm 25 = 0$
(c) $8x + 6y \pm 25 = 0$
(d) none of these

31. The tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Then its point of contact is
(a) $(3, -1)$ (b) $(-3, 0)$
(c) $(-1, -1)$ (d) $(-2, 1)$

32. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of this circle is
(a) $\frac{3}{2}$ (b) $\frac{3}{4}$
(c) $\frac{7}{10}$ (d) $\frac{4}{5}$

33. A circle passes through (0, 0), $(a, 0)$ and $(0, b)$. The coordinates of its centre are
(a) $\left(\frac{b}{2}, \frac{a}{2}\right)$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$
(c) (b, a) (d) (a, b)

34. The equation of the line parallel to tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) and passing through origin is
(a) $xy_1 + x_1y = 0$ (b) $xx_1 - yy_1 = 0$
(c) $xx_1 + yy_1 = 0$ (d) $xy_1 - x_1y = 0$

35. $3x + y = 0$ is a tangent to a circle with centre $(2, -1)$. Then the other tangent to the circle from the origin is
(a) $x + 3y = 0$ (b) $3x - y = 0$
(c) $x - 3y = 0$ (d) $x + y = 0$

36. The equation $(x^2 - 4)^2 + (y^2 - 9)^2 = 0$ represents points
(a) which lie on a circle with centre at (0, 0)
(b) which lie on a circle with centre at (2, 3)
(c) which are collinear
(d) none of these

37. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the ΔPAB as P moves on the circle is
(a) a parabola (b) a circle
(c) an ellipse (d) a pair of straight lines

38. The ΔPQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal to
(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/6$

39. The equations of tangents to the circle $x^2 + y^2 = 25$ which are inclined at angle of 30° to the x-axis are
(a) $y = x\sqrt{3} \pm 5$ (b) $\sqrt{3}\, y = x \pm 10$
(c) $\pm\sqrt{3}\, y = x + 10$ (d) none of these

40. The value of k for which the circles $x^2 + y^2 - 3x + ky - 5 = 0$ and $4x^2 + 4y^2 - 12x - y - 9 = 0$ become concentric is
(a) $-\frac{1}{8}$ (b) $\frac{1}{8}$
(c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

ANSWERS

1. (c)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (a), (c)
11. (a), (d)	**12.** (a), (c)	**13.** (a), (b)	**14.** (a)	**15.** (c)	**16.** (c)	**17.** (d)	**18.** (d)	**19.** (c)	**20.** (c)
21. (a)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (b)
31. (a)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (b)	**40.** (d)

Conic Sections (Parabola, Ellipse and Hyperbola)

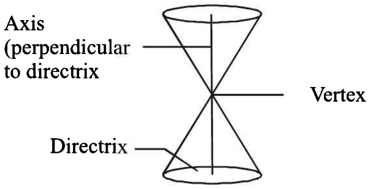
4

BRIEF REVIEW OF THE CONCEPTS

CONE

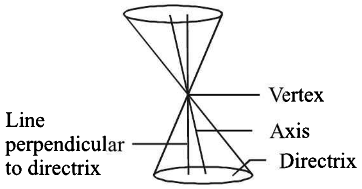
A cone is the surface or the solid figure formed by moving a line so that the line always intersects a plane (called the directrix) and passes through a fixed point (called the vertex) See fig. (a) and (b). In a right circular cone Fig (a) the axis is perpendicular to (forms a 90° angle with) the directrix.

Rotating a right triangle around one of its shorter sides (making the side the axis) will produce a right circular cone. In an oblique circular cone Fig. (b) the axis does not form a 90° angle with the directrix.



Right circular cone

(a)

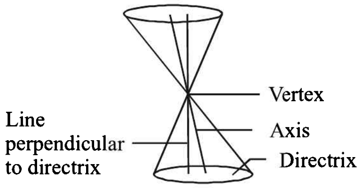


Oblique circular cone

(b)

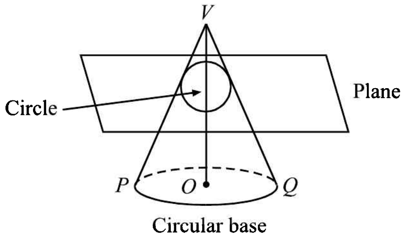
CONIC SECTIONS— AS SECTIONS OF A RIGHT CIRCULAR CONE

1. Section of a right circular cone by a plane, which is passing through its vertex, is a *pair of straight lines*. Lines always pass through the vertex of the cone.

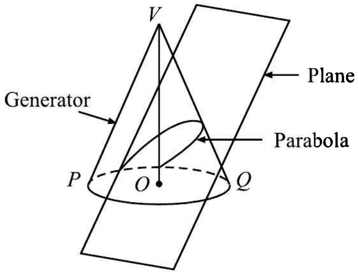


Oblique circular cone

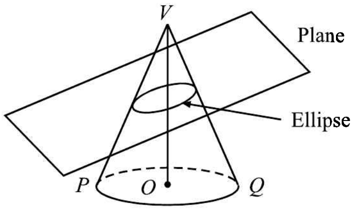
2. Section of a right circular cone by a plane, which is parallel to its base, is a *circle*.



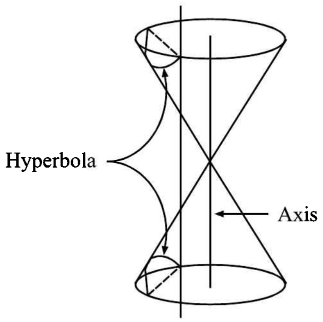
3. Section of a right circular cone by a plane, which is parallel to a generator of the cone, is a *parabola*.



4. Section of a right circular cone by a plane, which is not parallel to any generator and not parallel or perpendicular to the axis of the cone, is an *ellipse*.

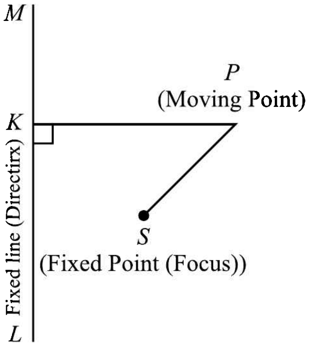


5. Section of a right circular cone by a plane, which is parallel to the axis of the cone, is a *hyperbola*.



CONIC SECTIONS OR CONICS – AS PLANE CURVES

Conic sections are two-dimensional or plane curves and therefore a desirable definition of conics avoids the notion of a cone, which is three-dimensional.



A conic section *or* **conic** *is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.*

The fixed point is called a **focus**, the fixed straight line is called a **directrix**, and the constant ratio is called the **eccentricity**, which is denoted by 'e'.

From the definition, we have

$$\frac{PS}{PK} = \text{constant} = e.$$

If $e = 1$, the curve is a **parabola**.

If $e < 1$, the curve is an **ellipse**.

If $e > 1$, the curve is a **hyperbola**.

The straight line passing through the focus and perpendicular to the directrix is called the **axis** of the conic.

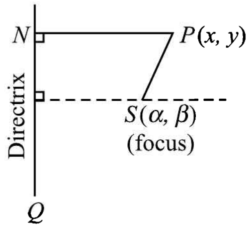
A point of intersection of conic with its axis is called a **vertex** of the conic.

Point to Note: It is assumed that the point P moves in the plane in which the fixed point and the fixed straight line lie.

We proceed to find the equation of the simplest of these curves, namely that in which the eccentricity is equal to *unity*. This curve as you know, is called a parabola.

EQUATION OF CONIC

Let $S(\alpha, \beta)$ be the focus and QN be the directrix whose equation is $Ax + By + C = 0$



Let $P(x, y)$ be any point on the conic. From P, draw PN ⊥ QN. If e is the eccentricity of the conic, then by definition

$$\frac{PS}{PN} = e \Rightarrow PS^2 = e^2PN^2$$

or $$(x - \alpha)^2 + (y - \beta)^2 = e^2\left(\frac{Ax + By + C}{\sqrt{A^2 + B^2}}\right)^2$$

This is the cartesian equation of the conic which, on simplification, takes the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g and h are constants.

Thus, the equation of a conic is an equation of second degree in x and y.

GENERAL EQUATION

The general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents

- a pair of straight lines if $\Delta = 0$

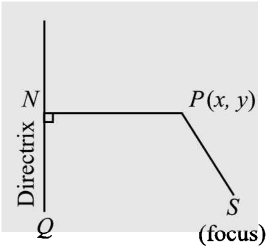
  where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

  or $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$,
- a circle if $\Delta \neq 0$, $a = b$ and $h = 0$,
- a parabola if $\Delta \neq 0$ and $h^2 = ab$,
- an ellipse if $\Delta \neq 0$ and $h^2 < ab$, and
- a hyperbola if $\Delta \neq 0$ and $h^2 > ab$.

CENTRE OF CONIC

Short-Cut Method

Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be the given conic.



- Differentiate S partially with respect to x treating y as constant, we get

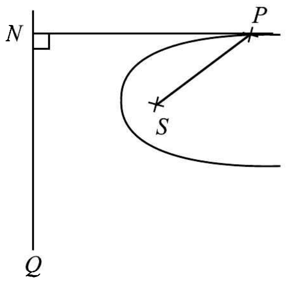
  $$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g.$$
- Differentiate S partially with respect to y treating x as constant, we get

  $$\frac{\partial S}{\partial x} = 2hx + 2by + 2f.$$
- Equating these two equations to zero and solving for x and y, we get the coordinates of centre as,

  $$(x, y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right).$$

PARABOLA

A *parabola* is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed straight line (called the directrix).



Let S be the focus, QN be the directrix and P be any point on the parabola. Then, by definition, $PS = PN$ where PN is the length of the perpendicular from P on the directrix QN.

Equation of a Parabola in the Standard Form

Let S be the focus and ZM the directrix of the parabola (Fig. 13.8). Draw $SZ \perp ZM$ and bisect SZ at A. Let $ZS = 2a (a > 0)$ so that $ZA = AS = a$

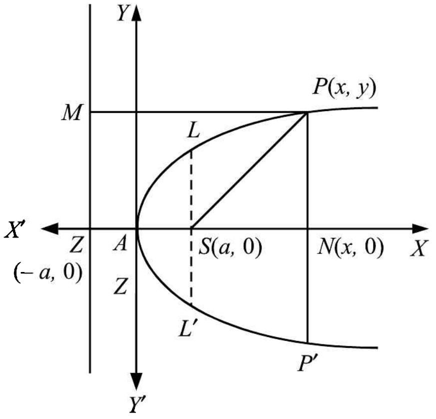
$\therefore$ By definition of parabola, A lies on the parabola.

Take ZS produced as the x-axis, A as origin and AY, a line through $A \perp$ to ZX as the y-axis.

$\therefore$ Coordinates of S are $(a, 0)$ and the co-ordinates of Z are $(-a, 0)$. Join SP and draw PM and PN perpendiculars to ZM and axis of x respectively.

ZM is a line $\parallel$ to y-axis at a distance a to its left.

$\therefore$ Its equation is



$x = -a$ or $x + a = 0$

By definition of parabola,

$SP = PM = ZN$ [$PM = ZN$]

or $SP^2 = PM^2 = ZN^2$ or $SP^2 = ZN^2$

or $(x - a)^2 + y^2 = (x + a)^2$

or $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

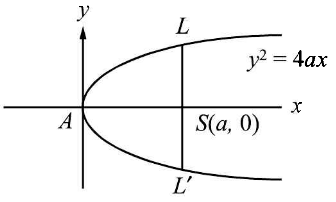
or $y^2 = 4ax (a > 0)$, which is the required equation of the parabola.

Cor. From the above equation of the parabola, we have

$$PN^2 = 4AS \cdot AN$$

Latus Rectum of the Parabola

Let the given parabola be $y^2 = 4ax$. In the figure LSS' (a line through focus $\perp$ to axis) is the latus rectum.



Also by definition,

$LSL' = 2(\sqrt{4a \cdot a}) = 4a$

= double ordinate (Any chord of the parabola $y^2 = 4ax$ which is $\perp$ to its axis is called the double ordinate) through the focus S.

Point to Note: Two parabolas are said to be equal when their latus recta are equal.

Parametric Equations of a Parabola

Clearly $x = at^2$, $y = 2at$ satisfy the equation $y^2 = 4ax$ for all real values of t. Hence the parametric equations of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$, where t is the parameter.

Also, $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ for all real values of t. This point is also described as the point 't' on the parabola.

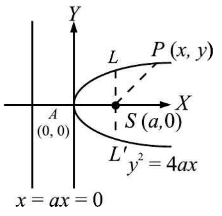
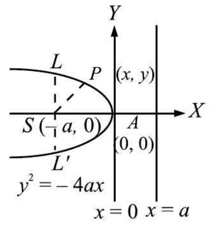
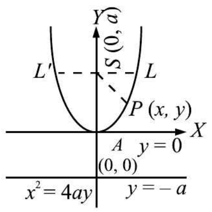
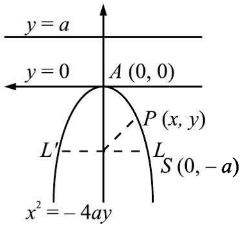
Focal Chord

Any chord to the parabola which passes through the focus is called a focal chord of the parabola.

Four standard forms of the parabola are shown in the table below:

Table 4.1

Four Standard Forms of the Parabola

Standard Equation	$y^2 = 4ax\ (a > 0)$	$y^2 = -4ax\ (a > 0)$	$x^2 = 4ax\ (a > 0)$	$x^2 = -4ax\ (a > 0)$
Shape of the parabola				
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Extermities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$

Continued on following page

Four Standard Forms of the Parabola

Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Equation of tangents at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Focal distance of a point $P(x, y)$	$x + a$	$x - a$	$y + a$	$y - a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

EQUATION OF A CHORD

1. The equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is:
$$y(y_1 + y_2) = 4ax + y_1y_2$$
2. The equation of chord joining the points $(a t_1^2, 2at_1)$ and $(a t_2^2, 2at_2)$ is:
$$y(t_1 + t_2) = 2(x + at_1t_2)$$
3. Length of the chord $y = mx + c$ to the parabola
$y^2 = 4ax$ is given by $\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$.

Condition for the chord to be a focal chord The chord joining the points $(a t_1^2, 2at_1)$ and $(a t_2^2, 2at_2)$ passes through focus provided $t_1t_2 = -1$.

Length of focal chord The length of a focal chord joining the points $(a t_1^2, 2at_1)$ and $(a t_2^2, 2at_2)$ is $(t_2 - t_1)^2$.

Note: The length of the focal chord through the point 't' on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.

Condition for tangency and point of contact The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

EQUATIONS OF TANGENT IN DIFFERENT FORMS

1. Point form The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is
$$yy_1 = 2a(x + x_1).$$

Note: The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of parabola is polynomial of second degree in x and y.

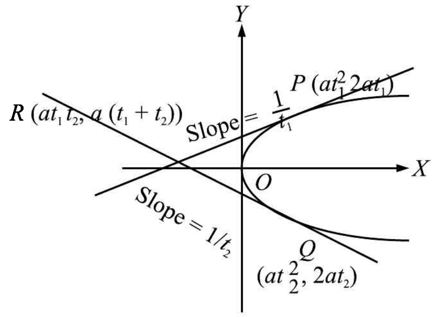
2. Parametric form The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is
$$ty = x + at^2.$$

3. Slope form The equation of tangent to the parabola $y^2 = 4ax$ in terms of slope 'm' is
$$y = mx + \frac{a}{m}.$$

The coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(a t_1^2, 2at_1)$ and $Q(a t_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



$$R \equiv (at_1t_2, a(t_1 + t_2)).$$

Points to Note:

- Angle between tangents at two points $P(a t_1^2, 2at_1)$ and $Q(a t_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is:
$$\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$$
- The *G. M.* of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The A.M. of the y-coordinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. Point form The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is
$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

2. Parametric form The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is
$$y + tx = 2at + at^3.$$

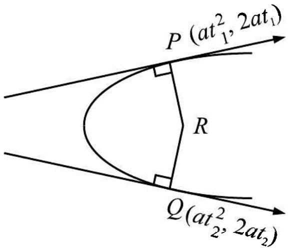
3. Slope form The equation of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is
$$y = mx - 2am - am^3.$$

Note: The coordinates of the point of contact are $(am^2, -2am)$.

Condition for normality The line $y = mx + c$ is a normal to the parabola

$$y^2 = 4ax \text{ if } c = -2am - am^3.$$

Point of intersection of normals The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

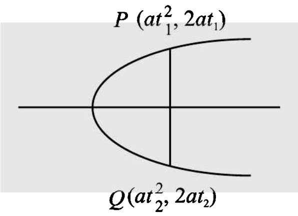


$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)].$$

Key Points to Remember

1. If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}.$$



Note that PQ is normal to the parabola at P and not at Q.

2. If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$.

POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, = \text{ or } < 0$, respectively.

NUMBER OF TANGENTS DRAWN FROM A POINT TO A PARABOLA

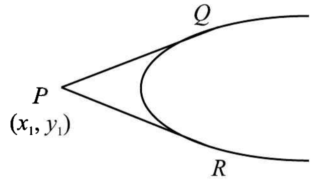
Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

Equation of the Pair of Tangents

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$,

where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2a(x + x_1)$



Some Useful Results on Parabola

1. The tangent at any point P on the parabola bisects the angle between the focal chord through P and the perpendicular from P on the directrix.
2. The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.
3. Tangents at the extremities of any focal chord intersect at right angles on the directrix.
4. Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.

ELLIPSE

An ellipse is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point (called focus) and a fixed line (called directrix) is a constant which is less than one. This ratio is called eccentricity and is denoted by e. For an ellipse, $e < 1$.

Let S be the focus, QN be the directrix and P be any point on the ellipse. Then, by definition, $\frac{PS}{PN} = e$ or PS $= ePN$, $e < 1$, where PN is the length of the perpendicular from P on the directrix QN.

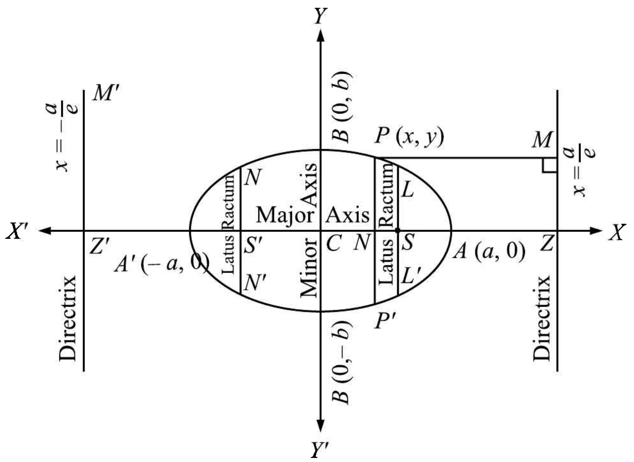
An alternate definition An ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.

EQUATION OF AN ELLIPSE IN STANDARD FORM

The standard form of the equation of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b),$$

where a and b are constants.



Some Terms and Properties Related to an Ellipse

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, has been shown in the figure given above.

1. Symmetry

(a) On replacing y by $-y$, the above equation remains unchanged. So, the curve is symmetrical about x-axis

(b) On replacing x by $-x$, the above equation remains unchanged. So, the curve is symmetrical about y-axis

2. Foci If S and S' are the two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively, then distance between foci is given by

$$SS' = 2ae.$$

3. Directrices If ZM and $Z'M'$ are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by

$$ZZ' = \frac{2a}{e}.$$

4. Axes The lines AA' and BB' are called the **major axis** and **minor axis** respectively of the ellipse.

The length of major axis $= AA' = 2a$

The length of minor axis $= BB' = 2b$

5. Centre The point of intersection C of the axes of the ellipse is called the centre of the ellipse. All chords, passing through C are bisected at C.

6. Vertices The end points A and A' of the major axis are known as the vertices of the ellipse

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

7. Focal chord A chord of the ellipse passing through its focus is called a focal chord.

8. Ordinate and double ordinate Let P be a point on the ellipse. From P, draw $PN \perp AA'$ (major axis of the ellipse)and produce PN to meet the ellipse at P'. Then PN is called an *ordinate* and PNP' is called the *double ordinate* of the point P.

9. Latus rectum If LL' and NN' are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis AA', passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \quad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \quad N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = NN'$.

10. By definition, $SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$

and $S'P = e\left(\frac{a}{e} + x\right) = a + ex.$

This implies that distances of any point $P(x, y)$ lying on the ellipse from foci are : $(a - ex)$ and $(a + ex)$. In other words

$$SP + S'P = 2a$$

i.e., sum of distances of any point $P(x, y)$ lying on the ellipse from foci is constant.

11. Eccentricity of the ellipse Since, $SP = ePM$, therefore,

$$
\begin{aligned}
SP^2 &= e^2PM^2 \\
\text{or} \quad (x - ae)^2 + (y - 0)^2 &= e^2\left(\frac{a}{e} - x\right)^2 \\
(x - ae)^2 + y^2 &= (a - ex)^2 \\
x^2 + a^2e^2 - 2aex + y^2 &= a^2 - 2aex + e^2x^2 \\
x^2(1 - e^2) + y^2 &= a^2(1 - e^2) \\
\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} &= 1.
\end{aligned}
$$

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

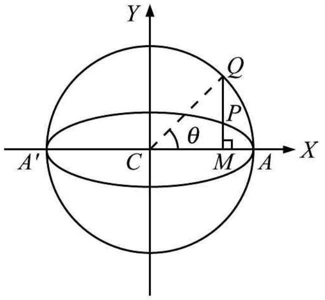
$$b^2 = a^2(1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}.$$

12. Auxiliary circle The circle drawn on major axis AA' as diameter is known as the Auxiliary circle.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then the equation of its auxiliary circle is:

$$x^2 + y^2 = a^2.$$

Let Q be a point on auxiliary circle so that QM, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding points on the ellipse and auxiliary circle respectively.



The angle θ is known as *eccentric angle* of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at θ with x-axis.

13. Parametric equation of the ellipse The coordinates $x = a\cos\theta$ and $y = b\sin\theta$ satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for all real values of θ. Thus, $x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.

Hence the coordinates of any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be taken as $(a\cos\theta, b\sin\theta)$. This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point $(a\cos\theta, b\sin\theta)$ on the ellipse.

14. Equation of Chord The equation of the chord joining the points $P \equiv (a\cos\theta_1, b\sin\theta_1)$ and $Q \equiv (a\cos\theta_2, b\sin\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 > 0, = 0$ or < 0.

CONDITION FOR TANGENCY AND POINTS OF CONTACT

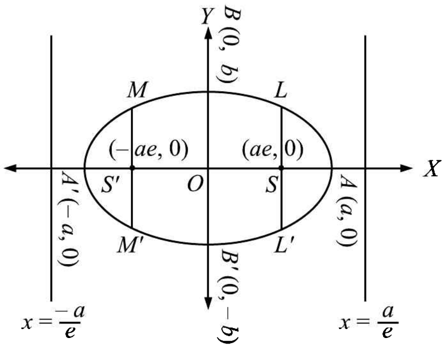
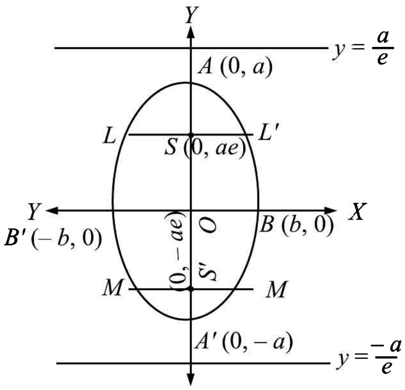
The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Two standard forms of the ellipse are shown below along with their properties:

Table 4.2

Two Standard Forms of the Ellipse

Standard Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \; (a > b)$ (Horizontal Form of an Ellipse)	$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1 \; (a, b)$ (Vertical Form of an Ellipse)
Shape of the Ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Eccentricity	$e = \sqrt{\frac{a^2-b^2}{a^2}}$	$e = \sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latra-recta	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{b^2}{a}, \pm ae\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = a - ey_1$ and $S'P = a + ey_1$.

Continued on following page

Sum of focal radii $SP + S'P =$	$2a$	$2a$
Distance between foci	$2ae$	$2ae$
Distance between directrices	$\frac{2a}{e}$	$\frac{2a}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm a$

EQUATION OF TANGENT IN DIFFERENT FORMS

1. Point form The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Note: The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$, and xy by $\frac{xy_1 + x_1y}{2}$. This method is used only when the equation of ellipse is a polynomial of second degree in x and y.

2. Parametric form The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

3. Slope form The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

Notes:

- **Number of tangents drawn from a point** Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.
- **Director circle** It is the locus of points from which perpendicular tangents are drawn to the ellipse.
  The equation of Director Circle of the ellipse
  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 + b^2.$$
- The product of perpendiculars from the foci on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to b^2.

EQUATION OF NORMAL IN DIFFERENT FORMS

1. Point form The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

2. Parametric form The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2.$$

or
$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2.$$

3. Slope form The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

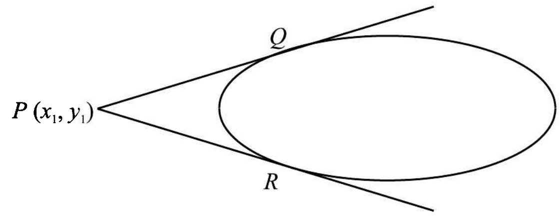
$$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$$

Notes:

- The coordinates of the points of contact are
  $$\left(\pm\frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2 + b^2m^2}}\right)$$
- **Condition for normality** The line $y = mx + c$ is a normal to the ellipse
  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2m^2)}.$$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point



$P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$SS_1 = T^2$$

where $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

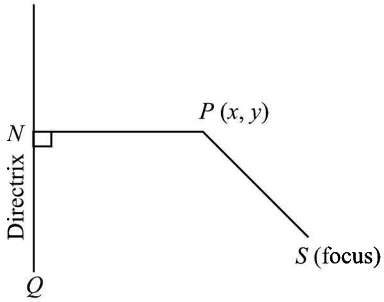
HYPERBOLA

A hyperbola is the locus of a point which moves in a plane so that the ratio of its distances from a fixed point (called focus) and a fixed line (called directrix) is a constant which is greater then one. This ratio is called eccentricity and is denoted by e. For a hyperbola, $e > 1$.

Let S be the focus, QN be the directrix and P be any point on the hyperbola. Then, by definition,

$$\frac{PS}{PN} = e \quad\text{or}\quad PS = ePN,\ e > 1,$$

where PN is the length of the perpendicular from P on the directrix QN.



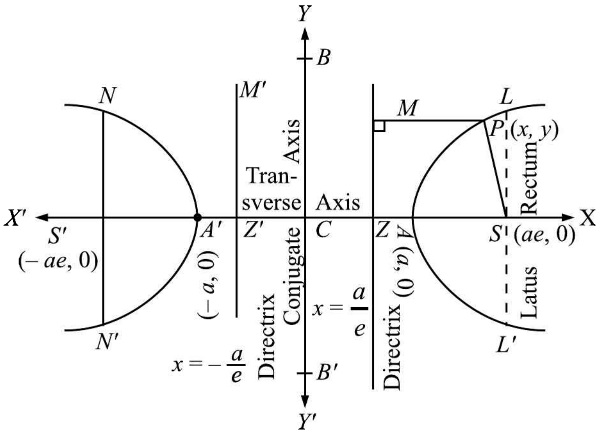
An alternate definition A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points (called foci) is constant.

EQUATION OF A HYPERBOLA IN STANDARD FORM

The general form of standard hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are constants.



Some Terms and Properties Related to a Hyperbola

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, has been shown in the figure above.

1. Symmetry Since only even powers of x and y occur in the above equation , so the curve is symmetrical about both the axes.

2. Foci If S and S' are the two foci of the hyperbola and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively, then distance between foci is given by $SS' = 2ae$.

3. Directrices ZM and $Z'M'$ are the two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by $ZZ' = \frac{2a}{e}$.

4. Axes The lines AA' and BB' are called the **transverse axis** and **conjugate axis** respectively of the hyperbola.

The length of transverse axis = $AA' = 2a$

The length of conjugate axis = $BB' = 2b$.

5. Centre The point of intersection C of the axes of the hyperbola is called the centre of the hyperbola. All chords, passing through C, are bisected at C.

6. Vertices The points $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$, where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.

7. Focal chord A chord of the hyperbola passing through its focus is called a focal chord.

8. Focal distances of a point The difference of the focal distances of any point on the hyperbola is constant and equal to the length of the transverse axis of the hyperbola. If P is any point on the hyperbola, then

$$S'P - SP = 2a = \text{Transverse axis.}$$

9. Latus rectum If LL' and NN' are the latus rectum of the hyperbola then these lines are perpendicular to the transverse axis AA', passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \qquad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \qquad N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

$$\text{Length of latus rectum} = LL' = \frac{2b^2}{a} = NN'.$$

10. Eccentricity of the hyperbola We know that

$$SP = e\,PM \quad \text{or} \quad SP^2 = e^2\,PM^2$$

or

$$\begin{aligned}(x - ae)^2 + (y - 0)^2 &= e^2\left(x - \frac{a}{e}\right)^2\\ (x - ae)^2 + y^2 &= (ex - a)^2\\ x^2 + a^2e^2 - 2aex + y^2 &= e^2x^2 - 2aex + a^2\\ x^2(e^2 - 1) - y^2 &= a^2(e^2 - 1)\\ \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} &= 1.\end{aligned}$$

On comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$b^2 = a^2(e^2 - 1) \quad \text{or} \quad e = \sqrt{1 + \frac{b^2}{a^2}}$$

11. Parametric equations of the hyperbola Since the coordinates $x = a\sec\theta$ and $y = b\tan\theta$ satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for all real values of θ, therefore, $x = a\sec\theta, y = b\tan\theta$ are the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.

Hence, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $(a\sec\theta, b\tan\theta)$. This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point $(a\sec\theta, b\tan\theta)$ on the hyperbola.

12. Equation of chord The equation of the chord joining the points

$P \equiv (a\sec\theta_1, b\tan\theta_1)$ and $Q \equiv (a\sec\theta_2, b\tan\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right).$$

OR

$$\begin{vmatrix} x & y & 1 \\ a\sec q_1 & b\tan q_1 & 1 \\ a\sec q_2 & b\tan q_2 & 1 \end{vmatrix} = 0$$

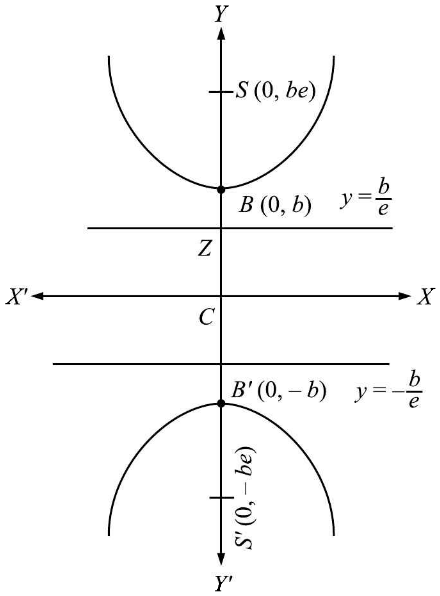
CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \left(\text{i.e. } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\right)$$



Properties of hyperbola and conjugate are given below in the table:

Table 4.3 Properties of Hyperbola and its Conjugate

	Hyperbola	*Conjugate Hyperbola*
Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
centre	(0, 0)	(0, 0)
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$	$e=\sqrt{\frac{a^2+b^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametrc coordinates	$(a\sec\theta, b\tan\theta)$	$(b\sec\theta, a\tan\theta)$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii ($S'P - SP$)	$2a$	$2b$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0, =0$ or <0.

CONDITION FOR TANGENCY AND POINTS OF CONTACT

The condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Equation of Tangent in Different Forms

1. Point form The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Note: The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$. This method is used only when the equation of hyperbola is a polynomial of second degree in x and y.

2. Parametric form The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

3. Slope form The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Notes:

- **Number of tangents from a point** Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.
- **Director circle** It is the locus of points from which ⊥ tangents are drawn to the hyperbola. The equation of director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

Equations of Normal in Different Forms

1. Point form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2.$$

2. Parametric form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2.$$

3. Slope form The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}.$$

Note:

The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2 - b^2m^2}}\right)$$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $SS_1 = T^2$

where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

ASYMPTOTES OF HYPERBOLA

The lines $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ i.e., $y = \pm\frac{bx}{a}$ are called the asymptotes of the hyperbola.

The curve comes close to these lines as $x \to \infty$ or $x \to -\infty$ but never meets them. In other words, asymptote to a curve touches the curve at infinity.

Notes:

- The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- Asymptotes are the diagonals of the rectangle passing through A, B, A', B' with sides parallel to axes.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The asymptotes pass through the centre of the hyperbola.
- The bisector of the angle between the asymptotes are the coordinate axes.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is a constant equal to $\frac{a^2b^2}{a^2 + b^2}$.
- Any line drawn parallel to the asymptote of the hyperbola would meet the curve only at one point.
- A hyperbola and its conjugate hyperbola have the same asymptotes.

RECTANGULAR HYPERBOLA

If asymptotes of the standard hyperbola are perpendicular to each other, then it is known as Rectangular Hyperbola. Thus,

$$2\tan^{-1}\frac{b}{a}=\frac{\pi}{2}\Rightarrow b=a \quad\text{or}\quad x^2-y^2=a^2$$

is general form of the equation of the rectangular hyperbola.

If we take the coordinate axes along the asymptotes of a rectangular hyperbola, then equation of rectangular hyperbola becomes : $xy=c^2$, where c is any constant.

In **parametric form** , the equation of rectangular hyperbola is

$x=ct$, $y=c/t$, where t is the parameter.

The point $(ct, c/t)$ on the hyperbola $xy=c^2$ is generally referred as the point 't'.

Properties of Rectangular Hyperbola, $x^2 - y^2 = a^2$

1. The equations of asymptotes of the rectangular hyperbola are $y=\pm x$.
2. The transverse and conjugate axes of a rectangular hyperbola are equal in length.
3. Eccentricity, $e=\sqrt{1+\dfrac{b^2}{a^2}}=\sqrt{2}$.

Properties of Rectangular Hyperbola $xy = c^2$

1. Equation of the chord joining 't_1' and 't_2' is:
$$x+yt_1t_2-c(t_1+t_2)=0$$
2. Equation of tangent at (x_1, y_1) is:
$$xy_1+x_1y=2c^2 \text{ or } \frac{x}{x_1}+\frac{y}{y_1}=2.$$
3. Equation of tangent at 't' is: $\dfrac{x}{t}+yt=2c$.
4. Point of intersection of tangents at 't_1' and 't_2' is:
$$\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1t_2}\right)$$
5. Equation of normal at (x_1, y_1) is: $xx_1-yy_1=x_1^2-y_1^2$.
6. Equation of normal at 't' is: $xt^3-yt-ct^4+c=0$

Solved Problems

1. If QQ' is a double ordinate of a parabola $y^2=9x$, then the locus of its point of trisection is

(a) $y^2=x$ (b) $y^2=3x$
(c) $y^2=6x$ (d) none of these

Solution (a) Let R and T be the points of trisection of double ordinate QQ'. Let (h, k) be the coordinates of R. Then, $AL=h$ and $RL=k$.

$\therefore\quad RT=RL+LT=k+k=2k.$

Since $RQ=TR=Q'T=2k$

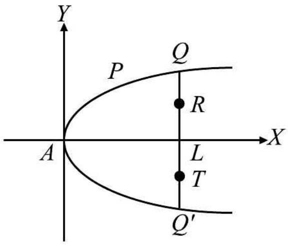
$\therefore\quad LQ=LR+RQ=k+2k=3k.$

Thus, the coordinates of Q are $(h, 3k)$.

Since $(h, 3k)$ lies on the parabola $y^2=9x$

$\therefore\quad 9k^2=9h$ or $k^2=h$.

Hence, the locus of (h, k) is $\boldsymbol{y^2=x}$.

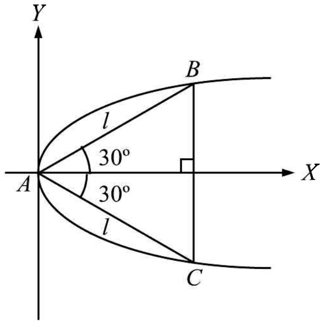


2. The length of the side of an equilateral triangle, inscribed in the parabola $y^2=8x$ so that one angular point is at the vertex, is

(a) $16\sqrt{3}$ (b) $8\sqrt{3}$
(c) $4\sqrt{3}$ (d) none of these

Solution (a) Let ABC be the equilateral triangle inscribed in the parabola

$$y^2=8x \qquad ...(1)$$



Let l be the length of the side of equilateral triangle.

$\therefore\quad AB=BC=CA=l.$

$\therefore$ The coordinates of B are $(l\cos 30^\circ, l\sin 30^\circ)$

i.e., $\left(\dfrac{l\sqrt{3}}{2}, \dfrac{l}{2}\right)$.

Since the point B lies on (1),

$$\therefore\quad \left(\frac{l}{2}\right)^2=8\left(\frac{l\sqrt{3}}{2}\right) \quad\text{or}\quad l=\mathbf{16\sqrt{3}}.$$

3. The normal to the parabola $y^2=8x$ at $(2, 4)$ meets the parabola again at

(a) (18, 12) (c) (18, – 12)
(c) (– 18, 12) (d) none of these

Solution (b) Equation of parabola is $y^2=8x$ ...(1)

Here $4a=8$, $\therefore\ a=2$.

$\therefore$ Equation of normal at (2, 4) is

$$y-4=-\frac{4}{2\times 2}(x-2)$$

$$\left[(y-y_1)=-\frac{y_1}{2a}(x-x_1)\right]$$

$$\Rightarrow\quad y-4=-(x-2) \text{ or } x+y-6=0 \qquad ...(2)$$

From (2), $y = 6 - x$.

Putting this value of y in (1),

$$(6 - x)^2 = 8x \text{ or } x^2 - 20x + 36 = 0$$

or $(x - 2)(x - 18) = 0.\ \therefore\ x = 2, 18.$

When $x = 2$, $y = 6 - x = 6 - 2 = 4$

and when $x = 18$, $y = 6 - x = 6 - 18 = -12$.

$\therefore$ Normal (2) meets the parabola (1) in (2, 4) and (18, – 12). Hence, the normal at (2, 4) meets the parabola again in **(18, –12).**

4. The equation of the normal to the parabola $y^2 = 4x$, which passes through the point (3, 0), is

(a) $y = 0$ (b) $y = x - 3$

(c) $y = -x + 3$ (d) all of these

Solution (d) The given parabola is $y^2 = 4x$ ...(1)

Here $4a = 4$, $\therefore$ $a = 1$.

$\therefore$ Equation of any normal to (1) is

$y = mx - 2m - m^3$ ...(2) $[y = mx - 2am - am^3]$

If it passes through (3, 0), then $0 = 3m - 2m - m^3$

or $m^3 - m = 0$ or $m(m^2 - 1) = 0\ \therefore\ m = 0, \pm 1$.

Putting these values of m in (2), the equations of the normals are $\boldsymbol{y = 0, y = x - 3}$ **and** $\boldsymbol{y = -x + 3}$.

5. If the point $(at^2, 2at)$ be one extremity of a focal chord of the parabola $y^2 = 4ax$, then the length of the chord is

(a) $a\left(t+\frac{1}{t}\right)^2$ (b) $a\left(t-\frac{1}{t}\right)^2$

(c) $2a\left(t-\frac{1}{t}\right)^2$ (d) none of these

Solution (a) Let the other extremity of the focal chord be $(at_1^2, 2at_1)$.

$\therefore$ $tt_1 = -1$ or $t_1 = \frac{-1}{t}$.

$\therefore$ The coordinates of the other extremity are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

Length of the chord = distance between the points $(at^2, 2at)$ and $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

$$= \sqrt{\left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2}$$

$$= a\sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + 4\left(t + \frac{1}{t}\right)^2}$$

$$= a\sqrt{\left(t + \frac{1}{t}\right)^2\left[\left(t - \frac{1}{t}\right)^2 + 4\right]}$$

$$= a\sqrt{\left(t + \frac{1}{t}\right)^2\left(t + \frac{1}{t}\right)^2} = \mathbf{a}\left(\mathbf{t} + \frac{\mathbf{1}}{\mathbf{t}}\right)^2$$

6. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents

(a) a pair of straight lines

(b) an ellipse

(c) a parabola

(d) a hyperbola

Solution (c) Given, $x = t^2 + t + 1$ and $y = t^2 - t + 1$

$\Rightarrow$ $x + y = 2(t^2 + 1)$ and $x - y = 2t$.

Eliminating t, we get

$$x + y = \frac{1}{2}(x - y)^2 + 2$$

or $x^2 - 2xy + y^2 - 2x - 2y + 4 = 0$

Here $a = 1, h = -1, b = 1, g = -1, f = -1, c = 4$.

$$\therefore\ \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$
$$= 1 \cdot 1 \cdot 4 + 2(-1)(-1)(-1) - 1 \cdot 1 - 1 \cdot 1 - 4 \cdot 1$$
$$= 4 - 2 - 1 - 1 - 4 = -4 \neq 0.$$

Also, $h^2 - ab = (-1)^2 - 1 \cdot 1 = 0$.

Hence, the given curve represents a parabola.

7. If (4, 0) is the vertex and y-axis, the directrix of a parabola, then its focus is

(a) (8, 0) (b) (4, 0)

(c) (0, 8) (d) (0, 4)

Solution (a) Since y-axis is the directrix and (4, 0) is the vertex of the parabola,

$\therefore$ Distance between the vertex and the directrix = 4.

$\therefore$ Distance of focus from the directrix = $2 \times 4 = 8$.

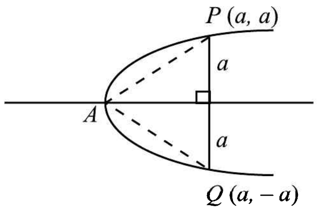
$\therefore$ Coordinates of the focus, which is on x-axis, are (8, 0).

8. The angle subtended by the double ordinate of length $2a$ of the parabola $y^2 = ax$, at the vertex is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) none of these

Solution (c) Let PQ be a double ordiante of length $2a$ of the parabola $y^2 = ax$. Then, the coordinates of P and Q are (a, a) and $(a, -a)$ respectively.



Now, slope of $AP = m_1 = \frac{a - 0}{a - 0} = 1$ and

slope of $AQ = m_2 = \frac{-a - 0}{a - 0} = -1$.

Since $m_1 \times m_2 = -1$, $\therefore$ $\angle PAQ$ $\therefore$ $\pi/2$ i.e., the double ordinate subtends a right angle at the vertex.

9. Two tangents are drawn from the point (– 2, – 1) to the parabola $y^2 = 4x$. If α is the angle between them, then $\tan\alpha$ =

(a) 3 (b) 1/3

(c) 2 (d) 1/2

Solution (a) Given parabola is $y^2 = 4x$.

Here $4a = 4$, $\therefore$ $a = 1$.

Equation of pair of tangents drawn from the point (– 2, – 1) to the parabola is $SS_1 = T^2$

i.e., $(y^2 - 4x)[(-1)^2 - 4(-2)] = [y(-1) - 2 \cdot 1(x - 2)]^2$

$\Rightarrow$ $9(y^2 - 4x) = (-2x - y + 4)^2$

$\Rightarrow$ $4x^2 - 8y^2 + 4xy + 20x - 8y + 16 = 0$

Since α is the angle between the tangents,

$$\therefore\ \tan\alpha = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right| = \left|\frac{2\sqrt{4-4(-8)}}{4-8}\right|$$

$$= |-3| = \mathbf{3}.$$

10. The normals to the parabola $y^2 = 4ax$ from the point $(5a, 2a)$ are

(a) $y = x - 3a$ (b) $y = -2x + 12a$

(c) $y = -3x + 33a$ (d) $y = x + 3a$

Solution (a), (b) Equation of normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3 \qquad \text{...(1)}$$

Since it passes through the point $(5a, 2a)$,

$$\therefore\ 2a + 5at = 2at + at^3 \Rightarrow t^3 - 3t - 2 = 0$$

or $$(t-2)(t+1)^2 = 0$$

$$\therefore\ t = -1, -1, 2.$$

Putting $t = -1$ and $t = 2$ in (1), the normals from $(5a, 2a)$ are $\mathbf{y = x - 3a}$ and $\mathbf{y = -2x + 12a}$.

11. The portion of a tangent to a parabola $y^2 = 4ax$ cut off between the directrix and the curve subtends an angle θ at the focus, where $\theta =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

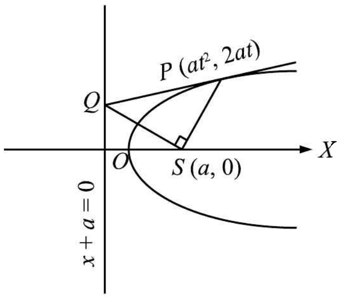
(c) $\frac{\pi}{2}$ (d) none of these

Solution (c) The equation of the tangent at $P(at^2, 2at)$ to

$$y^2 = 4ax \text{ is } ty = x + at^2 \qquad \text{...(1)}$$

It meets the directrix $x = -a$.

$$\therefore\ ty = -a + at^2 \Rightarrow y = \frac{a(t^2-1)}{t}.$$



Thus, (1) meets the directrix at $Q\left(-a, \frac{a(t^2-1)}{t}\right)$.

Now, slope of PS is $m_1 = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2-1}$ and

slope of QS is $m_2 = \frac{a(t^2-1)/t - 0}{-a-a} = -\frac{(t^2-1)}{2t}$.

Since $m_1m_2 = -1$, therefore PQ subtends a right angle at the focus.

12. The length of the latus rectum of the parabola $25[(x-2)^2 + (y-4)^2] = (4x - 3y + 12)^2$ is

(a) $\frac{16}{5}$ (b) $\frac{8}{5}$

(c) $\frac{12}{5}$ (d) none of these

Solution (a) The given equation of the parabola can be written as

$$(x-2)^2 + (y-4)^2 = \left(\frac{4x-3y+12}{\sqrt{(4)^2+(-3)^2}}\right)^2.$$

$\therefore$ The coordinates of focus are (2, 4) and the equation of directrix is $4x - 3y + 12 = 0$.

The distance of the focus from the directrix

$$= \frac{|4(2) - 3(4) + 12|}{\sqrt{4^2 + (-3)^2}} = \frac{8}{5}.$$

$$\therefore \text{ The length of latus rectum } = 2\times\frac{8}{5} = \mathbf{\frac{16}{5}}.$$

13. A ray of light is coming along the line which is parallel to y-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $(x-4)^2 = 4(y+2)$. After reflection, the ray must pass through the point

(a) $(4, -1)$ (b) $(0, 1)$

(c) $(-4, 1)$ (d) none of these

Solution (a) The equation of axis of the parabola is $x - 4 = 0$ which is parallel to y-axis. So the ray of light is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection.

$\therefore$ The ray must pass through the point **(4, –1).**

14. If $y + 3 = m_1(x + 2)$ and $y + 3 = m_2(x + 2)$ are two tangents to the parabola $y^2 = 8x$, then

(a) $m_1 + m_2 = 0$ (b) $m_1m_2 = -1$

(c) $m_1m_2 = 1$ (d) none of these

Solution (b) Clearly the two tangents, having slopes m_1 and m_2, meet on the line $x = -2$, which is the directrix of the parabola $y^2 = 8x$, therefore the two tangents must be at right angles, i.e., $\mathbf{m_1m_2 = -1}$.

15. The parametric representation $(3 + t^2, 3t - 2)$ represents a parabola with

(a) focus at $(-3, -2)$ (b) vertex at $(3, -2)$

(b) directrix $x = -5$ (d) all of these

Solution (b) We have, $x = 3 + t^2$ and $y = 3t - 2$

$$\Rightarrow\ x - 3 = t^2 \text{ and } y + 2 = 3t$$

$$\Rightarrow\ (y+2)^2 = 9(x-3)$$

which is a parabola with **vertex at (3, –2),** focus at $\left(\frac{21}{4}, -2\right)$ and directrix $x = \frac{3}{4}$.

16. If PSQ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$, then the length SQ is

(a) 6 (b) 4

(c) 3 (d) none of these

Solution (c) Since the semi latus rectum of a parabola is the harmonic mean between the segment of any focal chord of a parabola, therefore SP, 4, SQ are in H.P.

$$\Rightarrow\ 4 = 2\frac{SP\cdot SQ}{SP+SQ} \Rightarrow 4 = 2\cdot\frac{6\cdot SQ}{6+SQ} \Rightarrow SQ = \mathbf{3}.$$

17. If the two parabolas $y^2 = 4a(x-2)$ and $x^2 = 4a(y-3)$ touch each other, then their point of contact lies on a

(a) circle (b) parabola

(c) ellipse (d) hyperbola

Solution (d) Let $P(x_1, y_1)$ be the point of contact of the two given parabolas

$$y^2 = 4a(x-2) \qquad \text{...(1)}$$

and $$x^2 = 4a(y-3) \qquad \text{...(2)}$$

Equation of tangent at P to (1) is

$$yy_1 = 2a(x + x_1) - 8a$$

or $$2ax - y_1 y + (2ax_1 - 8a) = 0 \qquad \text{...(3)}$$

Equation of tangent at P to (2) is

$$xx_1 = 2a\,(y + y_1) - 12a$$

or $$x_1 x - 2ay - (2ay_1 - 12a) = 0 \qquad \text{...(4)}$$

Since (3) and (4) represent the same line,

$$\therefore \quad \frac{2a}{x_1} = \frac{-y_1}{-2a} \Rightarrow x_1 y_1 = 4a^2.$$

$\therefore$ point of contact (x_1, y_1) lies on the curve $xy = 4a^2$. which is a **hyperbola.**

18. If ASB is a focal chord of a parabola such that $AS = 2$ and $SB = 4$, then the latus rectum of the parabola is

(a) $\frac{8}{3}$ (b) $\frac{16}{3}$

(c) $\frac{25}{3}$ (d) none of these

Solution (b) We know that semi-latus rectum of a parabola is the harmonic mean of segments of a focal chord. $\therefore$ Semi latus rectum $= 2\dfrac{AS \cdot SB}{AS + SB} = \dfrac{16}{6}$.

$\therefore$ Latus rectum $= \dfrac{32}{6}$ i.e., $\mathbf{\dfrac{16}{3}}$.

19. A line bisecting the ordinate PN of a point $P(at^2, 2at)$, $t > 0$, on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at Q. If NQ meets the tangent at the vertex at the point T, then the coordinates of T are

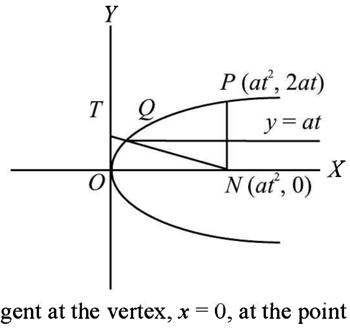
(a) $\left(0, \frac{4}{3}at\right)$ (b) $(0, 2\,at)$

(c) $\left(\frac{1}{4}at^2, at\right)$ (d) $(0, at)$

Solution (a) Equation of the line parallel to the axis and bisecting the ordinate PN of the point $P(at^2, 2at)$ is $y = at$ which meets the parabola $y^2 = 4ax$ at the point $Q\left(\frac{1}{4}at^2, at\right)$.

Coordinates of N are $(at^2, 0)$.

Equation of NQ is $y = \dfrac{0 - at}{at^2 - \frac{1}{4}at^2}(x - at^2)$,



which meets the tangent at the vertex, $x = 0$, at the point

$$y = \frac{4}{3}at.$$

20. The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is

(a) 90° (b) 30°

(c) $\tan^{-1}1/2$ (d) 45°

Solution (a) Equations of tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is $SS_1 = T^2$

i.e., $[y^2 - 4a(x - a)]\,[0 - 4a\,(0 - a)] = [y \times 0 - 2a\,(x - 0)]^2$

$\Rightarrow$ $(y^2 - 4ax + 4a^2)4a^2 = 4a^2x^2$ or $y^2 - 4ax + 4a^2 = x^2$

or $x^2 - y^2 + 4ax - 4a^2 = 0$.

Since the sum of coefficients of x^2 and y^2 is zero, $\therefore$ the angle between the tangents is 90°.

21. The equation of the normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the end of the latus rectum in the first quadrant, is

(a) $x + ey - ae^3 = 0$ (b) $x - ey + ae^3 = 0$

(c) $x - ey - ae^3 = 0$ (d) none of these

Solution (c) The end of the latus rectum in the first quadrant is $\left(ae, \dfrac{b^2}{a}\right)$.

Equation of normal at $\left(ae, \dfrac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2 \qquad \left[\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2\right]$$

or $$\frac{a}{e}x - ay = a^2e^2 \qquad \left[\because e^2 = \frac{a^2 - b^2}{a^2}\right]$$

or $\mathbf{x - ey - ae^3 = 0.}$

22. If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, then

(a) $e^4 + e^2 - 1 = 0$ (b) $e^4 - e^2 + 1 = 0$

(c) $e^4 - e^2 - 1 = 0$ (d) none of these

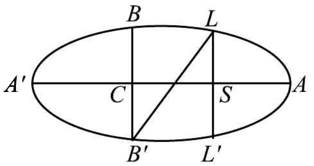
Solution (a) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let the normal at the extremity L of the latus rectum passes through the extremity B' of the minor axis.

Coordinates of L are $\left(ae, \dfrac{b^2}{a}\right)$ and coordinates of B' are $(0, -b)$.

Equation of the normal at L is



$$\frac{a^2 \cdot x}{ae} - \frac{b^2 \cdot y}{b^2/a} = a^2 - b^2 \qquad \left[\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2\right]$$

or $$\frac{ax}{e} - ay = a^2 - b^2.$$

If it passes through $B'(0, -b)$, then $0 + ab = a^2 - b^2$

$\Rightarrow$ $a^2b^2 = (a^2 - b^2)^2$

But $b^2 = a^2(1 - e^2)$.

∴ $a^2 \times a^2(1 - e^2) = [a^2 - a^2(1 - e^2)]^2$

⇒ $a^4(1 - e^2) = a^4(1 - 1 + e^2)^2$

⇒ $1 - e^2 = e^4$ or $\mathbf{e^4 + e^2 - 1 = 0.}$

23. The eccentricity of the curve represented by the equation $x^2 + 2y^2 - 2x + 3y + 2 = 0$ is

(a) 0 (b) 1/2

(c) $1/\sqrt{2}$ (d) $\sqrt{2}$

Solution (c) The given equation can be written in the form

$$\frac{(x-1)^2}{1/8} + \frac{(y+3/4)^2}{1/16} = 1.$$

which represents an ellipse.

Here, $a^2 = 1/8$ and $b^2 = 1/16$.

$$\therefore \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{\frac{1}{8} - \frac{1}{16}}{1/8} = \frac{1}{2} \quad \text{i.e.,} \quad e = \mathbf{\frac{1}{\sqrt{2}}}.$$

24. The sum of the focal distances from any point on the ellipse $9x^2 + 16y^2 = 144$ is

(a) 32 (b) 18

(c) 16 (d) 8

Solution (d) The given equation of ellipse can be written in the form $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Here $a^2 = 16$ and $b^2 = 9$. ∴ $a = 4$.

Sum of the focal distances from any point on the ellipse $= 2a = \mathbf{8}$.

25. If $P(x, y)$, $F_1(3, 0)$, $F_2(-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

(a) 8 (b) 6

(c) 10 (d) 12

Solution (c) $PF_1 + PF_2$

$$= \sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2}$$

$$= \sqrt{(x-3)^2 + \frac{(400 - 16x^2)}{25}} + \sqrt{(x+3)^2 + \frac{(400 - 16x^2)}{25}}$$

$$= \frac{1}{5}\,[\,(25 - 3x) + (25 + 3x)] = \mathbf{10.}$$

26. Eccentric angle of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

Solution (a), (c) The equation of ellipse can be written in the form

$$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{2})^2} = 1.$$

Let the eccentric angle of the point be θ, then its coordinates are $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$.

Since the distance of the point from the centre is 2 units

∴ $(\sqrt{6}\cos\theta - 0)^2 + (\sqrt{2}\sin\theta - 0)^2 = 4$

⇒ $6\cos^2\theta + 2(1 - \cos^2\theta) = 4 \Rightarrow 4\cos^2\theta = 2$

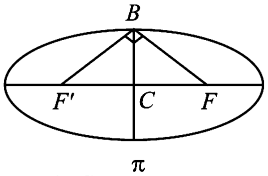
⇒ $\cos\theta = \pm\frac{1}{\sqrt{2}}$. ∴ $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

27. An ellipse has CB as a semi minor axis, F, F' are its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\sqrt{\frac{3}{2}}$ (d) none of these

Solution (a) Since $\angle FBF' = \frac{\pi}{2}$



∴ $\angle FBC = \angle F'BC = \frac{\pi}{4}$

∴ $CB = CF \Rightarrow b = ae$

$\Rightarrow b^2 = a^2e^2 \Rightarrow a^2(1 - e^2) = a^2e^2$

$\Rightarrow 2e^2 = 1 \Rightarrow e = \mathbf{1/\sqrt{2}}$.

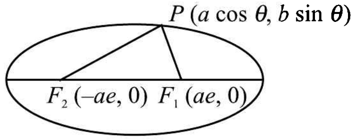
28. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is

(a) $2abe$ (b) abe

(c) $\frac{1}{2}abe$ (d) none of these

Solution (b) Let $P \equiv (a\cos\theta, b\sin\theta)$

Then, A = area of ΔPF_1F_2



$$= \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$= \left|\frac{1}{2}\cdot 2ae\cdot b\sin\theta\right| = abe\,|\sin\theta|.$$

Clearly, A is maximum when $|\sin\theta| = 1$.

∴ Maximum value of A = ***abe.***

29. The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ and $25x^2 + 9y^2 = 450$ passing through (3, 5) is

(a) 0 (b) 2

(c) 3 (d) 4

Solution (c) Since $3 \times 3^2 + 5 \times 5^2 - 32 > 0$, the point (3, 5) lies outside the ellipses $3x^2 + 5y^2 = 32$.

Also, $25 \times 3^2 + 9 \times 5^2 - 450 = 0$,

$\therefore$ the point (3, 5) lies on the ellipse $25x^2 + 9y^2 = 450$. So the required number of tangents is **3.**

30. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then its eccentric angle θ is equal to

(a) 0 (b) 90°

(c) 45° (d) 60°

Solution (c) Equation of any tangent to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad ...(1)$$

Also, $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the given ellipse.

Comparing coefficients in (1) and (2), we get

$$\frac{\cos\theta / a}{1/a} = \frac{\sin\theta / b}{1/b} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} = \sin\theta$$

$$\therefore \quad \theta = \mathbf{45°}.$$

31. If the angle between the straight lines joining foci and the ends of minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is 90°, then the eccentricity is

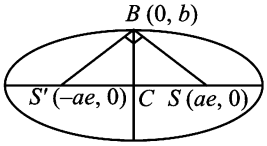
(a) $\frac{1}{2}$ (b) $\sqrt{\frac{3}{2}}$

(c) $\frac{1}{\sqrt{2}}$ (d) none of these

Solution (c) Slope of BS is $m_1 = \frac{b-0}{0-ae} = -\frac{b}{ae}$ and slope of BS' is

$$m_2 = \frac{b-0}{0+ae} = \frac{b}{ae}.$$

Since $\angle SBS' = 90°$,



$$\therefore \quad m_1 m_2 = -1 \Rightarrow \frac{-b}{ae} \times \frac{b}{ae} = -1 \Rightarrow b^2 = a^2e^2$$

$$\Rightarrow \quad a^2(1 - e^2) = a^2e^2 \Rightarrow 1 - e^2 = e^2 \Rightarrow 2e^2 = 1.$$

$$\therefore \quad e = \mathbf{\frac{1}{\sqrt{2}}}.$$

32. The line $y = 2t^2$ meets the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in real points if

(a) $|t| \le 1$ (b) $|t| > 1$

(c) $|t| < 3$ (d) none of these

Solution (a) Putting $y = 2t^2$ in the equation of the given ellipse

$\frac{x^2}{9} + \frac{y^2}{4} = 1$, we get

$$\frac{x^2}{9} + \frac{4t^4}{4} = 1 \Rightarrow x^2 = 9(1 - t^4) = 9(1 - t^2)(1 + t^2).$$

This will give real values of x if $1 - t^2 \ge 0$ i.e., $|t| \le$ **1.**

33. If e, e' be the eccentricities of a hyperbola and its conjugate, then

(a) $e^2 + e'^2 = 1$ (b) $\frac{1}{e^2} - \frac{1}{e'^2} = 1$

(c) $\frac{1}{e^2} + \frac{1}{e'^2} = 1$ (d) none of these

Solution (c) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Then, the conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

Their eccentricities e and e' are given by

$$b^2 = a^2(e^2 - 1) \text{ and } a^2 = b^2(e'^2 - 1)$$

Multiplying the corresponding sides, we have

$$a^2b^2 = a^2b^2(e^2 - 1)(e'^2 - 1)$$

$$\Rightarrow \quad 1 = (e^2 - 1)(e'^2 - 1) \Rightarrow 1 = e^2e'^2 - e'^2 - e^2 + 1$$

$$\Rightarrow \quad e^2 + e'^2 = e^2e'^2 \Rightarrow \mathbf{\frac{1}{e^2} + \frac{1}{e'^2} = 1.}$$

34. The number of tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ through (4, 1) is

(a) 1 (b) 2

(c) 0 (d) 3

Solution (b) Since $\left.\frac{x^2}{4} - \frac{y^2}{3} - 1\right]_{(4,1)} = \frac{16}{4} - \frac{1}{3} - 1 > 0$,

$\therefore$ the point (4, 1) lies outside the hyperbola, hence the number of tangents through (4, 1) is **two.**

35. If θ is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity e, then sec θ/2 =

(a) e (b) $e/2$

(c) $e/3$ (d) none of these

Solution (a) Equations of asymptotes of the given hyperbola are

$$\frac{x}{a} + \frac{y}{b} = 0 \text{ and } \frac{x}{a} - \frac{y}{b} = 0.$$

Slope of asymptote $= \frac{b}{a} = \tan\alpha$ (say)

If θ is the angle between the two asymptotes, then

$$\tan\theta = \frac{\frac{b}{a} + \frac{b}{a}}{1 - b^2/a^2} = \frac{\tan\alpha + \tan\alpha}{1 - \tan^2\alpha}$$

$$= \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha$$

$$\therefore \quad \frac{\theta}{2} = \alpha \Rightarrow \tan\frac{\theta}{2} = \tan\alpha = b/a.$$

$$\therefore \quad \sec\frac{\theta}{2} = \sqrt{1+\tan^2\theta/2} = \sqrt{1+\frac{b^2}{a^2}} = e.$$

36. The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is

(a) 2 (b) $2/\sqrt{3}$
(c) 4 (d) 4/3

Solution (a)

37. If the equation $4x^2 + ky^2 = 18$ represents a rectangular hyperbola, then k =

(a) 4 (b) – 4
(c) 3 (d) none of these

Solution (b)

38. The asymptotes of the hyperbola $xy - 3x + 4y + 2 = 0$ are

(a) $x = -4$ (b) $x = 4$
(c) $y = -3$ (d) $y = 3$

Solution (a), (d)

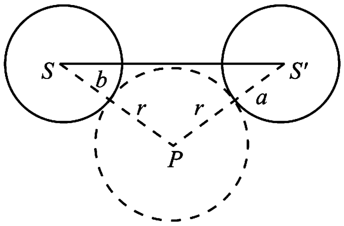
39. The locus of the centre of a circle which touches two given circles externally is

(a) an ellipse (b) a parabola
(c) a hyperbola (c) none of these

Solution (c) Let S and S' be the centres and a and b be the radii of the given circles.

Let P be the centre and r be the radius of the circle which touches the given circles externally.

Then $S'P = r + a$ and $SP = r + b$



$\therefore \quad S'P - SP = (r + a) - (r + b) = a - b$ = constant.

Hence the locus of P is a **hyperbola** whose foci are S and S'.

40. If a circle makes intercepts of length 5 and 3 on two perpendicular lines, then the locus of the centre of the circle is

(a) a parabola (b) an ellipse
(c) a hyperbola (d) none of these

Solution (c)

41. If the line $ax + by + c = 0$ is a normal to the hyperbola $xy = 1$, then

(a) $a > 0, b < 0$ (b) $a > 0, b > 0$
(c) $a < 0, b < 0$ (d) $a < 0, b > 0$

Solution (a), (d) Equation of normal at the point $\left(t, \frac{1}{t}\right)$ to the hyperbola $xy = 1$ is

$$xt^3 - yt - t^4 + 1 = 0.$$

Its slope $= t^2 = \frac{-a}{b}$.

$$\therefore \quad \frac{-a}{b} > 0 \Rightarrow \frac{a}{b} < 0$$

$\therefore$ **$a > 0, b < 0$** or **$a < 0, , b > 0$.**

42. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents

(a) no locus if $k > 0$ (b) an ellipse if $k < 0$
(c) a point if $k = 0$ (d) a hyperbola if $k > 0$

Solution (c)

43. The curve for which the slope of the tangent at any point equals the ratio of the abscissa to the ordinate of the point is

(a) a circle
(b) an ellipse
(c) a rectangular hyperbola
(d) none of these

Solution (c)

44. A point moves in a plane so that its distance PA and PB from two fixed points A and B in the plane satisfy the relation $PA - PB = k (k \neq 0)$, then the locus of P is

(a) a hyperbola
(b) a branch of the hyperbola
(c) a parabola
(d) an ellipse

Solution (b)

45. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is

(a) 1 (b) 5
(c) 7 (d) 9

Solution (c)

Problems for Practice

Level of Difficulty I

1. If the focal distance of a point on a parabola $y^2 = 4x$ is 6, then the coordinates of the point are

(a) $(5, 2\sqrt{5})$ (b) $(5, -2\sqrt{5})$
(c) $(5, \sqrt{5})$ (d) none of these

2. For the parabola $x^2 + 4x + 4y + 16 = 0$, the equation of the axis and the directrix are given by

(a) $x + 2 = 0, y - 2 = 0$
(b) $x - 2 = 0, y + 2 = 0$
(c) $x + 2 = 0, y + 2 = 0$
(d) none of these

3. The equation of the parabola, whose focus is (3, – 4) and directrix is the line $x + y - 2 = 0$, is

(a) $x^2 + 2xy + y^2 - 8x + 20y + 46 = 0$
(b) $x^2 - 2xy + y^2 - 8x + 20y + 46 = 0$
(c) $x^2 - 2xy + y^2 + 8x - 20y + 46 = 0$
(d) none of these

4. The equation of the tangent to the parabola $y^2 = 6x$ at the point whose ordinate is 6, is

(a) $x + 2y + 6 = 0$ (b) $2x - y + 6 = 0$
(c) $x - 2y + 6 = 0$ (d) none of these

5. The equation of the tangent to the parabola $y^2 = 16x$ inclined at an angle of 60° to x-axis, is
(a) $3x - \sqrt{3}y + 4 = 0$ (b) $3x + \sqrt{3}y + 4 = 0$
(c) $\sqrt{3}x - y + 4 = 0$ (d) none of these

6. The equation of the tangent to the parabola $y^2 = 16x$, perpendicular to the line $2x - y + 5 = 0$ is
(a) $2x - y + 2 = 0$ (b) $2x + y + 2 = 0$
(c) $x + 2y - 16 = 0$ (d) $x + 2y + 16 = 0$

7. The equation of the normal to the parabola $y^2 = 4x$ which is parallel to the line $y - 2x + 5 = 0$ is
(a) $2x + y - 12 = 0$ (b) $2x - y - 12 = 0$
(c) $x + 2y - 12 = 0$ (d) none of these

8. The value of k for which the line $x + y + 1 = 0$ touches the parabola $y^2 = kx$ is
(a) -4 (b) 4 (c) 2 (d) -2

9. The straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if
(a) $c = am - \frac{a}{m}$ (b) $c = m - \frac{a}{m}$
(c) $c = am + \frac{a}{m}$ (d) none of these

10. The angle between the two tangents drawn from the point (1, 4) to the parabola $y^2 = 12x$ is
(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}\left(\frac{1}{3}\right)$
(c) $\tan^{-1}(2)$ (d) none of these

11. If (4, 0) is the vertex and y-axis, the directrix of a parabola, then its focus is
(a) (8, 0) (b) (4, 0) (c) (0, 8) (d) (0, 4)

12. The line $y = mx + 1$ is a tangent to the parabola $y^2 = 4x$, if
(a) $m = 1$ (b) $m = 2$ (c) $m = 4$ (d) $m = 3$

13. The slope of the normal at the point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is
(a) $1/t$ (b) t (c) $-t$ (d) $-1/t$

14. If the points $(au^2, 2au)$ and $(av^2, 2av)$ are the extremities of a focal chord of the parabola $y^2 = 4ax$, then
(a) $uv - 1$ (b) $uv + 1 = 0$
(c) $u + v = 0$ (d) $u - v = 0$

15. The line $y = mx + c$ touches the parabola $x^2 = 4ay$ if
(a) $c = -am$ (b) $c = -a/m$
(c) $c = -am^2$ (d) $c = a/m^2$

16. If the parabola $x^2 = ay$ makes an intercept of length $\sqrt{40}$ on the line $y - 2x = 1$, then a is equal to
(a) 1 (b) -2 (c) -1 (d) 2

17. With respect to the parabola $y^2 = 2x$, the points $P(4, 2)$ and $Q(1, 4)$ are such that
(a) P and Q both lie inside the parabola
(b) P lies inside whereas Q lies outside the parabola
(c) P lies outside whereas Q lies inside the parabola
(d) P and Q both lie outside the parabola

18. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is
(a) $\frac{1}{8}$ (b) 8 (c) 4 (d) $\frac{1}{4}$

19. If $x + y = k$ is normal to $y^2 = 12x$, then k is
(a) 3 (b) 9 (c) -9 (d) -3

20. If t is the parameter for one end of a focal chord of the parabola $y^2 = 4ax$, then its length is
(a) $a\left(t - \frac{1}{t}\right)$ (b) $a\left(t + \frac{1}{t}\right)$
(c) $a\left(t - \frac{1}{t}\right)^2$ (d) $a\left(t + \frac{1}{t}\right)^2$

21. If (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then its equation is
(a) $x^2 + 8y = 32$ (b) $y^2 + 8x = 32$
(c) $x^2 - 8y = 32$ (d) $y^2 - 8x = 32$.

22. Equation of the parabola whose vertex is (–3, –2), axis is horizontal and which passes through the point (1, 2) is
(a) $y^2 + 4y + 4x - 8 = 0$
(b) $y^2 + 4y - 4x + 8 = 0$
(c) $y^2 + 4y - 4x - 8 = 0$
(d) none of these

23. At what point on the parabola $y^2 = 4x$, the normal makes equal angles with the axes?
(a) (4, 4) (b) (9, 6) (c) (4, –4) (d) (1, –2)

24. The circle on focal radii of a parabola as diameter touches the
(a) axis (b) directrix
(c) tangent at the vertex
(d) none of these

25. If the line $3x - 4y + 5 = 0$ is a tangent to the parabola $y^2 = 4ax$, then a is equal to
(a) $-\frac{5}{4}$ (b) $-\frac{4}{3}$ (c) $\frac{5}{4}$ (d) $\frac{15}{16}$

26. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is
(a) 0 (b) 1 (c) 2 (d) infinite

27. The equation of the hyperbola, referred to its axes as axes of coordinates, given that the distances of one of its vertices from the foci are 9 and 1 units, is
(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (b) $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(c) $\frac{x^2}{16} - \frac{y^2}{9} = -1$ (d) none of these

Level of Difficulty II

28. If the tangent at the point $(2\sec\theta, 3\tan\theta)$ of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is parallel to $3x - y + 4 = 0$, then the value of θ is
(a) 45° (b) 60° (c) 30° (d) 75°

29. If e, e' be the eccentricities of two conics $S = 0$ and $S' = 0$ and if $e^2 + e'^2 = 3$, then both S and S' can be

(a) hyperbolas (b) ellipses
(c) parabolas (d) none of these

30. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

(a) $\dfrac{x}{x_1 + x_2} + \dfrac{y}{y_1 + y_2} = 1$

(b) $\dfrac{x}{x_1 - x_2} + \dfrac{y}{y_1 - y_2} = 1$

(c) $\dfrac{x}{y_1 + y_2} + \dfrac{y}{x_1 + x_2} = 1$

(d) $\dfrac{x}{y_1 - y_2} + \dfrac{y}{x_1 - x_2} = 1$

31. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b - 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b + 3c)^2 = 0$
(d) $d^2 + (3b - 2c)^2 = 0$

32. The eccentricity of an ellipse, with its centre at the origin, is 1/2. If one of the directrices is $x = 4$, the the equation of the ellipse is

(a) $4x^2 + 3y^2 = 12$ (b) $3x^2 + 4y^2 = 12$
(c) $3x^2 + 4y^2 = 1$ (d) $4x^2 + 3y^2 = 1$

33. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis, is

(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x + 3)$
(b) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x + 1)$

34. An ellipse slides between two lines at right angles to one another. The locus of its centre is

(a) a parabola (b) an ellipse
(c) a circle (d) None of these

35. S and T are the foci of an ellipse and B is an end of the minor axis. If STB is an equilateral triangle, the eccentricity of the ellipse is

(a) 1/4 (b) 1/3 (c) 1/2 (d) 2/3

36. Let E be the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then

(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

37. If the distance of a point on the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{2} = 1$ from the centre is 2, then the eccentric angle is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{2}$

38. Two perpendicular tangents to $y^2 = 4ax$ always intersect on the line

(a) $x = a$ (b) $x + a = 0$
(c) $x + 4a = 0$(d) $x + 2a = 0$

39. The point $(2K, K)$ lies inside the region bounded by the parabola $x^2 = 4y$ and its latus rectum. Then,

(a) $0 \leq a \leq 1$
(b) $0 < a < 1$
(c) $0 < a \leq 1$
(d) none of these

40. If b, k are intercepts of a focal chord of the parabola $y^2 = 4ax$ then k is equal to

(a) $\dfrac{ab}{b - a}$ (b) $\dfrac{b}{b - a}$

(c) $\dfrac{a}{b - a}$ (d) $\dfrac{ab}{a - b}$

ANSWERS

1. (a), (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (a)
11. (a)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (a), (b)	**17.** (b)	**18.** (c)	**19.** (b)	**20.** (d)
21. (a)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (d)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (a)
31. (c)	**32.** (b)	**33.** (c)	**34.** (c)	**35.** (c)	**36.** (d)	**37.** (b)	**38.** (b)	**39.** (b)	**40.** (a)

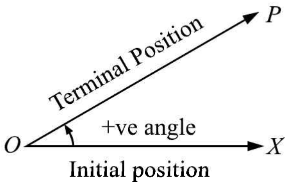
Trigonometric Ratios and Identities

5

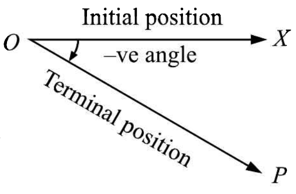
BRIEF REVIEW OF THE CONCEPTS

ANGLE

Let a revolving line starting from *OX* revolves about its end point *O* on a plane in the direction of arrow and occupy the position *OP*. It is said to trace out an angle *XOP*. Here *OX* is called the *initial* position and *OP*, the *terminal* position. The fixed point *O* is called the *vertex*.



(a)



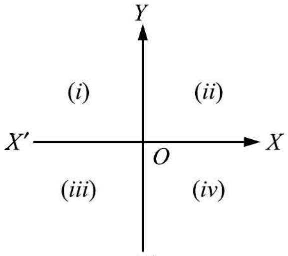
(b)

An angle is considered as the figure traced by rotating a given ray about its end point. The angle *XOP* is said to be positive if it is traced out by a line revolving in the anti-clockwise direction, as in (a), and negative if it is traced out by a line revolving in the clockwise direction, as in (b).

QUADRANTS

Let *X' OX*, *Y' OY* be two perpendicular lines meeting in the point *O*. These divide the plane into four parts called quadrants:

1. *XOY* is called the first quadrant.
2. *YOX'* is called the second quadrant.
3. *X'OY'* is called the third quadrant.
4. *Y'OX* is called the fourth quadrant.



Angle in a Particular Quadrant

If the final position of the revolving line tracing out an angle lies in a particular quadrant, the angle is said to lie in that quadrant.

MEASUREMENT OF AN ANGLE

We know that different units are used to measure the same quantity. For example, different units (kilogram, pound, etc.) are used to measure the same weight. In the same manner, different units are employed in the measurement of an angle in different systems.

In trigonometry, following three systems are used for measuring an angle according to different units employed:

1. Sexagesimal (or English or F.P.S.) system;
2. Centesimal (or French or C.G.S.) system and
3. Circular system.

Sexagesimal System

In this system, an angle is measured in terms of degrees, minutes and seconds. We know that there are 4 right angles in a complete revolution and one complete revolution = 360 degrees (360°). Thus we have

1 right angle = 90 degrees (90°)
1 degree = 60 minutes (60′)
1 minute = 60 seconds (60″).

Centesimal System

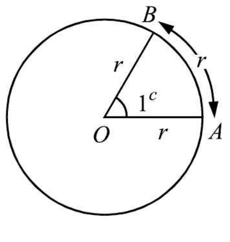
In this system, an angle is measured in terms of grades, minutes and seconds. Thus we have

1 right angle = 100 grades (100^g).
1 grade = 100 minutes (100′).
1 minute = 100 seconds (100″).

Circular System

In this system, the angle is measured in radians.

Radian: Radian is the angle subtended at the centre of circle by an arc, whose length is equal to the radius. Let *O* be the centre of a circle of radius *r*, cut off an arc *AB* = *r*, then ∠*AOB* = 1 radian and is written as 1^c.

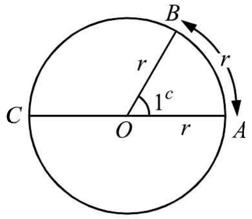


Points to Note:

- '*c*' used in the notation of radian is the first letter of the word circular system.
- When no unit is mentioned with an angle, it is always understood to be in radians.

Radian is a constant angle

Proof: Let *O* be the centre of a circle of radius *r*. Let *AB* be the arc such that arc *AB* = *r* = radius of the circle. Then ∠*AOB* = 1 radian (by definition). Produce *AO* to meet the circumference in *C*. Since the angles at the centre of a circle are proportional to the arcs on which they stand,



$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC} \Rightarrow \frac{1 \text{ radian}}{2 \text{ rt.}\angle\text{s}} = \frac{r}{\pi r} = \frac{1}{\pi}$$

[$\angle AOC$ = 2 rt. $\angle$s and arc ABC = (1/2) circumference of the circle = (1/2) $(2\pi r) = \pi r$]

$$\therefore \quad 1 \text{ radian} = \frac{2 \text{ rt.}\angle\text{s}}{\pi}, \qquad \text{...(1)}$$

which is independent of r, the radius of the circle. Hence radian is constant angle.

Cor. 1. To find a relation between the three systems

From (1), π radians = 2 rt. angles = 180°

[$\because$ 1 rt. angle = 90°]

Again, π radians = 2 rt. angles = 200^g

[$\because$ 1 rt. angle = 100^g]

Thus, π radians = $180° = 200^g$,
which is the required relation.

Cor. 2. To find the value of a radian in sexagesimal and centesimal measures

1. In sexagesimal measure, π radians = 180°

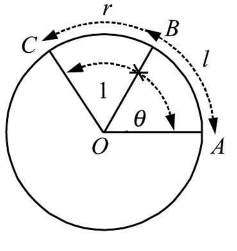
$$\therefore \ 1 \text{ radian} = \frac{180°}{\pi} = \frac{180°}{3{\cdot}1416} = 57°\,17'45'' \text{ approximately.}$$

2. In centesimal measure, π radians = 200^g

$$\therefore \ 1 \text{ radian} = \frac{200^g}{\pi} = \frac{200^g}{3.1416} = 63^g\,66'20'' \text{ approximately.}$$

Arc–angle Relation

To prove that the number of radians in an angle subtended by an arc of a circle at the centre = arc/radius or θ = l/r, where θ in the angle subtended by an arc.



Let O be the centre of the circle, whose radius is r. Let l be the length of the arc AB of the circle subtending an angle (=θ radians) at the centre O. Cut off arc BC = radius, and join OC. Then $\angle BOC$ = 1 radian, (by definition). Now, since angles at the centre of a circle are proportional to arcs subtending them,

$$\therefore \quad \frac{\angle AOB}{\angle BOC} = \frac{AB}{BC}, \text{ i.e., } \frac{\angle AOB}{1 \text{ radian}} = \frac{l}{r}$$

$$\text{or} \quad \angle AOB = \frac{l}{r} \text{ radians or } \theta = \frac{l}{r}$$

[$\because$ $\angle AOB$ = θ radians]

$$\therefore \quad \text{Number of radians in } \angle AOB = \frac{\text{length of arc } AB}{\text{radius of the circle}}.$$

Circular Measure of an Angle

The circular measure of an angle is the number of radians it contains. An angle of one radian is denoted by 1^c.

Points to Note:

- An angle can have any magnitude.
- In the result θ = l/r; θ is always in radians and units of l and r are same.
- Following table gives the circular measure of some standard angles:

Angles in degree	0°	30°	45°	60°	90°	120°	135°
Circular measure	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
Angles in degree	150°	180°	270°	360°			
Circular measure	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π			

- The symbol π (read as pie) is the ratio of the circumference of the circle to its diameter. π is an irrational number and its value is generally taken as 22/7 unless otherwise mentioned.

RELATION BETWEEN SIDES AND INTERIOR ANGLES OF A REGULAR POLYGON

1. Sum of interior angles of polygon of n sides
   $= (2n - 4) \times 90°$
2. Each interior angle of a regular polygon of n sides

$$= \frac{2n-4}{n} \times 90°.$$

FUNDAMENTAL IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta = 1 - \sin^2\theta$
   or $\sin^2\theta = 1 - \cos^2\theta$
2. $1 + \tan^2\theta = \sec^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$
3. $1 + \cot^2\theta = \text{cosec}^2\theta$ or $\text{cosec}^2\theta - \cot^2\theta = 1$.

Note:

Since $\sin^2\theta + \cos^2\theta = 1$, $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$

Note: $\Rightarrow$ $\quad -1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$;
or $\quad 0 \le \sin^2\theta \le 1$, $0 \le \cos^2\theta \le 1$.

Since $\text{cosec}\,\theta = 1/\sin\theta$, $\text{cosec}\,\theta \ge 1$ or $\text{cosec}\,\theta \le -1$

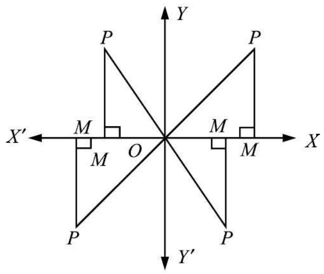
Also, since $\sec\theta = 1/\cos\theta$, $\sec\theta \ge 1$ or $\sec\theta \le -1$.

SIGNS OF TRIGONOMETRIC RATIOS IN DIFFERENT QUADRANTS

The following table describes the signs of various t-ratios in different quadrants. Also, refer to, the figure given below the table.

Table 5.1

Quadrant	***I***	***II***	***III***	***IV***
$MP = y$	+ve	+ve	–ve	–ve
$OM = x$	+ve	–ve	–ve	+ve
$\sin\theta = \frac{y}{r}$	$\frac{+\text{ve}}{+\text{ve}}$ = +ve	$\frac{+\text{ve}}{+\text{ve}}$ = +ve	$\frac{-\text{ve}}{+\text{ve}}$ = –ve	$\frac{-\text{ve}}{+\text{ve}}$ = –ve
$\cos\theta = \frac{x}{r}$	$\frac{+\text{ve}}{+\text{ve}}$ = +ve	$\frac{-\text{ve}}{+\text{ve}}$ = –ve	$\frac{-\text{ve}}{+\text{ve}}$ = –ve	$\frac{+\text{ve}}{+\text{ve}}$ = +ve
$\tan\theta = \frac{y}{x}, x \neq 0$	$\frac{+\text{ve}}{+\text{ve}}$ = +ve	$\frac{+\text{ve}}{-\text{ve}}$ = –ve	$\frac{-\text{ve}}{-\text{ve}}$ = +ve	$\frac{-\text{ve}}{+\text{ve}}$ = –ve



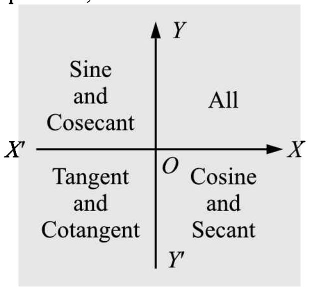
The signs of other *t*-ratios can be found by using reciprocal relations, i.e.,

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$. So, we have

Quadrant: →	I	II	III	IV
t-ratios which are +ve	All	sinθ cosecθ	tanθ cotθ	cosθ secθ

Key Points to Remember

1. In the first quadrant in, **all** are +**ve**.
2. In the second quadrant, **sin** and **cosec** are +**ve**
3. In the third quadrant, **tangent** and **cotangent** are +**ve**
4. In the fourth quadrant, **cosine** and **secant** are +**ve**.



Simple rule to remember:

add – sugar – to – coffee
orv
after – school – to – college

In the above, ‘*a*’ stands for ‘all’, ‘*s*’ stands for ‘sine’, ‘*t*’ stands for ‘tan’ and ‘*c*’ stands for ‘cos’. The reciprocals of these ratios are also positive in the respective quadrants.

Table 5.2 Domain and Range of Trigonometric Ratios

Functions	***Domain***	***Range***
$\sin x$, $\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} \middle\vert n \in \mathrm{I}\right\}$	$(-\infty, \infty)$
$\cot x$	$(-\infty, \infty) - \{n\pi \mid n \in I\}$	$(-\infty, \infty)$
$\sec x$	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} \middle\vert n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$(-\infty, \infty) - \{n\pi \mid n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

Table 5.3 Trigonometric Ratios of Standard Angles

T-Ratios \ ***Angles***	***0°***	***30°***	***45°***	***60°***	***90°***
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined
$\operatorname{cosec} x$	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
$\cot x$	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Table 5.3 Trigonometric Ratios of Allied Angles (their Sum or Difference is a Multiple of 90°)

	− θ	90° − θ	90° + θ	180° − θ	180° + θ	270° − θ	270° + θ	360° − θ	360° + θ
sinθ	− sinθ	cosθ	cosθ	sinθ	− sinθ	− cosθ	− cosθ	−sinθ	sinθ
cosθ	cosθ	sinθ	− sinθ	− cosθ	− cosθ	− sinθ	sinθ	cosθ	cosθ
tanθ	− tanθ	cotθ	− cotθ	− tanθ	tanθ	cotθ	−cotθ	−tanθ	tanθ

Working Rules to Find Allied Angles

Case I: When the angle is $n\pi \pm \theta$, where $n \in I$ and θ is acute.

- There is no change in trigonometric function i.e., sin remains sin, cos remains cos and tan remains tan. Angle associated becomes θ.
- The sign is affixed according to the quadrant in which the angle lies.

Case II: When the angle is $\frac{n\pi}{2} \pm \theta$, where *n* is an odd integer and θ is acute.

- The trigonometric function is replaced by its cofunction i.e., sin changes to cos, tan changes to cot and sec changes to cosec and vice-versa.
  Angle associated becomes θ.
- The sign is affixed according to the quadrant in which the angle lies.
  Note that the sign is always decided on the basis of the operating function.

Key Results on Allied Angles

1. $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.
2. $\sin(n\pi + \theta) = (-1)^n \sin\theta$, $\cos(n\pi + \theta) = (-1)^n \cos\theta$
3. $\sin\left(\dfrac{n\pi}{2} + \theta\right) = \begin{cases} (-1)^{\frac{n-1}{2}} \cos q, \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}} \sin q, \text{if } n \text{ is even}. \end{cases}$

$$\cos\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n+1}{2}}\sin\theta, \text{ if } n \text{ is odd,}\\ (-1)^{\frac{n}{2}}\cos\theta, \text{ if } n \text{ is even.}\end{cases}$$

ADDITION AND SUBTRACTION FORMULAE

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
3. $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$
4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
5. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
6. $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$
7. $\cot(A+B) = \dfrac{\cot A\cot B-1}{\cot A+\cot B}$
8. $\cot(A-B) = \dfrac{\cot A\cot B+1}{\cot B-\cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
10. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
11. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$
12. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$
13. $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$
14. $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$
15. $\tan\left(\dfrac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A}$.
16. $\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1\cos A_2 \ldots \cos A_n (S_1 - S_3 + S_5 - \ldots)$
17. $\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (1 - S_2 + S_4 - S_6 + \ldots)$
18. $\tan(A_1 + A_2 + \ldots + A_n) = \dfrac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$

    where $S_1 = \Sigma \tan A_1$, $S_2 = \Sigma \tan A_1\tan A_2$, $S_3 = \Sigma \tan A_1\tan A_2\tan A_3$ and so on.
19. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$

    $$= \frac{\sin\left(\alpha+(n-1)\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}\sin\left(\frac{n\beta}{2}\right)$$
20. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

    $$= \frac{\cos\left(\alpha+(n-1)\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}\sin\left(\frac{n\beta}{2}\right)$$

TRANSFORMATION FORMULAE

Product into Sum or Difference

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B), A > B$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B), A > B$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Sum and Difference into Product

1. $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
2. $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
3. $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
4. $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
5. $\tan C + \tan D = \dfrac{\sin(C+D)}{\cos C\cos D}$
6. $\tan C - \tan D = \dfrac{\sin(C-D)}{\cos C\cos D}$
7. $\cot C + \cot D = \dfrac{\sin(C+D)}{\sin C\sin D}$
8. $\cot C - \cot D = \dfrac{\sin(D-C)}{\sin C\sin D}$

TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

(An Angle of the form $n\theta$, $n \in$ I)

1. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

    $= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
3. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
4. $\cot 2\theta = \dfrac{\cot^2\theta-1}{2\cot\theta}$
5. $1 + \cos 2\theta = 2\cos^2\theta$, $\cos^2\theta = \dfrac{1}{2}(1 + \cos 2\theta)$
6. $1 - \cos 2\theta = 2\sin^2\theta$, $\sin^2\theta = \dfrac{1}{2}(1 - \cos 2\theta)$
7. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$
8. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos^3\theta = \dfrac{1}{4}(\cos 3\theta + 3\cos\theta)$
9. $\tan 3\theta = \dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

10. $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$

11. $\cos A \cos 2A \cos 2^2A \ldots \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \sin A}$

TRIGONOMETRIC RATIOS OF SUBMULTIPLE ANGLES

(An Angle of the form $\frac{\theta}{n}$, $n \in$ I)

1. $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2}$

2. $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$

$= \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

3. $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

4. $\cot\theta = \dfrac{\cot^2\frac{\theta}{2} - 1}{2\cot\theta/2}$

5. $\cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$

6. $\sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$

7. $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

8. $\cot^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{1-\cos\theta}$

9. $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$

10. $\dfrac{1+\cos\theta}{\sin\theta} = \cot\dfrac{\theta}{2}$

TRIGONOMETRIC RATIOS OF SOME SPECIAL ANGLES

1. $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

2. $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

3. $\tan 15^\circ = 2 - \sqrt{3}$

4. $\cot 15^\circ = 2 + \sqrt{3}$

5. $\sin 22\dfrac{1^\circ}{2} = \dfrac{1}{2}(\sqrt{2-\sqrt{2}})$

6. $\cos 22\dfrac{1^\circ}{2} = \dfrac{1}{2}(\sqrt{2+\sqrt{2}})$

7. $\tan 22\dfrac{1^\circ}{2} = \sqrt{2} - 1$

8. $\cot 22\dfrac{1^\circ}{2} = \sqrt{2} + 1$

9. $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

10. $\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

11. $\sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

12. $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$

13. $\sin 9^\circ = \dfrac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4}$

14. $\cos 9^\circ = \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}$

15. $\tan 18^\circ = \dfrac{\sqrt{25-10\sqrt{5}}}{5}$

16. $\tan 36^\circ = \sqrt{5-2\sqrt{5}}$

GREATEST AND LEAST VALUES OF THE EXPRESSION

$a\sin\theta + b\cos\theta$

Let $a = r\cos\alpha$, $b = r\sin\alpha$, then

$a^2 + b^2 = r^2$

or $r = \sqrt{a^2+b^2}$

Then $a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$= r\sin(\theta+\alpha)$

But $-1 \le \sin(\theta+\alpha) \le 1$, so

$-r \le r\sin(\theta+\alpha) \le r$

or $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$.

Thus, the greatest and least values of $a\sin\theta + b\cos\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

Key Results to Remember

If $A + B + C = \pi$, then

1. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
2. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$
3. $\tan A + \tan B + \tan C = \tan A\tan B\tan C$
4. $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$
5. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$
6. $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1.$

Solved Problems

1. If $\operatorname{cosec}\theta = x + \dfrac{1}{4x}$ then the value of $\operatorname{cosec}\theta + \cot\theta$ is
(a) $2x$ (b) $-2x$
(c) $\dfrac{1}{2x}$ (d) $-\dfrac{1}{2x}$

Solution (a) Let $\operatorname{cosec}\theta + \cot\theta = k$

$\Rightarrow \quad \operatorname{cosec}\theta - \cot\theta = \dfrac{1}{k}$

On adding, we get

$$2\operatorname{cosec}\theta = k + \frac{1}{k} \quad \text{or} \quad 2\left(x + \frac{1}{4x}\right) = k + \frac{1}{k}$$

$$\Rightarrow \quad 2x + \frac{1}{2x} = k + \frac{1}{k} \Rightarrow \mathbf{k = 2x.}$$

2. If $3\sin\theta + 5\cos\theta = 5$, then the value of $5\sin\theta - 3\cos\theta$ is
(a) 3 (b) -3
(c) 5 (d) -5

Solution (a), (b) We have, $3\sin\theta + 5\cos\theta = 5 \quad \ldots(1)$

Let $5\sin\theta - 3\cos\theta = x \quad \ldots(2)$

Squaring and adding (1) and (2), we get

$$(9\sin^2\theta + 25\cos^2\theta + 30\sin\theta\cos\theta) + (25\sin^2\theta + 9\cos^2\theta - 30\sin\theta\cos\theta) = 25 + x^2$$

$\Rightarrow \quad 9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$

$\Rightarrow \quad 34 = 25 + x^2 \Rightarrow x^2 = 9$

$\therefore \quad x = \pm 3$, *i.e.*, $5\sin\theta - 3\cos\theta = \mathbf{\pm 3.}$

3. If $\tan^2\theta = 1 - e^2$, then $\sec\theta + \tan^3\theta \operatorname{cosec}\theta =$
(a) $(1 - e^2)^{3/2}$ (b) $(2 - e^2)^{1/2}$
(c) $(2 - e^2)^{3/2}$ (d) none of these

Solution (c) We have, $\sec\theta + \tan^3\theta \operatorname{cosec}\theta$

$$= \sec\theta\left(1 + \tan^3\theta\,\frac{\operatorname{cosec}\theta}{\sec\theta}\right)$$

$= \sec\theta\,(1 + \tan^3\theta\cdot\cot\theta) = \sec\theta(1 + \tan^2\theta)$

$= \sec\theta\,\sec^2\theta = \sec^3\theta$

$= (\sec^2\theta)^{\frac{3}{2}} = (1 + \tan^2\theta)^{\frac{3}{2}}$

$= (1 + 1 - e^2)^{\frac{3}{2}} = \mathbf{(2 - e^2)^{\frac{3}{2}}} \quad [\because \tan^2\theta = 1 - e^2]$

4. The equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only if
(a) $x + y = 2xy$ (b) $x + y = -2xy$
(c) $x + y = 1$ (d) $x = y$

Solution (d) We have, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

$\because \quad \sec^2\theta \geq 1 \Rightarrow \dfrac{4xy}{(x+y)^2} \geq 1 \Rightarrow (x + y)^2 \leq 4xy$

$\Rightarrow (x + y)^2 - 4xy \leq 0 \Rightarrow (x - y)^2 \leq 0$

But for real values of x and y, $(x - y)^2 \nless 0$

$\therefore \quad (x - y)^2 = 0 \Rightarrow x = y$. Also $x + y \neq 0 \Rightarrow x \neq 0, y \neq 0$.

$\therefore$ Given equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only when $\mathbf{x = y}$.

5. The value of tan 1° tan 2° ... tan 89° is
(a) 0 (b) 1
(c) -1 (d) none of these

Solution (b) The given expression

$= (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ) \ldots (\tan 44^\circ \tan 46^\circ)\tan 45^\circ$

$= [\tan 1^\circ \tan(90^\circ - 1^\circ)]\cdot[\tan 2^\circ \tan(90^\circ - 2^\circ)] \ldots [\tan 44^\circ \tan(90^\circ - 44^\circ)]\tan 45^\circ$

$= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ) \ldots (\tan 44^\circ \cot 44^\circ)\tan 45^\circ$

$= \mathbf{1.} \quad [\because \tan\theta\cot\theta = 1 \text{ and } \tan 45^\circ = 1]$

6. If $\sin x + \sin^2 x = 1$, then $\cos^8 x + 2\cos^6 x + \cos^4 x$ is equal to
(a) 0 (b) -1
(c) 2 (d) 1

Solution (d) We have $\sin x + \sin^2 x = 1$

$\Rightarrow \quad \sin x = 1 - \sin^2 x$

$\therefore \quad \sin x = \cos^2 x$

$\therefore \quad \cos^8 x + 2\cos^6 x + \cos^4 x$

$= \sin^4 x + 2\sin^3 x + \sin^2 x$

$= (\sin^2 x)^2 + 2\sin^2 x \times \sin x + (\sin x)^2$

$= (\sin^2 x + \sin x)^2$

$= (1)^2 = \mathbf{1.} \quad [\because \sin x + \sin^2 x = 1 \text{ (given)}]$

7. The value of $\cos^2\dfrac{\pi}{16} + \cos^2\dfrac{3\pi}{16} + \cos^2\dfrac{5\pi}{16} + \cos^2\dfrac{7\pi}{16}$ is
(a) 1 (b) 2
(c) 3 (d) none of these

Solution (b) The given expression

$$= \cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \cos^2\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) + \cos^2\left(\frac{\pi}{2} - \frac{\pi}{16}\right)$$

$$= \cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \sin^2\frac{3\pi}{16} + \sin^2\frac{\pi}{16}$$

$$= \left(\cos^2\frac{\pi}{16} + \sin^2\frac{\pi}{16}\right) + \left(\cos^2\frac{3\pi}{16} + \sin^2\frac{3\pi}{16}\right)$$

$= 1 + 1 = \mathbf{2.}$

8. If $\sin\theta + \cos\theta = m$ and $\sec\theta + \operatorname{cosec}\theta = n$, then
(a) $2n = m(n^2 - 1)$ (b) $2m = n(m^2 - 1)$
(c) $2n = m(m^2 - 1)$ (d) none of these

Solution (b) We have, $\sin\theta + \cos\theta = m \quad \ldots(1)$

and $\quad \sec\theta + \operatorname{cosec}\theta = n$

$$\Rightarrow \quad \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = n \Rightarrow \frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta} = n$$

$$\Rightarrow \quad \frac{m}{\cos\theta\sin\theta} = n \quad \text{[Using (1)]}$$

$$\Rightarrow \quad \cos\theta\,\sin\theta = \frac{m}{n} \quad \ldots(2)$$

Squaring (1), we get

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = m^2 \quad \Rightarrow 1 + 2\cdot\dfrac{m}{n} = m^2$

$\Rightarrow \quad \dfrac{2m}{n} = m^2 - 1$

$\Rightarrow \quad \mathbf{2m = n\,(m^2 - 1).}$

9. The value of $\sin\left[n\pi + (-1)^n\dfrac{\pi}{4}\right]$, $n \in I$ is

(a) 0 (b) $\dfrac{1}{\sqrt{2}}$

(c) $-\dfrac{1}{\sqrt{2}}$ (d) none of these

Solution (b) $\sin\left[n\pi + (-1)^n\dfrac{\pi}{4}\right] = (-1)^n \sin\left[(-1)^n\dfrac{\pi}{4}\right]$

$\left[\because \sin(n\pi + \theta) = (-1)^n \sin\theta = (-1)^n(-1)^n \sin\dfrac{\pi}{4}\right]$

$[\because \sin(-\theta) = -\sin\theta$

$\therefore \quad \sin[(-1)^n\theta] = (-1)^n\sin\theta]$

$= (-1)^{2n}\sin\dfrac{\pi}{4} = \sin\dfrac{\pi}{4} = \mathbf{\dfrac{1}{\sqrt{2}}}.$

10. The value of $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}$ is

(a) 0 (b) 1

(c) $\dfrac{1}{2}$ (d) $-\dfrac{1}{2}$.

Solution (d) We have, $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right) + \left(\sin\frac{5\pi}{7} - \sin\frac{3\pi}{7}\right) + \left(\sin\frac{7\pi}{7} - \sin\frac{5\pi}{7}\right)\right]$$

$= -\mathbf{\dfrac{1}{2}}. \qquad \left[\because \sin\dfrac{7\pi}{7} = \sin\pi = 0\right]$

11. If $\alpha + \beta = 90°$, then the maximum value of $\sin\alpha\sin\beta$ is

(a) 1 (b) $\dfrac{1}{2}$

(c) $\dfrac{3}{2}$ (d) none of these

Solution (b) Let $y = \sin\alpha\,\sin\beta$

Now $y = \dfrac{1}{2}(2\sin\alpha\,\sin\beta)$

$= \dfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$= \dfrac{1}{2}[\cos(\alpha - \beta) - \cos 90°]$

$= \dfrac{1}{2}\cos(\alpha - \beta)$

Since maximum value of $\cos(\alpha - \beta)$ is 1

$\therefore$ Maximum value of $y = \mathbf{\dfrac{1}{2}}$.

12. If in a ΔABC, $\cos A = \dfrac{\sin B}{2\sin C}$, then it is

(a) an isosceles triangle

(b) an equilateral triangle

(c) a right angled triangle

(d) none of these

Solution (a) We have, $\cos A = \dfrac{\sin B}{2\sin C}$

$\Rightarrow \quad 2\cos A\,\sin C = \sin B$

$\Rightarrow \quad \sin(A + C) - \sin(A - C) = \sin B$

$\Rightarrow \quad \sin B - \sin(A - C) = \sin B$

$[\because$ in ΔABC, $A + C = 180° - B$

$\therefore \sin(A + C) = \sin B]$

$\Rightarrow \quad \sin(A - C) = 0 \Rightarrow A - C = 0$ or $A = C$.

Hence the triangle is **isosceles.**

13. If $\sin\theta = n\sin(\theta + 2\alpha)$, then $\tan(\theta + \alpha) =$

(a) $\dfrac{1-n}{1+n}\tan\alpha$ (b) $\dfrac{n+1}{n-1}\tan\alpha$

(c) $\dfrac{1+n}{1-n}\tan\alpha$ (d) none of these

Solution (c) Given, $\sin\theta = n\sin(\theta + 2\alpha)$

$\Rightarrow \quad \dfrac{1}{n} = \dfrac{\sin(\theta + 2\alpha)}{\sin\theta} \Rightarrow \dfrac{1+n}{1-n} = \dfrac{\sin(\theta + 2\alpha) + \sin\theta}{\sin(\theta + 2\alpha) - \sin\theta}$

[by componendo and dividendo]

$$= \frac{2\sin\frac{\theta + 2\alpha + \theta}{2}\cos\frac{\theta + 2\alpha - \theta}{2}}{2\cos\frac{\theta + 2\alpha + \theta}{2}\cdot\sin\frac{\theta + 2\alpha - \theta}{2}}$$

$= \dfrac{2\sin(\theta + \alpha)\cos\alpha}{2\cos(\theta + \alpha)\sin\alpha} = \tan(\theta + \alpha)\cot\alpha.$

$\Rightarrow \tan(\theta + \alpha) = \mathbf{\dfrac{1+n}{1-n}\tan\alpha.}$

14. The value of

$\left(1 + \cos\dfrac{\pi}{10}\right)\left(1 + \cos\dfrac{3\pi}{10}\right)\left(1 + \cos\dfrac{7\pi}{10}\right)\left(1 + \cos\dfrac{9\pi}{10}\right)$ is

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{16}$

(c) $\dfrac{1}{32}$ (d) none of these

Solution (b) The given expression

$= \left(1 + \cos\dfrac{\pi}{10}\right)\left(1 + \cos\dfrac{3\pi}{10}\right)\left(1 - \cos\dfrac{3\pi}{10}\right)\left(1 - \cos\dfrac{\pi}{10}\right)$

$$\left[\because \cos\frac{7\pi}{10} = \cos\left(\pi - \frac{3\pi}{10}\right) = -\cos\frac{3\pi}{4}\right.$$

$$\left.\text{and } \cos\frac{9\pi}{10} = \cos\left(\pi - \frac{\pi}{10}\right) = -\cos\frac{\pi}{10}\right]$$

$$= \left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right) = \sin^2\frac{\pi}{10}\cdot\sin^2\frac{3\pi}{10}$$

$$= \sin^2 18° \cdot \sin^2 54° = \left(\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{5}+1}{4}\right)^2 = \mathbf{\frac{1}{16}}.$$

15. The value of $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$ is

(a) 1 (b) 2
(c) − 1 (d) none of these

Solution (b) The given expression

$$= \frac{1}{2}\left(2\cos^2\frac{\pi}{8} + 2\cos^2\frac{3\pi}{8} + 2\cos^2\frac{5\pi}{8} + 2\cos^2\frac{7\pi}{8}\right)$$

$$= \frac{1}{2}\left(1+\cos\frac{\pi}{4} + 1+\cos\frac{3\pi}{4} + 1+\cos\frac{5\pi}{4} + 1+\cos\frac{7\pi}{4}\right)$$

$$[\because 2\cos^2\theta = 1 + \cos 2\theta]$$

$$= \frac{1}{2}\left[4+\cos\frac{\pi}{4} + \cos\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi + \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left(4+\cos\frac{\pi}{4} - \cos\frac{\pi}{4} - \cos\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = \frac{1}{2} \times 4 = \mathbf{2}.$$

16. The value of

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) \text{ is}$$

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$
(c) $\frac{1}{16}$ (d) none of these

Solution (b) The given expression

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\sin\frac{\pi}{8}\right)\left(1-\sin\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$\left[\begin{array}{l}\because \cos\frac{3\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\frac{\pi}{8}, \\ \cos\frac{5\pi}{8} = \cos\left(\frac{\pi}{2} + \frac{\pi}{8}\right) = -\sin\frac{\pi}{8}, \\ \cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}\end{array}\right]$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\sin^2\frac{\pi}{8}\right) = \sin^2\frac{\pi}{8}\ \cos^2\frac{\pi}{8}$$

$$= \frac{1}{4}\left[2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right]^2 = \frac{1}{4}\sin^2\frac{\pi}{4} = \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \mathbf{\frac{1}{8}}.$$

17. The value of $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°}$ is

(a) 1 (b) 2
(c) 4 (d) none of these

Solution (c) We have,

$$\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = \frac{1\cdot\cos 10° - \sqrt{3}\sin 10°}{\sin 10°\cdot\cos 10°}$$

$$= \frac{2(\cos 60°\cdot\cos 10° - \sin 60°\cdot\sin 10°)}{\sin 10°\cdot\cos 10°}$$

[Putting $1 = r\cos\alpha$, $\sqrt{3} = r\sin\alpha$
$\Rightarrow r^2 = 1 + 3 = 4$ i.e., $r = 2$
and $\tan\alpha = \sqrt{3}$ i.e., $\alpha = 60°$]

$$= \frac{2\cos(60°+10°)}{\frac{1}{2}\sin 20°} = 4\cdot\frac{\cos 70°}{\sin 20°}$$

$$= 4\cdot\frac{\sin 20°}{\sin 20°} = \mathbf{4}.$$

18. Which of the following is correct?

(a) sin 1° > sin 1 (b) sin 1° < sin 1
(c) sin 1° = sin 1 (d) $\sin 1° = \frac{\pi}{180}\sin 1$

Solution (b) Since 1 radian = 57° 30′ approx.
and sin 57° 30′ > sin 1°, ∴ **sin 1° < sin 1.**

19. Given $\frac{\pi}{2} < \alpha < \pi$, then the expression

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} + \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} =$$

(a) $\frac{1}{\cos\alpha}$ (b) $-\frac{2}{\cos\alpha}$
(c) $\frac{2}{\cos\alpha}$ (d) none of these

Solution (b) The given expression

$$= \frac{(1-\sin\alpha)+(1+\sin\alpha)}{\sqrt{[(1+\sin\alpha)(1-\sin\alpha)]}}$$

$$= \frac{2}{\sqrt{(\cos^2\alpha)^2}} = \frac{2}{|\cos\alpha|} = -\frac{2}{\cos\alpha}$$

$$\left[\because \frac{\pi}{2} < \alpha < \pi \Rightarrow \cos\alpha < 0 \Rightarrow |\cos\alpha| = -\cos\alpha\right].$$

20. The inequality $2^{\sin\theta} + 2^{\cos\theta} \geq 2^{\left(1-\frac{1}{\sqrt{2}}\right)}$ holds for

(a) $0 \leq \theta < \pi$ (b) $\pi \leq \theta < 2\pi$
(c) for all real θ (d) none of these

Solution (c) We have,

$$\frac{1}{2}[2^{\sin\theta} + 2^{\cos\theta}] \geq \sqrt{2^{\sin\theta}\cdot 2^{\cos\theta}}$$

[∵ A.M. ≥ G.M.]

$$\Rightarrow 2^{\sin\theta} + 2^{\cos\theta} \geq 2.\ 2^{(\sin\theta + \cos\theta)/2} \quad \text{...(1)}$$

Now $(\sin\theta + \cos\theta) = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

$\geq -\sqrt{2}$ for all real θ.

$\therefore \quad 2^{\sin\theta} + 2^{\cos\theta} \geq 2 \cdot 2^{(\sin\theta + \cos\theta)/2}$

$> 2 \cdot 2^{-\sqrt{2}/2} = 2^{1-(1/\sqrt{2})}$

Thus, $2^{\sin\theta} + 2^{\cos\theta} \geq 2^{1-(1/\sqrt{2})}$, **for all real θ.**

21. $\sin^6 x + \cos^6 x$ lies between

(a) $\frac{1}{4}$ and 1 (b) $\frac{1}{4}$ and 2

(c) 0 and 1 (d) none of these

Solution (a) We have,

$\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)$

$= 1 - 3\sin^2 x\cos^2 x$

$= 1 - \frac{3}{4}\cdot 4\sin^2 x\cos^2 x = 1 - \frac{3}{4}(\sin 2x)^2$

⇒ maximum value of $\sin^6 x + \cos^6 x$ is $1 - \frac{3}{4} \times 0 = \mathbf{1}$

and minimum value is $1 - \frac{3}{4} \times 1 = \mathbf{\frac{1}{4}}$.

22. If $x + \frac{1}{x} = 2\cos\theta$ then $x^n + \frac{1}{x^n}$ is equal to

(a) $2\sin n\theta$ (b) $\cos n\theta$

(c) $\sin n\theta$ (d) $2\cos n\theta$, $n \in Z^+$

Solution (d) We have,

$x + \frac{1}{x} = 2\cos\theta \quad \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

$\Rightarrow x^n = \cos n\theta \pm i\sin n\theta$

Now, $\frac{1}{x} = \frac{1}{\cos\theta \pm i\sin\theta} = \cos\theta \pm i\sin\theta$

$\therefore \quad \frac{1}{x^n} = \cos n\theta \pm i\sin n\theta$

$\therefore \quad x^n + \frac{1}{x^n} = \mathbf{2\cos n\theta}$.

23. The minimum value of the expression

$\sin\alpha + \sin\beta + \sin\gamma$,

when α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is

(a) – 3 (b) negative

(c) positive (d) zero

Solution (c) We have,

$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$\because$ each of $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ is less than $\frac{\pi}{2}$

$\therefore \cos\frac{\alpha}{2}, \cos\frac{\beta}{2}, \cos\frac{\gamma}{2}$ are all +ve

$\therefore$ min. value of $4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ is +ve.

Hence, minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is **+ve.**

24. The greatest value of $\cos(xe^{|x|} + 7x^2 - 3x)$, $x \in [-1, \infty)$ is

(a) 0 (b) 1

(c) – 1 (d) none of these

Solution (b) Since $\cos\theta \leq 1$ for all real θ,

∴ the given expression ≤ 1 for all x.

Hence, the greatest value = **1.**

25. In a ΔABC, if angle C is obtuse, then

(a) $\tan A\tan B < 1$ (b) $\tan A\tan B \leq 1$

(c) $\tan A\tan B > 1$ (d) none of these

Solution (a) Since $A + B + C = \pi$,

$\therefore \quad A + B = \pi - C \quad \Rightarrow \tan(A + B) = \tan(\pi - C)$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} > 0 \quad \text{...(1)}$

[∵ angle C is obtuse ∴ $\tan C < 0$]

But since C is obtuse angle, so A and B will both be less than $\frac{\pi}{2}$.

∴ Both $\tan A$ and $\tan B$ are positive.

Hence from (1), $1 - \tan A\tan B > 0$

$\Rightarrow \quad \mathbf{\tan A\tan B < 1.}$

26. If $\sin(x - y)$, $\sin x$ and $\sin(x + y)$ are in H.P. then

$\left|\sin x \cdot \sec\frac{y}{2}\right|$ is equal to

(a) 2 (b) $\sqrt{2}$

(c) 1 (d) none of these

Solution (b) Since $\sin(x - y)$, $\sin x$ and $\sin(x + y)$ are in H.P.,

$\therefore \quad \sin x = \frac{2\sin(x-y)\sin(x+y)}{\sin(x-y)+\sin(x+y)}$

$= \frac{2(\sin^2 x - \sin^2 y)}{2\sin x\cos y}$

$\Rightarrow \quad \sin^2 x \cdot \cos y = \sin^2 x - \sin^2 y$

$\Rightarrow \quad \sin^2 x(1 - \cos y) = \sin^2 y$

$\Rightarrow \quad \sin^2 x \cdot 2\sin^2\frac{y}{2} = 4\sin^2\frac{y}{2}\cos^2\frac{y}{2}$

$\Rightarrow \quad \left|\sin x \cdot \sec\frac{y}{2}\right| = \mathbf{\sqrt{2}}$.

27. If $A = \cos^2\theta + \sin^4\theta$, then for all values of θ

(a) $1 \leq A \leq 2$ (b) $\frac{13}{16} \leq A \leq 1$

(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

Solution (d) We have, $A = \cos^2\theta + \sin^2\theta\sin^2\theta$

$\Rightarrow \quad A \leq \cos^2\theta + \sin^2\theta \cdot 1 \qquad (\because \sin^2\theta \leq 1)$

$\Rightarrow \quad A \leq 1.$

Again, $A = (1 - \sin^2\theta) + \sin^4\theta$

$\Rightarrow \quad A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{4}\right) \Rightarrow A \geq \frac{3}{4}$.

Hence $\mathbf{\frac{3}{4} \le A \le 1}$.

28. If $n = \frac{\pi}{4\alpha}$, then $\tan\alpha \cdot \tan 2\alpha \cdot \tan 3\alpha \ldots \tan(2n-1)\alpha$ is equal to

(a) 1 (b) – 1
(c) ∞ (d) none of these

Solution (a) We have,

$$\tan\alpha \cdot \tan(2n-1)\alpha = \tan\alpha \cdot \tan\left(\frac{\pi}{2\alpha} - 1\right)\alpha$$

$$= \tan\alpha \cdot \tan\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha \cot\alpha = 1.$$

∴ The given expression = **1.**

29. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$

(a) –1 (b) 0
(c) 1 (d) 2

Solution (b) We have, $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3} = k$ (say)

$$\Rightarrow \frac{1}{x} = \frac{1}{k},\ \frac{1}{y} = \frac{\cos\frac{2\pi}{3}}{k},\ \frac{1}{z} = \frac{\cos\frac{4\pi}{3}}{k}$$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k}\left(1 + \cos\frac{2\pi}{3} + \cos\frac{4\pi}{3}\right)$$

$$= \frac{1}{k}\left(1 - \frac{1}{2} - \frac{1}{2}\right) = 0$$

$\Rightarrow xy + xz + yz = \mathbf{0}.$

30. The ratio of the greatest value of $2 - \cos x + \sin^2 x$ to its least value is

(a) $\frac{1}{4}$ (b) $\frac{9}{4}$
(c) $\frac{13}{4}$ (d) none of these

Solution (c) We have,

$$2 - \cos x + \sin^2 x = 2 - \cos x + 1 - \cos^2 x$$

$$= -(\cos^2 x + \cos x) + 3$$

$$= -\left[\left(\cos x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 3 = \frac{13}{4} - \left(\cos x + \frac{1}{2}\right)^2.$$

∴ Maximum value occurs at $\cos x = -\frac{1}{2}$ and it is $\frac{13}{4}$ and minimum value occurs at $\cos x = 1$ and it is 1

∴ The required ratio is $\mathbf{\frac{13}{4}}$.

31. The expression $2^{\sin\theta} + 2^{-\cos\theta}$ is minimum when θ is equal to

(a) $2n\pi + \frac{\pi}{4}, n \in I$ (b) $2n\pi + \frac{7\pi}{4}, n \in I$

(c) $n\pi \pm \frac{\pi}{4}, n \in I$ (d) none of these

Solution (b) Since A.M. ≥ G.M.

$$\therefore \frac{2^{\sin\theta} + 2^{-\cos\theta}}{2} \ge \sqrt{2^{\sin\theta} \cdot 2^{-\cos\theta}}$$

$$\Rightarrow 2^{\sin\theta} + 2^{-\cos\theta} \ge 2^{1 + \frac{1}{2}(\sin\theta - \cos\theta)}$$

$$= 2^{1 + \frac{1}{\sqrt{2}}\sin\left(\theta - \frac{\pi}{4}\right)}$$

Thus, $2^{\sin\theta} + 2^{-\cos\theta}$ is minimum

when $\sin\left(\theta - \frac{\pi}{4}\right) = -1$ *i.e.*, $\theta - \frac{\pi}{4} = 2n\pi + \frac{3\pi}{2}$

or $\theta = \mathbf{2n\pi + \frac{7\pi}{4}, n \in I}$.

32. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$

(a) ≥ 0 only when θ ≥ 0
(b) ≤ 0 for all real θ
(c) ≥ 0 for all real θ
(d) ≤ 0 only when θ ≤ 0

Solution (c) We have, $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$$= \sin\theta(2\sin 2\theta\cos\theta)$$

$$= \sin^2 2\theta \ge \mathbf{0}, \text{ for all real } \theta.$$

33. If $u_n = 2\cos n\theta$ then $u_1 u_n - u_{n-1}$ is equal to

(a) u_{n-2} (b) u_{n+1}
(c) 0 (d) none of these

Solution (b) $u_1 u_n - u_{n-1}$

$$= 2\cos\theta \cdot 2\cos n\theta - 2\cos(n-1)\theta$$

$$= 2[\cos(n-1)\theta + \cos(n+1)\theta] - 2\cos(n-1)\theta$$

$$= 2\cos(n+1)\theta = \mathbf{u_{n+1}}.$$

34. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to

(a) 1 (b) 4
(c) 2 (d) none of these

Solution (c) Given : $\sin\theta + \operatorname{cosec}\theta = 2$

Squaring both sides, we get

$$\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta = 4$$

$$\Rightarrow \sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta \cdot \frac{1}{\sin\theta} = 4$$

$$\Rightarrow \sin^2\theta + \operatorname{cosec}^2\theta = 4 - 2 = \mathbf{2}.$$

35. For $m \ne n$, if $\tan m\theta = \tan n\theta$, then the different values of θ are in

(a) A. P. (b) H. P.
(c) G. P. (d) none of these

Solution (a) We have,

$$\tan m\theta = \tan n\theta \Rightarrow m\theta = n\theta + k\pi, k \in Z$$

$$\therefore \theta = \frac{k\pi}{m-n}, k \in Z.$$

These values of θ are in **A. P.** with first term $\frac{\pi}{m-n}$ and common difference $\frac{\pi}{m-n}$.

36. If $\tan\theta + \sec\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

Solution (c) We have, $\sec\theta + \tan\theta = \sqrt{3}$.

Since $\sec^2\theta - \tan^2\theta = 1 \quad \therefore \quad \sec\theta - \tan\theta = \frac{1}{\sqrt{3}}$

$$\therefore \quad 2\tan\theta = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \quad \therefore \quad \theta = 30^\circ = \frac{\pi}{6}.$$

37. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then the value of $\cos\left(\theta + \frac{\pi}{4}\right)$ is

(a) $\frac{2}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) $-\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

Solution (d) We have,

$$\sin(\pi\cos\theta) = \cos(\pi\sin\theta) = \sin\left(\frac{\pi}{2} + \pi\sin\theta\right)$$

$$\Rightarrow \quad \pi\cos\theta = \frac{\pi}{2} + \pi\sin\theta \Rightarrow \pi(\cos\theta - \sin\theta) = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta\cdot\cos\frac{\pi}{4} - \sin\theta\cdot\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$\left[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \quad \cos\left(\theta + \frac{\pi}{4}\right) = \mathbf{\frac{1}{2\sqrt{2}}}.$$

38. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 =$

(a) 0 (b) 1

(c) 2 (d) 3

Solution (a) Since, $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$

$$\therefore \quad \sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 1$$

$$\Rightarrow \quad \theta_1 = \theta_2 = \theta_3 = \frac{\pi}{2}$$

$$\therefore \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = \mathbf{0}.$$

39. Let $0 < x < \frac{\pi}{4}$. Then $(\sec 2x - \tan 2x)$ equals

(a) $\tan\left(x - \frac{\pi}{4}\right)$ (b) $\tan\left(\frac{\pi}{4} - x\right)$

(c) $\tan\left(x + \frac{\pi}{4}\right)$ (d) $\tan^2\left(x + \frac{\pi}{4}\right)$

Solution (b) We have, $\sec 2x - \tan 2x$

$$= \frac{1}{\cos 2x} - \frac{\sin 2x}{\cos 2x} = \frac{1 - \sin 2x}{\cos 2x} = \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \tan x}{1 + \tan x} = \mathbf{\tan}\left(\frac{\pi}{4} - x\right).$$

40. Which of the following is correct?

(a) tan 1 = tan 2 (b) tan 1 = $\frac{1}{4}$ tan 2

(c) tan 1 > tan 2 (d) tan 1 < sin 2

Solution (c) Since 1 radian = 57° (nearly)

∴ 2 radians = 114° (nearly)

This lies in 2nd quadrant

∴ tan 1 > 0 and tan 2 < 0

∴ **tan 1 > tan 2.**

41. $\sin^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$

(c) $x = y$ (d) $x \neq 0, y \neq 0$

Solution (c) We know that $\sin^2\theta \geq 1$

$$\frac{4xy}{(x+y)^2} \geq 1 \Rightarrow 4xy \geq (x+y)^2 \Rightarrow (x-y)^2 \leq 0$$

$$\Rightarrow x - y = 0 \Rightarrow y = x.$$

42. If $\sin(\alpha + \beta) = 1$, $\sin(\alpha - \beta) = \frac{1}{2}$ then $\tan(\alpha + 2\beta)\tan(2\alpha + \beta)$ is equal to

(a) 1 (b) –1

(c) zero (d) none of these

Solution (a) $\sin(\alpha + \beta) = 1 \Rightarrow \alpha + \beta = \frac{\pi}{2}$

$$\sin(\alpha - \beta) = \frac{1}{2} \Rightarrow \alpha - \beta = \frac{\pi}{6}$$

$$\therefore \quad \alpha = \frac{\pi}{3}, \beta = \frac{\pi}{6}$$

So, $\tan(\alpha + 2\beta)\tan(2\alpha + \beta) = \tan\frac{2\pi}{3}\tan\frac{5\pi}{6}$

$$= \left(-\cot\frac{\pi}{6}\right)\left(-\cot\frac{\pi}{3}\right) = 1.$$

43. The equation $a\sin x + b\cos x = c$, where $|c| > \sqrt{a^2 + b^2}$ has

(a) a unique solution

(b) infinite number of solutions

(c) no solution

(d) none of these

Solution (c) Since, $-\sqrt{a^2 + b^2} \leq a\sin x + b\cos x \leq \sqrt{a^2 + b^2}$, therefore.

$a\sin x + b\cos x = c$ has no solution.

44. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -21/65$ and $\cos\alpha + \cos\beta = -27/65$, then the value of $\cos\frac{\alpha - \beta}{2}$ is

(a) $\frac{6}{65}$ (b) $\frac{3}{\sqrt{130}}$

(c) $-\frac{3}{\sqrt{130}}$ (d) $\frac{-6}{65}$

Solution (c) $\left(\frac{-21}{65}\right)^2 + \left(-\frac{27}{65}\right)^2 = (\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2$

$\Rightarrow \frac{1170}{4225} = 2 + 2\cos(\alpha - \beta) = 4\cos^2\frac{\alpha-\beta}{2}$

$\Rightarrow \frac{\alpha-\beta}{2} = -\frac{1170}{4\times(65)^2}$

$= \frac{1170}{(130)^2} = \frac{9}{130}$

$\Rightarrow \cos\frac{\alpha-\beta}{2} = -\frac{3}{\sqrt{130}}$. $[\because \pi < \alpha - \beta < 3\pi]$.

Problems for Practice

Level of Difficulty I

1. If $\tan\theta = -\frac{4}{3}$ then $\sin\theta$ is

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) none of these

2. If $\cos(\alpha + \beta) = \frac{4}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$ and α_1, β lie between 0 and $\frac{\pi}{4}$, then $\tan 2\alpha =$

(a) $\frac{56}{33}$ (b) $\frac{33}{56}$ (c) $\frac{33}{48}$ (d) none of these

3. The expression $\frac{\sec 8A - 1}{\sec 4A - 1}$ is equal to

(a) $\frac{\tan 8A}{\tan 2A}$ (b) $\frac{\tan 2A}{\tan 8A}$

(c) $\frac{\tan 4A}{\tan 2A}$ (d) $\frac{\tan 2A}{\tan 4A}$

4. If $\operatorname{cosec}\theta + \cot\theta = \frac{11}{2}$, then $\tan\theta =$

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{44}$

5. If α and β be between 0 and $\frac{\pi}{2}$ and if $\cos(\alpha + \beta) = \frac{12}{13}$ and $\sin(\alpha - \beta) = \frac{3}{5}$, then $\sin 2\alpha$ is equal to

(a) $\frac{64}{65}$ (b) $\frac{56}{65}$ (c) 0 (d) $\frac{16}{15}$

6. The expression $\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$ is equal to

(a) $\cos^2 x$ (b) $1 + \cos x$

(c) $\cos 2x$ (d) $2\cos x$

7. The value of $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$ is

(a) 1 (b) 0

(c) −1 (d) none of these

8. If $\sin\alpha = -\frac{3}{5}$, where $\pi < \alpha < \frac{3\pi}{2}$, then $\cos\frac{\alpha}{2} =$

(a) $-\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$

(c) $\frac{3}{\sqrt{10}}$ (d) $-\frac{3}{\sqrt{10}}$.

9. If $A + B = 45°$, then $(1 + \tan A)(1 + \tan B) =$

(a) 1 (b) − 1 (c) 2 (d) none of these

10. If $A + B + C = \pi$, then

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin A \sin C} + \frac{\cos C}{\sin A \sin B} =$$

(a) 1 (b) 2 (c) −1 (d) −2

11. If $A + B + C = 180°$ and $\tan 3A + \tan 3B + \tan 3C = k \tan 3A \tan 3B \tan 3C$ then $k =$

(a) 1 (b) 2 (c) 4 (d) none of these

12. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is

(a) $\frac{1}{\sqrt{2}}$ (b) 0 (c) 1 (d) none of these

13. The value of $\tan 3A - \tan 2A - \tan A$ is equal to

(a) $\tan 3A \tan 2A \tan A$

(b) $-\tan 3A \tan 2A \tan A$

(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$

(d) None of these

14. If $\sin\alpha = \frac{12}{13}\left(0 < \alpha < \frac{\pi}{2}\right)$ and $\cos\beta = -\frac{3}{5}\left(\pi < \beta < \frac{3\pi}{2}\right)$, the value of $\sin(\alpha + \beta)$ is

(a) $-\frac{56}{65}$ (b) $\frac{16}{65}$ (c) $\frac{56}{65}$ (d) $-\frac{16}{65}$

15. The value of $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°} =$

(a) 3 (b) 1 (c) 0 (d) 2

16. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, then the value of $A + B$ is

(a) $\frac{\pi}{4}$ (b) zero (c) π (d) $\frac{\pi}{6}$

17. Let $0 < x \le \frac{\pi}{4}$, then $(\sec 2x - \tan 2x)$ equals

(a) $\tan^2\left(x+\frac{\pi}{4}\right)$ (b) $\tan\left(x+\frac{\pi}{4}\right)$

(c) $\tan\left(\frac{\pi}{4}-x\right)$ (d) $\tan\left(\frac{\pi}{4}-x\right)$

18. The greatest value of $\sin x \cos x$ is

(a) -1 (b) 2 (c) $\frac{1}{2}$ (d) 1

19. $\log\tan 1° + \log\tan 2° + \ldots + \log\tan 89° =$

(a) 1 (b) 0 (c) $\frac{\pi}{4}$ (d) none of these

20. If $\tan\frac{\pi}{9}$, x and $\tan\frac{5\pi}{18}$ are in A. P. and $\tan\frac{\pi}{9}$, y and $\tan\frac{7\pi}{18}$ are also in A. P., then

(a) $2x = y$ (b) $x > y$

(c) $x = y$ (d) none of these

Level of Difficulty II

21. The expression $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n =$

(a) $2\cot^n\left(\frac{A-B}{2}\right)$ if n is even

(b) 0 if n is even

(c) $2\cot^n\left(\frac{A-B}{2}\right)$ if n is even

(d) 0 if n is odd

22. If $m = \operatorname{cosec}\theta - \sin\theta$ and $n = \sec\theta - \cos\theta$, then $m^{2/3} + n^{2/3} =$

(a) $(mn)^{-2/3}$ (b) $(mn)^{2/3}$

(c) $(mn)^{-1/3}$ (d) $(mn)^{1/3}$

23. If $3\tan\theta\tan\phi = 1$, then $\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} =$

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{1}{3}$ (d) 3

24. The value of the $\sin 20° \sin 40° \sin 80°$ is

(a) $\frac{\sqrt{3}}{8}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) none of these

25. The value of $\cos^2 73° + \cos^2 47° + \cos 73° \cos 47°$ is

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{3}{4}$ (d) none of these

26. The value of $\sqrt{3}\operatorname{cosec} 20° - \sec 20°$ is

(a) 4 (b) 12 (c) -2 (d) none of these

27. The value of $\sin 36° \sin 72° \sin 108° \sin 144°$ is

(a) $\frac{1}{16}$ (b) $\frac{3}{16}$ (c) $\frac{5}{16}$ (d) none of these

28. If $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)}$, then $\tan A$, $\tan B$, $\tan C$ are in

(a) A. P. (b) G. P.

(c) H. P. (d) none of these

29. The value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 90°$ is

(a) $7\frac{1}{2}$ (b) $8\frac{1}{2}$ (c) $9\frac{1}{2}$ (d) none of these

30. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$ $-\,2\left[\sin^6\left(\frac{3\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to

(a) 1 (b) $\sin 4\alpha + \cos 6\alpha$

(c) 3 (d) none of these

31. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, then the value of $A + B$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

32. In a triangle $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then the values of $\tan A$, $\tan B$ and $\tan C$ are

(a) 1, 2, 3 (b) 2, 1, 3 (c) 1, 2, 0 (d) none of these

33. Which of the following is a rational number?

(a) $\sin 15°$ (b) $\cos 15°$

(c) $\sin 15° \cos 15°$ (d) $\sin 15° \cos 75°$

34. If $\sin\theta = \frac{1}{3}$, then the quadratic equation whose roots are $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$ is

(a) $x^2 - 6x + 1 = 0$ (b) $x^2 + 6x + 1 = 0$

(c) $x^2 - 6x - 1 = 0$ (d) $x^2 + 6x - 1 = 0$

35. If in a ΔABC, $\tan A + \tan B + \tan C$ has the value 6 then the value of $\cot A \cdot \cot B \cdot \cot C$ is equal to

(a) 6 (b) 1 (c) $\frac{1}{6}$ (d) none of these

36. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to

(a) $-\frac{5}{3}$ (b) $-\frac{3}{5}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

37. $\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40°$ is equal to

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

38. If $\sin(A + B + C) = 1$, $\tan(A - B) = \frac{1}{\sqrt{3}}$ and $\sec(A + C) = 2$, then

(a) $A = 90°, B = 60°, C = 30°$

(b) $A = 120°, B = 60°, C = 0°$

(c) $A = 60°, B = 30°, C = 0°$

(d) none of these

39. $\sin x + \sqrt{3}\cos x$ is maximum when

(a) $x = 60°$ (b) $x = 45°$

(c) $x = 30°$ (d) $x = 0°$

40. The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) none of these

ANSWERS

1. (d)
2. (a)
3. (a)
4. (c)
5. (b)
6. (d)
7. (b)
8. (a)
9. (c)
10. (b)
11. (a)
12. (b)
13. (a)
14. (a)
15. (d)
16. (a)
17. (c)
18. (c)
19. (b)
20. (a)
21. (a),(d)
22. (a)
23. (b)
24. (a)
25. (c)
26. (a)
27. (c)
28. (b)
29. (c)
30. (a)
31. (d)
32. (a),(b)
33. (c)
34. (a)
35. (c)
36. (b)
37. (c)
38. (c)
39. (c)
40. (c)

Trigonometric Equations

6

BRIEF REVIEW OF THE CONCEPTS

TRIGONOMETRIC EQUATION

An equation involving one or more trigonometric ratios of unknown angles is called a trigonometric equation.

For example, $2\cos\theta + 3\cos 2\theta = 0$, $\cos^2\theta + \sin\theta = \frac{1}{3}$, etc. are trigonometric equations in unknown angle θ.

SOLUTION OF A TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given equation is called a solution of the equation. For example, consider the equation $\sqrt{2}\sin\theta = 1$. The of value the angle, $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ satisfy this equation. Therefore, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are solutions of the given equation $\sqrt{2}\sin\theta = 1$.

GENERAL SOLUTION

Since the trigonometric functions are periodic, a solution generalised by means of periodicity is known as the general solution.

For example, general solution of the equation: $\sin\theta = 0$ is $\theta = n\pi$, where n is any integer.

Key Points to Remember

Trigonometric Equation	*General Solution*
• $\sin\theta = 0$	$\theta = n\pi,\ n \in I$
• $\cos\theta = 0$	$\theta = (2n + 1)(\pi/2),\ n \in I$
• $\tan\theta = 0$	$\theta = n\pi,\ n \in I$
• $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha,\ n \in I$
• $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha,\ n \in I$
• $\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha,\ n \in I$
• $\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in I$
• $\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in I$
• $\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in I$
• $a\cos\theta + b\cos\theta = c$	$\theta = 2n\pi + \alpha \pm \beta$ where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$

Point to Note:

If $|c| \le \sqrt{a^2+b^2}$ is not statisfied, then no real solution exists.

Short-cut Methods for Solving Trigonometric Equations

(i) Squaring should be avoided as far as possible. If squaring is done, then check for extra solutions.

For example, consider the equating $\sin\theta + \cos\theta = 1$.

On squaring, we get

$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$

$\Rightarrow \theta = \frac{n\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

The values of the angle, $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ do not satisfy the given equation. So, we get extra solutions. Thus, if squaring is must, verify each of the solutions.

(ii) Never cancel a common factor containing 'θ' from the two sides of an equation.

For example, consider the equation $\tan\theta = \sqrt{2}\sin\theta$. If we divide both sides by $\sin\theta$, we get $\cos\theta = \frac{1}{\sqrt{2}}$, which is clearly not equivalent to the given equation as the solutions obtained by $\sin x = 0$ are lost. Thus, **instead of dividing an equation by a common factor, take this factor out as a common factor from all terms of the equation.**

(iii) Make sure that the answer should not contain any value of unknown 'θ' which makes any of the terms undefined.

(iv) If $\tan\theta$ or $\sec\theta$ is involved in the equation, θ should not be an odd multiple of $\pi/2$.

(v) If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be a multiple of π or 0.

(vi) The value of $\sqrt{f(\theta)}$ is always positive. For example,

$\sqrt{\cos^2\theta} = |\cos\theta|$ and not $\pm\cos\theta$.

(vii) All the solutions should satisfy the given equation and lie in the domain of the variable of the given equation.

Solved Problems

1. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$, is

(a) 3 (b) 2

(c) 1 (d) 0

Solution (b) We have, $\tan x + \sec x = 2\cos x$

$\Rightarrow \dfrac{\sin x + 1}{\cos x} = 2\cos x$ [$\cos x \neq 0$ i.e., $\sin x \neq 1$ or -1]

$\Rightarrow 2\cos^2 x = \sin x + 1$

$\Rightarrow 2[1 - \sin^2 x] - (1 + \sin x) = 0$

$\Rightarrow (1 + \sin x)[2(1 - \sin x) - 1] = 0$

$\Rightarrow (1 + \sin x)(1 - 2\sin x) = 0$

$\Rightarrow \sin x = \dfrac{1}{2}$ [$\because \sin x \neq -1$]

$\Rightarrow x = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ in $[0, 2\pi]$

Hence, the number of solutions is **2.**

2. The smallest positive angle which satisfies the equation $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ is

(a) $\dfrac{5\pi}{6}$ (b) $\dfrac{2\pi}{3}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{6}$

Solution (a) We have, $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2(1 - \cos^2\theta) + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$

$\Rightarrow (\cos\theta - \sqrt{3})(2\cos\theta + \sqrt{3}) = 0$

$\Rightarrow \cos\theta = -\dfrac{\sqrt{3}}{2}$

($\because \cos\theta = \sqrt{3}$ is not possible as $\cos\theta \leq 1$)

$\Rightarrow \theta = \dfrac{5\pi}{6}$.

3. If $5\cos 2\theta + 2\cos^2\dfrac{\theta}{2} + 1 = 0$, $-\pi < \theta < \pi$, then θ is equal to

(a) $\cos^{-1}\left(\dfrac{3}{5}\right)$ (b) $\dfrac{\pi}{3}, \pi - \cos^{-1}\left(\dfrac{3}{5}\right)$

(c) $\dfrac{\pi}{3}, \cos^{-1}\left(\dfrac{3}{5}\right)$ (d) $\dfrac{\pi}{3}$

Solution (b) The given equation can be written as

$5[2\cos^2\theta - 1] + (1 + \cos\theta) + 1 = 0$

$\Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow (5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = \dfrac{1}{2}$ or $\cos\theta = -\dfrac{3}{5}$

$\Rightarrow \theta = \dfrac{\pi}{3}$ or $\theta = \boldsymbol{\pi - \cos^{-1}\left(\dfrac{3}{5}\right)}$ in $-\pi < \theta < \pi$.

4. The general value of θ satisfying the equation $3\tan(\theta - 15°) = \tan(\theta + 15°)$, is

(a) $n\pi + (-1)^n\dfrac{\pi}{4}$ (b) $\dfrac{n\pi}{2} + (-1)^n\dfrac{\pi}{4}$

(c) $\dfrac{n\pi}{2} + (-1)^n\dfrac{\pi}{6}$ (d) none of these

Solution (b) We have, $3\tan(\theta - 15°) = \tan(\theta + 15°)$

$\Rightarrow \dfrac{\tan(\theta + 15°)}{\tan(\theta - 15°)} = \dfrac{3}{1}$

$\Rightarrow \dfrac{\tan(\theta + 15°) + \tan(\theta - 15°)}{\tan(\theta + 15°) - \tan(\theta - 15°)} = \dfrac{4}{2}$

[By componendo and dividendo]

$\Rightarrow \dfrac{\sin(\theta + 15° + \theta - 15°)}{\sin(\theta + 15° - \theta + 15°)} = 2$

$\Rightarrow 2\sin 2\theta = 2 \Rightarrow \sin 2\theta = 1 = \sin\dfrac{\pi}{2}$

$\Rightarrow 2\theta = n\pi + (-1)^n\dfrac{\pi}{2}$

$\therefore \theta = \dfrac{n\pi}{2} + (-1)^n\dfrac{\pi}{4}$, $n \in Z$.

5. The set of values of x for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is

(a) ϕ (b) $\left\{\dfrac{\pi}{4}\right\}$

(c) $\left\{n\pi + \dfrac{\pi}{4} : n = 1, 2, 3 ...\right\}$

(d) $\left\{2n\pi + \dfrac{\pi}{4} : n = 1, 2, 3 ...\right\}$

Solution (a) We have,

$\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1 \Rightarrow \tan(3x - 2x) = 1$

$\Rightarrow \tan x = 1 \quad \therefore x = n\pi + \pi/4$

But this value does not satisfy the given equation as

$\tan 2x = \tan \pi/2 = \infty$

6. The general solution of the equation $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$ is

(a) $\theta = \dfrac{n\pi}{4}$ (b) $\theta = \dfrac{n\pi}{12}$

(c) $\theta = \dfrac{n\pi}{6}$ (d) none of these

Solution (b) We write the given equation as

$\tan\theta + \tan 4\theta = -\tan 7\theta(1 - \tan\theta \tan 4\theta)$

$\Rightarrow (\tan\theta + \tan 4\theta)/(1 - \tan\theta \tan 4\theta) = -\tan 7\theta$

$\Rightarrow \tan(\theta + 4\theta) = -\tan 7\theta \Rightarrow \tan 5\theta = \tan(-7\theta)$

$\therefore 5\theta = n\pi + (-7\theta)$ or $12\theta = n\pi$

$\therefore \theta = \boldsymbol{n\pi/12}$, $n \in Z$.

7. Solution of the system of equations

$x + y = \frac{\pi}{4}$, $\tan x + \tan y = 1$ is

(a) $x = \frac{\pi}{2} - n\pi, y = n\pi$

(b) $x = \frac{\pi}{4} - n\pi, y = n\pi$

(c) $x = \frac{\pi}{4} - n\pi, y = 2n\pi$

(d) none of these

Solution (b) We have, $\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$\Rightarrow \quad 1 = \frac{1}{1 - \tan x \tan y}$ [Using given equations]

$\Rightarrow \quad \tan x \tan y = 0$

$\Rightarrow$ Either $\tan x = 0$ or $\tan y = 0$

$\therefore$ Either $x = n\pi$ and $y = \frac{\pi}{4} - x = \frac{\pi}{4} - n\pi$

or $\quad y = n\pi$ and $x = \frac{\pi}{4} - n\pi$.

8. The equation $2\cos^2\left(\frac{x}{2}\right) . \sin^2 x = x^2 + \frac{1}{x^2}, 0 \le x \le \frac{\pi}{2}$ has

(a) one real solution
(b) no solution
(c) more than one real solution
(d) none of these

Solution (b) Since $x^2 + x^{-2} = (x - x^{-1})^2 + 2 \ge 2$

and $\quad 2\cos^2\frac{x}{2}\sin^2 x \le 2$,

$\therefore$ the given equation is valid only if

$$2\cos^2\frac{x}{2}\sin^2 x = 2$$

$\Leftrightarrow \quad \cos\frac{x}{2} = \operatorname{cosec} x = 1$, which cannot be true.

9. The number of solutions of the equation $3\sin^2 x - 7\sin x + 2 = 0$, in the interval $[0, 5\pi]$, is

(a) 0 (b) 5

(c) 6 (d) 10

Solution (c) We have, $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \quad \sin x = \frac{1}{3}$ or $\sin x = 2$ (Not possible)

As $\quad \sin x = \frac{1}{3}$ is positive,

$\therefore$ no. of solutions between 0 and $\pi = 2$,
no. solutions between 2π and $3\pi = 2$,
no. of solutions between 4π and $5\pi = 2$.

$\therefore$ Total number of solutions = **6.**

10. The values of x, $0 \le x \le \frac{\pi}{2}$ which satisfy the equation

$81^{\sin^2 x} + 81^{\cos^2 x} = 30$ are

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{7\pi}{18}$

Solution (a), (c) Let $81^{\sin^2 x} = y$ then $81^{\cos^2 x} = 81^{(1-\sin^2 x)} = 81y^{-1}$,

so that the given equation can be written as

$$y^2 - 30y + 81 = 0 \Rightarrow y = 3 \text{ or } y = 27$$

$\Rightarrow \quad 81^{\sin^2 x} = 3$ or 27

$\Rightarrow \quad \sin x = 1/2$ or $\sqrt{3}/2$, [$\because 0 \le x \le \pi/2$]

so that $x = \pi/\mathbf{6}$ or $\pi/\mathbf{3}$.

11. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

(a) 0 (b) 1

(c) 2 (d) infinite

Solution (a) Since for any real value of x,

$$5^x + 5^{-x} \ge 2.$$

$\therefore \quad \sin(e^x) \ge 2$,

which is not possible for any real value of x. Thus, the given equation has **no solution.**

12. The general solution of the equation $\sin^{50} x - \cos^{50} x = 1$ is

(a) $2n\pi + \frac{\pi}{2}$ (b) $2n\pi + \frac{\pi}{3}$

(c) $n\pi + \frac{\pi}{2}$ (d) $n\pi + \frac{\pi}{3}$

Solution (c) We have, $\sin^{50} x - \cos^{50} x = 1$

$\Rightarrow \quad \sin^{50} x = 1 + \cos^{50} x$

Since $\sin^{50} x \le 1$ and $1 + \cos^{50} x \ge 1$, therefore, the two sides are equal only if

$\sin^{50} x = 1 = 1 + \cos^{50} x$ i.e., $\sin^{50} x = 1$ and $\cos^{50} x = 0$

$\therefore \quad x = \boldsymbol{n\pi} + \frac{\pi}{2}, n \in Z.$

13. If $\sin\theta = k$ for exactly one value of θ, $\theta \in / \left[0, \frac{7\pi}{3}\right]$, then the value of k is

(a) 1 (b) -1

(c) $1/\sqrt{2}$ (d) 0

Solution (a), (b) Clearly $-1 \le k \le 1$. For any value of k other than 1 and -1, $\sin\theta$ has two values (in quadrant, I II or III, IV).

$\therefore \quad k = \mathbf{1}$ or $\mathbf{-1}$.

14. If $0 \le x \le 2\pi$ and $|\cos x| \le \sin x$, then

(a) $x \in \left[0, \frac{\pi}{4}\right]$

(b) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

(c) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

(d) none of these

Solution (c) We have, $|\cos x| \le \sin x \Rightarrow \sin x \ge 0$

($\because |\cos x| \ge 0$)

$\Rightarrow x \notin (\pi, 2\pi)$

If $x = 2\pi$, $|\cos 2\pi| \le \sin 2\pi$ which is not possible,

∴ $x \in [0, \pi]$.

If $x \in \left[0, \frac{\pi}{2}\right]$, then $|\cos x| \le \sin x \Rightarrow \cos x \le \sin x$

$$\Rightarrow x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

If $x \in \left(\frac{\pi}{2}, \pi\right)$, then

$|\cos x| \le \sin x \Rightarrow -\cos x \le \sin x \Rightarrow \tan x \le -1$

(∵ $\cos x < 0$)

$\Rightarrow x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$. ∴ $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$.

15. The smallest positive angle satisfying the equation $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution (b) We have, $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$

$\Rightarrow 4\sin^2\theta - 8\cos\theta + 1 = 0$

$\Rightarrow 4(1 - \cos^2\theta) - 8\cos\theta + 1 = 0$

$\Rightarrow 4\cos^2\theta + 8\cos\theta - 5 = 0$

$\Rightarrow (2\cos\theta - 1)(2\cos\theta + 5) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ [∵ $2\cos\theta + 5 \ne 0$]

$\Rightarrow \theta = \frac{\pi}{3}$.

16. Number of ordered pairs (a, x) satisfying the equation $\sec^2(a + 2)x + a^2 - 1 = 0$; $-\pi < x < \pi$ is

(a) 2 (b) 1

(c) 3 (d) infinite

Solution (c) Given equation is

$$\sec^2(a + 2)x + a^2 - 1 = 0$$

$\Rightarrow \tan^2(a + 2)x + a^2 = 0$

$\Rightarrow \tan^2(a + 2)x = 0$ and $a = 0$

$\Rightarrow \tan^2 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, -\frac{\pi}{2}$

∴ $(0, 0)$, $(0, \pi/2)$, $(0, -\pi/2)$ are ordered pairs satisfying the equation.

17. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$ then $\sin 2\alpha$ is equal to

(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$

(c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

Solution (b) Since, α is a root of

$$25\cos^2\theta + 5\cos\theta - 12 = 0$$

∴ $$25\cos^2\alpha + 5\cos\alpha - 12 = 0$$

$\Rightarrow \cos\alpha = \frac{-5 \pm \sqrt{25 + 1200}}{50} \Rightarrow \cos\alpha = -\frac{4}{5}$

$\Rightarrow \sin 2\alpha = 2\sin\alpha\cos\alpha = -\frac{24}{25}$

18. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$, has a solution for

(a) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$ (b) all real values of a

(c) $|a| \le \frac{1}{\sqrt{2}}$ (d) $|a| \ge \frac{1}{\sqrt{2}}$

Solution (c) We have, $\sin^{-1}x = 2\sin^{-1}a$

Since $-\frac{\pi}{2} \le 2\sin^{-1}x \le \frac{\pi}{2}$

∴ $-\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4} \Rightarrow \sin\left(-\frac{\pi}{4}\right) \le a \le \sin\frac{\pi}{4}$

$\Rightarrow -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}} \Rightarrow |a| \le \frac{1}{\sqrt{2}}$

19. In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the equation $\log_{\sin\theta}(\cos 2\theta) = 2$ has

(a) no solution

(b) a unique solution

(c) two solutions

(d) infinitely many solutions

Solution (b) We have, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

∴ $-1 \le \sin\theta \le 1$, here $0 < \sin\theta < 1$.

Now, $\log_{\sin\theta}\cos 2\theta = 2$

$\Rightarrow \cos 2\theta = \sin^2\theta$

$\Rightarrow 1 - 2\sin^2\theta = \sin^2\theta$

$\Rightarrow 3\sin^2\theta = 1 \Rightarrow \sin^2\theta = \frac{1}{3}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$ {∵ $0 < \sin\theta < 1$}

The given equation has a **unique solution.**

20. The number of values of x in $[0, 2\pi]$ satisfying the equation $|\cos x - \sin x| \ge \sqrt{2}$, is

(a) 0 (b) 1

(c) 2 (d) 3

Solution (c) Given equation is $|\cos x - \sin x| \ge \sqrt{2}$.

Since $|\cos x - \sin x| \le \sqrt{1 + 1} = \sqrt{2}$,

∴ we must have $|\cos x - \sin x| = \sqrt{2}$

$\Rightarrow \left|\cos\left(x + \frac{\pi}{4}\right)\right| = 1 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 1, -1.$

∴ $x + \frac{\pi}{4} = 0, 2\pi, 4\pi, 6\pi, \ldots \pi, 3\pi, 5\pi, \ldots.$

∴ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

21. The equation $a\sin x + b\cos x = c$ where

$$|c| > \sqrt{a^2+b^2}\text{, has}$$

(a) one solution
(b) two solutions
(c) no solution
(d) infinite number of solutions

Solution (c) Given equation is $a\sin x + b\cos x = c$

$$\Rightarrow \frac{a\sin x}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\left[\text{dividing by } \sqrt{a^2+b^2}\right] \qquad \ldots(1)$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$, then $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$

∴ From (1), we get

$$\sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} > 1, \quad \left(\because |c| > \sqrt{a^2+b^2}\right)$$

which is impossible.

∴ The given equation has **no solution.**

22. The most general values of θ for which $\sin\theta - \cos\theta = \min\limits_{a\in R}(1, a^2 - 6a + 10)$ are given by

(a) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$ (b) $n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{4}$

(c) $2n\pi + \frac{\pi}{4}$ (d) none of these

Solution (b) We have,

$$\sin\theta - \cos\theta = \min_{a\in R}\{1, a^2 - 6a + 11\}.$$

Since $a^2 — 6a + 11 = (a-3)^2 + 2 > 2$ for all a

$$\therefore \quad \sin\theta - \cos\theta = 1 \quad \Rightarrow \sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}$$

$$\therefore \quad \theta = \boldsymbol{n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{4}}\text{, where } n \in Z.$$

23. The equation $\cos 2x + a\sin x = 2a - 7$ has a solution for

(a) all a (b) $a > 6$
(c) $a < 2$ (d) $a \in [2, 6]$

Solution (d) Given equation is $\cos 2x + a\sin x = 2a - 7$

$$\Rightarrow \quad 1 - 2\sin^2 x + a\sin x = 2a - 7$$

$$\Rightarrow \quad 2\sin^2 x - a\sin x + 2a - 8 = 0$$

$$\Rightarrow \quad \sin x = \frac{a \pm \sqrt{a^2 - 16(a-4)}}{4} = \frac{a \pm (a-8)}{4}$$

$$\Rightarrow \quad \sin x = \frac{a-4}{2} \quad \text{or} \quad \sin x = 2 \text{ (not possible)}$$

$$\therefore \quad \sin x = \frac{a-4}{2}.$$

$$\because \quad -1 \le \sin x \le 1, \therefore -1 \le \frac{a-4}{2} \le 1 \Rightarrow \mathbf{2 \text{ £ } \boldsymbol{a} \text{ £ } 6.}$$

24. The least difference between the roots, in the first quadrant $\left(0 \le x \le \frac{\pi}{2}\right)$, of the equation

$$4\cos x\,(2 - 3\sin^2 x) + (\cos 2x + 1) = 0\text{, is}$$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution (a) We have,

$$4\cos x\,(2 - 3\sin^2 x) + (\cos 2x + 1) = 0$$

$$\Rightarrow \quad 4\cos x\,(3\cos^2 x - 1) + 2\cos^2 x = 0$$

$$\Rightarrow \quad 2\cos x\,(6\cos^2 x + \cos x - 2) = 0$$

$$\Rightarrow \quad 2\cos x\,(3\cos x + 2)(2\cos x - 1) = 0$$

⇒ either $\cos x = 0$ which gives $x = \pi/2$

or $\cos x = -2/3$,

which gives no value of x for which $0 \le x \le \pi/2$

or $\cos x = 1/2$, which gives $x = \pi/3$.

So, the required difference = $\pi/2 - \pi/3 = \boldsymbol{\pi/6}$**.**

25. Let α, β be any two positive values of x for which $2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in G. P. .The minimum value of $|\alpha - \beta|$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) none of these

Solution (d) Since

$2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in G.P.

$$\therefore \quad \cos^2 x = 2\cos x\,(1 - 3\cos^2 x)$$

$$\Rightarrow \quad 6\cos^3 x + \cos^2 x - 2\cos x = 0$$

$$\Rightarrow \quad \cos x\,(2\cos x - 1)(3\cos x + 2) = 0$$

$$\Rightarrow \quad \cos x = 0, \frac{1}{2}, -\frac{2}{3} \quad \therefore \quad x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$$

$$\therefore \quad \text{If } \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}, \quad (\because \alpha, \beta \text{ are are positive})$$

then $|\alpha - \beta| = \frac{\pi}{6}$.

Problems for Practice

Level of Difficulty I

1. The solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$ lies in the interval

   (a) $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ (b) $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

   (c) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (d) $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

2. If $4\sin^2\theta = 1$, then the values of θ are

   (a) $2n\pi \pm \frac{\pi}{3}$ (b) $n\pi \pm \frac{\pi}{3}$

   (c) $n\pi \pm \frac{\pi}{6}$ (d) $2n\pi \pm \frac{\pi}{6}$

3. $\cot\theta = \sin 2\theta$ ($\theta \neq n\pi$, n integer), if θ equals

   (a) 45° or 90° (b) 45° or 60°

   (c) 90° only (c) 45° only

4. If $\cot\theta - \tan\theta = \sec\theta$, then θ is equal to

   (a) $2n\pi + \frac{3\pi}{2}$ (b) $n\pi + (-1)^n \frac{\pi}{6}$

   (c) $n\pi + \frac{\pi}{2}$ (d) none of these

5. The most general value of θ which satisfies both the equations

   $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$, is

   (a) $2n\pi + \frac{\pi}{6}$ (b) $2n\pi + \frac{7\pi}{6}$

   (c) $2n\pi + \frac{11\pi}{6}$ (d) none of these

6. The values of θ satisfying $\sin^2\theta - \cos\theta = \frac{1}{4}$ in the interval $0 \leq \theta \leq 2\pi$, are

   (a) $\frac{\pi}{3}, \frac{4\pi}{3}$ (b) $\frac{\pi}{3}, \frac{7\pi}{3}$

   (c) $\frac{\pi}{3}, \frac{5\pi}{3}$ (d) $\frac{\pi}{6}, \frac{5\pi}{6}$

7. If $4\cos^2\theta + \sqrt{3} = 2(\sqrt{3}+1)\cos\theta$, then $\theta =$

   (a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{4}$

   (c) $2n\pi \pm \frac{\pi}{6}$ (d) none of these

8. The general solution of the equation

   $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$ is

   (a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$

   (c) $n\pi + \frac{\pi}{3}$ (d) $n\pi - \frac{\pi}{3}$

9. The most general value of θ which satisfies both the equations

   $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$ is

   (a) $2n\pi + \frac{3\pi}{4}$ (b) $2n\pi + \frac{7\pi}{4}$

   (c) $2n\pi + \frac{5\pi}{4}$ (d) none of these

10. The value of θ satisfying $\cos\theta + \sqrt{3}\sin\theta = 2$ is

    (a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

11. The general value of θ, obtained from the equation $\cos 2\theta = \sin\alpha$, is

    (a) $\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

    (b) $\theta = \frac{n\pi + (-1)^n\pi}{2}$

    (c) $\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$

    (d) $2\theta = \frac{\pi}{2} - \alpha$

12. If $\cos p\theta = \cos q\theta$, $p \neq q$, then

    (a) $\theta = 2n\pi$ (b) $\theta = \frac{2n\pi}{p \pm q}$

    (c) $\theta = \frac{n\pi}{p+q}$ (d) none of these

13. Let $2\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radians). Then x lies in the interval

    (a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(-1, \frac{5\pi}{6}\right)$

    (c) $(-1, 2)$ (d) $\left(\frac{\pi}{6}, 2\right)$

14. The number of distinct values of θ satisfying $0 \leq \theta \leq \pi$ and satisfying the equation $\sin\theta + \sin 5\theta = \sin 3\theta$, is

    (a) 6 (b) 7 (c) 8 (d) 9

15. The number of solutions of the equation

    $\tan\theta = \tan 3\theta$, $\tan\theta \neq 0$, $\theta \in [-\pi, \pi]$, is

    (a) 0 (b) 1 (c) 2 (d) infinite

16. The general solution of the equation

    $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ is

    (a) $x = 2n\pi \pm \frac{7\pi}{6}$ (b) $x = 2n\pi \pm \frac{5\pi}{3}$

    (c) $x = 2n\pi \pm \frac{5\pi}{6}$ (d) none of these

17. If $\sin A = \sin B$ and $\cos A = \cos B$, then

    (a) $A = n\pi + B$ (b) $A = 2n\pi + B$

    (c) $A = 2n\pi - B$ (d) none of these

18. The general solution of the equation
$7\cos^2\theta + 3\sin^2\theta = 4$ is
(a) $\theta = 2n\pi \pm \frac{\pi}{3}$ (b) $\theta = 2n\pi \pm \frac{2\pi}{3}$
(c) $\theta = n\pi \pm \frac{\pi}{3}$ (d) none of these

19. The solution of the equation $\tan 2\theta \tan\theta = 1$ is
(a) $\theta = \frac{n\pi}{3} + \frac{\pi}{6}$ (b) $\theta = \frac{n\pi}{3} + \frac{\pi}{4}$
(c) $\theta = \frac{n\pi}{3} - \frac{\pi}{6}$ (d) none of these

20. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, in the variable x has a real root. Then p can take any value in the interval
(a) $(0, \pi)$ (b) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
(b) $(-\pi, 0)$ (c) $(0, 2\pi)$

Level of Difficulty II

21. The solution of the equation
$\sin 2x + \sin 4x = 2\sin 3x$ is
(a) $x = \frac{n\pi}{3}$ (b) $x = n\pi$
(c) $x = 2n\pi$ (d) none of these

22. The solution set of $2\sin^2\theta = 3\cos\theta$, in the interval $0 \le \theta \le 2\pi$, is
(a) $\left\{\frac{\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}(-2)\right\}$
(c) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (d) none of these

23. The general solution of the equation
$\sin x + \cos x = \sqrt{2}\cos A$ is
(a) $x = 2n\pi + \frac{\pi}{4} \pm A$ (b) $x = 2n\pi + \frac{\pi}{3} \pm A$
(c) $x = 2n\pi + \frac{\pi}{6} \pm A$ (d) none of these

24. The general solution of $4\sin^4 x + \cos^4 x = 1$ is
(a) $(2n + 1)\frac{\pi}{2}$ (b) $n\pi$
(c) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$ (d) none of these

25. General solution of the equation
$(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$ is
(a) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (b) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
(c) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

26. The quadratic equation $8\sec^2\theta - 6\sec\theta + 1 = 0$ has
(a) exactly two roots
(b) exactly four roots
(c) infinitely many roots
(d) no roots

27. For $m \ne n$, if $\tan m\theta = \tan n\theta$, then the different values of θ are in
(a) A. P. (b) H. P.
(c) G. P. (d) no particular sequence

28. The number of the solutions of the equation
$\cos\left(\pi\sqrt{x-4}\right)\cos(\pi\sqrt{x}) = 1$ is
(a) >2 (b) 2 (c) 1 (d) 0

29. The equation $\sin^4 x + \cos^4 x = a$ has a solution for
(a) all of values of a (b) $a = 1$
(c) $a = \frac{1}{2}$ (d) $\frac{1}{2} < a < 1$

30. The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$, where x and y are real, is
(a) $x = \frac{\pi}{3} - n\pi,\ y = n\pi$
(b) ϕ
(c) $x = n\pi, y = \frac{\pi}{3} - n\pi,$
(d) none of these

31. The equation $\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$ possesses a solution if
(a) $k > -3$ (b) $k < -2$
(c) $-3 \le k \le -2$ (d) k is any positive integer

32. Solution of the equation $4\cot 2\theta = \cot^2\theta - \tan^2\theta$ is
(a) $\theta = n\pi \pm \frac{\pi}{2}$ (b) $\theta = n\pi \pm \frac{\pi}{3}$
(c) $\theta = n\pi \pm \frac{\pi}{4}$ (d) none of these

33. The equation $\cos 2x + a\sin x = 2a - 7$ has a solution for
(a) all a (b) $a > 6$
(c) $a < 2$ (d) $a \in [2, 6]$

34. The set of all x in $(-\pi, \pi)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by
(a) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (b) $\left(-\frac{\pi}{10}, \pi\right)$
(c) $(-\pi, \pi)$ (d) $\left(-\pi, \frac{3\pi}{10}\right)$

35. The least positive non-integral solution of the equation $\sin\pi(x^2 + x) = \sin\pi x^2$ is
(a) rational
(b) irrational of the form $\sqrt{p}$
(c) irrational of the form $\frac{\sqrt{p} - 1}{4}$, where p is an odd integer
(d) irrational of the form $\frac{\sqrt{p} + 1}{4}$, where p is an even integer

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (a),(c)	**8.** (b),(c)	**9.** (c)	**10.** (d)
11. (a)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (b)	**18.** (a),(b)	**19.** (a)	**20.** (a)
21. (a),(c)	**22.** (c)	**23.** (a)	**24.** (b),(c)	**25.** (a)	**26.** (d)	**27.** (a)	**28.** (c)	**29.** (b,c,d)	**30.** (b)
31. (c)	**32.** (c)	**33.** (d)	**34.** (a)	**35.** (c)					

Inverse Trigonometric Functions

7

BRIEF REVIEW OF THE CONCEPTS

INVERSE FUNCTIONS

If $f: X \rightarrow Y$ is a function which is both one-one and onto, then its inverse function $f^{-1}: Y \rightarrow X$ is defined as:

$$y = f(x) \Leftrightarrow f^{-1}(y) = x, \forall x \in X, \forall y \in Y.$$

INVERSE TRIGONOMETRIC FUNCTIONS

Consider the sine function with domain R and range $[-1, 1]$. This function is many-one and onto. So, its inverse does not exist. If we restrict its domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1] \text{ given by } \sin\theta = x$$

is one-one and onto and therefore it is invertible.

The inverse of sine function is defined as

$$\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

such that $\sin^{-1}x = \theta \Leftrightarrow \sin\theta = x$.

Thus, if x is a real number between -1 and 1, then $\sin^{-1}x$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x, i.e.,

$$\sin^{-1}x = \theta \Leftrightarrow x = \sin\theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

and $-1 \leq x \leq 1$.

The least numerical value among all the values of the angle whose sine is x, is called the principal value of $\sin^{-1}x$.

Similar definitions for $\cos^{-1}x$, $\tan^{-1}x$ and so on can be given.

DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Function	*Domain*	*Range (Principal Value)*
1. $y = \sin^{-1}x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
2. $y = \cos^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
3. $y = \tan^{-1}x$	$(-\infty, \infty)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4. $y = \text{cosec}^{-1}x$	$x \geq 1$ or $x \leq -1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$
5. $y = \sec^{-1}x$	$x \geq 1$ or $x \leq -1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
6. $y = \cot^{-1}x$	$(-\infty, \infty)$	$0 < y < \pi$

KEY FORMULAE TO REMEMBER

1. **(a)** $\sin^{-1}(\sin\theta) = \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(b) $\cos^{-1}(\cos\theta) = \theta, 0 \leq \theta \leq \pi$

(c) $\tan^{-1}(\tan\theta) = \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(d) $\cot^{-1}(\cot\theta) = \theta, 0 < \theta < \pi$

(e) $\sec^{-1}(\sec\theta) = \theta, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$

(f) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$

2. **(a)** $\sin(\sin^{-1}x) = x, -1 \leq x \leq 1$

(b) $\cos(\cos^{-1}x) = x, -1 \leq x \leq 1$

(c) $\tan(\tan^{-1}x) = x, -\infty < x < \infty$

(d) $\cot(\cot^{-1}x) = x, -\infty < x < \infty$

(e) $\sec(\sec^{-1}x) = x, x \leq -1 \text{ or } x \geq 1$

(f) $\text{cosec}(\text{cosec}^{-1}x) = x, x \leq -1 \text{ or } x \geq 1$

3. **(a)** $\sin^{-1}(-x) = -\sin^{-1}x, -1 \leq x \leq 1$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, -1 \leq x \leq 1$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, -\infty < x < \infty$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x, -\infty < x < \infty$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, x \leq -1 \text{ or } x \geq 1$

(f) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, x \leq -1 \text{ or } x \geq 1.$

4. **(a)** $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}, x \leq -1 \text{ or } x \geq 1$

(b) $\sec^{-1}x = \cos^{-1}\frac{1}{x}, x \leq -1 \text{ or } x \geq 1$

(c) $\cot^{-1}x = \begin{cases} \tan^{-1}\dfrac{1}{x}, & x>0 \\ \pi+\tan^{-1}\dfrac{1}{x}, & x<0 \end{cases}$

5. **(a)** $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$, $-1 \le x \le 1$

(b) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}$, $x \le -1$ or $x \ge 1$

(c) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$, $-\infty < x < \infty$

6. **(a)** If $xy < 1$, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$

(b) If $xy > 1$, $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x>0,\ y>0 \\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x<0,\ y<0 \end{cases}$$

(c) If $xy = 1$, $\tan^{-1}x + \tan^{-1}y = \begin{cases} \dfrac{\pi}{2}, & x>0, y>0 \\ -\dfrac{\pi}{2}, & x<0,\ y<0 \end{cases}$

(d) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$

7. **(a)** If $x, y \ge 0$, $x^2 + y^2 \le 1$

$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(b) If $x, y \ge 0$, $x^2 + y^2 > 1$

$\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(c) If $x, y \ge 0$, $x^2 + y^2 \le 1$

$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \pm \sqrt{1-x^2}\sqrt{1-y^2})$

(d) If $x, y \ge 0$, $x^2 + y^2 > 1$

$\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

8. **(a)** $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$= \sec^{-1}\dfrac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\dfrac{1}{x}$

(b) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$= \sec^{-1}\dfrac{1}{x}$

$= \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

(c) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\dfrac{1}{x} = \sec^{-1}\sqrt{1+x^2}$

$= \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

9. **(a)** $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}$, $|x| \le 1$

(b) $2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$, $x \ge 0$

(c) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$, $|x| < 1$

(d) $\pi - 2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}$, $|x| > 1$

10. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-xz-yz}\right)$

Solved Problems

1. The value of $\sin[\cot^{-1}\{\tan(\cos^{-1}x)\}]$ is

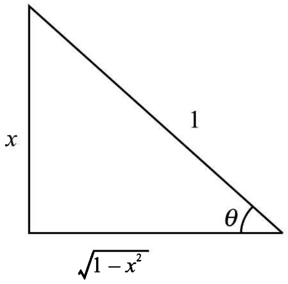
(a) $\dfrac{\sqrt{1-x^2}}{x}$ (b) $\dfrac{x}{\sqrt{1-x^2}}$

(c) x (d) none of these

Solution (c) Let $y = \sin[\cot^{-1}\{\tan(\cos^{-1}x)\}]$

$$= \sin\left[\cot^{-1}\left\{\tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right)\right\}\right]$$

$$\left(\because \cos^{-1}x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}\right)$$



$$= \sin\left[\cot^{-1}\frac{\sqrt{1-x^2}}{x}\right] \qquad \text{...(1)}$$

Let $\cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \theta$, then $\cot\theta = \dfrac{\sqrt{1-x^2}}{x} \Rightarrow \sin\theta = x$

$\therefore$ From (1), $y = \sin\theta = \boldsymbol{x}$.

2. $\cos^{-1}\dfrac{1}{2} + 2\sin^{-1}\dfrac{1}{2} =$

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{2\pi}{3}$

Solution (d) $\cos^{-1}\dfrac{1}{2} + 2\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{3} + 2\left(\dfrac{\pi}{6}\right) = \dfrac{2\pi}{3}$.

3. tan ($\cos^{-1}x$) is equal to

(a) $\pm\sqrt{\dfrac{1-x^2}{x}}$, $x \neq 0$ (b) $\dfrac{\sqrt{1+x^2}}{x}$, $x \neq 0$

(c) $\dfrac{x}{\sqrt{1+x^2}}$ (d) $\dfrac{\sqrt{1-x^2}}{x}$, $x \neq 0$

Solution (d) tan ($\cos^{-1}x$) = tan θ, where $\theta = \cos^{-1}x$

$$= \frac{\sin\theta}{\cos\theta} = \frac{\boldsymbol{\sqrt{1-x^2}}}{\boldsymbol{x}} \quad \boldsymbol{(x \neq 0)} \qquad (\because \cos\theta = x)$$

4. $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$ is equal to

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$

(c) $\dfrac{3\pi}{4}$ (d) 0

Solution (b) We have

$\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$

$$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} - \cos^{-1}x\right)\right] + \cos^{-1}\left[\sin\left(\frac{\pi}{2} - \sin^{-1}x\right)\right]$$

$= \sin^{-1}[\sin(\cos^{-1}x)] + \cos^{-1}[\cos(\sin^{-1}x)]$

$= \cos^{-1}x + \sin^{-1}x = \dfrac{\pi}{2}$.

5. If $\tan^{-1}\left(\dfrac{a}{x}\right) + \tan^{-1}\left(\dfrac{b}{x}\right) = \dfrac{\pi}{2}$, then $x =$

(a) $\sqrt{ab}$ (b) $\sqrt{2ab}$

(c) $2ab$ (d) ab

Solution (a) We have, $\tan^{-1}\left(\dfrac{a}{x}\right) + \tan^{-1}\left(\dfrac{b}{x}\right) = \dfrac{\pi}{2}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}}\right) = \frac{\pi}{2} \Rightarrow \frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}} = \tan\frac{\pi}{2} = \infty$$

$$\Rightarrow 1 - \frac{ab}{x^2} = 0 \Rightarrow x^2 = ab \Rightarrow x = \boldsymbol{\sqrt{ab}}.$$

6. If $\sin^{-1}\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{4} - \ldots\right) + \cos^{-1}\left(x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{4} - \ldots\right) = \dfrac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals

(a) $\dfrac{1}{2}$ (b) 1

(c) $-\dfrac{1}{2}$ (d) -1

Solution (b) Since $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ for $|x| \leq 1$.

$$\therefore \quad x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots$$

$$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}} \qquad (\because 0 < |x| < \sqrt{2})$$

$$\Rightarrow \frac{x}{2+x} = \frac{x^2}{2+x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3$$

$\Rightarrow x^2 = x \Rightarrow x = 0, 1.$

But $x \neq 0$, $\therefore$ $x = \mathbf{1}$.

7. If we consider only the principal values of the inverse trigonometric functions, then the value of $\tan\left(\cos^{-1}\dfrac{1}{5\sqrt{2}} - \sin^{-1}\dfrac{4}{\sqrt{17}}\right)$ is

(a) $\dfrac{\sqrt{29}}{3}$ (b) $\dfrac{29}{3}$

(c) $\dfrac{\sqrt{3}}{29}$ (d) $\dfrac{3}{29}$

Solution (d) $\tan\left(\cos^{-1}\dfrac{1}{5\sqrt{2}} - \sin^{-1}\dfrac{4}{\sqrt{17}}\right)$

$$= \tan\left[\tan^{-1}\left(\frac{7}{1}\right) - \tan^{-1}\left(\frac{4}{1}\right)\right]$$

$$= \tan(\tan^{-1}7 - \tan^{-1}4) = \tan\left(\tan^{-1}\frac{7-4}{1+7\times4}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{29}\right) = \boldsymbol{\frac{3}{29}}.$$

8. The solution set of the equation $\tan^{-1}x - \cot^{-1}x = \cos^{-1}(2-x)$ is

(a) [0, 1] (b) [– 1, 1]

(c) [1, 3] (d) none of these

Solution (c) $\tan^{-1}x$ and $\cot^{-1}x$ exist for all $x \in R$.

$\cos^{-1}(2-x)$ exists if $-1 \leq 2 - x \leq 1$ *i.e.*, $1 \leq x \leq 3$

So, the given equation holds for $\mathbf{1 \leq x \leq 3}$.

9. If $\sum_{i=1}^{20} \sin^{-1}x_i = 10\pi$ then $\sum_{i=1}^{20} x_i$ is equal to

(a) 20 (b) 10

(c) 0 (d) none of these

Solution (a) Since $-\dfrac{\pi}{2} \leq \sin^{-1}x_i \leq \dfrac{\pi}{2}$,

$\therefore \quad \sin^{-1} x_i = \dfrac{\pi}{2}, \ 1 \le i \le 20$

$\therefore \quad x_i = 1, \ 1 \le i \le 20$. Thus, $\sum_{i=1}^{20} x_i = \mathbf{20.}$

10. If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$, then $\sum_{i=1}^{2n} x_i$ is

(a) n (b) $2n$

(c) $\dfrac{n(n+1)}{2}$ (d) none of these

Solution (b) Since $0 \le \cos^{-1} x_i \le \pi$, $\therefore \ \cos^{-1} x_i = 0$ for all i.

$\therefore \quad x_i = 1$ for all i $\therefore \ \sum_{i=1}^{2n} x_i = \mathbf{2n.}$

11. If $\sin^{-1}\alpha + \sin^{-1}\beta + \sin^{-1}\gamma = \dfrac{3\pi}{2}$ then $\alpha\beta + \alpha\gamma + \beta\gamma$ is equal to

(a) 1 (b) 0

(c) 3 (d) -3

Solution (c) Since $-\dfrac{\pi}{2} \le \sin^{-1}x \le \dfrac{\pi}{2}$,

$\therefore \quad \sin^{-1}\alpha = \dfrac{\pi}{2}, \ \sin^{-1}\beta = \dfrac{\pi}{2}, \ \sin^{-1}\gamma = \dfrac{\pi}{2}$

$\therefore \quad \alpha = \beta = \gamma = 1$. Thus, $\alpha\beta + \alpha\gamma + \beta\gamma = \mathbf{3.}$

12. $\sec^{-1} (\sin x)$ is real if

(a) $x \in (-\infty, \infty)$ (b) $x \in [-1, 1]$

(c) $x = (2n + 1)\dfrac{\pi}{2}, \ n \in Z$

(d) $x = n\pi, \ n \in Z$

Solution (c) $\sec^{-1} (\sin x)$ is real if $\sin x \le -1$ or $\sin x \ge 1$. But $-1 \le \sin x \le 1$.

$\therefore \quad \sec^{-1} (\sin x)$ is real only if

$\sin x = -1$ or $1 \Rightarrow \boldsymbol{x = (2n + 1)\dfrac{\pi}{2}, \ n \in Z.}$

13. The value of $\sin\left[\sin^{-1}\dfrac{\sqrt{5}}{4} + \tan^{-1}\sqrt{\dfrac{5}{11}}\right]$ is

(a) $\dfrac{\sqrt{5}}{4\sqrt{11}}$ (b) $\dfrac{4}{\sqrt{35}}$

(c) $\dfrac{\sqrt{55}}{8}$ (d) none of these

Solution (c) We have, $\sin\left[\sin^{-1}\dfrac{\sqrt{5}}{4} + \tan^{-1}\sqrt{\dfrac{5}{11}}\right]$

$$= \sin\left[\sin^{-1}\frac{\sqrt{5}}{4} + \sin^{-1}\frac{\sqrt{5/11}}{\sqrt{1+\frac{5}{11}}}\right]$$

$$= \sin\left[\sin^{-1}\frac{\sqrt{5}}{4} + \sin^{-1}\frac{\sqrt{5}}{4}\right]$$

$$= \sin\left[2\sin^{-1}\frac{\sqrt{5}}{4}\right] = \sin\sin^{-1}\left[2\frac{\sqrt{5}}{4}\sqrt{1-\frac{5}{16}}\right] = \boldsymbol{\frac{\sqrt{55}}{8}}.$$

14. $\cos\left[\tan^{-1}\left[\sin(\cot^{-1}x)\right]\right] =$

(a) $\sqrt{\dfrac{x^2+2}{x^2+3}}$ (b) $\sqrt{\dfrac{x^2+2}{x^2+1}}$

(c) $\sqrt{\dfrac{x^2+1}{x^2+2}}$ (d) none of these

Solution (c) Let $\cot^{-1}x = \theta \Rightarrow \cot\theta = x$

$$\therefore \quad \sin(\cot^{-1}x) = \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}}$$

Thus, $\cos\left[\tan^{-1}\left[\sin(\cot^{-1}x)\right]\right]$

$$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \cos\phi$$

$$\left(\text{Put } \tan^{-1}\left\{\frac{1}{\sqrt{1+x^2}}\right\} = \phi \Rightarrow \tan\phi = \frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sec\phi} = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} = \boldsymbol{\sqrt{\frac{1+x^2}{2+x^2}}}.$$

15. If $\cos^{-1}\dfrac{3}{5} - \sin^{-1}\dfrac{4}{5} = \cos^{-1}x$, then $x =$

(a) 0 (b) 1

(c) -1 (d) none of these

Solution (b) We have, $\cos^{-1}\dfrac{3}{5} - \sin^{-1}\dfrac{4}{5} = \cos^{-1}x$

$$\Rightarrow \quad \sin^{-1}\left(\sqrt{1-\frac{9}{25}}\right) - \sin^{-1}\frac{4}{5} = \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{4}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1}x$$

$\Rightarrow \quad \cos^{-1}x = 0 \Rightarrow x = \cos 0 = 1 \ \therefore \ \boldsymbol{x = 1.}$

16. Solution of the equation $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\dfrac{1}{2}\right)$ is

(a) $x = \pm\dfrac{\sqrt{7}}{3}$ (b) $x = \pm\dfrac{\sqrt{5}}{3}$

(c) $x = \pm\dfrac{3\sqrt{5}}{2}$ (d) none of these

Solution (b) We have, $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\dfrac{1}{2}\right)$

$$\Rightarrow \quad \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin(\tan^{-1}2)$$

$$\Rightarrow \quad \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin\left(\sin^{-1}\frac{2}{\sqrt{1+4}}\right)$$

$$\Rightarrow \quad \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow 5\,(1 - x^2) = 4x^2$$

$$\Rightarrow \quad 9x^2 = 5, \therefore\ x = \pm\left(\frac{\sqrt{5}}{3}\right).$$

17. $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) =$

(a) 1 (b) 7

(c) 4 (d) none of these

Solution (b)

18. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x =$

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$

(c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

Solution (a)

19. If $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}$, $x \in N$, then the maximum value of x is

(a) 2 (b) 5

(c) 7 (d) none of these

Solution (a)

20. Let $f(x) = \cot^{-1}x + \operatorname{cosec}^{-1}x$. Then $f(x)$ is real for

(a) $x \in [-1, 1]$ (b) $x \in (-\infty, -1] \cup [1, \infty)$

(c) $x \in (-\infty, \infty)$ (d) none of these

Solution (b)

21. $\tan^{-1}x > \cot^{-1}x$ holds for

(a) $x > 1$ (b) $x < 1$

(c) $x = 1$ (d) all values of x

Solution (a)

22. If $a \le \tan^{-1}x + \cot^{-1}x + \sin^{-1}x \le b$, then

(a) $a = 0$ (b) $b = \frac{\pi}{2}$

(c) $a = \frac{\pi}{4}$ (d) $b = \pi$

Solution (a), (d)

23. If $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$, then x is equal to

(a) 1 (b) -1

(c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

Solution (c), (d)

24. $\tan^{-1}n + \cot^{-1}(n + 1) =$

(a) $\cot^{-1}(n^2 + n + 1)$ (b) $\cot^{-1}(n^2 - n + 1)$

(c) $\tan^{-1}(n^2 + n + 1)$ (d) none of these

Solution (c)

25. If $\tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(-\frac{\pi}{3} + \theta\right) = a\tan 3\theta$, then $a =$

(a) $\frac{1}{3}$ (b) 1

(c) 3 (d) none of these

Solution (c)

Problems for Practice

Level of Difficulty I

1. $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$ is equal to

(a) $\frac{6}{25}$ (b) $\frac{24}{25}$ (c) $\frac{4}{5}$ (d) $-\frac{24}{25}$

2. The principal value of $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) $\frac{4\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $-\frac{5\pi}{3}$ (d) $\frac{5\pi}{3}$

3. The value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) 0

4. The value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

5. The value of $\sin\left[\text{arc}\cos\left(-\frac{1}{2}\right)\right]$ is

(a) $\frac{1}{\sqrt{2}}$ (b) 1

(c) $\frac{\sqrt{3}}{2}$ (d) none of these

6. The value of $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$ is

(a) 1 (b) -1 (c) 0 (d) none of these

7. The value of tan $\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$ is

(a) $\frac{7}{17}$ (b) $-\frac{7}{17}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

8. $\cot^{-1} 9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4}$ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) none of these

9. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right) =$

(a) 1 (b) 7 (c) 4 (d) none of these

10. If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$, then

(a) $x = \frac{a+b}{1-ab}$ (b) $x = \frac{a-b}{1+ab}$

(c) $x = \frac{b-a}{1+ab}$ (d) none of these

11. Solution of the equation
$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$ is

(a) $x = 0$ (b) $x = \pm\frac{1}{2}$

(c) $x = \pm\frac{1}{3}$ (d) none of these

12. If $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$, then $x =$

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b-a}{1+ba}$

(c) $\frac{a+b}{1-ab}$ (d) none of these

13. $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8} =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

14. The principal value of $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) none of these

15. $\tan(\cos^{-1}x)$ is equal to

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{\sqrt{1+x^2}}$

(c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\frac{x}{\sqrt{1-x^2}}$

16. $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right] =$

(a) $\frac{13}{6}$ (b) $\frac{17}{6}$ (c) $-\frac{13}{6}$ (d) $-\frac{17}{6}$

17. The value of $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{5}$ (d) none of these

18. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

19. $\sin\left[\cot^{-1}\{\cos(\tan^{-1}x)\}\right] =$

(a) $\sqrt{\frac{1+x^2}{2+x^2}}$ (b) $\sqrt{\frac{1-x^2}{2+x^2}}$

(c) $\sqrt{\frac{1+x^2}{2-x^2}}$ (d) $\sqrt{\frac{2+x^2}{1+x^2}}$

20. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then $x =$

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

Level of Difficulty II

21. Solution of the equation $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$ is

(a) $x = 3$ (b) $x = \frac{1}{\sqrt{5}}$

(c) $x = 0$ (d) none of these

22. Solution of the equation $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$ is

(a) $x = 2n\pi + \frac{\pi}{4}$ (b) $x = n\pi + \frac{\pi}{4}$

(c) $x = n\pi + \frac{\pi}{2}$ (d) none of these

23. If $x = \tan^{-1}\frac{1}{7}$ and $y = \tan^{-1}\frac{1}{3}$, then

(a) $\cos 2x = \sin 4y$
(b) $\cos 4y = \cos 2x$
(c) $\cos 2y = \sin 4x$
(d) none of these

24. The value of $\cot^{-1} 3 + \sec^{-1}\frac{\sqrt{5}}{2}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none of these

25. The value of $\sin^2\left(\cos^{-1}\frac{1}{2}\right)+\cos^2\left(\sin^{-1}\frac{1}{3}\right)$ is

(a) $\frac{17}{36}$ (b) $\frac{59}{36}$

(c) $\frac{36}{59}$ (d) none of these

ANSWERS

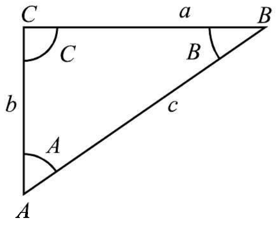
1. (d)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (a),(b)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (c),(d)
21. (a)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (b)					

8 Properties and Solutions of Triangles

BRIEF REVIEW OF THE CONCEPTS

PROPERTIES OF TRIANGLES

Let ABC be a triangle. By the angle A we shall mean that angle between the sides AB and AC which lies between 0° and 180°. The angles B and C are similarly defined. Thus we shall always have $A > 0, B > 0, C > 0$ and $A + B + C = 180°$. The sides BC, CA, AB opposite to the vertices A, B, C will be denoted by a, b, c respectively.



Thus, we shall have

$$0 < a < b + c,$$
$$0 < b < c + a,$$
$$0 < c < a + b.$$

The length of the radius of the circumscribed circle (or the circum-circle) will be denoted by R. The area of the triangle will be denoted by Δ and the semi-perimeter by s, so that $2s = a + b + c$.

KEY FORMULAE TO REMEMBER

1. **Law of Sines or Sine Rule** The sides of a triangle are proportional to the sines of the opposite angles. That is, in a ΔABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

2. **Law of Cosines or Cosine Formula** In any ΔABC,

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. **Projection Formula** In any ΔABC,

(a) $a = b\cos C + c\cos B$
(b) $b = a\cos C + c\cos A$
(c) $c = a\cos B + b\cos A$

4. **Law of Tangents or Tangent Rule (Napier's Analogy)** In any ΔABC,

(a) $\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2}$

(b) $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2}$

(c) $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}.$

5. **Half Angle Formulae or Semi-sum Formulae** In any triangle ABC, if $a + b + c = 2s$, then

(a) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ac}}$

$\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(b) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$

$\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(c) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$

$\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

6. **Area of a Triangle** The area of any triangle ABC is given by

$$\Delta = \frac{1}{2}\,bc\sin A = \frac{1}{2}\,ca\sin B = \frac{1}{2}\,ab\sin C$$

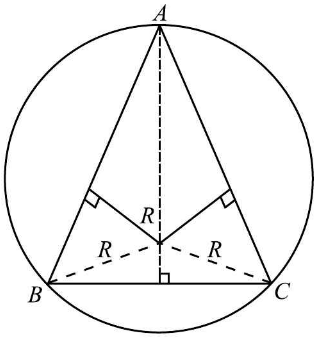
7. **Heron's Formula** In a ΔABC, if $a + b + c = 2s$, then its area is given by,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

CIRCLES CONNECTED WITH A GIVEN TRIANGLE

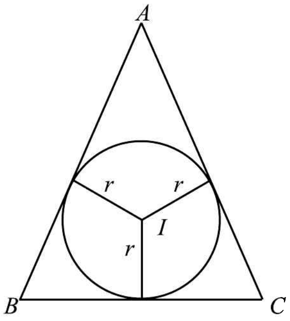
Circumcircle of a Triangle

The circle which passes through three vertices of a triangle is called the **circumcircle** of the triangle.



The centre of the circle is called **circum-centre**, usually denoted as O and its radius is called **circumradius**, usually denoted by R.

The circumcentre is the point of intersection of right bisectors of the sides of a triangle.



In any triangle ABC,

1. $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$
2. $R = \dfrac{abc}{4\Delta}$

Incircle of a Triangle

The circle which is drawn within a triangle such that the three sides touch this circle is called the **incircle**. The centre of this circle is called **incentre**, usually denoted by I, and its radius is called **inradius**, usually denoted by r.

The incentre is the point of concurrence of the bisectors of the three (internal) angles of the triangle.

In any triangle ABC,

1. $r = \dfrac{\Delta}{s}$
2. $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$
3. $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}},\ r = \dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}},\ r = \dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$
4. $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

Escribed Circles of a Triangle

The circle which lies outside the triangle and touches the side BC and also the sides AB and AC produced is called **escribed circle** or **excircle** opposite to angle A. Its centre is called **excentre** and its radius is called **exradius**. Similarly, there are two more excircles, one opposite to angle B and one opposite to angle C. The three excentres are usually denoted as I_1, I_2, I_3 and the three ex-radii are usually denoted as r_1, r_2, r_3.

Excentre is the point of concurrence of internal bisector of angle A and external bisectors of angles B and C.

Key Points to Remember

In any ΔABC,

1. $r_1 = \dfrac{\Delta}{s-a},\ r_2 = \dfrac{\Delta}{s-b},\ r_3 = \dfrac{\Delta}{s-c}$
2. $r_1 = s\tan\dfrac{A}{2},\ r_2 = s\tan\dfrac{B}{2},\ r_3 = s\tan\dfrac{C}{2}$
3. $r_1 = a\dfrac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}},\ r_2 = b\dfrac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}},$

   $r_3 = c\dfrac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$
4. $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2},$

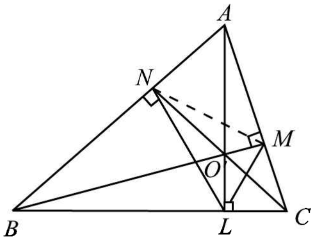
   $r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2},$

   $r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}.$

Orthocentre of a Triangle

Let the perpendiculars AL, BM and CN from the vertices A, B and C on the opposite sides BC, CA and AB of ΔABC, respectively, meet at O'. Then O' is the orthocentre of the ΔABC.

The triangle LMN is called the **pedal triangle** of the ΔABC.



THE DISTANCES OF THE ORTHOCENTRE FROM THE VERTICES

The distances of the orthocentre of the triangle from the vertices are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

Key Points to Remember

1. Circumradius of the pedal triangle = $\frac{R}{2}$.
2. Area of the pedal triangle = 2 $\Delta\cos A \cos B \cos C$.
3. Lengths of the medians AL, BM and CN of ΔABC are given by

$$AL = \frac{1}{2}\sqrt{b^2+c^2+2bc\cos A} = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

$$BM = \frac{1}{2}\sqrt{c^2+a^2+2ac\cos B} = \frac{1}{2}\sqrt{2c^2+2a^2-b^2}$$

$$CN = \frac{1}{2}\sqrt{a^2+b^2+2ab\cos C} = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

4. Circumcentre, Centroid and orthocentre are collinear and G divides OO' in the ratio 1 : 2.
5. Distance between the circumcentre O and the incentre I is

$$OI = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

6. $OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

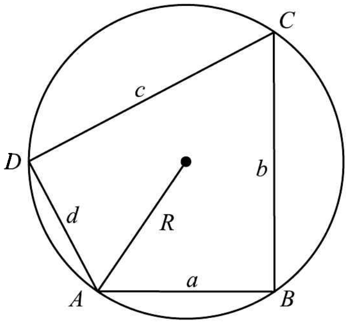
$$OI_2 = R\sqrt{1+8\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$$

$$OI_3 = R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$$

where I_1, I_2, I_3 are the centres of the escribed circles opposite to the angles A, B and C respectively and O is the circumcentre.

CYCLIC QUADRILATERAL

A quadrilateral is a *cyclic quadrilateral* if its vertices lie on a circle.



Key Points to Remember

1. Area of cyclic quadrilateral $ABCD$ is
$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$
where $2s = a + b + c + d$.
2. Circumradius of cyclic quadrilateral
$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ad+bc)(ac+bd)}{(s-a)(s-b)(s-c)(s-d)}}$$
3. $\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$ and similarly other angles.
4. **Ptolemy's Theorem** If $ABCD$ is a cyclic quadrilateral, then
$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

i.e., in a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the lengths of opposite sides.

REGULAR POLYGON

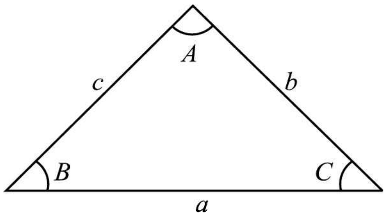
A polygon is called a *regular polygon* if all its sides are equal and its angles are equal.

Key Points to Remember

1. If a polygon has 'n' sides, sum of its internal angles is $(n-2)$ π and each angle is $(n-2)\frac{\pi}{n}$.
2. In a regular polygon the centroid, the circumcentre and the incentre are same.
3. Area of a regular polygon = $\frac{na^2}{4}\cot\frac{\pi}{n} = \frac{nR^2}{2}\sin\frac{2\pi}{n}$ $= nr^2\tan\frac{\pi}{n}$, where a is length of side, n is number of sides of polygon, R is radius of circumscribing circle and r is radius of incircle of the polygon.
4. **Radius of circumscribing circle**
$$R = \frac{a}{2\sin\frac{\pi}{n}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n},$$
where a is the length of each side of regular polygon of n sides.
5. **Radius of inscribed circle**
$$r = \frac{a}{2}\cot\frac{\pi}{n}$$

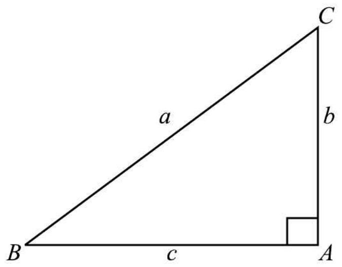
SOLUTION OF TRIANGLES

A triangle has six elements: the three sides a, b, c and the three angles A, B, C. If three of the elements are given, atleast one of which must be a side, then the other three elements can be uniquely determined. The process of finding the remaining three elements of the triangle is called the *solution of the triangle*.



Solution of a Right Angled Triangle

It is possible to find the remaining parts of a right triangle if in addition to the right angle, one side and any other part (side or angle) are known.



1. If $A = 90°$, side b and angle B are given, then
$$\frac{b}{\sin B} = \frac{a}{\sin 90°} \Rightarrow a = \frac{b}{\sin B}$$
Also, $\frac{c}{b} = \cot B$, so that $c = b \cot B$.
2. If the sides b and c are given, then $a = \sqrt{b^2+c^2}$,
$$\tan B = \frac{b}{c} \text{ and } C = 90 - B.$$

Solution of an Oblique Triangle

A triangle which is not a right angled triangle is called *oblique triangle*. Following cases are possible in solving oblique triangles.

CASE 1: When the three sides are given.

In this case only one triangle is possible and so angles can be determined uniquely. The following formulae are generally used :

1. $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$
2. $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$
3. $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
4. $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
5. $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$
6. $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$
7. $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$
8. $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$
9. $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
10. $\cos A = \frac{b^2+c^2-a^2}{2bc}$
11. $\cos B = \frac{a^2+c^2-b^2}{2ac}$
12. $\cos C = \frac{a^2+b^2-c^2}{2ab}$

CASE 2: When two sides and the included angle between them are given.

In this case only one triangle is possible and so remaining two angles and one side can be uniquely determined. Let b, c be the two given sides and A the given included angle.

1. To find angles B and C, proceed as follows:
$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{b-c}{(b+c)\tan\frac{A}{2}}$$
or $\log \tan \frac{B-C}{2} = \log (b-c) - \log (b+c) - \log \frac{A}{2}$.

Since b, c and A are given, the value of $\log \tan \frac{B-C}{2}$ can be found. Again $\frac{B-C}{2} \left(= 90° - \frac{A}{2}\right)$ is known as A is given.

By adding and subtracting, we obtain values of B and C.
2. To find side a, we use the sine formula:
$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } a = \frac{b\sin A}{\sin B}$$
or $\log a = \log b + \log \sin A - \log \sin B$

which gives the value of a.

Here it is assumed that $b > c$. But if $c > b$, the Napier's analogy, $\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2}$ can be applied.

CASE 3: When two angles and one side are given.

In this case only one triangle is possible and so remaining one angle and two sides can be uniquely determined. Let one side a and two angles B and C of ΔABC are given. Then third angle A is given by
$$A = 180° - (B + C).$$
The other sides can be found out by using sine formula.
$$b = a\,\frac{\sin B}{\sin A} \text{ and } c = a\,\frac{\sin C}{\sin A}.$$

CASE 4: When three angles are given.

In this case sides cannot be determined uniquely. Only ratio of sides can be determined by sine rule and hence there will be infinite number of such triangles.

CASE 5: When two sides and the angle opposite to one of them are given.

In this case either no triangle or one triangle or two triangles are possible. For this reason, it is called the **ambiguous case**.

Let b, c and B be given.

The following possibilities are there:

1. when B is acute and $b < c \sin B$, no triangle is possible.
2. When B is acute and $b = c \sin B$, then only one triangle is possible which is right angled.
3. When B is acute and $b > c \sin B$, two triangles are possible if $b < c$ and only one triangle if $b \geq c$.
4. When B is obtuse, there is no triangle, if $b < c$ and only one triangle if $b > c$.

Key Points to Remember

1. The mid point of the hypotenuse of a right angled triangle is equidistant from the 3 vertices of the triangle.
2. In a right angled triangle, the orthocentre coincides with the vertex containing the right angle.
3. In a right angled triangle, the mid point of the hypotenuse is the circumcentre of the triangle.

Solved Problems

1. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$ then $n =$

(a) 5 (b) 7

(c) 6 (d) 4

Solution (b) $T_n = {}^nC_3 \quad \therefore \quad T_{n+1} - T_n = 21$

$\Rightarrow \quad {}^{n+1}C_3 - {}^nC_3 = 21 \quad \Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 21$

$\Rightarrow \quad {}^nC_2 = 21 \Rightarrow \dfrac{n(n-1)}{2} = 21 \Rightarrow n^2 - n - 42 = 0$

$\Rightarrow \quad (n-7)(n+6) = 0 \Rightarrow n = 7$ as $n \geq 1$

2. In a triangle ABC, $2ca\sin\dfrac{A-B+C}{2}$ is equal to

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

Solution (b) We have $A + C = \pi - B$

and $\dfrac{A-B+C}{2} = \dfrac{\pi}{2} - B$

$\therefore$ Given expression $2ac\cos B = 2ca\dfrac{c^2+a^2-b^2}{2ca}$

$= a^2 + c^2 - b^2$

3. In a trangle ABC, $a = 4$, $b = 3$, $\angle A = 60°$, then c is the root of the equation

(a) $c^2 - 3c - 7 = 0$ (c) $c^2 + 3c + 7 = 0$

(c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

Solution (a) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\Rightarrow \quad \cos 60° = \dfrac{9+c^2-16}{6c} \Rightarrow c^2 - 3c - 7 = 0$

4. In a ΔABC, $\tan\dfrac{A}{2} = \dfrac{5}{6}$, $\tan\dfrac{C}{2} = \dfrac{2}{5}$, then

(a) a, c, b are in A. P.

(b) a, b, c are in A. P.

(c) b, a, c are in A. P.

(d) a, b, c are in A. P.

Solution (b) $\tan\dfrac{A}{2} = \dfrac{5}{6}$, $\tan\dfrac{C}{2} = \dfrac{2}{5}$

$\Rightarrow \quad \tan\dfrac{A}{2}\tan\dfrac{C}{2} = \dfrac{1}{3}$

$\Rightarrow \quad \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)} \cdot \dfrac{(s-b)(s-a)}{s(s-c)}}$

$\Rightarrow \quad \dfrac{s-b}{s} = \dfrac{1}{3} \Rightarrow 3s - 3b = s \Rightarrow 2s = 3b$

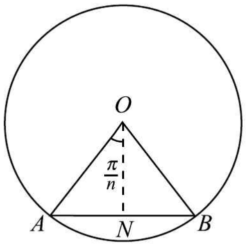
$\Rightarrow 2b = c + a \Rightarrow a, b, c$ are in A. P.

5. The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side a, is

(a) $a\cot\left(\dfrac{\pi}{n}\right)$ (b) $\dfrac{a}{2}\cot\left(\dfrac{\pi}{2n}\right)$

(c) $a\cot\left(\dfrac{\pi}{2n}\right)$ (d) $\dfrac{a}{4}\cot\left(\dfrac{\pi}{2n}\right)$

Solution (b) $AB = a$



$ON \perp AB$ and $AN = BN$

In ΔAON, $\tan\dfrac{\pi}{n} = \dfrac{AN}{ON}$

$\Rightarrow ON = AN\cot\dfrac{\pi}{n} = \dfrac{a}{2}\cot\dfrac{\pi}{n}$

and $\sin\dfrac{\pi}{n} = \dfrac{AN}{OA}$

$\Rightarrow \quad OA = AN\operatorname{cosec}\dfrac{\pi}{n} = \dfrac{a}{2}\operatorname{cosec}\dfrac{\pi}{n}$

Sum of the radii $= ON + OA$

$= \dfrac{a}{2}\cot\dfrac{\pi}{n} + \dfrac{a}{2}\operatorname{cosec}\dfrac{\pi}{n} = \dfrac{a}{2}\left[\dfrac{\cos\pi/n}{\sin\pi/n} + \dfrac{1}{\sin\pi/n}\right]$

$= \dfrac{a}{2}\left[\dfrac{1+\cos\dfrac{\pi}{n}}{\sin\dfrac{\pi}{n}}\right] = \dfrac{a}{2}\left[\dfrac{1+2\cos^2\dfrac{\pi}{2n}-1}{2\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n}}\right] = \dfrac{a}{2}\cot\dfrac{\pi}{2n}$

6. If in a triangle ABC

$$a\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$$

then the sides a, b and c

(a) are in A. P. (b) are in G. P.

(c) are in H. P. (d) satisfy $a + b = c$

Solution (a) $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$

$\Rightarrow \quad a\left[\dfrac{s(s-c)}{ab}\right] + c\left[\dfrac{s(s-a)}{bc}\right] = \dfrac{3b}{2}$

$\Rightarrow \quad \dfrac{s[s-c+s-a]}{b} = \dfrac{3b}{2}$

$\Rightarrow \quad 2s[2s - c - a] = 3b^2$

$\Rightarrow \quad 2s[a + b + c - c - a] = 3b^2$

$\Rightarrow \quad (a + b + c)b = 3b^2 \Rightarrow a + b + c = 3b$

$\Rightarrow \quad 2b = a + c$ i.e., a, b, c are in A. P.

7. In a triangle ABC, medians AD are BE are drawn. If $AD = 4$, $\angle DAB = \pi/6$ and $\angle ABE = \pi/3$, then the area of the ΔABC is

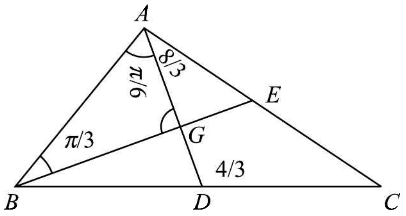
(a) $16/3\sqrt{3}$ (b) $32/3\sqrt{3}$

(c) $32/3$ (d) $64/3$

Solution (b) Given $AD = 4$ and $BD = DC$

In ΔABG, $\tan\frac{\pi}{3} = \frac{AG}{BG}$

$$\Rightarrow \quad BG = AG\cot\frac{\pi}{3} = \frac{8}{3}\times\frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}$$



Area of $\Delta ADB = \frac{1}{2}\times BG\times AD$

$$= \frac{1}{2}\times\frac{8}{3\sqrt{3}}\times 4 = \frac{16}{3\sqrt{3}}$$

Since median divides a triangle into two triangles of equal area. Therefore

Area of $\Delta ABC = 2 \times$ area of ΔADB

$$= 2\times\frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}}$$

8. The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0 < \alpha < \pi/2$. Then the greatest angle of the triangle is

(a) 90° (c) 60°

(c) 120° (c) 150°

Solution (c) Let $a = \sin\alpha$, $b = \cos\alpha$, $c = \sqrt{1+\sin\alpha\cos\alpha}$

then $\cos C = \frac{a^2+b^2-c^2}{2ab} = -\frac{\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha} = -\frac{1}{2}$

$\therefore \quad C = 120°$

9. In ΔABC, $\angle A = 45°$, $\angle C = 60°$, then $a + c\sqrt{2} =$

(a) b (b) $2b$

(c) $\sqrt{2}\,b$ (c) $\sqrt{3}\,b$

Solution (b) In any triangle $A + B + C = 180°$

$\because \quad A = 45°, C = 60°, 45° + B + 60° = 180°$

or $\quad \angle B = 75°$ ...(1)

$\because \quad a + c\sqrt{2} = k\sin A + \sqrt{2}\,k\sin C$

$= k(\sin 45° + \sqrt{2}\sin 60°)$

$$= k\left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}\sqrt{3}}{2}\right) = 2k\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$= 2k\sin 75° = 2k\sin B$ [using (1)]

$\therefore \quad a + c\sqrt{2} = \mathbf{2b.}$

10. If the angles of a triangle are in the ratio of 2 : 3 : 7, then the sides are in the ratio of

(a) $\sqrt{2} : 2 : (\sqrt{3}+1)$ (b) $2 : \sqrt{2} : (\sqrt{3}+1)$

(c) $\sqrt{2} : (\sqrt{3}+1) : 2$ (d) $2 : (\sqrt{3}+1) : \sqrt{2}$

Solution (a) Let a, b, c sides and A, B, C angles of triangle.

Given $A : B : C = 2 : 3 : 7$

$\therefore \quad A = 2x, B = 3x, C = 7x$

$A + B + C = 180°$ or $2x + 3x + 7x = 180°$

or $\quad x = 15°$ or $A = 30°, B = 45°, C = 105°$

By Sine rule

$a : b : c = \sin A : \sin B : \sin C$

$\therefore \quad a : b : c = \sin 30° : \sin 45° : \sin(60° + 45°)$

$$= \frac{1}{2} : \frac{1}{\sqrt{2}} : \left(\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)$$

$\therefore \quad a : b : c = \mathbf{\sqrt{2} : 2 : (\sqrt{3} + 1)}$

on multiplying by $2\sqrt{2}$.

11. In a ΔABC, $2ac\sin\frac{1}{2}(A - B + C) =$

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

Solution (b) We have, $A - B + C = 180° - 2B$

$$\therefore \quad 2ac\sin\left[\frac{1}{2}(A-B+C)\right] = 2ac\sin(90° - B)$$

$= 2ac\cos B$

$= \mathbf{c^2 + a^2 - b^2}$ [using law of cosines]

12. If the angles A, B, C of a triangle ABC are in $A.P.$ and the sides a, b and c opposite to these angles are in $G.P.$, then a^2, b^2, c^2 are in

(a) G. P. (b) A. P.

(c) H. P. (d) none of these

Solution (b) Given A, B, C are in $A.P.$

$\therefore \quad B = \frac{A+C}{2}$ or $A + C = 2B$ ...(1)

In ΔABC, $A + B + C = 180°$, $\therefore 3B = 180°$,

or $\quad B = 60°$

also, $\quad a$, b, c are in $G.P.$, $\therefore b^2 = ac$.

Using $\cos B = \frac{c^2+a^2-b^2}{2ac}$, we have

$\cos 60° = \frac{c^2+a^2-b^2}{2b^2}$ or $c^2 + a^2 = 2b^2$

$\therefore \quad a^2, b^2, c^2$ are in $\boldsymbol{A.P.}$

13. If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$ then its area is

(a) 8 sq. units (b) 9 sq. units

(c) 6 sq. units (d) none of these

Solution (b) $\cos(A - B) = \frac{4}{5} \Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

$$\therefore \quad \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

or $\quad \frac{1}{3} = \frac{6-3}{6+3}\cot\frac{C}{2}$ or $\cot\frac{C}{2} = 1$ or $C = \frac{\pi}{2}$

Hence area of $\Delta ABC = \frac{1}{2} \times 6 \times 3 = \mathbf{9}$ **sq. units.**

14. If r_1, r_2, r_3 in a triangle be in H.P., then the sides are in

(a) H. P. (b) A. P.

(c) G. P. (d) none of these

Solution (b) Since r_1, r_2, r_3 are in H.P.

$\therefore \quad \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P.

$\therefore \quad \frac{1}{r_2} - \frac{1}{r_1} = \frac{1}{r_3} - \frac{1}{r_2}$ where $r_1 = \frac{\Delta}{s-a}$ and so on

$\therefore \quad \frac{s-b}{\Delta} - \frac{s-a}{\Delta} = \frac{s-c}{\Delta} - \frac{s-b}{\Delta}$

$\Rightarrow \quad a - b = b - c \Rightarrow 2b = a + c$

$\Rightarrow \quad$ **a, b, c are in A. P.**

15. In a ΔABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$ and D divides BC internally in the ratio 1 : 3. Then, $\frac{\sin\angle BAD}{\sin\angle CAD}$ equals

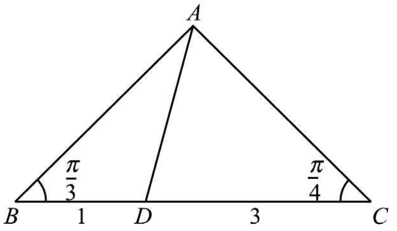
(a) $\frac{\sqrt{2}}{3}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{6}}$ (d) $\frac{1}{3}$

Solution (c) By sine formula, In ΔABC, we have

$$\frac{BD}{\sin\angle BAD} = \frac{AD}{\sin\frac{\pi}{3}} \qquad ...(1)$$

By sine formula, in ΔABD, we have



$$\frac{DC}{\sin\angle DAC} = \frac{AD}{\sin\frac{\pi}{4}} \qquad ...(2)$$

Divide (1) by (2), we get

$$\frac{BD}{DC}\cdot\frac{\sin\angle DAC}{\sin\angle BAD} = \frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{3}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{6}}$$

$$\therefore \quad \frac{\sin\angle BAD}{\sin\angle DAC} = \frac{\sqrt{6}}{2}\cdot\frac{1}{3} = \frac{\sqrt{6}}{2}\cdot\frac{1}{3} = \mathbf{\frac{1}{\sqrt{6}}}.$$

16. In an equilateral Δ, $r : R : r_1 =$

(a) 1 : 2 : 3 (b) 3 : 2 : 1

(c) 3 : 1 : 2 (d) none of these

Solution (a) $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$$= 4R\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{R}{2}$$

[Since $A = B = C = 60°$]

$$r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

$$= 4R\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3R}{2}$$

$\therefore \quad r : R = 1 : 2$ and $R : r_1 = 2 : 3$

$\therefore \quad r : R : r_1 = \mathbf{1 : 2 : 3.}$

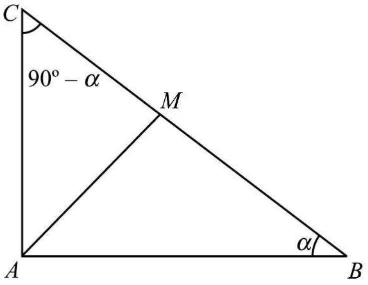
17. In a right angled triangle, the hypotenuse is four times as long as the perpendicular drawn to it from the opposite vertex. One of the acute angle is

(a) 45° (b) 30°

(c) 15° (d) none of these

Solution (c) $\frac{AM}{MB} = \tan\alpha \Rightarrow MB = AM\cot\alpha$

$$\frac{AM}{MC} = \cot\alpha \Rightarrow MC = AM\tan\alpha$$



$$(MB + MC) = AM[\cot\alpha + \tan\alpha]$$

i.e., $\quad BC = AM\left[\frac{1}{\tan\alpha} + \tan\alpha\right]$

i.e., $\quad 4AM = 2\,AM\left(\frac{1+\tan^2\alpha}{2\tan\alpha}\right)$

i.e., $\quad 2 = \frac{1}{\sin 2\alpha} \Rightarrow \sin 2\alpha = \frac{1}{2}$

$\Rightarrow \quad 2\alpha = 30° \Rightarrow \alpha = \mathbf{15°.}$

18. If H is the orthocentre of $\Delta\, ABC$, then AH is equal to

(a) $c\cot A$ (b) $b\cot A$

(c) $a\cot B$ (d) $a\cot A$

Solution (d) From ΔAHB, $\frac{AH}{\sin(90°-A)} = \frac{c}{\sin(A+B)}$

$$\Rightarrow \quad AH = \frac{c\cos A}{\sin(180°-C)} = \frac{c\cos A}{\sin C} = \frac{a}{\sin A}\cos A$$

$$\left[\text{since } \frac{a}{\sin A} = \frac{c}{\sin C}\right]$$

$$= \mathbf{a\cot A.}$$

19. If, in a $\Delta\, ABC$, $(a + b + c)(b + c - a) = \lambda bc$, then

(a) $\lambda < 0$ (b) $\lambda > 4$

(c) $\lambda > 0$ (d) $0 < \lambda < 4$

Solution (d) Given equation becomes $(b + c)^2 - a^2 = \lambda bc$

or $\quad b^2 + c^2 - a^2 = (\lambda - 2)\, bc$

or $$\frac{b^2+c^2-a^2}{2bc} = \frac{\lambda-2}{2} \Rightarrow \frac{\lambda-2}{2} = \cos A$$

But $-1 < \cos A < 1$

$\therefore \quad -1 < \frac{\lambda-2}{2} < 1 \Rightarrow \mathbf{0 < \lambda < 4.}$

20. If the sides of a right angled triangle are in *A.P.*, then tangents of the acute angles of the triangle are

(a) $\sqrt{\sqrt{3}+\frac{1}{2}}, \sqrt{\sqrt{3}-\frac{1}{2}}$

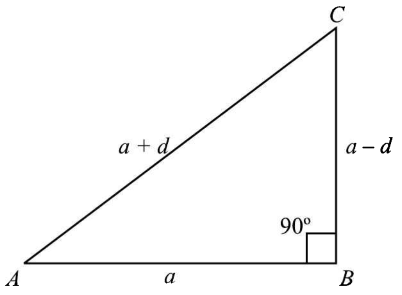
(b) $\sqrt{\sqrt{5}-\frac{1}{2}}, \sqrt{\sqrt{5}+\frac{1}{2}}$

(c) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (d) $\frac{3}{4}, \frac{4}{3}$

Solution (d) Let the sides of the given triangle be

$a-d, a, a+d$

Now $(a+d)^2 = (a-d)^2 + a^2$



or $4ad = a^2$ or $a = 4d$

Now $\tan A = \frac{a-d}{a} = \frac{4d-d}{4d} = \mathbf{\frac{3}{4}}$

and $\tan C = \frac{a}{a-d} = \frac{4d}{4d-d} = \mathbf{\frac{4}{3}}$.

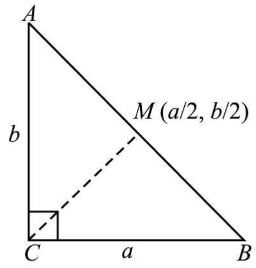
21. In a ΔABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then $2(r+R)$ is equal to

(a) $a+b$ (b) $b+c$

(c) $c+a$ (d) $a+b+c$

Solution (a) The hypotenuse of the right triangle *ABC* is *AB*. We take *C* at the origin and *CB* along *x*-axis and *CA* along *y*-axis. The mid-point $M\left(\frac{a}{2}, \frac{b}{2}\right)$ of *AB* is the circum-centre of the triangle.

Therefore,



$$R^2 = MC^2 = \frac{1}{4}(a^2+b^2) = \frac{1}{4}c^2 \Rightarrow R = \frac{c}{2}$$

Next, $r = (s-c)\tan\frac{C}{2} = (s-c)\tan\frac{\pi}{4} = s-c$

Thus, $2(r+R) = 2r + 2R = 2s - 2c + c$

$= a + b + c - 2c + c = \mathbf{a+b.}$

22. The angle C of the triangle ABC in which

$(c+a+b)(a+b-c) = ba$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Solution (a)

23. If in a triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then the sides a, b, c are in

(a) A. P. (b) G. P.

(c) H. P. (d) None of these

Solution (a)

24. If in a ΔABC, $2\cos A = \sin B \operatorname{cosec} C$, then

(a) $2a = bc$ (b) $c = a$

(c) $a = b$ (d) $b = c$

Solution (b)

25. If D is the mid-point of the side BC of a triangle ABC and AD is perpendicular to AC, then

(a) $b^2 = a^2 - c^2$ (b) $a^2 + b^2 = 5c^2$

(c) $3b^2 = a^2 - c^2$ (d) $3a^2 = b^2 - 3c^2$

Solution (c)

26. The area of the circle and the area of the regular polygon of n sides and of perimeter equal to that of circle are in the ratio of

(a) $\tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (b) $\cos\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

(c) $\sin\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (d) $\cot\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

Solution (a)

27. If the radius of the circumcircle of an isosceles ΔPQR is equal to PQ $(= PR)$, then angle P is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

Solution (d)

28. Let the angles A, B, C of ΔABC be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Then angle A is

(a) 75° (b) 45°

(c) 60° (d) none of these

Solution (a)

29. If the radius of the incircle of a triangle with its sides $5k$, $6k$ and $5k$ is 6, then k is equal to

(a) 3 (b) 4

(c) 5 (d) 6

Solution (b)

30. The perimeter of a ΔABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then the angle A is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) π

Solution (a)

Problems for Practice

Level of Difficulty I

1. If in a ΔABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c} = \frac{a}{bc}+\frac{b}{ca}$ then $\angle A =$

(a) 90° (b) 45° (c) 60° (d) none of these

2. In a ΔABC, $\operatorname{cosec}A(\sin B\cos C + \cos B\sin C) =$

(a) $\frac{c}{a}$ (b) $\frac{a}{c}$ (c) 1 (d) None of these

3. In a ΔABC, $a = 13$ cm, $b = 12$ cm and $c = 5$ cm. The distance of A from BC is

(a) $\frac{144}{13}$ (b) $\frac{65}{12}$ (c) $\frac{60}{13}$ (d) $\frac{25}{13}$

4. If the sides of a triangle are 13, 14, 15 then radius of its incircle is

(a) $\frac{65}{4}$ (b) $\frac{67}{8}$ (c) 24 (d) 4

5. If the lengths of the sides of a triangle are 3, 4 and 5 units then R is

(a) 3.5 (b) 3.0 (c) 2.0 (d) 2.5

6. ΔABC is right angled at C, then $\tan A + \tan B =$

(a) $\frac{b^2}{ac}$ (b) $a + b$ (c) $\frac{a^2}{bc}$ (d) $\frac{c^2}{ab}$

7. If R is the radius of the circumcircle of the ΔABC, and Δ is its area then

(a) $R = \frac{a+b+c}{\Delta}$ (b) $R = \frac{a+b+c}{4\Delta}$

(c) $R = \frac{abc}{4\Delta}$ (d) $R = \frac{abc}{\Delta}$

8. In a ΔABC, $\angle A = \frac{\pi}{2}$, then $\cos^2 B + \cos^2 C$ equals

(a) – 2 (b) – 1 (c) 1 (d) zero

9. In ΔABC, if $a = 16$, $b = 24$, $c = 20$, then $\sin\frac{A}{2} =$

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) none of these

10. In ΔABC, $a = 18$, $b = 24$, $c = 30$, then $\sin A =$

(a) $\frac{1}{5}$ (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) none of these

11. In ΔABC, if $a = 8$, $b = 15$, $c = 17$, then $\sin\frac{A}{2}$ and $\cos A$ are equal to

(a) $\frac{1}{\sqrt{17}}, \frac{15}{17}$ (b) $\frac{2}{\sqrt{17}}, \frac{13}{17}$

(c) $\frac{2}{\sqrt{17}}, \frac{11}{17}$ (d) none of these

12. In ΔABC, $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2} =$

(a) $3s$ (b) $2s$ (c) s (d) none of these

13. In any ΔABC, $\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} =$

(a) Δ (b) 2Δ (c) 3Δ (d) none of these

14. In any ΔABC, $2R^2\sin A\sin B\sin C =$

(a) 2Δ (b) 3Δ (c) Δ (d) none of these

15. In any ΔABC, $rr_1 + r_2r_3 =$

(a) ab (b) ac (c) bc (d) none of these

Level of Difficulty II

16. In a ΔABC, $\angle A > \angle B$. If the measure of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$, then the measure of angle C is

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

17. If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$ then its area is

(a) 8 sq. units (b) 9 sq. units

(c) 6 sq. units (d) none of these

18. In a triangle ABC, $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, how many such triangles are possible

(a) 1 (b) 0 (c) 2 (d) infinite

19. If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle on the opposite sides, then

$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} =$

(a) $\frac{1}{r}$ (b) $\frac{2}{r}$ (c) $\frac{3}{r}$ (d) none of these

20. In any ΔABC, if $a\cos A = b\cos B$, then triangle ABC is

(a) right angled (b) equilateral

(c) isosceles (d) none of these

21. If in ΔABC, $3\sin A = 6\sin B = 2\sqrt{3}\sin C$, then angle A is

(a) 0° (b) 30° (c) 60° (d) 90°

22. In a triangle, the lengths of two larger sides are 10 and 9 respectively. If the angles are in A.P., then the length of the third side can be

(a) $5-\sqrt{6}$ (b) $3\sqrt{3}$
(c) 5 (d) $5+\sqrt{6}$

23. In an equilateral Δ, $r : R : r_1$ =
(a) 1 : 2 : 3 (b) 3 : 2 : 1
(c) 3 : 1 : 2 (d) none of these

24. If the sides of a triangle are 13, 14, 15 then radius of its incircle is
(a) $\frac{65}{4}$ (b) $\frac{67}{8}$ (c) 24 (d) 4

25. In a ΔABC, if $3\tan\frac{A}{2}\tan\frac{C}{2}=1$, then sides a, b, c are in
(a) A. P. (b) G. P. (c) H. P. (d) none of these

26. In any ΔABC, $\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=$
(a) Δ (b) 2Δ (c) 3Δ (d) none of these

27. If Δ is the area of the ΔABC, s its semi-perimeter, then $abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=$
(a) $2s\Delta$ (b) $s\Delta$ (c) $3s\Delta$ (d) none of these

28. In any ΔABC, $2R^2\sin A\sin B\sin C$ =
(a) 2Δ (b) 3Δ (c) Δ (d) none of these

29. In a ΔABC, $\angle B=\frac{\pi}{3}$ and $\angle C=\frac{\pi}{4}$ and D divides BC internally in the ratio 1 : 3. Then, $\frac{\sin\angle BAD}{\sin\angle CAD}$ equals
(a) $\frac{\sqrt{2}}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{6}}$ (d) $\frac{1}{3}$

30. In a ΔABC, a = 13 cm, b = 12 cm and c = 5 cm. The distance of A from BC is
(a) $\frac{144}{13}$ (b) $\frac{65}{12}$ (c) $\frac{60}{13}$ (d) $\frac{25}{13}$

ANSWERS

1. (a)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (b)
11. (a)	**12.** (c)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (d)	**17.** (b)	**18.** (b)	**19.** (a)	**20.** (c)
21. (d)	**22.** (a),(d)	**23.** (c)	**24.** (b)	**25.** (d)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (c)

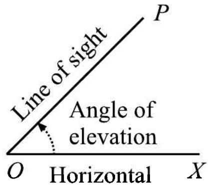
9

Heights and Distances

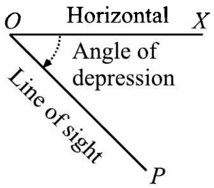
BRIEF REVIEW OF THE CONCEPTS

If *OX* be a horizontal line through *O*, the eye of the observer and *P* be an object in the vertical plane through *OX*, then $\angle XOP$ is called:

1. **The angle of elevation,** if *P* is *above OX* as in Fig (a); and



(a)



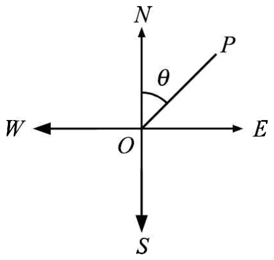
(b)

2. **The angle of depression,** if *P* is *below OX as in* Fig. (b).

The straight line *OP* (joining the eye of the observer to The object) is called the *line of sight* of the observer.

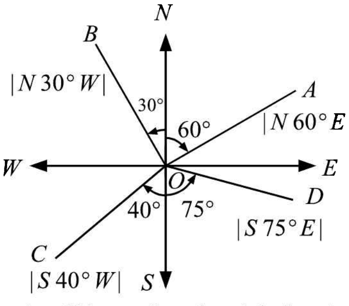
BEARINGS OF A POINT

Let *EW* be a line in the east-west direction and *NS* be a line perpendicular to it in the north-south direction. Let the two lines intersect at *O*.



Let *P* be any point. The acute angle which *OP* makes with *NS* is called the bearing of the point *P* from *O*. The bearing of a point is briefly indicated by giving the size of the acute angle and specify-ing whether it is measured from *ON* or *OS* and whether to the east or west.

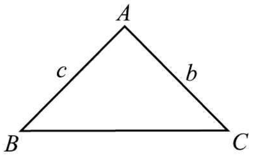
In the Fig.



1. *OA* is in the direction 60° east of north and the bearing of *A* is written as *N* 60° *E*.
2. *OB* is in the direction 30° west of north and the bearing of *B* is written as *N* 30° *W*.
3. *OC* is in the direction 40° west of south and the bearing of *C* is written as *S* 40° *W*.
4. *OD* is in the direction 75° east of south and the bearing of *D* is written as *S* 75° *E*.

SOME USEFUL RESULTS

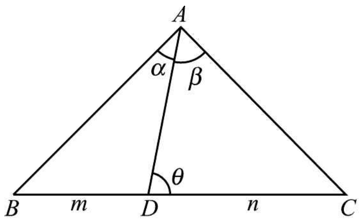
1. In any triangle *ABC*



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. In any triangle *ABC*, if $BD : DC = m : n$ and $\angle BAD = \alpha$, $\angle CAD = \beta$ and $\angle ADC = \theta$, then

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$



3. In a triangle, the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.
4. Few important values to memorise:

$$\sqrt{2} = 1.414,\ \sqrt{3} = 1.732,\ \sqrt{5} = 2.236$$

$$\sin 0° = 0,\ \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}},\ \sin 18° = \frac{\sqrt{5}-1}{4},$$

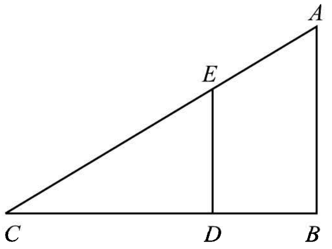
$$\sin 30° = \frac{1}{2},$$

$$\sin 36° = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}},\ \sin 45° = \frac{1}{\sqrt{2}},$$

$$\sin 54° = \frac{1+\sqrt{5}}{4},$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2},\ \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \sin 90^\circ = 1.$$

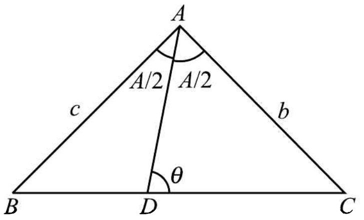
5.



In a triangle ABC, if $DE \parallel AB$, then $\frac{AB}{DE} = \frac{BC}{DC} = \frac{AC}{EC}$.

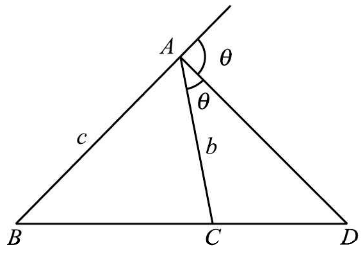
6. In a triangle ABC, if AD is the angle bisector of $\angle BAC$, then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}.$$

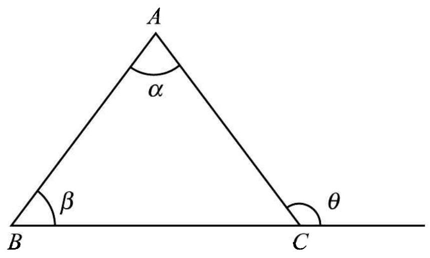


7. In a triangle ABC, if AD is the external angle bisector of $\angle A$, then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}.$$

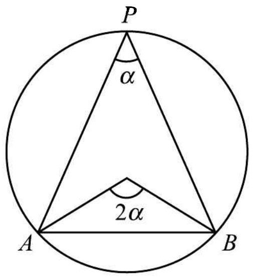


8. In a triangle ABC, the exterior angle is equal to the sum of interior opposite angles.

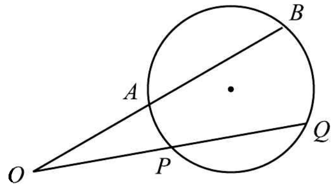


$$\theta = \alpha + \beta.$$

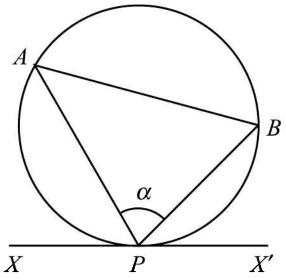
9. The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.



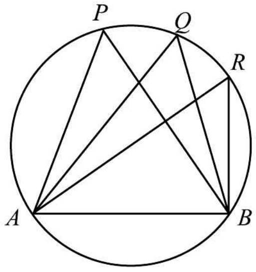
10. If two secants AB and PQ of a circle meet at point 'O', then $OA.OB = OP.OQ$.



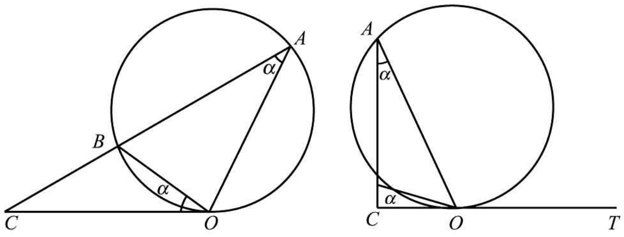
11. If the line joining two points A and B subtends the greatest angle α, at a point P on a given line XX' then the circle, through A, B and P will touch the straight line XX' at the point P.



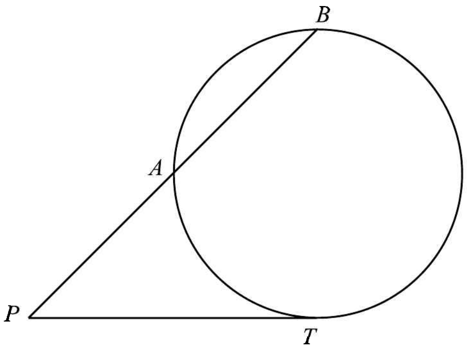
12. Angles in the same segment of a circle are equal i.e, $\angle APB = \angle AQB = \angle ARB$.



13. Angles in the alternate segments of a circle are equal.



14. PAB is a secant of a circle and PT is tangent then $PA \cdot PB = PT^2$



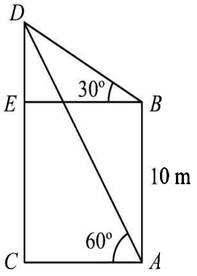
Solved Problems

1. The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. The height of the tower is

(a) 10 m (b) 15 m

(c) 20 m (d) none of these

Solution (b) Let AB and CD be the pole and tower respectively, then



$$\angle DAC = 60^\circ$$

and $$\angle DBE = 30^\circ$$

In ΔACD, $\tan 60^\circ = \dfrac{CD}{AC} \Rightarrow AC = \dfrac{CD}{\sqrt{3}}$ ...(1)

In ΔDBE, $\tan 30^\circ = \dfrac{DE}{BE} = \dfrac{DE}{CA}$ or $\dfrac{1}{\sqrt{3}} = \dfrac{DE}{AC}$.

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{DE}{\frac{CD}{\sqrt{3}}} = \frac{DE}{CD}\sqrt{3} \text{ or } \frac{CD}{DE} = 3$$

$$\Rightarrow \frac{DE + EC}{DE} = 3$$

$$\Rightarrow 1 + \frac{EC}{DE} = 3 \Rightarrow \frac{EC}{DE} = 2$$

$$\Rightarrow DE = \frac{EC}{2} = \frac{10}{2} = 5 \text{ m}$$

$$\therefore CD = DE + EC = 10 + 5 = \mathbf{15\ m.}$$

2. AB is a vertical pole and C is its middle point. The end A is on the level ground and P is any point on the level ground other than A. The portion CB subtends an angle β at P. If $AP : AB = 2 : 1$ then $\beta =$

(a) $\tan^{-1}\dfrac{4}{9}$ (b) $\tan^{-1}\dfrac{1}{9}$

(c) $\tan^{-1}\dfrac{5}{9}$ (d) $\tan^{-1}\dfrac{2}{9}$

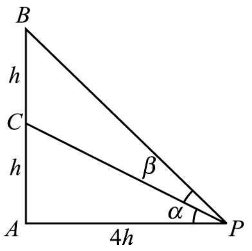
Solution (d) Given that $AC = CB = h$ (say)

$$\angle CPA = \alpha, \angle BPC = \beta$$

and $$\frac{AP}{AB} = \frac{2}{1}.$$

$$\Rightarrow AP = 2AB = 4h$$

In ΔAPC, $\tan\alpha = \dfrac{h}{4h}$



$$\Rightarrow \tan\alpha = \frac{1}{4}$$

In ΔABP, $\tan(\alpha + \beta) = \dfrac{2h}{4h} = \dfrac{1}{2}$

$$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \frac{1}{2} \Rightarrow \frac{\frac{1}{4} + \tan\beta}{1 - \frac{1}{4}\tan\beta} = \frac{1}{2}$$

$$\Rightarrow \frac{1 + 4\tan\beta}{4 - \tan\beta} = \frac{1}{2} \Rightarrow 2 + 8\tan\beta = 4 - \tan\beta$$

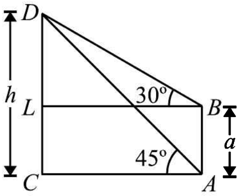
$$\Rightarrow 9\tan\beta = 2 \Rightarrow \beta = \mathbf{\tan^{-1}\frac{2}{9}}.$$

3. The angle of elevation of the top of a tower from the top and bottom of a building of height 'a' are 30° and 45° respectively. If the tower and the building stand at the same level, the height of the tower is

(a) $\frac{a(3+\sqrt{3})}{2}$ (b) $a(\sqrt{3}+1)$

(c) $a\sqrt{3}$ (d) $a(\sqrt{3}-1)$

Solution (a) CD is tower of height h. AB is building of height a



In ΔBLD, $\tan 30° = \frac{h-a}{LB}$

$$\therefore \quad LB = \frac{(h-a)}{\tan 30°}$$

$$= \sqrt{3}\,(h-a) \qquad \text{...(1)}$$

In ΔACD, $\tan 45° = \frac{h}{LB}$ or $h(\sqrt{3}-1) = \sqrt{3}\,a$

$$\therefore \quad h = \frac{\sqrt{3}a}{\sqrt{3}-1} = \frac{\sqrt{3}(\sqrt{3}+1)a}{2}$$

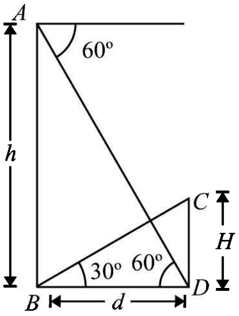
$$\therefore \quad h = \left(\frac{3+\sqrt{3}}{2}\right)a.$$

4. A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point, h vertically above the first, the angle of depression of the foot of the tower is 60°. The height of the tower is

(a) $\frac{h}{3}$ (b) $\frac{h}{3d}$

(c) $3h$ (d) $\frac{3h}{d}$

Solution (a) Let CD be the tower



From ΔBCD, $\frac{H}{d} = \tan 30°$ ...(1)

and from ΔABD, $\frac{h}{d} = \tan 60°$ ...(2)

$$\therefore \quad \frac{H/d}{h/d} = \frac{\tan 30°}{\tan 60°}$$

or $H = \frac{h}{3}$.

5. A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance 'a' from the foot of the tower. If the pole mounted on the tower also subtends an equal angle at O, the height of the pole is

(a) $b\,\frac{a^2-b^2}{a^2+b^2}$ (b) $b\,\frac{a^2+b^2}{a^2-b^2}$

(c) $a\,\frac{a^2-b^2}{a^2+b^2}$ (d) $a\,\frac{a^2+b^2}{a^2-b^2}$

Solution (b)

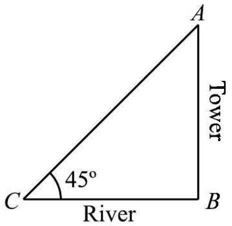
6. A person standing on the bank of a river finds that the angle of elevation of the top of a tower on the opposite bank is 45°. Then which of the following statements is correct?

(a) Breadth of the river is twice the height of the tower

(b) Breadth of the river and the height of the tower are the same

(c) Breadth of the river is half of the height of the tower

(d) none of these



Solution (b)

7. At a point A, the angle of elevation of a tower is such that its tangent is $\frac{5}{12}$; on walking 120m nearer the tower the tangent of the angle of elevation is $\frac{3}{4}$. The height of the tower is

(a) 225 metres (b) 200 metres

(c) 230 metres (d) none of these

Solution (a)

8. If upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 10m, the height of the tree is

(a) $20\sqrt{3}$ m (b) $10\sqrt{3}$ m

(c) $15\sqrt{3}$ m (d) none of these

Solution (b)

9. A tower subtends an angle α at a point on the same level as the root of the tower and at a second point, b metres above the first, the angle of depression of the foot of the tower is β. The height of the tower is

(a) $b\cot\alpha\,\tan\beta$ (b) $b\tan\alpha\,\tan\beta$

(c) $b\tan\alpha\,\cot\beta$ (d) none of these

Solution (c)

10. The upper 3/4th portion of a vertical pole subtends an angle $\tan^{-1}3/5$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is

(a) 20 m (b) 40 m
(c) 60 m (d) 80 m

Solution (b)

11. A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 metres away from the tree the angle of elevation becomes 30°. The breadth of the river is

(a) 40 m (b) 30 m
(c) 20 m (d) 60 m.

Solution (c)

Problems for Practice

Level of Difficulty I

1. An aeroplane flying at a height of 300 metres above the ground passes vertically above another plane at an instant when the angle of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Then the height of the lower plane from the ground is (in metres)

(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$
(c) 50 (d) $150(\sqrt{3}+1)$

2. A man of height 6 ft. observes the top of a tower and the foot of the tower at angles of 45° and 30° of elevation and depression respectively. The height of the tower is

(a) $(1+\sqrt{3})$m (b) $3(1+\sqrt{3})$m
(c) $6(1+\sqrt{3})$m (d) none of these

3. If a flagstaff 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is

(a) 30° (b) 60° (c) 45° (d) none of these

4. From the top of a light house 60m high with its base at sea level the angle of depression of a boat is 15°. The distance of the boat from the light house is

(a) $30\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m (b) $30\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m
(c) $60\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m (d) $60\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m

5. A person, standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°, when he retreats 40m from the bank, he finds the angle to be 30°. The height of the tree and the breadth of the river are

(a) $10\sqrt{3}$ m, 10 m (b) $20\sqrt{3}$ m, 10m
(c) $20\sqrt{3}$ m, 20 m (d) none of these

6. A chimney is such that on walking towards it 50 m in a horizontal line through its base the angular elevation of its top changes from 30° to 45°. The height of the chimney is

(a) $15(\sqrt{3}+1)$m (b) $25(\sqrt{3}+1)$m
(c) $30(\sqrt{3}+1)$m (d) none of these

7. An observer on the top of a cliff 200 m above the sea level, observes the angles of depression of two ships on opposite sides of the cliff to be 45° and 30° respectively. The distance between the ships if the line joining them points to the base of cliff, is

(a) $100(\sqrt{3}+1)$m (b) $200(\sqrt{3}+1)$m
(c) $150(\sqrt{3}+1)$m (d) None of these

8. From the top of a cliff 300 metres high, the top of a tower was observed at an angle of depression 30° and from the foot of the tower the top of the cliff was observed at an angle of elevation 45°. The height of the tower is

(a) $50(3-\sqrt{3})$m (b) $200(3-\sqrt{3})$m
(c) $100(3-\sqrt{3})$m (d) none of these

9. If the elevation of the sun is 30°, then the length of the shadow cast by a tower of 150 ft height is

(a) $75\sqrt{3}$ ft. (b) $200\sqrt{3}$ ft.
(c) $150\sqrt{3}$ ft. (d) none of these

10. At the foot of the mountain the elevation of its summit is 45°; after ascending 1000m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. The height of the mountain is

(a) $\frac{\sqrt{3}+1}{2}$ m (b) $\frac{\sqrt{3}-1}{2}$ m
(c) $\frac{\sqrt{3}+1}{2\sqrt{3}}$ m (d) none of these

Level of Difficulty II

11. The longer side of a parallelogram is 10 cm and the shorter is 6 cm. If the longer diagonal makes an angle 30° with the longer side, the length of the longer diagonal is

(a) $5\sqrt{3}+\sqrt{11}$ (b) $4\sqrt{3}+\sqrt{11}$
(c) $5\sqrt{3}+\sqrt{13}$ (d) none of these

12. The angle of elevation of a tower from a point *A* due south of it is *x* and from a point *B* due east of *A* is *y*. If $AB = l$, then the height *h* of the tower is given by

(a) $\frac{l}{\sqrt{\cot^2 y-\cot^2 x}}$
(b) $\frac{l}{\sqrt{\tan^2 y-\tan^2 x}}$
(c) $\frac{2l}{\sqrt{\cot^2 y-\cot^2 x}}$
(d) none of these

13. The horizontal distance between two tower is 60 m. The angular elevation of the top of the taller tower as seen from the top of the shorter one is 30°. If the height of the taller tower is 150 m, the height of the shorter one is

(a) 116 m (b) 200 m (c) 216 m (d) none of these

14. *PQ* is a post of given height *a*, and *AB* is a tower at some distance; α and β are the angles of elevation of *B*, the top of the tower, at *P* and *Q* respectively. The height of the tower and its distance from the post are

(a) $\dfrac{a\sin\alpha\cos\beta}{\sin(\alpha-\beta)}$ (b) $\dfrac{a\sin\alpha\cos\beta}{\sin(\alpha-\beta)}$

(c) $\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$ (d) none of these

15. A man observes that when he moves up a distance *c* metres on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30°, and when he moves up further a distance *c* metres the angle of depression of that point is 45°. The angle of inclination of the slope with the horizontal is

(a) 60° (b) 45° (c) 75° (d) 30°

ANSWERS

1. (a) **2.** (c) **3.** (b) **4.** (c) **5.** (c) **6.** (b) **7.** (b) **8.** (c) **9.** (c) **10.** (a)
11. (a) **12.** (a) **13.** (a) **14.** (a) **15.** (c)

10

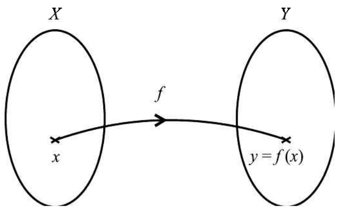
Functions

BRIEF REVIEW OF THE CONCEPTS

FUNCTION OR MAPPING

Let X and Y be any two non-empty sets and there be corres-pondence or association between the elements of X and Y such that for every element $x \in X$, there exists a unique element $y \in Y$, written as $y = f(x)$. Then we say that f is a mapping or function from X to Y, and is written as

$$f : X \rightarrow Y \text{ such that } y = f(x),\ x \in X,\ y \in Y.$$



Real Function

If $f : X \rightarrow Y$ be a function from a non-empty set X to another non-empty set Y, where $X, Y \subseteq R$ (set of all real numbers), then we say that f is a **real valued function** or in short a **real function**.

Features of a Mapping $f : X \rightarrow Y$

1. For each element $x \in X$, there exist a unique element $y \in Y$.
2. The element $y \in Y$ is called the image of x under the mapping f.
3. If there is an element in X which has more than one image in Y, then $f : X \in Y$ is not a function. But distinct elements of X may be associated to the same element of Y.
4. If there is an element in X which does not have an image in Y, then $f : X \rightarrow Y$ is not a function.

Point to Note: Throughout this chapter a 'function' will mean a 'real function'.

Value of a Function

The value of a function $y = f(x)$ at $x = a$ is denoted by $f(a)$. It is obtained by putting $x = a$ in $f(x)$.

Notes:

- If for some value of x say $x = a$, the denominator vanishes, we say that $f(a)$ is undefined (or does not exist).
- If for some value of x say $x = a$, $f(a)$ takes the form $\frac{0}{0}$, we say that $f(a)$ is indeterminate.

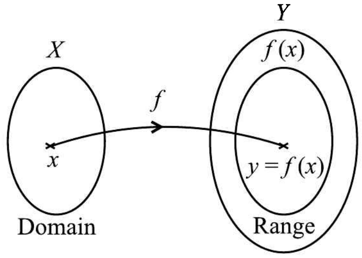
Domain and Range of a Function

If $f : X \rightarrow Y$ be a function, then the set X is said to be the domain of f and range of f

= set of all image points in Y under the map f.

$= f(X) = \{f(x) : f(x) \in Y;\ x \in X\}$

The set Y is also called the co-domain of f. Clearly $f(X) \subseteq Y$.



Intervals in *R*

The set of all numbers lying between two given real numbers is called an interval in R.

Let a and b be any two real numbers such that $a < b$, then we define the following types of intervals.

1. **Closed interval [*a*, *b*]**

   = closed interval from a to b

   $= \{x : x \in R;\ a \leq x \leq b\}$

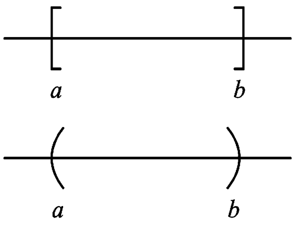
   = set of all real numbers lying between a and b including the end points a and b.

2. **Open interval (*a*, *b*) or]*a*, *b*[**

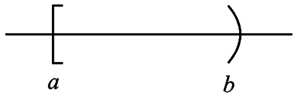
   = open interval from a to b

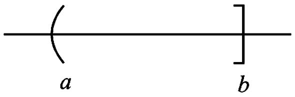
   $= \{x : x \in R;\ a < x < b\}$

   = set of all real numbers lying between a and b, excluding the end points a and b.



3. **Closed-open interval [*a*, *b*) and open–closed interval (*a*, *b*]**





$[a, b) = \{x : x \in R;\ a \leq x < b\}$ $(a, b] = \{x : x \in R;\ a < x \leq b\}$

4. **Real number set R as an open interval**

We introduce two special numbers $-\infty$ and $+\infty$, where

$-\infty = a$ number less than any real number,

$+\infty = a$ number greater than any real number.

$-\infty < x$ for all $x \in R$, and $x < \infty$ for all $x \in R$.

Hence the set R can be thought of as the open interval $(-\infty, \infty)$, so that

$$R = (-\infty, \infty) = \{x : x \in R;\ -\infty < x < \infty\}.$$

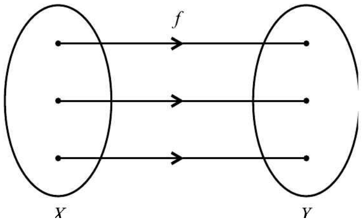
Also, the infinite intervals in R can be given by

$$(-\infty, a),\ (a, +\infty),\ (-\infty, a],\ [a, +\infty).$$

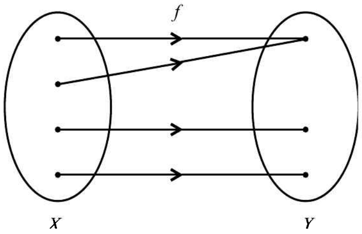
Point to Note: The numbers $+\infty$ and $-\infty$ do not follow the ordinary rules of arithmetic.

TYPES OF FUNCTIONS

One-One or injective function A function $f : X \to Y$ is said to be one-one or injective if distinct elements of X have distinct images in Y.



Many-One function A function $f : X \to Y$ is said to be many-one if there exists atleast two distinct elements in X whose images are same



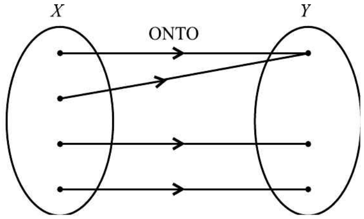
Short-cut Method to Check Whether the Function $f : X \to Y$ is One-one or Many-one

- (a) Consider any two points $x, y \in X$.
  (b) Put $f(x) = f(y)$ and solve the equation.
  (c) If we get $x = y$ only, then f is one-one, otherwise it is many-one.
- If a function is either strictly increasing or strictly decreasing in the whole domain (or equivalently, $f'(x) > 0$ or $f'(x) < 0$, $\forall x \in X$), then it is one-one, otherwise it is many-one.
- If any straight line parallel to x-axis intersects the graph of the function atmost at one point, then the function is one-one, otherwise it is many-one (i.e., it intersects the graph of the function in atleast two points).
- Any continuous function $f(x)$ which has atleast one local maxima or local minima is many-one.
- All even functions are many-one.
- All polynomials of even degree defined on R have atleast one local maxima or minima and hence are many one on the domain R. Polynomials of odd degree can be one-one or many-one.

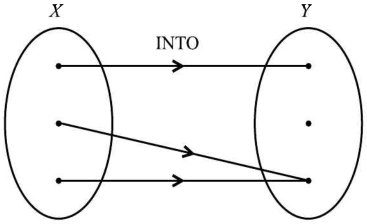
Point to Note: If X and Y are any two finite sets having $\boldsymbol{m}$ and $\boldsymbol{n}$ elements respectively, then the number of one-one functions from X to Y would be

$$= \begin{Bmatrix} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{Bmatrix}$$

Onto or surjective function A function $f : X \to Y$ is said to be onto or surjective if every element of Y is the image of some element of X under the map f.



A function $f : X \to Y$ is an into function if it is not an onto function.



Key Points to Remember

1. If X and Y are any two finite sets having $\boldsymbol{m}$ and $\boldsymbol{n}$ elements respectively, where $1 \leq n \leq m$, then the number of onto functions from X to Y is given by

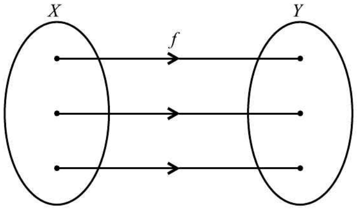
$$\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m$$

2. Any polynomial function f is onto if degree is odd and into if degree of f is even.

Short-cut Method to Check Whether the Function $f : X \to Y$ is Onto or Into

- Find the range of the function f.
- If range of $f = Y$, then f is onto, otherwise it is into.

Bijective Function A function $f : X \to Y$ is said to be bijective, if f is both one-one and onto.



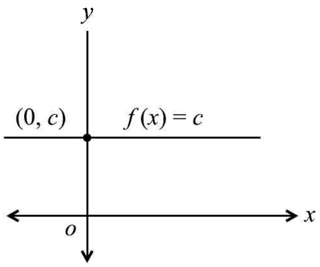
Key Point to Remember

If X and Y are any two finite sets having the same number of elements, say n, then the number of bijective functions from X to Y is $n!$.

Some Important Functions

Constant function A function $f : R \to R$ defined as $f(x) = c$, $\forall\, x \in R$, where c is a constant, is called a constant function. Its domain is R and range is singleton set $\{c\}$.

The graph of a constant function is a straight line parallel to x-axis when x is the independent variable.



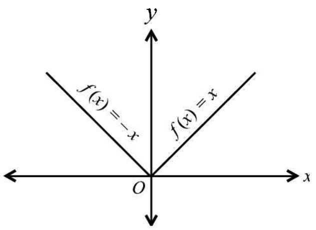
Identity function The function $f : R \to R$ defined as $f(x) = x$, $\forall\, x \in R$, is called the identity function. Its domain is R and range is also R.

The graph of the identity function is a straight line passing through origin and inclined at an angle of 45° with x-axis.

Modulus function or absolute value function The function $f : R \to R$, defined as

$$f(x) = |x| = \begin{cases} x, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

is called the absolute value function or modulus function. Its domain is R and its range is $[0, \infty)$. The graph of the modulus function is as shown in the figure below.

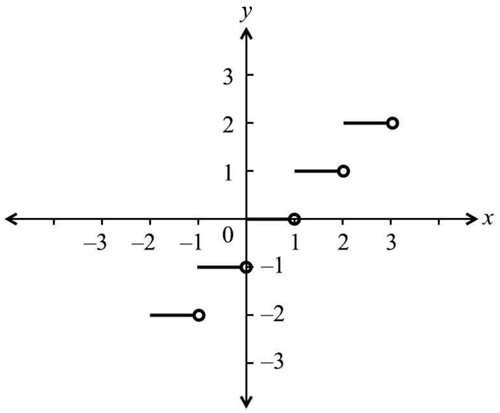


Greatest integer (step or integral) function The function $f : R \to R$ defined as $f(x) = [x]$ is called the greatest integer function,

where $[x]$ = integral part of x or greatest integer not greater than x or greatest integer less than or equal to x.

i.e., $f(x) = n$, where $n \le x < n + 1$, $n \in Z$ (the set of integers).

Its domain is R and range is Z. The graph of the greatest integer function is as shown below:



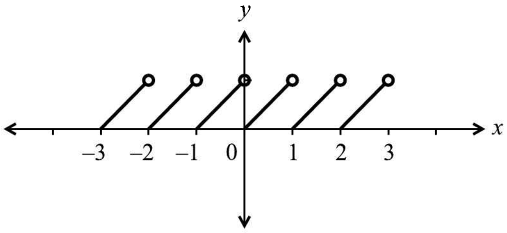
Note:

- $[x] \le x < [x] + 1$
- $[x + y] = \begin{cases} [x] + [y] & , \text{if } \{x\} + \{y\} < 1 \\ [x] + [y] + 1, & \text{if } \{x\} + \{y\} \ge 1 \end{cases}$

where $\{x\}$ denotes the fractional part of x.

- $n \le x < n + 1 \Leftrightarrow [x] = n$
- $n_1 \le [x] \le n_2 \Rightarrow n_1 \le x < n_2 + 1$
- $x - 1 < [x] \le x$
- $[[x]] = [x]$
- $[n + x] = n + [x]$, where n is any integer
- $[x] + [-x] = \begin{cases} 0 & \text{if } x \in Z \\ -1 & \text{if } x \notin Z \end{cases}$

Fractional-part function The function $f : R \to R$ defined as $f(x) = x - [x]$ or $f(x) = \{x\}$, where $\{x\}$ denotes the fractional part of x, is called the fractional-part function. Its domain is R and range is $[0, 1)$. The graph of the fractional part function is as shown below:



Notes:

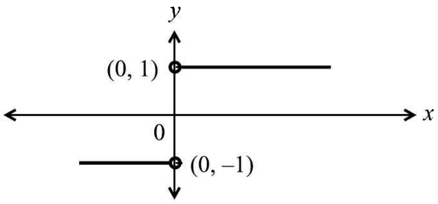
- If x is an integer, then $x = [x]$ $\Rightarrow \{x\} = 0$ $\Rightarrow \{[x]\} = 0$
- $[\{x\}] = 0$
- $0 \le \{x\} < 1$
- $\{x\} + \{-x\} = \begin{cases} 0, \text{if } x \in \text{integer} \\ 1, \text{if } x \notin \text{integer} \end{cases}$

Signum function The function $f : R \to R$ defined as,

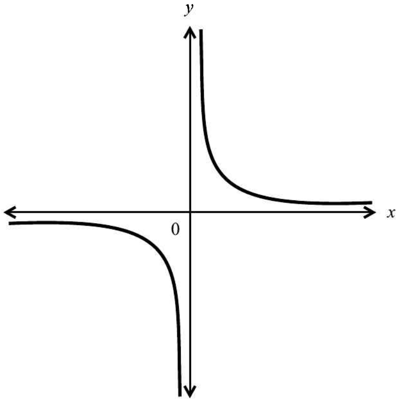
$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is called the signum function.

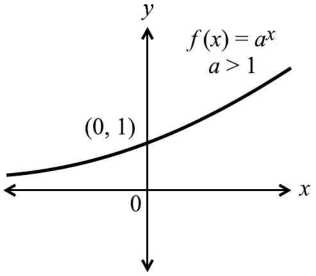
Its domain is R and range is the set $\{-1, 0, 1\}$. The graph of the signum function is as shown below:

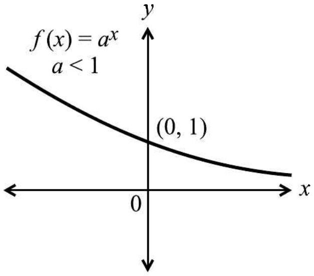


Reciprocal function The function $f: R \backslash \{0\} \to R$ defined by $f(x) = \frac{1}{x}$, is called the reciprocal function. Its domain as well as range is $R \backslash \{0\}$. The graph of the reciprocal function is as shown below:



Exponential function Let $a\ (\neq 1)$ be a positive real number. Then the function $f: R \to R$, defined by $f(x) = a^x$, is called the exponential function. Its domain is R and range is $(0, \infty)$. The graph of the exponential function is as shown below:

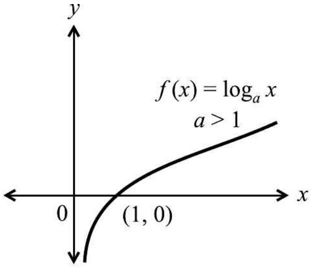


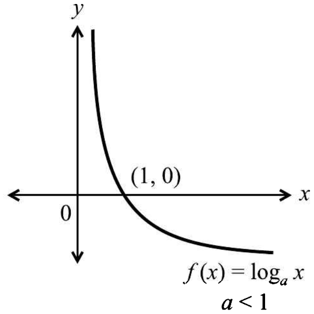


Note:

- $a^x = e^{x \log_e a}$, $(a > 0)$.
- $a^{\log_a x} = x$, $(a > 0, a \neq 1)$.
- $\log_a^{b} = \frac{\log_c^b}{\log_c^a}$, $c > 0$ and $c \neq 1$.
- $\log_b^{a} = \frac{1}{\log_a^b}$, provided $a \neq 1$ and $b \neq 1$.

Logarithmic function Let $a\ (\neq 1)$ be a positive real number. Then the function $f: (0, \infty) \to \mathrm{R}$, defined by $f(x) = \log_a x$, is called the logarithmic function. Its domain is $(0, \infty)$ and range is R. The graph of the logarithmic function is as shown below:





Note:

- $\log_a a = 1$, $\log_a 1 = 0$
- $\log_a 0 = \begin{cases} -\infty, \text{if } a > 1 \\ +\infty, \text{if } 0 < a < 1 \end{cases}$
- $\log_e x$ is also denoted as: *lnx*.

Polynomial function A function $f: R \to R$, defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where $n \in N$ and $a_0, a_1, a_2, \ldots, a_n \in R$, is called a polynomial function.

If $a_n \neq 0$, then n is called the degree of the polynomial. The domain of a polynomial function is R.

Rational function A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials over the set of real numbers and $q(x) \neq 0$, is called a rational function. Its domain is $R \backslash \{x \mid q(x) = 0\}$.

Trigonometric functions

Function	*Domain*	*Range*
• $y = \sin x$	R	$[-1, 1]$
• $y = \cos x$	R	$[-1, 1]$
• $y = \tan x$	$R \backslash \left\{(2n+1)\frac{\pi}{2} \middle\vert n \in \mathrm{I}\right\}$	R
• $y = \cot x$	$R \backslash \{n\pi \mid n \in \mathrm{I}\}$	R
• $y = \sec x$	$R \backslash \left\{(2n+1)\frac{\pi}{2} \middle\vert n \in \mathrm{I}\right\}$	$(-\infty, -1] \cup [1, \infty)$
• $y = \operatorname{cosec} x$	$R \backslash \{n\pi \mid n \in \mathrm{I}\}$	$(-\infty, -1] \cup [1, \infty)$

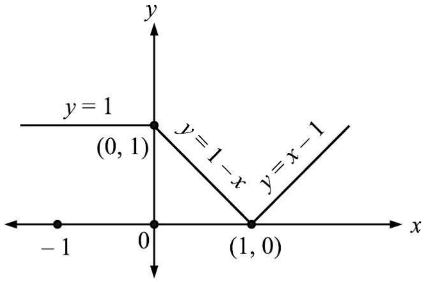
Inverse Trigonometric Functions

Function	*Domain*	*Range*
• $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
• $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$[0, \pi]$
• $y = \tan^{-1} x$	$-\infty < x < \infty$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
• $y = \cot^{-1} x$	$-\infty < x < \infty$	$(0, \pi)$
• $y = \operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
• $y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Two Ways of Defining a Function

(i) Uniform definition: If a function is defined as $y = f(x)$, $x \in [a, b]$, we say that it is uniformly defined.

(ii) Piecewise definition: If a function $y = f(x)$, $x \in [a, b]$ assumes different forms in different subsets of $[a, b]$, we say that it is piecewise defined.



Explicit and Implicit Functions

Explicit function: A function y is said to be an explicit function of x, if the dependent variable y can be expressed totally in terms of the independent variable x.

Implicit function: When the variables x and y occur together in an equation $f(x,y) = 0$, in which y cannot be expressed explicitly in terms of x, then y is said to be an implicit function of x.

OPERATIONS ON FUNCTIONS

Let f and g be two real functions with domain D_1 and D_2 respectively. Then,

1. The sum function $(f + g)$ is defined by
$$(f + g)(x) = f(x) + g(x), \forall x \in D_1 \cap D_2$$
The domain of $f + g$ is $D_1 \cap D_2$
2. The difference function $(f - g)$ is defined by
$$(f - g)(x) = f(x) - g(x), \forall x \in D_1 \cap D_2$$
The domain of $f - g$ is $D_1 \cap D_2$
3. The product function fg is defined by
$$(fg)(x) = f(x) \cdot g(x), \forall x \in D_1 \cap D_2$$
The domain of fg is $D_1 \cap D_2$
4. The quotient function $\left(\frac{f}{g}\right)$ is defined by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in D_1 \cap D_2 \backslash \{x : g(x) = 0\}$$
The domain of $\frac{f}{g}$ is $D_1 \cap D_2 \backslash \{x : g(x) = 0\}$
5. The scalar multiple function cf is defined by
$$(cf)(x) = c \cdot f(x), \forall x \in D_1$$
The domain of cf is D_1

COMPOSITION OF FUNCTIONS

Let f and g be two real functions with domain D_1 and D_2 respectively.

If range of $f \subseteq$ domain of g, then composite function (gof) is defined by

$$(gof)(x) = g(f(x)), \forall x \in D_1$$

Also, if range of $g \subseteq$ domain of f, then composite function (fog) is defined by

$$(fog)(x) = f(g(x)), \forall x \in D_2.$$

Key Point to Remember

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

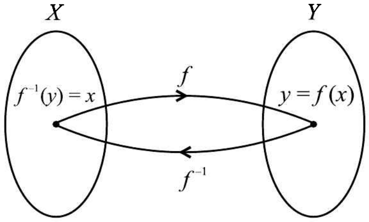
1. If both f and g are one-one, then so is $g \circ f$.
2. If both f and g are onto, then is also onto $g \circ f$.
3. If $g \circ f$ is one-one, then f is one-one but g may not be one-one.
4. If $g \circ f$ is onto, then g is onto but f may not be onto.
5. If f and g are bijective, then is also bijective $g \circ f$.
6. It may happen that gof may exist and $f \circ g$ may not exist. Moreover, even if both $g \circ f$ and $f \circ g$ exist, they may not be equal. (see illustration 14).

INVERSE FUNCTIONS

If the function $f : X \to Y$ is both one-one and onto, then we define inverse function $f^{-1} : Y \to X$ by the rule

$$y = f(x) \Leftrightarrow f^{-1}(y) = x, \forall x \in X, \forall y \in Y$$

as shown in the figure below:



Short-Cut Method to Find the Inverse of a Function

Let $f : X \to Y$ be a bijective function.

- Put $f(x) = y$.
- Solve the equation $y = f(x)$ to obtain x in terms of y.

Interchange x and y to obtain the inverse of f.

ODD AND EVEN FUNCTIONS

Odd function A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for every real number x in the domain of f.

Even function A function $f(x)$ is said to be even if $f(-x) = f(x)$ for every real number x in the domain of f.

Key Points to Remember

1. The graph of an odd function is symmetric about origin and it is placed either in the first and third quadrant or in the second and fourth quadrant.
2. The graph of an even function is symmetric about the y-axis.
3. To express a given function $f(x)$ as the sum of an even and odd function, we write
$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)].$$
where $\frac{1}{2}[f(x) + f(-x)]$ is an even function and
$\frac{1}{2}[f(x) - f(-x)]$ is an odd function.

4. $f(x) = 0$ is the only function which is both even and odd.
5. If $f(x)$ is an odd function, then $f'(x)$ is an even function provided $f(x)$ is differentiable on R.
6. If $f(x)$ is an even function, then $f'(x)$ is an odd function provided $f(x)$ is differentiable on R.
7. If f and g are even functions, then fog is also an even function, provided fog is defined.
8. If f and g are odd functions, then fog is also an odd function, provided fog is defined.
9. If f is an even function and g is an odd function, then fog is an even function.
10. If f is an odd function and g is an even function, then fog is an even function.
11. For a real domain, even functions are not one-one.

PERIODIC FUNCTION

A function $f(x)$ is said to be a periodic function of x, provided there exists a real number $T > 0$ such that

$$f(x + T) = f(x), \forall x \in R.$$

The smallest positive real number T, satisfying the above condition is known as the *period or the fundamental period* of $f(x)$.

Short-Cut Method to Check the Periodicity of a Function

- Put $f(T + x) = f(x)$ and solve this equation to find the positive values of T independent of x.
- If no positive value of T independent of x is obtained, then $f(x)$ is a non-periodic function.
- If positive values of T independent of x are obtained, then $f(x)$ is a periodic function and the least positive value of T is the period of the function $f(x)$.

Key Points to Remember

1. Constant function is periodic with no fundamental period.
2. If $f(x)$ is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.
3. If $f(x)$ is periodic with period T_1 and $g(x)$ is periodic with period T_2, then $f(x) + g(x)$ is periodic with period equal to l.c.m of T_1 and T_2, provided there is no positive k such that $f(k + x) = g(x)$ and $g(k + x) = f(x)$.
4. If $f(x)$ is periodic with period T, then $kf(ax + b)$, is also periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.
5. $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π.
6. $\tan x$ and $\cot x$ are periodic functions with period π.
7. $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x|$ and $|\operatorname{cosec} x|$ are periodic functions with period π.
8. $\sin^n x$, $\cos^n x$, $\sec^n x$ and $\operatorname{cosec}^n x$ are periodic functions with period 2π when n is odd or π when n is even.
9. $\tan^n x$ and $\cot^n x$ are periodic functions with period π.
10. If $f(x)$ is a periodic function with period T and $g(x)$ is any function such that range of $f \subset$ domain *of* g, then gof is also periodic with period T.

Short-cut Method to Find the Domain of a Function

Algebraic Functions

- Denominator should be non-zero.
- Expression under the square root should be non-negative.

Trigonometric Functions

- $\sin x$ and $\cos x$ are defined for all real values of x.
- $\tan x$ and $\sec x$ are defined for all real values of x except $x = (2n + 1)\frac{\pi}{2}$, where $n \in Z$.
- $\cot x$ and $\operatorname{cosec} x$ are defined for all real values of x except $x = n\pi$, where $n \in Z$.

Inverse Trigonometric Functions

- $\sin^{-1} x$ and $\cos^{-1} x$ are defined for $-1 \leq x \leq 1$.
- $\tan^{-1} x$ and $\cot^{-1} x$ are defined for all real values of x.
- $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are defined for $x \leq -1$ or $x \geq 1$.

Logarithmic Functions

- $\log_b a$ is defined when $a > 0$, $b > 0$ and $b \neq 1$.

Exponential Functions

- a^x is defined for all real values of x, where $a > 0$.

Key Points to Remember

1. $(x - a)(x - b) > 0 \Rightarrow x < a$ or $x > b$, for $a < b$
2. $(x - a)(x - b) < 0 \Rightarrow a < x < b$, for $a < b$
3. $|x| < a \Rightarrow -a < x < a$
4. $|x| > a \Rightarrow x < -a$ or $x > a$
5. $\log_b a > k \Rightarrow \begin{cases} a > b^k, \text{ if } b > 1 \\ a < b^k, \text{ if } b > 1 \end{cases}$
6. $\sqrt{x^2} = |x|$
7. $\sqrt[n]{x^n} = |x|$, if n is even and $\sqrt[n]{x^n} = x$, if n is odd.

Short-cut Method to Find the Range of a Function $y = f(x)$

- Find the domain of the function $y = f(x)$.
- If the domain is an infinite interval, solve the equation $y = f(x)$ and find x in terms of y to get $x = g(y)$. Find the real values of y for which x is real. The set of values of y so obtained constitutes the range of f. Note that if finite number of values of x are excluded from the domain, find the values of y for these values of x and exclude these values of y from the range of f found earlier.
- If the domain is a finite interval, find the least and greatest value of y for values of x in the domain. If a is the least value and b the greatest value of y, then range $(f) = [a, b]$.

Solved Problems

1. The domain of the function $f(x) = \sqrt{\log_{10}\dfrac{3-x}{x}}$ is

(a) $\left(0, \dfrac{3}{2}\right)$ (b) $(0, 3)$

(c) $\left(-\infty, \dfrac{3}{2}\right]$ (d) $\left(0, \dfrac{3}{2}\right]$

Solution (d) $\sqrt{\log_{10}\left(\dfrac{3-x}{x}\right)}$ is defined for

$$\log_{10}\left(\frac{3-x}{x}\right) \geq 0 \Rightarrow \frac{3-x}{x} \geq 10^0 = 1 \Rightarrow 3 - x \geq x$$

$$\Rightarrow 2x \leq 3 \Rightarrow x \leq \frac{3}{2} \qquad ...(1)$$

Also, $\log_{10}\left(\dfrac{3-x}{x}\right)$ is defined for

$$\frac{3-x}{x} > 0 \text{ or } \frac{x(3-x)}{x^2} > 0$$

i.e., $x(x-3) < 0 \Rightarrow 0 < x < 3$ ...(2)

From (1) and (2), we get domain of $f = \left(0, \dfrac{3}{2}\right]$.

2. The domain of the function $f(x) = \sqrt{x - \sqrt{1-x^2}}$ is

(a) $\left[-1, -\dfrac{1}{\sqrt{2}}\right] \cup \left[\dfrac{1}{\sqrt{2}}, 1\right]$

(b) $[-1, 1]$

(c) $\left(-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{\sqrt{2}}, +\infty\right)$

(d) $\left[\dfrac{1}{\sqrt{2}}, 1\right]$

Solution (d) For $f(x)$ to be defined, we must have

$$x - \sqrt{1-x^2} \geq 0 \text{ or } x \geq \sqrt{1-x^2} > 0$$

$\therefore$ $x^2 \geq 1 - x^2$ or $x^2 \geq \dfrac{1}{2}$.

Also, $1 - x^2 \geq 0$ or $x^2 \leq 1$.

Now, $x^2 \geq \dfrac{1}{2} \Rightarrow \left(x - \dfrac{1}{\sqrt{2}}\right)\left(x + \dfrac{1}{\sqrt{2}}\right) \geq 0$

$$\Rightarrow x \leq -\frac{1}{\sqrt{2}} \text{ or } x \geq \frac{1}{\sqrt{2}}.$$

Also, $x^2 \leq 1 \Rightarrow (x-1)(x+1) \leq 0 \Rightarrow -1 \leq x \leq 1$

Thus, $x > 0$, $x^2 \geq \dfrac{1}{2}$ and $x^2 \leq 1 \Rightarrow x \in \left[\dfrac{1}{\sqrt{2}}, 1\right]$.

3. The domain of the function

$$f(x) = \sin^{-1}\left\{\log_2\left(\frac{1}{2}x^2\right)\right\} \text{ is}$$

(a) $[-2, -1) \cup [1, 2]$ (b) $(-2, -1] \cup [1, 2]$

(c) $[-2, -1] \cup [1, 2]$ (d) $(-2, -1) \cup (1, 2)$

Solution (c) For $f(x)$ to be defined, we must have

$$-1 \leq \log_2\left(\frac{1}{2}x^2\right) \leq 1 \quad \Rightarrow 2^{-1} \leq \frac{1}{2}x^2 \leq 2^1$$

[$\because$ the base = 2 > 1]

$$\Rightarrow 1 \leq x^2 \leq 4 \qquad ...(1)$$

Now, $1 \leq x^2 \Rightarrow x^2 - 1 \geq 0$ i.e., $(x-1)(x+1) \geq 0$

$$\Rightarrow x \leq -1 \text{ or } x \geq 1 \qquad ...(2)$$

Also, $x^2 \leq 4 \Rightarrow x^2 - 4 \leq 0$ i.e., $(x-2)(x+2) \leq 0$

$$\Rightarrow -2 \leq x \leq 2 \qquad ...(3)$$

From (2) and (3), we get the domain of f

$$= ((-\infty, -1] \cup [1, \infty)) \cap [-2, 2]$$

$$= [-2, -1] \cup [1, 2].$$

4. The domain of the function

$$f(x) = \frac{1}{\sqrt{x^{12} - x^9 + x^4 - x + 1}} \text{ is}$$

(a) $(-\infty, -1)$ (b) $(1, \infty)$

(c) $(-1, 1)$ (d) $(-\infty, \infty)$

Solution (d) $f(x)$ is defined for

$x^{12} - x^9 + x^4 - x + 1 > 0$

$$\Rightarrow x^4(x^8 + 1) - x(x^8 + 1) + 1 > 0$$

$$\Rightarrow (x^8 + 1)\, x\, (x^3 - 1) + 1 > 0.$$

If $x \geq 1$ or $x \leq -1$, then the above expression is positive.

If $-1 < x \leq 0$, the above inequality still holds.

If $0 < x < 1$, then $x^{12} - xp(x^8 + 1) + (x^4 + 1) > 0$

[$\because$ $x^4 + 1 > x^8 + 1$ and so $x^4 + 1 > x(x^8 + 1)$].

The domain of $f = (-\infty, \infty)$.

5. The domain of the function $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$ is given by

(a) $[-1, 2) \cup [3, \infty)$ (b) $(-1, 2) \cup [3, \infty)$

(c) $[-1, 2] \cup [3, \infty)$ (d) none of these

Solution (a) We have,

$$f(x) = \sqrt{\frac{(x+1)(x-3)}{(x-2)}} = \frac{\sqrt{(x+1)(x-2)(x-3)}}{(x-2)}.$$

$f(x)$ is defined for $(x+1)(x-2)(x-3) \geq 0$

or $[x - (-1)](x-2)(x-3) \geq 0 \Rightarrow -1 \leq x \leq 2$ or $x \geq 3$.

Also, $x - 2 \neq 0$ *i.e.,* $x \neq 2$.

Domain of f $= ([-1, 2] \cup [3, \infty)) \backslash \{2\}$

$$= [-1, 2) \cup [3, \infty).$$

6. The domain of $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(a) $R - \{-1, -2\}$ (b) $(-2, +\infty)$

(c) $R - \{-1, -2, -3\}$ (d) $(-3, +\infty) - \{-1, -2\}$

Solution (d) For $f(x)$ to be defined,

$x + 3 > 0$ and $x^2 + 3x + 2 \neq 0$

$\Rightarrow$ $x > -3$ and $(x+1)(x+2) \neq 0$, *i.e.*, $x \neq -1, -2$.

$\therefore$ domain $= (-3, +\infty) - \{-1, -2\}$.

7. The domain of the function

$$f(x) = {}^{24-x}C_{3x-1} + {}^{40-6x}C_{8x-10} \text{ is,}$$

(a) $\{2, 3\}$ (b) $\{1, 2, 3\}$

(c) $\{1, 2, 3, 4\}$ (d) none of these

Solution (a) ${}^{24-x}C_{3x-1}$ is defined if,

$24 - x > 0$, $3x - 1 \geq 0$ and $24 - x \geq 3x - 1$

$\Rightarrow$ $x < 24$, $x \geq \frac{1}{3}$ and $x \leq \frac{25}{4}$

$$\Rightarrow \frac{1}{3} \leq x \leq \frac{25}{4} \qquad \text{...(1)}$$

${}^{40-6x}C_{8x-10}$ is defined if

$40 - 6x > 0$, $8x - 10 \geq 0$ and $40 - 6x \geq 8x - 10$

$\Rightarrow$ $x < \frac{20}{3}$, $x \geq \frac{5}{4}$ and $x \leq \frac{25}{7}$

$$\Rightarrow \frac{5}{4} \leq x \leq \frac{25}{7} \qquad \text{...(2)}$$

From (1) and (2) we get $\frac{5}{4} \leq x \leq \frac{25}{7}$

But $24 - x \in N$, $\therefore$ x must be an integer, $\therefore$ $x = 2, 3$.

Hence domain $(f) = \{2, 3\}$.

8. The domain of the function $f(x) = \sqrt{\cos^{-1}\left(\frac{1-|x|}{2}\right)}$ is

(a) $(-\infty, -3) \cup (3, \infty)$ (b) $[-3, 3]$

(c) $(-\infty, -3] \cup [3, \infty)$ (d) ϕ

Solution (b) $\cos^{-1}\left(\frac{1-|x|}{2}\right)$ is defined if $-1 \leq \frac{1-|x|}{2} \leq 1$

$\Rightarrow$ $-2 \leq 1 - |x| \leq 2$ $\Rightarrow$ $-3 \leq -|x| \leq 1$

$\Rightarrow$ $-1 \leq |x| \leq 3$

$|x| \geq -1$ is true for all real values of x.

$|x| \leq 3$ $\Rightarrow$ $-3 \leq x \leq 3$.

Also $\sqrt{\cos^{-1}\left(\frac{1-|x|}{2}\right)}$ is defined if $\cos^{-1}\left(\frac{1-|x|}{2}\right) \geq 0$

$\Rightarrow$ $\frac{1-|x|}{2} \geq \cos 0 = 1$ $\Rightarrow$ $1 - |x| \geq 2$

$\Rightarrow$ $|x| \leq -1$ (Absurd)

$\therefore$ Domain $(f) = [-3, 3]$.

9. The domain of the function

$$f(x) = \log_{1/2}\left(x - \frac{1}{2}\right) + \log_2\sqrt{4x^2 - 4x + 5} \text{ is}$$

(a) $\left[\frac{1}{2}, \infty\right)$ (b) $\left(\frac{1}{2}, \infty\right)$

(c) $(-\infty, \infty)$ (d) none of these

Solution (b) $\log_{1/2}\left(x - \frac{1}{2}\right)$ is defined if $x - \frac{1}{2} > 0$

i.e., $x > \frac{1}{2}$

$\log_2\sqrt{4x^2 - 4x + 5}$ is defined if $4x^2 - 4x + 5 > 0$

$\Rightarrow 4\left[\left(x - \frac{1}{2}\right)^2 + 1\right] > 0$ which is true for all real x.

Domain of $f = \left(\frac{1}{2}, \infty\right)$.

10. The domain of the function

$$f(x) = \sqrt[3]{1-3x} + 3\cos^{-1}\left(\frac{2x-1}{3}\right) + e^{3\tan x} \text{ is}$$

(a) $[-1, 2]$ (b) $(-1, 2)$

(c) $(-\infty, \infty)$ (d) none of these

Solution (a) $\sqrt[3]{1-3x}$ and $e^{3\tan x}$ is defined for all real x.

$3\cos^{-1}\left(\frac{2x-1}{3}\right)$ is defined if $-1 \leq \frac{2x-1}{3} \leq 1$

$\Rightarrow$ $-3 \leq 2x - 1 \leq 3$ $\Rightarrow$ $-1 \leq x \leq 2$

Domain of $f = [-1, 2]$.

11. The domain of the function

$$f(x) = \cos\left[\log\left(\frac{\sqrt{16-x^2}}{3-x}\right)\right] \text{ is}$$

(a) $(-4, 4)$ (b) $(-4, 3)$

(c) $(-\infty, -4) \cup (3, \infty)$ (d) none of these

Solution (b) $f(x)$ is defined if $\frac{\sqrt{16-x^2}}{3-x} > 0$

$\Rightarrow$ $16 - x^2 > 0$ and $3 - x > 0$

$\Rightarrow$ $(x-4)(x+4) < 0$ and $x < 3$

$\Rightarrow$ $-4 < x < 4$ and $x < 3$ or $-4 < x < 3$

Domain of $f = (-4, 3)$.

12. The domain of the function $f(x) = \log_2 \log_3 \log_4 x$ is

(a) $[4, \infty)$ (b) $(4, \infty)$

(c) $(-\infty, 4)$ (d) none of these

Solution (b) $f(x)$ is defined if $\log_3 \log_4 x > 0$, $\log_4 x > 0$ and $x > 0$

$\Rightarrow$ $\log_4 x > 3^0 = 1$, $x > 4^0$ and $x > 0$

$\Rightarrow$ $x > 4^1$, $x > 1$ and $x > 0$ $\Rightarrow$ $x > 4$

Domain of $f = (4, \infty)$.

13. The domain of the function

$$f(x) = \cos^{-1}\left(\frac{2-|x|}{4}\right) + [\log(3-x)]^{-1} \text{ is}$$

(a) $[-6, 3)\backslash\{2\}$ (b) $[-6, 2) \cup (2, 3]$

(c) $[-6, 3]$ (d) $[-6, 3)$

Solution (a) $\cos^{-1}\left(\frac{2-|x|}{4}\right)$ is defined for $-1 \leq \frac{2-|x|}{4} \leq 1$

$\Rightarrow$ $-4 \leq 2 - |x| \leq 4$ $\Rightarrow$ $-6 \leq -|x| \leq 2$

$\Rightarrow -2 \le |x| \le 6 \Rightarrow |x| \le 6 \Rightarrow -6 \le x \le 6$ ...(1)

$\dfrac{1}{\log(3-x)}$ is defined if $\log(3-x) \ne 0$ and $3 - x > 0$

$\Rightarrow 3 - x \ne e^0 = 1$ and $x < 3 \Rightarrow x \ne 2$ and $x < 3$ ...(2)

From (1) and (2), we get domain of f

$$= [-6, 6] \cap ((-\infty, 3) \setminus \{2\}) = [-6, 3) \setminus \{2\}.$$

14. The domain of the function

$$f(x) = \cos^{-1}\left(\frac{3}{4 + 2\sin x}\right) \text{ is}$$

(a) $\left[-\frac{\pi}{6} + 2n\pi, \frac{\pi}{6} + 2n\pi\right]$

(b) $\left(-\frac{\pi}{6} + 2n\pi, \frac{\pi}{6} + 2n\pi\right)$

(c) $\left(-\frac{\pi}{6} + 2n\pi, \frac{\pi}{6} + 2n\pi\right]$

(d) $\left[-\frac{\pi}{6} + 2n\pi, \frac{\pi}{6} + 2n\pi\right)$

Solution (a) $f(x)$ is defined if $-1 \le \left(\dfrac{3}{4 + 2\sin x}\right) \le 1$

Since $4 + 2 \sin x > 0$ for all real x, therefore

$\dfrac{3}{4 + 2\sin x} \le 1 \Rightarrow 3 \le 4 + 2\sin x \quad \Rightarrow \sin x \ge -\dfrac{1}{2}$

$\Rightarrow -\dfrac{\pi}{6} + 2n\pi \le x \le \dfrac{\pi}{6} + 2n\pi,\ n \in I$

Domain of $f = \left[-\dfrac{\pi}{6} + 2n\pi, \dfrac{\pi}{6} + 2n\pi\right]$.

15. The domain of the function $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^2}}}$ is

(a) $(-\infty, 1)$ (b) $(-1, \infty)$

(c) $[0, 1]$ (d) $[-1, 1]$

Solution (d) $f(x)$ is defined if

$1 - \sqrt{1 - \sqrt{1 - x^2}} \ge 0,\ 1 - \sqrt{1 - x^2} \ge 0$ and $1 - x^2 \ge 0$

$1 - x^2 \ge 0 \Rightarrow (x + 1)(x - 1) \le 0 \Rightarrow -1 \le x \le 1.$

Clearly for these values, the other two inequalities hold.

Thus domain of $f = [-1, 1]$.

16. The domain of the function $f(x) = \sqrt{x^2 - [x]^2}$, where $[x]$ denotes the greatest integer less than or equal to x, is

(a) $(0, \infty)$ (b) $(-\infty, 0)$

(c) $(-\infty, \infty)$ (d) none of these

Solution (d) $f(x)$ is defined if $x^2 - [x]^2 \ge 0 \Rightarrow x^2 \ge [x]^2$, which is true for all positive real x and all negative integers x.

17. The domain of the function $f(x) = \sqrt{e^{\sin^{-1}(\log_{16} x^2)}}$ is

(a) $\left[\frac{1}{4}, 4\right]$ (b) $\left[-4, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 4\right]$

(c) $\left[-4, -\frac{1}{4}\right]$ (d) none of these

Solution (b) $f(x)$ is defined if

$-1 \le \log_{16} x^2 \le 1 \quad \Rightarrow 16^{-1} \le x^2 \le 16^1$

$\Rightarrow \dfrac{1}{16} \le x^2 \le 16$

$x^2 \ge \dfrac{1}{16} \quad \Rightarrow \left(x - \dfrac{1}{4}\right)\left(x + \dfrac{1}{4}\right) \ge 0$

$\Rightarrow x \le -\dfrac{1}{4}$ or $x \ge \dfrac{1}{4}$ ...(1)

$x^2 \le 16 \Rightarrow (x - 4)(x + 4) \le 0 \Rightarrow -4 \le x \le 4$ ...(2)

From (1) and (2), we get domain of f

$$= \left[-4, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 4\right].$$

18. The domain of the function $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ is

(a) $(-\infty, -2) \cup [4, \infty)$ (b) $(-\infty, -2] \cup [4, \infty)$

(c) $(-\infty, -2) \cup (4, \infty)$ (d) none of these

Solution (a) $f(x)$ is defined for

$[x]^2 - [x] - 6 > 0 \quad \Rightarrow ([x] - 3)([x] + 2) > 0$

$\Rightarrow [x] < -2$ or $[x] > 3$

But $[x] < -2 \Rightarrow [x] = -3, -4, -5, \ldots$

$\therefore$ $x < -2.$

Also, $[x] > 3 \Rightarrow [x] = 4, 5, 6, \ldots$

$\therefore$ $x \ge 4$

Domain of $f = (-\infty, -2) \cup [4, \infty)$.

19. The domain of the function

$$f(x) = \cot^{-1}\left(\frac{x}{\sqrt{x^2 - [x^2]}}\right),\ x \in R \text{ is}$$

(a) $R - \{\pm\sqrt{n}, n \in N\}$ (b) $R - \{\sqrt{n}, n \ge 0, n \in I\}$

(c) R (d) $R - \{0\}$

Solution (b) Domain of $\cot^{-1}x$ is R and $\dfrac{x}{\sqrt{x^2 - [x^2]}}$ is defined if $x^2 \ne [x^2]$ ($\because x^2 \ge [x^2]$)

$\Rightarrow$ $x^2 \ne 0$ or +ve integer.

Hence, domain $= R - \{\sqrt{n} : n \ge 0, n \in I\}$.

20. The domain of the function

$$f(x) = \log_3\left[-\log_{1/2}\left(1 + \frac{1}{x^{1/5}}\right) - 1\right] \text{ is}$$

(a) $(-\infty, 1)$ (b) $(0, 1)$

(c) $(1, \infty)$ (d) none of these

Solution (b) $f(x)$ is defined if

$-\log_{1/2}\left(1 + \dfrac{1}{x^{1/5}}\right) - 1 > 0,\ 1 + \dfrac{1}{x^{1/5}} > 0,\ x \ne 0$

$\Rightarrow \log_{1/2}\left(1 + \dfrac{1}{x^{1/5}}\right) < -1,\ x^{1/5} + 1 > 0,\ x \ne 0$

$\Rightarrow 1 + \dfrac{1}{x^{1/5}} > \left(\dfrac{1}{2}\right)^{-1},\ x > (-1)^5,\ x \ne 0$

⇒ $\frac{1}{x^{1/5}} > 1, x > -1$ and $x \neq 0$

⇒ $0 < x < 1$ and $x > -1 \Rightarrow 0 < x < 1$.

∴ Domain $(f) = (0, 1)$.

21. The range of the function $y = \frac{1}{2 - \sin 3x}$ is

(a) $\left(\frac{1}{3}, 1\right)$ (b) $\left[\frac{1}{3}, 1\right)$

(c) $\left[\frac{1}{3}, 1\right]$ (d) none of these

Solution (c) We have, $y = \frac{1}{2 - \sin 3x}$ ...(1)

Domain of $y = (-\infty, \infty)$

From (1), $2 - \sin 3x = \frac{1}{y} \Rightarrow \sin 3x = \frac{2y-1}{y}$

$$\Rightarrow x = \frac{1}{3} \sin^{-1}\left(\frac{2y-1}{y}\right)$$

$$\left[\begin{array}{l} \because y = \frac{1}{2 - \sin 3x} \therefore y > 0 \\ \text{as } -1 \le \sin 3x \le 1 \end{array}\right]$$

For x to be real,

$-1 \le \frac{2y-1}{y} \le 1 \Rightarrow -y \le 2y - 1 \le y \quad (\because y > 0)$

$\Rightarrow 2y - 1 \ge -y$ and $2y - 1 \le y$

$\Rightarrow y \ge \frac{1}{3}$ and $y \le 1$

∴ Range of $y = \left[\frac{1}{3}, 1\right]$.

22. The value of the function $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$ lies in the interval

(a) $(-\infty, \infty)\backslash\left\{\frac{1}{5}, 1\right\}$ (b) $(-\infty, \infty)$

(c) $(-\infty, \infty)\backslash\{1\}$ (d) none of these

Solution (b) $f(x)$ is defined if $x^2 + x - 6 \neq 0$

i.e., $(x + 3)(x - 2) \neq 0$ i.e., $x \neq -3, 2$

∴ Domain $(f) = (-\infty, \infty)\backslash\{-3, 2\}$.

Let $y = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow x^2y + xy - 6y = x^2 - 3x + 2$

$\Rightarrow x^2(y - 1) + x(y + 3) - (6y + 2) = 0$

For x to be real, $(y + 3)^2 + 4(y - 1)(6y + 2) \ge 0$

$\Rightarrow 25y^2 - 10y + 1 \ge 0$ i.e., $(5y - 1)^2 \ge 0$

which is true for all real y.

Range of $f = (-\infty, \infty)$.

23. The range of the function $f(x) = \tan\sqrt{\frac{\pi^2}{9} - x^2}$ is

(a) $\left[0, \sqrt{3}\right]$ (b) $(0, \sqrt{3})$

(c) $[0, \sqrt{3})$ (d) $(0, \sqrt{3}]$

Solution (a) For $f(x)$ to be defined, $\frac{\pi^2}{9} - x^2 \ge 0$

$\Rightarrow -\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

∴ Domain of $f = \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

The greatest value of $f(x) = \tan\sqrt{\frac{\pi^2}{9} - 0}$, when $x = 0$ and the least value of $f(x) = \tan\sqrt{\frac{\pi^2}{9} - \frac{\pi^2}{9}}$, when $x = \frac{\pi}{3}$.

∴ The greatest value of $f(x) = \sqrt{3}$ and the least value of $f(x) = 0$

∴ Range of $f = [0, \sqrt{3}]$.

24. The range of the function $f(x) = \sin x - \cos x$ is

(a) $(-\sqrt{2}, \sqrt{2})$ (b) $[-\sqrt{2}, \sqrt{2}]$

(c) $[0, \sqrt{2}]$ (d) none of these

Solution (b) We have, $f(x) = \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$.

Clearly, $f(x)$ is defined for all real x.

∴ Domain of $f = (-\infty, \infty)$.

Let $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

$\Rightarrow x - \frac{\pi}{4} = \sin^{-1}\frac{y}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4} + \sin^{-1}\frac{y}{\sqrt{2}}$.

For x to be real, $-1 \le \frac{y}{\sqrt{2}} \le 1 \Rightarrow -\sqrt{2} \le y \le \sqrt{2}$.

∴ Range of $y = [-\sqrt{2}, \sqrt{2}]$.

25. The period of the function $f(x) = |\sin 4x| + |\cos 4x|$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}$

(c) $\frac{\pi}{4}$ (d) none of these

Solution (b) Since, $|\sin x| + |\cos x|$ is a periodic function with period $\frac{\pi}{2}$, $\therefore f(x) = |\sin 4x| + |\cos 4x|$ is a periodic function with period $\frac{1}{4}\frac{\pi}{2}$ i.e., $\frac{\pi}{8}$.

26. The period of the function

$$f(x) = \cos\left(\frac{\pi x}{n!}\right) - \sin\left(\frac{\pi x}{(n+1)!}\right) \text{ is}$$

(a) $2(n + 1)!$ (b) $2(n!)$

(c) $(n + 1)$ (d) not periodic

Solution (a) Since $\sin x$ and $\cos x$ are periodic functions with period 2π.

∴ Period of $\cos\left(\frac{\pi x}{n!}\right) = \frac{2\pi}{\pi / n!} = 2(n!)$

and period of $\sin\left(\frac{\pi x}{(n+1)!}\right) = \frac{2\pi}{\pi/(n+1)!} = 2(n + 1)!$

∴ Period of $f(x)$ = L.C.M. of $\{2(n!), 2(n + 1)!\}$
$= 2(n + 1)!$.

27. The period of the function $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is

(a) 2
(b) 1
(c) 4
(d) none of these

Solution (b)

28. If the period of the function $f(x) = \sin(\sqrt{[n]}\,x)$, where $[n]$ denotes the greatest integer less than or equal to n, is 2π, then

(a) $1 \le n < 2$
(b) $1 < n < 2$
(c) $1 \le n \le 2$
(d) none of these

Solution (a)

29. The value of $n \in I$ for which the function

$$f(x) = \frac{\sin nx}{\sin\left(\frac{x}{n}\right)} \text{ has } 4\pi \text{ as its period is}$$

(a) 2
(b) 3
(c) 4
(d) 5

Solution (a)

30. Range of the function f defined by $f(x) = \left[\frac{1}{\sin\{x\}}\right]$ (where [.] and {.} respectively denote the greatest integer and the fractional part functions) is

(a) Z, the set of integers
(b) N, the set of natural numbers
(c) W, the set of whole numbers
(d) $\{2, 3, 4, \ldots\}$

Solution (b)

31. The number of solutions of the equation $a^{f(x)} + g(x) = 0$, $a>0$, $g(x) \ne 0$ and has minimum value $\frac{1}{2}$ is

(a) One
(b) Two
(c) Zero
(d) Intinitely many

Solution (c)

32. If $e^x + e^{f(x)} = e$, then range of the function of f is

(a) $(-\infty, 1]$
(b) $(-\infty, 1)$
(c) $(1, \infty)$
(d) $[1, \infty)$

Solution (b)

33. The domain of the function $f(x) = \log_{\left[x+\frac{1}{2}\right]} |x^2 - x - 6|$, where $[\cdot]$ denotes the greatest integer function, is

(a) $\left(\frac{3}{2}, 3\right) \cup (3, \infty)$
(b) $\left[\frac{3}{2}, 3\right) \cup (3, \infty)$
(c) $\left[\frac{3}{2}, \infty\right)$
(d) none of these

Solution (b)

34. Period of the function $\log(\sin^{-1}(x - [x])$([.] denotes the greatest integer function) is

(a) 1
(b) 2π
(c) $\pi/2$
(d) none of these

Solution (a)

35. The function $f: R \to R$, defined by

$$f(x) = x - [x], \forall x \in R \text{ is}$$

(a) one-one
(b) onto
(c) Both one-one and onto
(d) neither one-one nor onto

Solution (b)

36. Let $A = \{x \in R \mid -1 \le x \le 1\} = B$. Then the mapping $f: A \to B$ given by $f(x) = x\,|x|$ is

(a) Injective but not surjective
(b) Surjective but not injective
(c) Bijective
(d) none of these

Solution (c)

37. The function $f: R \to R$, defined by $f(x) = [x]$, $\forall x \in R$, is

(a) one-one
(b) onto
(c) Both one-one and onto
(d) neither one-one nor onto

Solution (d)

38. The inverse of the function $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$ is

(a) $\frac{1}{2}\log_a\left(\frac{1-x}{1+x}\right)$
(b) $\frac{1}{2}\log_a\left(\frac{1+x}{1-x}\right)$
(c) $\log_a\left(\frac{1+x}{1-x}\right)$
(d) none of these

Solution (b)

39. Let $f: R \to R$ be a function defined by

$$f(x) = x + \sqrt{x^2}, \text{ then } f \text{ is}$$

(a) injective
(b) surjective
(c) bijective
(d) none of these

Solution (d)

40. Let $f: R \to R$ be a function defined by $f(x) = \frac{x^2 - 8}{x^2 + 2}$, then f is

(a) one-one but not onto
(b) one-one and onto
(c) onto but not one-one
(d) neither one-one nor onto

Solution (d)

Problems for Practice

Level of Difficulty I

1. The domain of the function $f(x) = \sqrt{2-2x-x^2}$ is

(a) $-2-\sqrt{3} \le x \le -2+\sqrt{3}$

(b) $-2 \le x \le 2$

(c) $-1-\sqrt{3} \le x \le -1+\sqrt{3}$

(d) $-\sqrt{3} \le x \le \sqrt{3}$

2. Domain of $\sqrt{4x-x^2}$ is

(a) $R\backslash[0, 4]$ (b) $R\backslash(0, 4)$

(c) $(0, 4)$ (d) $[0, 4]$

3. The domain of the function $y = f(x) =$ is

(a) $(-\infty, 1)$ (b) $(0, \infty)$

(c) $(1, \infty)$ (d) $(0, 1)$

4. The domain of the function

$$f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x) \text{ is}$$

(a) $(1, 4)$ (b) $[1, 4]$

(c) $[1, 4)$ (d) $(1, 4]$

5. The domain of the function

$$f(x) = \frac{1}{\log_3(x-2)} + \sqrt{5-x} \text{ is}$$

(a) $[2, 5]\backslash\{3\}$ (b) $(2, 5]\backslash\{3\}$

(c) $[2, 5)\backslash\{3\}$ (d) $(2, 5)\backslash\{3\}$

6. The domain of definition of $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$ is

(a) $(-\infty, \infty)\backslash[-1, 1]$

(b) $(-\infty, \infty)\backslash[-2, 2]$

(c) $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

(d) none of these

7. The domain of the function $f(x) = \log_{10}|4-x^2|$ is

(a) $(-\infty, \infty)\backslash\{-2, 2\}$

(b) $(0, \infty)$

(c) $(-\infty, 0)$

(d) none of these

8. The domain of the function

$f(x) = \log_{10}[1 - \log_{10}(x^2 - 5x + 16)]$ is

(a) $(2, 3)$ (b) $[2, 3]$ (c) $(2, 3]$ (d) $[2, 3)$

9. The range of the function $y = \dfrac{x}{1+x^2}$ is

(a) $\left[0, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\left[-\frac{1}{2}, 0\right]$ (d) none of these

10. The range of the function $y = \sin^{-1}\left(\dfrac{x^2}{1+x^2}\right)$ is

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$

(c) $\left[0, \frac{\pi}{2}\right]$ (d) none of these

11. The range of the function $y = 3\sin\sqrt{\dfrac{\pi^2}{16} - x^2}$ is

(a) $\left[0, \frac{3}{\sqrt{2}}\right]$ (b) $\left[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]$

(c) $\left[-\frac{3}{\sqrt{2}}, 0\right]$ (d) none of these

12. The period of the function $f(x) = \cos\left(\dfrac{8x+5}{4\pi}\right)$ is

(a) 2π (b) π (c) π^2 (d) none of these

13. The period of the function $f(x) = 2\sin x + 3\cos 2x$ is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) none of these

14. The period of the function $f(x) = a\sin kx + b\cos kx$ is

(a) $\frac{2\pi}{k}$ (b) $\frac{2\pi}{|k|}$ (c) $\frac{\pi}{|k|}$ (d) none of these

15. The period of the function $f(x) = \sin^4 2x + \cos^4 2x$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{4}$ (d) none of these

16. The function $f(x) = \sec[\log(x + \sqrt{1+x^2})]$ is

(a) even (b) odd

(c) constant (d) None of these

17. The number of bijective functions from a set A to itself when A contains 106 elements is

(a) 106 (b) $(106)^2$ (c) $(106)!$ (d) 2^{106}

18. The number of surjections from $A = \{1, 2, ..., n\}$, $n \ge 2$ onto $B = \{a, b\}$ is

(a) ${}^nP_2$ (b) $2^n - 2$ (c) $2^n - 1$ (d) none of these

19. If $A = \left\{x : -\frac{2}{5} \le x \le \frac{\pi-2}{5}\right\}$, $B = \{y : -1 \le y \le 1\}$ and $f(x) = \cos(5x+2)$, then the mapping $f: A \to B$ is

(a) one-one but not onto

(b) onto but not one-one

(c) Both one-one and onto

(d) neither one-one nor onto

20. The value of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \ne 0$ is the inverse of itself, is

(a) -2 (b) -1 (c) 1 (d) 2

Level of Difficulty II

21. The domain of the function $f(x) = \log_x \cos x$ is

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\backslash\{1\}$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\backslash\{1\}$

(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) none of these

22. The domain of the function

$f(x) = \log_3[-(\log_3 x)^2 + 5\log_3 x - 6]$ is

(a) $(0, 9) \cup (27, \infty)$

(b) $[9, 27]$ (c) $(9, 27)$

(d) none of these

23. The domain of definition of the function $y(x)$ given by the equation $2^x + 2^y = 2$ is

(a) $0 < x \le 1$ (b) $0 \le x \le 1$

(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

24. The domain of the function $f(x) = x^{\frac{1}{\log x}}$ is

(a) $(0, \infty)\backslash\{1\}$ (b) $(0, \infty)$
(c) $[0, \infty)$ (d) $[0, \infty)\backslash\{1\}$

25. The function $f(x) = k|\cos x| + k^2|\sin x| + \phi(k)$ has period $\frac{\pi}{2}$ if k is equal to

(a) 1 (b) 2 (c) 3 (d) none of these

26. The period of the function $f(x) = \frac{|\sin x|-|\cos x|}{|\sin x+\cos x|}$ is

(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) none of these

27. The period of the function

$$f(x) = \begin{Bmatrix} 1, \text{ when } x \text{ is a rational} \\ 0, \text{ when } x \text{ is irrational} \end{Bmatrix} \text{ is}$$

(a) 1 (b) 2
(c) non-periodic (d) none of these

28. The period of the function

$$f(x) = \cos\frac{2\pi x}{5} + \sin\frac{\pi x}{4} \text{ is}$$

(a) 5 (b) 8 (c) 12 (d) 40

29. Let $f(x) = (-1)^{[x]}$ (where [.] denotes the greatest integer function), then

(a) Range of f is $\{-1, 1\}$
(b) f is an even function
(c) f is an odd function
(d) $\lim\limits_{x \to n} f(x)$ exists, for every integer n

30. A function whose graph is symmetrical about the origin is given by

(a) $f(x) = (3^x + 3^{-x})$
(b) $f(x) = \cos[\log(x + \sqrt{1+x^2})]$
(c) $f(x + y) = f(x) + f(y) \ \forall\ x, y \in R$
(d) none of these

31. The inverse of the function $y = [1 - (x - 3)^4]^{1/7}$ is

(a) $3 + (1 - x^7)^{1/4}$ (b) $3 - (1 - x^7)^{1/4}$
(c) $3 - (1 + x^7)^{1/4}$ (d) none of these

32. If $f(x) = \frac{1}{1-x}$, $x \neq 0, 1$, then the graph of the function $y = f[f\{f(x)\}]$, $x > 1$ is

(a) a straight line (b) a circle
(c) an ellipse (d) a pair of straight lines

33. If $f(x) = \frac{1-x}{1+x}$; $x \neq 0$ then $f[f(x)] + f\left[f\left(\frac{1}{x}\right)\right]$

(a) < 2 (b) ≥ 2
(c) $= 2$ (d) none of these

34. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $\left(0, \frac{1}{2}\right]$
(c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

35. The domain of definition of the funtion

$$f(x) = \sqrt{\log_{10}\left(\frac{5x - x^2}{4}\right)} \text{ is}$$

(a) [1, 4] (b) [1, 0] (c) [0, 5] (d) [5, 0]

36. A function f from the set of natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases} \text{ is}$$

(a) one-one but not onto
(b) onto but not one-one
(c) one-one and onto both
(d) neither one-one nor onto

37. Let $f(x)$ be a polynomial funtion of second degree. If $f(1) = f(-1)$ and a, b, c are in AP, then $f'(a)$, $f'(b)$ and $f'(c)$ are in

(a) A. P. (b) G. P.
(c) H. P. (d) arithmetico-geometric progression

38. If $f : R \to R$ satisfies $f(x + y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum\limits_{r=1}^{n} f(r)$ is

(a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$
(c) $7n(n + 1)$ (d) $\frac{7n(n+1)}{2}$

39. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then

(a) $f(x) = f(-x)$
(b) $f(2 + x) = f(2 - x)$
(c) $f(x + 2) = f(x - 2)$
(d) $f(x) = -f(-x)$

40. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)
(c) [2, 3] (d) [1, 2)

ANSWERS

1. (c)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (b)
11. (a)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (b)	**19.** (c)	**20.** (b)
21. (d)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (d)	**28.** (d)	**29.** (a)	**30.** (c)
31. (a)	**32.** (a)	**33.** (b)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (a)	**38.** (d)	**39.** (b)	**40.** (b)

Limits

11

BRIEF REVIEW OF THE CONCEPTS

LIMIT DEFINED

Let a function f be defined at every point in the neighbourhood of a (an open interval about a) except possibly at a. If as x approaches closer and closer to a, but not equal to a, then the value of the function $f(x)$ approaches a real number l. The number l is referred to as the limit of $f(x)$ as x tends to a and we write it as

$$\lim_{x \to a} f(x) = l$$

Note that $f(x)$ approaches l means the absolute difference between $f(x)$ and l, *i.e.*, $|f(x) - l|$ can be made as small as we please.

When the values of $f(x)$ do not approach a single finite value as x approaches a, we say that the limit does not exist.

Righthand Limit We say that right hand limit of $f(x)$ as x tends to 'a' exists and is equal to l_1 if as x approaches 'a' through values greater than 'a', the values of $f(x)$ approach a definite unique real number l_1 and we write

$$\lim_{x \to a^+} f(x) = l_1 \quad \text{or} \quad f(a + 0) = l_1.$$

Working Rule

To evaluate $\lim_{x \to a^-} f(x)$

1. Put $x = a + h$ in $f(x)$ to get $\lim_{h \to 0} f(a + h)$
2. Take the limit as $h \to 0$.

Lefthand Limit We say that left hand limit of $f(x)$ as x tends to 'a' exists and is equal to l_2 if as x approaches 'a' through values less than 'a', the values of $f(x)$ approach a definite unique real number l_2 and we write

$$\lim_{x \to a} f(x) = l_2 \quad \text{or} \quad f(a - 0) = l_2.$$

Working Rule

To evaluate $\lim_{x \to a^-} f(x)$.

1. Put $x = a - h$ in $f(x)$ to get $\lim_{h \to 0} f(a - h)$.
2. Take the limit as $h \to 0$.

ALGEBRA OF LIMITS

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then following results are true:

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m.$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m.$

3. $\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x) = kl,$

where k is a constant.

4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = lm.$

5. $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{l}{m}$ (provided $m \neq 0$).

6. $\lim_{x \to a} (fog)(x) = \lim_{x \to a} f[g(x)] = f\left(\lim_{x \to a} g(x)\right) = f(m).$

In particular,

(i) $\lim_{x \to a} \log g(x) = \log\left(\lim_{x \to a} g(x)\right) = \log m.$

(ii) $\lim_{x \to a} e^{g(x)} = e^{\lim_{x \to a} g(x)} = e^m.$

7. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = l^n,$ for all $n \in N.$

8. *Sandwich Theorem (or Squeeze Principle).*

If f, g and h are functions such that $f(x) \leq g(x) \leq h(x)$ for all x in some neighbourhood of the point a (except possibly at $x = a$) and if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l.$

EVALUATION OF LIMITS

The problems on limits can be divided into the following categories:

Algebraic Limits

The following methods are useful for evaluating limits of algebraic functions:

Method of Factorization If $f(x)$ and $g(x)$ are polynomials and $g(a) \neq 0$, then we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}.$$

Now, if $f(a) = 0 = g(a)$, then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. We cancel this common factor $(x - a)$ from both the numerator and denominator and again put $x = a$ in the given expression. If we get a meaningful number then that number is the limit of the given expression, otherwise we repeat this process till we get a meaningful number.

Method of Rationalization This method is useful where radical signs (*i.e.*, expressions of the form $\sqrt{a} \pm \sqrt{b}$) are involved either in the numerator or in the denominator or both. The numerator or (and) the denominator (as required) is (are) rationalised and limit taken after cancelling out the common factors.

Standard Formula $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, where $n \in Q$, the set of rational numbers.

Limit of an Algebraic Function when $x \to \infty$

In order to find the limit of a function of the type $\dfrac{f(x)}{g(x)}$ as $x \to \infty$, where $f(x)$ and $g(x)$ are algebraic functions of x, it is convenient to divide all the terms of $f(x)$ and $g(x)$ by the highest power of x in numerator and denominator both and use the following standard limits:

(i) $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

(ii) $\lim\limits_{x \to \infty} \dfrac{1}{x^p} = 0$, if $p > 0$.

Some Useful Limits

(i) $$\lim_{n \to \infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } -1 < a < 1 \\ \text{does not exist}, & \text{if } a \le -1 \end{cases}$$

(ii) $$\lim_{x \to \infty} \frac{a_0x^p + a_1x^{p-1} + \ldots\ldots + a_{p-1}x + a_p}{b_0x^q + b_1x^{q-1} + \ldots\ldots + b_{q-1}x + b_q} = \begin{cases} \dfrac{a_0}{b_0}, & \text{if } p = q \\ 0, & \text{if } p < q \\ \infty, & \text{if } p > q \end{cases}$$

Working Rule

(i) $\Sigma n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

(ii) $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

(iv) $\Sigma ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$;

provided $r < 1$.

Trigonometric Limits

For finding the limits of trigonometric functions, we use trigonometric transformations and simplify. The following results are quite useful.

1. (i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (ii) $\lim\limits_{x \to 0} \cos x = 1$

(iii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$ (iv) $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1$

(v) $\lim\limits_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1$ (vi) $\lim\limits_{x \to 0} \dfrac{\sin x^0}{x} = \dfrac{\pi}{180}$.

2. $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$, where $a \neq 0$, on taking $x = a+h$.

Some Useful Expansions

(i) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$ to ∞

(ii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$ to ∞

(iii) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots$ to ∞

(iv) $\sin^{-1}x = x$
$+ \dfrac{1^2 \cdot x^3}{3!} + \dfrac{1^2 \cdot 3^2}{5!} x^5 + \dfrac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \ldots$ to ∞

(v) $(\sin^{-1}x)^2$
$= \dfrac{2}{2!}x^2 + \dfrac{2 \cdot 2^2}{4!}x^4 + \dfrac{2 \cdot 2^2 \cdot 4^2}{6!}x^6 + \ldots$ to ∞

(vi) $\tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$ to ∞.

Exponential and Logarithmic Limits

For finding the limits of exponential and logarithmic functions, the following results are useful:

(i) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ (ii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$, $a > 0$

(iii) $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \log_e\left(\dfrac{a}{b}\right)$; $a, b > 0$

(iv) $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x} = n$

(v) $\lim\limits_{x \to 0} (1+x)^{1/x} = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e$

(vi) $\lim\limits_{h \to 0} (1+ah)^{1/h} = e^a$ (vii) $\lim\limits_{x \to \infty} \dfrac{\log x}{x^m} = 0$, $(m > 0)$

(viii) $\lim\limits_{x \to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$, $(a > 0, a \neq 1)$

(ix) $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x = e^a$

(x) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{f(x)}\right)^{f(x)} = e$, where $f(x) \to \infty$ as $x \to \infty$.

(xi) $\lim\limits_{x \to a} (1 + f(x)^{1/f(x)} = e$.

Some Useful Expansions

(i) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ to ∞

(ii) $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots$ to ∞

(iii) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ to ∞, $-1 < x \le 1$

(iv) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$ to ∞, $-1 \le x < 1$

(v) $a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2!} + \dots$ to ∞

(vi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ to ∞, $-1 < x < 1$,

n being any negative integer or fraction.

The expansion formulae mentioned above can be used with advantage in simplification and evaluation of limits.

For example, $\lim\limits_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

$$= \lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x \to 0} \left[\frac{1}{4!} + \text{terms containing } x \text{ and its powers}\right]$$

$$= \frac{1}{4!} = \frac{1}{24}.$$

Working Rule

(i) If $\lim\limits_{x \to a} f(x) = A > 0$ and $\lim\limits_{x \to a} g(x) = B$, then

$$\lim_{x \to a} [f(x)]^{g(x)} = A^B.$$

(ii) If $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim\limits_{x \to a} g(x)[f(x) - 1]}.$$

EVALUATION OF LIMITS USING L'HOSPITAL'S RULE

Besides the methods given above to evaluate limits, there is yet another method for finding limits, usually known as L'Hospital's Rule as given below for in deter minant forms:

(i) $\left(\frac{0}{0}\right)$ form : If $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, then

$\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

(ii) $\left(\frac{\infty}{\infty}\right)$ form : If $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, then

$\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

Note that sometimes we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.

Working Rule

(i) L'Hospital's Rule is applicable only when $\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(ii) If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, simplify the given expression till it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hospital's rule.

(iii) For applying L'Hospital's rule differentiate the numerator and denominator separately.

Solved Problems

1. The value of $\lim\limits_{x \to 0} \frac{e^x - (1+x)}{x^2}$ is

(a) $\frac{1}{2}$ (b) 1

(c) 0 (d) $\frac{1}{4}$

Solution (a) $\lim\limits_{x \to 0} \frac{e^x - (1+x)}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} = \frac{1}{2} \times 1 = \frac{1}{2}.$$

2. $\lim\limits_{x \to 0} \frac{(1-x)^n - 1}{x}$ is

(a) $(n-1)!$ (b) $n!$

(c) n (d) $-n$

Solution (d) $\lim\limits_{x \to 0} \frac{(1-x)^n - 1}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{n(1-x)^{n-1}(-1)}{1} = -n.$$

[Using L'Hospital's Rule]

3. If α is a repeated root of $ax^2 + bx + c = 0$, then

$\lim\limits_{x \to \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2}$ is

(a) a (b) b

(c) c (d) 0

Solution (a) $\lim\limits_{x\to\alpha}\dfrac{\tan(ax^2+bx+c)}{(x-\alpha)^2}$

$$\left(\frac{0}{0}\text{ form as } a\alpha^2+b\alpha+c=0\right)$$

$$=\lim_{x\to\alpha}\frac{(2ax+b)\sec^2(ax^2+bx+c)}{2(x-\alpha)}$$

$$\left(\frac{0}{0}\text{ form as } \alpha \text{ being a repeated root of } ax^2+bx+c=0,\ 2a\alpha+b=0\right)$$

$$=\lim_{x\to\alpha}\frac{2a\sec^2(ax^2+bx+c)+(2ax+b)^2\times 2\sec^2(ax^2+bx+c)\tan(ax^2+bx+c)}{2}=\frac{2a}{2}=a.$$

4. $\lim\limits_{x\to0}\dfrac{e^x+e^{-x}+2\cos x-4}{x^4}$ is equal to

(a) 0 (b) 1

(c) $\dfrac{1}{6}$ (d) $-\dfrac{1}{6}$

Solution (c) $\lim\limits_{x\to0}\dfrac{e^x+e^{-x}+2\cos x-4}{x^4}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to0}\frac{e^x-e^{-x}-2\sin x}{4x^3}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{e^x+e^{-x}-2\cos x}{12x^2}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{e^x-e^{-x}+2\sin x}{24x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{e^x+e^{-x}+2\cos x}{24}=\frac{4}{24}=\frac{1}{6}.$$

5. $\lim\limits_{x\to0^+}\log_{\tan x}(\sin x)$ is equal to

(a) 1 (b) -1

(c) 0 (d) none of these

Solution (a) $\lim\limits_{x\to0^+}\log_{\tan x}(\sin x)=\lim\limits_{x\to0^+}\dfrac{\log_e(\sin x)}{\log_e(\tan x)}$

$$\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to0^+}\frac{\cos x/\sin x}{\sec^2 x/\tan x}=\lim_{x\to0}\frac{1}{\sec^2 x}=1.$$

6. If $f(x)$, $g(x)$ be differentiable functions and $f(1)=g(1)=2$ then $\lim\limits_{x\to1}\dfrac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)}$ is equal to

(a) 0 (b) 1

(c) 2 (d) none of these

Solution (c)

$$\lim_{x\to1}\frac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to1}\frac{f(1)g'(x)-f'(x)g(1)}{g'(x)-f'(x)}=2\lim_{x\to1}\frac{g'(x)-f'(x)}{g'(x)-f'(x)}=2.$$

7. Let $f(x)$ be a twice differentiable function and $f''(0)=5$, then $\lim\limits_{x\to0}\dfrac{3f(x)-4f(3x)+f(9x)}{x^2}$ is equal to

(a) 30 (b) 120

(c) 40 (d) none of these

Solution (b) $\lim\limits_{x\to0}\dfrac{3f(x)-4f(3x)+f(9x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to0}\frac{3f'(x)-12f'(3x)+9f'(9x)}{2x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{3f''(x)-36f''(3x)+81f''(9x)}{2}$$

$$=\frac{3f''(0)-36f''(0)+81f''(0)}{2}=24f''(0)$$

$= 24 \cdot 5 = 120.$

8. If $f(9)=9$ and $f'(9)=1$, then $\lim\limits_{x\to9}\dfrac{3-\sqrt{f(x)}}{3-\sqrt{x}}$ is equal to

(a) 0 (b) 1

(c) -1 (d) none of these

Solution (b) $\lim\limits_{x\to9}\dfrac{3-\sqrt{f(x)}}{3-\sqrt{x}}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to9}\frac{0-\dfrac{1}{2\sqrt{f(x)}}\cdot f'(x)}{0-\dfrac{1}{2\sqrt{x}}}\quad\text{[Using L'Hospital's Rule]}$$

$$=\lim_{x\to9}\frac{\sqrt{x}}{\sqrt{f(x)}}\cdot f'(x)=\frac{3}{3}\cdot f'(9)=1.$$

9. If $f(5)=7$ and $f'(5)=7$, then $\lim\limits_{x\to5}\dfrac{xf(5)-5f(x)}{x-5}$ is given by

(a) -28 (b) 28

(c) 35 (d) -35

Solution (a) $\lim\limits_{x\to5}\dfrac{xf(5)-5f(x)}{x-5}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to5}\frac{f(5)-5f'(x)}{1-0}\quad\text{[Using L'Hospital's Rule]}$$

$=f(5)-5f'(5)=7-5\cdot7=-28.$

10. If $\lim\limits_{x\to0}\dfrac{\sin 2x+a\sin x}{x^3}$ be finite, then the value of a and the limit are given by

(a) -2, 1 (b) -2, -1

(c) 2, 1 (d) 2, -1

Solution (b) Let $k=\lim\limits_{x\to0}\dfrac{\sin 2x+a\sin x}{x^3}$ $\left(\dfrac{0}{0}\right)$

$$=\lim_{x\to0}\frac{2\cos 2x+a\cos x}{3x^2}$$

[Using L'Hospital's Rule]

We require 2 cos2x + a cos x = 0 for x = 0 as denominator is zero. ∴ a = – 2.

$$\text{Hence} \quad k = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1.$$

11. If $g(x) = -\sqrt{25-x^2}$, then $\lim_{x\to 1}\dfrac{g(x)-g(1)}{x-1}$ is equal to

(a) $\dfrac{3}{\sqrt{24}}$ (b) $\dfrac{1}{\sqrt{24}}$

(c) $-\dfrac{1}{\sqrt{24}}$ (d) none of these

Solution (b) $\lim_{x\to 1}\dfrac{g(x)-g(1)}{x-1}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 1}\frac{g'(x)-0}{1-0} \qquad \text{[Using L'Hospital's Rule]}$$

$$= \lim_{x\to 1}\frac{-1}{2\sqrt{25-x^2}}\cdot(-2x) = \lim_{x\to 1}\frac{x}{\sqrt{25-x^2}} = \frac{1}{\sqrt{24}}.$$

12. If α and β be the roots of $ax^2 + bx + c = 0$, then $\lim_{x\to\alpha}(1 + ax^2 + bx + c)^{1/(x-\alpha)}$ is

(a) log |a (α – β)| (b) $e^{a(\alpha-\beta)}$

(c) $e^{a(\beta-\alpha)}$ (d) none of these

Solution (b) $\lim_{x\to\alpha}(1 + ax^2 + bx + c)^{1/(x-\alpha)}$

$$= e^{\lim_{x\to\alpha}\frac{1}{(x-\alpha)}\left[(1+ax^2+bx+c)-1\right]}$$

[Using $\lim_{x\to a}$ $[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$

provided $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$]

$$= e^{\lim_{x\to\alpha}\frac{(ax^2+bx+c)}{(x-\alpha)}} = e^{\lim_{x\to\alpha}\frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}}$$

[∵ α, β are roots of $ax^2 + bx + c = 0$]

$= e^{a(\alpha-\beta)}$.

13. If f (a) = 2, f'(a) = 1, g(a) = – 1, g'(a) = 2, then $\lim_{x\to a}\dfrac{g(x)f(a)-g(a)f(x)}{x-a}$ is equal to

(a) 3 (b) 5

(c) – 3 (d) 0

Solution (b) $\lim_{x\to a}\dfrac{g(x)f(a)-g(a)f(x)}{x-a}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to a}\frac{f(a)g'(x)-g(a)f'(x)}{1-0} \qquad \text{[Using L'Hospital's Rule]}$$

$= f(a)\,g'(a) - g(a)\,f'(a) = 2 \times 2 - (-1)\cdot 1 = 5.$

14. $\lim_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}$, $a \neq n\pi$, n is an integer, equals

(a) $e^{\cot a}$ (b) $e^{\tan a}$

(c) $e^{\sin a}$ (d) $e^{\cos a}$

Solution (a) $\lim_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}$

$$= \lim_{x\to a}\left(1+\frac{\sin x-\sin a}{\sin a}\right)^{\frac{1}{x-a}}$$

$$= \lim_{x\to a}\left[\left\{1+\left(\frac{\sin x-\sin a}{\sin a}\right)\right\}^{\frac{\sin a}{\sin x-\sin a}}\right]^{\frac{\sin x-\sin a}{(x-a)\sin a}}$$

$$= \lim_{e^{x\to a}}\frac{\sin x-\sin a}{(x-a)\sin a}$$

$$= \lim_{e^{x\to a}}\frac{2}{x-a}\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\cdot\frac{1}{\sin a}$$

$$= \lim_{e^{x\to a}}\cos\left(\frac{x+a}{2}\right)\left[\sin\left(\frac{x-a}{2}\right)\Big/\left(\frac{x-a}{2}\right)\right]\frac{1}{\sin a}$$

$$= e^{\frac{\cos a}{\sin a}} = e^{\cot a}.$$

15. $\lim_{x\to 0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{1/x}$ is equal to

(a) e (b) e^2

(c) e^3 (d) e^{-1}

Solution (b) $\lim_{x\to 0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{1/x} = \lim_{x\to 0}\left[\dfrac{1+\tan x}{1-\tan x}\right]^{1/x}$

$$= e^{\lim_{x\to 0}\frac{1}{x}\left[\frac{1+\tan x}{1-\tan x}-1\right]}$$

$$\left[\because \frac{1+\tan x}{1-\tan x}\to 1 \text{ and } \frac{1}{x}\to\infty \text{ as } x\to 0\right]$$

$$= e^{\lim_{x\to 0}\frac{2\tan x}{x(1-\tan x)}} = e^{\lim_{x\to 0}2\left(\frac{\tan x}{x}\right)\cdot\frac{1}{1-\tan x}} = e^2.$$

16. $\lim_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ is equal to

(a) $e^{\pi/2}$ (b) $e^{2/\pi}$

(c) $e^{-2/\pi}$ (d) $e^{-\pi/2}$

Solution (b) $\lim_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\cdot\left[2-\frac{x}{a}-1\right]}$

$$\left[\text{Using } \lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a}g(x)[f(x)-1]}\right.$$

$$\left.\text{as } f(x)\to 1 \text{ and } g(x)\to\infty \text{ as } x\to a\right]$$

$$= e^{\lim_{x\to a}\left(1-\frac{x}{a}\right)\tan\left(\frac{\pi x}{2a}\right)} = e^{\lim_{x\to a}\frac{(1-x/a)}{\cot(\pi x/2a)}}$$

$$= e^{\lim_{x\to a}\frac{-1/a}{-\mathrm{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}} = e^{\lim_{x\to a}\frac{2}{\pi}\sin^2\left(\frac{\pi x}{2a}\right)} = e^{2/\pi}.$$

17. $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{\frac{a}{x}}$ is equal to

(a) e^{-a^2b} (b) e^{ab^2}

(c) e^{a^2b} (d) e^{-b^2a}

Solution (c) $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{\frac{a}{x}}$

$$= e^{\lim\limits_{x\to 0}\frac{a}{x}(\cos x + a\sin bx - 1)}$$

$$\left[\text{Using } \lim_{x\to a}[f(x)]^{\phi(x)} = e^{\lim\limits_{x\to a}\phi(x)[f(x)-1]}\right.$$

as $f(x) \to 1$ and $\phi(x) \to \infty$ as $x \to a$]

$$= e^{\lim\limits_{x\to 0}\frac{a(-\sin x + ab\cos bx)}{1}} = e^{a^2b}.$$

18. $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$ is equal to

(a) e^{-5} (b) e^5

(c) 0 (d) none of these

Solution (b) $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x+1}{5}}\right)^{\frac{x+1}{5}}\right]^{\frac{5(x+4)}{x+1}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{5(x+4)}{(x+1)}} = e^{\lim\limits_{x\to\infty}\frac{5\left(1+\frac{4}{x}\right)}{\left(1+\frac{1}{x}\right)}} = e^5.$$

19. If a, b, c, d are positive, then $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx} =$

(a) $e^{d/b}$ (b) $e^{c/a}$

(c) $e^{(c+d)/a+b}$ (d) e

Solution (a) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx}$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{a+bx}\right)^{a+bx}\right]^{\frac{c+dx}{a+bx}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{c+dx}{a+bx}} = e^{\lim\limits_{x\to\infty}\frac{\frac{c}{x}+d}{\frac{a}{x}+b}} = e^{d/b}.$$

20. The value of $\lim\limits_{n\to\infty}\left(1+\tan\dfrac{b}{n}\right)^{n}$ is

(a) e^b (b) $e^{b/2}$

(c) e (d) none of these

Solution (a) $\lim\limits_{n\to\infty}\left(1+\tan\dfrac{b}{n}\right)^{n} = e^{\lim\limits_{n\to\infty} n\left(1+\tan\frac{b}{n}-1\right)}$

$$= e^{\lim\limits_{n\to\infty} n\tan\frac{b}{n}} = e^{b\lim\limits_{n\to\infty}\frac{\tan b/n}{b/n}} = e^b.$$

21. If $f(x) = \begin{cases}\dfrac{\sin[x]}{[x]}, & [x] \neq 0\\ 0 & ,[x]=0\end{cases}$, where $[x]$ denotes the greatest integer $\leq x$, then $\lim\limits_{x\to 0} f(x)$ equals

(a) 0 (b) – 1

(c) 1 (d) none of these

Solution (d) $\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0}\dfrac{\sin[-h]}{[-h]}$

$$= \lim_{h\to 0}\frac{\sin(-1)}{(-1)} = \sin 1.$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin[h]}{[h]}$$

$= 1$ $[\because h \to 0 \Rightarrow [h] \to 0]$

$\therefore$ $\lim\limits_{x\to 0} f(x)$ does not exist.

22. Let $f(x) = x - [x]$ where $[x]$ denotes the greatest integer $\leq x$ and $g(x) = \lim\limits_{n\to\infty}\dfrac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}$, then $g(x) =$

(a) 0 (b) 1

(c) -1 (d) none of these

Solution (c) As $0 \leq x - [x] < 1$ $\forall$ $x \in R$, $0 \leq f(x) < 1$.

$\therefore$ $\lim\limits_{n\to\infty}\{f(x)\}^{2n} = 0.$

Thus, for $x \in R$, $g(x) = \lim\limits_{n\to\infty}\dfrac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}$

$$= \frac{0-1}{0+1} = -1.$$

23. If $f(x) = \begin{bmatrix}\dfrac{\tan^{-1}([x]+x)}{[x]-2x}, & [x]\neq 0\\ 0 & ,[x]=0\end{bmatrix}$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0} f(x)$ is equal to

(a) $-\dfrac{1}{2}$ (b) 1

(c) $\dfrac{\pi}{4}$ (d) does not exist

Solution (d) LHL $= \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{\tan^{-1}([-h]-h)}{[-h]+2h} = \lim_{h\to 0}\frac{\tan^{-1}(-1-h)}{(2h-1)}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}(1+h)}{(1-2h)} = \frac{\pi/4}{1} = \frac{\pi}{4}.$$

$$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\tan^{-1}([h]+h)}{[h]-2h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}(h)}{-2h} = -\frac{1}{2}.$$

Since L.H.L $\neq$ RHL $\therefore$ $\lim\limits_{x\to 0} f(x)$ does not exist.

24. If $f(x) = \begin{cases} \dfrac{\tan[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$ where

[x] denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0} f(x)$ equals

(a) 1 (b) – 1

(c) 0 (d) does not exist

Solution (d) L.H.L $= \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0}\dfrac{\tan[-h]}{[-h]}$

$$= \lim_{h\to 0}\frac{\tan(-1)}{(-1)} = \tan 1$$

$$\text{R.H.L} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\tan[h]}{[h]} = 0.$$

Since L.H.L ≠ RHL ∴ $\lim\limits_{x\to 0} f(x)$ does not exist.

25. Let $f(x) = \begin{cases} \sin x, & x \neq n\pi, \text{ where } n \in \mathbb{Z} \text{ and} \\ 2, & x = n\pi \end{cases}$

$g(x) = \begin{cases} x^2 + 1, & x \neq 2, \\ 3, & x = 2. \end{cases}$ then $\lim\limits_{x\to 0} g[f(x)]$ is

(a) 1 (b) 0

(c) 3 (d) does not exist

Solution (a) $g[f(x)] = \begin{cases} [f(x)]^2 + 1, & f(x) \neq 2 \\ 3, & f(x) = 2 \end{cases}$

$\therefore \quad g[f(x)] = \begin{cases} \sin^2 x + 1, & x \neq n\pi \\ 3, & x = n\pi \end{cases}$

$$\text{RHL} = \lim_{h\to 0} g[f(0+h)] = \lim_{h\to 0}(\sin^2 h + 1) = 1.$$

$$\text{LHL} = \lim_{h\to 0} g[f(0-h)] = \lim_{h\to 0}(\sin^2 h + 1) = 1.$$

$\therefore \quad \lim\limits_{x\to 0} g[f(x)] = 1.$

26. The value of $\lim\limits_{x\to\infty} x\left[\tan^{-1}\dfrac{x+1}{x+2} - \dfrac{\pi}{4}\right]$ is

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$

(c) 1 (d) – 1

Solution (b) $\lim\limits_{x\to\infty} x\left[\tan^{-1}\dfrac{x+1}{x+2} - \dfrac{\pi}{4}\right]$

$$= \lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}1\right] = \lim_{x\to\infty} x\tan^{-1}\left(\frac{\dfrac{x+1}{x+2} - 1}{1 + \dfrac{x+1}{x+2}}\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{-1}{2x+3}\right) = -\lim_{x\to\infty}\frac{\tan^{-1}\left(\dfrac{1}{2x+3}\right)}{\left(\dfrac{1}{2x+3}\right)}\cdot\frac{1}{\left(2+\dfrac{3}{x}\right)}$$

$$= -1 \times \frac{1}{2} = -\frac{1}{2}.$$

27. $\lim\limits_{n\to\infty}\dfrac{n^k\sin^2(n!)}{n+2}$, $0 < k < 1$, is equal to

(a) ∞ (b) 1

(c) 0 (d) none of these

Solution (c) $\lim\limits_{n\to\infty}\dfrac{n^k\sin^2(n!)}{n+2} = \lim\limits_{n\to\infty}\dfrac{n^k\sin^2(n!)}{n\left(1+\dfrac{2}{n}\right)}$

$$= \lim_{n\to\infty}\frac{\sin^2(n!)}{n^{1-k}\left(1+\dfrac{2}{n}\right)} = \frac{\text{a finite quantity}}{\infty}$$

[∵ $\sin^2(n!)$ always lies between 0 and 1. Also, since $1 - k > 0$, ∴ $n^{1-k} \to \infty$ as $n \to \infty$]

= 0.

28. $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(a) exists and it equals $\sqrt{2}$

(b) exists and it equals $-\sqrt{2}$

(c) does not exist because $(x - 1) \to 0$

(d) does not exist because left hand limit is not equal to right hand limit

Solution (d) $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1} = \lim\limits_{x\to 1}\dfrac{\sqrt{2\sin^2(x-1)}}{x-1}$

$$= \lim_{x\to 1}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)}$$

$$\text{LHL} = \lim_{x\to 1^-}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)} = \lim_{h\to 0} - \frac{\sqrt{2}\,|\sin(1-h-1)|}{(1-h-1)}$$

$$= \lim_{h\to 0}\frac{\sqrt{2}\,|-\sin h|}{-h} = -\sqrt{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$= -\sqrt{2}\cdot 1 = -\sqrt{2}.$$

$$\text{RHL} = \lim_{x\to 1^+}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)} = \lim_{h\to 0}\frac{\sqrt{2}\,|\sin(1+h-1)|}{(1+h-1)}$$

$$= \lim_{h\to 0}\frac{\sqrt{2}\,|\sin h|}{h} = \sqrt{2}\lim_{h\to 0}\frac{\sin h}{h} = \sqrt{2}\cdot 1 = \sqrt{2}.$$

Since LHL ≠ RHL,

$\therefore \quad \lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$ does not exist.

29. The value of $\lim\limits_{n\to\infty}\left[\dfrac{1}{n} + \dfrac{e^{1/n}}{n} + \dfrac{e^{2/n}}{n} + \ldots + \dfrac{e^{(n-1)/n}}{n}\right]$ is

(a) 1 (b) 0

(c) $e - 1$ (d) $e + 1$

Solution (c) $\lim\limits_{n\to\infty}\left[\dfrac{1}{n} + \dfrac{e^{1/n}}{n} + \dfrac{e^{2/n}}{n} + \ldots + \dfrac{e^{(n-1)/n}}{n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1 + e^{1/n} + (e^{1/n})^2 + \ldots + (e^{1/n})^{n-1}}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1\cdot\left[(e^{1/n})^n-1\right]}{n(e^{1/n}-1)} = (e-1)\lim_{n\to\infty}\frac{1}{\left(\dfrac{e^{1/n}-1}{1/n}\right)}$$

$$= (e-1)\times 1 = (e-1).$$

30. $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}(1+\cos^{2m} n!\pi x)$ is equal to

(a) 2 (b) 1

(c) 0 (d) none of these

Solution (a), (b) We know that $|\cos\theta| \le 1$ for all θ.
So, if $|\cos n!\,\pi x| < 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = (1+0) = 1$$

and if $|\cos n!\,\pi x| = 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = \lim_{m\to\infty}\lim_{n\to\infty}(1+1^{2m})$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}(1+1) = 2.$$

31. The values of constants a and b so that

$\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right) = 0$ are

(a) $a = 1, b = -1$
(b) $a = -1, b = 1$
(c) $a = 0, b = 0$
(d) $a = 2, b = -1$

Solution (a) We have, $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right) = 0$

$$\Rightarrow \lim_{x\to\infty}\frac{(x^2+1)-(ax+b)(x+1)}{x+1} = 0$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-(a+b)x-b+1}{x+1} = 0$$

$\Rightarrow$ $1 - a = 0$ and $a + b = 0$
$\Rightarrow$ $a = 1$ and $b = -1$.

32. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}\,x}$ is

(a) λ (b) -1

(c) zero (d) does not exist

Solution (d) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}\,x}$

$$= \lim_{x\to 0}\frac{\sqrt{1-(1-2\sin^2 x)}}{\sqrt{2}\,x}$$

$$\lim_{x\to 0}\frac{\sqrt{2\sin^2 x}}{\sqrt{2}\,x} = \lim_{x\to 0}\frac{|\sin x|}{x}$$

So, $f(x) = \dfrac{|\sin x|}{x}$

Now, $f(0+0) = \lim\limits_{h\to 0}\dfrac{|\sin(0+h)|}{0+h}$

$$= \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$f(0-0) = \lim_{h\to 0}\frac{|\sin(0-h)|}{-h}$$

$$= \lim_{h\to 0}\frac{\sin h}{-h} = -1$$

$\therefore$ $f(0+0) \ne f(0-0)$

33. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$ is

(a) e^4 (b) e^2

(c) e^3 (d) e

Solution (a)

34. Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if

$$\lim_{x\to a}\frac{f(a)\,g(x)-f(a)-g(a)\,f(x)+g(a)}{g(x)-f(x)} = 4,$$

then the value of k is

(a) 4 (b) 2

(c) 1 (d) 0

Solution (a)

35. If $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b, are

(a) $a \in \mathbf{R}$, $b = 2$
(b) $a = \mathbf{1}$, $b \in \mathbf{R}$
(c) $a \in R$, $b \in \mathbf{R}$
(d) $a = \mathbf{1}$ and $b = 2$.

Solution (b)

36. $\lim\limits_{n\to\infty}\left[\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\ldots+\dfrac{1}{(2n+1)(2n+3)}\right]$ is equal to

(a) 1 (b) $\dfrac{1}{2}$

(c) $-\dfrac{1}{2}$ (d) none of these

Solution (b)

37. The value of $\lim\limits_{x\to\infty}\left[\dfrac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}\right]^{nx}$ is

(a) $n!$ (b) n

(c) $(n-1)!$ (d) 0

Solution (a)

38. $\lim\limits_{x\to -1}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1} x}}{\sqrt{x+1}}$ is equal to

(a) $\dfrac{1}{\sqrt{2\pi}}$ (b) $\dfrac{1}{\sqrt{\pi}}$

(c) $\dfrac{1}{\sqrt{2}}$ (d) none of these

Solution (a)

39. $\lim\limits_{x\to 0}\left[\dfrac{\sin([x-3])}{[x-3)}\right]$, where [.] represents greatest integer function, is

(a) 0 (b) 1
(c) does not exist (d) sin 1

Solution (c)

40. $\lim\limits_{x\to 1}\dfrac{x\sin(x-[x])}{x-1}$, where [·] denotes the greatest integer function, is equal to

(a) 1 (b) –1
(c) ∞ (d) does not exist

Solution (d)

Problems for Practice

Level of Difficulty I

1. $\lim\limits_{x\to e}\dfrac{\log x-1}{x-e}$ is equal to

(a) $\dfrac{1}{e}$ (b) 1 (c) 0 (d) $\dfrac{1}{2}$

2. $\lim\limits_{x\to 1}(1-x)\tan\left(\dfrac{\pi x}{2}\right)$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{2}{\pi}$ (c) $-\dfrac{\pi}{2}$ (d) $-\dfrac{2}{\pi}$

3. The value of $\lim\limits_{x\to 1}\dfrac{x^7-2x^5+1}{x^3-3x^2+2}$ is

(a) 0 (b) 1 (c) – 1 (d) none of these

4. The value of $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/3}-(1-x)^{1/3}}{x}$ is

(a) $\dfrac{2}{3}$ (b) $\dfrac{1}{3}$ (c) 1 (d) none of these

5. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$ is equal to

(a) 1 (b) – 1 (c) 2 (d) 0

6. $\lim\limits_{x\to 0^+}\log_{\tan x}(\tan 2x)$ is equal to

(a) 1 (b) – 1 (c) 0 (d) none of these

7. For $x\in R$, $\lim\limits_{x\to\infty}\left(\dfrac{x-3}{x+2}\right)^x$

(a) e (b) e^{-1} (c) e^{-5} (d) e^5

8. $\lim\limits_{n\to\infty}\left(\cos\dfrac{x}{n}\right)^n$ is equal to

(a) e^1 (b) e^{-1} (c) 1 (d) none of these

9. The value of $\lim\limits_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is

(a) $\dfrac{1}{2}$ (b) 1 (c) 0 (d) none of these

10. $\lim\limits_{n\to\infty}\left[\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{n(n+1)}\right]$ is equal to

(a) 1 (b) – 1 (c) 0 (d) none of these

11. For $x\in R$, $\lim\limits_{x\to\infty}\left(\dfrac{x-3}{x+2}\right)^x$ is

(a) e (b) e^{-1} (c) e^{-5} (d) e^5

12. Let $f(2) = 4$ and $f'(2) = 4$

Then, $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}$ is given by

(a) 2 (b) –2 (c) –4 (d) 3

13. $\lim\limits_{n\to\infty}\dfrac{1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)}{n^3}$ is equal to

(a) 1 (b) – 1 (c) $\dfrac{1}{3}$ (d) none of these

14. The value of $\lim\limits_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$ is

(a) $4\sqrt{2}$ (log 3)^2 (b) $8\sqrt{2}$ (log 3)^2
(c) $2\sqrt{2}$ (log 3)^2 (d) none of these

15. $\lim\limits_{x\to 1}(1+\cos\pi x)\cot^2\pi x$ is equal to

(a) 1 (b) – 1 (c) $\dfrac{1}{2}$ (d) none of these

Level of Difficulty II

16. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\sin x}-\sqrt[3]{1-\sin x}}{x}$ is

(a) $\dfrac{2}{3}$ (b) $-\dfrac{2}{3}$ (c) $\dfrac{3}{2}$ (d) $-\dfrac{3}{2}$

17. $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{1/\sin x}$ is equal to

(a) 0 (b) 1 (c) – 1 (d) none of these

18. The value of $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$ is

(a) 1 (b) – 1 (c) 0 (d) none of these

19. The value of $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+1}{4x^2-1}\right)^{\frac{x^3}{1+x}}$ is

(a) 0 (b) ∞ (c) 1 (d) – 1

20. The value of $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$ is

(a) 0 (b) 1 (c) – 1 (d) ∞

21. $\lim\limits_{x\to 2}\dfrac{2^x - x^2}{x^x - 2^2}$ is equal to

(a) $\dfrac{\log 2 - 1}{\log 2 + 1}$ (b) $\dfrac{\log 2 + 1}{\log 2 - 1}$

(c) 1 (d) – 1

22. $\lim\limits_{n\to\infty}\dfrac{n!}{(n+1)! - n!}$ is equal to

(a) 0 (b) ∞ (c) 1 (d) none of these

23. The value of $\lim\limits_{x\to\infty}\dfrac{x^5}{5^x}$ is

(a) 1 (b) – 1 (c) 0 (d) none of these

24. The value of $\lim\limits_{n\to\infty}\dfrac{3^n + 2^n}{3^n - 2^n}$ is

(a) – 1 (b) 1 (c) 0 (d) ∞

25. $\lim\limits_{n\to\infty}\cos\left[\pi\sqrt{n^2+n}\right]$, $n \in Z$ is equal to

(a) 0 (b) 1 (c) – 1 (d) None of these

26. If $a = \min\{x^2 + 4x + 5, x \in R\}$ and $b = \lim\limits_{\theta\to 0}\dfrac{1-\cos 2\theta}{\theta^2}$, then the value of $\sum\limits_{r=0}^{n} a^r \cdot b^{n-r}$ is

(a) $\dfrac{2^{n+1}-1}{4\cdot 2^n}$ (b) $2^{n+1} - 1$

(c) $\dfrac{2^{n+1}-1}{3\cdot 2^n}$ (d) none of these

27. $\lim\limits_{n\to\infty}\dfrac{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1)}{n^3}$ is equal to

(a) 1 (b) – 1 (c) $\dfrac{1}{3}$ (d) none of these

28. The value of $\lim\limits_{x\to 1}\dfrac{x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x - n}{x-1}$ is

(a) $\dfrac{n(n+1)}{2}$ (b) 0 (c) 1 (d) n

29. The value of $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{x^4}$ is

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{8}$ (c) $\dfrac{1}{16}$ (d) none of these

30. $\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}}$ is equal to

(a) $\dfrac{4}{3}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{3}$ (d) none of these

ANSWERS

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (a)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (a)
11. (c)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (a)	**20.** (b)
21. (a)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (a)

Continuity and Differentiability

12

BRIEF REVIEW OF THE CONCEPTS

CONTINUITY OF A FUNCTION AT A POINT

A function $f(x)$ is said to be *continuous* at an interior point $x = a$ of its domain if $\lim_{x \to a} f(x) = f(a)$. In other words, a function $f(x)$ is said to be *continuous* at a point $x = a$ provided lefthand limit, righthand limit and value of the function are equal:

A function $f(x)$ is *continuous* at a point $x = a$ if

$$\lim_{h \to 0} f(a - h) = \lim_{h \to 0} f(a + h) = f(a).$$

CONTINUITY OF A FUNCTION ON AN INTERVAL

Continuity on an open interval A function $f(x)$ is said to be continuous on an open interval (a, b) if it is continuous at each point of (a, b).

Continuity on a closed interval A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

1. $f(x)$ is continuous from right at $x = a$, i.e.,
$$\lim_{h \to 0} f(a + h) = f(a).$$
2. $f(x)$ is continuous from left at $x = b$, i.e.,
$$\lim_{h \to 0} f(b - h) = f(b).$$
3. $f(x)$ is continuous at each point of the open interval (a, b).

DISCONTINUITY OF A FUNCTION

A function $f(x)$, which is not continuous at a point $x = a$, is said to be *discontinuous* at that point.

Key Points to Remember

The discontinuity may arise due to any of the following situations:

1. $\lim_{h \to 0} f(a - h) \neq \lim_{h \to 0} f(a + h)$, i.e., LHL and RHL exist, but are not equal.
2. $\lim_{h \to 0} f(a - h) = \lim_{h \to 0} f(a + h) \neq f(a)$, i.e., LHL and RHL exist and are equal, but are different from $f(a)$.
3. $f(a)$ is not defined.
4. Atleast one of the limits $\lim_{h \to 0} f(a - h)$ or $\lim_{h \to 0} f(a + h)$ does not exist or atleast one of these limits is ∞ or $-\infty$.

Note: For continuity of $f(x)$ at the end points of an interval $[a, b]$ we must have

- $\lim_{h \to 0} f(a + h) = f(a)$ at $x = a$.
- $\lim_{h \to 0} f(b - h) = f(b)$ at $x = b$.

TYPES OF DISCONTINUITY

Removable discontinuity If $\lim_{h \to 0} f(a - h)$ and $\lim_{h \to 0} f(a + h)$ exist and are equal, but are not equal to $f(a)$, then the function $f(x)$ is said to have a *removable discontinuity* at $x = a$. However, by suitably defining the function at $x = a$, $f(x)$ can be made continuous at $x = a$.

Discontinuity of the first kind If $\lim_{h \to 0} f(a - h)$ and $\lim_{h \to 0} f(a + h)$ exist but are not equal, then the function $f(x)$ is said to have a *discontinuity of the first kind* at $x = a$.

If $\lim_{h \to 0} f(a - h)$ exists but is not equal to $f(a)$, then the function $f(x)$ is said to have a *discontinuity of the first kind from the left* at $x = a$.

Similarly, if $\lim_{h \to 0} f(a + h)$ exists but is not equal to $f(a)$, then the function $f(x)$ is said to have a *discontinuity of the first kind from the right* at $x = a$.

Discontinuity of the second kind If atleast one of the limits $\lim_{h \to 0} f(a - h)$ or $\lim_{h \to 0} f(a + h)$ does not exist *or atleast one of these limits is* ∞ or $-\infty$, then the function $f(x)$ is said to have a *discontinuity of the second kind at* $x = a$.

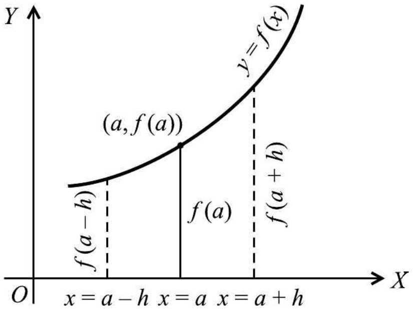
If $\lim_{h \to 0} f(a - h)$ does not exist or is equal to ∞ or $-\infty$, then the function $f(x)$ is said to have a *discontinuity of the second kind from the left* at $x = a$. Discontinuity of the second kind from the right is similarly defined.

GEOMETRICAL MEANING OF CONTINUITY

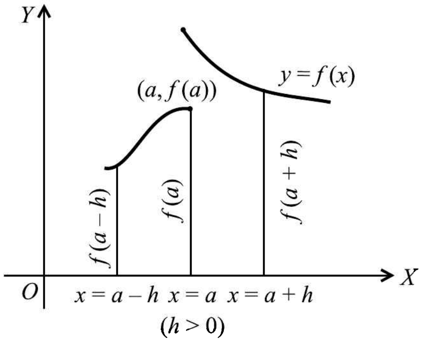
1. A function $f(x)$ will be continuous at a point $x = a$, if there is no break or cut or gap in the graph of the function $y = f(x)$

at the point $(a, f(a))$. Otherwise, it is discontinuous at that point.

2. A function $f(x)$ will be continuous on the closed interval $[a, b]$ if the graph of the function $y = f(x)$ is an unbroken line (curved or straight) from the point $(a, f(a))$ to the point $(b, f(b))$.



$f(x)$ has a continuous graph at $x = a$



$f(x)$ has a discontinuous graph at $x = a$

Key Points on Continuous Functions

1. If f and g are continuous at $x = a$, then
   (a) $f + g$ is continuous at $x = a$.
   (b) $f - g$ is continuous at $x = a$.
   (c) fg is continuous at $x = a$.
   (d) f/g is continuous at $x = a$, provided $g(a) \neq 0$.
   (e) kf is continuous at $x = a$, where k is any real constant.
   (f) $[f(x)]^{m/n}$ is continuous at $x = a$, provided $[f(x)]^{m/n}$ is defined on an interval containing a, and m and n are integers.
2. If f is continuous at a and g is continuous at $f(a)$, then gof is continuous at a.
3. If f is continuous at $x = a$ and g is discontinuous at $x = a$, then $f + g$ and $f - g$ are discontinuous at $x = a$, whereas fg may be continuous at $x = a$.
4. If f is continuous at $x = a$ and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that $\forall\ x \in (a - \delta, a + \delta)$, $f(x)$ has the same sign as $f(a)$.
5. If f is a continuous function defined on $[a, b]$ such that $f(a) \cdot f(b) < 0$, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b).
6. If f is a continuous function defined on $[a, b]$ and k is any real number between $f(a)$ and $f(b)$, then there exists atleast one solution of the equation $f(x) = k$ in the open interval (a, b).
7. If a function f is continuous on a closed interval $[a, b]$, then it is bounded on $[a, b]$ i.e., there exists real numbers k and K such that
   $$k \leq f(x) \leq K \text{ for all } x \in [a, b].$$
8. Every polynomial is continuous at every point of the real line.
9. Every rational function is continuous at every point where its denominator is different from zero.
10. Logarithmic functions, Exponential functions, Trigonometric functions, Inverse circular functions and Absolute value functions are continuous in their domain.

DIFFERENTIABILITY OF A FUNCTION AT A POINT

Let f be a function defined on an interval $I \subseteq R$. We say that f is *differentiable* at an interior point $c \in I$ provided

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists and is finite.}$$

We denote this limit by $f'(c)$, called the *derivative of f at c*.

In view of the definition of limit of a function f, one may observe that $f'(c)$ exists provided

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

The limit on the left, denoted by $Lf'(c)$, is called the *left hand derivative* of f at c and the limit on the right, denoted by $Rf'(c)$, is called the *right hand derivative* of f at c.

Thus, $f(x)$ is differentiable at $x = c$ if $Lf'(c) = Rf'(c)$.

DIFFERENTIABILITY OF A FUNCTION ON AN INTERVAL

A function $f(x)$ is said to be *differentiable* on an open interval (a, b) if $f(x)$ is differentiable at every point of this interval (a, b).

It is *differentiable* on a closed interval $[a, b]$ if it is differentiable on the open interval (a, b) and the limits

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

and $$Lf'(b) = \lim_{h \to 0} \frac{f(b-h) - f(b)}{(b-h) - b} \text{ exist.}$$

Key Points on Differentiability

1. Every polynomial function, exponential function and constant function is differentiable at each point of the real line.
2. Logarithmic functions, Trigonometric functions and Inverse Trigonometric functions are differentiable in their domain.
3. The sum, difference, product and quotient of two differentiable functions is differentiable.
4. The composition of differentiable functions is a differentiable function.
5. If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
6. If $f(x)$ and $g(x)$ both are not differentiable at a point, then the sum function $f(x) + g(x)$ and the product function $f(x) \cdot g(x)$ can still be differentiable at that point.

Relation Between Continuity and Differentiability

1. If a function $f(x)$ is differentiable at a point $x = a$ then it is continuous at $x = a$.

2. If $f(x)$ is only continuous at a point $x = a$, there is no guarantee that $f(x)$ is differentiable there.
3. If $f(x)$ is not differentiable at $x = a$ then it may or may not be continuous at $x = a$.
4. If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.
5. If left hand derivative and right hand derivative of $f(x)$ at $x = a$ are finite (they may or may not be equal) then $f(x)$ is continuous at $x = a$.

Solved Problems

1. If $f(x) = \begin{cases} \dfrac{e^{[x]+x}-1}{[x]+x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ then

(a) $\lim\limits_{x \to 0^+} f(x) = 1$

(b) $\lim\limits_{x \to 0^-} f(x) = \dfrac{1}{e} - 1$

(c) $f(x)$ is continuous at $x = 0$

(d) $f(x)$ is discontinuous at $x = 0$.

Solution (a), (d) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} f(0 + h)$

$$= \lim_{h \to 0} \frac{e^{[h]+h}-1}{[h]+h} = \lim_{h \to 0} \frac{e^{h}-1}{h} = \mathbf{1}$$

and $$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{e^{[-h]-h}-1}{[-h]-h}$$

$$= \lim_{h \to 0} \frac{e^{-1-h}-1}{(-1-h)} = \frac{e^{-1}-1}{-1}$$

$$= 1 - \frac{1}{e}.$$

Since $\lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0} f(x)$, therefore, **$f(x)$ is not continuous at $x = 0$.**

2. If $f(x) = \begin{cases} \dfrac{1-\cos 10x}{x^2}, & x < 0 \\ a & , x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25}, & x > 0 \end{cases}$, then the value of a so that $f(x)$ may be continuous at $x = 0$ is

(a) 25 (b) 50

(c) – 25 (d) none of these

Solution (b) We have,

$$\lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{625+\sqrt{h}}-25}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\left[\sqrt{625+\sqrt{h}}+25\right]}{(625+\sqrt{h}-625)}$$

$$= \lim_{h \to 0} \left[\sqrt{625+\sqrt{h}}+25\right]$$

$$= 25 + 25 = 50,$$

$$\lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{1-\cos 10h}{h^2} = \lim_{h \to 0} \frac{2\sin^2 5h}{h^2}$$

$$= 2 \lim_{h \to 0} \left(\frac{\sin 5h}{5h}\right)^2 \times 25 = 50$$

and $f(0) = a$.

Since $f(x)$ is continuous at $x = 0$

$$\therefore \quad \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(0 - h) = f(0)$$

$$\Rightarrow \quad a = \mathbf{50}.$$

3. If $f(x) = \begin{cases} \dfrac{[x]-1}{x-1}, & x \neq 1 \\ 0 & , x = 1 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at $x = 1$

(b) differentiable but not continuous at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) neither continuous nor differentiable at $x = 1$

Solution (d) We have, $f(x) \begin{cases} \dfrac{-1}{x-1} & , 0 < x < 1 \\ \dfrac{1-1}{x-1} = 0 & , 1 < x < 2 \\ 0 & , x = 1 \end{cases}$

$$\lim_{h \to 0} f(1 - h) = \lim_{h \to 0} \frac{-1}{(1-h)-1} = \lim_{h \to 0} \frac{1}{h} \to \infty$$

$\therefore$ **$f(x)$ is not continuous and hence not differentiable at $x = 1$.**

4. If $f(x) = \begin{cases} x[x], & 0 \leq x < 2 \\ (x-1)[x], & 2 \leq x \leq 3 \end{cases}$, where [.] denotes the greatest integer function, then

(a) both $f'(1)$ and $f'(2)$ do not exist

(b) $f'(1)$ exists but $f'(2)$ does not exist

(c) $f'(2)$ exists but $f'(1)$ does not exist

(d) both $f'(1)$ and $f'(2)$ exists

Solution (a) We have,

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)[1-h]-1[1]}{-h}$$

$$= \lim_{h\to 0} \frac{0-1}{-h} \to \infty$$

$\therefore$ **$f'(1)$ does not exist.**

$$\text{Also, } Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)[2-h]-(2-1)[2]}{-h}$$

$$= \lim_{h\to 0} \frac{2-h-2}{-h} = 1$$

$$\text{and } Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)[2+h]-(2-1)[2]}{h}$$

$$= \lim_{h\to 0} \frac{2+2h-2}{h} = 2.$$

$\therefore$ **$f'(2)$ also does not exist.**

5. If $f(x) = \begin{cases} x^p \cos\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, then at $x = 0$, $f(x)$ is

(a) continuous if $p > 0$ and differentiable if $p > 1$
(b) continuous if $p > 1$ and differentiable if $p > 0$
(c) continuous and differentiable if $p > 0$
(d) none of these

Solution (a) Continuity at $x = 0$:

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (-h)^p \cos\frac{1}{h}$$

$$= 0 \text{ if } p > 0$$

$$\text{R.H.L.} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} h^p \cos\frac{1}{h}$$

$$= 0 \text{ if } p > 0$$

and $f(0) = 0.$

$\therefore$ **$f(x)$ is continuous at $x = 0$ if $p > 0$**

Differentiability at $x = 0$:

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^p \cos\frac{1}{h} - 0}{-h}$$

$$= \lim_{h\to 0} (-h)^{p-1} \cos\frac{1}{h} = 0 \text{ if } p-1 > 0,$$

i.e., $p > 1$;

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^p \cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h^{p-1} \cos\frac{1}{h} = 0 \text{ if } p > 1$$

$\therefore$ **$f(x)$ is differentiable at $x = 0$ if $p > 1$.**

6. If $f(x) = x\sin\frac{1}{x}$, $x \neq 0$, then the value of the function at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0
(b) -1
(c) 1
(d) indeterminate

Solution (a) For $f(x)$ to be continuous at $x = 0$, we must have

$$f(0) = \lim_{x\to 0} f(x)$$

$$\Rightarrow \quad f(0) = \lim_{x\to 0} x\sin\frac{1}{x} = 0 \times a \text{ finite quantity} = 0$$

Hence $f(0) = $ **0.**

7. Let $g(x) = xf(x)$, where $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

At $x = 0$,

(a) g is differentiable but g' is not continuous
(b) g is differentiable while f is not
(c) both f and g are differentiable
(d) g is differentiable and g' is continuous.

Solution (a), (b) We have, $g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

For $x \neq 0$,

$$g'(x) = x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + 2x\sin\frac{1}{x}$$

$$= -\cos\frac{1}{x} + 2x\sin\frac{1}{x}.$$

For $x = 0$

$$g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin\frac{1}{x} - 0}{x}$$

$$= \lim_{x\to 0} x\sin\frac{1}{x} = \ .$$

$$\therefore \quad g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

g' is not continuous at $x = 0$ as $\cos\frac{1}{x}$ is not continuous at $x = 0$. Also, **f is not differentiable at $x = 0$.**

8. If $f(x) = \begin{cases} \frac{1}{1+e^{1/x}}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at $x = 0$
(b) continuous but not differentiable at $x = 0$
(c) differentiable but not continuous at $x = 0$
(d) none of these

Solution (d) $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+0} = 1$

$$\text{and } \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{1}{1+e^{1/h}} = \lim_{h\to 0} \frac{e^{-1/h}}{e^{-1/h}+1}$$

$$= \frac{0}{1} = 0.$$

Since $\lim_{h\to 0} f(0-h) \neq \lim_{h\to 0} f(0+h)$, therefore, $f(x)$ is not continuous and hence not differentiable at $x = 0$.

9. If $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at $x = 0$

(b) continuous but not differentiable at $x = 0$

(c) differentiable but not continuous at $x = 0$

(d) none of these

Solution (d) $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \dfrac{e^{-1/h}-1}{e^{-1/h}+1} = \dfrac{0-1}{0+1}$

$$= -1$$

and $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \dfrac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0} \dfrac{1-e^{-1/h}}{1+e^{-1/h}}$

$$= \frac{1-0}{1+0} = 1.$$

Since $\lim_{h\to 0} f(0-h) \neq \lim_{h\to 0} f(0+h)$, therefore, $f(x)$ is not continuous and hence not differentiable at $x = 0$.

10. The function $f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is

(a) discontinuous at only one point

(b) discontinuous exactly at two points

(c) continuous everywhere

(b) none of these

Solution (a) The only doubtful point is $x = 0$.

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (-h+1)^{2-\left(\frac{1}{h}-\frac{1}{h}\right)}$$

$$= \lim_{h\to 0} (1-h)^2 = 1$$

$$\text{R.H.L.} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (h+1)^{2-\left(\frac{1}{h}+\frac{1}{h}\right)}$$

$$= \lim_{h\to 0} (1+h)^{2-\frac{2}{h}} = \lim_{h\to 0} (1+h)^2 [(1+h)^{1/h}]^{-2}$$

$$= 1 \times e^{-2} = e^{-2}.$$

Since LHL $\neq$ RHL, $\therefore$ **$f(x)$ is not continuous at $x = 0$.**

11. If $f(x) = [x-2]$, then

(a) $f'(2.5) = \dfrac{1}{2}$ and $f'(5) = 3$

(b) $f'(2.5) = 0$ and $f'(5) = 3$

(c) $f'(2.5) = 0$ and $f'(5)$ does not exist

(d) both $f'(2.5)$ and $f'(5)$ do not exist

Solution (c) We have

$$Lf'(2\cdot 5) = \lim_{h\to 0} \frac{f(2\cdot 5-h)-f(2\cdot 5)}{-h}$$

$$= \lim_{h\to 0} \frac{[2\cdot 5-h-2]-[2\cdot 5-2]}{-h}$$

$$= \lim_{h\to 0} \frac{0}{-h} = 0$$

and $Rf'(2\cdot 5) = \lim_{h\to 0} \dfrac{f(2\cdot 5+h)-f(2\cdot 5)}{h}$

$$= \lim_{h\to 0} \frac{[2\cdot 5+h-2]-[2\cdot 5-2]}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$\therefore$ $f'(2\cdot 5) = 0$.

Also, $Lf'(5) = \lim_{h\to 0} \dfrac{f(5-h)-f(5)}{-h}$

$$= \lim_{h\to 0} \frac{[5-h-2]-[5-2]}{-h}$$

$$= \lim_{h\to 0} \frac{2-3}{-h} \to \infty.$$

Hence, **$f'(2\cdot 5) = 0$ while $f'(5)$ does not exist.**

12. If $f(x) = [\tan x]$, $x \in \left(0 \dfrac{\pi}{3}\right)$, then $f'\left(\dfrac{\pi}{4}\right)$ is equal to

(a) 1 (b) 0

(c) does not exist (d) none of these

Solution (c) $Lf'\left(\dfrac{\pi}{4}\right) = \lim_{h\to 0} \dfrac{f\left(\frac{\pi}{4}-h\right)-f\left(\frac{\pi}{4}\right)}{-h}$

$$= \lim_{h\to 0} \frac{\left[\tan\left(\frac{\pi}{4}-h\right)\right]-\left[\tan\frac{\pi}{4}\right]}{-h} = \lim_{h\to 0} \frac{0-1}{-h} \to \infty$$

$\therefore$ $f'\left(\dfrac{\pi}{4}\right)$ **does not exist.**

13. If $f(x) = \dfrac{1}{1-x}$, then the points of discontinuity of the function $f[f\{f(x)\}]$ are

(a) $\{0, -1\}$ (b) $\{0, 1\}$

(c) $\{1, -1\}$ (d) none of these

Solution (b) We have, $f(x) = \dfrac{1}{1-x}$.

As at $x = 1$, $f(x)$ is not defined, $x = 1$ is a point of discontinuity of $f(x)$.

If $x \neq 1$, $f[f(x)] = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-1/(1-x)} = \dfrac{x-1}{x}$.

$\therefore$ $x = 0, 1$ are points of discontinuity of $f[f(x)]$.

If $x \neq 0, x \neq 1$,

$$f[f\{f(x)\}] = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{(x-1)}{x}} = x.$$

14. If $f(x) = \begin{cases} \dfrac{e^{[x]+|x|}-2}{[x]+|x|-1}, & x \neq 0 \\ & x = 0 \end{cases}$, ([.] denotes the greatest integer function) then

(a) $f(x)$ is continuous at $x = 0$

(b) $\lim_{x\to 0^+} f(x) = -1$

(c) $\lim_{x\to 0^-} f(x) = 1$

(d) none of these

Solution (d) $f(x) = \begin{cases} \dfrac{e^{[x]+|x|}-2}{[x]+|x|}, & x \neq 0 \\ -1, & x = 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{[x]+|x|}-2}{[x]+|x|} = \frac{e^{-1}-2}{-1}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{[x]+|x|}-2}{[x]+|x|}$$

$$= \lim_{x\to 0^+} \frac{e^{x}-2}{x} \to -\infty .$$

15. If $\operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$, then the function

$f(x) = \operatorname{sgn}(\operatorname{sgn}(x))$ is

(a) continuous as well as differentiable at $x = 0$

(b) continuous but not differentiable at $x = 0$

(c) differentiable but not continuous at $x = 0$

(d) neither differentiable nor continuous at $x = 0$

Solution (d) We have,

$$\operatorname{sgn}[\operatorname{sgn}(x)] = \begin{cases} \operatorname{sgn}\left(\dfrac{|x|}{x}\right) & , x \neq 0 \\ \operatorname{sgn}(0) & , x = 0 \end{cases} = \begin{cases} \left|\frac{|x|}{x}\right| \Big/ \frac{|x|}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} \dfrac{x}{|x|}, x \neq 0 \\ 0 \quad , x = 0 \end{cases}$$

Continuity at $x = 0$:

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h}{|-h|} = -1$$

and $f(0) = 0$

$\therefore$ **$f(x)$ is not continuous and hence not differentiable at $x = 0$.**

16. If $f(x) = x^5 \operatorname{sgn} x$, where $\operatorname{sgn} x = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(a) differentiable as well as continuous at $x = 0$

(b) continuous but not differentiable at $x = 0$

(c) differentiable but not continuous at $x = 0$

(d) neither differentiable nor continuous at $x = 0$

Solution (a) Since

$$\operatorname{sgn} x = \begin{Bmatrix} \dfrac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} \quad \text{or} \quad \operatorname{sgn} x = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$

$$\text{Therefore, } f(x) = x^5 \operatorname{sgn} x = \begin{cases} x^5 & , x > 0 \\ 0 & , x = 0 \\ -x^5 & , x < 0 \end{cases}$$

Clearly, **$f(x)$ is continuous as well as differentiable at $x = 0$.**

17. If $f(x) = |x - 2|$ and $g(x) = f[f(x)]$, then $g'(x)$ for $x > 20$ is

(a) 1 (b) 2

(c) – 1 (d) none of these

Solution (a) For $x > 20$, $f(x) = x - 2$

and $g(x) = f[f(x)] = f(x - 2) = x - 2 - 2 = x - 4$.

$\therefore$ $g'(x) = \mathbf{1}$ for $x > 20$.

18. Let $f(x) = |x|$ and $g(x) = [x]$, where [.] denotes the greatest integer function. Then $(fog)'(-2)$ is

(a) 0 (b) does not exist

(c) – 1 (d) 1

Solution (b) $(fog)(x) = f(g(x)) = f([x]) = |[x]|$.

$$\text{Now, } L(fog)'(-2) = \lim_{h\to 0} \frac{(fog)(-2-h)-(fog)(-2)}{-h}$$

$$= \lim_{h\to 0} \frac{|[-2-h]|-|[-2]|}{-h}$$

$$= \lim_{h\to 0} \frac{|-3|-|-2|}{-h} = \lim_{h\to 0} \frac{-1}{h} \to -\infty$$

$\therefore$ **$(fog)'(-2)$ does not exist.**

19. The Dirichlet function, defined as

$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$, is

(a) continuous for all real x

(b) continuous only at some values of x

(c) discontinuous for all real x

(d) discontinuous only at some values of x

Solution (c) Let x_0 be any arbitrary real number.

Case I. x_0 is rational

Then $f(x_0) = 1$.

In any vicinity of a rational point there are irrational points, where $f(x) = 0$. Hence, in any vicinity of x_0 there are points x for which

$$|\Delta y| = |f(x_0) - f(x)| = 1.$$

Case II. x_0 is irrational

Then $f(x_0) = 0$.

In any vicinity of an irrational point there are rational points at which $f(x) = 1$. Hence, it is possible to find the values of x for which

$$|\Delta y| = |f(x_0) - f(x)| = 1.$$

Thus, in both cases, the difference Δy does not tend to zero as $\Delta x \to 0$. Therefore, x_0 is a point of discontinuity. Since x_0 is an arbitrary point, the Dirichlet function **$f(x)$ is discontinuous at each point.**

20. Let $f(x) = \begin{cases} ax^2 + 1, x > 1 \\ x + a \ , x \leq 1 \end{cases}$. Then $f(x)$ is derivable at $x = 1$, if

(a) $a = 2$ (b) $a = 1$

(c) $a = 0$ (d) $a = \dfrac{1}{2}$

Solution (d) $Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0} \frac{(1-h+a)-(1+a)}{-h} = \lim_{h\to 0} \frac{-h}{-h} = 1.$$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[a(1+h)^2+1]-(1+a)}{h}$$

$$= \lim_{h\to 0} \frac{ah^2+2ah}{h} = \lim_{h\to 0} (ah + 2a) = 2a$$

Since $f'(1)$ exists, $\therefore$ $Lf'(1) = Rf'(1)$ i.e., $2a = 1$

$\Rightarrow$ $\boldsymbol{a = \frac{1}{2}}$.

21. Let $f(x + y) = f(x)f(y)$ for all x, y, where $f(0) \neq 0$. If $f'(0) = 2$, then $f(x)$ is equal to

(a) Ae^x (b) Ae^{2x}

(c) $2x$ (d) none of these

Solution (b) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x+0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)\cdot f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}\cdot f(x)$$

$$= f'(0)\cdot f(x) = 2f(x). \quad (\because f'(0) = 2)$$

Now, $\frac{df}{dx} = 2f$ or $\frac{df}{f} = 2\,dx \Rightarrow d(\log f - 2x) = 0$

$\therefore$ $\log f - 2x = c$, $\therefore$ $f = e^{2x+c} = e^c\cdot e^{2x} = Ae^{2x}$,

where $A = e^c$ = constant.

22. Let $f(x + y) = f(x)\cdot f(y)$ for all x, y where $f(0) \neq 0$. If $f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ is equal to

(a) 6 (b) 0

(c) 1 (d) none of these

Solution (a) $f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$

$$= \lim_{h\to 0} \frac{f(5+h)-f(5+0)}{h}$$

$$= \lim_{h\to 0} \frac{f(5)\cdot f(h)-f(5)\cdot f(0)}{h}$$

$(\because f(x + y) = f(x)\cdot f(y)$ for all $x, y)$

$$= \left(\lim_{h\to 0} \frac{f(h)-f(0)}{h}\right)\cdot f(5)$$

$= f'(0)\cdot f(5) = 3\cdot 2 = \mathbf{6}$.

23. Let $f : R \to R$ be a function such that

$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then

(a) $f(x)$ is a quadratic function

(b) $f(x)$ is continuous but not differentiable

(c) $f(x)$ is differentiable in R

(d) $f(x)$ is bounded in R

Solution (c) We have,

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}, f(0) = 0 \text{ and } f'(0) = 3$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+f(3h)}{3}-\frac{f(3x)+f(0)}{3}}{h}$$

$$= \lim_{h\to 0} \frac{f(3h)-f(0)}{3h} = 3$$

$\therefore$ $f(x) = 3x + c$, $\because$ $f(0) = 0 \Rightarrow c = 0$

$\therefore$ $f(x) = 3x$.

24. If a function f: $R \to R$ be such that $f(x + y) = f(x)\cdot f(y)$ for all $x, y \in R$ where $f(x) = 1 + x\,\phi(x)$ and $\lim_{x\to 0} \phi(x) = 1$, then

(a) $f'(x)$ does not exist

(b) $f'(x) = 2f(x)$ for all x

(c) $f'(x) = f(x)$ for all x

(d) none of these

Solution (c) $Rf'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)\,f(h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{1+h\,\phi(h)-1}{h}$$

$$= f(x) \lim_{h\to 0} \phi(h)$$

$$= f(x)\cdot 1 = f(x)$$

and $Lf'(x) = \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$

$$= \lim_{h\to 0} \frac{f(x)\,f(-h)-f(x)}{-h}$$

$$= f(x) \lim_{h\to 0} \frac{f(-h)-1}{-h}$$

$$= f(x) \lim_{h\to 0} \frac{1-h\,\phi(-h)-1}{-h}$$

$$= f(x)\cdot 1 = f(x).$$

Hence **$f'(x)$ exists and is equal to $f(x)$.**

25. Let $f(x + y) = f(x)\cdot f(y)$ for all $x, y \in R$ and $f(x) = 1 + x\,\phi(x)\log 2$ where $\lim_{x\to 0} \phi(x) = 1$. Then $f'(x)$ is equal to

(a) $\log 2^{f(x)}$ (c) $\log[f(x)]^2$

(c) $\log 2$ (d) none of these

Solution (a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{f(x) \cdot f(h) - f(x)}{h}$

$[\because f(x+y) = f(x) \cdot f(y)]$

$= f(x) \lim_{h \to 0} \left(\dfrac{f(h) - 1}{h} \right)$

$= f(x) \lim_{h \to 0} \dfrac{1 + h\,\phi(h) \log 2 - 1}{h}$

$[\because f(x) = 1 + x\,\phi(x) \log 2]$

$= f(x) \log 2 \lim_{h \to 0} \phi(h)$

$= f(x) \cdot \log 2 \cdot 1 \quad \left[\because \lim_{h \to 0} \phi(h) = 1\right]$

$= \mathbf{\log 2^{f(x)}}$.

26. Let $f(x + y) = f(x) \cdot f(y)$ and $f(x) = 1 + x\,g(x)\,G(x)$ where $\lim_{x \to 0} g(x) = a$ and $\lim_{x \to 0} G(x) = b$. Then $f'(x) = kf(x)$, where k is equal to

(a) $\dfrac{a}{b}$ (b) $1 + ab$

(c) ab (d) none of these

Solution (c) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x + y) = f(x) \cdot f(x)]$

$= f(x) \cdot \lim_{h \to 0} \left(\dfrac{f(h) - 1}{h} \right)$

$= f(x) \lim_{h \to 0} \dfrac{1 + h\,g(h)\,G(h) - 1}{h}$

$= f(x) \lim_{h \to 0} g(h)\,G(h)$

$= f(x) \lim_{h \to 0} g(h) \lim_{h \to 0} G(h) = ab\,f(x)$. $\quad \therefore\ k = ab$

27. If $x + 4|y| = 6y$, then y as a function of x is

(a) not continuous at $x = 0$

(b) not defined for all real x

(c) $\dfrac{dy}{dx} = \dfrac{1}{2}$ for $x < 0$ (d) derivable at $x = 0$

Solution (c) We have, $4x + 3|y| = 5y$

$\Rightarrow 4x + 3y = 5y$ if $y \geq 0$

and $4x - 3y = 5y$ if $y < 0 \Rightarrow y = \begin{cases} 2x, & x \geq 0 \\ \dfrac{1}{2}x, & x < 0 \end{cases}$

Clearly, y is continuous at $x = 0$ but not differentiable at $x = 0$.

Also, $\dfrac{dy}{dx} = \begin{cases} 2, & x \geq 0 \\ \mathbf{\dfrac{1}{2}}, & \mathbf{x < 0} \end{cases}$.

28. The function $f(x) = \max.\ \{(1 - x), (1 + x), 2\}$, $x \in (-\infty, \infty)$, is

(a) continuous at all points

(b) differentiable at all points

(c) differentiable at all points except at $x = 1$ and $x = -1$.

(d) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous.

Solution (a), (c) $f(x) = \begin{cases} 1 - x, & x \leq -1 \\ 2, & -1 < x \leq 1 \\ 1 + x, & x > 1 \end{cases}$

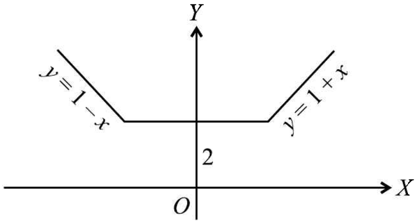
$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (1 - x) = 2 = \lim_{x \to -1^+} f(x)$

and $\lim_{x \to 1} f(x) = 2$, so **f is continuous at all points.**

$f'(-1^-) = \lim_{h \to 0^-} \dfrac{f(-1 + h) - f(-1)}{h}$

$= \lim_{h \to 0^-} \dfrac{1 + 1 - h - 2}{h} = -1.$

$f'(-1^+) = 0$. Similarly, $f'(1^-) = 0$ and $f'(1+) = 1$, so **f is differentiable everywhere except at $x = -1, 1$.**



29. Let $f : R \to R$ be a function defined by $f(x) = \max\ \{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$

(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$.

Solution (d) If $x < -1$, then $x > x^3$. So, $f(x) = x$.

If $x = -1$, then $x = x^3$. So, $f(x) = x$.

If $-1 < x < 0$, then $x < x^3$. So, $f(x) = x^3$.

If $x = 0$, then $x = x^3$. So, $f(x) = x^3$.

If $0 < x < 1$, then $x > x^3$. So, $f(x) = x$.

If $x = 1$, then $x = x^3$. So, $f(x) = x$.

If $x > 1$, then $x < x^3$. So, $f(x) = x^3$.

Thus, $f(x) = \begin{matrix} x, & x \leq -1 \\ x^3, & -1 < x \leq 0 \\ x, & 0 < x \leq 1 \\ x^3, & x > 1. \end{matrix}$

Clearly, **$f(x)$ is not differentiable at $x = -1, 0, 1$.**

30. If $f(x) = \begin{cases} x e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable for all x

(b) continuous for all x but not differentiable at $x = 0$

(c) differentiable for all x but not continuous at $x = 0$

(d) none of these

Solution (b) We have,

$$f(x) = \begin{cases} xe^{-\left(\frac{-1}{x}+\frac{1}{x}\right)} & , x<0 \\ xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} & , x>0 \\ 0 & , x=0 \end{cases} = \begin{cases} x, x<0 \\ xe^{-2/x}, x>0 \\ 0 \quad , x=0 \end{cases}$$

$f(x)$ is differentiable as well as continuous everywhere except possibly at $x = 0$.

Now,

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{he^{-2/h}-0}{h} = \lim_{h\to 0} e^{-2/h} = 0$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h-0}{-h} = 1.$$

Since $Lf'(0) \neq Rf'(0)$, therefore **$f(x)$ is not differentiable at $x = 0$** but **$f(x)$ is continuous at $x = 0$** [as $Lf'(0)$ and $Rf'(0)$ are finite].

Problems for Practice

Level of Difficulty I

1. Let $f(x) = \dfrac{\tan\left(\dfrac{\pi}{4}-x\right)}{\cot 2x}, x \neq \dfrac{\pi}{4}$. The value which should be assigned to f at $x = \dfrac{\pi}{4}$, so that it is continuous everywhere is

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) none of these

2. If $f(x) = \begin{cases} \dfrac{1-\sin x}{(\pi-2x)^2} & , \text{when } x \neq \dfrac{\pi}{2} \\ \lambda & , x = \dfrac{\pi}{2} \end{cases}$, then $f(x)$ will be a continuous function at $x = \dfrac{\pi}{2}$ when λ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) none of these

3. Let $\phi(x) = \begin{cases} \dfrac{1-\cos \lambda x}{x\sin x} & , x \neq 0 \\ \dfrac{1}{2} & , x = 0 \end{cases}$. If $\phi(x)$ is continuous at x = 0, then λ is

(a) ± 1 (b) 0 (c) 2 (d) none of these

4. If the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\log(1+3x)-\log(1-2x)}{x}, & x \neq 0 \\ a & , x = 0 \end{cases}$$

is continuous at $x = 0$, then the value of a is

(a) 5 (b) 1 (c) -1 (d) none of these

5. The values of a, b and c which make the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & , x<0 \\ c & , x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} & , x>0 \end{cases}$$

continuous at $x = 0$ are

(a) $a = \frac{-3}{2}, c = \frac{1}{2}, b \neq 0$

(b) $a = \frac{3}{2}, c = \frac{1}{2}, b \neq 0$

(c) $a = \frac{-3}{2}, c = \frac{1}{2}, b \neq 0$

(d) none of these

6. The function $f(x) = \begin{cases} \dfrac{x(e^{1/x}-e^{-1/x})}{e^{1/x}+e^{-1/x}}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) none of these

7. The value of the derivative of $|x - 1| + |x - 3|$ at $x = 2$ is

(a) -2 (b) 0 (c) 2 (d) not defined

8. If $f(x + y) = 2 f(x)\cdot f(y)$ for all x, y, where $f'(0) = 3$ and $f(4) = 2$, then $f'(4)$ is equal to

(a) 6 (b) 12 (c) 4 (d) none of these

9. The function $f(x) = \begin{cases} |x-3|, x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, x<1 \end{cases}$ is

(a) continuous at $x = 1$ but not derivable at $x = 1$

(b) continuous and derivable at $x = 1$

(c) not derivable at $x = 1$

(d) not continuous at $x = 1$

10. If $g(x) = (x^2 + 2x + 3) f(x)$, $f(0) = 5$ and $\lim\limits_{x\to 0} \dfrac{f(x)-5}{x} = 4$, then $f'(0)$ is equal to

(a) 22 (b) 20 (c) 18 (d) none of these

11. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(a) $\sqrt{e}$ (b) $1/\sqrt{e}$ (c) 2 (d) none of these

12. Let $f(x) = \dfrac{1-\tan x}{4x-\pi}$, $x \neq \dfrac{\pi}{4}$, $x \in \left[0, \dfrac{\pi}{2}\right]$. $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$, then $f\left(\dfrac{\pi}{4}\right)$ is

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) 1 (d) -1

13. If $f(x) = \begin{cases} \dfrac{x}{1+|x|}, & |x| \geq 1 \\ \dfrac{x}{1-|x|}, & |x| < 1 \end{cases}$, then $f(x)$ is

(a) discontinuous and non-differentiable at $x = -1$, 1 and 0

(b) discontinuous and non-differentiable at $x = -1$, where as continuous and differentiable $x = 0$ and $x = 1$

(c) discontinuous and non-differentiable at $x = -1$ and $x = 1$, whereas continuous and differentiable at $x = 0$

(d) none of these

14. The function $f(x) = (1 + x)^{\cot x}$ is not defined at $x = 0$. The value of $f(0)$ so that $f(x)$ becomes continuous at $x = 0$, is

(a) 1 (b) 0 (c) e (d) none of these

15. If $f(x) = \begin{cases} ax^2 + b, & 0 \leq x < 1 \\ x + 3, & 1 < x \leq 2 \\ 4, & x = 1 \end{cases}$ then the value of (a, b) for which $f(x)$ cannot be continuous at $x = 1$ is

(a) (2, 2) (b) (3, 1) (c) (4, 0) (d) (5, 2)

Level of Difficulty II

16. If $f(x) = \begin{cases} x, \text{ when } x \text{ is rational} \\ 1 - x, \text{ when } x \text{ is irrational} \end{cases}$, then

(a) $f(x)$ is continuous for all real x

(b) $f(x)$ is discontinuous for all real x

(c) $f(x)$ is continuous only at $x = 1/2$

(d) $f(x)$ is discontinuous only at $x = 1/2$.

17. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and fog = I (identity function). Then $f'(b)$ is equal to

(a) 2 (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{2}$ (d) none of these

18. Let $f(x) = \begin{cases} (1 + |\sin x|)^{a/|\sin x|}, & \dfrac{-\pi}{6} < x < 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \dfrac{\pi}{6} \\ b, & x = 0 \end{cases}$. The values of a and b so that $f(x)$ may be continuous at $x = 0$ are

(a) $a = \dfrac{-2}{3}$, $b = e^{2/3}$ (b) $a = \dfrac{2}{3}$, $b = e^{-2/3}$

(c) $a = \dfrac{2}{3}$, $b = e^{2/3}$ (d) none of these

19. Let a function $f : R \to R$ satisfy the equation $f(x + y) = f(x) + f(y)$ for all x, y. If the function $f(x)$ is continuous at $x = 0$, then

(a) $f(x) = 0$ continuous for all x

(b) $f(x)$ is continuous for all positive real x

(c) $f(x)$ is continuous for all x

(d) none of these

20. If $f(x)$ is continuous in [0, 1] and $f\left(\dfrac{1}{2}\right) = 2$, then $\lim\limits_{n \to \infty} f\left(\dfrac{\sqrt{n}}{2\sqrt{n}+1}\right)$ is equal to

(a) 0 (b) ∞ (c) 2 (d) none of these

21. If $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$, then at $x = 0$,

(a) $f(x)$ is differentiable as well as continuous

(b) $f(x)$ is differentiable but not continuous

(c) $f(x)$ is continuous but not differentiable

(d) $f(x)$ is neither continuous not differentiable

22. If $f(x) = \log_{10} x$, then at $x = 1$

(a) $f(x)$ is continuous as well as differentiable

(b) $f(x)$ is continuous but not differentiable

(c) $f(x)$ is differentiable but not continuous

(d) $f(x)$ is neither continuous nor differentiable

23. Let $f(x) = \cos x$ and $g(x) = [x + 2]$, where [.] denotes the greatest integer function. Then $(gof)'\left(\dfrac{\pi}{2}\right)$ is

(a) 1 (b) 0 (c) -1 (d) does not exist

24. Let $f(x)$ be a function satisfying $f(x + y) = f(x)\,f(y)$ for all $x, y \in R$. If $f(x) = 1 + x\,\phi(x) + x^2\,\phi(x)\,\psi(x)$, where $\lim\limits_{x \to 0} \phi(x) = a$ and $\lim\limits_{x \to 0} \psi(x) = b$, then $f'(x)$ is equal to

(a) $(a + b)\,f(x)$ (b) $af(x)$

(b) $bf(x)$ (d) none of these

25. The function $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, is discontinuous at

(a) all x (b) all integer points

(c) no x (d) x which is not an integer.

26. The left-hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k$, k an integer and $[x]$ = greatest integer $\leq x$, is

(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}\cdot(k-1)\pi$

(c) $(-1)^k\cdot k\pi$ (d) $(-1)^{k-1}\cdot k\pi$.

27. If the function

$$f(x) = \begin{cases} (1 - |\tan x|)^{\frac{a}{|\tan x|}} & , \dfrac{-\pi}{4} < x < 0 \\ b & , x = 0 \\ e^{\frac{\sin 3x}{\sin 2x}} & , 0 < x < \dfrac{\pi}{4} \end{cases}$$

is continuous at $x = 0$, then

(a) $a = \frac{-3}{2}$ $b = \frac{3}{2}$ (b) $a = \frac{3}{2}$ $b = e^{3/2}$

(c) $a = \frac{-3}{2}$, $b = e^{3/2}$ (d) none of these

28. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n$, then at $x = 0$, $f(x)$

(a) has no limit
(b) is discontinuous
(c) is continuous but not differentiable
(d) is differentiable

29. The values of constants a and b so as to make the function $f(x) = \begin{cases} \frac{1}{|x|} & ,|x| \geq 1 \\ ax^2 + b, & |x| < 1 \end{cases}$ continuous as well as differentiable for all x, are

(a) $a = \frac{-1}{2}$, $b = \frac{3}{2}$

(b) $a = \frac{1}{2}$, $b = \frac{3}{2}$

(c) $a = \frac{-1}{2}$, $b = \frac{-3}{2}$

(d) none of these

30. Let f be a function defined and continuous on [2, 5]. If $f(x)$ takes rational values for all x and $f(4) = 8$ then the value of $f(3.7)$ is

(a) 0 (b) 8
(c) – 1 (d) none of these

ANSWERS

1. (b)	**2.** (c)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (b)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (d)	**16.** (c)	**17.** (c)	**18.** (c)	**19.** (c)	**20.** (c)
21. (c)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (c)	**26.** (a)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (b)

13 Differentiation

BRIEF REVIEW OF THE CONCEPTS

DERIVATIVE OF A FUNCTION

Let $y = f(x)$ be a function defined on the interval $[a, b]$. Let for a small increment δx in x, the corresponding increment in the value of y be δy. Then

$$y = f(x) \text{ and } y + \delta y = f(x + \delta x)$$

On subtraction, we get

$$\delta y = f(x + \delta x) - f(x)$$

or
$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Taking limit on both sides when $\delta x \to 0$ we have,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x},$$

if this limit exists, is called the *derivative* or *differential coefficient* of y with respect to x and is written as $\frac{dy}{dx}$ or $f'(x)$. Thus

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Derivative at a Point

The value of $f'(x)$ obtained by putting $x = a$, is called the derivative of $f(x)$ at $x = a$ and it is denoted by $f'(a)$ or $\left\{\frac{dy}{dx}\right\}_{x=a}$.

Standard Derivatives

The following formulae can be applied directly for finding the derivative of a function:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 1$
9. $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ x > 0$
10. $\frac{d}{dx}(x^n) = nx^{n-1}$
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
12. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2},\ -\infty < x < \infty$
14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$
15. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1$
16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2},\ -\infty < x < \infty$
17. $\frac{d}{dx}(|x|) = \frac{x}{|x|}$ or $\frac{|x|}{x},\ x \neq 0.$

RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero, i.e.,
$$\frac{d}{dx}(c) = 0.$$
2. The derivative of constant times a function is constant times the derivative of the function, i.e.,

$$\frac{d}{dx}\{c\cdot f(x)\} = c\frac{d}{dx}\{f(x)\}.$$

3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}.$$

Note: In general, if $f_1(x), f_2(x), \ldots. f_n(x)$ are n differentiable functions, then we have

$$\frac{d}{dx}[f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)] \pm \ldots. \pm \frac{d}{dx}[f_n(x)].$$

PRODUCT RULE OF DIFFERENTIATION

If $f(x)$ and $g(x)$ are differentiable functions of x, then

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x).$$

Note: The rule can be extended for the product of any finite number of functions. Thus, if $f(x)$, $g(x)$ and $h(x)$ are differentiable functions of x, then

$$\frac{d}{dx}[f(x)\,g(x)\,h(x)] = f(x)\,g(x)h'(x) + f(x)\,g'(x)h(x) + f'(x)\,g(x)\,h(x)$$

QUOTIENT RULE OF DIFFERENTIATION

If $f(x)$ and $g(x)$ are two differentiable functions of x, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Differentiation of a Function of a Function (Chain Rule)

If y is a differentiable function of u and u is a differentiable function of x, then $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

Key Points on Chain Rule

1. The chain rule can be extended further as:
   If y is a function of u, u is a function of v and v is a function of x, then
   $$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx} \text{ and so on.}$$
2. If $y = u^n$, where u is a function of x, then
   $$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = nu^{n-1}\times\frac{du}{dx}.$$
   $$\left[\because \frac{dy}{du} = nu^{n-1}\right]$$

Aliter:

Let $y = \left(\frac{3+4x}{2-x}\right)^2$. Then,

$$\frac{dy}{dx} = 2\left(\frac{3+4x}{2-x}\right)^1\cdot\frac{d}{dx}\left(\frac{3+4x}{2-x}\right)$$

$$= 2\frac{3+4x}{2-x}\cdot\frac{(2-x)\cdot 4 - (3+4x)(-1)}{(2-x)^2}$$

$$= 2\frac{3+4x}{2-x}\cdot\frac{11}{(2-x)^2} = \frac{22(3+4x)}{(2-x)^3}.$$

DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes x and y are separately given as functions of a single variable t (called a parameter), i.e., $x = f(t)$ and $y = g(t)$. In this case,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}.$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}.$$

DERIVATIVE OF IMPLICIT FUNCTIONS

The derivative of an implicit function, given by the relation $f(x, y) = 0$ in which y is not expressible explicitly in terms of x, can be found by the following steps:

Working Rule

Step 1. Differentiate each term of the equation $f(x, y) = 0$ w.r.t. x, keeping in mind that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$; $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$ and so on.

Step 2. Collect the terms containing $\frac{dy}{dx}$ on one side and the terms not involving $\frac{dy}{dx}$ on the other side.

Step 3. Divide by coefficient of $\frac{dy}{dx}$ to get $\frac{dy}{dx}$ as a function of x or y or both.

Shorter Method for Finding the Derivative of an Implicit Function

Step 1. Take all the terms of the function to be differentiated to the left hand side and put left hand side equal to $\phi(x, y)$.

Step 2.

$$\frac{dy}{dx} = -\frac{\text{derivatie of } f(x,y) \text{ w.r.t. } x \text{ treating } y \text{ as constant}}{\text{derivatie of } f(x,y) \text{ w.r.t. } y \text{ treating } x \text{ as constant}}.$$

DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

If $y = f(x)$ and $z = g(x)$, then in order to find the derivative of $f(x)$ w.r.t. $g(x)$, we use the formula

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}.$$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithms before differentiation is called logarithmic differentiation. When the function to be differentiated involves a function in its power or when the function is the product or quotient of a number of functions, we first take log on both sides and then differentiate each logarithmic term separately.

Properties of Logarithms

1. $\log_e (mn) = \log_e m + \log_e n$
2. $\log_e \left(\frac{m}{n}\right) = \log_e m - \log_e n$
3. $\log_e (m)^n = n \log_e m$
4. $\log_e e = 1$
5. $\log_n m = \frac{\log_e m}{\log_e n}$
6. $\log_n m \cdot \log_m n = 1.$

Shorter Methods of Finding the Derivative of a Logarithmic Function

If $y = [f(x)]^{g(x)}$, then to find $\frac{dy}{dx}$, in addition to the method discussed above, we can also apply any of the following two methods:

Method 1

Step 1. Express $y = [f(x)]^{g(x)} = e^{g(x) \log f(x)}$

$[\because a^x = e^{x \log a}]$

Step 2. Differentiate w.r.t. x to obtain $\frac{dy}{dx}$.

Method 2

Step 1. Evaluate

A = Differential coefficient of y treating $f(x)$ as constant.

Step 2. Evaluate

B = Differential coefficient of y treating $g(x)$ as constant.

Step 3. $\frac{dy}{dx} = A + B.$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

For problems involving inverse trigonometric functions, first try for a suitable substitution to simplify it and then differentiate. If no such substitution is found, then differentiate directly.

Important Substitutions to Reduce the Function to a Simpler Form

Expressions	Substitutions
$\sqrt{a^2 - x^2}$	Put $x = a \sin \theta$ or $x = a \cos \theta$
$\sqrt{x^2 - a^2}$	Put $x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$
$\sqrt{a^2 + x^2}$	Put $x = a \tan \theta$ or $x = a \cot \theta$
$\frac{a-x}{a+x}$ or $\frac{a+x}{a-x}$	Put $x = a \tan \theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	Put $x = a \cos \theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	Put $x^2 = a^2 \cos \theta$

Key Trigonometric and Inverse Trigonometric Transformations

1. $1 + \cos mx = 2 \cos^2 \frac{mx}{2}$
2. $1 - \cos mx = 2 \sin^2 \frac{mx}{2}$
3. $\sin mx = \frac{2 \tan \frac{mx}{2}}{1 + \tan^2 \frac{mx}{2}}$
4. $\cos mx = \frac{1 - \tan^2 \frac{mx}{2}}{1 + \tan^2 \frac{mx}{2}} = \frac{\cot^2 \frac{mx}{2} - 1}{\cot^2 \frac{mx}{2} + 1}$
5. $\tan \left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
6. $\tan \left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$
7. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy}\right)$,

   provided $x, y > 0$ and $xy < 1$
8. $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \left(\frac{x+y}{1-xy}\right)$,

   provided $x, y > 0$ and $xy > 1$
9. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy}\right)$; if $x, y > 0$
10. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \operatorname{cosec}^{-1} x$
11. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$,

    provided $x, y \geq 0$ and $x^2 + y^2 \leq 1$
12. $\sin^{-1} x \pm \sin^{-1} y = \pi - \sin^{-1} (x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$,

    if $x, y \geq 0$ and $x^2 + y^2 > 1$
13. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1} (xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$,

    if $x, y > 0$ and $x^2 + y^2 \leq 1$
14. $\cos^{-1} x \pm \cos^{-1} y = \pi - \cos^{-1} (xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$,

    if $x, y > 0$ and $x^2 + y^2 > 1$

15. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\tan^{-1}x$

16. $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

17. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}} = \dfrac{1}{2}\cos^{-1}x$

SUCCESSIVE DIFFERENTIATION

Let $y = f(x)$ be a function of x, then $\dfrac{dy}{dx}$ is again a function of x and is called the first derivative of y w.r.t. x. If the first derivative is differentiable, its derivative is called second derivative of the original function and is denoted by $\dfrac{d^2y}{dx^2}$ or y_2. If the second derivative is differentiable, its derivative is called the third derivative of the original function and is denoted by $\dfrac{d^3y}{dx^3}$ or y_3 and so on. This process of differentiating a function more than once is called *successive differentiation.*

DIFFERENTIATION OF A FUNCTION GIVEN IN THE FORM OF A DETERMINANT

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note: The differentiation of a determinant can be done in columns also.

Solved Problems

1. If S_n denotes the sum of n terms of a G.P. whose common ratio is r, then $(r-1)\dfrac{dS_n}{dr}$ is equal to

(a) $(n-1)S_n + nS_{n-1}$
(b) $(n-1)S_n - nS_{n-1}$
(c) $(n-1)S_n$
(d) none of these

Solution (b) We have, $S_n = \dfrac{a(r^n-1)}{r-1}$

$\Rightarrow (r-1)\ S_n = ar^n - a$

Differentiating both sides with respect to r, we get

$$(r-1)\frac{dS_n}{dr} + S_n = nar^{n-1} - 0$$

$$\Rightarrow \quad (r-1)\frac{dS_n}{dr} = nar^{n-1} - S_n$$

$= n\,[n^{\text{th}}$ term of G.P.$] - S_n$

$= n(S_n - S_{n-1}) - S_n$

$= \mathbf{(n-1)\,S_n - n\,S_{n-1}}$.

2. If $y = \displaystyle\sum_{r=1}^{x}\tan^{-1}\frac{1}{1+r+r^2}$ then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{1+x^2}$ (b) $\dfrac{1}{1+(1+x)^2}$
(c) 0 (d) none of these

Solution (b) We have,

$$y = \sum_{r=1}^{x}\tan^{-1}\frac{1}{1+r+r^2} = \sum_{r=1}^{x}\tan^{-1}\left(\frac{(r+1)-r}{1+(r+1)r}\right)0$$

$$= \sum_{r=1}^{x}\left[\tan^{-1}(r+1) - \tan^{-1}r\right]$$

$= [\tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \ldots + \tan^{-1}x$
$\quad - \tan^{-1}(x-1) + \tan^{-1}(x+1) - \tan^{-1}x]$

$= [\tan^{-1}(x+1) - \tan^{-1}1]$

$$\therefore \quad \frac{dy}{dx} = \mathbf{\frac{1}{1+(x+1)^2}}.$$

3. If $y^2 = P(x)$, a polynomial of degree $n \ge 3$, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right) =$

(a) $-P(x)\cdot P'''(x)$ (b) $P(x)\cdot P'''(x)$
(c) $P(x)\cdot P''(x)$ (d) none of these

Solution (b) We have,

$$y^2 = P(x) \quad \ldots(1)$$

$$\Rightarrow 2y\frac{dy}{dx} = P'(x) \quad \ldots(2)$$

$$\Rightarrow 2\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\cdot\frac{d^2y}{dx^2} = P''(x)$$

$$\Rightarrow 2y^2\left(\frac{dy}{dx}\right)^2 + 2y^3\cdot\frac{d^2y}{dx^2} = y^2P''(x)$$

$$\Rightarrow 2y^3\,\frac{d^2y}{dx^2} = y^2P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$

$$= y^2P''(x) - \frac{1}{2}\,[P'(x)]^2 \qquad \text{[from (2)]}$$

$$\Rightarrow 2\,\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$$

$$= 2y\,\frac{dy}{dx}\,P''(x) + y^2P'''(x) - \frac{1}{2}\,2P'(x)\cdot P''(x)$$

$$= P'(x)\,P''(x) + y^2P'''(x) - P'(x)\,P''(x)$$

$$\left[\because 2y\frac{dy}{dx} = P'(x)\right]$$

$$= y^2P'''(x)$$

$$= \boldsymbol{P(x)\,P'''(x)}. \qquad [\because y^2 = P(x)]$$

4. The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 1 (d) none of these

Solution (a) Let $y = f(\tan x)$ and $u = g(\sec x)$

$$\Rightarrow \frac{dy}{dx} = f'(\tan x)\sec^2 x$$

and $$\frac{du}{dx} = g'(\sec x)\cdot\sec x\tan x$$

$$\therefore \frac{dy}{du} = \frac{dy}{dx}\bigg/\frac{du}{dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$$

$$\therefore \left.\frac{dy}{du}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}}$$

$$= \frac{f'(1)}{g'(\sqrt{2})\cdot\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\times 2}{4} = \boldsymbol{\frac{1}{\sqrt{2}}}.$$

5. If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to

(a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) 1 (d) none of these

Solution (b) When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$,

$$\therefore \quad |\cos x| = -\cos x$$

$$\therefore \quad f(x) = -\cos x \Rightarrow f'(x) = \sin x$$

$$\therefore \quad f'\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} = \boldsymbol{\frac{1}{\sqrt{2}}}.$$

6. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) 1 (b) -1

(c) 0 (d) none of these

Solution (a) When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore \cos x - \sin x > 0.$

When $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\cos x < \sin x \;\therefore\; \cos x - \sin x < 0.$

Also, when $\frac{\pi}{2} < x < \pi$, $\cos x - \sin x < 0.$

$\therefore$ When $\frac{\pi}{4} < x < \pi$, $\cos x - \sin x < 0$

$\therefore |\cos x - \sin x| = -(\cos x - \sin x)$, when $\frac{\pi}{4} < x < \pi$

$$\therefore \quad f(x) = -\cos x + \sin x, \text{ when } \frac{\pi}{4} < x < \pi$$

$$\Rightarrow \quad f'(x) = \sin x + \cos x$$

$$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = \mathbf{1}.$$

7. Let f be a twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = [f(x)]^2 + [g(x)]^2$ and $h(5) = 11$ then $h(10) =$

(a) 11 (b) 0

(c) -1 (d) none of these

Solution (a) We have, $h(x) = [f(x)]^2 + [g(x)]^2$

Differentiating both sides w.r.t. x, we get

$$h'(x) = 2f(x)\cdot f'(x) + 2g(x)\,g'(x)$$

$$= 2f(x)\,g(x) + 2g(x)\,g'(x) \qquad [\because f'(x) = g(x)]$$

$$= 2g(x)\,[f(x) + g'(x)] \qquad \text{...(1)}$$

But $g(x) = f'(x) \Rightarrow g'(x) = f''(x) = -f(x)$

$$[\because f''(x) = -f(x)]$$

$\therefore$ From (1),

$$h'(x) = 2g(x)\,[f(x) - f(x)] = 0$$

$\Rightarrow h(x) =$ constant for all x.

Given $h(5) = 11$, $\therefore h(10) = \mathbf{11}$.

8. A function $f(x)$ is so defined that for all x, $[f(x)]^n = f(nx)$. If $f'(x)$ denotes derivative of $f(x)$ w.r.t. x, then $f'(x)\cdot f(nx) =$

(a) $f(x)$ (b) 0

(c) $f(x)\cdot f'(nx)$ (d) none of these

Solution (c) We have, $[f(x)]^n = f(nx)$

Differentiating w.r.t. x, we get

$$n[f(x)]^{n-1}\cdot f'(x) = f'(nx)\cdot n$$

$$\Rightarrow [f(x)]^{n-1}\cdot f'(x) = f'(nx) \Rightarrow [f(x)]^n\cdot f'(x) = f'(nx)\cdot f(x)$$

[Multiplying both sides by $f(x)$]

$$\Rightarrow \quad f(nx)\cdot f'(x) = \boldsymbol{f'(nx)\cdot f(x)} \quad [\because [f(x)]^n = f(nx)]$$

9. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, then the values of a, b, c and d such that $f'(x) = x\cos x$ for all x are

(a) $b = c = 0, a = d = 1$ (b) $b = d = 0, a = c = 1$

(c) $c = d = 0, a = b = 1$ (d) none of these

Solution (a) We have,

$f'(x) = a\sin x + (ax+b)\cos x + c\cos x - (cx+d)\sin x$

But $f'(x) = x\cos x$ for all x (given).

$\therefore\ x\cos x = (a-d)\sin x + (b+c)\cos x + ax\cos x - cx\sin x$

Equating the coefficients of $\sin x$, $\cos x$, $x\cos x$ and $x\sin x$, we get

$$a - d = 0,\ b + c = 0,\ a = 1,\ c = 0$$

$$\therefore\quad \mathbf{b = c = 0 \text{ and } a = d = 1.}$$

10. If $x = f(t)$, $y = \phi(t)$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{f_1\phi_2 - \phi_1 f_2}{f_1^2}$ (b) $\dfrac{f_1\phi_2 - \phi_1 f_2}{f_1^3}$

(c) $\dfrac{\phi_1 f_2 - f_1\phi_2}{f_1^3}$ (d) none of these

Solution (b) We have,

$$\frac{dx}{dt} = \frac{df}{dt} = f_1,\quad \frac{dy}{dt} = \frac{d\phi}{dt} = \phi_1$$

$$\therefore\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi_1}{f_1}.$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\phi_1}{f_1}\right) = \frac{d}{dt}\left(\frac{\phi_1}{f_1}\right)\cdot\frac{dt}{dx}$$

$$= \frac{f_1\phi_2 - \phi_1 f_2}{f_1^2}\times\frac{1}{f_1} = \frac{f_1\phi_2 - \phi_1 f_2}{f_1^3}.$$

where the suffixes denote differentiation with respect to t.

11. If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$, then $(x^2+4)\left(\dfrac{dy}{dx}\right)^2$ is equal to

(a) $n^2(y^2-4)$ (b) $n^2(4-y^2)$
(c) $n^2(y^2+4)$ (d) none of these

Solution (c) $\dfrac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \tan\theta(\sec\theta + \cos\theta)$

$$\frac{dx}{d\theta} = n\,[\sec^n\theta\tan\theta + \cos^{n-1}\theta\sin\theta]$$
$$= n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore\quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$\Rightarrow\quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$= n^2\cdot\frac{(\sec^n\theta - \cos^n\theta)^2 + 4}{(\sec\theta - \cos\theta)^2 + 4}$$

$$= \frac{n^2(y^2+4)}{(x^2+4)}$$

$$\Rightarrow\quad (x^2+4)\left(\frac{dy}{dx}\right)^2 = \mathbf{n^2(y^2+4).}$$

12. If f, g, h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$$

then Δ' (the derivative of Δ w.r.t. x) is given by

(a) $\begin{vmatrix} f' & g' & h' \\ f & g & h \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$

(b) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2f'')' & (x^2g'')' & (x^2h'')' \end{vmatrix}$

(c) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$

(d) none of these

Solution (c) We have,

$$\Delta = \begin{vmatrix} f & g & h \\ xf'+f & xg'+g & xh'+h \\ x^2f''+4xf'+2f & x^2g''+4xg'+2g & x^2h''+4xh'+2h \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f''+4xf' & x^2g''+4xg' & x^2h''+4xh' \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$]

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$ [Applying $R_3 \to R_3 - 4R_2$]

$$= x^3\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$\therefore\quad \Delta' = \begin{vmatrix} \mathbf{f} & \mathbf{g} & \mathbf{h} \\ \mathbf{f'} & \mathbf{g'} & \mathbf{h'} \\ \mathbf{(x^3f'')'} & \mathbf{(x^3g'')'} & \mathbf{(x^3h'')'} \end{vmatrix}$$

13. If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\dfrac{n\pi}{2} & 4 \\ \sin x & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix}$ then the value of

$\dfrac{d^n}{dx^n}[f(x)]_{x=0}$ is

(a) 0 (b) 1
(c) -1 (d) none of these

Solution (a)

$$\frac{d^n}{dx^n}[f(x)] = \begin{vmatrix} n! & n! & 2 \\ \cos\left(x+\dfrac{n\pi}{2}\right) & \cos\dfrac{n\pi}{2} & 4 \\ \sin\left(x+\dfrac{n\pi}{2}\right) & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix}$$

$$\therefore \quad \frac{d^n}{dx^n}[f(x)]_{x=0} = \begin{vmatrix} n! & n! & 2 \\ \cos\frac{n\pi}{2} & \cos\frac{n\pi}{2} & 4 \\ \sin\frac{n\pi}{2} & \sin\frac{n\pi}{2} & 8 \end{vmatrix} = \mathbf{0}$$

[∵ C_1 and C_2 are identical]

14. If $f(x) = (1 - x)^n$, then the value of

$$f(0) + f'(0) + \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!} \text{ is equal to}$$

(a) 2^n (b) 0

(c) 2^{n-1} (d) none of these

Solution (b) We have,

$$f(x) = (1 - x)^n, f'(x) = -n(1 - x)^{n-1},$$
$$f''(x) = n(n - 1)(1 - x)^{n-2},$$
$$f'''(x) = -n(n - 1)(n - 2)(1 - x)^{n-3}\ldots$$
$$f^n(x) = (-1)^n n(n - 1)(n - 2) \ldots 1$$
$$\Rightarrow \quad f(0) = 1, f'(0) = -n, f''(0) = n(n - 1),$$
$$f'''(0) = -n(n - 1)(n - 2) \ldots f^n(0) = (-1)^n n!$$

Therefore,

$$f(0) + f'(0) + \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!}$$
$$= 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \ldots + \frac{(-1)^n n!}{n!}$$
$$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\, {}^nC_n$$
$$= (1 - 1)^n = \mathbf{0}.$$

15. If $y = f(x^3)$, $z = g(x^5)$, $f'(x) = \tan x$ and $g'(x) = \sec x$, then the value of $\frac{dy}{dz}$ is

(a) $\frac{3}{5x^2}\cdot\frac{\tan x^3}{\sec x^5}$ (b) $\frac{5x^2}{3}\cdot\frac{\sec x^5}{\tan x^3}$

(c) $\frac{3x^2}{5}\cdot\frac{\tan x^3}{\sec x^5}$ (d) none of these

Solution (a) We have, $y = f(x^3) \Rightarrow \frac{dy}{dx}$

$$= f'(x^3)\cdot 3x^2 = 3x^2 \tan x^3$$

Also, $z = g(x^5)$

$$\Rightarrow \quad \frac{dz}{dx} = g'(x^5)\cdot 5x^4 = 5x^4 \sec x^5$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{3x^2 \tan x^3}{5x^4 \sec x^5} = \mathbf{\frac{3}{5x^2}\cdot\frac{\tan x^3}{\sec x^5}}.$$

16. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then the value of $[f(10)]^2 - [\phi(10)]^2$ is

(a) 0 (b) 9

(c) 41 (d) none of these

Solution (b) $\frac{d}{dx}\{[f(x)]^2 - [f(x)]^2\}$

$$= 2[f(x)\cdot f'(x) - \phi(x)\cdot\phi'(x)]$$
$$= 2[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)]$$

[∵ $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$]

$$= 0.$$

$$\Rightarrow \quad [f(x)]^2 - [\phi(x)]^2 = \text{constant}$$
$$\therefore [f(10)]^2 - [\phi(10]^2 = [f(3)]^2 - [\phi(3)]^2$$
$$= [f(3)]^2 - [f'(3)]^2$$
$$= 25 - 16 = \mathbf{9}.$$

17. If $8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = -1$ is equal to

(a) 0 (b) $\frac{1}{14}$

(c) $-\frac{1}{14}$ (d) none of these

Solution (c) We have,

$$8f(x) + 6f\left(\frac{1}{x}\right) = x + 5 \text{ for all } x \quad \ldots(1)$$

Therefore,

$$8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5 \quad \ldots(2)$$

$\left[\text{Putting } x = \frac{1}{x}\right]$

From (1) and (2), we have

$$f(x) = \frac{1}{28}\left(8x - \frac{6}{x} + 10\right)$$

$$\therefore \quad y = x^2 f(x) = \frac{1}{28}(8x^3 - 6x + 10x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{28}(24x^2 + 20x - 6)$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=-1} = \frac{1}{28}(24 - 20 - 6) = -\mathbf{\frac{1}{14}}.$$

18. Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ $(x \neq 0)$, then the value of $f'(27)$ is

(a) 15 (b) 45

(c) 0 (d) none of these

Solution (a) We have, for all x $(x \neq 0)$

$$f(x^3) = x^5$$

Differentiating w.r.t. x, we get

$$f'(x^3)\cdot 3x^2 = 5x^4 \Rightarrow f'(x^3) = \frac{5}{3}x^2$$

$$\therefore \quad f'(27) = f'(3^3) = \frac{5}{3}(3)^2 = \mathbf{15}.$$

19. Let $f(x) = 2^{2x-1}$ and $\phi(x) = -2^x + 2x \log 2$. If $f'(x) > \phi'(x)$, then

(a) $0 < x < 1$ (b) $0 \le x < 1$

(c) $x > 0$ (d) $x \ge 0$

Solution (c) Since $f'(x) > \phi'(x)$

$\Rightarrow \; 2^{2x-1}\, 2\log 2 > -2^x \log 2 + 2\log 2$

$\Rightarrow \; 2^{2x} > -2^x + 2$

$\Rightarrow \; 2^{2x} + 2^x - 2 > 0$

$\Rightarrow \; (2^x - 1)(2^x + 2) > 0$

$\Rightarrow \; 2^x - 1 > 0$ $\quad [\because 2^x + 2 > 0$ for all $x]$

$\Rightarrow \; 2^x > 1 \; \therefore \; \mathbf{x > 0}$.

20. Let $f(x)$ be a polynomial function satisfying $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. If $f(4) = 65$ and l_1, l_2, l_3 are in G.P. then $f'(l_1)$, $f'(l_2), f'(l_3)$ are in

(a) A. P. (b) G. P.

(c) H. P. (d) none of these

Solution (b) Since $f(x)$ is a polynomial function satisfying

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right),$$

$\therefore \quad f(x) = x^n + 1$ or $f(x) = -x^n + 1$

If $f(x) = -x^n + 1$, then $f(4) = -4^n + 1 \neq 65$

So, $\quad f(x) = x^n + 1$.

Since $\quad f(4) = 65 \;\therefore\; 4^n + 1 = 65 \Rightarrow n = 3$

$\therefore \quad f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2$

$\therefore \quad f'(l_1) = 3l_1^2,\; f'(l_2) = 3l_2^2,\; f'(l_3) = 3l_3^2$

Since l_1, l_2, l_3 are in G. P.,

$\therefore \quad f'(l_1), f'(l_2), f'(l_3)$ are also in **G.P.**

21. If $f(x) = \log\left(\frac{m(x)}{n(x)}\right)$, $m(1) = n(1) = 1$ and $m'(1) = n'(1) = 2$, then $f'(1)$ is equal to

(a) 0 (b) 1

(c) – 1 (d) none of these

Solution (a) $f'(x) = \dfrac{n(x)}{m(x)}\left[\dfrac{n(x)m'(x) - m(x)n'(x)}{[n(x)]^2}\right]$

$$\Rightarrow \quad f'(1) = \frac{n(1)}{m(1)}\left[\frac{n(1)m'(1) - m(1)n'(1)}{[n(1)]^2}\right] = \mathbf{0}.$$

22. If $\dfrac{d}{dx}\left(\dfrac{1 + x^4 + x^8}{1 + x^2 + x^4}\right) = ax^3 + bx$ then

(a) $a = 4, b = 2$ (b) $a = 4, b = -2$

(c) $a = -2, b = 4$ (d) none of these

Solution (b) We have,

$$\frac{d}{dx}\left[\frac{(1 + x^2 + x^4)(1 - x^2 + x^4)}{(1 + x^2 + x^4)}\right] = ax^3 + bx$$

$\Rightarrow \quad \dfrac{d}{dx}(1 - x^2 + x^4) = ax^3 + bx$

$\Rightarrow \quad -2x + 4x^3 = ax^3 + bx \Rightarrow \mathbf{a = 4}$ **and** $\mathbf{b = -2}$.

23. Let $f(x) = (x^3 + 2)^{30}$. If $f^n(x)$ is a polynomial of degree 20, where $f^n(x)$ denotes the nth derivative of $f(x)$ w.r.t. x, then the value of n is

(a) 60 (b) 40

(c) 70 (d) none of these

Solution (c) $f(x)$ is a polynomial of degree 90. $f'(x)$ reduces the degree of $f(x)$ by one. Thus, in order to get a polynomial of degree 20, we must reduce the degree of $f(x)$ by 70. Hence, $f(x)$ should be differentiated 70 times to get a polynomial of degree 20.

$\therefore \quad \mathbf{n = 70}$.

24. If $f(x) = \sin\left(\frac{\pi}{2}[x] - x^5\right)$, $1 < x < 2$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'\left(\sqrt[5]{\frac{\pi}{2}}\right)$ is equal to

(a) $5\left(\frac{\pi}{2}\right)^{4/5}$ (b) $-5\left(\frac{\pi}{2}\right)^{4/5}$

(c) 0 (d) none of these

Solution (b) Since, $1 < x < 2$. $\therefore [x] = 1$

$\therefore \quad f(x) = \sin\left(\frac{\pi}{2} - x^5\right) = \cos x^5$

$\Rightarrow \quad f'(x) = -\sin x^5 \cdot 5x^4$

$$\Rightarrow \quad f'\left(\sqrt[5]{\frac{\pi}{2}}\right) = -5\left(\frac{\pi}{2}\right)^{4/5} \cdot \sin\frac{\pi}{2} = \mathbf{-5}\left(\frac{\pi}{2}\right)^{4/5}.$$

25. If $f(x) + f(y) + f(z) + f(x) \cdot f(y) \cdot f(z) = 14$ for all $x, y, z \in R$, then

(a) $f(0) = 2$ (b) $f'(x) = 0$, for all $x \in R$

(c) $f'(x) > 0$, for all $x \in R$

(d) none of these

Solution (a), (b) We have,

$$f(x) + f(y) + f(z) + f(x) \cdot f(y) \cdot f(z) = 14 \quad \text{...(1)}$$

for all $x, y, z \in R$

Putting $x = y = z = 0$, we get

$3f(0) + [f(0)]^3 = 14 \Rightarrow [f(0)]^3 + 3f(0) - 14 = 0$

$\Rightarrow \; \mathbf{f(0) = 2}$.

Now, putting $y = z = x$ in (1), we get

$3f(x) + [f(x)]^3 = 14$

Differentiating w.r.t. x, we get

$3f'(x) + 3[f(x)]^2 \cdot f'(x) = 0$

$\Rightarrow \; 3f'(x)\,[1 + \{f(x)\}^2] = 0$

$\Rightarrow \; \mathbf{f'(x) = 0}$, for all x.

26. Let $f(x)$ be a polynomial of degree 3 such that $f(3) = 1$, $f'(3) = -1$, $f''(3) = 0$ and $f'''(3) = 12$. Then the value of $f'(1)$ is

(a) 12 (b) 23

(c) – 13 (d) none of these

Solution (b) Let $f(x) = a(x-3)^3 + b(x-3)^2 + c(x-3) + d$

Then,

$$f(3) = 1 = d \Rightarrow d = 1$$
$$f'(3) = -1 = c \Rightarrow c = -1$$
$$f''(3) = 0 = 2b \Rightarrow b = 0$$
$$f'''(3) = 12 = 6a \Rightarrow a = 2.$$
$$\therefore \quad f(x) = 2(x-3)^3 - x + 4 \Rightarrow f'(x) = 6(x-3)^2 - 1$$
$$\therefore \quad f'(1) = 6(4) - 1 = \mathbf{23.}$$

27. Let $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. If $\phi(x) = [go(fh)](x)$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to

(a) 4
(b) 0
(c) -4
(d) none of these

Solution (c) We have, $(fh)(x) = f(x) \cdot h(x) = \sin x \cos x$

$$\therefore \quad [go(fh)](x) = g[(fh)(x)] = g[f(x) \cdot h(x)]$$
$$= g(\sin x \cos x) = 2 \sin x \cos x = \sin 2x$$

i.e., $\phi(x) = \sin 2x$

$$\Rightarrow \quad \phi'(x) = 2\cos 2x \text{ and } \phi''(x) = -4\sin 2x$$
$$\therefore \quad \phi''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = \mathbf{-4.}$$

28. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$$ is

(a) 2^n
(b) 2^{n-1}
(c) 0
(d) 1

Solution (c)

$f(x) = x^n \quad \Rightarrow f(1) = 1$

$f'(x) = nx^{n-1} \quad \Rightarrow f'(1) = n$

$f''(x) = n(n-1)x^{n-2} \quad \Rightarrow f''(1) = n(n-1)$

… … … … … … … … … … …

… … … … … … … … … … …

$f^n(x) = n(n-1)(n-2) \ldots 2.1$

$\Rightarrow f^n(1) = n(n-1)(n-1) \ldots 2.1$

We have,

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$$
$$= 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \ldots + \frac{(-1)^n n.(n-1)(n-2)\ldots 21}{n!}$$
$$= (1-1)^n = 0$$

29. If $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$,

then $\dfrac{y'}{y} =$

(a) $\frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$
(b) $\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$
(c) $\frac{1}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$
(d) none of these

Solution (a)

30. If α is a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree > 2, then the determinant

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$ is divisible by

(a) $A(x)$
(b) $B(x)$
(c) $C(x)$
(d) $f(x)$

Solution (d) Let $\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

So $\Delta(\alpha) = 0 = \Delta'(\alpha)$, therefore α is a repeated root of $\Delta(x)$ and α is a repeated root of the quadratic equation $f(x) = 0$, so Δ is divisible by $\boldsymbol{f(x)}$.

31. If $\cos y = x\cos(a + y)$ and $\dfrac{dy}{dx} = \dfrac{k}{1 + x^2 - 2x\cos a}$ then the value of k is

(a) $\sin a$
(b) $\cos a$
(c) 1
(d) $-\sin a$

Solution (a) We have, $\cos y = x\cos(a + y)$

$$\Rightarrow \quad \cos y = x(\cos a \cos y - \sin a \sin y)$$
$$\Rightarrow \quad \cot y = x(\cos a \cot y - \sin a)$$
$$\Rightarrow \quad \cot y\,(1 - x\cos a) = -x\sin a$$
$$\Rightarrow \quad y = \cot^{-1}\left(\frac{x\sin a}{x\cos a - 1}\right)$$
$$\therefore \quad \frac{dy}{dx} = \frac{-1}{1 + \left(\dfrac{x\sin a}{x\cos a - 1}\right)^2} \cdot \left[\frac{(x\cos a - 1)\sin a - x\sin a\cos a}{(x\cos a - 1)^2}\right]$$
$$= \frac{\sin a}{(x\cos a - 1)^2 + x^2\sin^2 a} = \frac{\sin a}{1 + x^2 - 2x\cos a}$$

Thus, $k = \mathbf{\sin a}$.

32. If $f(x) = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \sin x \sin\left(x + \frac{\pi}{3}\right)$ and $\gamma\left(\frac{5}{4}\right) = 3$ then $(gof)(x)$ is equal to

(a) 1
(b) 2
(c) 3
(d) none of these

Solution (c) $f'(x)$

$$= -2\cos x \sin x - 2\cos\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right) + \cos x \sin\left(x+\frac{\pi}{3}\right) + \sin x \cos\left(x+\frac{\pi}{3}\right)$$

$$= -\sin 2x - \sin\left(2x+\frac{2\pi}{3}\right) + \sin\left(x+x+\frac{\pi}{3}\right)$$

$$= -2\sin\left(2x+\frac{2\pi}{3}\right)\cos\frac{\pi}{3} + \sin\left(2x+\frac{2\pi}{3}\right)$$

$$= -\sin\left(2x+\frac{2\pi}{3}\right) + \sin\left(2x+\frac{2\pi}{3}\right) = 0.$$

$\Rightarrow$ $f(x)$ = constant for all x.

But, $f(0) = \cos^2 0 + \cos^2\frac{\pi}{3} + \sin 0 \cdot \sin\frac{\pi}{3} = \frac{5}{4}$

$\therefore$ $f(x) = \frac{5}{4}$ for all x.

Thus, $(gof)(x) = g[f(x)] = g\left(\frac{5}{4}\right) = \mathbf{3}$.

33. If $f(x) = \cos x \cos 2x \cos 4x \cos 8x$, then $f'\left(\frac{\pi}{4}\right)$ is

(a) -1 (b) 2

(c) $\sqrt{2}$ (d) none of these

Solution (c) $f(x) = \dfrac{2\sin x\cos x\cos 2x\cos 4x\cos 8x}{2\sin x}$

$$= \frac{\sin 16x}{2^4 \sin x}$$

$$\Rightarrow \quad f'(x) = \frac{1}{16}\left[\frac{\sin x\cdot\cos 16x\cdot 16 - \sin 16x\cdot\cos x}{\sin^2 x}\right]$$

$$\therefore \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{16}\left[\frac{\frac{1}{\sqrt{2}}\cdot 1\cdot 16 - \frac{1}{\sqrt{2}}\cdot 0}{\left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{\mathbf{2}}.$$

34. If $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x)(\cos 3x + i \sin 3x) \ldots (\cos nx + i \sin nx)$ and $f(1) = 1$ then $f''(1)$ is equal to

(a) $\dfrac{n(n+1)}{2}$ (b) $\left(\dfrac{n(n+1)}{2}\right)^2$

(c) $-\left(\dfrac{n(n+1)}{2}\right)^2$ (d) none of these

Solution (c) $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x)(\cos 3x + i \sin 3x) \ldots (\cos nx + i \sin nx)$

$$= \cos(x + 2x + 3x + \ldots + nx) + i\sin(x + 2x + 3x + \ldots + nx)$$

$$= \cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x$$

$$\Rightarrow \quad f'(x) = \frac{n(n+1)}{2}\left[-\sin\frac{n(n+1)}{2}x + i\cos\frac{n(n+1)}{2}x\right]$$

$$\Rightarrow \quad f''(x) = -\left(\frac{n(n+1)}{2}\right)^2\left(\cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x\right)$$

$$= -\left(\frac{n(n+1)}{2}\right)^2 f(x)$$

$$\therefore \quad f''(1) = -\left(\frac{n(n+1)}{2}\right)^2 f(1) = -\left(\frac{\mathbf{n(n+1)}}{\mathbf{2}}\right)^2.$$

35. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant. Then $\dfrac{d^3}{dx^3}[f(x)]$ at $x = 0$ is

(a) p (b) $p + p^2$

(c) $p + p^3$ (d) independent of p

Solution (d) We have,

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

and $$f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\therefore f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

[$\because$ R_1 and R_2 are identical]

36. If $f(x) = x^m$, m being a non-negative integer, then the value of m for which $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$, for all $\alpha, \beta > 0$, is

(a) 1 (b) 2

(c) 0 (d) none of these

Solution (b), (c) We have, $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$

$\Rightarrow (\alpha + \beta)^{m-1} = \alpha^{m-1} + \beta^{m-1}$

Since, for $m > 2$, the above equality is not valid

$\therefore$ we must have $\boldsymbol{m = 2}$. Also, for $\boldsymbol{m = 0}$, $f'(x) = 0$ for all x. So the equality is trivially true.

37. If $f(x) = \sqrt{x^2 + 6x + 9}$, then $f'(x)$ is equal to

(a) 1 for $x < -3$ (b) -1 for $x < -3$

(c) 1 for all $x \in R$ (d) none of these

Solution (b) $f(x) = \sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = |x + 3|$

$$= \begin{Bmatrix} x+3, & x+3 \geq 0 \text{ i.e. } x \geq -3 \\ -x-3, & x+3 < 0 \text{ i.e. } x < -3 \end{Bmatrix}$$

$$\therefore \quad f'(x) = \begin{Bmatrix} 1 & \text{if } x \geq -3 \\ \mathbf{-1} & \textbf{if } \boldsymbol{x < -3} \end{Bmatrix}.$$

38. If $f(x) = |x - 1|$ and $g(x) = f[f\{f(x)\}]$, then for $x > 2$, $g'(x)$ is equal to

(a) -1 if $2 \leq x < 3$ (b) 1 if $2 \leq x < 3$

(c) 1 for all $x > 2$ (d) none of these

Solution (a) We have,

$f(x) = x - 1$ ($\because$ $x > 2$)

$$f[f(x)] = f(x-1) = |(x-1)-1| = |x-2|$$

$$= (x-2) \qquad (\because x > 2)$$

$$\therefore \quad g(x) = f[f\{f(x)\}] = f(x-2)$$

$$= |(x-2)-1| = |x-3|$$

$$= \begin{Bmatrix} x-3, & \text{if } x \geq 3 \\ -x+3, & \text{if } 2 \leq x < 3 \end{Bmatrix}$$

$$\therefore \quad g'(x) = \begin{Bmatrix} 1, & \text{if } x \geq 3 \\ \mathbf{-1}, & \textbf{if } \mathbf{2 \leq x < 3} \end{Bmatrix}.$$

39. If $f(x) = |(x-4)(x-5)|$, then $f'(x)$ is equal to

(a) $-2x+9$, for all $x \in R$

(b) $2x-9$ if $4 < x < 5$

(c) $-2x+9$ if $4 < x < 5$

(d) none of these

Solution (c) $f(x) = |(x-4)(x-5)|$

$$= \begin{Bmatrix} (x-4)(x-5) & \text{if } (x-4)(x-5) \geq 0 \\ -(x-4)(x-5) & \text{if } (x-4)(x-5) < 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} x^2-9x+20 & \text{if } x \leq 4 \text{ or } x \geq 5 \\ -(x^2-9x+20) & \text{if } 4 < x < 5 \end{Bmatrix}$$

$$\therefore \quad f'(x) = \begin{Bmatrix} 2x-9 & \text{if } x \leq 4 \text{ or } x \geq 5 \\ \mathbf{-2x+9} & \textbf{if } \mathbf{4 < x < 5} \end{Bmatrix}$$

40. If $f(x) = |x-3|$ and $\phi(x) = (fof)(x)$, then for $x > 10$, $\phi'(x)$ is equal to

(a) 1 (b) 0

(c) -1 (d) none of these

Solution (a) Since $x > 10$, therefore, $f(x) = x-3$

$\therefore \phi(x) = (fof)(x) = f[f(x)] = f(x-3) = x-3-3 = x-6$

$\Rightarrow \phi'(x) = \mathbf{1}$.

Problems for Practice

Level of Difficulty I

1. If $y = (1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$, then $\dfrac{dy}{dx} =$

(a) 1 (b) -1

(c) x (d) $\sqrt{x}$

2. Differential coefficient of $\log_{10} x$ with respect to $\log_x 10$ is

(a) $-\dfrac{(\log 10)^2}{(\log x)^2}$ (b) $\dfrac{(\log_x 10)^2}{(\log 10)^2}$

(c) $\dfrac{(\log_{10} x)^2}{(\log 10)^2}$ (d) $-\dfrac{(\log x)^2}{(\log 10)^2}$

3. If $f(x) = \left(\dfrac{x^a}{x^b}\right)^{a+b} \cdot \left(\dfrac{x^b}{x^c}\right)^{b+c} \cdot \left(\dfrac{x^c}{x^a}\right)^{c+a}$, then $f'(x)$ is equal to

(a) 1 (b) 0

(c) x^{a+b+c} (d) none of these

4. If $f(x) = \left(\dfrac{\sin^m x}{\sin^n x}\right)^{m+n} \cdot \left(\dfrac{\sin^n x}{\sin^p x}\right)^{n+p} \cdot \left(\dfrac{\sin^p x}{\sin^m x}\right)^{p+m}$, then $f'(x)$ is equal to

(a) 0 (b) 1

(c) $\cos^{m+n+p} x$ (d) none of these

5. If $y = \dfrac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}} + \dfrac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}} + \dfrac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$

then $\dfrac{dy}{dx} =$

(a) 0

(b) 1

(c) $(\alpha+\beta+\gamma)\, x^{\alpha+\beta+\gamma-1}$

(d) none of these

6. If $y = f\left(\dfrac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\dfrac{dy}{dx}$ is equal to

(a) $\sin\left(\dfrac{2x-1}{x^2+1}\right)^2 \cdot \left(\dfrac{2+2x+x^2}{(x^2+1)^2}\right)$

(b) $\sin\left(\dfrac{2x-1}{x^2+1}\right)^2 \left(\dfrac{2+2x-2x^2}{(x^2+1)^2}\right)$

(c) $\sin\left(\dfrac{2x-1}{x^2+1}\right)^2 \cdot \left(\dfrac{2+2x-x^2}{(x^2+1)^2}\right)$

(d) none of these

7. If g is the inverse of f and $f'(x) = \dfrac{1}{1+x^3}$, then $g'(x)$ is equal to

(a) $1 + [g(x)]^3$ (b) $\dfrac{1}{1+[g(x)]^3}$

(c) $[g(x)]^3$ (d) none of these

8. If $y = a\cos(\log x) + b\sin(\log x)$, then

$x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} =$

(a) 0 (b) y (c) $-y$ (d) none of these

9. If $y = (\sin^{-1}x)^2$, then $(1-x^2)\dfrac{d^2y}{dx^2}$ is equal to

(a) $x\dfrac{dy}{dx} + 2$ (b) $x\dfrac{dy}{dx} - 2$

(c) $-x\dfrac{dy}{dx} + 2$ (d) none of these

10. If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$ and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}, \text{ then } F'(a) \text{ is equal to}$$

(a) a (b) $-a$ (c) 0 (d) none of these

11. If $f(x)$, $g(x)$, $h(x)$ are polynomials in x of degree 2 and

$$F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}, \text{ then } F'(x) \text{ is equal to}$$

(a) 1 (b) 0 (c) -1 (d) none of these

12. If $y = (x + \sqrt{1+x^2})^n$, then

$(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2x^2y$

13. If $\sin y = x \sin(a + y)$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{\sin a}{\sin a \sin^2(a+y)}$ (b) $\dfrac{\sin^2(a+y)}{\sin a}$

(c) $\sin a \sin^2(a+y)$ (d) $\dfrac{\sin^2(a-y)}{\sin a}$

14. If $x = e^{y + e^{y + \ldots \text{ to } \propto}}$, $x > 0$ then $\dfrac{dy}{dx}$ is

(a) $\dfrac{1-x}{x}$ (b) $\dfrac{1}{x}$ (c) $\dfrac{x}{1+x}$ (d) $\dfrac{1+x}{x}$

15. If $y = \sin^{-1}\left[\sqrt{x-ax} - \sqrt{a-ax}\right]$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{1}{\sin\sqrt{a-ax}}$ (b) $\sin\sqrt{x} \cdot \sin\sqrt{a}$

(c) $\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$ (d) zero

16. If $\phi(x) = \log_5 \log_3 x$, then $\phi'(e)$ is equal to

(a) $e \log 5$ (b) $-e \log 5$

(c) $\dfrac{1}{e \log 5}$ (d) none of these

17. If $y = \sec^{-1}\dfrac{\sqrt{x}-1}{x+\sqrt{x}} + \sin^{-1}\dfrac{x+\sqrt{x}}{\sqrt{x}-1}$, then $\dfrac{dy}{dx}$ is equal to

(a) x (b) 1 (c) 0 (d) none of these

18. If $y = \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)$, $a > b > 0$, then

(a) $\dfrac{dy}{dx} = \dfrac{1}{a + b\cos x}$

(b) $\dfrac{d^2y}{dx^2} = \dfrac{b \sin x}{(a + b\cos x)^2}$

(c) $\dfrac{dy}{dx} = \dfrac{1}{a - b\cos x}$

(d) $\dfrac{d^2y}{dx^2} = \dfrac{-b\sin x}{(a - b\cos x)^2}$

19. If $y = x^{(\log x)^{\log\log x}}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{y\log y}{x\log x}(2 \log\log x + 1)$

(b) $\dfrac{x\log x}{y\log y}(2 \log\log x + 1)$

(c) $\dfrac{2y\log y}{x\log x}(\log\log x + 1)$

(d) none of these

20. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, where $0 < \theta < \dfrac{\pi}{2}$, then $\dfrac{d^2y}{dx^2}$ at $x = \dfrac{\pi}{4}$ is equal to

(a) $\dfrac{4\sqrt{2}}{a\pi}$ (b) $\dfrac{8\sqrt{2}}{a\pi}$

(c) $\dfrac{4}{a\pi\sqrt{2}}$ (d) none of these

Level of Difficulty II

21. If $y = e^{ax}\sin bx$, then $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx}$ is equal to

(a) $-(a^2 + b^2)y$ (b) $(a^2 + b^2)y$

(c) $-y$ (d) none of these

22. If $x = f(t)\cos t - f'(t)\sin t$, $y = f(t)\sin t + f'(t)\cos t$

then $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$ is equal to

(a) $f(t) - f''(t)$(b) $[f(t) - f''(t)]^2$

(c) $[f(t) + f''(t)]^2$ (d) none of these

23. If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of $f(x)$ on the interval [0, 7] is

(a) 1 (b) -1

(c) 0 (d) does not exist

24. If $y = x^{n-1}\log x$, then $x^2y_2 + (3 - 2n)xy_1$ is equal to

(a) $-(n - 1)^2 y$ (b) $(n - 1)^2y$

(c) $-n^2y$ (d) n^2y

25. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, then the values of a, b, c and d such that $f'(x) = x\cos x$ for all x are

(a) $b = c = 0, a = d = 1$

(b) $b = d = 0, a = c = 1$

(c) $c = d = 0, a = b = 1$

(d) none of these

26. If the parametric equation of a curve is given by $x = \cos\theta + \log\tan\dfrac{\theta}{2}$ and $y = \sin\theta$, then the points for which $\dfrac{d^2y}{dx^2} = 0$ are given by

(a) $\theta = n\pi,\ n \in I$

(b) $\theta = (2n + 1)\,\dfrac{\pi}{2},\ n \in I$

(c) $\theta = (2n + 1)\,\pi,\ n \in I$

(d) $\theta = 2n\pi,\ n \in I.$

27. If $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$, then $\dfrac{df^{-1}(x)}{dx}$ is equal to

(a) $-\dfrac{3}{(1-x)^2}$ (b) $\dfrac{3}{(1-x)^2}$

(c) $\dfrac{1}{(1-x)^2}$ (d) none of these

28. If $y = \tan^{-1}\left[\dfrac{\log(e/x^2)}{\log(ex^2)}\right] + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$, then $\dfrac{d^2y}{dx^2} =$

(a) 2 (b) 1 (c) 0 (d) –1

29. If the function $y\,(x)$ represented by $x = \sin t$, $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}},\ t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ satisfies the equation $(1 - x^2)\,y'' - xy' = ky$, then k is equal to

(a) 1 (b) –2 (c) 2 (d) none of these

30. If $f(x) = \log|2x|,\ x \neq 0$, then $f'(x)$ is equal to

(a) $\dfrac{1}{x}$ (b) $-\dfrac{1}{x}$ (c) $\dfrac{1}{|x|}$ (d) none of these

31. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\dfrac{dy}{dx}$ is equal to

(a) – 1 (b) 1 (c) 0 (d) none of these

32. If $f(x) = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, then $(1 - x^2)\,f'(x) - xf(x) =$

(a) 1 (b) –1 (c) 0 (d) none of these

33. If $y = \cos^{-1}\left(\dfrac{2\cos x - 3\sin x}{\sqrt{13}}\right)$, then $\dfrac{dy}{dx}$ is equal to

(a) 1 (b) 0

(c) constant (≠ 1) (d) none of these

34. If $y = \sin x$, then $\dfrac{d^2}{dy^2}(\cos^7 x)$ is equal to

(a) $35\cos^3 x - 42\cos^5 x$

(b) $35\cos^3 x + 42\cos^5 x$

(c) $42\cos^3 x - 35\cos^5 x$

(d) none of these

35. If $f(x) = \left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$, then $f'(0)$ is equal to

(a) $\left(2\log\dfrac{a}{b} + \dfrac{a^2 - b^2}{ab}\right)\left(\dfrac{a}{b}\right)^{a+b}$

(b) $\left(2\log\dfrac{a}{b} + \dfrac{b^2 - a^2}{ab}\right)\left(\dfrac{a}{b}\right)^{a+b}$

(c) $\left(2\log\dfrac{a}{b} + \dfrac{a^2 + b^2}{ab}\right)\left(\dfrac{a}{b}\right)^{a+b}$

(d) none of these

ANSWERS

1. (b)	**2.** (d)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (c)
11. (b)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (a),(b)	**19.** (a)	**20.** (b)
21. (a)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (a)
31. (b)	**32.** (a)	**33.** (a)	**34.** (a)	**35.** (b)					

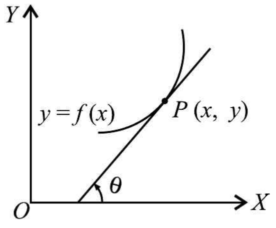
Applications of Derivatives 14

BRIEF REVIEW OF THE CONCEPTS

TANGENTS AND NORMALS

Geometrical meaning of derivative at a point The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

Consider a curve $y = f(x)$ and a point $P(x, y)$ on this curve. If tangent to the curve at $P(x, y)$ makes an angle θ with the positive direction of x-axis, then, at the point $P(x, y)$: $\frac{dy}{dx} = \tan\theta = m$ = gradient or slope of tangent to the curve at $P(x, y)$.



Equation of Tangent

The equation of a tangent to a curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

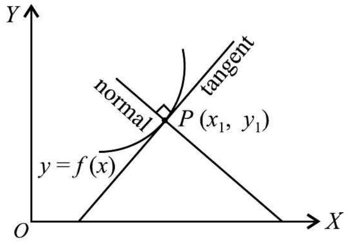
$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$$

[Using point slope form of equation of the straight line]

Equation of Normal

The equation of normal to a curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}} (x - x_1)$$



[∵ the normal to the curve at the point $P(x_1, y_1)$ is a line perpendicular to the tangent at the point $P(x_1, y_1)$ and passing through it. Therefore, slope of normal at $P(x_1, y_1)$

$$= \frac{-1}{\text{Slope of tangent at P}(x_1, y_1)} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}.$$

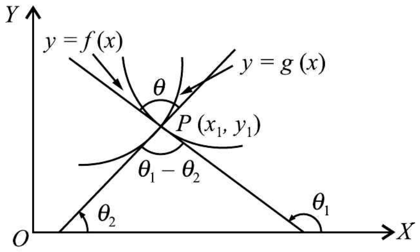
Key Points to Remember

1. If $\frac{dy}{dx} > 0$, the tangent makes an acute angle with the x-axis.
2. If $\frac{dy}{dx} < 0$, the tangent makes an obtuse angle with the x-axis.
3. If $\frac{dy}{dx} = 0$, the tangent is parallel to x-axis.
4. If the tangent is perpendicular to x-axis, then
$$\frac{dy}{dx} = \infty, \text{ i.e., } \frac{dx}{dy} = 0.$$
5. If the tangent is equally inclined to the axes, then
$$\frac{dy}{dx} = \tan 45° \text{ or } \tan 135° = \pm 1.$$
6. The slope of a line having equation $ax + by + c = 0$ is given by
$$m = -\frac{a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}.$$
7. The two lines having slopes m_1 and m_2 are
   (i) perependicular if $m_1 m_2 = -1$ and
   (ii) parallel if $m_1 = m_2$.

Thus, if m be the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$ and parallel to it is m.

Angle of Intersection of Two Curves

Let $y = f(x)$ and $y = g(x)$ be two curves intersecting at a point $P(x_1, y_1)$. Then, the angle of intersection of these two curves is defined as the angle between the tangents to the two curves at their point of intersection.



If θ is the required angle of intersection, then,

$$\theta = \pm(\theta_1 - \theta_2),$$

where θ_1 and θ_2 are the inclinations of tangents to the curves $y = f(x)$ and $y = g(x)$ respectively at the point P.

Short-Cut Method to Find the Angle of Intersection

- Find $f'(x)$ and $g'(x)$.
- If $f'(x) \times g'(x) = -1$, then the two curves are said to cut each other orthogonally, wherever they cut.
- If the product is not -1, solve the equation of the two curves to get their point of intersection. If (α, β) be their point of intersection, then find $f'(\alpha)$ and $g'(\alpha)$. Let $m_1 = f'(\alpha)$ and $m_2 = g'(\alpha)$.
- If θ is the angle between the tangents, then

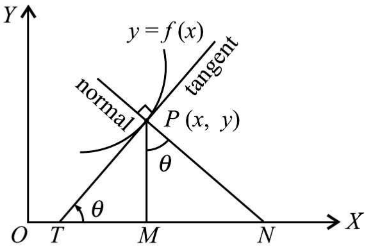
$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{f'(\alpha) - g'(\alpha)}{1 + f'(\alpha)g'(\alpha)}.$$

Repeat this process for other points of intersection.

Notes: The two curves are said to touch each other at their point of intersection (α, β), if the slope of their tangents at (α, β) are equal.

Length of Tangent, Length of Normal, Sub-tangent and Subnormal

Let the tangent and normal at the point $P(x, y)$ on the curve meet the axis of x at the points T and N respectively. Let M be the foot of the ordinate at P. Then,



1. Length of the tangent $= PT = |y \operatorname{cosec}\theta|$

$$= \left|y\sqrt{1+\cot^2\theta}\right| = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$

2. Length of the normal $= PN = |y\sec\theta|$

$$= \left|y\sqrt{1+\tan^2\theta}\right| = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

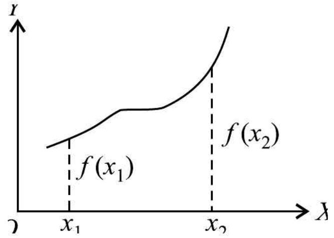
3. Subtangent $= TM = |y\cot\theta| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

4. Subnormal $= MN = |y\tan\theta| = \left|y\left(\frac{dy}{dx}\right)\right|$.

INCREASING AND DECREASING FUNCTIONS (MONOTONICITY)

Increasing function A function $f(x)$ is said to be an increasing function on an interval I, if

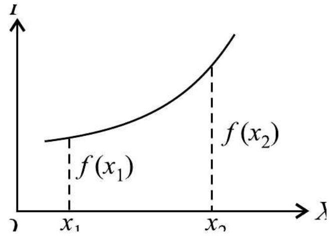
$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2),\ \forall\ x_1, x_2 \in I.$$



Increasing function

Strictly increasing function A function $f(x)$ is said to be a strictly increasing function on an interval I, if

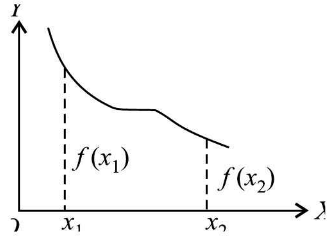
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ \forall\ x_1, x_2 \in I.$$



Strictly increasing function

Decreasing function A function $f(x)$ is said to be a decreasing function on an interval I, if

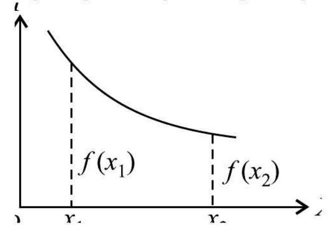
$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2),\ \forall\ x_1, x_2 \in I$$



Decreasing Function

Strictly decreasing function A function $f(x)$ is said to be a strictly decreasing function on an interval I, if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in I.$$



Strictly decreasing function

Monotonic function A function $f(x)$ is said to be monotonic on an interval I if it is either increasing or decreasing on I.

Test for Monotonicity of Functions

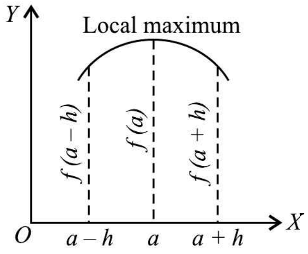
1. $f(x)$ is increasing in $[a, b]$ if $f'(x) \geq 0, \forall x \in [a, b]$.
2. $f(x)$ is strictly increasing in $[a, b]$ if $f'(x) > 0, \forall x \in [a, b]$.
3. $f(x)$ is decreasing in $[a, b]$ if $f'(x) \leq 0, \forall x \in [a, b]$.
4. $f(x)$ is strictly decreasing in $[a, b]$ if $f'(x) < 0, \forall x \in [a, b]$.

Key Points to Remember

1. If a function $f(x)$ is strictly increasing (strictly decreasing) on an interval I, then f^{-1} exists and is also strictly increasing (strictly decreasing).
2. If $f(x)$ is monotonic on an interval I, then $f(x)$ has at the most one zero in the interval I.
3. If the functions $f(x)$ and $g(x)$, both are increasing or decreasing on an interval I, then the composite function $(gof)(x)$ is an increasing function on I.
4. If the function $f(x)$ is increasing and $g(x)$ decreasing on an interval I, then the composite function $(gof)(x)$ is decreasing on the interval I.
5. A function may be increasing in some interval I_1 and decreasing in some other interval I_2.

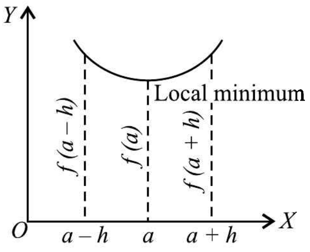
MAXIMA AND MINIMA OF FUNCTIONS

Local maximum A function $y = f(x)$ is said to have a local maximum value at a point $x = a$, if $f(x) \leq f(a), \forall x \in (a - h, a + h)$, for small $h > 0$, i.e., $f(a)$ is the greatest of all the values of $f(x)$ in the interval $(a - h, a + h)$.



The point $x = a$ is called a point of local maximum of the function $f(x)$.

Local minimum A function $y = f(x)$ is said to have a local minimum value at a point $x = a$, if $f(x) \geq f(a), \forall x \in (a - h, a + h)$, for small $h > 0$, i.e., $f(a)$ is the smallest of all the values of $f(x)$ in the interval $(a - h, a + h)$.



The point $x = a$ is called a point of local minimum of the function $f(x)$.

Key Points to Remember

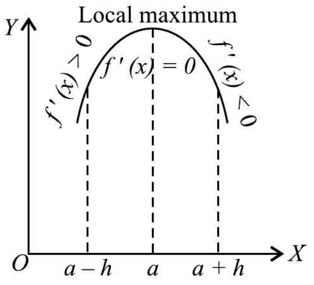
1. The points at which a function attains either the local maximum value or local minimum value are called the *extreme points* and both local maximum and local minimum values are called the extreme values of the function $f(x)$.
2. The local maximum and local minimum values are also known as *relative maximum and relative minimum values* respectively.

Working Rule to Determine the Points of Local Maxima and Local Minima

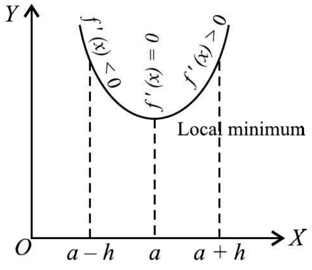
Method I (First Derivative Test)

- For the function $y = f(x)$, find $f'(x)$.
- Put $f'(x) = 0$ and solve this equation for x. Let its roots be a, b, c etc. These points are called *stationary points* or *critical points*.
- At $x = a$, determine the sign of $f'(x)$ for values of x slightly less than a and that for values of x slightly greater than a.

(a) If $f'(x)$ changes sign from positive to negative as x increases through a, then $x = a$ is a point of maximum.



(b) If $f'(x)$ changes sign from negative to positive as x increases through a, then $x = a$ is a point of minimum.



(c) $f'(x)$ does not change sign as x increases through a, then $x = a$ is neither a point of maximum nor a point of minimum. Such a point is called a point of inflexion.

We repeat this process for other values of x and examine them for maxima or minima.

Method II (Second Derivative Test)

- For the function $y = f(x)$, find $f'(x)$ and $f''(x)$.
- Put $f'(x) = 0$ and solve this equation for x. Let its roots be a, b, c etc.
- At $x = a$

(a) if $f''(a) < 0$, then $x = a$ is a point of local maxima;
(b) if $f''(a) > 0$, then $x = a$ is a point of local minima;
(c) if $f''(a) = 0$, we cannot say any thing.

General test In the cases where the second derivative vanishes, the method discussed above fails to give any result. In those cases, we make use of still higher derivatives, and the following working rule proves very useful.

Working Rule

If $f(x)$ has a derivative at $x = a$ such that

(i) $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$, and

(ii) $f^n(a)$ exists and is not zero,

then for n odd, $f(a)$ is not an extreme value, while for n even $f(a)$ is a maximum or minimum value according as $f^n(a)$ is negative or positive.

Greatest and Least Values of a Function in a Closed Interval (Absolute Maximum and Absolute Minimum)

If $f(x)$ is continuous in an interval $[a, b]$, then greatest or absolute maximum value of $f(x)$ = max. $\{f(a), f(b)$, values of $f(x)$ at all critical points in $(a, b)\}$.

Also, least or absolute minimum value of $f(x)$ = min. $\{f(a), f(b)$, values of $f(x)$ at all critical points in $(a, b)\}$

If a function is defined and continuous on an interval which is not a closed interval, then it cannot have any greatest or least value other than local maximum or local minimum values.

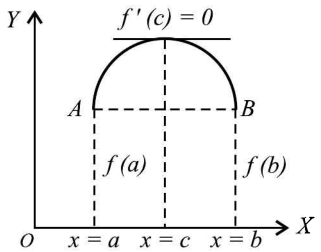
ROLLE'S AND LAGRANGE'S MEAN VALUE THEOREM

Rolle's Theorem

If a function f defined on the closed interval $[a, b]$, is

1. continuous on $[a, b]$,
2. derivable on (a, b) and
3. $f(a) = f(b)$, then there exists atleast one real number c between a and b $(a < c < b)$ such that $f'(c) = 0$.

Geometrical interpretation Let the curve $y = f(x)$, which is continuous on $[a, b]$ and derivable on (a, b), be drawn.



The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.

Algebraic interpretation Between two zeros a and b of $f(x)$ (i.e., between two roots a and b of $f(x) = 0$) there exists atleast one zero of $f'(x)$.

Key Points to Remember

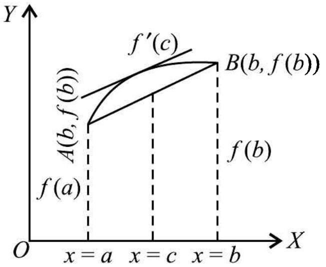
1. The value of c may not be unique i.e., there can be more than one such c.
2. Rolle's theorem holds only when all the three conditions are satisfied on the interval under consideration. Even if one of the conditions fails to hold, then the conclusion of the Rolle's Theorem is not applicable.
3. Every polyonomial function is continuous and differentiable for all real x.
4. The function $\log x$ is continuous on $(0, \infty)$.
5. The function e^x, $\sin x$ and $\cos x$ are continuous and differentiable for all real x, whereas $\tan x$ and $\sec x$ are not continuous at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ and $\cot x$ and $\operatorname{cosec} x$ are not continuous at $x = 0, \pm\pi, \pm 2\pi, \ldots$.
6. $|x - a|$ is not differentiable at $x = a$ (e.g., $|x|$ is not differentiable at $x = 0$).
7. If f and g are continuous on $[a, b]$, then the functions $f + g$, $f - g$, fg are also continuous on $[a, b]$, $\frac{f}{g}$ is also continuous on $[a, b]$, provided $g(x) \neq 0$ on $[a, b]$.
8. If the derivative of a function has finite and unique value on an interval, then the function is derivable on that interval.

Lagrange's Mean Value Theorem

If a function f defined on the closed interval $[a, b]$, is

1. continuous on $[a, b]$ and
2. derivable on (a, b), then there exists atleast one real number c between a and b $(a < c < b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Geometrical interpretation The theorem states that between two points A and B on the graph of f there exists atleast one point where the tangent is parallel to the chord AB.

Solved Problems

1. If the tangent at each point of the curve

$$y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$$

makes an acute angle with the positive direction of x-axis, then

(a) $a \geq 1$ (b) $-1 \leq a \leq 1$
(c) $a \leq -1$ (d) none of these

Solution (b) We have, $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$

$$\Rightarrow \frac{dy}{dx} = 2x^2 - 4ax + 2.$$

Since, the tangent makes an acute angle with the positive direction of x-axis, therefore,

$$\frac{dy}{dx} \geq 0 \Rightarrow 2x^2 - 4ax + 2 \geq 0 \text{ for all } x$$

$$\Rightarrow 16a^2 - 16 \leq 0$$

$$(\because \text{Disc.} = (4a)^2 - 4(2)(2) \leq 0)$$

$$\Rightarrow a^2 - 1 \leq 0 \text{ i.e., } (a-1)(a+1) \leq 0$$

$$\Rightarrow \mathbf{-1 \leq a \leq 1.}$$

2. If the slope of the curve $y = \frac{ax}{b-x}$ at the point (1, 1) is 2, then the values of a and b are

(a) 1, – 2 (b) – 1, 2
(c) 1, 2 (d) none of these

Solution (c) We have, $y = \frac{ax}{b-x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(b-x)a - ax\cdot(-1)}{(b-x)^2} = \frac{ab}{(b-x)^2}.$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,\,1)} = \frac{ab}{(b-1)^2} = 2 \text{ (given)} \qquad \text{...(1)}$$

Since the curve passes through the point (1, 1), therefore,

$$1 = \frac{a}{b-1} \text{ i.e., } a = b - 1 \qquad \text{...(2)}$$

Putting $a = b - 1$ in (1), we get

$$\frac{(b-1)b}{(b-1)^2} = 2 \Rightarrow b = 2. \quad \therefore \quad a = 2 - 1 = 1.$$

Hence, $\mathbf{a = 1, b = 2}$.

3. If m be the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$, then

(a) $m < 1$ (b) $|m| \leq 1$
(c) $|m| > 1$ (d) none of these

Solution (b) We have, $e^{2y} = 1 + 4x^2 \Rightarrow e^{2y}\cdot 2\frac{dy}{dx} = 8x$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{e^{2y}} = \frac{4x}{1+4x^2}.$$

$$\therefore \text{ Slope of tangent} = m = \frac{4x}{1+4x^2}$$

$$\Rightarrow |m| = \frac{4|x|}{1+4|x|^2} \leq 1$$

$$\left[\because (1-2|x|)^2 \geq 0 \quad \Rightarrow 1+4|x|^2 - 4|x| \geq 0 \quad \Rightarrow \frac{4|x|}{1+4|x|^2} \leq 1\right]$$

4. The tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ is equally inclined to the axes at the point

(a) (1, – 2) (b) (4, 4)
(c) (4, – 4) (d) (– 4, 4)

Solution (b) We have, $\sqrt{x} + \sqrt{y} = 4 \qquad \text{...(1)}$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Since the tangent is equally inclined to the axes,

$$\frac{dy}{dx} = \tan 45° \text{ or } \tan 135° \text{ i.e., } 1 \text{ or } -1. \text{ Thus}$$

$$-\frac{\sqrt{y}}{\sqrt{x}} = \pm 1. \text{ This gives } y = x.$$

From (1), $\sqrt{x} + \sqrt{x} = 4 \Rightarrow x = 4$. Also, $y = x = 4$.

The point is **(4, 4).**

5. The curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$, where a and b are constants, cut each other

(a) at an angle $\frac{\pi}{3}$ (b) at an angle $\frac{\pi}{4}$
(c) orthogonally (d) none of these

Solution (c) The two curves are

$$x^3 - 3xy^2 = a \qquad \text{...(1)}$$

and $$3x^2y - y^3 = b \qquad \text{...(2)}$$

Differentiating (1) with respect to x, we get

$$3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2-y^2}{2xy}$$

Differentiating (2) with respect to x, we get

$$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2xy}{x^2-y^2}$$

The product of $\frac{dy}{dx}$ for the two curves

$$= \left(\frac{x^2-y^2}{2xy}\right)\times\left(\frac{-2xy}{x^2-y^2}\right) = -1.$$

$\therefore$ The curves cut each other **orthogonally.**

6. The two curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ at the point (1, 2)

(a) intersect orthogonally
(b) intersect at an angle $\frac{\pi}{3}$
(c) touch each other
(d) none of these

Solution (c) We have,

$$y^2 = 4x \qquad \text{...(1)}$$

and $$x^2 + y^2 - 6x + 1 = 0 \qquad \text{...(2)}$$

Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(1,\,2)} = \frac{2}{2} = 1 = m_1 \text{ (say).}$$

Differentiating (2) with respect ot x, we get

$$2x + 2y\frac{dy}{dx} - 6 = 0 \Rightarrow \frac{dy}{dx} = \frac{3-x}{y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(1,\,2)} = \frac{3-1}{2} = 1 = m_2 \text{ (say)}$$

Since $m_1 = m_2$, therefore the two curves **touch each other** at (1, 2).

7. The line $\frac{x}{a}+\frac{y}{b} = 2$ touches the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n = 2$ at the point (a, b) for
(a) $n = 2$ only
(b) $n = -3$ only
(c) n is any real number
(d) none of these

Solution (c) We have, $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n = 2$

$$\Rightarrow \quad \frac{nx^{n-1}}{a^n}+\frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(a,\,b)} = -\frac{b^n\cdot a^{n-1}}{a^n b^{n-1}} = -\frac{b}{a}.$$

$\therefore$ The equation of tangent at (a, b) is

$$y - b = -\frac{b}{a}(x - a) \Rightarrow \frac{x}{a}+\frac{y}{b} = 2.$$

$\therefore$ The line $\frac{x}{a}+\frac{y}{b} = 2$ touches the curve at (a, b), **for all n.**

8. The function f whose graph passes through the point $\left(0, \frac{7}{3}\right)$ and whose derivative is $x\sqrt{1-x^2}$, is given by

(a) $f(x) = -\frac{1}{3}\left\{(1-x^2)^{\frac{3}{2}} - 8\right\}$

(b) $f(x) = \frac{1}{3}\left\{(1-x^2)^{\frac{3}{2}} + 8\right\}$

(c) $f(x) = -\frac{1}{3}\{\sin x + 7\}$

(d) none of these

Solution (a) We have, $f'(x) = x\sqrt{1-x^2}$

$$\Rightarrow \quad f(x) = \int x\sqrt{1-x^2}dx = -\frac{1}{2}\frac{(1-x^2)^{3/2}}{3/2} + c$$

$$\Rightarrow \quad f(x) = -\frac{1}{3}(1-x^2)^{3/2} + c$$

Since it passes through $\left(0, \frac{7}{3}\right)$

$$\therefore \quad \frac{7}{3} = -\frac{1}{3} + c \Rightarrow c = \frac{8}{3}$$

$$\therefore \quad \boldsymbol{f(x) = -\frac{1}{3}\left\{(1-x^2)^{3/2} - 8\right\}}.$$

9. If the line $ax + by + c = 0$ is a tangent to the curve $xy = 4$, then
(a) $a > 0, b > 0$
(b) $a > 0, b < 0$
(c) $a < 0, b > 0$
(d) $a < 0, b < 0$

Solution (a), (d) We have, $xy = 4 \Rightarrow x\cdot\frac{dy}{dx} + y\cdot 1 = 0$

i.e., $\frac{dy}{dx} = -\frac{y}{x} = -\frac{4}{x^2}$ $\quad (\because xy = 4)$

$\therefore$ Slope of tangent $= -\frac{4}{x^2}$.

Slope of the line $ax + by + c = 0$ is $= -\frac{a}{b}$.

Since the given line is a tangent to the curve

$$\therefore \quad -\frac{4}{x^2} = -\frac{a}{b} \Rightarrow \frac{a}{b} > 0$$

It is possible only when **$a > 0$, $b > 0$ or $a < 0$, $b < 0$.**

10. The sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies as
(a) $(\text{abscissa})^{-3}$
(b) $(\text{abscissa})^3$
(c) $(\text{ordinate})^{-3}$
(d) none of these

Solution (a) We have, $x^2y^2 = a^2(x^2 - a^2)$ ...(1)

$$\Rightarrow \quad x^2\cdot 2y\frac{dy}{dx} + y^2\cdot 2x = a^2\cdot 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a^2-y^2}{xy}.$$

$$\therefore \text{ Sub-normal } = y\frac{dy}{dx} = \frac{a^2-y^2}{x} = \frac{x^2(a^2-y^2)}{x^3}$$

$$= \frac{a^4}{x^3} \quad [\because \text{ from (1) } x^2(a^2-y^2) = a^4]$$

$\Rightarrow$ The sub-normal varies inversely as the cube of its abscissa.

11. For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is
(a) 1 : 1
(b) 2 : 1
(c) $x : y$
(d) $x^2 : y$

Solution (b) We have, $y^2 = 4ax$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \text{ i.e., } \frac{dy}{dx} = \frac{2a}{y}.$$

$$\text{Sub-tangent} = \frac{y}{dy/dx} = \frac{y}{2a/y} = \frac{y^2}{2a} = \frac{4ax}{2a} = 2x.$$

$\therefore$ Sub-tangent : Abscissa $= 2x : x = 2 : 1$.

12. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3) =$
(a) -1
(b) $-\frac{3}{4}$
(c) $\frac{4}{3}$
(d) 1

Solution (d) Slope of normal to $y = f(x)$ at (3, 4) is $\frac{-1}{f'(3)}$.

Thus, $\frac{-1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right) = \tan\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1.$

$\Rightarrow$ $f'(3) = $ **1.**

13. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

Solution (b), (c) We have $xy = 1 \Rightarrow y = \frac{1}{x}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2}.$

$\therefore$ The slope of the normal $= x^2$.

If $ax + by + c = 0$ is normal to the curve $xy = 1$

then $x^2 = -\frac{a}{b} \quad \therefore \quad -\frac{a}{b} > 0$

$\Rightarrow$ $\boldsymbol{a > 0, b < 0}$ **or** $\boldsymbol{a < 0, b > 0}$**.**

14. The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope, is

(a) $\pi/4$ (b) $\pi/2$
(c) π (d) $3\pi/2$

Solution (b) We have, $f'(x) = e^x \cos x + \sin x . e^x$

and $f''(x) = -\sin x\, e^x + \cos x\, e^x + \cos x\, e^x + \sin x\, e^x$.

Now, $f''(x) = 2 \cos x\, e^x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pi/2$.

Also, $f'''(x) = -2 \sin x\, e^x + 2 \cos x\, e^x = -$ve

$\therefore$ slope is maximum at $\boldsymbol{x = \pi/2}$**.**

15. If the curve $y = x^2 + bx + c$ touches the line $y = x$ at the point $(1, 1)$, then the values of x for which the curve has a negative gradient are

(a) $x < \frac{1}{2}$ (b) $x > \frac{1}{2}$
(c) $x < -\frac{1}{2}$ (d) $x > -\frac{1}{2}$

Solution (a) We have, $y = x^2 + bx + c$

$\Rightarrow \quad \frac{dy}{dx} = 2x + b.$

Since the curve touches the line $y = x$ at the point $(1, 1)$

$\therefore \quad (2x+b)]_{(1,\,1)} = 1$ i.e., $2 + b = 1 \Rightarrow b = -1$.

Also, the curve passes through the point $(1, 1)$

$\therefore \quad 1 = 1 + b + c$ i.e., $c = -b = 1$.

$\therefore \quad y = x^2 - x + 1 \Rightarrow \frac{dy}{dx} = 2x - 1.$

Now, $\frac{dy}{dx} < 0 \Rightarrow 2x - 1 < 0 \Rightarrow \boldsymbol{x < \frac{1}{2}}$.

16. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in

(a) G. P. (b) A. P.
(c) H. P. (d) none of these

Solution (a) We have, $y^2 = 4ax$

$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$ i.e., $\frac{dy}{dx} = \frac{2a}{y}.$

Sub-tangent $= \frac{y}{dy/dx} = \frac{y}{2a/y} = \frac{y^2}{2a}$

Sub-normal $= y\frac{dy}{dx} = y \times \frac{2a}{y} = 2a.$

Clearly, $\frac{y^2}{2a}$, y, $2a$ are in **G. P.**

17. If $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$ be a polynomial in $x \in R$ with $0 < a_1 < a_2 \ldots < a_n$, then $P(x)$ has

(a) no point of minimum
(b) only one point of minimum
(c) only two points of minimum
(d) none of these

Solution (b) We have,

$P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$

$\Rightarrow \quad P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_n\, x^{2n-1}.$

For maximum or minimum, $P'(x) = 0$

$\Rightarrow \quad x(2a_1 + 4a_2x^2 + \ldots + 2na_nx^{2n-2}) = 0$

$\Rightarrow \quad x = 0 \quad [\because$ each $a_i > 0$ and powers of x are even$]$

Now, $P''(x) = 2a_1 + 12a_2x^2 + \ldots + 2n(2n-1)\, a_nx^{2n-2}$

$\therefore \quad P''(x)]_{x=0} = 2a_1 > 0$ i.e., $P(x)$ has a minimum at $x = 0$ only.

18. The minimum value of $\log_a x + \log_x a$, $0 < x < a$, is

(a) 1 (b) 2
(c) -2 (d) none of these

Solution (b) Let $z = \log_a x$

$\Rightarrow \quad \frac{1}{z} = \frac{1}{\log_a x} = \log_x a$

$\therefore \quad \log_a x + \log_x a = z + \frac{1}{z} = \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2 + 2,$

whose minimum value is **2.**

19. The minimum value of $2 \log_{10} x - \log_x .01$, $x > 1$, is

(a) 1 (b) -1
(c) 2 (d) none of these

Solution (d) Let $z = 2 \log_{10}x$.

$\therefore \quad \log_x .01 = \log_x \frac{1}{100}$

$= \log_x 10^{-2} = -2 \log_x 10 = \frac{-2}{\log_{10} x}$

$= -\frac{4}{z}.$

$\therefore \quad 2 \log_{10}x - \log_x .01 = z + \frac{4}{z} = \left(\sqrt{z} - \frac{2}{\sqrt{z}}\right)^2 + 4,$

whose minimum value is 4.

20. The fraction exceeding its pth power by the greatest number possible, where $p \geq 2$, is

(a) $\left(\frac{1}{p}\right)^{1/p-1}$ (b) $\left(\frac{1}{p}\right)^{p-1}$
(c) $p^{1/p-1}$ (d) none of these

Solution (a) Let $y = x - x^p$, where x is the fraction

$\Rightarrow \frac{dy}{dx} = 1 - px^{p-1}.$

For maximum or minimum, $\frac{dy}{dx} = 0$

$\Rightarrow 1 - px^{p-1} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}.$

Now, $\frac{d^2y}{dx^2} = -p\,(p-1)\,x^{p-2}$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right]_{x=\left(\frac{1}{p}\right)^{1/(p-1)}} = -p\,(p-1)\left(\frac{1}{p}\right)^{p-2/(p-1)} < 0$$

$\therefore$ y is maximum at $x = \left(\frac{1}{p}\right)^{1/(p-1)}$.

21. The largest term in the sequence

$x_n = \frac{n^2}{n^3+200}$, $n \in N$, is

(a) $\frac{49}{543}$ (b) $\frac{8}{89}$

(c) $\frac{1}{52}$ (d) none of these

Solution (a) Let $f(n) = x_n = \frac{n^2}{n^3+200}$, $n \in N$

$$\Rightarrow \quad f'(n) = \frac{(n^3+200)\cdot 2n - n^2\cdot 3n^2}{(n^3+200)^2}$$

$$= \frac{n(400-n^3)}{(n^3+200)^2}.$$

But $f'(n) \neq 0$ for any $n \in N$. Hence $f(n)$ has no critical point.

But the function $f(n)$ is increasing for $n < 8$ and it starts decreasing for $n \geq 8$.

Here, $f(7) = \frac{49}{543}$ and $f(8) = \frac{8}{89}$.

Clearly, $f(7) > f(8)$.

Hence the largest value is $\frac{49}{543}$.

22. Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \tan^{-1}\alpha - 3x^2, & 0 < x < 1 \\ -6x, & x \geq 1 \end{cases}$$

$f(x)$ can have a maximum at $x = 1$ if the value of α is

(a) 0 (b) 2

(c) 1 (d) none of these

Solution (d) We have,

$$f'(x) = \begin{cases} -6x, & 0 < x < 1 \\ -6, & x \geq 1 \end{cases}$$

$\therefore$ $f'(1-h) = -6\,(1-h) < 0$

and $f'(1+h) = -6 < 0$.

Since $f'(x)$ does not change sign as x passes through 1, therefore, $f(x)$ does not have a maximum or minimum at $x = 1$, whatever be the value of α.

23. For the function $f(x) = \int_0^x \frac{\sin t}{t}\,dt$, where $x > 0$,

(a) maximum occurs at $x = n\pi$, n even

(b) minimum occurs at $x = n\pi$, n odd

(c) maximum occurs at $x = n\pi$, n odd

(d) minimum occurs at $x = n\pi$, n even

Solution (c), (d) We have,

$f'(x) = \frac{\sin x}{x}$ and $f''(x) = \frac{x\cos x - \sin x}{x^2}$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow$ $\frac{\sin x}{x} = 0 \Rightarrow \sin x = 0$; $x \neq 0$.

$\therefore$ $x = n\pi$; $n = 1, 2, 3, \ldots$ ($\because$ $x > 0$).

At $x = n\pi$, $f''(x) = \frac{n\pi\cos n\pi - \sin n\pi}{(n\pi)^2} = \frac{\cos n\pi}{n\pi}$

$= \frac{(-1)^n}{n\pi}$.

$\therefore$ Extreme points are $x = n\pi$, $n = 1, 2, 3, \ldots$, where the maximum occurs at $x = \pi, 3\pi, 5\pi, \ldots$ and the minimum occurs at $x = 2\pi, 4\pi, 6\pi, \ldots$

24. The minimum value of $e^{(x^4-x^3+x^2)}$ is

(a) e (b) e^2

(c) 1 (d) none of these

Solution (c) $e^{(x^4-x^3+x^2)}$ is minimum when $(x^4 - x^3 + x^2)$ is minimum.

Since $x^4 - x^3 + x^2 = x^2\,(x^2 - x + 1)$

$$= x^2\left[\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right]$$

≥ 0, $\forall\, x \in R$.

$\therefore$ The minimum value of $(x^4 - x^3 + x^2)$ is 0 for $x = 0$. Hence the minimum value of $e^{(x^4-x^3+x^2)}$ is $e^0 = \mathbf{1}$.

25. The minimum value of the function

$f(x) = 2\,|x-2| + 5\,|x-3|$, $\forall\, x \in R$ is

(a) 3 (b) 2

(c) 5 (d) 7

Solution (b) We have,

$f(x)$ $= 2\,|x-2| + 5\,|x-3|$

$= 2\,(2-x) + 5\,(3-x) = 19 - 7x$, if $x < 2$

$= 5$, if $x = 2$

$= 2\,(x-2) + 5\,(3-x) = 11 - 3x$, if $2 < x < 3$

$= 2\,(3-2) = 2$, if $x = 3$

$= 2\,(x-2) + 5\,(x-3) = 7x - 19$, if $x > 3$

Thus, we find that $f(x)$ has a minimum value **2** at $x = 3$.

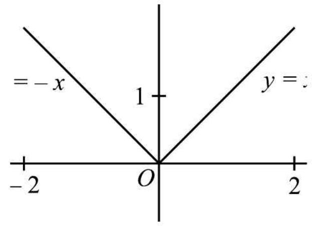
26. Let $f(x) = \begin{cases} |x|, & 0 < |x| \leq 2 \\ 1, & x = 0 \end{cases}$. Then at $x = 0$, f has

(a) a local maximum (b) no local maximum

(c) a local minimum (d) no extremum

Solution (a) We have,

$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ 1, & x = 0 \\ x, & 0 < x \leq 2 \end{cases}$$



The graph of $f(x)$ is:

Clearly, from the graph,

$f(0) = 1, f(0 - \varepsilon) < 1,$

$f(0 + \varepsilon) < 1$

where ε is small and positive.

$\therefore$ $f(x)$ has a **local maximum** at $x = 0$.

27. The minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is

(a) $\frac{1}{243}$ (b) -5

(c) $\frac{1}{5}$ (d) none of these

Solution (a) Let $y = 27^{\cos 2x} \cdot 81^{\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$

y will be minimum when $3\cos 2x + 4\sin 2x$ is minimum.

Let $Z = 3\cos 2x + 4\sin 2x$

Put $3 = r\cos\theta$, $4 = r\sin\theta$

Then, $r = \sqrt{3^2+4^2} = 5$ and $\tan\theta = \frac{4}{3}$

i.e., $\theta = \tan^{-1}\left(\frac{4}{3}\right)$.

$\therefore$ $Z = 5\cos(2x - \theta) \Rightarrow -5 \le Z \le 5.$

$\therefore$ Min. $Z = -5 \Rightarrow$ Min. $y = 3^{-5} = \mathbf{\frac{1}{243}}$.

28. $f(x) = 1 + [\cos x]\, x$, in $0 < x \le \frac{\pi}{2}$

(a) has a minimum value 0

(b) has a maximum value 2

(c) is continuous in $\left[0, \frac{\pi}{2}\right]$

(d) is not differentiable at $x = \frac{\pi}{2}$

Solution (c) Since $f(x) = 1$ in $0 < x < \frac{\pi}{2}$ (as $[\cos x] = 0$)

$\therefore$ $f(x)$ is **continuous in** $\left[0, \frac{\pi}{2}\right]$.

29. The points of extremum of the function

$F(x) = \int_1^x e^{-t^2/2}(1-t^2)\,dt$ are

(a) $x = 1, -1$ (b) $x = -2$

(c) $x = 0$ (d) $x = \frac{1}{2}$

Solution (a) We have, $F'(x) = e^{-x^2/2}(1 - x^2)$

$F'(x) = 0 \Rightarrow e^{-x^2/2}(1 - x^2) = 0 \Rightarrow 1 - x^2 = 0$

$\Rightarrow$ $x = \pm 1$.

$\therefore$ The points of extremum are $\mathbf{x = -1}$ **and** $\mathbf{x = 1}$.

30. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $(0, 1/2]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

Solution (d) We have,

$$f(x) = (1 + b^2)\left[x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2}\right] - \frac{b^2}{1+b^2} + 1$$

$$= (1 + b^2)\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{1+b^2} \ge \frac{1}{1+b^2}$$

$\therefore$ $m(b) = \frac{1}{1+b^2}$. So, range of $m(b) = \mathbf{(0, 1]}$.

31. If $f(x) = xe^{x(1-x)}$ then $f(x)$ is

(a) increasing on $\left[-\frac{1}{2}, 1\right]$

(b) decreasing on R

(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

Solution (a) We have,

$f'(x) = e^{x(1-x)} + x \cdot e^{x(1-x)} \cdot (1 - 2x)$

$= e^{x(1-x)}[1 + x(1 - 2x)] = e^{x(1-x)} \cdot (-2x^2 + x + 1).$

Now, the sign-scheme for $-2x^2 + x + 1$ is

–ve –1/2 +ve 1 –ve

$\therefore$ $f'(x) \ge 0$ if $x \in \left[-\frac{1}{2}, 1\right]$ because $e^{x(1-x)}$ is always positive.

So, $f(x)$ is increasing on $\left[-\mathbf{\frac{1}{2}}, \mathbf{1}\right]$.

32. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 + y^2 - xy) \le 2$ then the maximum value of xy is

(a) 2500 (b) 3000

(c) 1200 (d) 3500

Solution (a) $\log_{10}\frac{(x^3+y^3)}{x^2+y^2-xy} \le 2$ and $x + y > 0$

$\Rightarrow$ $0 < x + y \le 100$

$\Rightarrow$ maximum value of xy = **2,500.**

33. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is an increasing function in the set of real numbers if a and b satisfy the condition

(a) $a^2 - 3b - 15 > 0$ (b) $a^2 - 3b + 15 > 0$

(c) $a^2 - 3b + 15 < 0$ (d) $a > 0$ $b > 0$

Solution (c) We have, $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$

$\Rightarrow$ $f'(x) = 3x^2 + 2ax + b + 5\sin 2x$

Since $f(x)$ is an increasing function

$\therefore$ $3x^2 + 2ax + b - 5 > 0,$ $\forall x \in R$

$\Rightarrow$ $4a^2 - 4 \cdot 3 \cdot (b - 5) < 0$

$\therefore$ $\mathbf{a^2 - 3b + 15 < 0.}$

34. If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has, in the interval $(0, 1)$

(a) atleast one root (b) atmost one root

(c) no root (d) none of these

Solution (a) Let $f(x) = ax^3 + bx^2 + cx$, $x \in [0, 1]$.

$\therefore$ $f'(x) = 3ax^2 + 2bx + c.$

Since $f(x)$ is a polynomial function of x, it is continuous and differentiable for all $x \in [0, 1]$.

Also, $f(0) = 0$; $f(1) = a + b + c = 0$.

$\therefore \quad f(0) = f(1)$.

Applying Rolle's theorem, $f'(k) = 0$ for atleast one value k, $0 < k < 1$. Hence k is a root of the equation

$3ax^2 + 2bx + c = 0$, where $0 < k < 1$.

35. If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$ has a positive root $x = \alpha$, then the equation

$na_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is

(a) smaller than α (b) greater than α
(c) equal to α
(d) greater than or equal to α

Solution (a) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x$.

Then $f(\alpha) = 0$ (Given). Also $f(0) = 0$.

Moreover, $f(x)$ is continuous and differentiable in $[0, \alpha]$ as it is a polynomial function of x. Hence, by Rolle's theorem, there exists a 'c' in $(0, \alpha)$ such that $f'(x) = 0$ for $x = c$ i.e.,

$na_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + ... + 2a_2 x + a_1 = 0$.

36. The equation $x \log x = 3 - x$ has, in the interval $(1, 3)$,

(a) exactly one root (b) atmost one root
(c) atleast one root (d) no root

Solution (c) Let $f(x) = (x - 3) \log x$

Then, $f(1) = -2 \log 1 = 0$ and $f(3) = (3 - 3) \log 3 = 0$.

As, $(x - 3)$ and $\log x$ are continuous and differentiable in $[1, 3]$, therefore $(x - 3) \log x = f(x)$ is also continuous and differentiable in $[1, 3]$. Hence, by Rolle's theorem, there exists a value of x in $(1, 3)$ such that

$$f'(x) = 0 \Rightarrow \log x + (x-3)\frac{1}{x} = 0 \Rightarrow x \log x = 3 - x.$$

37. Between any two real roots of the equation $e^x \sin x = 1$, the equation $e^x \cos x = -1$ has

(a) atleast one root (b) exactly one root
(c) atmost one root (d) no root

Solution (a) Let α, β $(\alpha < \beta)$ be any two real roots of

$$f(x) = e^{-x} - \sin x.$$

Then, $f(\alpha) = 0 = f(\beta)$

Moreover, $f(x)$ is continuous and differentiable for $x \in [\alpha, \beta]$.

Hence, from Rolle's theorem, there exists atleast one x in (α, β) such that

$$f'(x) = 0 \Rightarrow -e^{-x} - \cos x = 0$$

$$\Rightarrow -e^{-x}(1 + e^x \cos x) = 0 \Rightarrow e^x \cos x = -1.$$

38. If $f(x)$ is differentiable in the interval $[2, 5]$, where $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then there exists a number c, $2 < c < 5$ for which $f'(c) =$

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$
(c) $\frac{1}{10}$ (d) none of these

Solution (c) As $f(x)$ is differentiable in $[2, 5]$, therefore, it is also continuous in $[2, 5]$. Hence, by mean value theorem, there exists a real number c in $(2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} \Rightarrow f'(c) = \frac{\frac{1}{2} - \frac{1}{5}}{3} = \mathbf{\frac{1}{10}}.$$

39. If a, b, c be non-zero real numbers such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = 0,$$

then the equation $ax^2 + bx + c = 0$ will have

(a) one root between 0 and 1 and other root between 1 and 2
(b) both the roots between 0 and 1
(c) both the roots between 1 and 2
(d) none of these

Solution (a) Let $f(y) = \int_0^y (1 + \cos^8 x)(ax^2 + bx + c)\,dx$

$$\Rightarrow \quad f'(y) = (1 + \cos^8 y)(ay^2 + by + c) \qquad ...(1)$$

Now, $f(1) = \int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = 0$

and $f(2) = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = 0$

Also, $f(0) = 0$.

$\therefore \quad f(0) = f(1) = f(2)$.

Now by Rolle's theorem for $f(x)$ in $[0, 1]$.

$f'(\alpha) = 0$, for atleast one α, $0 < \alpha < 1$

and by Rolle's theorem for $f(x)$ in $[1, 2]$,

$f'(\beta) = 0$, for atleast one β, $1 < \beta < 2$.

From (1), $f'(\alpha) = 0 \Rightarrow (1 + \cos^8\alpha)(a\alpha^2 + b\alpha + 2) = 0$.

But $1 + \cos^8\alpha \neq 0$,

$\therefore \quad a\alpha^2 + b\alpha + c = 0$,

i.e., α is a root of the equation $ax^2 + bx + c = 0$.

Similarly $f'(\beta) = 0 \Rightarrow a\beta^2 + b\beta + c = 0$,

i.e., β is a root of the equation $ax^2 + bx + c = 0$.

But the equation $ax^2 + bx + c = 0$, being a quadratic equation, cannot have more than two roots.

$\therefore$ The equation $ax^2 + bx + c = 0$ has one root α **between 0 and 1** and other root β **between 1 and 2.**

40. Let f be a function which is continuous and differentiable for all real x. If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2, 4]$, then

(a) $f(4) < 8$ (b) $f(4) \geq 8$
(c) $f(4) \geq 12$ (d) none of these

Solution (b) By mean value theorem, there exists a real number $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \Rightarrow f'(c) = \frac{f(4) + 4}{2}$$

Since $f'(x) \geq 6 \ \forall\ x \in [2, 4]$

$$\therefore \quad f'(c) \geq 6 \Rightarrow \frac{f(4)+4}{2} \geq 6 \quad \Rightarrow f(4) + 4 \geq 12$$

$$\Rightarrow \boldsymbol{f(4) \geq 8.}$$

41. The set of values of x for which $\log(1 + x) < x$, is

(a) $x < 0$ (b) $x > 0$

(c) $0 < x < 1$ (d) none of these

Solution (b) Let $f(x) = \log(1 + x) - x$

$$\Rightarrow \quad f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$$

$\Rightarrow \quad f'(x) < 0$, for $x > 0$

$\Rightarrow \quad f(x)$ is decreasing for $x > 0$

$\Rightarrow \quad f(x) < f(0)$, for $x > 0$

$\Rightarrow \quad \log(1 + x) - x < 0$, for $x > 0$

i.e., $\log(1 + x) < x$, for $\boldsymbol{x > 0}$.

42. If $0 < x < \frac{\pi}{2}$, then

(a) $\frac{2}{\pi} > \frac{\sin x}{x}$ (b) $\frac{2}{\pi} < \frac{\sin x}{x}$

(c) $\frac{\sin x}{x} < 1$ (d) $\frac{\sin x}{x} > 1$

Solution (b), (c) Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$\text{Then} \qquad f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{\cos x(x - \tan x)}{x^2} < 0 \text{ if } x \in \left(0, \frac{\pi}{2}\right).$$

$$\left(\because \tan x > x \text{ and } \cos x > 0 \text{ when } 0 < x < \frac{\pi}{2}\right)$$

$\therefore \quad f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$. Since $0 < x < \frac{\pi}{2}$,

$$\therefore \quad f\left(\frac{\pi}{2}\right) < f(x) < f(0) \Rightarrow \boldsymbol{\frac{2}{\pi} < \frac{\sin x}{x} < 1.}$$

43. If $0 < x < \frac{\pi}{2}$, then

(a) $\cos(\sin x) > \cos x$

(b) $\cos(\sin x) < \cos x$

(c) $\cos(\sin x) > \sin(\cos x)$

(d) $\cos(\sin x) < \sin(\cos x)$

Solution (a), (c) We know that $\sin x < x$ if $0 < x < \frac{\pi}{2}$ ...(1)

Since $\cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$, $\cos(\sin x) > \cos x$.

Also, since $0 < x < \frac{\pi}{2}$, $\therefore\ 0 < \cos x < 1 < \frac{\pi}{2}$

$\therefore\ \sin(\cos x) < \cos x$ [Using (1)]

Hence $\boldsymbol{\cos(\sin x) > \cos x > \sin(\cos x)}$ if $0 < x < \frac{\pi}{2}$.

44. If $ax + \frac{b}{x} \geq c$ for all positive x, where $a, b > 0$, then

(a) $ab < \frac{c^2}{4}$ (b) $ab \geq \frac{c^2}{4}$

(c) $ab \geq \frac{c}{4}$ (d) none of these

Solution (b) Let $f(x) = ax + \frac{b}{x} - c$; $x > 0$; $a, b > 0$

$$\Rightarrow \quad f'(x) = a - \frac{b}{x^2} = \frac{ax^2 - b}{x^2}.$$

$$f'(x) = 0 \ \Rightarrow a - \frac{b}{x^2} = 0 \ \Rightarrow x = \left(\frac{b}{a}\right)^{1/2}.$$

But $ax + \frac{b}{x} \geq c$; $\therefore\ f(x) \geq 0$ for all $x > 0$

$$\therefore \quad f\left[\left(\frac{b}{a}\right)^{1/2}\right] \geq 0 \Rightarrow a\left(\frac{b}{a}\right)^{1/2} + b\left(\frac{a}{b}\right)^{1/2} - c \geq 0$$

$$\Rightarrow 2\sqrt{ab} \geq c \Rightarrow \boldsymbol{ab \geq \frac{c^2}{4}}.$$

45. The function $f(x) = |x + 2| + |x - 1|$ is

(a) increasing in $(1, \infty)$

(b) increasing in $[1, \infty)$

(c) decreasing in $(-\infty, -2]$

(d) decreasing in $(-\infty, -2)$

Solution (a), (d) We have,

$$f(x) = |x + 2| + |x - 1| = \begin{Bmatrix} -2x - 1, & x < -2 \\ 3, & -2 \leq x \leq 1 \\ 2x + 1, & x > 1 \end{Bmatrix}$$

$$\therefore \quad f'(x) = \begin{Bmatrix} -2, & x < -2 \\ \text{does not exist}, & x = -2 \\ 0, & -2 < x < 1 \\ \text{does not exist}, & x = 1 \\ 2, & x > 1 \end{Bmatrix}$$

Thus, $\boldsymbol{f(x)}$ **is increasing in $\boldsymbol{(1, \infty)}$ and decreasing in $\boldsymbol{(-\infty, -2)}$.**

46. The values of k for which the function $f(x) = kx^3 - 9x^2 + 9x + 3$ may be increasing on R are

(a) $k > 3$ (b) $k < 3$

(c) $k \leq 3$ (d) none of these

Solution (a) We have, $f(x) = kx^3 - 9x^2 + 9x + 3$

$\Rightarrow \quad f'(x) = 3kx^2 - 18x + 9.$

Since $f(x)$ is increasing on R, therefore,

$f'(x) > 0 \ \forall\ x \in R$

$\Rightarrow \quad 3kx^2 - 18x + 9 > 0 \ \forall\ x \in R$

$\Rightarrow \quad kx^2 - 6x + 3 > 0 \ \forall\ x \in R$

$\Rightarrow \quad k > 0$ and $36 - 12k < 0$

[$\because\ ax^2 + bx + c > 0 \ \forall\ x \in R$
$\Rightarrow a > 0$ and discrininant < 0]

$\Rightarrow \quad k > 3.$

Hence, $f(x)$ is increasing on R, if $\boldsymbol{k > 3}$.

47. The function $f(x) = \dfrac{|x-1|}{x^2}$

(a) increases in $(-\infty, 0) \cup (1, 2)$
(b) increases in $(0, 1) \cup (2, \infty)$
(c) decreases in $(0, 1) \cup (2, \infty)$
(d) decreases in $(-\infty, \infty) \cup (1, 2)$

Solution (a), (c) $f(x) = \dfrac{|x-1|}{x^2} = \begin{cases} \dfrac{x-1}{x^2}, & x \geq 1 \\ \dfrac{1-x}{x^2}, & x < 1, x \neq 0 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} \dfrac{x-2}{x^3}, & x < 1, x \neq 0 \\ \text{does not exist}, & x = 1 \\ \dfrac{2-x}{x^3}, & x > 1 \end{cases}$$

Clearly, $f'(x) > 0$ for $x < 0$ or $1 < x < 2$

and $f'(x) < 0$ for $0 < x < 1$ or $x > 2$.

Thus, $f(x)$ is **increasing for $(-\infty, 0) \cup (1, 2)$ and decreasing for $(0, 1) \cup (2, \infty)$.**

48. If $f(x) = 2x^3 + 9x^2 + \lambda x + 20$ is a decreasing function of x in the largest possible interval $(-2, -1)$ then λ is equal to

(a) 12 (b) -12
(c) 6 (d) none of these

Solution (a) Since $f(x)$ is decreasing in the interval $(-2, -1)$, therefore,

$f'(x) < 0 \Rightarrow 6x^2 + 18x + \lambda < 0$.

The value of λ should be such that the equation

$6x^2 + 18x + \lambda = 0$ has roots -2 and -1.

Therefore, $(-2)(-1) = \dfrac{\lambda}{6} \Rightarrow \boldsymbol{\lambda = 12}$.

49. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then

(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general

Solution (a), (c) We have,

$$h'(x) = f'(x)\,[1 - 2f(x) + 3\,(f(x))^2]$$

$$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$$

$$= 3f'(x)\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$. Thus, **$h(x)$ increases (decreases) whenever $f(x)$ increases (decreases).**

50. The equation $x + e^x = 0$ has

(a) only one real root (b) only two real roots
(c) no real root (d) none of these

Solution (a) Let $f(x) = x + e^x = 0$.

Since $f(-\infty) = -\infty$ and $f(+\infty) = \infty$,

$\therefore \quad f(x) = 0$ has a real root.

Let the real root be α. Then $f(\alpha) = 0$.

Now, $f'(x) = 1 + e^x > 0, \forall x \in R$

$\therefore \quad f(x)$ is an increasing function $\forall x \in R$.

$\therefore \quad$ for any other real number β,

$f(\beta) > f(\alpha)$ or $f(\beta) < f(\alpha)$.

But $f(\alpha) = 0$; so, $f(\beta) \neq 0$.

$\therefore \quad f(x) = 0$ has no other real root.

Hence, the equation has only **one real root.**

Problems for Practice

Level of Difficulty I

1. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $t = 2$ is

(a) $\dfrac{7}{6}$ (b) $\dfrac{5}{6}$ (c) $\dfrac{6}{7}$ (d) 1

2. The points on the curve $y^2 = 4a\left(x + a\sin\dfrac{x}{a}\right)$ at which the tangent is parallel to x-axis, lie on

(a) a straight line (b) a parabola
(c) a circle (d) an ellipse

3. The angle between the tangents to the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the points $(a, 0)$ and $(0, b)$ is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{3}$ (d) none of these

4. The equation of the tangent to the curve $y = \sqrt{9 - 2x^2}$ at the point where the ordinate and the abscissa are equal, is

(a) $2x + y - 3\sqrt{3} = 0$ (b) $2x + y + 3\sqrt{3} = 0$
(c) $2x - y - 3\sqrt{3} = 0$ (d) none of these

5. The angle of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ at origin is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) none of these

6. If the normal at the point "t" on the curve $xy = c^2$ meets the curve again at "t_2", then

(a) $t_1^3 t_2 = 1$
(b) $t_1^3 t_2 = -1$
(c) $t_1 t_2^3 = -1$
(d) $t_1 t_2^3 = 1$.

7. The angle made by the tangent at any point on the curve $x = a\,(t + \sin t \cos t)$, $y = a\,(1 + \sin t)^2$, with x-axis, is

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{2} + \frac{t}{2}$
(c) $\frac{\pi}{4} + \frac{t}{2}$
(d) none of these

8. The curves $y^2 = 2x$ and $2xy = k$ cut at right angles if

(a) $k^2 = 8$
(b) $k^2 = 4$
(c) $k^2 = 2$
(d) none of these

9. The line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point

(a) $(-a, ba)$
(b) $\left(a, \frac{a}{b}\right)$
(c) $\left(a, \frac{b}{a}\right)$
(d) none of these

10. The curve $y = ax^3 + bx^2 + cx$ is inclined at 45° to x-axis at $(0, 0)$ but it touches x-axis at $(1, 0)$, then the values of a, b, c are given by

(a) $a = 1, b = -2, c = 1$
(b) $a = 1, b = 1, c = -2$
(c) $a = -2, b = 1, c = 1$
(d) $a = -1, b = 2, c = 1$

11. The shortest distance of the point $(0, 0)$ from the curve $y = \frac{1}{2}(e^x + e^{-x})$ is

(a) 2 (b) 1 (c) 3 (d) none of these

12. If $f(x) = |x| + |x - 1| + |x - 2|$, then

(a) $f(x)$ has minima at $x = 1$
(b) $f(x)$ has maxima at $x = 0$
(c) $f(x)$ has neither maxima nor minima at $x = 0$
(d) $f(x)$ has neither maxima nor minima at $x = 2$.

13. If $h(x) = f(x) + f(-x)$, then $h(x)$ has got an extreme value at a point where $f'(x)$ is

(a) even function
(b) odd function
(c) zero
(d) none of these

14. The function $f(x) = \sum_{k=1}^{5} (x - k)^2$ assumes minimum value for x given by

(a) 3 (b) 2 (c) $\frac{5}{2}$ (d) 5

15. The value of c in Lagrange's theorem for the function

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

in the interval $[-1, 1]$ is

(a) 0
(b) $\frac{1}{2}$
(c) $-\frac{1}{2}$
(d) non-existent in the interval

16. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has atleast one real root lying between

(a) 0 and 1
(b) 1 and 3
(c) 0 and 3
(d) none of these

17. The function $f(x) = 2x^2 - \log|x|$, $x \neq 0$ is increasing in the interval

(a) $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$
(b) $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$
(c) $\left(-\frac{1}{2}, \frac{1}{2}\right)$
(d) none of these

18. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$

(a) cut at right angles
(b) touch each other
(c) cut at an angle $\frac{\pi}{3}$
(d) cut at an angle $\frac{\pi}{4}$

19. The function $f(x) = \cot^{-1}x + x$ increases in the interval

(a) $(1, \infty)$
(b) $(-1, \infty)$
(c) $(-\infty, \infty)$
(d) $(0, \infty)$

20. The greatest value of $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$ on $[0, 1]$ is

(a) 1 (b) 2 (c) 3 (d) 1/3

Level of Difficulty II

21. The real number x when added to its inverse gives the minimum value of the sum at x equal to

(a) 2 (b) 1 (c) -1 (d) -2

22. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $\left(\frac{-9}{8}, \frac{9}{2}\right)$
(b) $(2, -4)$
(c) $(2, 4)$
(d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

23. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point $(2, 1)$ and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x + 1)^3$
(b) $(x - 1)^3$
(c) $(x - 1)^2$
(d) $(x + 1)^2$

24. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(a) $(0, 0)$
(b) $(0, a)$
(c) $(a, 0)$
(d) (a, a)

25. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval

(a) $(2, 3)$
(b) $(1, 2)$
(c) $(0, 1)$
(d) $(1, 3)$

26. The angle of intersection of the curves $y = 2\sin^2 x$ and $y = \cos 2x$ at $x = \frac{\pi}{6}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none of these

27. The equation of the normal to the curve $y = e^{-2|x|}$ at the point where the curve cuts the line $x = \frac{1}{2}$ is

(a) $2e\,(ex + 2y) = e^2 - 4$

(b) $2e\,(ex - 2y) = e^2 - 4$

(c) $2e\,(ey - 2x) = e^2 - 4$

(d) none of these

28. The sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as

(a) $(\text{abscissa})^2$ (b) $(\text{ordinate})^2$

(c) abscissa (d) ordinate

29. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has, in the interval (0, 1),

(a) exactly one root (b) atleast one root

(c) atmost one root (d) no root

30. The function $f(x) = x^{1/x}$ is increasing in the interval

(a) (e, ∞) (b) $(-\infty, e)$

(c) $(-e, e)$ (d) none of these

31. $(1 + x)^p \le 1 + x^p$, where

(a) $p > 1$ (b) $0 \le p \le 1$

(c) $x > 0$ (d) $x < 0$

32. If $0 < \alpha < \beta < \frac{\pi}{2}$ then

(a) $\frac{\tan\beta}{\tan\alpha} < \frac{\alpha}{\beta}$ (b) $\frac{\tan\beta}{\tan\alpha} > \frac{\alpha}{\beta}$

(c) $\frac{\tan\alpha}{\tan\beta} < \frac{\alpha}{\beta}$ (d) $\frac{\tan\alpha}{\tan\beta} > \frac{\alpha}{\beta}$

33. If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$ for $0 \le x \le 1$, then

(a) $g(x)$ increases in $\left(-\infty, \frac{1}{2}\right)$

(b) $g(x)$ increases in $\left(0, \frac{1}{2}\right)$

(c) $g(x)$ decreases in $\left(\frac{1}{2}, 1\right)$

(d) $g(x)$ decreases in $\left(\frac{1}{2}, \infty\right)$

34. If the function $f(x) = 3\cos|x| - 6ax + b$ increases for all $x \in R$, then the range of values of a is given by

(a) $a > -\frac{1}{2}$ (b) $a < -\frac{1}{2}$

(c) $a \le b$ (d) $a \ge b$

35. Let $f(x) = \cot^{-1}[g(x)]$, where $g(x)$ is an increasing function for $0 < x < \pi$. Then $f(x)$ is

(a) increasing in $(0, \pi)$

(b) decreasing in $(0, \pi)$

(c) increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(d) none of these

36. Given that $f'(x) > g'(x)$ for all real x and $f(0) = g(0)$, then

(a) $f(x) > g(x)\ \forall\, x \in (0, \infty)$

(b) $f(x) < g(x)\ \forall\, x \in (-\infty, 0)$

(c) $f(x) < g(x)\ \forall\, x \in (0, \infty)$

(d) $f(x) > g(x)\ \forall\, x \in (-\infty, 0)$

37. If $f'(x) = \frac{1}{1 + x^2}$ for all x and $f(0) = 0$, then

(a) $f(2) < 0.4$ (b) $f(2) > 2$

(c) $0.4 < f(2) < 2$ (d) $f(2) = 2$

38. A function f is such that $f'(4) = f''(4) = 0$ and f has minimum value 10 at $x = 4$. Then $f(x) =$

(a) $4 + (x - 4)^4$

(b) $10 + (x - 4)^4$

(c) $(x - 4)^4$

(d) none of these

39. The range of values of k for which the function

$f(x) = (k^2 - 7k + 12)\cos x + 2(k - 4)x + \log 2$

does not possess critical points, is

(a) (1, 5) (b) (1, 5) – {4}

(c) (1, 4) (d) none of these

40. Let $f(x) = 1 + 3x^2 + 3^2x^4 + \ldots + 3^{30} \cdot x^{60}$. Then $f(x)$ has

(a) atleast one maximum

(b) exactly one maximum

(c) atleast one minimum

(d) exactly one minimum

ANSWERS

1. (c)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (a)
11. (b)	**12.** (a,c,d)	**13.** (a)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (b)
21. (b)	**22.** (d)	**23.** (b)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (b)
31. (b),(c)	**32.** (b)	**33.** (b),(c)	**34.** (b)	**35.** (b)	**36.** (a),(b)	**37.** (c)	**38.** (b)	**39.** (b)	**40.** (d)

Indefinite Integration

15

BRIEF REVIEW OF THE CONCEPTS

MATHEMATICAL NOTATION

Integration is the inverse process of differentiation. That is, the process of finding a function, whose differential coefficient is known, is called *integration*.

If the differential coefficient of $F(x)$ is $f(x)$, i.e.,

$$\frac{d}{dx}[F(x)] = f(x),$$

then we say that the *antiderivative* or *integral* of $f(x)$ is $F(x)$, written as

$$\int f(x)\, dx = f(x),$$

Here $\int \{\}\, dx$ is the notation of integration. $f(x)$ is the integrand, x is the variable of integration and dx denotes the integration with respect to x.

INDEFINITE INTEGRAL

We know that if $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)\, dx = F(x)$.

Also, for any arbitrary constant C,

$$\frac{d}{dx}[F(x) + C] = \frac{d}{dx}[F(x)] + 0 = f(x).$$

$$\therefore \quad \int f(x)\, dx = F(x) + C,$$

This shows that $F(x)$ and $F(x) + C$ are both integrals of the same function $f(x)$. Thus, for different values of C, we obtain different integrals of $f(x)$. This implies that the integral of $f(x)$ is not definite. By virtue of this property $F(x)$ is called the indefinite integral of $f(x)$.

Properties of Indefinite Integration

1. $\frac{d}{dx}\left[\int f(x)\, dx\right] = f(x)$
2. $\int f'(x)\, dx = \int \frac{d}{dx}[f(x)]\, dx = f(x)$
3. $\int k\, f(x)\, dx = k \int f(x)\, dx$, where k is any constant
4. If $f_1(x)$, $f_2(x)$, $f_3(x)$, ... (finite in number) are functions of x, then

$$\int \left[f_1(x) \pm f_2(x) \pm f_3(x) \ldots\right] dx$$

$$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \int f_3(x)\, dx \pm \ldots$$

5. If $\int f(x)\, dx = F(x)$, then

$$\int f(ax \pm b)\, dx = \frac{1}{a} F(ax \pm b).$$

STANDARD FORMULAE OF INTEGRATION

The following results are a direct consequence of the definition of an integral.

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$
2. $\int \frac{1}{x}\, dx = \log|x| + C$
3. $\int e^x\, dx = e^x + C$
4. $\int a^x\, dx = \frac{a^x}{\log_e a} + C.$
5. $\int \sin x\, dx = -\cos x + C$
6. $\int \cos x\, dx = \sin x + C$
7. $\int \sec^2 x\, dx = \tan x + C$
8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
9. $\int \sec x \tan x\, dx = \sec x + C$
10. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C.$
11. $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C.$
12. $\int \cot x\, dx = \log|\sin x| + C$
13. $\int \sec x\, dx = \log|\sec x + \tan x| + C$
14. $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$
15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C\ ;\ |x| < 1$
16. $\int \frac{dx}{(1+x^2)} = \tan^{-1}x + C$
17. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C\ ;\ |x| > 1$

METHODS OF INTEGRATION

Method of Transformation

When the integrand is a trigonometric function, we transform the given function into standard integrals or their algebraic sum by using trigonometric formulae:

Some Useful Trigonometric Identities

1. $\sin^2 mx = \frac{1-\cos 2mx}{2}$
2. $\cos^2 mx = \frac{1+\cos 2mx}{2}$
3. $\sin mx = 2\sin\frac{\sin mx}{2}\cos\frac{mx}{2}$
4. $\sin^3 mx = \frac{3\sin mx - \sin 3mx}{4}$
5. $\cos^3 mx = \frac{3\cos mx + \cos 3mx}{4}$
6. $\tan^2 mx = \sec^2 mx - 1$
7. $\cot^2 mx = \operatorname{cosec}^2 mx - 1$
8. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
9. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
10. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

Method of Substitution

By suitable substitution, the variable x in $\int f(x)\,dx$ is changed into another variable t so that the integrand $f(x)$ is changed into $F(t)$ which is some standard integral or algebraic sum of standard integrals.

There is no general rule for finding a proper substitution and the best guide in this matter is experience. However, the following suggestions will prove useful.

1. If the integrand is of the form $f'(ax+b)$, then we put

$$ax + b = t \text{ and } dx = \frac{1}{a}\,dt.$$

$$\text{Thus, } \int f'(ax+b)\,dx = \int f'(t)\frac{dt}{a}$$

$$= \frac{1}{a}\int f'(t)\,dt$$

$$= \frac{f(t)}{a} = \frac{f(ax+b)}{a}.$$

2. When the integrand is of the form $x^{n-1}f'(x^n)$, we put

$$x^n = t \text{ and } nx^{n-1}\,dx = dt.$$

$$\text{Thus, } \int x^{n-1} f'(x^n)\,dx = \int f'(t)\frac{dt}{n}$$

$$= \frac{1}{n}\int f'(t)\,dt$$

$$= \frac{1}{n}f(t) = \frac{1}{n}f(x^n).$$

3. When the integrand is of the form $[f(x)]^n.f'(x)$, we put

$$f(x) = t \text{ and } f'(x)\,dx = dt.$$

$$\text{Thus, } \int [f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1}$$

$$= \frac{[f(x)]^{n+1}}{n+1}$$

4. When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put

$$f(x) = t \text{ and } f'(x)\,dx = dt.$$

$$\text{Thus, } \int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log t = \log f(x).$$

Method of Integration by Parts

The process of integration of the product of two functions is known as integration by parts.

For example, if u and v are two functions of x, then

$$\int (uv)\,dx = u.\int v dx - \int\left(\frac{du}{dx}\cdot\int v dx\right)dx.$$

In words, integral of the product of two functions = first function × integral of the second – integral of (differential of first × integral of the second function).

Key Points to Remember

1. Choose the first and second function in such a way that the derivative of the first function and the integral of the second function can be easily found.
2. In case of integrals of the form $\int f(x).\,x^n dx$, take x^n as the first function and $f(x)$ as the second function.
3. In case of integrals of the form $\int (\log x)^n\cdot dx$, take 1 as the second function and $(\log x)^n$ as the first function.
4. Rule of integration by parts may be used repeatedly, if required.
5. If the two functions are of different type, we can choose the first function as the one whose initial comes first in the word "ILATE", where

   I — Inverse Trigonometric function
   L — Logarithmic function
   A — Algebraic function
   T — Trigonometric function
   E — Exponential function.
6. In case, both the functions are trigonometric, take that function as second function whose integral is simpler. If both the functions are algebraic, take that function as first function whose derivative is simpler.
7. If the integral consists of an inverse trigonometric function of an algebraic expression in x, first simplify the integrand by a suitable trigonometric substitution and then integrate the new integrand.

Method of Partial Fractions for Rational Functions

Integrals of the type $\int \frac{p(x)}{g(x)}\,dx$ can be integrated by resolving the integrand into partial fractions. We proceed as follows:

- Check degree of $p(x)$ and $g(x)$.
- If degree of $p(x)$ > degree of $g(x)$, then divide $p(x)$ by $g(x)$ till its degree is less, i.e., put in the form

$$\frac{p(x)}{g(x)} = r(x) + \frac{f(x)}{g(x)}$$

where degree of $f(x)$ < degree of $g(x)$.

CASE 1: When the denominator contains non-repeated linear factors. That is

$$g(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n).$$

In such a case write $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \dots + \frac{A_n}{x-\alpha_n}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 2: When the denominator contains repeated as well as non-repeated linear factors. That is

$$g(x) = (x - \alpha_1)^2 (x - \alpha_3) \dots (x - \alpha_n).$$

In such a case express $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \frac{A_3}{x-\alpha_3} + \dots + \frac{A_n}{(x-\alpha_n)}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 3: When the denominator contains a non repeated quadratic factor which cannot be factorised further:

$$g(x) = (ax^2 + bx + c)(x - \alpha_3)(x - \alpha_4) \dots (x - \alpha_n).$$

In such a case express $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3}{x-\alpha_3} + \dots + \frac{A_n}{x-\alpha_n}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 4: When the denominator contains a repeated quadratic factor which cannot be factorised further. That is

$$g(x) = (ax^2 + bx + c)^2 (x - \alpha_5)(x - \alpha_6) \dots (x - \alpha_n)$$

In such a case write $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3 x + A_4}{(ax^2 + bx + c)^2} + \frac{A_5}{x-\alpha_5} + \dots + \frac{A_n}{(x-\alpha_n)}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

***Notes*:** Corresponding to repeated linear factor $(x - a)^r$ in the denominator, a sum of r partial fractions of the type

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_r}{(x-a)^r} \text{ is taken.}$$

CASE 5: If the integrand contains only even powers of x

1. Put $x^2 = z$ in the integrand.
2. Resolve the resulting rational expression in z into partial fractions
3. Put $z = x^2$ again in the partial fractions and then integrate both sides.

Key Points to Remember

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$
3. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$
4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$
5. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$
6. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$
7. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$
8. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$
9. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

Integrals of the Form

1. $\int f(a^2 - x^2)\, dx$, 2.. $\int f(a^2 + x^2)\, dx$,

3. $\int f(x^2 - a^2)\, dx$, 4. $\int f\left(\frac{a-x}{a+x}\right) dx$,

Working Rule

Integral	Substitution
$\int f(a^2 - x^2)\, dx$	$x = a \sin\theta$ or $x = a \cos\theta$
$\int f(x^2 + a^2)\, dx$	$x = a \tan\theta$ or $x = a \cot\theta$
$\int f(x^2 - a^2)\, dx$	$x = a \sec\theta$ or $x = a \operatorname{cosec}\theta$
$\int f\left(\frac{a-x}{a+x}\right) dx$ or $\int f\left(\frac{a+x}{a-x}\right) dx$	$x = a \cos 2\theta$

Integrals of the Form $\int \sin^m x \cos^n x\, dx$

Working Rule

- If the power of $\sin x$ is an odd positive integer, put $\cos x = z$.
- If the power of $\cos x$ is an odd positive integer, put $\sin x = z$.

- If the power of $\sin x$ and $\cos x$ are both odd positive integers, put $\sin x = z$ or $\cos x = z$.
- If the power of $\sin x$ and $\cos x$ are both even positive integers, use De' Moivre's theorem as follows:

  Let $\cos x + i \sin x = z$. Then $\cos x - i\sin x = z^{-1}$

  Adding these, we get

  $$z + \frac{1}{z} = 2\cos x \text{ and } z - \frac{1}{z} = 2i \sin x$$

  By De' Moivre's theorem, we have

  $$z^n + \frac{1}{z^n} = 2\cos nx \text{ and } z^n - \frac{1}{z^n} = 2i\sin nx \quad \ldots(1)$$

  $$\therefore \sin^m x \cdot \cos^n x = \frac{1}{(2i)^m} \cdot \frac{1}{2^n}\left(z+\frac{1}{z}\right)^n\left(z-\frac{1}{z}\right)^m$$

  $$= \frac{1}{2^{m+n}} \cdot \frac{1}{i^m}\left(z+\frac{1}{z}\right)^n\left(z-\frac{1}{z}\right)^m.$$

  Now expand each of the factors on the R.H.S. using Binomial theorem. Then group the terms equidistant from the beginning and the end. Thus express all such pairs as the sines or cosines of multiple angles. Further integrate term by term.
- If the sum of powers of $\sin x$ and $\cos x$ is an even negative integer, put $\tan x = z$.

Integral of the Form

1. $\displaystyle\int \frac{dx}{a + b\cos x}$
2. $\displaystyle\int \frac{dx}{a + b\sin x}$
3. $\displaystyle\int \frac{dx}{a + b\cos x + c\sin x}$

Working Rule

- Put $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ so that the given integrand becomes a function of $\tan\dfrac{x}{2}$.
- Put $\tan\dfrac{x}{2} = z \Rightarrow \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = dz$
- Integrate the resulting rational algebraic function of z
- In the answer, put $z = \tan\dfrac{x}{2}$.

Integrals of the Form

1. $\displaystyle\int \frac{dx}{a + b\cos^2 x}$
2. $\displaystyle\int \frac{dx}{a + b\sin^2 x}$
3. $\displaystyle\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$

Working Rule

- Divide the numerator and denominator by $\cos^2 x$.
- In the denominator, replace $\sec^2 x$, if any, by $1 + \tan^2 x$.
- Put $\tan x = z \Rightarrow \sec^2 x\, dx = dz$.
- Integrate the resulting rational algebraic function of z.
- In the answer, put $z = \tan x$.

Integrals of the form: $\displaystyle\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx$

Working Rule

- Put Numerator = λ (denominator) + μ (derivative of denominator)

  $$a\cos x + b\sin x = \lambda(c\cos x + d\sin x) + \mu(-c\sin x + d\cos x).$$
- Equate coefficients of $\sin x$ and $\cos x$ on both sides and find the values of λ and μ.
- Split the given integral into two integrals and evaluate each integral separately, i.e.,

  $$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx = \lambda\int 1\,dx + \mu\int \frac{-c\sin x + d\cos x}{a\cos x + b\sin x}dx$$

  $$= \lambda x + \mu \log|a\cos x + b\sin x|.$$
- Substitute the values of λ and μ found in second step.

Integrals of the form:

Working Rule

- Put Numerator = l (denominator) + m (derivative of denominator) + n

  $$a + b\cos x + c\sin x = l(e + f\cos x + g\sin x) + m(-f\sin x + g\cos x) + n$$
- Equate coefficients of $\sin x$, $\cos x$ and constant term on both sides and find the values of l, m, n.
- Split the given integral into three integrals and evaluate each integral separately, i.e.,

  $$\int \frac{a + b\cos x + c\sin x}{e + f\cos x + g\sin x}dx$$

  $$= l\int 1dx + m\int \frac{-f\sin x + g\cos x}{e + f\cos x + g\sin x}dx + n\int \frac{dx}{e + f\cos x + g\sin x}$$

  $$= lx + m\log|e + f\cos x + g\sin x| + n\int \frac{dx}{e + f\cos x + g\sin x}$$
- Substitute the values of l, m, n found in second step.

Integrals of the Form

1. $\displaystyle\int \frac{dx}{ax^2 + bx + c}$
2. $\displaystyle\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

3. $\int \sqrt{ax^2 + bx + c}\, dx$

Working Rule

- Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.
- Complete the square in the terms involving x, i.e., write $ax^2 + bx + c$ in the form $a\,[x \pm \alpha)^2 \pm \beta^2]$.
- The integrand is converted to one of the nine special integrals.
- Integrate the function.

Integrals of the form: 1. $\int \frac{px+q}{ax^2+bx+c}\,dx$ 2. $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ 3. $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Integral	Working Rule
$\int \frac{px+q}{ax^2+bx+c}\,dx$	Put $px + q = \lambda\,(2ax + b) + \mu$ or $px + q = \lambda$ (derivative of quadratic) + μ. Comparing the coefficient of x and constant term on both sides, we get $p = 2a\lambda$ and $q = b\lambda + \mu \Rightarrow \lambda = \frac{p}{2a}$ and $\mu = \left(q - \frac{bp}{2a}\right)$ Then the integral becomes $\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$ $= \frac{p}{2a}\log\,\lvert ax^2+bx+c\rvert + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$
$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$	In this case the integral becomes $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$ $= \frac{p}{a}\sqrt{ax^2+bx+c} + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Integrals of the form: $\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx$, where $P(x)$ is a polynomial in x of degree n.

Working Rule

Write $\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx = (a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1})\sqrt{ax^2+bx+c} + k\int \frac{dx}{\sqrt{ax^2+bx+c}}$

where k, a_0, a_1, ... a_{n-1} are constants to be determined by differentiating the above relation and equating the coefficients of various powers of x on both sides.

Integrals of the form: $\int \frac{x^2+1}{x^4+kx^2+1}\,dx$ or $\int \frac{x^2-1}{x^4+kx^2+1}\,dx$, where k is a constant positive, negative or zero.

Working Rule

- Divide the numerator and denominator by x^2.
- Put $x - \frac{1}{x} = z$ or $x + \frac{1}{x} = z$ whichever substitution, on differentiation gives, the numerator of the resulting integrand.
- Evaluate the resulting integral in z
- Express the result in terms of x.

Integrals of the form: $\int \frac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral	*Substitution*
$\int \frac{1}{(ax+b)\sqrt{cx+d}}\,dx$	$cx + d = z^2$
$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	$px + q = z^2$
$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px + q = \frac{1}{z}$
$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$	$x = \frac{1}{z}$.

Integrals of the form: $\int f\left(x, (ax+b)^{\alpha/n}\right)\, dx$, where α and n are integers.

Working Rule

Put $ax + b = z^n$.

Integrals of the Form $\int f\left(x, (ax+b)^{\alpha/n}, (ax+b)^{\beta/m}\right)$ where α, β, m, n are integers.

Working Rule

Put $ax + b = z^k$, where k = l.c.m. (m, n).

Integrals of the Form

1. $\displaystyle\int \frac{x^m}{(a+bx)^p}\, dx$
2. $\displaystyle\int \frac{dx}{x^m (a+bx)^p}$
3. $\displaystyle\int x^m (a+bx^n)^p\, dx$

Integral	*Substitution*
$\displaystyle\int \frac{x^m}{(a+bx)^p}\, dx$, m is a + *ve* integer	Put $a + bx = z$
$\displaystyle\int \frac{dx}{x^m (a+bx)^p}$, where either ($m$ and p are positive integers) or (m and p are fractions, but $m + p$ = integer > 1)	Put $a + bx = zx$
$\int x^m (a+bx^n)^p\, dx$, where m, n, p are rationals.	
(a) p is a + *ve* integer	Apply Binomial theorem to $(a + bx^n)^p$.
(b) p is a − *ve* integer	Put $x = z^k$ where k = common denominator of m and n.
(c) $\dfrac{m+1}{n}$ is an integer	Put $a + bx^n = z^k$, where k = denominator of p.
(d) $\dfrac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$ where k = denominator of fraction p.

Integrals of the Form: $\int e^x \left[f(x) + f'(x)\right] dx$

Working Rule

- Split the integral into two integrals.
- Integrate only the first integral by parts, i.e.,

$$\int e^x \left[f(x) + f'(x)\right] dx = \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$

$$= \left[f(x).e^x - \int f'(x).e^x\, dx\right] + \int e^x f'(x)\, dx$$

$$= e^x f(x) + C.$$

Integrals of the Form: Where the initial integrand reappears after integrating by parts.

Working Rule

- Apply the method of integration by parts twice.
- On integrating by parts second time, we will obtain the given integrand again. Put it equal to I.
- Transpose and collect terms involving I on one side and evaluate I.

Solved Problems

1. $\displaystyle\int \frac{\cot x}{\sqrt{\sin x}}\, dx$ is equal to

(a) $2\sqrt{\sin x} + C$ (b) $\dfrac{1}{2\sqrt{\sin x}} + C$

(c) $\dfrac{-2}{\sqrt{\sin x}} + C$ (d) $\dfrac{2}{\sqrt{\sin x}} + C$

Solution (c) $\displaystyle\int \frac{\cot x}{\sqrt{\sin x}}\, dx = \int \frac{\cos x}{(\sin x)^{3/2}}\, dx = \int \frac{dz}{z^{3/2}}$

[Putting $\sin x = z \Rightarrow \cos x\, dx = dz$]

$$= \frac{-2}{\sqrt{z}} + C = \frac{-2}{\sqrt{\sin x}} + C.$$

2. $\displaystyle\int e^{\tan^{-1} x}\left(1 + \frac{x}{1+x^2}\right) dx$ is equal to

(a) $\dfrac{1}{2} x e^{\tan^{-1} x}$ (b) $\dfrac{1}{2} e^{\tan^{-1} x}$

(c) $x e^{\tan^{-1} x}$ (d) $e^{\tan^{-1} x}$

Solution (c) $\displaystyle\int e^{\tan^{-1} x}\left(1 + \frac{x}{1+x^2}\right) dx$

$$= \int e^{\tan^{-1} x}\, dx + \int e^{\tan^{-1} x} \cdot \frac{x}{1+x^2}\, dx$$

$$= e^{\tan^{-1} x} \cdot x - \int e^{\tan^{-1} x} \cdot \frac{x}{1+x^2}\, dx + \int e^{\tan^{-1} x} \cdot \frac{x}{1+x^2}\, dx$$

$$= x e^{\tan^{-1} x}.$$

3. $\displaystyle\int e^{3\log x} \left(x^4 + 1\right)^{-1} dx$ is equal to

(a) $\dfrac{1}{4} \log (x^4 + 1) + C$

(b) $-\log (x^4 + 1) + C$

(c) $\log(x^4+1)+C$

(d) none of these

Solution (a) $\int e^{3\log x}.\ (x^4+1)^{-1}\,dx$

$$= \int e^{\log x^3}\cdot\frac{dx}{x^4+1} = \int\frac{x^3}{x^4+1}\,dx$$

$$= \frac{1}{4}\int\frac{4x^3}{x^4+1}\,dx = \mathbf{\frac{1}{4}\log(x^4+1)+C}.$$

4. The value of $\int\frac{(x-x^3)^{1/3}}{x^4}\,dx$ is

(a) $\frac{1}{8}\left(1-\frac{1}{x^2}\right)^{4/3}+1+C$

(b) $\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3}+C$

(c) $\frac{-3}{8}\left(\frac{1}{x^2}-1\right)^{4/3}+C$

(d) none of these

Solution (c) $\int\frac{\left(x-x^3\right)^{1/3}}{x^4}\,dx = \int\frac{1}{x^3}\left(\frac{1}{x^2}-1\right)^{1/3}dx$

$$= \frac{-1}{2}\int t^{1/3}\,dt\ \left[\text{Putting } \frac{1}{x^2}-1=t\Rightarrow\frac{-2}{x^3}\,dx=dt\right]$$

$$= \frac{-1}{2}\cdot\frac{t^{4/3}}{4/3}+C = \mathbf{\frac{-3}{8}\left(\frac{1}{x^2}-1\right)^{4/3}}+C.$$

5. $\int|x|^3\,dx$ is equal to

(a) $\frac{-x^4}{4}$ (b) $\frac{|x|^4}{4}$

(c) $\frac{x^4}{4}$ (d) none of these

Solution (a), (c) $\int|x|^3\,dx = \pm\int x^3\,dx = \pm\mathbf{\frac{x^4}{4}}$.

6. $\int\frac{dx}{x^2\left(1+x^4\right)^{3/4}}$ is equal to

(a) $-\frac{\left(1+x^4\right)^{1/4}}{x}+C$ (b) $\frac{\left(1+x^4\right)^{1/4}}{x}+C$

(c) $-\frac{\left(1+x^4\right)^{3/4}}{x}+C$ (d) none of these

Solution (a) $\int\frac{dx}{x^2(1+x^4)^{3/4}} = \int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

$$= \frac{-1}{4}\int\frac{dt}{t^{3/4}}\ \left[\text{Putting } 1+\frac{1}{x^4}=t\Rightarrow\frac{dx}{x^5}=\frac{-1}{4}\,dt\right]$$

$$= \frac{-1}{4}\cdot\frac{t^{1/4}}{1/4}+C = -\left(1+\frac{1}{x^4}\right)^{1/4}+C$$

$$= -\mathbf{\frac{(1+x^4)^{1/4}}{x}}+C.$$

7. If $\int x^{13/2}\cdot(1+x^{5/2})^{1/2}\,dx$

$= A\,(1+x^{5/2})^{7/2}+B\,(1+x^{5/2})^{5/2}+C\,(1+x^{5/2})^{3/2}$, then

(a) $A=-\frac{4}{35}, B=-\frac{8}{25}, C=\frac{4}{15}$

(b) $A=\frac{4}{35}, B=-\frac{8}{25}, C=-\frac{4}{15}$

(c) $A=\frac{4}{35}, B=-\frac{8}{25}, C=\frac{4}{15}$

(d) none of these

Solution (c) $\int x^{13/2}.\left(1+x^{5/2}\right)^{1/2}dx$

$$= \int x^5.x^{3/2}.(1+x^{5/2})^{1/2}\,dx = \int x^5.\frac{4}{5}z.z\,dz$$

$$\left[\begin{aligned}&\text{Putting } 1+x^{5/2}=z^2\Rightarrow\frac{5}{2}x^{3/2}\,dx=2z\,dz\\&\text{i.e. } x^{3/2}\,dx=\frac{4}{5}z\,dz\end{aligned}\right]$$

$$= \frac{4}{5}\int z^2\cdot(z^2-1)^2\,dz = \frac{4}{5}\int z^2\left(z^4-2z^2+1\right)dz$$

$$= \frac{4}{5}\left[\frac{z^7}{7}-\frac{2z^5}{5}+\frac{z^3}{3}\right]+C$$

$$= \frac{4}{35}(1+x^{5/2})^{7/2}-\frac{8}{25}(1+x^{5/2})^{5/2}+\frac{4}{15}(1+x^{5/2})^{3/2}+C.$$

$\therefore$ $\mathbf{A=\frac{4}{35}}$, $\mathbf{B=-\frac{8}{25}}$ **and** $\mathbf{C=\frac{4}{15}}$.

8. $\int\frac{dx}{\cos x\sqrt{\cos 2x}}$ is equal to

(a) $\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}\tan^{\frac{5}{2}}x\right)+C$

(b) $\sqrt{2}\left(\sqrt{\cot x}+\frac{1}{5}\tan^{\frac{5}{2}}x\right)+C$

(c) $\sqrt{2}\left(\sqrt{\tan x}-\frac{1}{5}\tan^{\frac{5}{2}}x\right)+C$

(d) none of these

Solution (d) $\int\frac{dx}{\cos x\sqrt{\cos 2x}} = \int\frac{dx}{\cos x\sqrt{\frac{1-\tan^2 x}{1+\tan^2 x}}}$

$$= \int\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \int\frac{dz}{\sqrt{1-z^2}}$$

[Putting $\tan x=z\Rightarrow\sec^2 x\,dx=dz$]

$= \mathbf{\sin^{-1}z+C=\sin^{-1}(\tan x)}+C.$

9. $\int\frac{dx}{\cos^3 x\sqrt{\sin 2x}}$ is equal to

(a) $\sqrt{2}\left(\sqrt{\cot x}+\frac{1}{5}\tan^{5/2}x\right)+C$

(b) $\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x\right)+C$

(c) $\sqrt{2}\left(\sqrt{\tan x}-\frac{1}{5}\tan^{5/2}x\right)+C$

(d) none of these

Solution (b) $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}} = \int \frac{dx}{\cos^3 x\sqrt{\frac{2\tan x}{1+\tan^2 x}}}$

$$= \int \frac{\sec^4 x}{\sqrt{2\tan x}}\,dx = \frac{1}{\sqrt{2}}\int \frac{(1+z^4).2z\,dz}{z}$$

[Putting $\tan x = z^2 \Rightarrow \sec^2 x\,dx = 2z\,dz$]

$$= \sqrt{2}\left(z + \frac{z^5}{5}\right) + C = \boldsymbol{\sqrt{2}\left(\sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x\right) + C.}$$

10. $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$ is equal to

(a) $\log\left|\frac{x}{x+\cos x}\right| + C$ (b) $\log\left|\frac{x+\cos x}{x}\right| + C$

(c) $\log\left|\frac{1}{x+\cos x}\right| + C$ (d) $\log|\,x + \cos x\,| + C$

Solution (a) $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$

$$= \int \frac{(x+\cos x) - x(1-\sin x)}{x(x+\cos x)}\,dx$$

$$= \int\left(\frac{1}{x} - \frac{1-\sin x}{x+\cos x}\right)dx$$

$$= \log|x| - \log|x+\cos x| + C = \boldsymbol{\log\left|\frac{x}{x+\cos x}\right| + C.}$$

11. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ is equal to

(a) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

(b) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} + \sqrt{ax} + C$

(c) $(x+a)\cot^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

(d) none of these

Solution (a) Put $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\,d\theta$

$$\therefore \int \sin^{-1}\sqrt{\frac{x}{a+x}} = \int \theta.\ 2a\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\,d\theta\right]$$

$$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\,d\theta$$

$$= a\theta\tan^2\theta - a(\tan\theta - \theta) + C$$

$$= a\theta(1 + \tan^2\theta) - a\tan\theta + C$$

$$= \boldsymbol{(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C.}$$

12. For the function $f(x) = 1 + 3^x\log 3$, the antiderivative F assumes the value 7 for $x = 2$. At what value of x does the curve $y = F(x)$ cut the abscissa?

(a) $x = 3$

(b) $x = 1$

(c) $x = 0$

(d) none of these

Solution (b) $f(x) = 1 + 3^x\log 3.$

$\therefore$ $F(x) = \int(1 + 3^x\log 3)\,dx = x + 3^x + c$

Since $F(2) = 2 + 9 + c = 7 \Rightarrow c = -4$

$\therefore$ $F(x) = x + 3^x - 4 = \boldsymbol{0 \Rightarrow x = 1.}$

13. $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to

(a) $\frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + C$ (b) $\frac{2(1-\sqrt{x})}{\sqrt{1-x}} + C$

(c) $\frac{(\sqrt{x}-1)}{\sqrt{1-x}} + C$ (d) none of these

Solution (a) $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$

$$= \int \frac{2\sin\theta\cos\theta\,d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta - \sin^4\theta}}$$

[Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$]

$$= 2\int\frac{d\theta}{1+\sin\theta} = 2\int\frac{1-\sin\theta}{\cos^2\theta}\,d\theta = 2(\tan\theta - \sec\theta)$$

$$= 2\left(\frac{\sqrt{x}}{\sqrt{1-x}} - \frac{1}{\sqrt{1-x}}\right) + C = \boldsymbol{\frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + C.}$$

14. $\int \frac{\sin^3 x\,dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}$ is equal to

(a) $\sec^{-1}(\sec x + \cos x) + C$

(b) $\sec^{-1}(\sec x - \cos x) + C$

(c) $\sec^{-1}(\cos x - \tan x) + C$

(d) none of these

Solution (a) $\int \frac{\sin^3 x}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}\,dx$

$$= \int \frac{\sin^3 x}{\cos x(\sec x + \cos x)\cos x\sqrt{\sec^2 x + 1 + \cos^2 x}}\,dx$$

$$= \int \frac{\sin^3 x\,dx}{\cos^2 x(\sec x + \cos x)\sqrt{(\sec x + \cos x)^2 - 1}}$$

$$= \int \frac{dz}{z\sqrt{z^2-1}}$$

$$\left[\text{Putting } \sec x + \cos x = z \Rightarrow \frac{\sin^3 x}{\cos^2 x}\,dx = dz\right]$$

$$= \boldsymbol{\sec^{-1} z + c = \sec^{-1}(\sec x + \cos x) + C.}$$

15. If $f(x) = \int \frac{(x^2 + \sin^2 x)}{1+x^2}\sec^2 x\,dx$ and $f(0) = 0$, then $f(1) =$

(a) $1 - \frac{\pi}{4}$ (b) $\frac{\pi}{4} - 1$

(c) $\tan 1 - \frac{\pi}{4}$ (d) none of these

Solution (c) We have, $f(x) = \int \frac{\left(x^2+\sin^2 x\right)}{1+x^2} \sec^2 x\,dx$

$$= \int \frac{x^2+\left(1-\cos^2 x\right)}{1+x^2} \cdot \sec^2 x\, dx$$

$$= \int\left(\sec^2 x - \frac{1}{1+x^2}\right) dx$$

$$= \tan x - \tan^{-1} x + C.$$

$\because$ $f(0) = 0$, $\therefore$ $C = 0$.

Thus, $f(x) = \tan x - \tan^{-1} x$.

Hence, $f(1) = \tan 1 - \tan^{-1} 1 = \mathbf{\tan 1 - \frac{\pi}{4}}$.

16. If $\int \frac{\sqrt{\cot x}}{\sin x \cos x}\, dx = P\sqrt{\cot x} + Q$, then P equals

(a) 1 (b) 2

(c) –1 (d) –2

Solution (d) $\int \frac{\sqrt{\cot x}}{\sin^2 x \frac{\cos x}{\sin x}}\, dx = P\sqrt{\cot x} + Q$

$I = \int \frac{\operatorname{cosec}^2 x\, dx}{\sqrt{\cot x}}$, let $\cot x = t$, $\operatorname{cosec}^2 x\, dx = -dt$

$$= -\int \frac{dt}{\sqrt{t}} = -2\sqrt{t} + c = -2\sqrt{\cot x} + c$$

$\therefore$ $\mathbf{P = -2.}$

17. $\int e^x \left(\frac{x+2}{x+4}\right)^2 dx$ is equal to

(a) $\frac{xe^x}{x+4} + C$ (b) $e^x\left(\frac{x+2}{x+4}\right) + C$

(c) $\frac{e^x}{x+4} + C$ (d) none of these

Solution (a) $\int e^x \left(\frac{x+2}{x+4}\right)^2 dx = \int e^x\left(\frac{x^2+4x+4}{(x+4)^2}\right) dx$

$$= \int e^x\left(\frac{x}{x+4} + \frac{4}{(x+4)^2}\right) dx$$

$$= \int e^x\left[\frac{x}{x+4} + \frac{d}{dx}\left(\frac{x}{x+4}\right)\right] dx$$

$$= \mathbf{e^x \cdot \frac{x}{x+4} + C.}$$

18. $\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\, dx$ is equal to

(a) $\frac{3}{2} x^{2/3} + 6 \tan^{-1} x^{1/6} + C$

(b) $\frac{3}{2} x^{2/3} - 6 \tan^{-1} x^{1/6} + C$

(c) $\frac{-3}{2} x^{2/3} + 6 \tan^{-1} x^{1/6} + C$

(d) none of these

Solution (a) Put $x = z^6 \Rightarrow dx = 6z^5\, dz$

$$\therefore \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\, dx = \int \frac{\left(z^6+z^4+z\right)6z^5 dz}{z^6\left(1+z^2\right)}$$

$$= 6\int \frac{z^5+z^3+1}{z^2+1}\, dz = 6\int\left(z^3 + \frac{1}{z^2+1}\right) dz$$

$$= \frac{3}{2} z^4 + 6 \tan^{-1} z + C$$

$$= \mathbf{\frac{3}{2} x^{2/3} + 6 \tan^{-1} x^{1/6} + C.}$$

19. $\int\left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$ is equal to

(a) $\sin^{-1}(\sin x - \cos x) + C$

(b) $\sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

(c) $\sqrt{2}\cos^{-1}(\sin x - \cos x) + C$

(d) none of these

Solution (b) $\int\left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

$$= \int\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(1-2\sin x\cos x)}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\, dx = \sqrt{2}\int \frac{dz}{\sqrt{1-z^2}}$$

[Putting $\sin x - \cos x = z \Rightarrow (\sin x + \cos x)\, dx = dz$]

$$= \sqrt{2}\sin^{-1} z + C$$

$$= \mathbf{\sqrt{2}\sin^{-1}(\sin x - \cos x) + C.}$$

20. $\int x^{1/3}(2 + x^{2/3})^{1/4}\, dx$ is equal to

(a) $\frac{2}{3}(2 + x^{2/3})^{9/4} + \frac{12}{5}(2 + x^{2/3})^{5/4} + C$

(b) $\frac{2}{3}(2 + x^{2/3})^{9/4} - \frac{12}{5}(2 + x^{2/3})^{5/4} + C$

(c) $\frac{1}{3}(2 + x^{2/3})^{9/4} - \frac{12}{5}(2 + x^{2/3})^{5/4} + C$

(d) none of these

Solution (b) Put $2 + x^{2/3} = z^4 \Rightarrow dx = 6z^3.\, x^{1/3}\, dz$

$$\therefore \int x^{1/3}\left(2 + x^{2/3}\right)^{1/4} dx = \int x^{1/3}.\, z(6z^3.\, x^{1/3})\, dz$$

$$= 6\int z^4\left(z^4-2\right) dz = 6\int\left(\frac{z^9}{9} - \frac{2z^5}{5}\right) + C$$

$$= \mathbf{\frac{2}{3}(2 + x^{2/3})^{9/4} - \frac{12}{5}(2 + x^{2/3})^{5/4} + C.}$$

21. If $\int f(x)\, dx = g(x) + C$, then $\int f(ax+b)\, dx$ is equal to

(a) $g(ax + b) + C$ (b) $ag(ax + b) + C$

(c) $\frac{1}{a}[g(ax+b) + C]$ (d) none of these

Solution (c) Put $ax + b = z \Rightarrow dx = \frac{1}{a}\ dz$

$$\therefore \quad \int f(ax+b)\ dx = \frac{1}{a}\int f(z)\ dz = \frac{1}{a}\ [g(z) + C]$$

$$= \frac{1}{\mathbf{a}}\ [\boldsymbol{g}\ (\boldsymbol{ax} + \boldsymbol{b}) + \boldsymbol{C}].$$

22. If $\int \frac{dx}{\sqrt{2ax - x^2}} = (f o g)(x) + C$, then

(a) $f(x) = \sin^{-1}x,\ g(x) = \frac{x+a}{a}$

(b) $f(x) = \sin^{-1}x,\ g(x) = \frac{x-a}{a}$

(c) $f(x) = \cos^{-1}x,\ g(x) = \frac{x-a}{a}$

(d) $f(x) = \tan^{-1}x,\ g(x) = \frac{x-a}{a}.$

Solution (b) $\int \frac{dx}{\sqrt{2ax - x^2}} = \int \frac{dx}{\sqrt{a^2 - (x^2 - 2ax + a^2)}}$

$$= \int \frac{dx}{\sqrt{a^2 - (x-a)^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C.$$

$\therefore \quad f(x) = \sin^{-1}x$ and $g(x) = \boldsymbol{\frac{x-a}{a}}.$

23. Let $f(x)$ be a polynomial of degree three satisfying $f(0) = -1$ and $f(1) = 0$. Also, 0 is a stationary point of $f(x)$. If $f(x)$ does not have an extremum at $x = 0$, then $\int \frac{f(x)}{x^3 - 1}dx$ is equal to

(a) $\frac{x^2}{2} + C$ (b) $x + C$

(c) $\frac{x^3}{6} + C$ (d) none of these

Solution (b) Let $f(x) = ax^3 + bx^2 + cx + d.$

Since $f(0) = -1$ and $f(1) = 0$

$\Rightarrow d = -1$ and $a + b + c + d = 0$

$\Rightarrow d = -1$ and $a + b + c = 1.$ ...(1)

Since 0 is a stationary point of $f(x)$,

$\therefore f'(0) = 0$

$\Rightarrow 3a\ (0)^2 + 2b\ (0) + c = 0$

$\Rightarrow c = 0.$

Since $f(x)$ does not have an extremum at $x = 0$,

$\therefore f''(0) = 0$

$\Rightarrow b = 0.$ and

$\therefore$ from (1), $a = 1.$

So, $f(x) = x^3 - 1.$

$$\therefore \quad \int \frac{f(x)}{x^3 - 1}\ dx = \int 1\,dx = \boldsymbol{x + C}.$$

24. If $\int f(x)\ dx = F(x)$, then $\int x^3 f(x^2)\ dx$ is equal to

(a) $\frac{1}{2}\left[x^2 (F(x))^2 - \int (F(x))^2\ dx\right]$

(b) $\frac{1}{2}\left[x^2.F(x^2) - \int F(x^2)\,d(x^2)\right]$

(c) $\frac{1}{2}\left[x^2 F(x) - \frac{1}{2}\int (F(x))^2\ dx\right]$

(d) none of these

Solution (b) $\int x^3 f(x^2)\ dx = \int x^2 f(x^2)\cdot\ x\ dx$

$$= \frac{1}{2}\int z\, f(z)\ dz$$

$$\left[\text{Putting } x^2 = z \Rightarrow xdx = \frac{1}{2}dz\right]$$

$$= \frac{1}{2}\quad \left[z\,F(z) - \int 1\cdot F(z)\,dz\right]$$

$$\left[\because \int f(x)\,dx = F(x)\right]$$

$$= \frac{1}{2}\ z\ F(z) - \frac{1}{2}\ \int F(z)\ dz$$

$$= \boldsymbol{\frac{1}{2}\ x^2\ F(x^2) - \frac{1}{2}\ \int F(x^2)\ d\ (x^2)}.$$

25. $\int \frac{[f(x)\,g'(x) + g(x)\,f'(x)]}{f(x).g(x)}$ [log $f(x)$ + log $g(x)$] dx is equal to

(a) $f(x)\ g(x)\ \log\ (f(x)\ g(x)) + C$

(b) $\frac{1}{2}\ [\log f(x)\ g(x)]^2 + C$

(c) $[\log f(x)\ g(x)^2] + C$

(d) $\log f(x).g(x) + C$

Solution (b) $\int \frac{[f(x)\,g'(x) + g(x)\,f'(x)]}{f(x)\cdot g(x)}$

[log $f(x)$ + log $g(x)$] dx

$$= \int \frac{[f(x)\,g'(x) + g(x)\,f'(x)]}{f(x)\cdot g(x)}\quad \log f(x)\ g(x)\ dx$$

$$= \int \frac{\log t}{t}\ dt$$

[Putting $f(x).\ g(x) = t \Rightarrow [f(x)\ g'(x) + g(x)\ f'(x)]\ dx = dt$]

$$= \frac{1}{2}\ (\log t)^2 + C = \boldsymbol{\frac{1}{2}\ [\log f(x)\cdot\ g(x)]^2 + C}.$$

26. $\int \frac{\log\,(x+1) - \log x}{x\,(x+1)}\ dx$ is equal to

(a) $-\frac{1}{2}\left[\log\left(\frac{x+1}{x}\right)\right]^2 + C$

(b) $C - [\{\log\ (x + 1)\}^2 - (\log x)^2]$

(c) $-\log\left[\log\left(\frac{x+1}{x}\right)\right] + C$

(d) $-\log\left(\frac{x+1}{x}\right) + C$

Solution (a) $\int \frac{\log(x+1)-\log x}{x(x+1)}\,dx$

$= -\int (\log(x+1)-\log x)\cdot \frac{1}{-x(x+1)}\,dx$

$= \boldsymbol{\frac{-1}{2}\left[\log\left(\frac{x+1}{x}\right)\right]^2} + C$

27. $\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx$ is equal to

(a) $\log\left(e^x+\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(b) $\log\left(e^x+\sqrt{e^{2x}-1}\right) + \sec^{-1}(e^x) + C$

(c) $\log\left(e^x-\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C$

(d) none of these

Solution (a) $\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx = \int \sqrt{\frac{\sec\theta-1}{\sec\theta+1}}.\ \tan\theta\,d\theta$

[Put $e^x = \sec\theta \Rightarrow dx\tan\theta\,d\theta$]

$$= \int \sqrt{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta}\right)d\theta = \int \frac{(1-\cos\theta)}{\cos\theta}\,d\theta$$

$= \int (\sec\theta-1)\,d\theta = \log(\sec\theta+\tan\theta)-\theta+C$

$= \boldsymbol{\log\left(e^x+\sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + C.}$

28. $\int \frac{dx}{\sqrt{\sin^3 x\sin(x+\alpha)}}$ is equal to

(a) $2\ \mathrm{cosec}\,\alpha\ \sqrt{\cos\alpha+\sin\alpha\cot x} + C$

(b) $-2\ \mathrm{cosec}\,\alpha\ \sqrt{\cos\alpha+\sin\alpha\cot x} + C$

(c) $\mathrm{cosec}\,\alpha\ \sqrt{\cos\alpha+\sin\alpha\cot x} + C$

(d) None of these

Solution (b) $\int \frac{dx}{\sin^3 x\sin(x+\alpha)}$

$$= \int \frac{dx}{\sqrt{\sin^3 x.(\sin x\cos\alpha+\cos x\sin\alpha)}}$$

$$= \int \frac{\mathrm{cosec}^2 x}{\sqrt{\cos\alpha+\sin\alpha\cot x}}\,dx = \frac{-2}{\sin\alpha}\int \frac{z}{z}\,dz$$

[Putting $\cos\alpha + \sin\alpha\cot x = z^2$
$\Rightarrow -\sin\alpha\ \mathrm{cosec}^2 x\,dx = 2z\,dz$]

$= \boldsymbol{-2\,\mathrm{cosec}\,\alpha.}\ \sqrt{\cos\alpha+\sin\alpha\cot x} + C.$

29. Let $f(x) = \int \frac{dx}{(1+x^2)^{3/2}}$ and $f(0) = 0$, then $f(1) =$

(a) $\frac{-1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) none of these

Solution (b) We have,

$$f(x) = \int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

[Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$]

$= \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{1+x^2}} + C.$

Since $f(0) = 0$, therefore, $C = 0$.

Thus, $f(x) = \frac{x}{\sqrt{1+x^2}}$.

Hence, $f(1) = \boldsymbol{\frac{1}{\sqrt{2}}}$.

30. If $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx = \frac{f(x)}{x\sin x+\cos x} + \tan x + C$, then

(a) $f(x) = \frac{x}{\cos x}$

(b) $f(x) = \frac{\cos x}{x}$

(c) $f(x) = \frac{-x}{\cos x}$

(d) none of these

Solution (c) Put $\frac{1}{x\sin x+\cos x} = z$

$\Rightarrow \frac{-1}{(x\sin x+\cos x)^2}(\sin x + x\cos x - \sin x)\,dx = dz$

$\Rightarrow \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx = -dz,$

$\therefore \int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx = -z.$

$\therefore \int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$

$= \int \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$

$= \frac{x}{\cos x}\cdot(-z) - \int(-z)\left(\frac{\cos x.1 + x\sin x}{\cos^2 x}\right)dx$

$= \frac{-x}{\cos x\,(x\sin x+\cos x)} + \int \sec^2 x\,dx$

$= \frac{-x}{\cos x\,(x\sin x+\cos x)} + \tan x + C.$

$\therefore\ f(x) = \boldsymbol{\frac{-x}{\cos x}}.$

Problems for Practice

Level of Difficulty I

1. $\int \frac{dx}{x\left(x^n+1\right)}$ is equal to

(a) $\log x - \frac{1}{n}\log(x^n+1) + C$

(b) $\frac{1}{n}\log\left(\frac{x^n+1}{x^n}\right) + C$

(c) $\log x + \frac{1}{n}\log(x^n+1) + C$

(d) None of these

2. $\int \frac{e^x\left(1+\sin x\right)}{\left(1+\cos x\right)}\,dx$ is equal to

(a) $e^x \cot x$ (b) $\sin(\log x)$

(c) $e^x \tan\frac{x}{2}$ (d) $\log \tan x$

3. If $\int \frac{2^{1/x}}{x^2}\,dx = K.2^{1/x}$, then K is equal to

(a) $\frac{-1}{\log 2}$ (b) $-\log 2$

(c) -1 (d) $\frac{1}{2}$

4. $\int \left[\sin(\log x) + \cos(\log x)\right]\,dx$ is equal to

(a) $\sin(\log x) + \cos(\log x) + C$

(b) $x\sin(\log x) + C$

(c) $x\cos(\log x) + C$

(d) none of these

5. $\int 7^{7^{7^x}}.7^{7^x}\cdot 7^x\,dx$ is equal to

(a) $\frac{7^{7^{7^x}}}{(\log 7)^3} + C$ (b) $\frac{7^{7^{7^x}}}{(\log 7)^2} + C$

(c) $7^{7^{7^x}}\cdot(\log 7)^3 + C$ (d) none of these

6. $\int \log\left(x+\sqrt{x^2+a^2}\right)\,dx$ is equal to

(a) $x\log\left(x+\sqrt{x^2+a^2}\right)+\sqrt{x^2+a^2} + C$

(b) $x\log\left(x+\sqrt{x^2+a^2}\right)-2\sqrt{x^2+a^2} + C$

(c) $x\log\left(x+\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2} + C$

(d) none of these

7. $\int \frac{\sqrt{1-\sin x}}{1+\cos x}\cdot e^{-x/2}\,dx$ is equal to

(a) $\sec\frac{x}{2}\cdot e^{-x/2} + C$ (b) $-\sec\frac{x}{2}\cdot e^{-x/2} + C$

(c) $\tan\frac{x}{2}\cdot e^{-x/2} + C$ (d) none of these

8. $\int \frac{e^{\tan^{-1}x}}{\left(1+x^2\right)}(1+x+x^2)\,dx$ is equal to

(a) $\frac{e^{\tan^{-1}x}}{1+x^2}$ (b) $e^{\tan^{-1}x}\cdot(1+x^2)$

(c) $xe^{\tan^{-1}x}$ (d) none of these

9. $\int \sin x\, d\left(\cos x\right)$ is equal to

(a) $\frac{\sin 2x}{2} - x + C$

(b) $\frac{1}{2}\left(\frac{\sin 2x}{2} - x\right) + C$

(c) $\frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + C$

(d) none of these

10. $\int \frac{d^2}{dx^2}(\tan^{-1}x)\,dx$ is equal to

(a) $\frac{1}{1+x^2} + C$

(b) $\tan^{-1}x + C$

(c) $x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$

(d) none of these

11. If $\int \left(\sin 2x + \cos 2x\right)\,dx = \frac{1}{\sqrt{2}}\sin(2x-c) + a$, then

(a) $c = \frac{\pi}{2}$, a = arbitrary constant

(b) $c = \frac{\pi}{6}$, a = arbitrary constant

(c) $c = \frac{\pi}{4}$, a = arbitrary constant

(d) none of these

12. If $\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right) + b$, then

(a) $a = -\frac{\pi}{4}$, b = arbitrary constant

(b) $a = \frac{\pi}{4}$, b = arbitrary constant

(c) $a = \frac{\pi}{2}$, b = arbitrary constant

(d) none of these

13. $\int e^x\left(\frac{1-x}{1+x}\right)^2\,dx$ is equal to

(a) $e^x\left(\frac{1-x}{1+x^2}\right) + C$ (b) $e^x\left(\frac{x-1}{1+x^2}\right) + C$

(c) $e^x.\frac{1}{1+x^2} + C$ (d) none of these

14. If $\int \left(\log(\log x) + \frac{1}{(\log x)^2} \right) dx = x\,[f(x) - g(x)] + C$, then

(a) $f(x) = \log(\log x),\ g(x) = \frac{1}{\log x}$

(b) $f(x) = \log x,\ g(x) = \frac{1}{\log x}$

(c) $f(x) = \frac{1}{\log x},\ g(x) = \log(\log x)$

(d) none of these

15. $\int e^x \left(\frac{x^3 + x + 1}{\left(1 + x^2\right)^{3/2}} \right) dx$ is equal to

(a) $\frac{x^2 e^x}{\left(1 + x^2\right)^{1/2}} + c$ (b) $\frac{e^x}{x\left(1 + x^2\right)^{1/2}} + c$

(c) $\frac{e^x}{\left(1 + x^2\right)^{1/2}} + c$ (d) $\frac{x e^x}{\left(1 + x^2\right)^{1/2}} + c$

16. The value of $\int \frac{\sqrt{1 - x}}{x}\, dx$ is

(a) $2\sqrt{1 + x} + \ln \left| \frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1} \right| + c$

(b) $\ln \left(\frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1} \right) + c$

(c) $2\sqrt{1 + x} + c$

(d) $\frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1} + c$

17. $\int \frac{dx}{\sin x . \sin(x + \alpha)}$ is equal to

(a) $\operatorname{cosec}\alpha \log \left| \frac{\sin x}{\sin(x + \alpha)} \right| + C$

(b) $\operatorname{cosec}\alpha \log \left| \frac{\sin(x + \alpha)}{\sin x} \right| + C$

(c) $\operatorname{cosec}\alpha \log \left| \frac{\sec(x + \alpha)}{\sec x} \right| + C$

(d) $\operatorname{cosec}\alpha \log \left| \frac{\sec x}{\sec(x + \alpha)} \right| + C.$

18. $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, dx$ is equal to

(a) $2\sqrt{1 - x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + C$

(b) $2\sqrt{1 - x} - \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + C$

(c) $-2\sqrt{1 - x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + C$

(d) none of these

19. $\int \frac{\sqrt[2]{x}}{1 + \sqrt[4]{x^3}}\, dx$ is equal to

(a) $\frac{4}{3}\left[1 + x^{3/4} + \log(1 + x^{3/4})\right] + C$

(b) $\frac{4}{3}\left[1 + x^{3/4} - \log(1 + x^{3/4})\right] + C$

(c) $\frac{4}{3}\left[1 - x^{3/4} + \log(1 + x^{3/4})\right] + C$

(d) none of these

20. $\int \frac{1 + x}{1 + \sqrt[3]{x}}\, dx$ is equal to

(a) $\frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} + x + C$

(b) $\frac{3}{5}x^{5/3} + \frac{3}{4}x^{4/3} + x + C$

(c) $\frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} - x + C$

(d) none of these

Level of Difficulty II

21. $\int \frac{\sin^3 x\, dx}{\left(\cos^4 x + 3\cos^2 x + 1\right)\tan^{-1}\left(\sec x + \cos x\right)} =$

(a) $\tan^{-1}(\sec x + \cos x) + c$

(b) $\log \tan^{-1}(\sec x + \cos x) + c$

(c) $\frac{1}{\left(\sec x + \cos x\right)^2} + c$

(d) none of these

22. If $\left(\frac{3x - 4}{3x + 4} \right) = x + 2$, then $\int f(x)\, dx$ is equal to

(a) $e^{x-2} \ln \left| \frac{3x - 4}{3x + 4} \right| + c$ (b) $-\frac{8}{3}\ln|x - 1| + \frac{2}{3}x + c$

(c) $\frac{8}{3}\ln|x - 1| + \frac{x}{3} + c$ (d) none of these

23. $\int \sqrt[3]{x}\ \sqrt[7]{1 + \sqrt[3]{x^4}}\, dx$ is equal to

(a) $\frac{21}{32}\left(1 + \sqrt[3]{x^4}\right)^{8/7} + C$

(d) $\frac{32}{21}\left(1 + \sqrt[3]{x^4}\right)^{8/7} + C$

(c) $\frac{7}{32}\left(1 + \sqrt[3]{x^4}\right)^{8/7} + C$

(d) none of these

24. $\int \frac{1 + x^4}{(1 - x^4)^{3/2}}\, dx =$

(a) $\frac{1}{\sqrt{x^2 - \frac{1}{x^2}}} + c$ (b) $\frac{1}{\sqrt{\frac{1}{x^2} - x^2}} + c$

(c) $\frac{1}{\sqrt{\frac{1}{x^2} + x^2}} + c$ (d) none of these

25. $\int \frac{dx}{x\sqrt{1-x^3}} =$

(a) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+c$

(b) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right|+c$

(c) $\frac{1}{3}\log\left|\frac{1}{\sqrt{1-x^3}}\right|+c$

(d) $\frac{1}{3}\log\left|1-x^3\right|+c$

26. $\int\left(\sqrt{\frac{a+x}{a-x}}-\sqrt{\frac{a-x}{a+x}}\right)dx$ is equal to

(a) $-\sqrt{a^2-x^2}+C$

(b) $\sqrt{a^2-x^2}+C$

(c) $-\sqrt{x^2-a^2}+C$

(d) none of these

27. $\int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx$ is equal to

(a) $\frac{4}{3}\left(\frac{2+x}{2-x}\right)^{2/3}+C$

(b) $\frac{3}{4}\left(\frac{2+x}{2-x}\right)^{2/3}+C$

(c) $\frac{3}{4}\left(\frac{2-x}{2+x}\right)^{2/3}+C$

(d) none of these

28. $\int \frac{x^4+1}{x^6+1}\,dx$ is equal to

(a) $\tan^{-1}x+\frac{1}{3}\tan^{-1}x^3+C$

(b) $\tan^{-1}x-\frac{1}{3}\tan^{-1}x^3+C$

(c) $-\tan x-\frac{1}{3}\tan^{-1}x^3+C$

(d) none of these

29. $\int \frac{(x+1)}{x\left(1+xe^x\right)^2}\,dx$ is equal to

(a) $\log\left(\frac{xe^x}{1+xe^x}\right)+\frac{1}{1+xe^x}+C$

(b) $\log\left(\frac{x}{1+xe^x}\right)+\frac{1}{1+xe^x}+C$

(c) $\log\left(\frac{1+xe^x}{xe^x}\right)+\frac{1}{1+xe^x}+C$

(d) none of these

30. $\int[1+\tan x.\tan(x+\alpha)\;dx$ is equal to

(a) $\cot\alpha\,.\log\left|\frac{\sin x}{\sin(x+\alpha)}\right|+C$

(b) $\tan\alpha.\log\left|\frac{\sin x}{\sin(x+\alpha)}\right|+C$

(c) $\cot\alpha.\log\left|\frac{\sin(x+\alpha)}{\sin x}\right|+C$

(d) none of these

31. $\int \frac{dx}{\sin^4 x+\cos^4 x}$ is equal to

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan 2x\right)+C$

(b) $\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan 2x\right)+C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\cot 2x\right)+C$

(d) none of these

32. If $\int \frac{(\sin\theta-\cos\theta)}{(\sin\theta+\cos\theta)\sqrt{\sin\theta\cos\theta+\sin^2\theta\cos^2\theta}}\,d\theta$

$=\operatorname{cosec}^{-1}[f(\theta)]+C$, then

(a) $f(\theta)=\sin 2\theta+1$

(b) $f(\theta)=1-\sin 2\theta$

(c) $f(\theta)=\sin 2\theta-1$

(d) none of these

33. The value of $\int \frac{dx}{x^n\left(1+x^n\right)^{1-n}}$, $n \in N$, is

(a) $\frac{1}{1-n}\left\{1+\frac{1}{x^n}\right\}^{1-\frac{1}{n}}+c$

(b) $\frac{1}{1+n}\left\{1-\frac{1}{x^n}\right\}^{1-\frac{1}{n}}+c$

(c) $-\frac{1}{1-n}\left\{1-\frac{1}{x^n}\right\}^{1-\frac{1}{n}}+c$

(d) $-\frac{1}{n-1}\left\{1+\frac{1}{x^n}\right\}^{1-\frac{1}{n}}+c$

34. $\int \frac{x^2}{\left(a+bx^2\right)^{5/2}}\,dx$ is equal to

(a) $-\frac{1}{3a}\left(\frac{x^2}{a+bx^2}\right)^{3/2}+C$

(b) $\frac{1}{3a}\left(\frac{x^2}{a+bx^2}\right)^{3/2}+C$

(c) $\frac{1}{2a}\left(\frac{x^2}{a+bx^2}\right)^{2/3}+C$

(d) none of these

35. Integral of $\frac{1}{1+(\log x)^2}$ with respect to $\log x$ is

(a) $\frac{\tan^{-1}(\log x)}{x} + C$ (b) $\tan^{-1}(\log x) + C$

(c) $\frac{\tan^{-1} x}{x} + C$ (d) none of these

36. The antiderivative of the function $(3x + 4)\,|\sin x|$, when $0 < x < \pi$, is given by

(a) $3\sin x - (3x + 4)\cos x$
(b) $3\sin x + (3x + 4)\cos x$
(c) $-3\sin x + (3x + 4)\cos x$
(d) none of these

37. The equation of a curve passing through origin is given by $y = \int x^3 \cos x^4\, dx$. If the equation of the curve is written in the form $x = g(y)$, then

(a) $g(y) = \sqrt[3]{\sin^{-1}(4y)}$

(b) $g(y) = \sqrt{\sin^{-1}(4y)}$

(c) $g(y) = \sqrt[4]{\sin^{-1}(4y)}$

(d) none of these

38. $\int x.\left(x^x\right)^x.\ (2\log x + 1)\, dx$ is equal to

(a) $x^{(x^x)} + C$ (b) $(x^x)^x + C$

(c) $x^x.\ \log x + C$ (d) none of these

39. $\int \frac{dx}{x^{1/2}\left(1+x^2\right)^{5/4}}$ is equal to

(a) $\frac{-2\sqrt{x}}{\sqrt[4]{1+x^2}} + C$ (b) $\frac{2\sqrt{x}}{\sqrt[4]{1+x^2}} + C$

(c) $\frac{-\sqrt{x}}{\sqrt[4]{1+x^2}} + C$ (d) $\frac{\sqrt{x}}{\sqrt[4]{1+x^2}} + C$

40. If $\int \frac{\cos 8x + 1}{\tan 2x - \cot 2x}\, dx = a\cos 8x + C$, then

(a) $a = \frac{-1}{16}$ (b) $a = \frac{1}{8}$

(c) $a = \frac{1}{16}$ (d) $a = \frac{-1}{8}$.

ANSWERS

1. (a)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (a)
11. (c)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (d)	**16.** (a)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (a)
21. (d)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (c)
31. (a)	**32.** (a)	**33.** (d)	**34.** (b)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (b)	**39.** (b)	**40.** (c)

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Definite Integral and Area

BRIEF REVIEW OF THE CONCEPTS

DEFINITE INTEGRAL

Let $F(x)$ be any antiderivative of $f(x)$, then for any two values of the independent variable x, say a and b, the difference $F(b) - F(a)$ is called the *definite integral* of $f(x)$ from a to b and is denoted

by $\int_a^b f(x)\,dx$. Thus

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, The numbers a and b are called the *limits of integration*; a is the lower limit and b is the upper limit. Usually $F(b) - F(a)$ is abbreviated by writing $F(x)\big|_a^b$.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $f(x)$ is continuous on $[a, b]$ and if $F(x)$ be antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The reason for using the term *definite integral* follows from the fact that the value of definite integral is independent of the particular choice of the antiderivative of $f(x)$. For if $F(x) + c$ is any other antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x) + c\,\Big|_a^b = [F(b) + c] - [F(a) + c]$$
$$= F(b) - F(a),$$

which is same as before.

Working Rule to Evaluate $\int_a^b f(x)\,dx$

Step 1: *Evaluate the indefinite integral* $\int_a^b f(x)\,dx$ *and omit the constant of integration. Let this be* $F(x)$.

Step 2: *Evaluate* $F(b)$ *and* $F(a)$.

Step 3: *Calculate* $F(b) - F(a)$.

Then, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Evaluation of Definite Integrals by Substitution

In this method, we make a substitution $f(x) = t$ to reduce the given integral to a known form of integral.

Note that when we make a substitution in a problem of definite integral, we have to change the limits according to the substitution.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ **2.** $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

4. $\int_o^a f(x)\,dx = \int_o^a f(a-x)\,dx$

5. $\int_o^a \dfrac{f(x)}{f(x) + f(a-x)}\,dx = \dfrac{a}{2}$

6. $\int_o^{2a} f(x)\,dx = \int_o^a f(x)\,dx + \int_o^a f(2a-x)\,dx$

7. $\int_o^{2a} f(x)\,dx = \begin{Bmatrix} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_o^a f(x)\,dx & \text{if } f(2a-x) = f(x) \end{Bmatrix}$

8. $\int_{-a}^a f(x)\,dx$

$$= \begin{Bmatrix} 2\int_o^a f(x)\,dx \text{ if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0 \qquad \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{Bmatrix}$$

9. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

10. $\int_a^b \dfrac{f(x)}{f(x) + f(a+b-x)} = \dfrac{b-a}{2}$

11. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

12. If $f(x) \leq g(x)$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

13. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

14. If $f(x)$ is continuous on $[a, b]$, m is the least and M is the greatest value of $f(x)$ on $[a, b]$, then

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$$

15. If $f(x)$ is a periodic function of period a, i.e.,

$f(a + x) = f(x)$, then

(a) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(b) $\int_a^{na} f(x)\,dx = (n-1)\int_0^a f(x)\,dx$

(c) $\int_{na}^{b+na} f(x)\,dx = \int_0^b f(x)\,dx$, where $b \in \mathrm{R}^+$

(d) $\int_b^{b+a} f(x)\,dx$ is independent of b.

(e) $\int_b^{b+na} f(x)dx = n\int_0^a f(x)dx$, where $n \in I$

In particular,

(i) if $b = 0$, $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$

(ii) if $n = 1$, $\int_b^{b+a} f(x)dx = \int_0^a f(x)dx$

16. For any two functions $f(x)$ and $g(x)$, integrable on the interval $[a, b]$, the Schwarz–Bunyakovsky inequality holds

$$\left|\int_a^b f(x)\cdot g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)\,dx\cdot\int_a^b g^2(x)\,dx}$$

17. If a function $f(x)$ is continuous on the interval $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a), \text{ where } a < c < b.$$

Useful Integral

$$\int_0^{\pi/2}\log \sin x\,dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}.$$

DIFFERENTIATION UNDER INTEGRAL SIGN

If the functions $g(x)$ and $h(x)$ are defined on $[a, b]$ and differentiable at all points $x \in (a, b)$ and $f(t)$ is continuous on $[a, b]$, then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = \frac{d}{dx}[h(x)]\cdot f[h(x)] - \frac{d}{dx}[g(x)]\cdot f[g(x)]$$

INTEGRATION OF MOD FUNCTIONS

$$\int_a^b |f(x)|\,dx$$

Short-cut Method

- Solve the equation $f(x) = 0$.
  Let the roots be α and β such that $a < \alpha < \beta < b$.
- Divide the interval (a, b) into subintervals (a, α), (α, β) and (β, b).
- Check the sign of $f(x)$ by taking any point in each of these subintervals.
- Use the property

$$\int_a^b f(x)\,dx = \int_a^\alpha f(x)\,dx + \int_\alpha^\beta f(x)\,dx + \int_\beta^b f(x)\,dx$$

with proper sign of $f(x)$ in each (a, α), (α, β), (β, b).

SUMMATION OF SERIES WITH THE HELP OF DEFINITE INTEGRAL AS THE LIMIT OF A SUM

If $f(x)$ is a continuous and single valued function defined on the interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ is defined as follows:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a+h) + f(a+2h) + \cdots + f(a+nh)]$$

where $nh = b - a$.

or $$\lim_{\substack{n\to\infty\\ h\to 0}} h\sum_{r=1}^{n} f(a+rh) = \int_a^b f(x)\,dx \qquad \text{...(1)}$$

Put $a = 0$ and $b = 1 \Rightarrow nh = 1 \Rightarrow h = \frac{1}{n}$.

Substitute $a = 0$, $b = 1$ and $h = \frac{1}{n}$ in (1), we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$$

Short-cut Method

- Find the rth term of the given series and express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)\cdot$ Then the given series can be written as

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

- Write the above series equal to a definite integral by replacing $\frac{1}{n}$ by dx, $\frac{r}{n}$ by x and

$$\lim_{n\to\infty}\sum \text{ by } \int.$$

Also, lower limit of integration $= \lim_{n\to\infty}\frac{r_1}{n}$

and upper limit of integration $= \lim_{n\to\infty}\frac{r_2}{n}$,

where r_1 and r_2 are the least and greatest values of r respectively.

Note: Here each term tends to zero when $n \to \infty$, so addition or omission of finite number of terms does not affect the required limit.

SOME USEFUL REDUCTION FORMULAE

1. If n is a positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{bmatrix} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{2}{3}, \text{ when } n \text{ is odd} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, \text{ when } n \text{ is even} \end{bmatrix}.$$

2. $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$$= \begin{bmatrix} \dfrac{m-1}{m+n}\cdot\dfrac{m-3}{m+n-2}\cdots\dfrac{2}{3+n}\cdot\dfrac{1}{1+n}; \text{ if } m \text{ is odd} \\ \text{and } n \text{ may be even or odd} \\ \dfrac{m-1}{m+n}\cdot\dfrac{m-3}{m+n-2}\cdots\dfrac{1}{2+n}\cdot\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3} \\ \text{if } m \text{ is even and } n \text{ is odd} \\ \dfrac{m-1}{m+n}\cdot\dfrac{m-3}{m+n-2}\cdots\dfrac{1}{2+n}\cdot\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, \\ \text{if } m \text{ is even and } n \text{ is even} \end{bmatrix}$$

These formulae can be expressed as a single formula:

$$\int_0^{\pi/2} \sin^m x\cdot\cos^n x\,dx$$

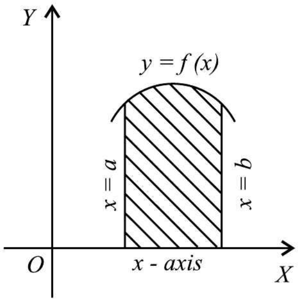
$$= \frac{[(m-1)(m-3)\ldots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots}$$

to be multiplied by $\dfrac{\pi}{2}$ when m and n are both even integers.

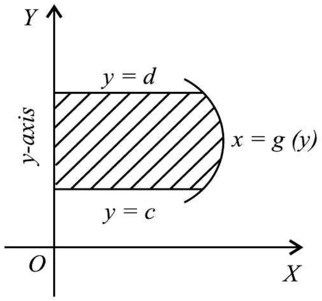
AREA OF PLANE REGIONS

1. The area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$. and $x = b$ (where $b > a$) is given by

$$A = \int_a^b y\ dx = \int_a^b f(x)\ dx$$

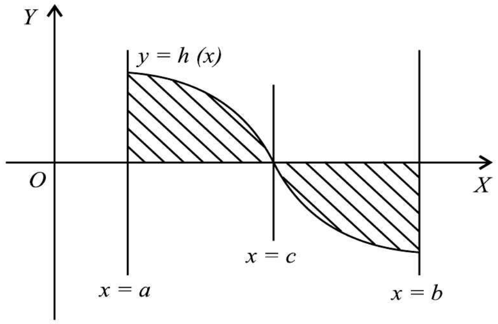


2. The area bounded by the curve $x = g(y)$, y-axis and the abscissae $y = c$ and $y = d$ (where $d > c$) is given by



$$A = \int_c^d x\,dy = \int_c^d g(y)\ dy.$$

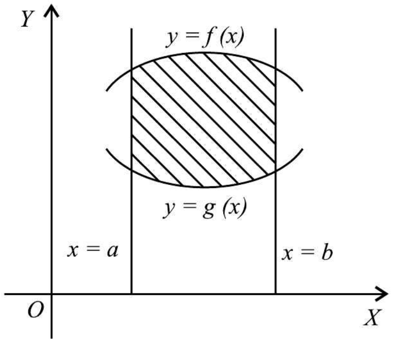
3. The area bounded by the curve $y = h(x)$, x-axis and the two ordinates $x = a$ and $x = b$ is given by



$$A = \left|\int_a^c y\,dx\right| + \left|\int_c^b y\,dx\right|$$

where c is a point in between a and b.

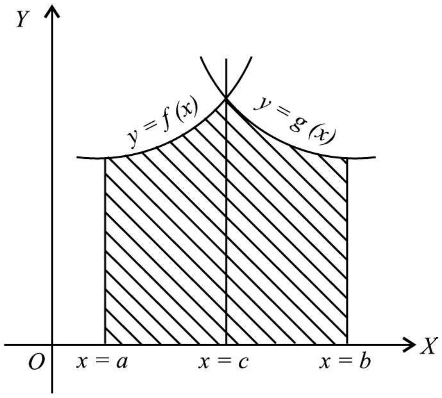
4. If we have two curves $y = f(x)$ and $y = g(x)$, such that $y = f(x)$ lies above the curve $y = g(x)$ and both are above the axis of x then the area bounded between them and the ordinates $x = a$ and $x = b$ $(b > a)$, is given by



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\ dx$$

i.e., upper curve area–lower curve area.

5. The area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$ is given by



$$A = \int_a^c f(x)\,dx + \int_c^b g(x)\ dx,$$

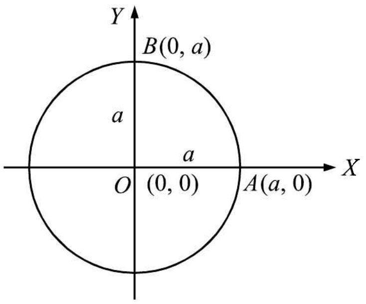
where $x = c$ is the point of intersection of the two curves.

Some Well-known Curves

1. **Circle**

(a) $x^2 + y^2 = a^2$

This represents a circle; the centre at (0, 0) and the radius a. It is symmetric about both the axes.



(b) $(x - \alpha)^2 + (y - \beta^2) = a^2$

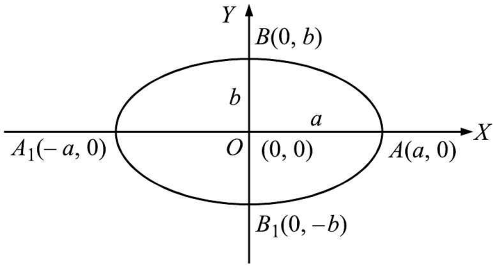
This is a circle; the centre at (α, β) and the radius a,

(c) $x^2 + y^2 + 2gx + 2fy + c = 0$ (general equation). This is a circle; centre at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

2. **Ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

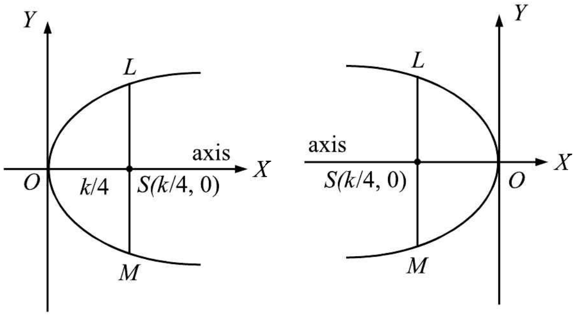
It is symmetric about both the axes, meeting x-axis at $(\pm a, 0)$ and y-axis at $(0, \pm b)$, where $b < a$. Here $AA_1 = 2a$ = length of major axis, $BB_1 = 2b$ = length of minor axis.



3. **Parabola**

$y^2 = kx$ or $x^2 = ky$ where $k > 0$ or $k < 0$. (standard equation)

(a)

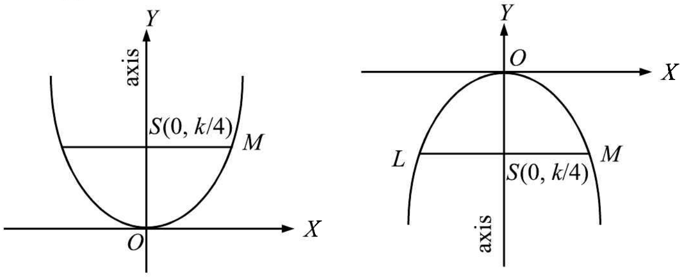


$y^2 = kx, k > 0$ $y^2 = kx, k > 0$

(a) **(b)**

It is symmetric about x-axis, where S is the focus, O is the vertex, LM is the latus rectum; $LM \perp OX$; O being the axis of the parabola.

(b)

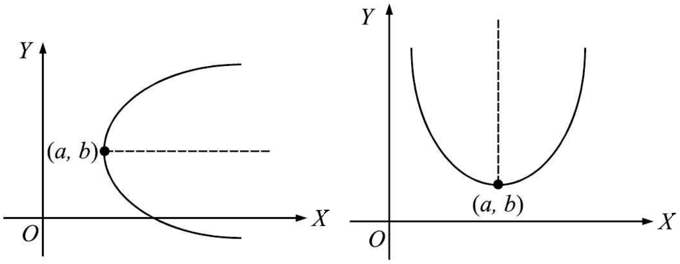


$x^2 = ky, k > 0$ $x^2 = ky, k < 0$

(a) **(b)**

It is symmetric about y-axis, where S is the focus, O is the vertex, LM is the latus-rectum, $LM \perp OY$.

(c)

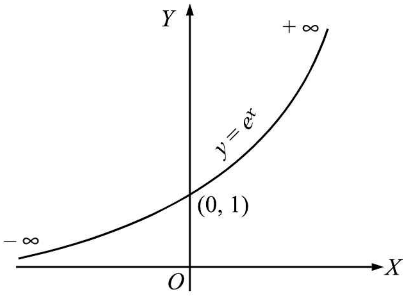


$(y - b)^2 = k(x - a), k, a, b > 0$ $(x - a)^2 = k(y - b), k, a, b > 0$

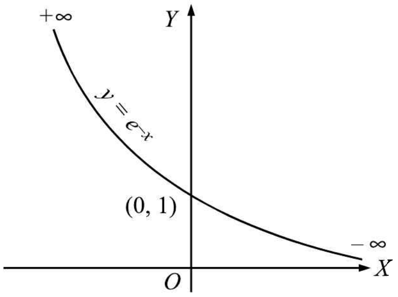
(a) **(b)**

(d) $x = ay^2 + by + c$ or $y = ax^2 + bx + c$ (general equation) change it in the form either (c) or (d).

4. **Exponential curves**

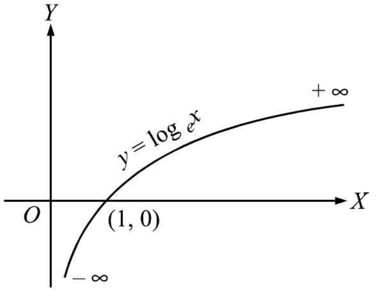


(a)

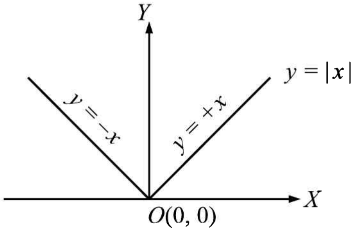


(b)

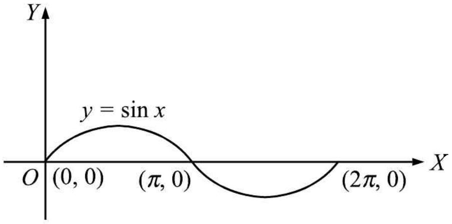
5. Logarithmic curves



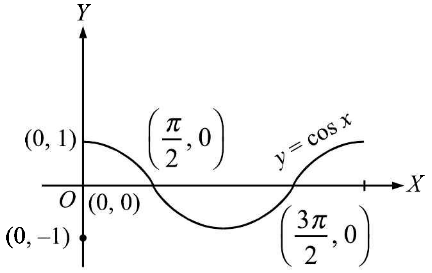
6. Modulus function



7. Trigonometric curves



(a)



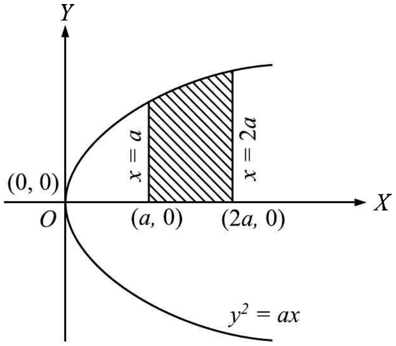
(b)

Equation (1) represents the parabola whose vertex is (0, 0) and axis is x-axis.

Equation (2) represents x-axis.

Equation (3) represents the line passing through (a, 0) and parallel to y-axis.

Equation (4) represents the line passing through (2a, 0) and parallel to y-axis.



Required area $= \int\limits_{x=a}^{x=2a} y\,dx = \int\limits_{a}^{2a} 2\sqrt{a}\sqrt{x}\,dx = 2\sqrt{a}\cdot\frac{2}{3}\left[x^{3/2}\right]_a^{2a}$

$= \frac{4}{3}(2\sqrt{2}-1)a^2$ sq.units.

Solved Problems

1. If $\int\limits_0^1 e^{x^2}(x-\alpha)\,dx = 0$, then

(a) $1 < \alpha < 2$ (b) $\alpha < 0$

(c) $0 < \alpha < 1$ (d) $\alpha = 0$

Solution (c) $\because \int\limits_0^1 e^{x^2}(x-\alpha)\,dx = 0$, $\therefore$ $e^{x2}(x - \alpha)$ must be +*ve* and –*ve* both for $x \in (0, 1)$ i.e., $e^x(x - \alpha) = 0$ for one $x \in (0, 1)$

$\therefore$ $\alpha \in$ **(0, 1).**

2. The value of the integer $\int\limits_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\,dx$, n integer, is

(a) 0 (b) π

(c) 2π (d) none of these

Solution (a) Let $I = \int\limits_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\,dx$

$$= \int\limits_0^{\pi} e^{\cos^2(\pi-x)}\cdot\cos^3(2n+1)(\pi-x)\,dx$$

$$= \int\limits_0^{\pi} e^{\cos^2 x}\cdot\cos^3\left[(2n+1)\pi-(2n+1)x\right]dx$$

$$= -\int\limits_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx = -I.$$

$\therefore$ $2I = 0 \Rightarrow I = \mathbf{0}.$

3. $\int\limits_0^{\infty}\left[\frac{2}{e^x}\right]dx$ (where [.] denotes the greatest integer function) equals

(a) $\log_e 2$ (b) e^2

(c) 0 (d) $\frac{2}{e}$

Solution (a) $\int_0^{\infty}\left[\frac{2}{e^x}\right]dx = \int_0^{\ln 2}\left[\frac{2}{e^x}\right]dx + \int_{\ln 2}^{\infty}\left[\frac{2}{e^x}\right]dx$

$= \int_0^{\ln 2} 1dx + 0 = \mathbf{\ln 2}$

4. If $\int_{-1}^{-4} f(x)\,dx = 4$ and $\int_{2}^{-4} (3 - f(x))\,dx = 7$, then the value of $\int_{-2}^{1} f(-x)\,dx$ is

(a) 30 (b) 29

(c) 28 (d) none of these

Solution (b) $\int_{-1}^{-4} f(x)\,dx = 4$ and $\int_{2}^{-4} (3 - f(x))\,dx = 7$,

$\Rightarrow \int_{-4}^{2} f(x)\,dx = 7 + 18 = 25$

$I = \int_{-2}^{1} f(-x)\,dx = \int_{-4}^{2} f(x)\,dx = \int_{-1}^{-4} f(x)\,dx + \int_{-4}^{2} f(x)\,dx$

$= 4 + 25 = \mathbf{29}.$

5. The value of the integral $\int_0^{\pi}\frac{\sin 2kx}{\sin x}\,dx$, where $k \in I$, is

(a) $\frac{\pi}{2}$ (b) π

(c) 0 (d) none of these

Solution (c) Let $I = \int_0^{\pi}\frac{\sin 2kx}{\sin x}\,dx = \int_0^{\pi}\frac{\sin 2k(\pi - x)}{\sin(\pi - x)}\,dx$

$= \int_0^{\pi}\frac{\sin(2k\pi - 2kx)}{\sin x}\,dx = -\int_0^{\pi}\frac{\sin 2kx}{\sin x}\,dx = -I$

$\therefore\quad 2I = 0 \Rightarrow I = \mathbf{0}.$

6. The value of the integral $\int_0^{\pi}\frac{x^2 \sin x}{(2x - \pi)(1 + \cos^2 x)}\,dx$ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$

(c) $\frac{\pi^2}{6}$ (d) none of these

Solution (a) Let $I = \int_0^{\pi}\frac{x^2 \sin x}{(2x - \pi)(1 + \cos^2 x)}\,dx$

$= \int_0^{\pi}\frac{(\pi - x)^2 \sin(\pi - x)}{(2\pi - 2x - \pi)(1 + \cos^2(\pi - x))}\,dx$

$= \int_0^{\pi}\frac{(\pi^2 - 2\pi x + x^2)\sin x}{(\pi - 2x)(1 + \cos^2 x)}\,dx$

$= \int_0^{\pi}\frac{(\pi^2 - 2\pi x)\sin x}{(\pi - 2x)(1 + \cos^2 x)}\,dx - I$

$\Rightarrow\quad 2I = \pi\int_0^{\pi}\frac{\sin x}{1 + \cos^2 x}\,dx = -\pi\int_1^{-1}\frac{dt}{1 + t^2}$

[Putting $\cos x = t \Rightarrow \sin x\,dx = -dt$]

$= \pi\left[\tan^{-1} t\right]_{-1}^{1} = \pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi^2}{2}$

$\therefore\quad I = \frac{\pi^2}{4}.$

7. Given $\int_1^{2} e^{x^2}\,dx = a$, the value of $\int_e^{e^4}\sqrt{\ln(x)}\,dx$ is

(a) $e^4 - e$ (b) $e^4 - a$

(c) $2e^4 - a$ (d) $2e^4 - e - a$

Solution (d) Given $\int_1^{2} e^{x^2}\,dx = a$, let $I = \int_e^{e^4}\sqrt{\ln(x)}\,dx$

Put $\ln(x) = t^2 \Rightarrow \frac{1}{x}\,dx = 2t\,dt$

$\therefore\quad I = \int_1^{2} e^{t^2}\cdot 2t^2\,dt = \left(t\,e^{t^2}\right)_1^2 - \int_1^{2} e^{t^2}\,dt$

$= \mathbf{2e^4 - e - a}.$

8. $\int_0^{\pi/3}\cos^4 3\phi.\ \sin^2 6\phi\ d\phi$ is equal to

(a) $\frac{\pi}{32}$ (b) $\frac{7\pi}{96}$

(c) $\frac{5\pi}{96}$ (d) none of these

Solution (c) $\int_0^{\pi/3}\cos^4 3\phi.\ \sin^2 6\phi\ d\phi$

$= \frac{1}{3}\int_0^{\pi}\cos^4\theta\cdot\sin^2 2\theta\,d\theta$

$\left[\text{Putting } 3\phi = \theta \Rightarrow d\phi = \frac{1}{3}d\theta\right]$

$= \frac{1}{3}\int_0^{\pi}\cos^4\theta\cdot(2\sin\theta\cos\theta)^2\,d\theta$

$= \frac{4}{3}\int_0^{\pi}\cos^6\theta\cdot\sin^2\theta\,d\theta$

$= \frac{4}{3}\times 2\int_0^{\pi/2}\cos^6\theta\cdot\sin^2\theta\,d\theta = \frac{8}{3}\cdot\frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5\pi}{96}.$

9. $\int_0^{1.5}[x^2]\,dx$, where [.] denotes the greatest in integer function, is equal to

(a) $\sqrt{2} - 2$ (b) $2 - \sqrt{2}$

(c) $2 + \sqrt{2}$ (d) none of these

Solution (b) $\int_0^{1.5}[x^2]\,dx = \int_0^1[x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{1.5}[x^2]\,dx$

$$\left[\begin{aligned}\because [x^2] &= 0, 0 \le x < 1\\ &\quad 1, 1 \le x < \sqrt{2}\\ &\quad 2, \sqrt{2} \le x < 1.5\end{aligned}\right]$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\cdot dx + \int_{\sqrt{2}}^{1.5} 2\ \ dx = (x)\Big|_1^{\sqrt{2}} + 2\,(x)\Big|_{\sqrt{2}}^{3/2}$$

$$= \sqrt{2} - 1 + 2\left(\frac{3}{2} - \sqrt{2}\right) = \mathbf{2} - \sqrt{\mathbf{2}}.$$

10. $\int_0^3\left[\sqrt{x}\right]\ dx$ is equal to

(a) 1 (b) 2

(c) – 1 (d) –2

Solution (b) $\int_0^3\left[\sqrt{x}\right]dx = \int_0^1\left[\sqrt{x}\right]dx + \int_1^3\left[\sqrt{x}\right]\ dx$

$$\left[\because \left[\sqrt{x}\right] = \begin{cases}0 \text{ if } 0 \le x < 1\\ 1 \text{ if } 1 \le x < 3\end{cases}\right]$$

$$= \int_0^1 0\,dx + \int_1^3 1\,dx = x\Big|_1^3 = \mathbf{2.}$$

11. $\int_0^{2\pi}\sqrt{\frac{1-\cos 2x}{2}}\ dx$ is equal to

(a) 2 (b) – 2

(c) 4 (d) – 4

Solution (c) $\int_0^{2\pi}\sqrt{\frac{1-\cos 2x}{2}}\,dx = \int_0^{2\pi}|\sin x|\,dx$

$$= \int_0^{\pi}|\sin x|\,dx + \int_{\pi}^{2\pi}|\sin x|\,dx$$

$$\left[\begin{aligned}&\because \sin x = 0 \text{ gives } x = n\pi, n = 0, \pm 1, \pm 2, \ldots;\\ &\text{out of which only } x = \pi \in (0, 2\pi)\cdot\end{aligned}\right]$$

$$= \int_0^{\pi}\sin x\,dx - \int_{\pi}^{2\pi}\sin x\ \ dx = -\ [\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$$

$= -\ (-\ 1\ -\ 1) + (1\ +\ 1) = \mathbf{4.}$

12. $\int_{-2}^{2}[x^2]\ dx$ is equal to

(a) $10 - 2\sqrt{3} - 2\sqrt{2}$ (b) $10 + 2\sqrt{3} - 2\sqrt{2}$

(c) $10 - 2\sqrt{3} + 2\sqrt{2}$ (d) none of these

Solution (a) $\int_{-2}^{2}[x^2]\ dx = 2\int_0^2[x^2]\ dx$

[∵ integrand is even]

$$= 2\left[\int_0^1[x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}}[x^2]\,dx + \int_{\sqrt{3}}^{2}[x^2]\,dx\right]$$

$$\left[\begin{aligned}&\because [x^2] = 0 \text{ if } 0 \le x < 1; 1 \text{ if } 1 \le x < \sqrt{2};\\ &\qquad 2 \text{ if } \sqrt{2} \le x < \sqrt{3}; 3 \text{ if } \sqrt{3} \le x < 2\end{aligned}\right]$$

$$= 2\left[\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx\right]$$

$$= 2\ (x)\Big|_1^{\sqrt{2}} + 4\,(x)\Big|_{\sqrt{2}}^{\sqrt{3}} + 6\,(x)\Big|_{\sqrt{3}}^{2}$$

$$= \left(\mathbf{10 - 2\sqrt{3} - 2\sqrt{2}}\right).$$

13. $\int_0^{\pi}|\sin x + \cos x|\ \ dx$ is equal to

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $3\sqrt{2}$ (d) none of these

Solution (b) $\int_0^{\pi}|\sin x + \cos x|\ \ dx$

$$= \int_0^{3\pi/4}|\sin x + \cos x|\,dx + \int_{3\pi/4}^{\pi}|\sin x + \cos x|\ \ dx$$

$$\left[\begin{aligned}&\because \sin x + \cos x = 0 \Rightarrow \cos x = -\sin x\\ &\Rightarrow \tan x - 1 \Rightarrow x = \frac{3\pi}{4} \in (0, \pi)\end{aligned}\right]$$

$$= \int_0^{3\pi/4}(\sin x + \cos x)\,dx - \int_{3\pi/4}^{\pi}(\sin x + \cos x)\ \ dx$$

$$= (-\cos x + \sin x)\Big|_0^{3\pi/4} - (-\cos x + \sin x)\Big|_{3\pi/4}^{\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 1\right] - \left[1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= \mathbf{2\sqrt{2}}\cdot$$

14. $\int_0^{100\pi}\sqrt{1-\cos 2x}\ \ dx$ is equal to

(a) $100\sqrt{2}$ (b) $50\sqrt{2}$

(c) $200\sqrt{2}$ (d) $400\sqrt{2}$

Solution (c) Let $I = \int_0^{100\pi}\sqrt{1-\cos 2x}\ \ dx = \sqrt{2}\int_0^{100\pi}|\sin x|\ \ dx$

Since $|\sin x|$ is a periodic function of the period π

$$\therefore\quad I = 100\ \sqrt{2}\int_0^{\pi}|\sin x|\ \ dx = 100\ \sqrt{2}\int_0^{\pi}\sin x\ \ dx$$

[∵ $\sin x \ge 0$ in $[0, \pi]$]

$$= 100\ \sqrt{2}\,[-\cos x]_0^{\pi} = \mathbf{200\sqrt{2}}.$$

15. $\int_{-1}^{2}[x-1]\ \ dx$ is equal to

(a) – 3 (b) 3

(c) –2 (d) 2

Solution (a) $\int_{-1}^{2}[x-1]\,dx$

$$= \int_{-1}^{0} [x-1]\,dx + \int_{0}^{1} [x-1]\,dx + \int_{1}^{2} [x-1]\,dx$$

$$\left[\because [x-1] = \begin{array}{l} -2 \text{ if } -1 \le x < 0; -1 \text{ if } 0 \le x < 1 \\ 0 \text{ if } 1 \le x < 2 \end{array}\right]$$

$$= \int_{-1}^{0} (-2)\,dx + \int_{0}^{1} (-1)\,dx + \int_{1}^{2} (0)\,dx = -2\,(x)\big|_{-1}^{0} - (x)\big|_{0}^{1}$$

$= \mathbf{-3.}$

16. $\displaystyle\int_{-20\pi}^{20\pi} |\cos x| \; dx$ is equal to

(a) 40 (b) 20

(c) 60 (d) none of these

Solution (d) Let $I = \displaystyle\int_{-20\pi}^{20\pi} |\cos x| \; dx = 2 \int_{0}^{20\pi} |\cos x| \; dx$

[$\because$ integrand is even]

Since $|\cos x|$ is a periodic function of period π,

$$\therefore \; I = 2 \cdot 20 \int_{0}^{\pi} |\cos x| \; dx$$

$$= 40 \int_{0}^{\pi/2} |\cos x|\,dx + 40 \int_{\pi/2}^{\pi} |\cos x| \; dx$$

$$= 40 \int_{0}^{\pi/2} \cos x\,dx - 40 \int_{\pi/2}^{\pi} \cos x \; dx$$

$$= 40[\sin x]_{0}^{\pi/2} - 40\,[\sin x]_{\pi/2}^{\pi}$$

$= 40\ (1 - 0) - 40\ (0 - 1) = \mathbf{80.}$

17. $\displaystyle\int_{\pi}^{10\pi} |\sin x|\,dx$ is equal to

(a) 18 (b) 20

(c) 40 (d) none of these

Solution (a) Since $|\sin x|$ is a periodic function of period π,

$$\therefore \; \int_{\pi}^{10\pi} |\sin x| \; dx = 9 \int_{0}^{\pi} |\sin x| \; dx = 9 \int_{0}^{\pi} \sin x \; dx$$

[$\because \sin x > 0$ in $(0, \pi)$]

$= -\,9\;[\cos x]_{0}^{\pi} = -\,9\,(-\,1 - 1) = \mathbf{18.}$

18. $\displaystyle\int_{0}^{100} (x - [x]) \; dx$ is equal to

(a) 50 (b) 100

(c) 200 (d) none of these

Solution (a) Since $x - [x]$ is a periodic function of period 1

$$\therefore \; \int_{0}^{100} (x - [x]) \; dx = 100 \int_{0}^{1} (x - [x]) \; dx$$

$$= 100 \int_{0}^{1} x\,dx - 100 \int_{0}^{1} [x] \; dx = 100 \left(\frac{x^2}{2}\right)\Bigg|_{0}^{1} - 100 \int_{0}^{1} 0\,dx$$

$= \mathbf{50.}$

19. If $f(x)$ be a periodic function of period a, then $\displaystyle\int_{a}^{na} f(x) \; dx$ is equal to

(a) $\displaystyle\int_{0}^{a} f(x) \; dx$ (b) $\displaystyle n \int_{0}^{a} f(x) \; dx$

(c) $\displaystyle (n - 1) \int_{0}^{a} f(x) \; dx$ (d) none of these

Solution (c) $\displaystyle\int_{a}^{na} f(x) \; dx$

$$= \int_{a}^{2a} f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \cdots + \int_{(n-1)a}^{na} f(x) \; dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx + \cdots + \int_{0}^{a} f(x)$$

$$= \boldsymbol{(n - 1) \int_{0}^{a} f(x) \; dx.}$$

20. $\displaystyle\int_{0}^{10\pi} (|\sin x| + |\cos x|) \; dx$ is equal to

(a) 20 (b) 40

(c) 10 (d) none of these

Solution (b) $\displaystyle\int_{0}^{10\pi} (|\sin x| + |\cos x|)\,dx$

$$= \int_{0}^{20\left(\frac{\pi}{2}\right)} (|\sin x| + |\cos x|) \; dx$$

$$= 20 \int_{0}^{\pi/2} (|\sin x| + |\cos x|) \; dx$$

$$\left[\begin{array}{c} \because |\sin x| + |\cos x| \text{ is a periodic} \\ \text{function of period } \dfrac{\pi}{2} \end{array}\right]$$

$$= 20 \int_{0}^{\pi/2} (\sin x + \cos x)\,dx$$

$$\left[\because \text{ in } \left(0, \frac{\pi}{2}\right) \text{ both } \sin x \text{ and } \cos x \text{ are positive}\right]$$

$= 20\ \left[-\cos x + \sin x\right]_{0}^{\pi/2} = 20\ (1 + 1) = \mathbf{40.}$

21. $\displaystyle\int_{-100\pi}^{100} \left(\sin^4 x + \cos^4 x\right) \; dx$ is equal to

(a) 100π (b) 150π

(c) 200π (d) none of these

Solution (b) $\displaystyle\int_{-100\pi}^{100} \left(\sin^4 x + \cos^4 x\right) \; dx$

$$= 2 \int_{0}^{100\pi} (\sin^4 x + \cos^4 x) \; dx$$

($\because$ integrand is even)

$$= 2 \int_{0}^{\frac{\pi}{2}(200)} (\sin^4 x + \cos^4 x) \; dx$$

$$= 2\cdot 200 \int_0^{\pi/2} (\sin^4 x + \cos^4 x)\ dx$$

$$\left[\because \sin^4 x + \cos^4 x \text{ is a periodic function of period } \frac{\pi}{2}\right]$$

$$= 400\left[\int_0^{\pi/2} \sin^4 x + \int_0^{\pi/2} \cos^4\left(\frac{\pi}{2} - x\right) dx\right]$$

$$= 800\cdot \int_0^{\pi/2} \sin^4 x\ \ dx = 800\cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \mathbf{150\pi}.$$

22. If $f(x)$ and $\phi(x)$ are continuous functions on the interval [0, 4] satisfying $f(x) = f(4 - x)$, $\phi(x) + \phi(4 - x) = 3$ and

$\int_0^4 f(x)\ dx = 2$, then $\int_0^4 f(x)\ \phi(x)\ dx =$

(a) 3 (b) 6

(c) 2 (d) none of these

Solution (a) $\int_0^4 f(x)\,\phi(x)\ dx = \int_0^4 f(4 - x)\,\phi(4 - x)\ dx$

$$= \int_0^4 f(x)\cdot(3 - \phi(x))\ dx$$

$$\left[\begin{array}{l}\because f(x) = f(4 - x)\\ \text{and } \phi(x) + \phi(4 - x) = 3\end{array}\right]$$

$$= 3\int_0^4 f(x)\, dx\ - I \Rightarrow 2\ \text{I} = 3\cdot 2 \quad \therefore\ I = \mathbf{3}.$$

23. If $2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1}{x} - 2\ \ x \neq 0$, then $\int_1^2 f(x)\ dx$ is equal to

(a) $\frac{-2}{5}\log 2 + \frac{1}{2}$ (b) $\frac{-2}{5}\log 2 - \frac{1}{2}$

(c) $\frac{2}{5}\log 2 + \frac{1}{2}$ (d) None of these

Solution (a) We have, $2\,f(x) + 3\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 2$

$\Rightarrow\ \ 2\,f\left(\frac{1}{x}\right) + 3\,f(x) = x - 2$

Solving the above two equations, we get

$$f(x) = \frac{-2}{5x} + \frac{3x}{5} - \frac{2}{5}$$

$$\therefore\ \int_1^2 f(x)\ dx = \int_1^2\left(\frac{-2}{5x} + \frac{3x}{5} - \frac{2}{5}\right) dx$$

$$= \left(\frac{-2}{5}\log x + \frac{3x^2}{10} - \frac{2}{5}x\right)\Bigg|_1^2$$

$$= \left(\frac{-2}{5}\log 2 + \frac{6}{5} - \frac{4}{5}\right) - \left(\frac{3}{10} - \frac{2}{5}\right)$$

$$= \left(\mathbf{\frac{-2}{5}\log 2 + \frac{1}{2}}\right).$$

24. Let $f(x)$ be a polynomial of degree 2 satisfying $f(0) = 1, f'(0) = -2$ and $f''(0) = 6$, then $\int_{-1}^2 f(x)\ dx$ is equal to

(a) 6 (b) 0

(c) 9 (d) none of these

Solution (c) Let $f(x) = ax^2 + bx + c$

$\Rightarrow\ \ f'(x) = 2ax + b\ $ and $f''(x) = 2a$.

We are given $f(0) = c = 1, f'(0) = b = -2$

and $\quad f''(0) = 2a = 6 \Rightarrow a = 3,\ b = -2\ $ and $c = 1$.

$\therefore\ \ f(x) = 3x^2 - 2x + 1$.

$$\therefore\ \int_{-1}^2 f(x)\ dx = \int_{-1}^2 (3x^2 - 2x + 1)\ dx$$

$$= [x^3 - x^2 + x]_{-1}^2$$

$$= (8 - 4 + 2) - (-1 - 1 - 1) = 6 + 3 = \mathbf{9}.$$

25. If $(-1, 2)$ and $(2, 4)$ are two points on the curve $y = f(x)$ and if $g(x)$ is the gradient of the curve at the point (x, y), then $\int_{-1}^2 g(x)\, dx$ is

(a) 2 (b) −2

(c) 0 (d) 1

Solution (a) Since $g(x) = f'(x)$

$$\therefore\ \int_{-1}^2 g(x)\ dx = \int_{-1}^2 f'(x)\, dx$$

$$= [f(x)]_{-1}^2 = f(2) - f(-1)$$

$$= 4 - 2\ [\text{As } f(-1) = 2 \text{ and } f(2) = 4] = \mathbf{2}.$$

26. The value of the integral $\int_{\pi/2}^{3\pi/2} [\sin x]\ dx$, where [.] denotes the greatest integer function, is

(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$

(c) 0 (d) π

Solution (b) $\int_{\pi/2}^{3\pi/2} [\sin x]\, dx = \int_{\pi/2}^{\pi} [\sin x]\, dx + \int_{\pi}^{3\pi/2} [\sin x]\ dx$

$$= \int_{\pi/2}^{\pi} 0\, dx + \int_{\pi}^{3\pi/2} (-1)\, dx = -\ [x]_{\pi}^{3\pi/2}$$

$$= -\left(\frac{3\pi}{2} - \pi\right) = \frac{-\pi}{2}.$$

27. The value of $\int_{-2}^2 \max\{(1 - x), (1 + x), 2\}\ dx$ is

(a) 8 (b) − 8

(c) 9 (d) − 9

Solution (c) For $-2 \le x \le -1$, we have $1 - x \ge 2$ and $1 - x > 1 + x$

$\therefore\ \ \max\ \{(1 - x), (1 + x), 2\} = 1 - x$.

For $-1 < x < 1$, we have $0 < 1 - x < 2$ and $0 < 1 + x < 2$

∴ max {(1 – x), (1 + x), 2} = 2.

For $1 \le x \le 2$, we have $1 + x \ge 2$ and $1 + x > 1 - x$

∴ max {(1 – x), (1 + x), 2} = 1 + x.

∴ $\int_{-2}^{2} \max\{(1-x),(1+x),2\}\;dx$

$$= \int_{-2}^{-1}(1-x)\,dx + \int_{-1}^{1}2\,dx + \int_{1}^{2}(1+x)\;dx$$

$$= \left[x-\frac{x^2}{2}\right]_{-2}^{-1} + [2x]_{-1}^{1} + \left[x+\frac{x^2}{2}\right]_{1}^{2} = \mathbf{9.}$$

28. If $f(a - x) = f(x)$ and $\int_0^{a/2} f(x)\;dx = p$, then $\int_0^a f(x)\;dx$ is equal to

(a) $2p$ (b) 0

(c) p (d) none of these

Solution (a) $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^{a} f(x)\,dx$

$$= p - \int_{a/2}^{0} f(a-z)\,dz$$

[Putting $x = a - z$ in the second integral so that $dx = -dz$]

$$= p + \int_0^{a/2} f(a-z)\,dz \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= p + \int_0^{a/2} f(a-x)\,dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

$$= p + \int_0^{a/2} f(x)\,dx \qquad [\because f(a-x) = f(x)]$$

$= p + p = \mathbf{2p.}$

29. The value of the integral $\int_{\log 1/3}^{\log 3} \log\left(x+\sqrt{1+x^2}\right)\;dx$ is

(a) log 3 (b) 2 log3

(c) 0 (d) none of these

Solution (c) Let $f(x) = \log\left(x+\sqrt{1+x^2}\right)$

Then, $f(-x) = \log\left(-x+\sqrt{1+x^2}\right)$

$$= \log\left(\frac{1}{x+\sqrt{1+x^2}}\right)$$

$$= -\log\left(x+\sqrt{1+x^2}\right) = -f(x)$$

∴ $f(x)$ is an odd function

∴ $\int_{\log 1/3}^{\log 3} \log\left(x+\sqrt{1+x^2}\right)\;dx = \mathbf{0.}$

30. If $I = \int_{-a}^{a}(\alpha\sin^5 x + \beta\tan^3 x + \gamma\cos x)\;dx$, where α, β, γ are constants, then the value of I depends on

(a) γ, a (b) α, β, γ, a

(c) α, β, a (d) none of these

Solution (a) We have, I

$$= \int_{-a}^{a}(\alpha\sin^5 x + \beta\tan^3 x + \gamma\cos x)\;dx$$

$$= \alpha\int_{-a}^{a}\sin^5 x\,dx + \beta\int_{-a}^{a}\tan^3 x\,dx + \gamma\int_{-a}^{a}\cos x\;dx$$

$$= \alpha \times 0 + \beta \times 0 + \gamma \times 2\int_0^a \cos x\;dx$$

[∵ $\sin^5 x$ and $\tan^3 x$ are odd functions]

∴ I depends on γ, $\boldsymbol{a}$.

31. If $f(a + b - x) = f(x)$, then $\int_a^b x\,f(x)\;dx$ equals

(a) $\dfrac{a+b}{2}\int_a^b f(x)\;dx$ (b) $\dfrac{a-b}{2}\int_a^b f(x)\;dx$

(c) 0 (d) none of these

Solution (a) Let $I = \int_a^b x\,f(x)\;dx$

$$= \int_a^b (a+b-x)\,f(a+b-x)\;dx$$

$$= \int_a^b [(a+b)-x]\,f(x)\;dx$$

$$= (a+b)\int_a^b f(x)\,dx - I$$

∴ $I = \dfrac{a+b}{2}\displaystyle\int_a^b f(x)\;dx.$

32. $\int_{-\pi}^{\pi}(cox\;px - \sin qx)^2\;dx$, where p and q are integers, is equal to

(a) – π (b) 0

(c) π (d) 2π

Solution (d) Let $I = \int_{-\pi}^{\pi}(cox\;px - \sin qx)^2\;dx$

$$= \int_{-\pi}^{\pi}\cos^2 px + \sin^2 qx - 2\sin qx\cos px)\;dx$$

$$= \int_{-\pi}^{\pi}(\cos^2 px + \sin^2 qx)\,dx - 2\int_{-\pi}^{\pi}\sin qx\cos px\;dx$$

$$= 2\int_0^{\pi}(\cos^2 px + \sin^2 qx) - 0$$

[∵ sin qx cos px is odd function]

$$= \int_0^{\pi}(2+\cos 2\,px - \cos 2\,qx)\ dx\ =\ \mathbf{2\pi}.$$

33. $\int_0^1 |\sin 2\pi x|\ dx$ is equal to

(a) 0

(b) $-\ 1/\pi$

(c) $1/\pi$

(d) $2/\pi$

Solution (d) Since $\sin 2px$ is positive for $0 < x \le \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$.

$$\therefore\quad |\sin 2\,\pi x| = \begin{bmatrix} \sin 2\pi x, \text{ for } 0 < x \le \frac{1}{2} \\ -\sin 2\pi x, \text{ for } \frac{1}{2} < x < 1 \end{bmatrix}$$

$$\therefore\ \int_0^1 |\sin 2\pi x|\ dx = \int_0^1 |\sin 2\pi x|\ dx = \int_0^1 \sin 2\pi x\ dx + \int_{1/2}^1 (-\sin 2\pi x)\ dx$$

dx

$$= \left[\frac{-\cos 2\pi\, x}{2\pi}\right]_0^{1/2} + \left[\frac{\cos 2\pi x}{2\pi}\right]_{1/2}^{1} = \frac{2}{\pi}\ .$$

34. $\lim\limits_{n\to\infty}\left[\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{1}{2n}\right]$ is equal to

(a) $\frac{1}{3}\log 2$

(b) $\frac{1}{2}\log 2$

(c) $\log 2$

(d) none of these

Solution (a) $\lim\limits_{n\to\infty}\left[\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^2}{r^3+n^3}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3+1} =$$

$$\int_0^1 \frac{x^2}{x^3+1}\,dx\ =\ \frac{1}{3}\left[\log(x^3+1)\right]_0^1$$

$$= \frac{1}{3}\ (\log 2 - \log 1) = \frac{\mathbf{1}}{\mathbf{3}}\ \mathbf{\log 2.}$$

35. If $x > 0$ and $\int_0^x [x]\,dx\ = [x]\left(\frac{1}{2}A + B\right)$, where [.] denotes the greatest integer function, then

(a) $A = [x] - 1$

(b) $B = x - [x]$

(c) $A = [x] + 1$

(d) $B = x + [x]$

Solution (a), (b) Let $[x] = l$ then $x = l + k,\ 0 \le k < 1$.

$$\therefore\quad \int_0^x [x]\,dx\ =\ \int_0^{l+k} [x]\,dx$$

$$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \cdots + \int_{l-1}^{l} [x]\,dx + \int_{l}^{l+k} [x]\,dx$$

$$= 0 + 1 + 2 + \cdots + (l-1) + l\cdot k$$

$$= \frac{1}{2}\ (l-1)\ l + lk = \frac{1}{2}\,[x]\ \big([x]-1\big) + [x]\big(x-[x]\big)$$

$$= [x]\ \left(\frac{1}{2}\big([x]-1\big) + \big(x-[x]\big)\right)$$

$\therefore$ $\mathbf{A = [x] - 1}$ and $\mathbf{B = x - [x]}$.

36. $\int_0^1 \frac{x^a - 1}{\log x}\ dx$, where $a > 0$, is equal to

(a) $\log(a+1)$

(b) $2\log(a+1)$

(c) $3\log a$

(d) $2\log a$

Solution (a) Let $f(a)\ =\ \int_0^1 \frac{x^a-1}{\log x}\ dx$

$$\Rightarrow\quad f'(a)\ =\ \int_0^1 \frac{x^a \log x}{\log x}\,dx\ =\ \int_0^1 x^a\ dx\ =\ \left.\frac{x^{a+1}}{a+1}\right|_0^1$$

$$= \frac{1}{a+1}$$

$$\therefore\quad f(a)\ =\ \int \frac{1}{a+1}\,da + \mathrm{C}\ = \log(a+1) + \mathrm{C}.$$

If $a = 0$, then $f(a) = 0$, $\therefore$ C = 0.

Hence, $f(a)$ = $\mathbf{\log(a+1)}$.

37. The value of the integral $\int_0^{100} 2^{x-[x]}\ dx$, where [.] denotes the greatest integer function, is

(a) $\log 2$

(b) $50\log 2$

(c) $100\log 2$

(d) none of these

Solution (c) $\int_0^{100} 2^{x-[x]}dx\ =\ \sum\limits_{k=1}^{100}\ \int_{k-1}^{k} 2^{x-[x]}\ \ dx$

$$= \sum_{k=1}^{100}\ \int_{k-1}^{k} 2^{x-(k-1)}\ dx\ =\ \sum_{k=1}^{100} 2^{-(k-1)} \int_{k-1}^{k} 2^x\ \ dx$$

$$= \sum_{k-1}^{100} 2^{(1-k)}\cdot(2^k - 2^{k-1})\ \log 2$$

$$= \log 2\ \sum_{k=1}^{100}(2-1)\ =\ \mathbf{100\ \log 2.}$$

38. If $\int_0^{\pi/2} \log\sin x\ \ dx\ =\ a$, then the value of

$\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\ \ dx$ in terms of a is

(a) $2a$

(b) $-\ 2a$

(c) $a/2$

(d) $a\log 2$

Solution (b) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\ \ dx$

$$= \int_0^{\pi/2} \frac{\log(1+\tan^2\theta)\sec^2\theta}{1+\tan^2\theta}\ d\theta$$

[Putting $x = \tan\theta$ so that $dx = \sec^2\theta\ d\theta$]

$$= \int_0^{\pi/2} \log \sec^2\theta \; d\theta = 2\int_0^{\pi/2} \log \sec\theta \; d\theta$$

$$= -\,2\int_0^{\pi/2} \log \cos\theta \; d\theta = -\,2\int_0^{\pi/2} \log \cos\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$= -\,2\int_0^{\pi/2} \log \sin\theta \; d\theta = -\,2\int_0^{\pi/2} \log \sin x \; dx = \boldsymbol{-2a}.$$

39. If $f(x)$ is a periodic function with period T, then $\int_{a+2T}^{b+2T} f(x) \; dx$ is equal to

(a) $\int_{a+T}^{b+T} f(x) \; dx$ (b) $\int_{a+T}^{b} f(x) \; dx$

(c) $\int_{a}^{b+T} f(x) \; dx$ (d) $\int_{a}^{b} f(x) \; dx]$

Solution (a), (d) $\int_{a+2T}^{b+2T} f(x) \; dx = \int_{a+T}^{b+T} f(y+T) \; dy$

[Putting $x = y + T$ so that $dx = dy$]

$$= \int_a^b f(z) \; dz$$

[Again, putting $z = y + T$ so that $dz = dy$]

$$= \int_a^b f(x) \; dx.$$

Hence, (a) and (d) are correct

40. The value of the integral $\int_a^{a+\frac{\pi}{2}} \left(|\sin x| + |\cos x|\right) \; dx$ is

(a) $a\pi$

(b) $2a\pi$

(c) $\frac{a\pi}{2}$

(d) independent of a

Solution (d) Since $(\,|\sin x| + |\cos x|$ is a periodic function with period $\frac{\pi}{2}$, therefore, $\int_a^{a+\frac{\pi}{2}} (\,|\sin x| + |\cos x|) \; dx$ is independent of a.

41. The value of the integral $\int_0^{2[x]} \left(x-[x]\right) \; dx$ is

(a) $[x]$ (b) $\frac{1}{2}\,[x]$

(c) $3\,[x]$ (d) $2\,[x]$

Solution (a) $\int_0^{2[x]} \left(x-[x]\right) \; dx = \int_0^{2[x]\cdot 1} \left(x-[x]\right) \; dx$

$$= 2\,[x] \int_0^1 \left(x-[x]\right) \; dx$$

[$\because$ $x - [x]$ is a periodic function of period 1]

$$= 2\,[x] \left(\frac{x^2}{2}\bigg|_0^1 - \int_0^1 [x] \, dx\right) = 2\,[x] \left(\frac{1}{2} - 0\right) = \boldsymbol{[x]}.$$

42. The value of

$$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{1}{2n}\right] \text{ is}$$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) none of these

Solution (a)

$$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{1}{2n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{n}{n^2+1^2} \; \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{1+\frac{r^2}{n^2}} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x^2} = dx$$

$$= \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}.$$

43. $\lim_{n\to\infty} \frac{(n\,!)^{1/n}}{n}$ is equal to

(a) e (b) $\frac{1}{e}$

(c) $e - 1$ (d) none of these

Solution (b) Let $y = \lim_{n\to\infty} \frac{(n\,!)^{1/n}}{n} = \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n}$

$$= \lim_{n\to\infty} \left[\frac{1\cdot 2\cdot 3\cdot 4 \cdots\cdots n}{n^n}\right]^{1/n}$$

$$\Rightarrow \quad \log y = \lim_{n\to\infty} \frac{1}{n} \cdot \log\left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\cdots\cdot\frac{n}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\frac{n}{n}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \log\frac{r}{n} = \int_0^1 \log x \; dx$$

$$= \left[x \log x\right]_0^1 - \int_0^1 x \cdot \frac{1}{x} \; dx = 0 - 1 = -\,1.$$

$$\therefore \quad y = e^{-1} = \boldsymbol{\frac{1}{e}}.$$

44. $\lim_{n\to\infty} \frac{1}{n^{p+1}} \left[1^p + 2^p + \cdots + n^p, \; p > -\,1\right.$, is equal to

(a) $\frac{1}{p+1}$ (b) $\frac{1}{p}$

(c) $\frac{1}{p+2}$ (d) none of these

Solution (a) $\lim_{n\to\infty} \frac{1}{n^{p+1}} \; [1^p + 2^p + \cdots + n^p]$

$$= \lim_{n\to\infty} \frac{1}{n^{p+1}}\sum_{r=1}^{n} r^p = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\left(\frac{r}{n}\right)^p$$

$$= \int_0^1 x^p\, dx = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{\mathbf{1}}{\mathbf{p+1}}.$$

45. If f and g are continuous functions on $[0, \pi]$ satisfying $f(x) + f(\pi - x) = g(x) + g(\pi - x) = 1$, then $\int_0^{\pi}[f(x) + g(x)]\, dx$ is equal to

(a) π (b) 2π

(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

Solution (a) $\int_0^{\pi} f(x)\cdot g(x)\, dx = \int_0^{\pi} f(\pi - x) g(\pi - x)\, dx$

$$= \int_0^{\pi}[1 - f(x)]\cdot[1 - g(x)]\, dx \text{ (given)}$$

$$= \int_0^{\pi}[1 - f(x) - g(x) + f(x)\cdot g(x)]\, dx$$

$$= \int_0^{\pi} 1\cdot dx - \int_0^{\pi}[f(x) + g(x)]\, dx + \int_0^{\pi} f(x)\cdot g(x)\, dx$$

$$\Rightarrow \int_0^{\pi}[f(x) + g(x)]\, dx = \int_0^{\pi} 1\cdot dx = \pi.$$

46. The value of the integral $\int_0^{41\pi/4} |\cos x|\, dx$ is

(a) $20 - \frac{1}{\sqrt{2}}$ (b) $20 + \frac{1}{\sqrt{2}}$

(c) $19 + \frac{1}{\sqrt{2}}$ (d) $19 - \frac{1}{\sqrt{2}}$

Solution (b) $\int_0^{41\pi/4} |\cos x|\, dx = \int_0^{10\pi} |\cos x|\, dx + \int_{10\pi}^{41\pi/4} |\cos x|$

$$= 10\int_0^{\pi} |\cos x|\, dx + \int_{10\pi}^{10\pi + \frac{\pi}{4}} |\cos x|\, dx$$

[Since $|\cos x|$ is a periodic function of period π]

$$= 10 \int_0^{\pi} |\cos x|\, dx + \int_0^{\pi/4} |\cos x|\, dx$$

$$= 10\left(\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right) + \sin x\Big|_0^{\pi/4}$$

$$= 10\left(\sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}\right) + \frac{1}{\sqrt{2}}$$

$$= 10\,(1 + 1) + \frac{1}{\sqrt{2}} = \mathbf{20} + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}.$$

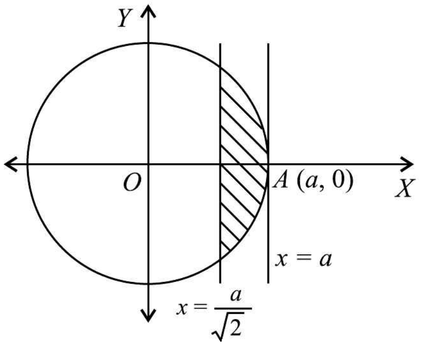
47. The area of the smaller part of the circle $x^2 + y^2 = a^2$, cut off by the line $x = \frac{a}{\sqrt{2}}$, is given by

(a) $\frac{a^2}{2}\left(\frac{\pi}{2} + 1\right)$ (b) $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$

(c) $a^2\left(\frac{\pi}{2} - 1\right)$ (d) none of these

Solution (b) Required area $= 2 \int_{a/\sqrt{2}}^{a} y\, dx$

$$= 2 \int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2}\, dx$$



$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/\sqrt{2}}^{a}$$

$$= 2\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - \left(\frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} + \frac{a^2}{2}\cdot\frac{\pi}{4}\right)\right]$$

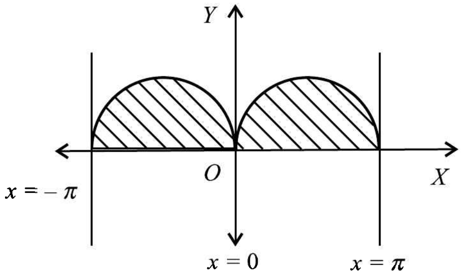
$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right).$$

48. The area bounded by the $y = |\sin x|$, x-axis and the lines $|x| = \pi$ is

(a) 2 (b) 1

(c) 4 (d) none of these

Solution (c) The required area



$$= \int_{-\pi}^{\pi} |\sin x|\, dx = 2\int_0^{\pi} \sin x\, dx = -2\,[\cos x]_0^{\pi}$$

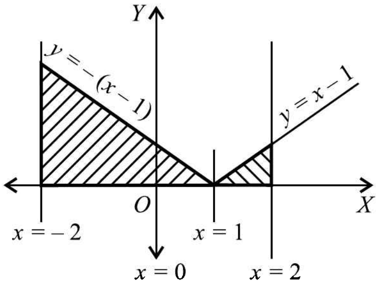
$$= -2\,(\cos\pi - \cos 0) = \mathbf{4}.$$

49. The area bounded by $y = |x - 1|$, $y = 0$ and $|x| = 2$ is

(a) 4 (b) 5

(c) 3 (d) none of these

Solution (b) The required area



$$= \int_{-2}^{2} |x-1|\, dx = -\int_{-2}^{1} (x-1)\, dx + \int_{1}^{2} (x-1)\, dx$$

$$= -\left[\frac{x^2}{2} - x\right]_{-2}^{1} + \left[\frac{x^2}{2} - x\right]_{1}^{2} = \mathbf{5.}$$

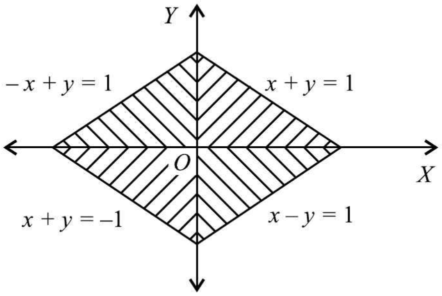
50. The area bounded by the lines $|x| + |y| = 1$, is

(a) 1 (b) 2

(c) 4 (d) none of these

Solution (b) The required area

$$= 4 \int_{0}^{1} (1-x)\; dx$$



$$= 4 \left[x - \frac{x^2}{2}\right]_0^1 = 4 \int_0^{\pi/4} \sin x \; = \mathbf{2.}$$

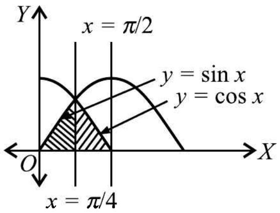
51. The area of one the curvilinear triangles formed by the curves $y = \sin x$, $y = \cos x$ and x-axis is

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$

(c) $2+2\sqrt{2}$ (d) none of these

Solution (b) Clearly the two curves $y = \sin x$ and $y = \cos x$ intersect at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

∴ Required area



$$= \int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$$

$$= -[\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2} = 2\left(1 - \frac{1}{\sqrt{2}}\right) = \mathbf{2 - \sqrt{2}}.$$

52. The area bounded by $y = \tan x$, $y = \cot x$, x-axis in $0 \le x \le \frac{\pi}{2}$ is

(a) 3 log 2

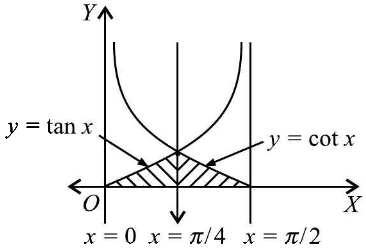
(b) log 2

(c) 2 log 2

(d) none of these

Solution (b) Clearly the two curves meet at the point $\left(\frac{\pi}{4}, 1\right)$

∴ The required area



$$= \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$$

$$= [\log \sec x]_0^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/2}$$

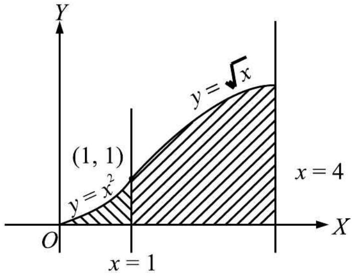
$$= \log \sqrt{2} - \log \frac{1}{\sqrt{2}} = 2 \log \sqrt{2} = \mathbf{\log 2.}$$

53. The area above x – axis, bounded by the line $x = 4$ and the curve $y = f(x)$, where $f(x) = x^2$, $0 \le x \le 1$ and $f(x) = \sqrt{x}$, $x \ge 1$, is

(a) 1 (b) 2

(c) 4 (d) 5

Solution (d) The required area



$$= \int_0^1 x^2\, dx + \int_1^4 \sqrt{x}\, dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^{3/2}}{3}\right]_1^4$$

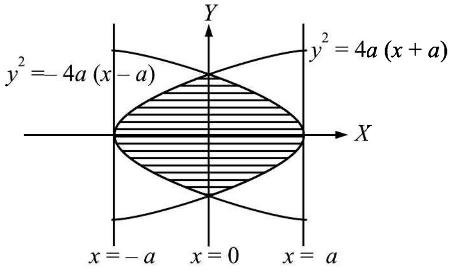
$$= \frac{1}{3} + \frac{2}{3}\,(8-1) = \mathbf{5.}$$

54. The area bounded by the parabolas $y^2 = 4a\,(x + a)$ and $y^2 = -4a\,(x - a)$ is

(a) $\frac{16}{3}a^2$ (b) $\frac{8}{3}a^2$

(c) $\frac{4}{3}a^2$ (d) none of these

Solution (a) The required area



$$= 4\int_0^a \sqrt{4a(a-x)}\ dx$$

$$= 4\cdot 2\ \sqrt{a}\cdot\left[\frac{-2(a-x)^{3/2}}{3}\right]_0^a = \frac{16}{3}\ a^2.$$

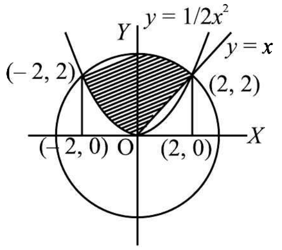
55. The area bounded by the parabolas $y = 4x^2$, $y = \dfrac{x^2}{9}$ and the line $y = 2$ is

(a) $\dfrac{20\sqrt{2}}{3}$ (b) $\dfrac{10\sqrt{2}}{3}$

(c) $\dfrac{40\sqrt{2}}{3}$ (d) none of these

Solution (a) The required area

$$= 2\int_0^2\left(3\sqrt{y} - \frac{1}{2}\sqrt{y}\right)\ dy = 2\left[2y^{3/2} - \frac{y^{3/2}}{3}\right]_0^2$$



$$= 2\left[4\sqrt{2} - \frac{2\sqrt{2}}{3}\right] = \mathbf{\frac{20\sqrt{2}}{3}}.$$

56. If $\int_0^1 f(x)\ dx = 1$, $\int_0^1 x f(x)\ dx = a$, $\int_0^1 x^2 f(x)\ dx = a^2$, then $\int_0^1 (a-x)^2\ f(x)\,dx$ equals

(a) $4a^2$ (b) 0

(c) $2a^2$ (d) none of the above

Solution (b) $\int_0^1 (a-x)^2\ f(x)\ dx$

$$= a^2\int_0^1 f(x)\,dx - 2a\int_0^1 x f(x)\,dx + \int_0^1 x^2\ f(x)\ dx$$

$$= a^2\ (1) - 2a\ (a) + a^2 = a^2 - 2a^2 + a^2 = \mathbf{0}.$$

57. If $u_{10} = \int_0^{\pi/2} x^{10}\sin x\ dx$, then the value of $u_{10} + 90u_8$ is

(a) $9\left(\dfrac{\pi}{2}\right)^9$ (b) $10\left(\dfrac{\pi}{2}\right)^9$

(c) $\left(\dfrac{\pi}{2}\right)^9$ (d) $9\left(\dfrac{\pi}{2}\right)^8$

Solution (b) $u_{10} = \int_0^{\pi/2} x^{10}\sin x\ dx$

$$= \left[-x^{10}\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} 10x^9\ \cos x\ dx$$

$$= 10\left[x^9\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} 9x^8\ \sin x\,dx\right]$$

$$= 10\left[\left(\frac{\pi}{2}\right)^9 - 9\int_0^{\pi/2} x^8\sin x\,dx\right] = 10\ \left(\frac{\pi}{2}\right)^9\ - 90u_8$$

$$\therefore\quad u_{10} + 90u_8 = \mathbf{10}\left(\frac{\pi}{2}\right)^9.$$

58. If $I_1 = \int_0^{3\pi} f\left(\cos^2 x\right)\ dx$ and $I_2 = \int_0^{\pi} f\left(\cos^2 x\right)\ dx$, then

(a) $I_1 = 5I_2$ (b) $I_1 = I_2$

(c) $I_1 = 3I_2$ (d) none of these

Solution (c) We know that if $f(x + m\pi) = f(x)$ for all integral values of m, then

$$\int_0^{n\pi} f(x)\ dx = n\int_0^{\pi} f(x)\ dx$$

Let $g(x) = f(\cos^2 x)$, then $g(x + m\pi) = [\cos^2(x + m\pi)]$

$= f\ (\cos^2 x) = g(x)$.

Hence $\int_0^{3\pi} f\ (\cos^2 x)\ dx = 3\int_0^{\pi} f(\cos^2 x)\ dx.$

$\therefore\quad \mathbf{I_1 = 3\ I_2.}$

59. The value of $\int_{-\pi/2}^{\pi/2}\left(\left[\dfrac{x}{\pi}\right] + 0.5\right)dx$ is

[where [.] denotes the greates integer function)

(a) π (b) $\pi/2$

(c) 0 (d) $-\pi/2$

Solution (c) Let $I = \int_{-\pi/2}^{\pi/2}\left(\left[\dfrac{x}{\pi}\right] + 0.5\right)dx$

Let $\dfrac{x}{\pi} = t$, $dx = \pi dt$

$$I = \int_{-1/2}^{1/2}([t] + 0.5)\pi\,dt$$

$$= \pi\left\{\int_{-1/2}^{0} -\frac{1}{2}\,dt + \int_0^{1/2}\frac{1}{2}\,dt\right\} = \pi\ \left\{-\frac{1}{4} + \frac{1}{4}\right\} = \mathbf{0}.$$

60. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and

$$F(t) = \int_0^t f(t-y)g(y)dy \text{, then}$$

(a) $F(t) = 1 - e^{-t}(1 + t)$

(b) $F(t) = e^t - (1 + t)$

(c) $F(t) = te^t$

(d) $F(t) = te^{-t}$

Solution (b) $F(t) = \int_0^t f(t-y)g(y)dy$

$$= \int_0^t e^{t-y}\cdot y\,dy = e^t.\int_0^t e^{-y}y\,dy$$

$$= e^t\left[(-ye^{-y})_0^t - \int_0^t 1(-e^{-y})dy\right]$$

$$= e^t[(-te^{-t} - 0) - (e^{-y})_0^t]$$

$$= e^t[-te^{-t} - (e^{-t} - e^0)]$$

$$= [1 - e^{-t}(1 + t)]e^t = e^t - (1 + t)$$

Problems for Practice

Level of Difficulty I

1. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) none of these

2. $\displaystyle\int_0^{2a} x^{9/2}(2a-x)^{-1/2}dx$ is equal to

(a) $\frac{63\pi}{4}a^5$ (b) $\frac{63\pi}{2}a^5$

(c) $63\pi\, a^5$ (d) $\frac{63\pi}{8}a^5$

3. $\displaystyle\int_0^2 \sin\frac{\pi[x]}{2}\,dx$ is equal to

(a) 1 (b) – 1 (c) 0 (d) none of these

4. If $f(x)$ is an even function, then $\displaystyle\int_0^x f(t)\,dt$ is

(a) odd

(b) even

(c) neither even nor odd

5. $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\int_0^{x^2}\sin\sqrt{t}\,dt$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{-1}{3}$ (d) $\frac{-2}{3}$

6. If ϕ and ϕ are two continuous functions, then the value of the integral $\displaystyle\int_{-\pi/4}^{\pi/4}\{f(x)+f(-x)\}\cdot\{g(x)-g(-x)\}\,dx$ is

(a) $\frac{\pi}{4}$ (b) 0 (c) $\frac{-\pi}{4}$ (d) none of these

7. The value of $\displaystyle\int_{-1/2}^{1/2}\cos x\log\frac{1+x}{1-x}\,dx$ is

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{-1}{2}$ (d) none of these

8. If $\displaystyle\int_0^{\pi} x f\left(\sin^3 x + \cos^2 x\right)\,dx = k\int_0^{\pi/2} f\left(\sin^3 x + \cos^2 x\right)\,dx$, then $k =$

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) none of these

9. The value of the integral

$$\int_{-2}^{3}\left[\cot^{-1}\left(\frac{x-1}{x+1}\right)+\cot^{-1}\left(\frac{x+1}{x-1}\right)\right] \text{ is}$$

(a) $\frac{5\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{2}$ (d) none of these

10. Let $I_1 = \displaystyle\int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x))\,dx$

and $I_2 = \displaystyle\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))\,dx$,

where f is a continuous function and z is any real number, then $I_1/I_2 =$

(a) 3/2 (b) 1/2 (c) 1 (d) none of these

11. The value of the interval

$$\int_1^2\sqrt{(2x+3)(3x^2+4)}\,dx \text{ cannot exceed}$$

(a) $\sqrt{48}$ (b) $\sqrt{66}$ (c) $\sqrt{73}$ (d) none of these

12. The value of the integral $\displaystyle\int_3^4\frac{[x^2]}{[x^2-14x+49]+[x^2]}\,dx$, where [.] denotes the greatest integer function, is

(a) 1 (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ (d) none of these

13. $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\cdots+\frac{1}{n}\right]$ is equal to

(a) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (b) $\frac{\pi}{4}-\frac{1}{2}\log 2$

(c) $-\left(\frac{\pi}{4}+\frac{1}{2}\log 2\right)$ (d) none of these

14. The value of the integral $\int_0^a \frac{e^x}{e^{[x]}}$ dx, $a \in Z_+$, is

(a) ae (b) $a(e+1)$

(c) $a(1-e)$ (d) $a(e-1)$

15. If $f(x)$ is function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all x ($x \neq 0$), then the value of the integral $\int_{\tan\theta}^{\cot\theta} f(x)\, dx$ is

(a) $\tan^2\theta$ (b) $2\tan\theta$ (c) 0 (d) none of these

16. If $\frac{d}{dx}\, g(x) = f(x)$, where $f(x)$ is continuous in $[a, b]$, then

$\int_a^b f(x)\cdot g(x)\, dx$ equals

(a) $\frac{1}{2}\left[[f(b)]^2-[f(a)]^2\right]$

(b) $f(b) - f(a)$

(c) $\frac{1}{2}\left[[g(b)]^2-[g(a)]^2\right]$

(d) $g(b) - g(a)$

17. $\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]$ is equal to

(a) $\int_0^1 f\left(\frac{1}{x}\right)\, dx$ (b) $\int_0^1 x f(x)\, dx$

(c) $\int_0^1 \frac{1}{x} f\left(\frac{1}{x}\right)\, dx$ (d) $\int_0^1 f(x)\, dx$

18. If $f(x) = \int_1^x \frac{\sin t}{t}\, dt$, then $\lim_{x\to\infty} f'(x) =$

(a) 1 (b) 0 (c) – 1 (d) ∞

19. $\lim_{n\to\infty}\frac{1}{n}\,(e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n})$ is equal to

(a) e (b) $e-1$ (b) $1-e$ (d) none of these

20. The smaller area enclosed by the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ is

(a) $\frac{a^2}{4}(\pi - 2)$ (b) $\frac{a^2}{4}(\pi + 2)$

(c) $\frac{a^2}{4}(2 - \pi)$ (d) none of these

Level of Difficulty II

21. $\int_0^{10\pi} |\sin x|\, dx$ is

(a) 20 (b) 8 (c) 10 (d) 18

22. $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then $\lim_{x\to\infty} n\,[I_n + I_{n+2}]$ equals

(a) $\frac{1}{2}$ (b) 1 (c) ∞ (d) zero

23. $\int_0^2 [x^2]\, dx$ is

(a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$

(c) $\sqrt{2} - 1$ (d) $-\sqrt{2} - \sqrt{3} + 5$

24. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx$ is

(a) $\frac{\pi^2}{4}$ (b) π^2 (c) zero (d) $\frac{\pi}{2}$

25. The area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$ is given by

(a) $\frac{9}{2}$ (b) $\frac{43}{6}$ (c) $\frac{35}{6}$ (d) None of these

26. If $f(a + b - x) = f(x)$, then $\int_a^b x f(x)\, dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$ (b) $\frac{a+b}{2}\int_a^b f(x)\, dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\, dx$ (d) $\frac{a+b}{2}\int_a^b f(a+b-x)\, dx$

27. The value of $\lim_{x\to 0}\frac{\int_0^{x^2} \sec^2 t\, dt}{x\sin x}$ is

(a) 3 (b) 2 (c) 1 (d) 1

28. $\lim_{n\to\infty}\frac{1+2^4+3^4+\ldots+n^4}{n^5}$

$-\lim_{n\to\infty}\frac{1+2^3+3^3+\ldots+n^3}{n^5}$ is

(a) $\frac{1}{30}$ (b) zero (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

29. Let $\frac{d}{dx}F(x)=\left(\frac{e^{\sin x}}{x}\right)$, $x > 0$.

If $\int_1^4 \frac{3}{x}e^{\sin x^3}dx = F(k) - F(1)$,

then one of the possible values of k, is

(a) 15 (b) 16 (c) 63 (d) 64

30. The area of the region bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$ is

(a) 2 sq. units

(b) 3 sq. units

(c) 4 sq. units

(d) 6 sq. units

31. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral $\int_0^1 f(x)\,g(x)dx$, is

(a) $e-\frac{e^2}{2}-\frac{5}{2}$ (b) $e+\frac{e^2}{2}-\frac{3}{2}$

(c) $e-\frac{e^2}{2}-\frac{3}{2}$ (d) $e+\frac{e^2}{2}+\frac{5}{2}$

32. $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}e^{r/n}$ is

(a) $1-e$ (b) $e-1$ (c) e (d) $e+1$

33. The value of $I = \int_0^{\pi/2}\frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}dx$ is

(a) 2 (b) 1 (c) 0 (d) 3

34. If $\int_0^{\pi} x f(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$, then A is

(a) $\pi/4$ (b) π (c) 0 (d) 2π

35. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx$, then the value of $\frac{I_2}{I_1}$ is

(a) –1 (b) –3 (c) 2 (d) 1

36. The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is

(a) 3 (b) 2 (c) 1 (d) 4

37. $\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin x}+1} =$

(a) 0 (b) 1 (c) $-\pi/2$ (d) $\pi/2$

38. $\int_{-\pi/4}^{\pi/4}\frac{x+\pi/4}{2-\cos 2x}dx$ is equal to

(a) $\frac{\pi^2}{6\sqrt{3}}$ (b) $\frac{\pi^2}{4\sqrt{3}}$ (c) $\frac{\pi^2}{2\sqrt{3}}$ (d) none of these

39. $\int_0^{\pi}|\cos\theta-\sin\theta|\,d\theta$ is equal to

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{3}$ (d) none of these

40. $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{n+n}\right]$ is equal to

(a) $3\log 2$ (b) $\log 2$

(c) $2\log 2$ (d) none of these

ANSWERS

1. (c)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (a)
11. (b)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (c)	**16.** (c)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (a)
21. (b)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (d)	**30.** (c)
31. (c)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (b)	**40.** (b)

17 Differential Equations

BRIEF REVIEW OF THE CONCEPTS

DIFFERENTIAL EQUATION

An equation involving an independent variable, a dependent variable and the derivatives of the dependent variable, is called a *differential equation.*

ORDER OF A DIFFERENTIAL EQUATION

The order of highest derivative appearing in a differential equation is called the *order* of the differential equation

DEGREE OF A DIFFERENTIAL EQUATION

The power of the highest order derivative appearing in a differential equation, after it is made free from radicals and fraction, is called the *degree* of the differential equation.

In the above illustrations,

differential equation (1) is of first order and first degree,

differential equation (2) is of order 3 and degree 1,

differential equation (3) is of order 2 and degree 1

and differential equation (4) is of order 3 and degree 2,

as after making it free from fractional exponent by squaring, it can be re-written as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^3y}{dx^3}\right)^2.$$

Linear and Non-linear Differential Equations

A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

Any relation between the dependent and independent variables (not involving the derivatives) which, when substituted in the differential equation, reduces it to an identity is called a *solution* of the differential equation.

General Solution

The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution*. Thus, the general solution of a differential equation of the nth order has n arbitrary constants.

Particular Solution

A solution obtained by giving particular values to arbitrary constants in the general solution is called a *particular solution*.

FORMATION OF A DIFFERENTIAL EQUATION

Let $f(x, y, c_1, c_2, \ldots c_n) = 0$ be the solution of a differential equation, where $c_1, c_2, \ldots, c_n$ are n arbitrary constants. If we eliminate these n constants, we obtain the differential equation of the nth order satisfied by the given solution values. Any equation taken together with n relations obtained by differentiating it n times helps us to eliminate the n constants.

Working Rule

- Write the given equation.
- Differentiate the given equation with respect to independent variable x as many times as the number of arbitrary constants.
- Eliminate the arbitrary constants with the help of the given equation and the equations obtained by differentiation to get the required differential equation.

SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

The following methods may be used to solve first order and first degree differential equations.

1. Variable Separable Differentiable Equations

A differential equation of the form

$$f(x) + g(y)\frac{dy}{dx} = 0 \qquad \ldots(1)$$

or $\quad f(x)\,dx + g(y)\,dy = 0,$

is said to have separated variables.

Integrating Equation (1), we obtain

$$\int f(x)\,dx + \int g(y)\frac{dy}{dx}\,dx = c,$$

where c is an arbitrary constant.

Hence $\int f(x)\,dx + \int g(y)\,dy = c$ is the solution of Equation (1).

***Notes*:**

- There is no need to add arbitrary constants of integration on both sides, since they can be combined to give just one arbitrary constant. Moreover the general solution of a differential equation of order one must have only one arbitrary constant.
- The constant of integration can be taken as c or $\log c$ or $\tan^{-1}c$ and so on depending on the nature of the problem.

Initial Value Problems

In many cases instead of finding the general solution, we find a particular solution satisfying a given initial condition, say, the condition that at some point x_0, the solution $y(x)$ has the value y_0. This is expressed as $y(x_0) = y_0$.

A first-order differential equation together with an initial condition is called an *initial value problem.*

2. Equations Reducible to Variable Separable Form

Sometimes in a given differential equation, the variables are not separable. But, some suitable substitution reduces it to a form in which the variables are separable. For example, the differential equations of the type $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by substituting $ax + by + c = t$. The reduced variable separable form is:

$$\frac{dt}{b\,f(t) + a} = dx.$$

Integrating both sides to obtain the solution of this differential equation.

HOMOGENEOUS DIFFERENTIAL EQUATIONS

Homogeneous Function

A function $f(x, y)$ in x and y is called *a homogeneous function* of degree n, if the degree of each term is n.

Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x, y)$ as well as $g(x, y)$ is a homogeneous function of same degree in x and y is called a *homogenous differential equation.*

Note that a function of $\frac{y}{x}$ is always a homogenous function.

Working Rule for Solving a Homogeneous Differential Equation

Let $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ be the given homogeneous differential equation.

- Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
- Putting these values in the given equation, we get

$$v + x\frac{dv}{dx} = F(v) \quad \Rightarrow \frac{dv}{[F(v) - v]} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{F(v) - v} = \log|x| + c.$$

- Replace v by $\frac{y}{x}$ to obtain the required solution.

4. Equations Reducible to the Homogeneous Form

Type I: Consider a differential equation of the form:

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}, \text{ where } \frac{a}{A} \neq \frac{b}{B} \qquad ...(1)$$

This is clearly non-homogeneous. In order to make it homogeneous, we proceed as follows:

We substitute $x = X + h$ and $y = Y + k$ in Equation (1), where h, k are constants to be determined suitably.

We have $\frac{dx}{dX} = 1$ and $\frac{dy}{dY} = 1$, so that

$$\frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}.$$

Now Equation (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)} \qquad ...(2)$$

Choose h and k so that

$$ah + bk + c = 0,$$
$$Ah + Bk + C = 0.$$

These equations give

$$h = \frac{bC - Bc}{aB - Ab},\ k = \frac{Ac - aC}{aB - Ab}. \qquad ...(3)$$

Now equation (2) becomes

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY},$$

which being a homogeneous equation can be solved by means of the substitution $Y = vX$.

Type II: Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}, \text{ where } \frac{a}{A} = \frac{b}{B}$$
$$= k \text{ (say)}$$

Since $aB - Ab = 0$, the above method fails in view of Equation (3).

We have $$\frac{dy}{dx} = \frac{k(Ax + By) + c}{Ax + By + C} \qquad ...(4)$$

Substitute $Ax + By = z$ so that

$$A + B\frac{dy}{dx} = \frac{dz}{dx}.$$

Now Equation (4) becomes

$$\frac{dz}{dx} = B.\frac{kz + c}{z + C} + A,$$

which is an equation with variables separable.

5. Linear Differential Equations

A differential equation of the form

$$\frac{dy}{dx}+Py = Q, \qquad \text{...(1)}$$

where P and Q are functions of x (or constants), is called a *linear differential equation of the first order*.

Working Rule For Solving $\frac{dy}{dx}+Py$ = Q.

- Find integrating factor $(I.\ F.) = e^{\int Pdx}$.
- The solution of the differential equation is

  $$y\ (I.F) = \int Q\ (I.F)\ dx + c,$$

  where c is constant of integration.

Remarks

1. Sometimes a first order differential equation which is not expressible as Equation (1) becomes a linear equation of the form

   $$\frac{dx}{dy}+Ry = S,$$

   where R and S are functions of y alone (or constants). The integrating factor in this case is given by

   $$I.F. = e^{\int R\,dy}.$$

   The solution of this equation is

   $$x\cdot(I.F.) = \int (S\times(I.F.))\ dy + C\ ,$$

   where C is the constant of integration.
2. The fact $e^{\log t} = t$ will be frequently used in the solution of linear equations.

6. Equations Reducible to the Linear Form

Consider a differential equation of the form:

$$\frac{dy}{dx}+Py = Qy^n, \qquad \text{...(1)}$$

where P and Q are functions of x. This equation can be reduced to the linear form as follows:

Dividing both sides of Equation (1) by y^n, we get

$$y^{-n}\ \frac{dy}{dx}+Py^{-n+1} = Q \qquad \text{...(2)}$$

Put $y^{-n+1} = z \Rightarrow (-\ n + 1)\ y^{-n}\ \frac{dy}{dx}=\frac{dz}{dx}$.

Substituting in Equation (2), we obtain

$$\frac{dz}{dx}+(1-n)\,Pz = (1 - n)\ Q,$$

which is now a linear equation with z as the dependent variable.

Solution by Inspection

The following derivatives must be remembered as they are very useful in solving some differential equations directly.

1. $d\ (x + y) = dx + dy$
2. $d\ (xy) = ydx + xdy$
3. $d\left(\frac{x}{y}\right)=\frac{y\ dx - x\ dy}{y^2}$
4. $d\left(\frac{y}{x}\right)=\frac{x\ dy - y\ dx}{x^2}$
5. $d\ (log\ xy) = \frac{y\ dx + x\ dy}{xy}$
6. $d\left(log\frac{y}{x}\right)=\frac{x\ dy - y\ dx}{xy}$
7. $d\ (x^m\ y^n) = x^{m-1}\ y^{n-1}\ (my\,dx + nx\,dy)$
8. $d\left(\frac{1}{2}\log\,(x^2+y^2)\right)=\frac{x\ dx + y\ dy}{x^2+y^2}$
9. $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)=\frac{x\ dy - y\ dx}{x^2-y^2}$
10. $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{x\ dy - y\ dx}{x^2+y^2}$
11. $\frac{d\,[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{[f(x,y)]^n}$
12. $d\left(\sqrt{x^2+y^2}\right)=\frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}}$

Solved Problems

1. The differential equation of all circles passing through the origin and having their centres on the x-axis is

(a) $y^2 = x^2 + 2xy\ \frac{dy}{dx}$ (b) $y^2 = x^2 - 2xy\ \frac{dy}{dx}$

(c) $x^2 = y^2 + xy\ \frac{dy}{dx}$ (d) none of these

Solution (a) The equation of circles passing through the origin and having their centres on the x-axis is

$$x^2 + y^2 + 2gx = 0, \qquad \text{...(1)}$$

where g is an arbitrary constant.

Differentiating with respect to x, we get

$$x + y\ \frac{dy}{dx} + g = 0.\ \text{ i.e., }\ g = -\left(x+y\frac{dy}{dx}\right).$$

Putting this value of g in (1), we get

$$x^2 + y^2 - 2x\left(x+y\frac{dy}{dx}\right) = 0,$$

i.e., $$y^2 = x^2 + 2xy\ \frac{dy}{dx}.$$

2. The degree of the differential equation, of which $y^2 = 4a\ (x + a)$ is a solution, is

(a) 1 (b) 2

(c) 3 (d) none of these

Solution (b) We have $y^2 = 4a\,(x + a)$ ...(1)

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 4a \text{ i.e., } a = \frac{y}{2}\frac{dy}{dx}.$$

Substituting the value of a in (1), we get

$$y^2 = 2y\frac{dy}{dx}\left[x + \frac{y}{2}\frac{dy}{dx}\right]$$

or $$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$$

or $$y\left[1-\left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx},$$

which is the required differential equation. The degree of the differential equation is **2.**

3. The differential equation of family of parabolas with foci at the origin and axis along the x-axis is

(a) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$

(b) $x\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} - y = 0$

(c) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} + y = 0$

(d) none of these

Solution (a) Let the directrix be $x = -2a$ and latus rectum be $4a$. Then, the equation of the parabola is

(distance from focus = distance from directrix),

$$x^2 + y^2 = (2a + x)^2 \text{ or } y^2 = 4a\,(a + x) \quad ...(1)$$

Differentiating with respect to x, we get

$$y\frac{dy}{dx} = 2a \text{ or } a = \frac{1}{2}y\frac{dy}{dx}.$$

Putting this value of a in (1), the differential equation is

$$y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right)$$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = \mathbf{0}.$$

4. The degree of differential equation

$$x = 1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \text{............ is}$$

(a) three
(b) one
(c) not defined
(d) none of these

Solution (b) $x = e^{\left(\frac{dy}{dx}\right)} \Rightarrow \frac{dy}{dx} = \ln x$

$\therefore$ order = 1 and degree = **1.**

5. The equation of the family of curves for which sub-normal is constant, is

(a) $y^2 = cx + k$
(b) $y^2 = 2cx + k$
(c) $x^2 = 2cy + k$
(d) none of these

Solution (b) Cartesian sub-normal = $y\frac{dy}{dx} = c$ (a constant)

or $y\,dy = c\,dx$.

Integrating , $\frac{y^2}{2} = cx + \frac{k}{2}$ i.e., $y^2 = \mathbf{2cx + k}$.

6. The equation of the curve through the point (1, 1) and whose slope is $\frac{2ay}{x(y-a)}$, is

(a) $y^a \,.\, x^{2a} = e^{y-1}$
(b) $y^a .\, x^{2a} = e^y$
(c) $y^{2a} .\, x^a = e^{y-1}$
(d) none of these

Solution (a) We have, slope = $\frac{dy}{dx} = \frac{2ay}{x(y-a)}$

$$\Rightarrow \frac{y-a}{y}dy = \frac{2a}{x}dx$$

Integrating both sides, we get

$$a \log|\,y\,| - y = -2a\log|\,x\,| + \log c$$

$$\Rightarrow y^a . x^{2a} = ce^y.$$

This passes through (1, 1), therefore $1 = ce$ i.e., $c = \frac{1}{e}$. So, the equation of the curve is $\boldsymbol{y^a \,.\, x^{2a} = e^{y-1}}$.

7. Solution of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is

(a) $x = y^2\,(c + y^2)$
(b) $x = y\,(c - y^2)$
(c) $x = 2y\,(c - y^2)$
(d) $x = y\,(c + y^2)$

Solution (d) We have,

$$(x + 2y^3)\frac{dy}{dx} = y \Rightarrow y\frac{dx}{dy} = x + 2y^3$$

$$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2,$$

which is a linear equation, if we take x as the dependent variable.

$$\text{I.F.} = e^{\int p\,dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}.$$

$\therefore$ The solution is $x.\frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}dy + c$

$$\Rightarrow x\cdot\frac{1}{y} = y^2 + c \quad \text{or} \quad \boldsymbol{x = y\,(c + y^2)}.$$

8. The equation of a curve passing through $\left(1, \frac{\pi}{4}\right)$ and having slope $\frac{\sin 2y}{x + \tan y}$ at (x, y) is

(a) $x = \tan y$
(b) $y = \tan x$
(c) $x = 2\tan y$
(d) $y = 2\tan x$

Solution (a) We have,

$$\frac{dy}{dx} = \frac{\sin 2y}{x + \tan y} \Rightarrow \frac{dx}{dy} - \frac{x}{\sin 2y} = \frac{\tan y}{\sin 2y}$$

$$I.F. = e^{-\int\frac{dy}{\sin 2y}} = e^{\log\sqrt{\cot y}} = \sqrt{\cot y}.$$

Hence the solution is

$$x\sqrt{\cot y} = \int \frac{\tan y}{\sin 2y}\cdot\sqrt{\cot y}\ dy + C.$$

$$= \int \frac{1}{2}\frac{\sec^2 y}{\sqrt{\tan y}}\ dy + C. = \boldsymbol{\sqrt{\tan y} + C}$$

Since the curve passes through $\left(1, \frac{\pi}{4}\right)$, therefore, $1 = 1 + C$ i.e., $C = 0$.

Thus, the equation of curve is $\boldsymbol{x = \tan y}$**.**

9. The general solution of the differential equation $\frac{dy}{dx} = y\tan x - y^2\sec x$ is

(a) $\tan x = (c + \sec x)y$ (b) $\sec y = (c + \tan y)x$

(c) $\sec x = (c + \tan x)y$ (d) none of these

Solution (c) We have $\frac{dy}{dx} = y\tan x - y^2\sec x$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x.$$

Putting $\frac{1}{y} = V \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dV}{dx}$,

we obtain $\frac{dV}{dx} + \tan x \cdot V = \sec x$ which is linear.

$$I.F. = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

Hence the solution is

$V\sec x = \int \sec^2 x\,dx + c$ or $\frac{1}{y}\sec x = \tan x + c$

or $\boldsymbol{\sec x = y\,(c + \tan x)}$**.**

10. The general solution of the differential equation $y(x^2y + e^x)dx - e^x dy = 0$ is

(a) $x^3y - 3e^x = cy$ (b) $x^3y + 3e^x = cy$

(c) $y^3x - 3e^y = cx$ (d) $y^3x + 3e^y = cx$

Solution (b) We have $y\,(x^2y + e^x)\,dx - e^x\,dy = 0$

$$\Rightarrow e^x\frac{dy}{dx} = x^2y^2 + ye^x$$

Dividing by y^2e^x, we get $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = x^2e^{-x}$

Put $\frac{1}{y} = V$ so that $\frac{-1}{y^2}\frac{dy}{dx} = \frac{dV}{dx}$.

We thus have $\frac{dV}{dx} + V = -x^2e^{-x}$, which is linear

$$\therefore\quad I.F. = e^{\int 1dx} = e^x.$$

Hence the solution is

$$V\cdot e^x = -\int x^2e^{-x}.e^x\,dx + \frac{c}{3}$$

or $\frac{1}{y}e^x = -\frac{x^3}{3} + \frac{c}{3}$ or $\boldsymbol{x^3y + 3e^x = cy}$**.**

11. Solution of the differential equation

$x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx$ is

(a) $\log x = \cos\left(\frac{y}{x}\right) + C$

(b) $\log y = \cos\left(\frac{x}{y}\right) + C$

(c) $\log x = \cos\left(\frac{x}{y}\right) + C$ (d) none of these

Solution (a) We have, $\frac{dy}{dx} = \frac{y\sin\frac{y}{x} - x}{x\sin\frac{y}{x}}$.

Put $y = Vx$ so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$.

Hence $V + x\frac{dV}{dx} = \frac{V\sin V - 1}{\sin V} = V - \frac{1}{\sin V}$

$$\Rightarrow x\frac{dV}{dx} = -\frac{1}{\sin V} \Rightarrow \int\frac{dx}{x} + \int\sin V\,dV = C.$$

$$\Rightarrow \log x - \cos V = C \Rightarrow \boldsymbol{\log x = \cos\frac{y}{x} + C}$$

12. The general solution of the differential equation

$$\frac{dy}{dx} + y\,g'(x) = g(x).\ g'(x),$$

where $g(x)$ is a given function of x, is

(a) $g(x) + \log[1 + y + g(x)] = C$

(b) $g(x) + \log[1 + y - g(x)] = C$

(c) $g(x) - \log[1 + y - g(x)] = C$

(d) none of these

Solution (b) We have, $\frac{dy}{dx} = (g(x) - y).\ g'(x)$

Put $g(x) - y = V \Rightarrow g'(x) - \frac{dy}{dx} = \frac{dV}{dx}$

Hence $g'(x) - \frac{dV}{dx} = V\cdot g'(x)$

$$\Rightarrow \frac{dV}{dx} = (1 - V)\,g'(x) \Rightarrow \frac{dV}{1 - V} = g'(x)\,dx$$

$$\Rightarrow \int\frac{dV}{1 - V} = \int g'(x)\,dx$$

$\Rightarrow$ $-\log(1 - V) = g(x) - C$

$\Rightarrow$ $g(x) + \log(1 - V) = C$

$\therefore$ $\boldsymbol{g(x) + \log[1 + y - g(x)] = C.}$

13. Solution of the equation $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$ is

(a) $y = x\tan\left(\frac{c + x^2 + y^2}{2}\right)$

(b) $x = y\tan\left(\frac{c + x^2 + y^2}{2}\right)$

(c) $y = x\tan\left(\frac{c - x^2 - y^2}{2}\right)$

(d) none of these

Solution (c) We have, $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

$\Rightarrow \quad \dfrac{1}{2}\, d\,(x^2 + y^2) + d\tan^{-1}\left(\dfrac{y}{x}\right) = 0$

Integrating, $\dfrac{1}{2}\,(x^2 + y^2) + \tan^{-1}\dfrac{y}{x} = \dfrac{c}{2}$

$\Rightarrow \quad x^2 + y^2 + 2\tan^{-1}\dfrac{y}{x} = c.$

$\therefore \quad \boldsymbol{y = x \tan\left(\dfrac{c - x^2 - y^2}{2}\right)}$ is the required solution.

14. The equation of the curve whose slope is $\dfrac{dy}{dx} = \dfrac{2y}{x}$; $x > 0, y > 0$ and which passes through the point (1, 1) is

(a) $x^2 = y$ (b) $y^2 = x$

(c) $x^2 = 2y$ (d) $y^2 = 2x$.

Solution (a) We have, $\dfrac{dy}{dx} = \dfrac{2y}{x} \Rightarrow \dfrac{dy}{y} = 2$

Integrating, we get

$\log y = 2 \log x + \log C \Rightarrow \log y = \log Cx^2$

$\therefore \quad y = Cx^2.$

Since the curve passes through the point (1, 1) $\therefore C = 1$.

So the curve is $\boldsymbol{y = x^2}$.

15. The solution of the equation

$y \sin x \dfrac{dy}{dx} = \cos x \left(\sin x - \dfrac{y^2}{2}\right),$

given $y = 1$ when $x = \dfrac{\pi}{2}$ is

(a) $y^2 = \sin x$ (b) $y^2 = 2 \sin x$

(c) $x^2 = \sin y$ (d) $x^2 = 2 \sin y$

Solution (a)

16. Solution of the equation $x\,dy = \left(y + x\dfrac{f(y/x)}{f'(y/x)}\right)dx$ is

(a) $f\left(\dfrac{x}{y}\right) = cy$ (b) $f\left(\dfrac{y}{x}\right) = cx$

(c) $f\left(\dfrac{y}{x}\right) = cxy$ (d) none of these

Solution (b)

17. The solution of the equation $\log \dfrac{dy}{dx} = 9x - xy + 6$, given that $y = 1$ when $x = 0$, is

(a) $3e^{6y} = 2e^{9x-6} + 6e^6$ (b) $3e^{6y} = 2e^{9x+6} - 6e^6$

(c) $3e^{6y} = 2e^{9x+6} + e^6$ (d) none of these

Solution (c)

18. The solution of the equation $e^{\frac{dy}{dx}} = x + 3$, given that when $x = -2$, $y = 3$, is

(a) $y = (x + 3) \log (x + 3) - x + 1$

(b) $y = (x + 3) \log (x + 3) + x + 1$

(c) $y = (x + 3) \log (x + 3) - x - 1$

(d) none of these

Solution (a)

19. The degree of the differential equation of all tangent lines to the parabola $y^2 = 4ax$ is

(a) 1 (b) 2

(c) 3 (d) none of these

Solution (b)

20. A solution of the differential equation

$\left(\dfrac{dy}{dx}\right)^2 - x\dfrac{dy}{dx} + y = 0$ is

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Solution (c)

21. The order of the differential equation whose general solution is given by

$y = c_1 \cos(2x + c_2) - (c_3 + c_4)\, a^{x + c_5} + c_6 \sin(x - c_7)$ is

(a) 3 (b) 4

(c) 5 (d) 2

Solution (c)

22. The degree of the differential equation

$\left(\dfrac{d^4y}{dx^4}\right)^{3/5} - 5\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 5 = 0$ is

(a) 2 (b) 3

(c) 4 (d) 5

Solution (b)

23. The order of the differential equation satisfying

$\sqrt{1 - x^4} + \sqrt{1 - y^4} = a\,(x^2 - y^2)$. is

(a) 1 (b) 2

(c) 3 (d) none of these

Solution (a)

24. The degree of the differential equation

$\dfrac{d^3y}{dx^3} + x\left(\dfrac{dy}{dx}\right)^4 = 4 \log\left(\dfrac{d^4y}{dx^4}\right)$ is

(a) 1 (b) 3

(c) 4 (d) none of these

Solution (d)

25. The order of the differential equation whose general solution is given by $y = (c_1 + c_2) \cos (x + c_3) - c_4 e^{x + c_5}$ where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is

(a) 5 (b) 4

(c) 3 (d) 2

Solution (c)

Problems for Practice

Level of Difficulty I

1. The order and degree of the differential equation $\left(1+3\frac{dy}{dx}\right)^{1/3}=4\frac{d^3}{dx}$ are

(a) $1, \frac{2}{3}$ (b) (3, 4) (c) (3, 3) (d) (1, 2)

2. The solution of the equation $\frac{d^2y}{dx^2} = e^{-2x}$ is $y =$

(a) $\frac{e^{-2x}}{4}$ (b) $\frac{e^{-2x}}{4} + cx + d$

(c) $\frac{1}{4}e^{-2x} + cx^2 + d$ (d) $\frac{1}{4}e^{-2x} + c + d$

3. The differential equation of all non-vertical lines in a plane is

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$

(c) $\frac{dy}{dx} = 0$ (d) $\frac{dx}{dy} = 0$

4. The degree and order of the differential equation of the family of all parabolas whose axis is x-axis, are respectively

(a) 2, 1 (b) 1, 2 (c) 3, 2 (d) 2, 3

5. The solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$, is

(a) $(x - 2) = ke^{-\tan^{-1}y}$

(b) $2x\, e^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$

(c) $x\, e^{\tan^{-1}y} = \tan^{-1}y + k$

(d) $x\, e^{2\tan^{-1}y} = e^{\tan^{-1}y} + k$

6. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is

(a) $(x^2 - y^2)y' = 2xy$ (b) $2(x^2 + y^2)y' = xy$

(c) $2(x^2 - y^2)y' = xy$ (d) $(x^2 + y^2)y' = 2xy$

7. The solution of the differential equation

$$ydx + (x + x^2y)dy = 0 \text{ is}$$

(a) $\frac{1}{xy} + \log y = c$ (b) $-\frac{1}{xy} + \log y = c$

(c) $-\frac{1}{xy} = c$ (d) $\log y = cx$

8. Integrating factor of differential equation $\cos x\frac{dy}{dx} + y\sin x = 1$ is

(a) $\sin x$ (b) $\sec x$

(c) $\tan x$ (d) $\cos x$

9. The integrating factor of the differential equation $\frac{dy}{dx}(x\log x) + y = 2\log x$ is given by

(a) $\log(\log x)$ (b) e^x

(c) $\log x$ (d) x

10. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is

(a) $\log y = kx$ (b) $y = kx$

(c) $y = \frac{k}{x}$ (d) $y = k\log x$

Level of Difficulty II

11. The differential equation of all parabolas whose axes are parallel to the axis of y, is

(a) $\frac{d^3y}{dx^3} = 1$ (b) $\frac{d^3y}{dx^3} = -1$

(c) $\frac{d^3y}{dx^3} = 0$ (d) none of these

12. The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represent

(a) circles (b) straight lines

(c) ellipse (d) parabola

13. The solution of $\frac{dy}{dx} = \frac{ax+g}{by+f}$ represents a circle when

(a) $a = b$ (b) $a = -b$

(c) $a = -2b$ (d) $a = 2b$

14. The solution of $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ is

(a) $\sin^{-1}x.\ \sin^{-1}y = C$

(b) $\sin^{-1}x = C\sin^{-1}y$

(c) $\sin^{-1}x - \sin^{-1}y = C$

(d) $\sin^{-1}x + \sin^{-1}y = C$

15. The solution of the differential equation

$$x^2\frac{dy}{dx} - xy = 1 + \cos\frac{y}{x} \text{ is}$$

(a) $\cos\frac{y}{x} = 1 + \frac{c}{x}$

(b) $x^2 = (c + x^2)\tan\frac{y}{x}$

(c) $\tan\frac{y}{2x} = c - \frac{1}{2x^2}$

(d) $\tan\frac{y}{x} = c + \frac{1}{x}$

16. Differential equation for $y = A\cos\alpha x + B\sin\alpha x$, where A and B are arbitrary constants is

(a) $\frac{d^2y}{dx^2} + \alpha y = 0$ (b) $\frac{d^2y}{dx^2} - \alpha y = 0$

(c) $\frac{d^2y}{dx^2} - \alpha^2 y = 0$ (d) $\frac{d^2y}{dx^2} + \alpha^2 y = 0$

17. If $2f(x) = f'(x)$ and $f(0) = 3$, then $f(2)$ equals

(a) $4e^3$ (b) $3e^4$ (c) $2e^3$ (d) $3e^2$

18. Solution of the equation $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ is

(a) $e^y = e^x - 1 + ce^{-e^x}$

(b) $e^y = e^x - 1 + ce^{e^x}$

(c) $e^x = e^y - 1 + ce^{-e^y}$

(d) none of these

19. The differential equation representing the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$, where c is a positive parameter, is of

(a) order 1 (b) order 2

(c) degree 3 (d) degree 4

20. Solution of the differential equation $x\,dy - y\,dx = 0$ represents

(a) parabola whose vertex is at origin

(b) circle whose centre is at origin

(c) a rectangular hyperbola

(d) straight line passing through origin

ANSWERS

1. (c)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (b)
11. (c)	**12.** (d)	**13.** (b)	**14.** (d)	**15.** (c)	**16.** (d)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (a)

18 Complex Numbers

BRIEF REVIEW OF THE CONCEPTS

IMAGINARY NUMBERS

Square root of a negative number is called an *imaginary number*.

Note: The symbol 'i' was first introduced by the famous mathematician, Leonhard Euler (1707–1783) in 1748, possibly because 'i' is the first letter of the Latin word 'imaginarius'.

INTEGRAL POWERS OF *i*

We have $i = \sqrt{-1}$, $i^2 = -1$. Therefore,

$$i^3 = i^2 \times i = (-1) \times i = -i,$$
$$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1.$$

We note that for any $n \in N$

1. $i^{2n} = (i^2)^n = (-1)^n = \begin{bmatrix} 1, \text{ when } n \text{ is even} \\ -1, \text{ when } n \text{ is odd} \end{bmatrix}$
2. $i^{2n+1} = (i^{2n})i = (-1)^n i = \begin{bmatrix} i, \text{ when } n \text{ is even} \\ -i, \text{ when } n \text{ is odd} \end{bmatrix}$.

Also, for any $n \in N$, the value of i^{-n} is found out by writing this as $\frac{1}{i^n}$ and solving i^n.

Thus, any integral power of i can be expressed in terms of ± 1 or $\pm i$.

Note:

For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either zero or positive. If both a and b are positive real numbers, then the calculation

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab} \text{ is wrong.}$$

The correct calculation is

$$\begin{aligned}\sqrt{-a} \times \sqrt{-b} &= (\sqrt{-1}\sqrt{a})\,(\sqrt{-1}\sqrt{b}) \\ &= (i\sqrt{a})\,(i\sqrt{b}) \\ &= i^2(\sqrt{a} \times \sqrt{b}) = (-1)\,(\sqrt{ab}) \\ &= -\sqrt{ab}.\end{aligned}$$

Thus, the calculation $\sqrt{2} \times \sqrt{-3} = \sqrt{(-2)\times(-3)}$

$= \sqrt{6}$ is wrong.

The correct result is

$$\begin{aligned}\sqrt{-2} \times \sqrt{-3} &= (i\sqrt{2})\,(i\sqrt{3}) \\ &= i^2(\sqrt{2} \times \sqrt{3}) \\ &= -\sqrt{6}.\end{aligned}$$

COMPLEX NUMBERS

An expression of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$, is called a *complex number*. It is usually denoted by z, i.e.,

$$z = x + iy.$$

x is called the *real* part and y the *imaginary* part of z and may be denoted by Re(z) and Im(z) respectively.

If $y = 0$, z is called *purely real* and if $x = 0$, z is called *purely imaginary*.

The set of complex numbers is denoted by C.

If $x = 0$ and $y = 0$, the complex number reduces to $0 + i{\cdot}0 = 0$, which is called the *zero complex number*.

Note: We observe that the system of complex numbers includes the system of real numbers, i.e., $R \subset C$.

Notes:
- Every real number is a complex number.
- 0 is both purely real and purely imaginary number.
- A complex number is an imaginary number if and only if its imaginary part is non-zero. Here real part may or may not be zero. $4 + 3i$ is an imaginary number but not purely imaginary.
- All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or may not be purely imaginary.

EQUALITY OF COMPLEX NUMBERS

Two complex numbers are said to be *equal* if and only if their real parts and imaginary parts are separately equal.

i.e., $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.

i.e., $z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2)$

and $Im(z_1) = Im(z_2)$.

Note: Inequality relation does not hold good in case of complex numbers having non-zero imaginary parts. For example, the statement $8 + 5i > 4 + 2i$ makes no sense.

ALGEBRA OF COMPLEX NUMBERS

1. Addition For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, their sum is defined as

$$z = z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$$

2. Subtraction For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the subtraction of z_2 from z_1 is defined as

$$z_1 - z_2 = z_1 + (-z_2) = (a_1 - a_2) + i\,(b_1 - b_2).$$

3. Multiplication Multiplication of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1 \cdot z_2 = (a + ib) \cdot (c + id) = (ac - bd) + i\,(ad + bc).$$

Note: The product of complex numbers can be easily computed if we actually carry out the multiplication as given below:

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2bd \\ &= ac + i(ad + bc) - bd \qquad (\because i^2 = -1) \\ &= (ac - bd) + i(ad + bc).\end{aligned}$$

4. Division Division of two complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2,$$

where $x_2 + iy_2 \neq 0$, is defined as

$$\begin{aligned}\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}.\end{aligned}$$

5. Multiplicative Inverse of a Non-zero Complex Number Multiplicative inverse of a non-zero complex number $z = a + ib$ is defined as

$$\begin{aligned}z^{-1} = \frac{1}{z} = \frac{1}{a+ib} &= \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\end{aligned}$$

i.e., $$z^{-1} = \frac{\text{Re}\,(z)}{|z|^2} + i\frac{[-\text{Im}\,(z)]}{|z|^2}.$$

CONJUGATE OF A COMPLEX NUMBER

Conjugate of a complex number $z = a + ib$ is defined as

$$\bar{z} = a - ib.$$

For example, $\overline{4+5i} = 4 - 5i$ and $\overline{4-5i} = 4 + 5i$.

Properties of Modulus

1. $\overline{(\bar{z})} = z$
2. $z = \bar{z}$ if and only if z is purely real
3. $z = -\bar{z}$ if and only if z is purely imaginary
4. $z + \bar{z} = 2\,\text{Re}\,(z)$ and $z - \bar{z} = 2i\,\text{Im}\,(z)$
5. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
6. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
7. $\overline{z_1z_2} = \bar{z}_1 \cdot \bar{z}_2$
8. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$
9. If $z = f\,(z_1)$, then $\bar{z} = f\,(\bar{z}_1)$
10. $\overline{\left(z^n\right)} = (\bar{z})^n$
11. $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}\,(\bar{z}_1 z_2) = 2\,\text{Re}\,(z_1\bar{z}_2)$

The complex number $\frac{a+ib}{c+id}$ in the form $A + iB$

We have, $$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

[Multiplying the Nu. and the Dn. by the conjugate of the Dn.]

$$\frac{(ac+bd) + i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$= A + iB, \text{ where } A = \frac{ac+bd}{c^2+d^2} \text{ and } B = \frac{bc-ad}{c^2+d^2}.$$

Important remark: To put the complex number $\frac{a+ib}{c+id}$ in the form $A + iB$ we should multiply the numerator and the denominator by the conjugate of the denominator.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$, denoted as mod (z) or $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}\,(z),\ b = \text{Im}\,(z).$$

Sometimes, $|z|$ is called absolute value of z. Note that $|z| \geq 0$.

For example, if $z = 3 + 2i$, then $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

Properties of Modulus

1. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$, *i.e.*, $x = 0, y = 0$
2. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$.
3. $z\bar{z} = |z|^2$
4. $-|z| \leq \text{Re}\,(z) \leq |z|$ and $-|z| \leq \text{Im}\,(z) \leq |z|$
5. $|z^n| = |z|^n$
6. $|z_1z_2| = |z_1|\,|z_2|$
7. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
8. $|z_1 \pm z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$
11. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\bar{z}_2)$
12. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\bar{z}_2)$

SQUARE ROOTS OF A COMPLEX NUMBER

Let $z = a + ib$ and let the square root of z be the complex number $x + iy$. Then

$$\sqrt{a+ib} = x + iy$$

or $$(a + ib) = (x + iy)^2 = (x^2 - y^2) + (2xy)\,i$$

Equating real and imaginary part, we get

$$a = x^2 - y^2 \qquad \text{...(1)}$$

and $$b = 2xy \qquad \text{...(2)}$$

Now, $$\begin{aligned}x^2 + y^2 &= \sqrt{(x^2 - y^2)^2 + 4x^2y^2} \\ &= \sqrt{a^2 + b^2} \qquad \text{...(3)}\end{aligned}$$

Solving the equations (1) and (3), we get

$$x = \pm\sqrt{\left(\frac{\sqrt{a^2+b^2} + a}{2}\right)}$$

and $$y = \pm\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}$$

From (2), we can determine the sign of xy. If $xy > 0$, then x and y will have same sign. Thus

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}+i\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}\right]$$

If $xy < 0$, then

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}-i\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}\right]$$

Thus, square roots of $z = a + ib$ are:

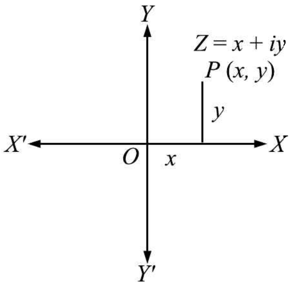
$$\pm\left[\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b > 0 \text{ and}$$

$$\pm\left[\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b < 0$$

ARGAND PLANE AND GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

Let O be the origin and OX and OY be the x-axis and y-axis respectively. Then, any complex number $z = x + iy = (x, y)$ may be represented by a unique point P whose coordinates are (x, y).

The representation of complex numbers as points in a plane forms an **Argand diagram**.

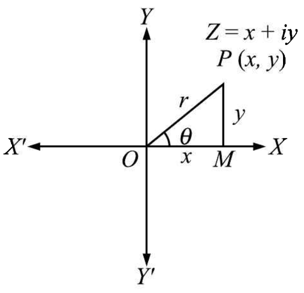


The plane on which complex numbers are represented is known as the **complex plane** or **Argand's plane** or **Gaussian plane**. The x-axis is called the **real axis** and y-axis the **imaginary axis**.

The complex number $z = x + iy$ is known as the **affix** of the point (x, y) which it represents.

POLAR FORM OF A COMPLEX NUMBER

Let O be the origin and OX and OY be the x-axis and y-axis respectively. Let $z = x + iy$ be a complex number represented by the point $P(x, y)$.



Draw $PM \perp OX$. Then, $OM = x$ and $PM = y$. Join OP. Let $OP = r$ and $\angle XOP = \theta$. Then

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

This form of z is called **polar or trigonometric form.**

Comparing real and imaginary parts, we get

$$x = r\cos\theta \quad \text{...(1)}$$

and $$y = r\sin\theta \quad \text{...(2)}$$

Squaring (1) and (2) and adding, we get

$$r^2 = x^2 + y^2 \text{ or } r = \sqrt{x^2+y^2} = |z|$$

Thus, r is known and is equal to the modulus of the complex number z.

Substituting the value of r in (1) and (2), we get

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \quad \text{...(3)}$$

Dividing (2) by (1), we get $\tan\theta = \frac{y}{x}$.

The form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ of the complex number z is called exponential form.

Any value of θ satisfying (3) is known as **amplitude or argument** of z and written as $\theta = \arg(z)$ or $\theta = \text{amp } z$.

The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$, is known as the **principal value of the argument.**

The general value of the argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of arg (z).

While reducing a complex number to polar form, we always take the principal value.

If $x > 0, y > 0$ (i.e., z is in first quadrant), then

$$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

If $x < 0, y > 0$ (*i.e.*, z is in second quadrant), then

$$\arg z = \theta = \pi - \tan^{-1}\left(\frac{y}{|x|}\right).$$

If $x < 0, y < 0$ (*i.e.*, z is in third quadrant), then

$$\arg z = \theta = -\pi + \tan^{-1}\left(\frac{y}{x}\right).$$

If $x > 0, y < 0$ (*i.e.*, z is in fourth quadrant), then

$$\arg z = \theta = -\tan^{-1}\left(\frac{|y|}{x}\right).$$

Notes:
- Argument of the complex number 0 is not defined.
- $\arg(x + i0) = \begin{cases}1, \text{if } x > 0\\ \pi, \text{if } x < 0\end{cases}$
- $\arg(0 + iy) = \begin{cases}\pi/2, \text{if } y > 0\\ 3\pi/2, \text{if } y < 0\end{cases}.$

Properties of Modulus

1. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
3. $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z$

4. $\arg(z^n) = n \arg z$
5. If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.
6. $\arg \bar{z} = - \arg z$

Particular Cases of Polar Form

1. $1 = 1 + i0 = \cos 0 + i \sin 0$
2. $-1 = -1 + i0 = \cos\pi + i\sin\pi$
3. $i = 0 + i1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
4. $-i = 0 + i(-1) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$
5. $1 - i = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$
6. $-1 - i = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$.

Logarithm of Complex Number

$$\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2 + \beta^2) + i\tan^{-1}\frac{\beta}{\alpha}\ (\alpha \neq 0)$$

$$\log(i\beta) = \log\beta + \frac{i\pi}{2}$$

Eular's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta,\ i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$$

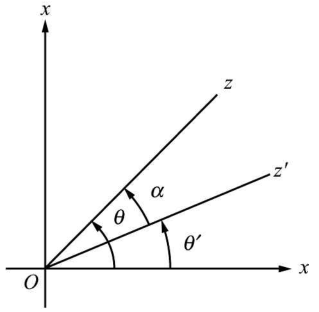
$$\log i = \log e^{\frac{i\pi}{2}} = \frac{i\pi}{2} = \frac{i\pi}{2},\ \log(\log i) = \log\left(\frac{i\pi}{2}\right)$$

$$= \log i + \log\left(\frac{\pi}{2}\right) = \frac{i\pi}{2} + \log(\pi/2).$$

Concept of Rotation

If z and z' are two complex numbers then argument of $\frac{z}{z'}$ is the angle through which Oz' must be turned in order that it may lie along Oz.

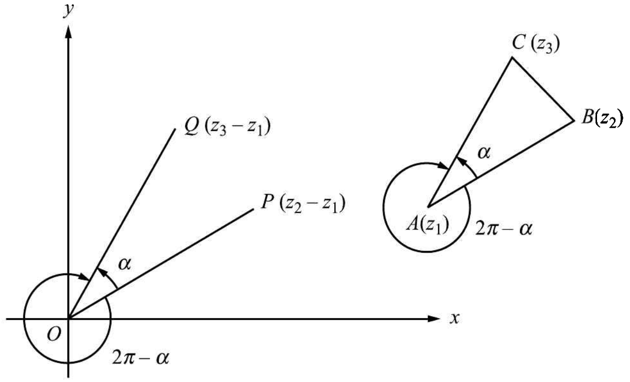
$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\theta'}} = \frac{|z|}{|z'|}e^{i(\theta - \theta')} = \frac{|z|}{|z'|}e^{i\alpha}.$$



In general, let z_1, z_2 z_3 be three vertices of a ΔABC described in the counter-clockwise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP} = (\cos\alpha + i\sin\alpha)$$

$$= \frac{CA}{BA}e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}.$$



Note that $\arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ is the angle through which OP must be rotated in the anti-clockwise direction so that it may lie along OQ.

Note: We can also write $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{-i(2\pi - \alpha)}$.

In this case, we are rotating OP in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

LAPLACE DEMOIVRE'S THEOREM (FOR INTEGRAL VALUES OF *n*)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, *where n is any integer.*

Notes:

- If n is any rational number, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.
- $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta$
  $= \cos n\theta - i\sin n\theta$.
- $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$
  $= \cos(-n\theta) + i\sin(-n\theta)$
  $= \cos n\theta - i\sin n\theta$.
- $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$.
- The theorem cannot be applied to $(\cos\theta + i\sin\theta)^n$ i.e., θ must be same with cos and sin both.
- The theorem is not directly applicable to $(\sin\theta + i\cos\theta)^n$, rather
  $$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$
  $$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right).$$
- $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$
  $= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$.

ROOTS OF A COMPLEX NUMBER

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right],$$

where $k = 0, 1, 2, 3, \ldots (n - 1)$.

Cube roots of unity

Let $z = 1^{1/3}$ or $z^3 - 1 = 0$

$\Rightarrow$ $(z - 1)(z^2 + z + 1) = 0.$

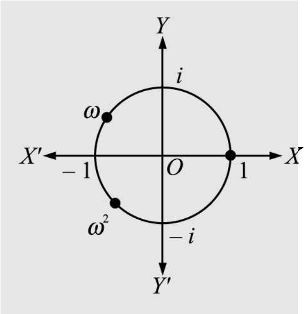
i.e., $z = 1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}.$

Put $\omega = \dfrac{-1+i\sqrt{3}}{2}$, then $\omega^2 = \dfrac{-1+i\sqrt{3}}{2}$. Thus

Cube roots of unity are 1, ω, ω^2.

Properties of Modulus

1. $1 + \omega + \omega^2 = 0$
2. $\omega^3 = 1$
3. $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$
4. $\bar{\omega} = \omega^2$ and $(\bar{\omega})^2 = \omega, \omega\bar{\omega} = \omega^3, \omega = e^{\frac{2\pi i}{3}}, \omega^2 = e^{-\frac{2\pi i}{3}}$
5. The cube roots of unity lie on the unit circle and divide the circumference into three equal parts.
6. If $a + b\omega + c\omega^2 = 0$, then



$a = b = c$ provided a, b, c are real.

Some Useful Relations

1. $x^2 + y^2 = (x + iy)(x - iy)$
2. $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
3. $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
4. $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$, in particular, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
5. $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$, in particular, $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
6. $x^2 + y^2 + z^2 - xy - xz - yz = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
7. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

*n*th ROOTS OF UNITY

Since $1 = \cos 0 + i \sin 0$, therefore,

$$(1)^{1/n} = (\cos 0 + i \sin 0)^{1/n}$$

$$= \cos \frac{2\pi r + 0}{n} + i \sin \frac{2\pi r + 0}{n}; \quad r = 0, 1, 2, \ldots, (n - 1)$$

$$= \cos \frac{2\pi r}{n} + i \sin \frac{2\pi r}{n}; \quad r = 0, 1, 2, \ldots, (n - 1)$$

$$= e^{i\frac{2r\pi}{n}}; \; r = 0, 1, 2, \ldots, (n - 1)$$

$$= 1, e^{(i2\pi/n)}, e^{(i4\pi/n)}, \ldots, e^{[i2(n-1)\pi/n]}$$

$$= 1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}, \text{ where } \alpha = e^{(i2\pi/n)}$$

Properties of Modulus

1. $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$
2. $\alpha \cdot \alpha \cdot \alpha^2 \cdot \ldots \alpha^{n-1} = (-1)^{n-1}$
3. The n, nth roots of unity lie on the unit circle $|z| = 1$ and form the vertices of a regular polygon of n sides.
4. nth roots of unity form a G. P. with common ratio $e^{(i2\pi/n)}$.

GEOMETRY OF COMPLEX NUMBERS

Distance Formula Distance between two points z_1 and z_2 is given by

$$|z_1 - z_2|$$

Section Formula If the line segment joining $A(z_1)$ and $B(z_2)$ is divided by the point $P(z)$ in the ratio $m_1 : m_2$, then

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

But if P divides AB externally in the ratio $m_1 : m_2$, then

$$z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

If P is mid point of AB, then $z = \dfrac{z_1 + z_2}{2}$.

Condition for Collinearity Three points z_1, z_2 and z_3 will be collinear if there exists a relation $az_1 + bz_2 + cz_3 = 0$ (a, b and c are real), such that $a + b + c = 0$. In other words,

three points z_1, z_2 and z_3 are collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$

EQUATION OF STRAIGHT LINE

Equation of straight line through z_1 and z_2 is given by

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

or $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

The general equation of straight line is

$\bar{a}z + a\bar{z} + b = 0$, where b is a real number.

EQUATION OF THE PERPENDICULAR BISECTOR

The equation of the perpendicular bisector of the line segment joining points $A(z_1)$ and $B(z_2)$ is

$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

EQUATION OF A CIRCLE

The equation of a circle with centre z_0 and radius r is $|z - z_0| = r$.

The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where b is a real number. The centre of this circle is '$- a$' and its radius is $\sqrt{a\bar{a} - b}$.

Key Results to Remember

1. $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ is a circle if $k \neq 1$ and is a line if $k = 1$.

2. The equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents a circle if $k \geq \dfrac{1}{2}\,|z_1 - z_2|^2$.

3. If $\arg\left[\dfrac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right] = \pm\pi, 0$, then the points z_1, z_2, z_3, z_4 are concyclic.

4. $|z - z_0| < r$ represents interior of the circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.

EQUATION OF ELLIPSE

If $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$, then the point z describes on ellipse having foci at z_1 and z_2 and $a \in R^+$.

EQUATION OF HYPERBOLA

If $|z - z_1| - |z - z_2| = 2a$, where $2a < |z_1 - z_2|$, then the point z describes a hyperbola having foci at z_1 and z_2 and $a \in R^+$.

Some Properties of Triangle

1. If z_1, z_2, z_3 are the vertices of triangle then centroid z_0 may be given as
$$z_0 = \frac{z_1 + z_2 + z_3}{3}$$

2. If z_1, z_2, z_3 are the vertices of an equilateral triangle then the circumcentre z_0 may be given as
$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

3. If z_1, z_2, z_3 are the vertices of an equilateral triangle then
$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$
or
$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$$

4. If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2, then
$$z_1^2 + z_2^2 + z_3^2 = 2z_2\,(z_1 + z_3).$$

5. If z_1, z_2, z_3 are the vertices of right angled isosceles triangle then
$$(z_1 - z_2)^2 = 2\,(z_1 - z_3)\,(z_3 - z_2).$$

Solved Problems

1. The least positive integer n such that $\left(\dfrac{2i}{1+i}\right)^n$ is a positive integer, is

(a) 16 (b) 8

(c) 4 (d) 2

Solution (a) We have,

$$\left(\frac{2i}{1+i}\right)^2 = \frac{4i^2}{(1+i)^2} = \frac{-4}{1+i^2+2i}$$
$$= \frac{-4}{2i} = \frac{-2}{i} = 2i$$
$$\therefore \left(\frac{2i}{1+i}\right)^4 = 4i^2 = -4 \;\therefore\; \left(\frac{2i}{1+i}\right)^8 = (-4)^2 = \mathbf{16}.$$

2. The smallest integer n for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$, is

(a) 2 (b) 4

(c) 8 (d) 12

Solution (b) Consider $\dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}$

$$= \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i$$
$$\therefore \left(\frac{1+i}{1-i}\right)^n = (i)^n = 1.$$

Now $(i)^n = 1 \Rightarrow n = 4, 8, 12, 16, \ldots$

Hence the smallest integer, $n = \mathbf{4}$.

3. The value of i^i is

(a) ω (b) $-\omega^2$

(c) $\dfrac{\pi}{2}$ (d) none of these

Solution (d) Let $A = i^i$

$\Rightarrow \log A = i \log i$

$\Rightarrow \log A = i \log (0 + i) = i\,[\log 1 + i \tan^{-1} \infty]$

$\Rightarrow \log A = i\left(0 + i\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2} \Rightarrow A = e^{-\pi/2}$.

4. The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) Re $(z) > 0$ (b) Re $(z) < 0$

(c) Re $(z) > 3$ (d) none of these

Solution (c) Given $|z - 4|^2 < |z - 2|^2$

$\Rightarrow |(x - 4) + iy|^2 < |(x - 2) + iy|^2$

$\Rightarrow (x - 4)^2 + y^2 < (x - 2)^2 + y^2$

$\Rightarrow -4x < -12 \Rightarrow 4x > 12;\ x > 3 \Rightarrow$ **Re $(z) > 3$.**

5. The complex number $z = x + iy$ which satisfy the equation $\left|\dfrac{z - 5i}{z + 5i}\right| = 1$ lie on

(a) the x-axis

(b) the line $y = 5$

(c) a circle through the origin

(d) none of these

Solution (a) $\left|\dfrac{z-5i}{z+5i}\right| = 1 \;\Rightarrow\; \left|\dfrac{x+i(y-5)}{x+i(y+5)}\right| = 1$

$\Rightarrow\;\; x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow\;\; 20y = 0;\ y = 0$, which is **$x$-axis.**

6. If $\left|\dfrac{z-5i}{z+6i}\right| = 1$, then locus of z is

(a) x-axis (b) y-axis

(c) $x = 1$ (d) $y = 1$

Solution (a) $\left|\dfrac{z-5i}{z+5i}\right| = 1 \Rightarrow \left|\dfrac{x+i(y-5)}{x+i(y+5)}\right| = 1$

$\Rightarrow\;\; x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow\;\; 20y = 0;\ y = 0$, which is **$x$-axis.**

7. If $z = x + iy$ and 'a' is a real number such that $|z - ai| = |z + ai|$, then locus of z is

(a) x-axis (b) y-axis

(c) $x = y$ (b) $x^2 + y^2 = 1$

Solution (a) We have, $|z - ai| = |z + ai|$

$\Rightarrow\;\; |x + i\,(y-a)|^2 = |x + i(y+a)|^2$

$\Rightarrow\;\; x^2 + (y-a)^2 = x^2 + (y+a)^2$

$\Rightarrow\;\; 4ay = 0;\ y = 0$, which is **$x$-axis.**

8. The locus represented by $|z - 1| = |z + i|$ is

(a) a circle of radius 1

(b) an ellipse with foci at 1 and $-i$

(c) a line through the origin

(d) a circle on the join of 1 and $-i$ as diameter

Solution (c) We have, $|z - 1| = |z + i|$

$\Rightarrow\;\; |(x-1) + iy| = |x + i\,(y+1)|$

$\Rightarrow\;\; (x-1)^2 + y^2 = x^2 + (y+1)^2$

$\Rightarrow\;\; x + y = 0$, which is **a line through the origin.**

9. The imaginary part of $\dfrac{2z+1}{iz+1}$ is -2, then the locus of the point representing z in the complex plane is

(a) circle (b) a straight line

(c) a parabola (d) none of these

Solution (b) Let $z = x + iy$

$$\therefore\ \frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+2iy}{(1-y)+ix}\cdot\left(\frac{1-y-ix}{1-y-ix}\right)$$

$$= \frac{(2x+1)(1-y)+2xy+i[-x(2x+1)+2y(1-y)]}{(1-y)^2+x^2}$$

$\because$ Imaginary part of $\left(\dfrac{2z+1}{iz+1}\right) = -2.$

$\therefore$ We have, $\dfrac{-x(2x+1)+2y(1-y)}{(1-y)^2+x^2} = -2$

or $\quad -2x^2 - 2y^2 - x + 2y = -2\,(1 + y^2 - 2y) - 2x^2$

i.e., $x + 2y - 2 = 0$, which is **a straight line.**

10. If 1, ω, ω^2 be the three cube roots of unity, then $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ ... to $2n$ factors =

(a) 1 (b) -1

(c) 0 (d) none of these

Solution (a) We have, $(1+\omega)(1+\omega^2)(1+\omega^4)$
$(1+\omega^8)$ ... to $2n$ factors

$= (1+\omega)(1+\omega^2)(1+\omega^3\cdot\omega)(1+\omega^6\cdot\omega^2)$
... to $2n$ factors

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)$ ... to $2n$ factors
$[\because\ \omega^3 = \omega^6 = \ldots = 1]$

$= [(1+\omega)(1+\omega)$ ... to n factors$]$
$[(1+\omega^2)(1+\omega^2)$ ... to n factors$]$

$= (1+\omega)^n(1+\omega^2)^n = [(1+\omega)(1+\omega^2)]^n$

$= (1+\omega+\omega^2+\omega^3)^n = (0+1)^n = \mathbf{1}.$
$[\because\ 1+\omega+\omega^2 = 0,\ \omega^3 = 1].$

11. The value of $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8$ is equal to

(a) 4 (b) 6

(c) 8 (d) 2

Solution (d) We have, $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8$

$$= \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^8 + \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^8$$

$= \cos 2\pi + i\sin 2\pi + \cos 2\pi - i\sin 2\pi$

$= 2\cos 2\pi = 2\,(1) = \mathbf{2}$

[By De-Moivre's theorem]

12. If $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right)$, $r = 1, 2, 3, \ldots$, then
$z_1z_2z_3 \ldots \infty =$

(a) i (b) $-i$

(c) 1 (d) -1

Solution (a) Since $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right)$,
$r = 1, 2, 3, \ldots$

we have, $z_1\cdot z_2\cdot z_3 \ldots\infty$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\ldots\infty$$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right)$$

$= \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = 0 + i\cdot 1 = \boldsymbol{i}.$

13. The value of $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ is

(a) 1 (b) – 1

(c) i (d) – i

Solution (c) We have,

$$\sum_{k=1}^{10}\left(-i^2\sin\frac{2\pi k}{11}-i\cos\frac{2\pi k}{11}\right) = \sum_{k=1}^{10}\left(-i^2\sin\frac{2\pi k}{11}-i\cos\frac{2\pi k}{11}\right)$$

$$= -i\sum_{k=1}^{10}\left(\cos\frac{2\pi k}{11}+i\sin\frac{2\pi k}{11}\right) = -i\sum_{k=1}^{10} e^{i\frac{2\pi k}{11}}$$

$$= -i\left[\sum_{k=1}^{10} e^{i\frac{2\pi k}{11}}-1\right]$$

$= -i$ (sum of 11th roots of unity – 1)

$= -i\,(0-1) = \boldsymbol{i}$**.**

14. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$

(c) $z_1 + z_2 = z_3 + z_4$ (d) none of these

Solution (b) Since the points z_1, z_2, z_3, z_4 are the vertices of a parallelogram, therefore, the mid points of the two diagonals must be same. Equating the complex numbers corresponding to the mid-points of the two diagonals, we have

$$\frac{z_1+z_3}{2}=\frac{z_2+z_4}{2}$$

or $\quad z_1 + z_3 = z_2 + z_4$.

15. The value of $\dfrac{4(\cos 75°+i\sin 75°)}{0.4(\cos 30°+i\sin 30°)}$ is

(a) $\dfrac{10}{\sqrt{2}}(1+i)$ (b) $\dfrac{10}{\sqrt{2}}(1-i)$

(c) $\dfrac{5}{\sqrt{2}}(1+i)$ (d) none of these

Solution (a) We have, $\dfrac{4(\cos 75°+i\sin 75°)}{0.4(\cos 30°+i\sin 30°)}$

$$= \frac{10(\cos 75°+i\sin 75°)(\cos 30°-i\sin 30°)}{\cos^2 30°-i^2\sin^2 30°}$$

$$= \frac{10[\cos(75°-30°)+i\sin(75°-30°)]}{\cos^2 30°+\sin^2 30°}$$

$$= 10\,[\cos 45° + i\sin 45°] = \boldsymbol{\frac{10}{\sqrt{2}}(1+i)}.$$

16. If $\sqrt[3]{a-ib} = x - iy$, then $\sqrt[3]{a+ib}$ =

(a) $x + iy$ (b) $x - iy$

(c) $y + ix$ (d) $y - ix$

Solution (a) We have, $\sqrt[3]{a+ib} = x - iy$

$\Rightarrow \quad a - ib = (x - iy)^3 = x^3 - 3x^2\cdot iy + 3x\,(iy)^2 - (iy)^3$

$= (x^3 - 3xy^2) - i\,(3x^2y - y^3)$

$\therefore \quad a + ib = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$= x^3 + 3x^2\cdot(iy) + 3x\,(iy)^2 + (iy)^3$

$= (x + iy)^3$

$\therefore \quad \sqrt[3]{a+ib} = \boldsymbol{x + iy}$**.**

17. If 1, ω, ω^2 are the three cube roots of unity, then $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ ... to $2n$ factors =

(a) 2^n (b) 2^{2n}

(c) 2^{4n} (d) none of these

Solution (b) We have,

$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$

$(1-\omega^8+\omega^{16})$... to $2n$ factors

$= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)$

$(1-\omega^2+\omega)$ ... to $2n$ factors.

$[\because \omega^4 = \omega,\ \omega^8 = \omega^2,\ \omega^{16} = \omega$ and so on$]$

$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$ ... to $2n$ factors

$= (2^2\omega^3)(2^2\omega^3)$ ... to n factors

$[\because (-2\omega)(-2\omega^2) = 2^2\omega^3 = 2^2]$

$= (2^2)^n = \boldsymbol{2^{2n}}$**.**

18. If α and β are the complex cube roots of unity, then $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1}$ =

(a) 0 (b) 1

(c) – 1 (d) none of these

Solution (a) Since α, β are the complex cube roots of unity, we may write

$\alpha = \omega$ and $\beta = \omega^2$.

Hence $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + \omega^8 + \omega^{-1}\omega^{-2}$

$= \omega^3\cdot\omega + \omega^6\cdot\omega^2 + (\omega^3)^{-1}$

$= \omega + \omega^2 + 1 = \mathbf{0}.\ [\because \omega^3 = 1]$

19. $\sqrt{-1-\sqrt{-1-\sqrt{-1-\text{... to }\infty}}}$ =

(a) 1 (b) – 1

(c) ω (d) ω^2

Solution (c), (d) Let $x = \sqrt{-1-\sqrt{-1-\sqrt{-1-\text{... to }\infty}}}$

Then $\quad x = \sqrt{-1-x}$ or $x^2 = -1 - x$ or $x^2 + x + 1 = 0$

$$\therefore \quad x = \frac{-1\pm\sqrt{1-4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-1\pm\sqrt{-3}}{2}$$

$$= \frac{-1\pm\sqrt{3}i}{2} = \omega \text{ or } \omega^2.$$

20. $\left(\dfrac{\sqrt{3}+i}{2}\right)^6 + \left(\dfrac{i-\sqrt{3}}{2}\right)^6$ =

(a) – 2 (b) 2

(c) – 1 (d) 1

Solution (a) We have,

$$\frac{\sqrt{3}+i}{2} = \frac{i\sqrt{3}+i^2}{2i} = -i\left(\frac{-1+\sqrt{3}i}{2}\right) = -i\omega$$

and $$\frac{i-\sqrt{3}}{2} = \frac{i^2-i\sqrt{3}}{2i} = -i\left(\frac{-1-\sqrt{3}i}{2}\right) = -i\omega^2$$

$$\left[\because \frac{1}{i} = -i\right]$$

Hence, $$\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 = (-i\omega)^6 + (-i\omega^2)^6$$

$= i^6(\omega^6 + \omega^{12})$

$= -1\,(1 + 1) = \mathbf{-2}.$

21. The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(a) $-1, \omega$ (b) $-1, \omega^2$

(c) ω, ω^2 (d) none of these

Solution (c) We have, $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow$ $(z + 1)(z^2 + z + 1) = 0.$

Its roots are -1, ω and ω^2. The root $z = -1$ does not satisfy the equation $z^{1985} + z^{100} + 1 = 0$ but $z = \omega$ and $z = \omega^2$ satisfy it. Hence ω and ω^2 are the common roots.

22. The solution of the equation $|z| - z = 1 + 2i$ is

(a) $\frac{3}{2} - 2i$ (b) $\frac{3}{2} + 2i$

(c) $2 - \frac{3}{2}i$ (d) none of these

Solution (a) We have, $|z| - z = 1 + 2i$

$\Rightarrow$ $\sqrt{x^2 + y^2} - (x + iy) = 1 + 2i$, where $z = x + iy$

$\Rightarrow$ $\sqrt{x^2 + y^2} - x = 1$ and $y = -2$

[Comparing real and imaginary parts]

$\Rightarrow$ $x = \frac{3}{2}$ and $y = -2$.

$\therefore$ The solution of the given equation is $\mathbf{\frac{3}{2} - 2i}$.

23. The locus of the complex number z in the Argand plane if $\left|\frac{1 - iz}{z - i}\right| = 1$, is

(a) a circle (b) x-axis

(c) y-axis (d) none of these

Solution (b) Let $z = x + iy$

Given, $\left|\frac{1 - iz}{z - i}\right| = 1 \Rightarrow \left|\frac{1 - i(x + iy)}{x + iy - i}\right| = 1$

$\Rightarrow$ $\left|\frac{1 + y - ix}{x + i(y - 1)}\right| = 1 \Rightarrow \frac{\sqrt{(1 + y)^2 + x^2}}{\sqrt{x^2 + (y - 1)^2}} = 1$

$\Rightarrow$ $(1 + y)^2 + x^2 = x^2 + (y - 1)^2$

$\Rightarrow$ $1 + y^2 + 2y + x^2 = x^2 + y^2 - 2y + 1$

$\Rightarrow$ $4y = 0$

$\Rightarrow$ $y = 0$, which is the equation of **x-axis.**

24. The equation $|z - 1|^2 + |z + 1|^2 = 4$ represents on the Argand plane

(a) a straight line

(b) an ellipse

(c) a circle with centre origin and radius 2

(d) a circle with centre origin and radius unity

Solution (d) We have, $|z - 1|^2 + |z + 1|^2 = 4$ ...(1)

$\Rightarrow$ $(x - 1)^2 + y^2 + (x + 1)^2 + y^2 = 4$

[Putting $z = x + iy$]

$\Rightarrow$ $2(x^2 + y^2 + 1) = 4,$

$\therefore$ $x^2 + y^2 = 1$ or $|z|^2 = 1$

$\Rightarrow$ $|z| = 1$ (since $|z|$ cannot be –ve)

Thus the equation (1) represents all points z on the circle with centre origin and radius unity.

25. The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) Re $(z) > 0$ (b) Re $(z) < 0$

(c) Re $(z) > 2$ (d) none of these

Solution (d) We have, $|z - 4| < |z - 2|$

$\Rightarrow$ $|z - 4|^2 < |z - 2|^2$

$\Rightarrow$ $|x + iy - 4|^2 < |x + iy - 2|^2$

[Putting $z = x + iy$]

$\Rightarrow$ $(x - 4)^2 + y^2 < (x - 2)^2 + y^2$

$\Rightarrow$ $x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$

$\Rightarrow$ $-4x < -12 \Rightarrow x > 3 \Rightarrow$ Re $(z) > 3$.

26. For any integer n, the argument of $z = \frac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

(e) all of the above

Solution (a) We have, $z = \frac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$

$$= \frac{\left(2e^{i\frac{\pi}{6}}\right)^{4n+1}}{\left(2e^{-i\frac{\pi}{3}}\right)^{4n}} = \frac{2^{4n+1}e^{i(4n+1)\frac{\pi}{6}}}{2^{4n}e^{-i4n\frac{\pi}{3}}}$$

$$= 2 \cdot e^{i(12n+1)\frac{\pi}{6}} = 2 \cdot e^{2n\pi i} \cdot e^{\frac{\pi i}{6}}$$

$$= 2 \cdot e^{\pi i/6} \qquad (\because e^{2n\pi i} = 1)$$

$\therefore$ $\arg z = \frac{\pi}{6}$.

27. If n is a positive integer, then $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n$ is equal to

(a) $2^n \cos\frac{n\pi}{6}$ (b) $2^{n+1} \cos\frac{n\pi}{6}$

(c) $2^{n-1} \cos\frac{n\pi}{6}$ (d) none of these

Solution (b) Let $\sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$ so that

$r^2 = 4$ and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow r = 2, \theta = \frac{\pi}{6}$

$\therefore$ $(\sqrt{3} + i)^n = 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$

or $(\sqrt{3} + i)^n = 2^n\left\{\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right\}$ ...(1)

Similarly,

$(\sqrt{3} - i)^n = 2^n\left\{\cos\left(\frac{n\pi}{6}\right) - i\sin\left(\frac{n\pi}{6}\right)\right\}$ ...(2)

Adding (1) and (2), we obtain

$$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n = 2\cdot 2^n\cos\left(\frac{n\pi}{6}\right)$$

$$= \mathbf{2^{n+1}\cos\left(\frac{n\pi}{6}\right)}.$$

28. If n is a positive integer, then $(1+i\sqrt{3})^n+(1-i\sqrt{3})^n=$

(a) $2^{n-1}\cos\frac{n\pi}{3}$ (b) $2^n\cos\frac{n\pi}{3}$

(c) $2^{n+1}\cos\frac{n\pi}{3}$ (d) none of these

Solution (c) Let $r\cos\theta = 1$ and $r\sin\theta = \sqrt{3}$

$\Rightarrow$ $r^2 = 4$ and $\tan\theta = \sqrt{3} \Rightarrow r = 2,\ \theta = \frac{\pi}{3}$

$\therefore$ $1 \pm i\sqrt{3} = 2\left(\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}\right).$

Now $(1+i\sqrt{3})^n+(1-i\sqrt{3})^n$

$$= 2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)+2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$$

$$= 2^n\left(2\cos\frac{n\pi}{3}\right) = \mathbf{2^{n+1}\cos\frac{n\pi}{3}}.$$

29. $\sum_{n=1}^{10} i^n$ is equal to

(a) 0 (b) $i-1$

(c) $i+1$ (d) -1

Solution (b)

$$\sum_{n=1}^{10} i^n = i+i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}$$

$$= i-1-i+1+i-1-i+1+i-1 = \mathbf{i-1}.$$

30. The locus of the point z satisfying the condition $\arg\frac{z-1}{z+1}=\frac{\pi}{3}$ is

(a) a straight line (b) circle

(c) a parabola (d) none of these

Solution (b) We have, $\arg\frac{z-1}{z+1}=\frac{\pi}{3}$

$\Rightarrow$ $\arg\frac{x+iy-1}{x+iy+1}=\frac{\pi}{3}$ [Putting $z = x+iy$]

$\Rightarrow$ $\tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1}=\frac{\pi}{3}$

$$\left(\because \operatorname{Arg}\frac{z_1}{z_2}=\operatorname{Arg} z_1-\operatorname{Arg} z_2\right)$$

$\Rightarrow$ $\tan^{-1}\dfrac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y^2}{x^2-1}}=\frac{\pi}{3} = \frac{\pi}{3}$

$\Rightarrow$ $\frac{2y}{x^2+y^2-1}\ \frac{\pi}{3}=\sqrt{3} = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow$ $x^2+y^2-\frac{2}{\sqrt{3}}y-1 = 0$, which is a **circle.**

31. If $|z-i| < 1$, then $|z+12-6i|$

(a) < 14 (b) < 16

(c) > 14 (d) $= 14$

Solution (a) Given, $|z-i| < 1$

Now, $|z+12-6i| = |(z-i)+(12-5i)|$

$\leq |z-i| + |12-5i|$

$[\because |z_1+z_2| \leq |z_1|+|z_2|]$

$< 1 + 13 = 14.$

Hence $\mathbf{|z+12-6i| < 14}$.

32. $\sqrt{i}-\sqrt{-i}$ is equal to

(a) $i\sqrt{2}$ (b) $\frac{1}{i\sqrt{2}}$

(c) 0 (d) $-i\sqrt{2}$

Solution (a), (d) We have, $i = 0+i\cdot 1 = \frac{1}{2}(0+2i)$

$$= \frac{1}{2}(1+i^2+2\cdot 1\cdot i) = \frac{1}{2}(1+i)^2$$

$\therefore$ $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

$\therefore$ $\sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1-i).$

Hence $\sqrt{i}-\sqrt{-i} = \pm\frac{1}{\sqrt{2}}[(1+i)-(1-i)] = \pm\sqrt{2}\,\mathbf{i}.$

33. If $iz^3+z^2-z+i = 0$, then

(a) $|z| < 1$ (b) $|z| > 1$

(c) $|z| = 1$ (d) $|z| = 0$

Solution (c) Given, $iz^3+z^2-z+i = 0$

$\Rightarrow$ $iz^2(z-i)-(z-i) = 0$

$\Rightarrow$ $(z-i)(iz^2-1) = 0 \Rightarrow z = i$ or $z^2 = \frac{1}{i} = -i.$

Now $z = -i \Rightarrow |z| = |i| = 1$

and $z^2 = -i \Rightarrow |z^2| = |-i| \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

Thus, in both cases $\mathbf{|z| = 1}$.

34. The greatest value of $|z+1|$ if $|z+4| \leq 3$ is

(a) 4 (b) 5

(c) 6 (d) none of these

Solution (c) We have, $|z+1| = |z+4-3|$

$= |(z+4)+(-3)|$

$\leq |z+4| + |-3| = |z+4| + 3$

$\leq 3+3 = \mathbf{6}.$ $(\because |z+4| \leq 3)$

Hence the greatest value of $|z+1|$ is **6.**

35. $\arg bi$ $(b > 0)$ is

(a) π (b) $\frac{\pi}{2}$

(c) $-\frac{\pi}{2}$ (d) 0

Solution (b) Since $b > 0$, bi represents a point on the positive side of the imaginary axis on which the argument of every point is $\frac{\pi}{2}$.

36. If $\omega = \left(\frac{z-i}{1+iz}\right)^n$, n integral, then ω lies on the unit circle for

(a) only even n (b) only odd n
(c) only positive n (d) all n

Solution (d) We have, $\omega = \left(\frac{z-i}{1+iz}\right)^n = \left(\frac{z-i}{i(z-i)}\right)^n$

$= \left(\frac{1}{i}\right)^n = (-i)^n$

$\therefore$ $|\omega| = |(-i)^n| = |-i|^n = 1$ for all n.

$\therefore$ ω lies on unit circle for **all n.**

37. The equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, $b \in R$ represents a circle (not point circle) if

(a) $|a|^2 > b$ (b) $|a|^2 < b$

(c) $|a| > b$ (d) $|a| < b$

Solution (a) We have, $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$

$\Rightarrow$ $z\bar{z} + a\bar{z} + \bar{a}z + a\bar{a} = a\bar{a} - b$

$\Rightarrow$ $(z + a)(\bar{z} + \bar{a}) = a\bar{a} - b \Rightarrow |z + a|^2 = |a|^2 - b.$

This represents a circle (not point circle) if $|a|^2 > b$.

38. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

Solution (b) We have, $|z| = \left|z + \frac{2}{z} - \frac{2}{z}\right| \le \left|z + \frac{2}{z}\right| + \frac{2}{|z|}$

$\Rightarrow$ $|z| \le 2 + \frac{2}{|z|} \Rightarrow |z|^2 \le 2|z| + 2$

$\Rightarrow$ $|z|^2 - 2|z| + 1 \le 1 + 2 \Rightarrow (|z| - 1)^2 \le 3$

$\Rightarrow$ $-\sqrt{3} \le |z| - 1 \le \sqrt{3} \Rightarrow 1 - \sqrt{3} \le |z| \le 1 + \sqrt{3}$

That is, the maximum value of $|z|$ is $1 + \sqrt{3}$.

39. Let z_k ($k = 0, 1, 2, ..., 6$) be the roots of the equation $(z + 1)^7 + z^7 = 0$, then $\sum_{k=0}^{6} \text{Re}(z_k)$ is equal to

(a) 0 (b) $\frac{3}{2}$

(c) $-\frac{7}{2}$ (d) $\frac{7}{2}$

Solution (c) Let $z_k = x_k + iy_k$, we have $(z_k + 1)^7 + z_k^7 = 0$

$\Rightarrow$ $(z_k + 1)^7 = -z_k^7 \Rightarrow |z_k + 1|^7 = |z_k|^7$

$\Rightarrow$ $|z_k + 1| = |z_k| \Rightarrow |x_k + iy_k + 1|^2 = |x_k + iy_k|^2$

$\Rightarrow$ $(x_k + 1)^2 + y_k^2 = x_k^2 + y_k^2$

$\Rightarrow$ $2x_k + 1 = 0$ or $x_k = -\frac{1}{2}$.

Thus, $\sum_{k=0}^{6} \text{Re}(z_k) = \sum_{k=0}^{6} x_k = -\frac{7}{2}$.

40. The equation $z^2 + \bar{z}^2 - 2|z|^2 + z + \bar{z} = 0$ represents a

(a) straight line (b) circle
(c) hyperbola (d) parabola

Solution (d) We have, $z^2 + \bar{z}^2 - 2|z|^2 + z + \bar{z} = 0$

$\Rightarrow$ $(x + iy)^2 + (x - iy)^2 - 2(x^2 + y^2) + x + iy + x - iy = 0$

[Putting $z = x + iy$]

$\Rightarrow$ $2x^2 + 2(iy)^2 - 2x^2 - 2y^2 + 2x = 0$

$\Rightarrow$ $-4y^2 + 2x = 0$ or $y^2 = \frac{1}{2}x$, which is a **parabola.**

41. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter, then the value of λ is

(a) 4 (b) 3

(c) 2 (d) $\sqrt{2}$

Solution (b) We have, $|z - \omega|^2 + |z - \omega^2|^2 = \lambda$

$\Rightarrow$ $\lambda = |\omega - \omega^2|^2 = |\omega^2 + \omega^4 - 2\omega^3|$

$= |\omega^2 + \omega - 2| = |-1 - 2| = \mathbf{3}.$

42. If the area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is 600 square units, then $|z|$ is equal to

(a) 10 (b) 20

(c) 30 (d) none of these

Solution (b) Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}|z|^2$.

$\therefore$ $\frac{3}{2}|z|^2 = 600 \Rightarrow |z| = \mathbf{20}.$

43. If $\text{Re}\left(\frac{z+2i}{z+4}\right) = 0$, then z lies on a circle with centre

(a) $(-2, -1)$ (b) $(-2, 1)$

(c) $(2, -1)$ (d) $(2, 1)$

Solution (a) Let $z = x + iy$

Then,

$$\frac{z+2i}{z+4} = \frac{x+iy+2i}{x+iy+4} = \frac{x+i(y+2)}{(x+4)+iy}$$

$$= \frac{[x+i(y+2)][(x+4)-iy]}{(x+4)^2+y^2}$$

$$= \frac{(x^2+4x+y^2+2y)+i(2x+4y+8)}{(x+4)^2+y^2}$$

Since $\text{Re}\left(\frac{z+2i}{z+4}\right) = 0 \Rightarrow x^2 + y^2 + 4x + 2y = 0$, which represents a circle with centre **(–2, –1).**

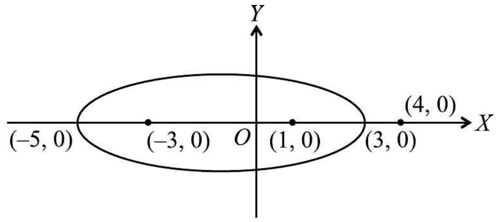
44. If $|z-1|+|z+3| \le 8$, then the range of values of $|z-4|$ is

(a) $(0, 8)$ (b) $[0, 8]$

(c) $[1, 9]$ (d) $[5, 9]$

Solution (c) Given $|z-1|+|z+3| \le 8$

$\therefore$ z lies inside or on the ellipse whose foci are $(1, 0)$ and $(-3, 0)$ and vertices are $(-5, 0)$ and $(3, 0)$.



Now, $|z-4|$ is distance of z from $(4, 0)$. Minimum distance is 1 and maximum is 9.

45. The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(a) i (b) $i-1$

(c) $-i$ (d) 0

Solution (b) $\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n(1+i)$

$= (1+i)\left\{\dfrac{i(1-i^{13})}{1-i}\right\} = (1+i)\left\{\dfrac{i(1-i)}{1-i}\right\}$

$= (1+i)\,i = \mathbf{-1+i}$.

46. The area of the triangle on the Argand plane formed by the complex numbers z, iz and $z + iz$ is

(a) $|z|^2$ (b) $\frac{1}{2}|z|^2$

(c) $\frac{1}{4}|z|^2$ (d) none of these

Solution (b) If $z = x + iy$, then the area of triangle ABC

$$= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$$

[$\because$ Vertices have (x, y), $(-y, x)$, $(x-y, x+y)$ as coordinates]

$$= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{2} = \frac{1}{2}(x^2 + y^2)$$

[$\because$ area is always positive]

$$= \mathbf{\frac{1}{2}}|z|^2.$$

47. If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$, then $b =$

(a) $\sqrt{3}$ (b) $\sqrt{2}$

(c) 1 (d) none of these

Solution (a) Since $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$

$\therefore$ $(\sqrt{3} - i)^{100} = 2^{99}(a - ib)$

$\therefore$ $(\sqrt{3} + i)^{100} - (\sqrt{3} - i)^{100} = 2^{99}(2ib) = 2^{100}(ib)$

$\Rightarrow$ $i^{100}[1 - \sqrt{3}\,i]^{100} - i^{100}[-1 + \sqrt{3}\,i]^{100} = 2^{100}(ib)$

or $(-2\omega)^{100} - (2\omega^2)^{100} = 2^{100}(ib)$

or $\omega - \omega^2 = (ib)$ or $\sqrt{3}\,i = ib$

$\therefore$ $b = \sqrt{3}$.

48. If $\log_{\sqrt{3}}\left(\dfrac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$, then the locus of z is

(a) $|z| < 5$ (b) $|z| = 5$

(c) $|z| > 5$ (d) none of these

Solution (a) $\log_{\sqrt{3}}\left(\dfrac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$

$\Rightarrow$ $\dfrac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2 \Rightarrow |z|^2 - |z| + 1 < 6 + 3|z|$

$\Rightarrow$ $|z|^2 - 4|z| - 5 < 0 \Rightarrow (|z| - 5)(|z| + 1) < 0$

$\Rightarrow$ $|z| - 5 < 0$, since $|z| + 1 > 0 \Rightarrow \mathbf{|z| < 5}$.

49. If $|z^2 - 1| = |z| + 1$, then z lies on

(a) a circle (b) the imaginary axis

(c) the rea axis (d) an ellipse

Solution (b)

50. If $\left(\dfrac{1+i}{1-i}\right)^x = 1$, then

(a) $x = 4n$, where n is any positive integer

(b) $x = 2n$, where n is any positive integer

(c) $x = 4n + 1$, where n is any positive integer

(d) $x = 2n + 1$, where n is any positive integer

Solution (a) $\left(\dfrac{1+i}{1-i}\right)^x = \left[\dfrac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x$

$$= \left[\frac{(1+i)^2}{1-i^2}\right]^e = \left[\frac{1-1+2i}{2}\right]^x$$

$\Rightarrow$ $\left(\dfrac{1+i}{1-i}\right)^x = (i)^x = 1$ (given)

$\Rightarrow$ $(i)^x = (i)^{4n}$,

where n is any positive integer.

$\Rightarrow$ $x = 4n$.

51. The locus of z which satisfies the inequality $\log_{0.3}|z-1| > \log_{0.3}|z-i|$ is given by

(a) $x + y > 0$ (b) $x - y < 0$

(c) $x + y < 0$ (d) $x - y > 0$

Solution (d) By the given condition, $|z-1| < |z-i|$

$\Rightarrow$ $|x - 1 + iy| < |x + i(y-1)|$

$\Rightarrow$ $(x-1)^2 + y^2 < x^2 + (y-1)^2$

$\Rightarrow$ $-2x < -2y \Rightarrow x > y \Rightarrow \mathbf{x - y > 0}$.

52. If α is an imaginary cube root of unity then for $n \in N$, the value of $\alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5}$ is

(a) -1 (b) 0
(c) 1 (d) $3.$

Solution (b) If α is an imaginary cube root of unity, then

$$\alpha^3 = 1,\ 1 + \alpha + \alpha^2 = 0$$

$$\therefore \quad \alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5}$$

$$= \alpha^{3n}[\alpha^1 + \alpha^3 + \alpha^5] = (\alpha^3)^n[\alpha + 1 + \alpha^3 \cdot \alpha^2]$$

$$= (1)^n [1 + \alpha + \alpha^2] \qquad (\because \alpha^3 = 1)$$

$$= 1 + \alpha + \alpha^2 = \mathbf{0}.$$

53. If $|z_1| = |z_2|$ and amp z_1 + amp $z_2 = 0$, then

(a) $z_1 = z_2$ (b) $\bar{z}_2 = z_2$
(c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

Solution (b)

54. $\tan\left[i \log \dfrac{a-ib}{a+ib}\right]$ is equal to

(a) $\dfrac{2ab}{a^2+b^2}$ (b) $\dfrac{a^2-b^2}{2ab}$

(c) $\dfrac{2ab}{a^2-b^2}$ (d) ab

Solution (c)

55. If arg $(z) < 0$, then arg $(-z)$ − arg (z) =
(a) π (b) $-\pi$
(c) $-\dfrac{\pi}{2}$ (d) $\dfrac{\pi}{2}$

Solution (a)

Problems for Practice

Level of Difficulty I

1. $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$ is
(a) positive (b) negative
(c) 0 (d) cannot be determined

2. The value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}} - 1$ is

(a) 2 (b) -2 (c) 1 (d) -1

3. The value of $\left[i^{19} + \left(\dfrac{1}{i}\right)^{25}\right]^2$ is

(a) 4 (b) -4 (c) 2 (d) -2

4. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to
(a) 1 (b) -1 (c) 0 (d) none of these

5. If n is any positive integer, then the value of $\dfrac{i^{4n+1} - i^{4n-1}}{2}$ equals
(a) 1 (b) -1 (c) i (d) $-i$

6. For a positive integer n, the expression

$(1-i)^n \left(1 - \dfrac{1}{i}\right)^n$ equals

(a) 0 (b) $2i^n$ (c) 2^n (d) 4^n

7. If $\dfrac{(a+ib)^2}{a-ib} - \dfrac{(a-ib)^2}{a+ib} = x + iy$, then x =

(a) $\dfrac{-2b^3}{(a^2+b^2)^2}$ (b) $\dfrac{6a^2b}{(a^2+b^2)^2}$
(c) 0 (d) none of these

8. The reciprocal of $3 + \sqrt{7}\, i$ is

(a) $\dfrac{3}{16} - \dfrac{\sqrt{7}}{16} i$ (b) $\dfrac{3}{4} - \dfrac{\sqrt{7}}{4} i$

(c) $\sqrt{7} + 3i$ (d) $3 - \sqrt{7}\, i$

9. $\dfrac{1-ix}{1+ix} = a - ib$ and $a^2 + b^2 = 1$, where a and b are real, then x =

(a) $\dfrac{2a}{(1+a)^2+b^2}$ (b) $\dfrac{2b}{(1+a)^2+b^2}$

(c) $\dfrac{2a}{(1+b)^2+a^2}$ (d) $\dfrac{2b}{(1+b)^2+a^2}$

10. The modulus of $\dfrac{(3+2i)^2}{(4-3i)}$ is

(a) $\dfrac{13}{5}$ (b) $\dfrac{11}{5}$

(c) $\dfrac{9}{5}$ (d) $\dfrac{7}{5}$

11. The multiplicative inverse of $\dfrac{3+4i}{4-5i}$ is

(a) $-\dfrac{8}{25} + \dfrac{31}{25} i$ (b) $\dfrac{8}{25} - \dfrac{31}{25} i$

(c) $-\dfrac{8}{25} - \dfrac{31}{25} i$ (d) none of these

12. If $\dfrac{(a+i)^2}{2a-1} = p + iq$, then $p^2 + q^2$ =

(a) $\dfrac{(a^2+1)^2}{2a^2+1}$ (b) $\dfrac{(a^2+1)^2}{2a^2-1}$

(c) $\dfrac{(a^2+1)^2}{4a^2-1}$ (d) none of these

13. The square roots of $7 + 24i$ are
(a) $\pm(3 + 4i)$
(b) $\pm(3 - 4i)$
(c) $\pm(4 + 3i)$
(d) $\pm(4 - 3i)$

14. The conjugate complex number of $\frac{2-i}{(1-2i)^2}$ is

(a) $\left(\frac{2}{25}\right)+\left(\frac{11}{25}\right)i$ (b) $\left(\frac{2}{25}\right)-\left(\frac{11}{25}\right)i$

(c) $\left(-\frac{2}{25}\right)+\left(\frac{11}{25}\right)i$ (d) $\left(-\frac{2}{25}\right)-\left(\frac{11}{25}\right)i$

15. $\left(\frac{-1+i\sqrt{3}}{2}\right)^{3n}+\left(\frac{-1-i\sqrt{3}}{2}\right)^{3n} =$

(a) 0 (b) 1 (c) 2 (d) 3

16. $(\cos 2\theta + i \sin 2\theta)^{-5} (\cos 3\theta - i \sin 3\theta)^6 (\sin\theta - i \cos\theta)^3 =$

(a) $\cos 25\theta + i\sin 25\theta$ (b) $\cos 25\theta - i\sin 25\theta$

(c) $\sin 25\theta + i\cos 25\theta$ (d) $\sin 25\theta - i\cos 25\theta$

17. $\left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{i-\sqrt{3}}{2}\right)^6 =$

(a) – 2 (b) 2 (c) – 1 (d) 1

18. The complex number z satisfying the equations $|z| - 4 = |z - i| - |z + 5i| = 0$, is

(a) $\sqrt{3} - i$ (b) $2\sqrt{3} - 2i$

(c) $-2\sqrt{3} - 2i$ (d) 0

19. If $z = \cos\theta + i \sin\theta$, then $\frac{z^{2n}-1}{z^{2n}+1} =$

(a) $i\cot n\theta$ (b) $i\tan n\theta$

(c) $\tan n\theta$ (d) $\cot n\theta$

20. If $a = \cos\theta + i \sin\theta$, then $\frac{1-a}{a-a} =$

(a) $\cot\frac{\theta}{2}$ (b) $\cot\theta$

(c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

21. If ω is an imaginary cube root of unity then $(1 + \omega - \omega^2)^7$ equals

(a) 128 ω (b) –128 ω

(c) 128 ω² (d) – 128 ω²

22. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$

(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

23. Let z_1 and $z_2$be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then

(a) $a^2 = b$ (b) $a^2 = b$

(c) $a^2 = 3b$ (d) $a^2 = 4b$

24. If 1, ω, ω² are the cube roots of untiy, then

$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$ is equal to

(a) 0 (b) 1

(c) ω (d) ω²

25. Let z, w be complex numbers such that $z + i\overline{w} = 0$ and $zw = \pi$ Then arg z equals

(a) 3π/4 (b) π/2 (c) π/4 (d) 5π/4.

Level of Difficulty II

26. The complex number z satisfying the equations $|z - i| = |z + 1| = 1$ is

(a) 0 (b) $1 + i$

(c) $-1 + i$ (d) $1 - i$

27. The locus of the complex number z in the Argand plane if $\left|\frac{z-3}{z+3}\right| = 2$, is

(a) a straight line (b) a single point

(c) a circle (d) none of these

28. If $\frac{z-1}{z+1}$ is purely imaginary, then

(a) $|z| > 1$ (b) $|z| < 1$

(c) $|z| = 1$ (d) none of these

29. If $z^4 = (z - 1)^4$, then the roots are represented in the argand plane by the points that are

(a) collinear

(b) concyclic

(c) vertices of a parallelogram

(d) none of these

30. If $z_r = \cos\frac{r\pi}{10} + i\sin\frac{r\pi}{10}$ then $z_1z_2z_3z_4 =$

(a) 0 (b) 1 (c) – 1 (d) none of these

31. If $z^2 - 2z\cos\theta + 1 = 0$, then $z^2 + z^{-2}$ is equal to

(a) $2\cos 2\theta$ (b) $2\sin 2\theta$

(c) $2\cos\theta$ (d) $2\sin\theta$

32. If $|z| = \text{Max.}\{|z - 1|, |z + 1|\}$, then

(a) $|z + \overline{z}| = \frac{1}{2}$ (b) $z + \overline{z} = 1$

(c) $|z + \overline{z}| = 1$ (d) none of these

33. If z_1, z_2, z_3, z_4 are represented by the vertices of a rhombus taken in the anticlockwise order, then

(a) $z_1 + z_2 = z_3 + z_4$ (b) $z_1 - z_2 + z_3 - z_4 = 0$

(c) amp $\frac{z_2 - z_4}{z_1 - z_3} = \frac{\pi}{2}$ (d) amp $\frac{z_1 - z_2}{z_3 - z_4} = \frac{\pi}{2}$

34. If a complex number lies in the third quadrant then its conjugate lies in the quadrant

(a) first (b) second

(c) third (d) none of these

35. The equation $|z - i| + |z + i| = k$ represents an ellipse if $k =$

(a) 1 (b) 2 (c) 4 (d) none of these

36. The region in the Argand diagram defined by $|z - 3| + |z + 3| < 6$ is the interior of the ellipse with major axis along

(a) real axis (b) imaginary axis

(c) $y = x$ (d) $y = -x$

37. If $|z + \bar{z}| + |z - \bar{z}| = 8$, then z lies on

(a) a circle (b) a straight line

(c) a square (d) none of these

38. The complex numbers given by $1 - 3i$, $4 + 3i$ and $3 + i$ represent the vertices of

(a) a right angled triangle

(b) an isosceles triangle

(c) an equilateral triangle

(d) none of these

39. If $\text{Im}\left(\dfrac{z+2i}{z+2}\right) = 0$, then z lies on the curve

(a) $x^2 + y^2 + 2x + 2y = 0$

(b) $x^2 + y^2 - 2x = 0$

(c) $x + y + 2 = 0$

(d) none of these

40. If $\dfrac{z-2}{z+2}$ $(z \neq -2)$ is purely imaginary then $|z|$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

41. The complex numbers $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other, for

(a) $x = n\pi$

(b) $x = 0$

(c) $x = \left(n + \dfrac{1}{2}\right)\pi$

(d) no value of x

42. If ω is the nth root of unity, then

$$(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) \text{ is}$$

(a) 2 (b) 0 (c) 1 (d) -1

43. If ω is an imaginary cube root of unity, then the value of $\dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} - \dfrac{1}{1+\omega}$ is

(a) 0 (b) 1 (c) -1 (d) -2

44. If $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle, then its radius is equal to

(a) $\dfrac{1}{3}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{2}{3}$ (d) 1

45. If z is complex number, then

(a) $|z^2| < |z|^2$

(b) $|z^2| \geq |z|^2$

(c) $|z^2| = |z|^2$

(d) $|z^2| > |z|^2$

46. If $n = 4m + 3$, m integral, then i^n is equal to

(a) $-i$ (b) i (c) 1 (d) -1

47. If $|z - 2| > |z - 4|$, then true statement is

(a) $x > -1$ (b) $x > 1$

(c) $x > -3$ (d) $x > 3$

48. If $z = x + iy$ then $z\bar{z} + 2(z + \bar{z}) + c = 0$ implies:

(a) circle (b) straight line

(c) parabola (d) point

49. The solution of the equation $|z| - z = 1 + 2i$ is

(a) $2 - \dfrac{3}{2}i$ (b) $\dfrac{3}{2} + 2i$

(c) $\dfrac{3}{2} - 2i$ (d) $-2 + \dfrac{3}{2}i$

50. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = \alpha + i\beta$ then $2 \cdot 5 \cdot 10 \ldots (1 + n^2) =$

(a) $\alpha - i\beta$ (b) $\alpha^2 - \beta^2$

(c) $\alpha^2 + \beta^2$ (d) none of these

ANSWERS

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (a)
11. (c)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (c)
21. (d)	**22.** (d)	**23.** (c)	**24.** (a)	**25.** (a)	**26.** (a),(c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (c)
31. (a)	**32.** (c)	**33.** (b),(c)	**34.** (b)	**35.** (c)	**36.** (a)	**37.** (c)	**38.** (d)	**39.** (c)	**40.** (b)
41. (d)	**42.** (b)	**43.** (a)	**44.** (c)	**45.** (c)	**46.** (a)	**47.** (d)	**48.** (a)	**49.** (c)	**50.** (b)

Sequences and Series

19

BRIEF REVIEW OF THE CONCEPTS

SEQUENCE

A succession of numbers $a_1, a_2, ..., a_n$ formed according to some definite rule is called a *sequence*.

A *sequence* is a function whose domain is the set N of natural numbers and range a subset of real numbers or complex numbers.

A sequence whose range is a subset of real numbers is called a *real sequence*. Since we shall be dealing with real sequences only, we shall use the term sequence to denote a real sequence.

Notation The different terms of a sequence are usually denoted by $a_1, a_2, a_3, ...$ or by $t_1, t_2, t_3, ...$ The subscript (always a natural number) denotes the position of the term in the sequence. The term at the *n*th place of a sequence, i.e., t_n is called the *general term of the sequence*.

A sequence is said to be *finite or infinite according* as it has finite or infinite number of terms.

PROGRESSIONS

If the terms of a sequence follow certain pattern, then the sequence is called a *progression*.

SERIES

By adding or subtracting the terms of a sequence, we obtain a *series*. A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

Arithmetic Progression (A.P.)

A sequence whose terms increase or decrease by a fixed number is called an arithmetic progression. The fixed number is called the *common difference* of the A. P.

In an A. P., the first term is usually denoted by a, the common difference by d and the *n*th term by t_n. Obviously

$$d = t_n - t_{n-1}.$$

Thus, an A. P. can be written as

$$a, a + d, a + 2d, ..., a + (n - 1) d, ...$$

For example,

(i) 1, 3, 5, 7, 9, ...

Since, 2nd term – Ist term = 3rd term – 2nd term
= 4th term – 3rd term
= ... = 2,

the sequence 1, 3, 5, 7, ... are in A. P. whose first term is 1 and common difference is 2.

(ii) 5, 3, 1, – 1, – 3, – 5, – 7, ... are in A. P. whose first term is 5 and common difference is –2.

The *n*th term of an Arithmetic Progression If a is the first term and d is the common difference of an A. P., then its *n*th term t_n is given by

$$t_n = a + (n - 1)d$$

Key Points to Remember

1. If an A. P. has n terms, then the *n*th term is called the last term of A. P. and it is denoted by l. That is
$$l = a + (n - 1) d.$$
2. Three numbers a, b, c are in A. P. if and only if
$$b - a = c - b, \text{ i.e., if and only if } a + c = 2b.$$
3. If a is the first term and d the common difference of an A. P. having m terms, then *n*th term from the end is $(m - n + 1)$th term from the beginning. Thus *n*th term from the end $= a + (m - n)d$.
4. Any three numbers in an A. P. can be taken as $a - d, a, a + d$. Any four numbers in an A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$. Similarly 5 numbers in A. P. can be taken as $a - 2d, a - d, a, a + d, a + 2d$.

Sum of *n* terms of an A. P. The sum of n terms of an A. P. with first term 'a' and common difference 'd' is given by

$$S_n = \frac{n}{2}[2a + (n - 1) d]$$

Key Points to Remember

1. If S_n is the sum of n terms of an A. P. whose first term is 'a' and last term is l, then
$$S_n = \frac{n}{2}(a + l).$$
2. If common difference d, number of terms n and the last term l, are given then
$$S_n = \frac{n}{2}[2l - (n - 1) d]$$
3. $t_n = S_n - S_{n-1}$.

Key Points to Remember

1. If $a_1, a_2, a_3, ..., a_n$ are in A. P., then
   (a) $a_1 + k, a_2 + k, ..., a_n + k$ are also in A. P.
   (b) $a_1 - k, a_2 - k, ..., a_n - k$ are also in A. P.
   (c) $ka_1, ka_2, ..., ka_n$ are also in A. P.

(d) $\frac{a_1}{k}, \frac{a_2}{k}, \ldots, \frac{a_n}{k}$, $k \neq 0$ are also in A. P.

2. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two A. P.s, then
   (a) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ are also in A. P.
   (b) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ are also in A. P.
   (c) $a_1b_1, a_2b_2, a_3b_3, \ldots$ are also in A. P.
   (d) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ may not be in A. P.
3. If $a_1, a_2, a_3, \ldots, a_n$ are in A. P., then
   (a) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$
   (b) $a_r = \frac{a_{r-k} + a_{r+k}}{2}$, $0 \leq k \leq n - r$.
4. If nth term of a sequence is a linear expression is n then the sequence is an A. P.
5. If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an A. P.

Arithmetic Mean (A. M.)

Single arithmetic mean A number 'A' is said to be the single A. M. between two given numbers a and b provided a, A, b are in A. P.

For example, since 2, 4, 6 are in A. P., therefore, 4 is the single A. M. between 2 and 6.

***n*-Arithmetic means** The numbers $A_1, A_2, \ldots, A_n$ are said to be the n arithmetic means between two given numbers a and b provided

$$a, A_1, A_2, \ldots, A_n, b \text{ are in A. P.}$$

For example, since 2, 4, 6, 8, 10, 12 are in A. P., therefore, 4, 6, 8, 10 are the four arithmetic means between 2 and 12.

Inserting single A.M. between two given numbers Let a and b be two given numbers and A be the A. M. between them. Then, a, A, b are in A. P. Thus

$$A - a = b - A \text{ or } 2A = a + b, \quad \text{or} \quad A = \frac{a+b}{2}.$$

Inserting *n*-arithmetic means between two given numbers Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between two given numbers a and b. Then $a, A_1, A_2, \ldots, A_n, b$ are in A.P.

Now, b = $(n + 2)$th term of A.P.

$$= a + (n + 2 - 1)\,d = a + (n + 1)\,d$$

or $d = \frac{b-a}{n+1}$, where d is common difference of A.P.

and

$$A_1 = a + d = a + \left(\frac{b-a}{n+1}\right),$$

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right).$$

Note: The sum of n arithmetic means between two given numbers is n times the single A.M. between them, i.e., if a and b are two given numbers and $A_1, A_2, \ldots, A_n$ are n arithmetic means between them, then

$$A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right).$$

GEOMETRIC PROGRESSION (G. P.)

A sequence (finite or infinite) of non-zero numbers in which every term, except the first one, bears a constant ratio with its preceding term, is called a geometric progression.

The constant ratio, also called the common ratio of the G.P., is usually denoted by r.

For example, in the sequence, 1, 2, 4, 8, ...,

$$\frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \ldots = 2, \text{ which is a constant.}$$

Thus, the sequence is a G.P. whose first term is 1 and the common ratio is 2.

*n*th Term of a G. P.

If a is the first term and r is the common ratio of a G. P., then its nth term t_n is given by

$$t_n = ar^{n-1}.$$

Notes:

- If a is the first term and r is the common ratio of a G.P., then the G.P. can be written as
  $$a, ar, ar^2, \ldots, ar^{n-1}, \ldots\ (a \neq 0).$$
- If a is the first term and r is the common ratio of a finite G.P. consisting of m terms, then the nth term from the end is given by ar^{m-n}.
- The nth term from the end of a G.P. with last term l and common ratio r is $\frac{l}{r^{(n-1)}}$.

Sum of *n* terms of a G. P.

The sum of first n terms of a G.P. with first term a and common ratio r is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1.$$

Notes:

- When $r = 1$,
  $S_n = a + a + a + \ldots$ upto n terms $= na$.
- If l is the last term of the G.P., then
  $$S_n = \frac{lr - a}{r - 1}, \; r \neq 1.$$

Sum of an infinite G. P. when |*r*| < 1

The sum of an infinite G.P. with first term a and common ratio r is

$$S_\infty = \frac{a}{1-r}; \text{ when } |r| < 1 \textit{ i.e., } -1 < r < 1.$$

Properties of G.P.

1. If $a_1, a_2, a_3, \ldots$ are in G.P., then
   - a1k, a2k, a3k, ... are also in G.P.
   - $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in G.P.
   - $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also in G.P.
   - $a_1^k, a_2^k, a_3^k \ldots$ are also in G.P.
2. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then
   - a1b1, a2b2, a3b3, ... are also in G.P.
   - $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in G.P.

- $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ may not be in G.P.

3. If $a_1, a_2, a_3, \ldots, a_n$ are in G.P., then

- $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = \cdots$
- $a_r = \sqrt{a_{r-k} a_{r+k}}$, $0 \le k \le n - r$.

4. If $a_1, a_2, a_3, \ldots$, is a G.P. of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$, is also as A.P. and vice-versa.

Geometric Mean (G. M.)

Single geometric mean A number G is said to be the single geometric mean between two given numbers a and b if a, G, b are in G.P.

For example, since 2, 4, 8 are in G.P., therefore, 4 is the G.M. between 2 and 8.

***n*-Geometric means** The numbers $G_1, G_2, \ldots, G_n$ are said to be the n geometric means between two given positive numbers a and b if $a, G_1, G_2, \ldots, G_n, b$ are in G.P.

For example, since 1, 2, 4, 8, 16 are in G.P., therefore, 2, 4, 8 are three geometric means between 1 and 16.

Inserting single G.M. between two given numbers Let a and b be two given positive numbers and G be the G.M. between them. Then a, G, b are in G.P. Thus

$$\frac{G}{a} = \frac{b}{G} \quad \text{or } G^2 = ab, \quad \text{or } G = \sqrt{ab} \qquad [\because G > 0]$$

Inserting *n*-geometric means between two given numbers Let $G_1, G_2, G_3, \ldots, G_n$ be the n geometric means between two given numbers a and b. Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

Now, b = $(n + 2)$th term of G.P.

$= ar^{n+1}$, where r is the common ratio

or $\quad r^{n+1} = \dfrac{b}{a}$ or $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

and

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

Properties of G.P.

1. The product of n geometric means between two given numbers is nth power of the single G.M. between them *i.e.,* if a and b are two given numbers and $G_1, G_2, \ldots, G_n$ are n geometric means between them, then

$$G_1 G_2 G_3 \ldots G_n = (\sqrt{ab})^n.$$

2. If A and G are respectively arithmetic and geometric means between two positive numbers a and b then
   (a) $A > G$
   (b) the quadratic equation having a, b as its roots is
   $$x^2 - 2Ax + G^2 = 0$$
   (c) the two positive numbers are $A \pm \sqrt{A^2 - G^2}$.

HARMONIC PROGRESSION (H.P.)

A sequence of non-zero numbers $a_1, a_2, a_3, \ldots$ is said to be a harmonic progression if the sequence

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \text{ is an A.P.}$$

For example, the sequence $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \ldots$ is a H.P., since the sequence obtained by taking reciprocals of its corresponding terms i.e., 1, 4, 7, 10, ... is an A.P.

A general H.P. is $\dfrac{1}{a}, \dfrac{1}{a+d}, \dfrac{1}{a+2d}, \ldots$

*n*th Term of an H.P.

$$n\text{th term of H.P.} = \frac{1}{n\text{th term of the corresponding A.P.}}$$

Key Points to Remember

1. Three numbers a, b, c are in H.P. if and only if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. i.e., $\frac{1}{a} + \frac{1}{c} = 2 \cdot \frac{1}{b}$, i.e., $b = \dfrac{2ac}{a+c}$.
2. No term of H.P. can be zero
3. There is no general formula for finding the sum of n terms of H.P.
4. Reciprocals of terms of H.P. are in A.P. and then properties of A.P. can be used.

Harmonic Mean (H. M.)

Single harmonic mean A number H is said to be the single harmonic mean between two given numbers a and b if a, H, b are in H.P.

For example, since $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are in H.P., therefore, $\frac{1}{3}$ is the H.M. between $\frac{1}{2}$ and $\frac{1}{4}$.

***n*-harmonic means** The numbers $H_1, H_2, \ldots, H_n$ are said to be the n harmonic means between two given numbers a and b if $a, H_1, H_2, \ldots, H_n, b$ are in H.P. That is,

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

Inserting single H.M. between two given numbers Let a and b be two given numbers and H be the H.M. between them. Then, a, H, b are in H.P. That is,

$$H = \frac{2ab}{a+b}.$$

Inserting *n*-harmonic means between two given numbers Let $H_1, H_2, \ldots, H_n$ be the n harmonic means between two given numbers a and b. Then, $a, H_1, H_2, \ldots, H_n, b$ are in H.P. Thus

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

Now, $\dfrac{1}{b}$ = $(n + 2)$th term of A.P.

$$= \frac{1}{a} + (n + 2 - 1)\, d,$$

where d is common difference of A.P. Thus, we get

$$d = \frac{\frac{1}{b} - \frac{1}{a}}{n+1} = \frac{a-b}{ab(n+1)}.$$

Now, $\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} = \frac{bn+a}{ab(n+1)}$

or $\quad H_1 = \frac{ab(n+1)}{bn+a}$

$$\frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + = \frac{2(a-b)}{ab(n+1)} = \frac{2(a-b)}{ab(n+1)}$$

or $\quad H_2 = \frac{ab(n+1)}{2a+(n-1)b}$

$\vdots \qquad \vdots$

In general, we have $H_n = \frac{ab(n+1)}{na+b}$.

RELATION BETWEEN A.M., G.M. AND H.M.

Let A, G, and H be arithmetic, geometric and harmonic means between two numbers a and b. Then,

1. $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$
2. $G^2 = AH$
3. $A \geq G \geq H$.

Note: If the two quantities are equal, i.e., $a = b$ then

$$A - G = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 = 0 \quad \therefore \quad A = G$$

From (3), $\frac{G}{H} = \frac{A}{G} = 1 \therefore G = H$

Hence $A = G = H$

Thus $A \geq G \geq H$, equality holds only when $a = b$.

SOME SPECIAL SEQUENCES

1. The sum of first n natural numbers

$$\Sigma n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

2. The sum of squares of first n natural numbers

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of cubes of the first n natural numbers

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Notes: If nth term of a sequence is

$$T_n = an^3 + bn^2 + cn + d,$$

then the sum of n terms is given by

$$S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d,$$

which can be evaluated using the above results.

ARITHMETICO-GEOMETRIC SEQUENCE (A.G.S.)

A sequence is said to be an arithmetico geometric sequence if its each term is the product of the corresponding terms of an A.P. and a G.P. In other words, if $a_1, a_2, a_3, \ldots$ is an A.P. and $b_1, b_2, b_3, \ldots$ is a G.P., then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots$ is an arithmetico-geometric sequence.

Properties of G.P.

1. The general form of an arithmetico-geometric sequence is

$$a,\ (a + d)\ r,\ (a + 2d)\ r^2,\ \ldots$$

The nth term of this sequence is: $[a + (n - 1)\ d]\ r^{n-1}$.

2. The sum of n terms of an A.G. sequence is given by

$$S_n = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)},$$

$r \neq 1$.

3. The sum of an infinite A.G. sequence is given by

$$S_\infty = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2}, \text{ where } |r| < 1.$$

METHOD OF DIFFERENCES

Suppose $a_1, a_2, a_3, \ldots$ is a sequence such that the sequence $a_2 - a_1, a_3 - a_2, \ldots$ is either an A.P. or a G.P. The nth term 'a'$_n$ of this sequence is obtained as follows:

$$S = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$S = \quad a_1 + a_2 + \ldots + a_{n-1} + a_n$$

$$\Rightarrow \quad a_n = a_1 + [a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})]$$

Since the terms within the brackets are either in an A.P. or in a G.P., we can find the value of a_n, the nth term, We can now find the sum of the n terms of the sequence as

$$S = \sum_{k=1}^{n} a_k.$$

Solved Problems

1. The number of terms common to two A.P.s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is

(a) 21	(b) 28
(c) 14	(d) none of these

Solution (c) By inspection, first common term to both the series is 23. Second common term = 51.

Third = 79 and so on. These nos. form an A.P. 23, 51, 79, ...

Since, $T_{15} = 23 + 14\ (28) = 23 + 392 = 415 > 407$

and $\quad T_{14} = 23 + 13\,(28) = 387 < 407$

∴ number of common terms = **14.**

2. The number of odd numbers between 60 and 360 is

(a) 148 (b) 150

(c) 153 (d) none of these

Solution (b) Let the number of odd numbers between 60 and 360 be n.

Here, the first term, $a = 61$; the common difference, $d = 2$; the nth term $= 359$.

Then $359 = 61 + (n - 1) \times 2$; or $n = 150$.

Hence, the number of odd numbers between 60 and 360 is **150.**

3. The number of numbers lying between 100 and 500 that are divisible by 7 but not by 21 is

(a) 57 (b) 19

(c) 38 (d) none of these

Solution (c) The numbers between 100 and 500 that are divisible by 7 are 105, 112, 119, 126, 133, 140, 147, ..., 483, 490, 497.

Let such numbers be n.

Then, $497 = 105 + (n - 1) \times 7$; or $n = 57$.

The numbers between 100 and 500 that are divisible by 21 are 105, 126, 147, ..., 483.

Let such numbers be m.

Then $483 = 105 + (m - 1) \times 21$; or $m = 19$.

Hence, the required number $= n - m = 57 - 19 =$ **38.**

4. The middle term in the following arithmetic progression 20, 16, 12, ..., – 176 is

(a) – 46 (b) – 76

(c) – 80 (d) none of these

Solution (b), (c) Here, the first term, $a = 20$; the common difference, $d = -4$ and the nth term $= -176$.

Then, $-176 = 20 + (n - 1)(-4)$; or $n = 50$.

Hence, the middle terms are 25th and 26th.

25th term $= 20 + 24 \times -4 = -\mathbf{76}$

26th term $= 20 + 25 \times -4 = -\mathbf{80}$**.**

5. In the series 3, 7, 11, 15, ... and 2, 5, 8, ... each continued to 100 terms, the number of terms that are identical is

(a) 21 (b) 27

(c) 25 (d) none of these

Solution (c) Let the nth term of the first series = the mth term of the second series.

$$\therefore \quad 3 + (n - 1) \cdot 4 = 2 + (m - 1) \cdot 3,$$

$$\text{or} \quad 4n = 3m \quad \text{or} \quad \frac{n}{3} = \frac{m}{4} = k \text{ (say)}$$

$$\therefore \quad n = 3k \text{ and } m = 4k.$$

As each series is continued to 100 terms,

$$n = 3k \le 100 \text{ and } m = 4k \le 100$$

∴ Possible values of k are 1, 2, 3, ..., 25 and corresponding to each value of k we get one identical term. Hence there are **25** identical terms.

6. The sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, is

(a) 865 (b) 900

(c) 930 (d) none of these

Solution (b) Since, in an A.P., the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term,

∴ We have, $a_5 + a_{20} = a_1 + a_{24}$, $a_{10} + a_{15} = a_1 + a_{24}$.

Hence, the given relation gives

$$3\,(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = 75,$$

$$\therefore \quad S_{24} = \left(\frac{24}{2}\right)(a_1 + a_{24}) = 12 \times 75 = \mathbf{900}.$$

7. The maximum sum of the series

$$20 + 19\frac{1}{3} + 18\frac{2}{3} + 18 + \ldots \text{ is}$$

(a) 310 (b) 290

(c) 320 (d) none of these

Solution (a) The given series is arithmetic whose first term = 20, common difference $= -\dfrac{2}{3}$.

As the common difference is negative, the terms will become negative after some stage. So the sum is maximum if only positive terms are added.

$$\text{Now } t_n = 20 + (n - 1)\left(-\frac{2}{3}\right) \ge 0 \text{ if } 60 - 2\,(n - 1) \ge 0;$$

$$\text{or} \quad 62 \ge 2n \quad \text{or} \quad 31 \ge n.$$

∴ The first 31 terms are non-negative.

$$\therefore \text{ Maximum sum} = S_{31} = \frac{31}{2}\left\{2 \times 20 + (31 - 1)\left(-\frac{2}{3}\right)\right\}$$

$$= \frac{31}{2}\{40 - 20\} = \mathbf{310}.$$

8. The sum of all natural numbers less than 200, that are divisible neither by 3 nor by 5, is

(a) 10730 (b) 10732

(c) 15375 (d) none of these

Solution (b) The required sum

$$= (1 + 2 + 3 + \ldots + 199) - (3 + 6 + 9 + \ldots + 198) - (5 + 10 + 15 + \ldots + 195) + (15 + 30 + 45 + \ldots + 195)$$

$$= \frac{199}{2}(1 + 199) - \frac{66}{2}(3 + 198) - \frac{39}{2}(5 + 195) + \frac{13}{2}(15 + 195)$$

$$= 199 \times 100 - 33 \times 201 - 39 \times 100 + 13 \times 105 = \mathbf{10732}.$$

9. The sum of 11 terms of an A.P. whose middle term is 30, is

(a) 320 (b) 330

(c) 340 (d) 350

Solution (b) Let 'a' be the first term and 'd' the common difference of the given A.P.

Middle term *i.e.*, 6th term

$$= a + (6 - 1)\,d$$

$$= a + 5d = 30 \quad \text{(Given)}$$

$$\text{Now,} \quad S_{11} = \frac{11}{2}[2a + (11 - 1)\,d]$$

$$= \frac{11}{2}\,(2a + 10d) = 11\,(a + 5d)$$

$$= 11 \times 30 = \mathbf{330.}$$

10. The interior angles of a polygon are in A.P. If the smallest angle is 120° and the common difference is 5, then the number of sides of the polygon is

(a) 16 (b) 8

(c) 9 (d) none of these

Solution (c) Sum of the interior angles of a polygon of n sides.

$$= (2n - 4)\,\frac{\pi}{2} = (n - 2)\,\pi = (n - 2)\cdot 180°$$

Also $a = 120°$, $d = 5°$

$$\therefore \quad \frac{\pi}{2}\,[2\cdot 120° + (n - 1)\,5°] = (n - 2) \times 180°$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n - 9)\,(n - 16) = 0$$

$\therefore$ $n = 9, 16$. But when

$n = 16, T_n = a + (16 - 1)\,d = 120° + 15 \times 5° = 195°.$

This is not possible as interior angle cannot be greater than 180°. $\therefore$ $n = \mathbf{9.}$

11. If the sum of any number of terms in a sequence is always three times the squared number of these terms, then the sequence is

(a) an A. P. (b) a G. P.

(c) an H. P. (d) none of these

Solution (a) Given, $S_n = 3n^2$

$$\therefore \quad T_n = S_n - S_{n-1} = 3n^2 - 3\,(n - 1)^2 = 6n - 3.$$

Putting $n = 1, 2, 3, \ldots$, the sequence is 3, 9, 15, 21,... which is clearly an **A.P.**

12. Let S_n denotes the sum of n terms of an A.P. whose first term is a. If the common difference

$$d = S_n - k\,S_{n-1} + S_{n-2} \text{ then } k =$$

(a) 1 (b) 2

(c) 3 (d) none of these

Solution (b) We have, $a_n = S_n - S_{n-1}$ ...(1)

and $a_{n-1} = S_{n-1} - S_{n-2}$ ...(2)

$$\therefore \quad d = a_n - a_{n-1}$$

$$= (S_n - S_{n-1}) - (S_{n-1} - S_{n-2})$$

[From (1) and (2)]

$$= S_n - 2S_{n-1} + S_{n-2}.$$

13. The sum of positive terms of the series

$$10 + 9\,\frac{4}{7} + 9\frac{1}{7} + \ldots \text{ is}$$

(a) $\dfrac{352}{7}$ (b) $\dfrac{437}{7}$

(c) $\dfrac{852}{7}$ (d) none of these

Solution (c) Here $a = 10$, $d = -\dfrac{3}{7}$.

Then, $t_n = 10 + (n - 1)\left(-\dfrac{3}{7}\right)$.

t_n is positive if $10 + (n - 1)\left(-\dfrac{3}{7}\right) \geq 0;$

or $70 - 3\,(n - 1) \geq 0$

or $73 \geq 3n;$ or $24\,\dfrac{1}{3} \geq n$

$\therefore$ First 24 terms are positive.

$\therefore$ Sum of the positive terms

$$= S_{24} = \frac{24}{2}\left[2\times 10 + 23\times\frac{-3}{7}\right]$$

$$= 12\left[20 - \frac{69}{7}\right] = \frac{\mathbf{852}}{\mathbf{7}}.$$

14. The minimum number of terms from the beginning of the series $20 + 22\,\frac{2}{3} + 25\,\frac{1}{3} + \ldots$, so that the sum may exceed 1568, is

(a) 25 (b) 27

(c) 28 (d) 29

Solution (d) It is in A.P. for which $a = 20, d = 2\,\dfrac{2}{3} = \dfrac{8}{3}$

Now, $S_n > 1568 \;\Rightarrow\; \dfrac{n}{2}\left[40 + (n-1)\dfrac{8}{3}\right] > 1568$

$$\Rightarrow \quad \frac{n}{2}\cdot\frac{112 + 8n}{3} > 1568 \Rightarrow n^2 + 14n > \frac{6}{8} \times 1568 = 1176$$

$$\Rightarrow \quad n^2 + 14n - 1176 > 0,$$

or $(n + 42)\,(n - 28) > 0$

As n is positive, $n - 28 > 0$ i.e., $n > 28$

$\therefore$ Minimum value of $n = \mathbf{29.}$

15. The maximum sum of the A.P. 40, 38, 36, 34, ... is

(a) 390 (b) 420

(c) 460 (d) none of these

Solution (d) It is in A.P. for which $a = 20, d = 2\,\dfrac{2}{3} = \dfrac{8}{3}$

Now, $S_n > 1568 \;\Rightarrow\; \dfrac{n}{2}\left[40 + (n-1)\dfrac{8}{3}\right] > 1568$

$$\Rightarrow \quad \frac{n}{2}\cdot\frac{112 + 8n}{3} > 1568 \Rightarrow n^2 + 14n > \frac{6}{8} \times 1568 = 1176$$

$$\Rightarrow \quad n^2 + 14n - 1176 > 0,$$

or $(n + 42)\,(n - 28) > 0$

As n is positive, $n - 28 > 0$ i.e., $n > 28$

$\therefore$ Minimum value of $n = 29.$

16. If S_1 is the sum of an arithmetic series of 'n' odd number of terms and S_2, the sum of the terms of the series in odd places, then $\dfrac{S_1}{S_2} =$

(a) $\dfrac{2n}{n+1}$ (b) $\dfrac{n}{n+1}$

(c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

Solution (a) Let the odd number of terms of an arithmetic series be

$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \ldots\ldots,\ a+(n-1)\,d$

Then, $S_1 = \frac{n}{2}\ \{2a+(n-1)\ d\}$

$$S_2 = a+(a+2d)+(a+4d)+\ldots \text{ to } \frac{n+1}{2} \text{ terms}$$

$$= \frac{n+1}{2\times 2}\left\{2a+\left(\frac{n+1}{2}-1\right)\times 2d\right\}$$

$$= \frac{n+1}{4}\ \{2a+(n-1)\ d\}.$$

$$\therefore\quad \frac{S_1}{S_2}=\frac{2n}{n+1}.$$

17. If a is the first term, d the common difference and S_k the sum to k terms of an A.P., then for $\frac{S_{kx}}{S_x}$ to be independent of x

(a) $a = 2d$ (b) $a = d$

(c) $2a = d$ (d) none of these

Solution (c) We have, $\frac{S_{kx}}{S_x}=\frac{\frac{kx}{2}[2a+(kx-1)d]}{\frac{x}{2}[2a+(x-1)d]}$

$$=\frac{k[(2a-d)+kxd]}{(2a-d)+xd}$$

For $\frac{S_{kx}}{S_x}$ to be independent of x, $2a-d=0$ or $\mathbf{2a=d}$.

18. Between two numbers whose sum is $2\frac{1}{6}$, an even number of arithmetic means are inserted. If the sum of these means exceeds their number by unity, then the number of means are

(a) 12 (b) 10

(c) 8 (d) none of these

Solution (a) Let $2n$ arithmetic means be $A_1, A_2, A_3, \ldots, A_{2n}$ between a and b.

Then $A_1+A_2+A_3+\ldots+A_{2n}=\frac{a+b}{2}\times 2n$

$$=\frac{\frac{13}{6}}{2}\times 2n=\frac{13n}{6}$$

Given, $A_1+A_2+A_3+\ldots+A_{2n}=2n+1$;

$\therefore\quad 2n+1=\frac{13n}{6}$; or $12n+6=13n$; $\therefore\ n=6$.

∴ The number of means $= 2n = 2\times 6 = \mathbf{12}$.

19. If a, b, c are in A.P. and p is the A.M. between a and b and q is the A.M. between b and c, then

(a) a is the A.M. between p and q

(b) b is the A.M. between p and q

(c) c is the A.M. between p and q

(d) none of these

Solution (b) $\because$ a, b, c are in A.P.,

$\therefore\quad 2b=a+c$ ...(1)

$\because\quad p$ is the A.M. between a and b

$\therefore\quad p=\frac{a+b}{2}$ ...(2)

$\because$ q is the A.M. between b and c

$\therefore\quad q=\frac{b+c}{2}$ ...(3)

Adding (2) and (3)

$$p+q=\frac{a+b}{2}+\frac{b+c}{2}=\frac{a+c+2b}{2}$$

$$=\frac{2b+2b}{2}=2b \quad \text{[Using (1)]}$$

$\therefore\quad 2b=p+q$ or $b=\frac{p+q}{2}$

Hence **b is the A.M. between p and q.**

20. If 11 A.M.s are inserted between 28 and 10, then the middle term in the series is

(a) 15 (b) 19

(c) 21 (d) none of these

Solution (b) Let $A_1, A_2, A_3, \ldots, A_{11}$ be 11 A.M.s between 28 and 10

$\therefore\quad 28, A_1, A_2, \ldots, A_{11}, 10$ are in A.P.

Let 'd' be the common difference of A.P.

Also the number of terms = 13.

$$10=a_{13}=a_1+12d=28+12d$$

$$\therefore\quad d=\frac{10-28}{12}=-\frac{18}{12}=-\frac{3}{2}$$

$\because$ Total number of terms = 13 (odd)

$\therefore$ Middle term $=\left(\frac{13+1}{2}\right)$th $=$ 7th $=a_7$

$$=a_1+6d=28+6\left(-\frac{3}{2}\right)=28-9=19.$$

Hence middle term of the A.P. = **19.**

21. If a, b, c are three consecutive terms of an A.P., then k^a, k^b, k^c are three consecutive terms of

(a) an A.P. (b) a G.P.

(c) an H.P. (d) none of these

Solution (b) Since a, b, c are in A.P.

$\Rightarrow\quad b-a=c-b\Rightarrow k^{b-a}=k^{c-b}$

$\Rightarrow\quad \frac{k^b}{k^a}=\frac{k^c}{k^b}\Rightarrow k^a, k^b, k^c$ are in **G.P.**

22. If p^{th}, q^{th} and r^{th} terms of a G.P. are themselves in G.P., then p, q, r are in

(a) A. P. (b) G. P.

(c) H. P. (d) none of these

Solution (a) Let A be the first term and R the common ratio of G.P.

Then $a_p=AR^{p-1}$

$a_q=AR^{q-1}$ and $a_r=AR^{r-1}$

Since a_p, a_q, a_r in the G.P.

$\Rightarrow (a_q)^2 = a_p \times a_r \Rightarrow (AR^{q-1})^2 = AR^{p-1} \times AR^{r-1}$

$\Rightarrow A^2 R^{2q-2} = A^2 R^{p+r-2}$

$\Rightarrow 2q - 2 = p + r - 2 \Rightarrow 2q = p + r \Rightarrow p, q, r$ in **A.P.**

23. If p, q, r are in A.P. and x, y, z are in G.P., then $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} =$

(a) 1 (b) 2

(c) – 1 (d) none of these

Solution (a) Let d be the common difference of A.P. and R ($\neq 0$), the common ratio of G.P., then

$$q = p + d, \; r = p + 2d$$

and $y = xR, \; z = xR^2$

so that $q - r = -d, \; r - p = 2d, \; p - q = -d$

$$\therefore \quad x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = x^{d} \cdot (xR)^{2d} \cdot (xR^2)^{-d}$$

$$= (x^{-d} \cdot x^{2d} \cdot x^{-d})\,(R^{2d} \cdot R^{-2d})$$

$$= (x^{-d+2d-d}) \cdot (R^{2d-2d})$$

$$= x^0 \cdot R^0 = 1 \times 1 = \mathbf{1}.$$

24. If the sum of the first three terms of a G.P. is 21 and the sum of the next three terms is 168, then the first term and the common ratio is

(a) 3, 4 (b) 2, 4

(c) 3, 2 (d) none of these

Solution (c) Let a be the first term and r, the common ratio of G.P.

Given $a_1 + a_2 + a_3 = 21 \quad \Rightarrow a + ar + ar^2 = 21$

$\Rightarrow a\,(1 + r + r^2) = 21$ ...(1)

Again $a_4 + a_5 + a_6 = 168$ (Given)

$\Rightarrow \quad ar^3 + ar^4 + ar^5 = 168$

$\Rightarrow \quad ar^3\,(1 + r + r^2) = 168$ ...(2)

Dividing (2) by (1), $r^3 = 8$,

$\therefore \quad r = \sqrt[3]{8} = \mathbf{2}$

$\therefore$ From (1), $a(1 + 2 + 4) = 21$

$\therefore \quad a = 21 \div 7 = \mathbf{3}.$

25. The product $9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27}$ ... to infinity is

(a) 9 (b) 3

(c) 81 (d) none of these

Solution (b) $9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27}$ ... to infinity

$$= 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots \text{ to } \infty} = 9^{\frac{1/3}{1-1/3}} = 9^{\frac{1/3}{2/3}} = 9^{\frac{1}{2}}1 = \mathbf{3.}$$

26. If, in a G.P., the $(p + q)$th term is a and the $(p - q)$th term is b, then pth term is

(a) $(ab)^{1/3}$ (b) $(ab)^{1/2}$

(c) $(ab)^{1/4}$ (d) none of these

Solution (b) Let the G.P. be $x, xy, xy^2, \ldots$

Then $t_{p+q} = xy^{p+q-1} = a$ ...(1)

and $t_{p-q} = xy^{p-q-1} = b$ ...(2)

Dividing (1) by (2), $y^{2q} = \dfrac{a}{b}$; $\therefore \; y = \left(\dfrac{a}{b}\right)^{\frac{1}{2q}}$

From (1), $x = a\left(\dfrac{b}{a}\right)^{\frac{p+q-1}{2q}}$

$$\therefore \quad t_p = xy^{p-1} = a \cdot \left(\frac{b}{a}\right)^{\frac{p+q-1}{2q}} \left(\frac{a}{b}\right)^{\frac{p-1}{2q}}$$

$$= a^{1-\frac{p+q-1}{2q}+\frac{p-1}{2q}} \cdot b^{\frac{p+q-1}{2q}-\frac{p-1}{2q}}$$

$$= a^{\frac{1}{2}} b^{\frac{1}{2}} = \sqrt{\boldsymbol{ab}} \; .$$

27. Let a be the first term and b be the nth term of a G.P. If P is the product of n terms, then $P^2 =$

(a) ab (b) $(ab)^n$

(c) $(ab)^{n/2}$ (d) none of these

Solution (b) If r be the common ratio of the given G.P., then

$b = ar^{n-1}$; $P = a \times ar \times ar^2 \times ar^3 \times \ldots \times ar^{n-1}$

$$\Rightarrow \quad P = a^n \cdot r^{1+2+3+\ldots+(n-1)} = a^n \cdot r^{\frac{n(n-1)}{2}}$$

$$\therefore \quad P^2 = a^{2n}\, r^{n(n-1)} = (a \cdot ar^{n-1})^n = \boldsymbol{(ab)^n}.$$

28. If $a^x = b^y = c^z = k$ and x, y, z are in G.P., then

(a) $\log_b a = \log_c b$ (b) $\log_c a = \log_a c$

(c) $\log_b a = \log_b c$ (d) none of these

Solution (a) Given, $a^x = b^y = c^z = k$ (say)

Taking, logarithm, we get

$x \log a = y \log b = z \log c = \log k.$

$$\therefore \quad x = \frac{\log k}{\log a},\; y = \frac{\log k}{\log b} \text{ and } z = \frac{\log k}{\log c}.$$

Since x, y, z are in G.P. $\therefore \; \dfrac{y}{x} = \dfrac{z}{y}$

$$\therefore \quad \frac{\log k}{\log b} \times \frac{\log a}{\log k} = \frac{\log k}{\log c} \times \frac{\log b}{\log k} \text{ or } \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

or $\quad \boldsymbol{\log_b a = \log_c b.}$ $\quad \left[\because \dfrac{\log a}{\log b} = \log_b a\right]$

29. The sum to n terms of the series

$\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \dfrac{15}{16} + \ldots$ is given by

(a) $2^n - 1$ (b) $2^n - n - 1$

(c) $2^{-n} + n - 1$ (d) $1 - 2^{-n}$

Solution (c) We have, $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \dfrac{15}{16} + \ldots$ to n terms

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \ldots \text{ to } n \text{ terms}$$

$$= n - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots n \text{ terms}\right)$$

$$= n - \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = n - 1 + \frac{1}{2^n} = \boldsymbol{n - 1 + 2^{-n}}.$$

30. If two geometric means g_1 and g_2 and one arithmetic mean A be inserted between two numbers, then

$$\frac{g_1^2}{g_2} + \frac{g_2^2}{g_1} =$$

(a) $4A$ (b) $3A$

(c) $2A$ (d) A

Solution (c) Let the two numbers be a and b.

$\therefore \quad A = \dfrac{a + b}{2}$ or $2A = a + b$ ...(*i*)

Again, a, g_1, g_2, b are in G.P.

∴ $\frac{g_1}{a}=\frac{g_2}{g_1}=\frac{b}{g_2}$

Now $\frac{g_1}{a}=\frac{g_2}{g_1}\Rightarrow\frac{g_1^2}{g_2}=a$

and $\frac{g_2}{g_1}=\frac{b}{g_2}\Rightarrow\frac{g_2^2}{g_1}=b$

∴ $a+b=\frac{g_1^2}{g_2}+\frac{g_2^2}{g_1}$ ...(*ii*)

∴ From (*i*) and (*ii*), we get

$\mathbf{2A}=\frac{g_1^2}{g_2}+\frac{g_2^2}{g_1}$.

31. If a, b, c are in A.P., x is the G.M. between a and b, y is the G.M. between b and c, then b^2 is

(a) A.M. between x^2 and y^2
(b) G.M. between x^2 and y^2
(c) H.M. between x^2 and y^2
(d) none of these

Solution (a) ∵ a, b, c are in A.P.

∴ $2b = a + c$

x is the G.M. between a and b, ∴ $x^2 = ab$.

y is the G.M. between b and c, ∴ $y^2 = bc$.

Now, $x^2 + y^2 = ab + bc = b(a + c) = b(2b) = 2b^2$

∴ $b^2=\frac{x^2+y^2}{2}$

∴ b^2 is the **A.M. between x^2 and y^2.**

32. The three numbers a, b, c between 2 and 18 are such that their sum is 25; the numbers 2, a, b are consecutive terms of an A.P. and the numbers b, c, 18 are consecutive terms of a G.P. The three numbers are

(a) 3, 8, 14 (b) 2, 9, 14
(c) 5, 8, 12 (d) none of these

Solution (c) We have, $a + b + c = 25$ ...(1)

∵ 2, a, b are in A.P., ∴ $2a = 2 + b$ ...(2)

∵ b, c 18 are in G.P., ∴ $c^2 = 18b$ ...(3)

From (1) and (2), $3b + 2c = 48$; or $3b = 48 - 2c$.

∴ From (3), $c^2 = 6(48 - 2c) = 288 - 12c$

or $c^2 + 12c - 288 = 0$;

or $c^2 + 24c - 12c - 288 = 0$

or $(c + 24)(c - 12) = 0$; ∴ $c = \mathbf{12}$, as $c \neq -24$.

∴ From (3), $b = \mathbf{8}$ and from (2), $a = \mathbf{5}$.

33. If a, b, c are in G.P. and $a^x = b^y = c^z$, then x, y, z are in

(a) A. P. (b) G. P.
(c) H. P. (d) none of these

Solution (c) Let $a^x = b^y = c^z = k$ (say)

∴ $a = (k)^{1/x}$, $b = (k)^{1/y}$, $c = (k)^{1/z}$ ...(1)

Substituting these values of a, b, c in (1), we have

$(k)^{2/y} = (k)^{1/x}\cdot(k)^{1/z} = (k)^{\frac{1}{x}+\frac{1}{z}}$

∴ $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$

∴ $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A.P.

Hence x, y, z are in **H.P.**

34. If a, b, c are in H.P., then $\frac{1}{b-a}+\frac{1}{b-c}=$

(a) $\frac{1}{a}+\frac{1}{b}$ (b) $\frac{1}{a}+\frac{1}{c}$
(c) $\frac{1}{b}+\frac{1}{c}$ (d) none of these

Solution (b) We have,

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{\frac{2ac}{a+c}-a}+\frac{1}{\frac{2ac}{a+c}-c}=$$

$$=\frac{a+c}{a(c-a)}+\frac{a+c}{-c(c-a)}=\frac{a+c}{c-a}\left(\frac{c-a}{ac}\right)$$

$$=\frac{a}{ac}+\frac{c}{ac}=\frac{\mathbf{1}}{\mathbf{c}}+\frac{\mathbf{1}}{\mathbf{a}}.$$

35. Given two numbers a and b. Let A denote the single A.M. and S denote the sum of n A.M.s between a and b, then $\frac{S}{A}$ depends on

(a) n, a, b (b) n, b
(c) n, a (d) n

Solution (d) ∵ A is the single A.M. between a and b.

∴ $A=\frac{a+b}{2}$.

Let $A_1, A_2, \ldots A_n$ be n A.M.s between a and b.

∴ $a, A_1, A_2, \ldots A_n, b$ is an A.P. with common difference

$d=\frac{b-a}{n+1}$.

Now, $S= A_1 + A_2 + \ldots + A_n = \frac{n}{2}(A_1 + A_n)$

$=\frac{n}{2}(a + d + b - d) = \frac{n}{2}(a + b)$

∴ $S=n\left(\frac{a+b}{2}\right)=nA\Rightarrow\frac{S}{A}=\mathbf{n}.$

36. Let a_n be the nth term of the G. P. of positive numbers. Let $\sum_{n=1}^{100}a_{2n}=\alpha$ and $\sum_{n=1}^{100}a_{2n}=\beta$, such that $\alpha\neq\beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$
(c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

Solution (a) Let a be the first term and r, the common ratio of the given G.P. Then

$\alpha=\sum_{n=1}^{100}a_{2n}\sum_{n=1}^{100}a_{2n}\Rightarrow\alpha=a_2+a_4+\ldots+a_{200}$

$$\Rightarrow \alpha = ar + ar^3 + \ldots + ar^{199}$$

$$\Rightarrow \alpha = ar\,(1 + r^2 + r^4 + \ldots + r^{198}) \quad \ldots(1)$$

and $\beta = \sum_{n=1}^{100} a_{2n-1} \Rightarrow \beta = a_1 + a_3 + \ldots + a_{199}$

$$\Rightarrow \beta = a + ar^2 + \ldots + ar^{198}$$

$$\Rightarrow \beta = a\,(1 + r^2 + \ldots + r^{198}) \quad \ldots(2)$$

From (1) and (2), we get $\frac{\alpha}{\beta} = r$.

37. The third term of a G.P. is 4. The product of first five terms is

(a) 4^3 (b) 4^5

(c) 4^4 (d) none of these

Solution (b) Let a be the first term and r, the common ratio of G.P.

Then, $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$, $a_4 = ar^3$, $a_5 = ar^4$

$\therefore\ a_1 \times a_2 \times a_3 \times a_4 \times a_5$

$= a \times ar \times ar^2 \times ar^3 \times ar^4$

$= a^5 r^{10} = (ar^2)^5 = \mathbf{4^5}$. $[\because a_3 = ar^2 = 4]$

38. If three positive numbers a, b, c are in H.P., then $a^n + c^n$

(a) $> 2b^n$ (b) $= 2b^n$

(c) $< 2b^n$ (d) $> b^n$

Solution (a) For the positive numbers a, c we have harmonic mean $H = b$ $\{\because a, b, c$ are in H.P.$\}$ and geometric mean $G = \sqrt{ac}$.

But $G > H$; $\therefore\ \sqrt{ac} > b$. ...(1)

For the positive numbers a^n, c^n, we have

geometric mean $= \sqrt{a^n c^n}$

arithmetic mean $= \frac{a^n + c^n}{2}$;

$\because$ A. M. > G. M.,

$\therefore\ \frac{a^n + c^n}{2} > \sqrt{a^n c^n}$ ...(2)

From (1) and (2), we get

$$\frac{a^n + c^n}{2} > (\sqrt{ac})^n > b^n,\ \therefore\ a^n + c^n > \mathbf{2b^n}.$$

39. If a, b, c are in A.P. as well as in G.P., then

(a) $a = b \neq c$ (b) $a \neq b = c$

(c) $a \neq b \neq c$ (d) $a = b = c$

Solution (d) Since a, b, c are in A.P.

$\therefore\ b = \frac{a+c}{2}$ ...(1)

Also a, b, c are in G.P., $\therefore\ b^2 = ac$ ...(2)

From (1) and (2), we have

$$\left(\frac{a+c}{2}\right)^2 = ac,$$

$\therefore\ (a + c)^2 - 4ac = 0$

$\Rightarrow\ (a - c)^2 = 0$,

$\therefore\ a = c$

$\therefore\ b = \frac{a+a}{2} = a$ [From (1)]

Hence $\mathbf{a = b = c}$.

40. If the A.M. and G.M. of the roots of a quadratic equation in x are p and q respectively, then its equation is

(a) $x^2 - 2px + q^2 = 0$

(b) $x^2 + 2px + q^2 = 0$

(c) $x^2 - px + q = 0$

(d) $x^2 - 2px + q = 0$

Solution (a) Let the roots be α and β.

Then, $\frac{\alpha+\beta}{2} = p$ and $\sqrt{\alpha\beta} = q$ (Given)

Now $\frac{\alpha+\beta}{2} = p \Rightarrow \alpha + \beta = 2p$

and $\sqrt{\alpha\beta} = q \Rightarrow \alpha\beta = q^2$.

Hence required equation is

$$x^2 - (\alpha + \beta)\,x + \alpha\beta = 0 \Rightarrow \mathbf{x^2 - 2px + q^2 = 0}.$$

41. If one A.M. A and two G.M.s p and q are inserted between two given numbers, then $\frac{p^2}{q} + \frac{q^2}{p} =$

(a) A (b) $2A$

(c) $3A$ (d) $4A$

Solution (b) Let the two numbers be a and b.

Then, $A = \frac{a+b}{2}$

Also, a, p, q, b are in G.P.

$\therefore\ \frac{p}{a} = \frac{q}{p} = \frac{b}{q}$; $\therefore\ \frac{q^2}{p} = a$ and $\frac{p^2}{q} = b$

$\therefore\ \frac{p^2}{q} + \frac{q^2}{p} = a + b = \mathbf{2A}$.

42. $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \cdot 16^{\frac{1}{32}} \ldots$ is equal to

(a) 1 (b) 2

(c) $\frac{3}{2}$ (d) $\frac{5}{2}$

Solution (b) We have, $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \cdot 16^{\frac{1}{32}} \ldots$

$$= 2^{\frac{1}{4}} \cdot 2^{\frac{2}{8}} \cdot 2^{\frac{3}{16}} \cdot 2^{\frac{4}{32}} \ldots = 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\ldots}$$

Let $S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots$ ...(1)

$\therefore\ \frac{1}{2}S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \ldots$ ...(2)

Subtracting (2) from (1), we get

$$\frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

$$= \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$$

$\therefore\ S = 1$.

Hence, the given expression $= 2^1 = \mathbf{2}$.

43. If $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$, then a, b, c are in

(a) A. P. (b) G. P.
(c) H. P. (d) none of these

Solution (c) We have,

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$$

$$\Rightarrow \frac{1}{b-a}-\frac{1}{c}=\frac{1}{a}-\frac{1}{b-c}\Rightarrow\frac{c-b+a}{(b-a)c}=\frac{b-c-a}{a(b-c)}$$

$$\Rightarrow \frac{1}{(b-a)c}=-\frac{1}{a(b-c)} \Rightarrow ab-ac=-bc+ac$$

$$\Rightarrow 2ac=ab+bc=b(a+c)\Rightarrow b=\frac{2ac}{a+c}$$

$\Rightarrow$ a, b, c are in **H.P.**

44. If one G.M. G and two A.M.s p and q be inserted between two given numbers, then $(2p-q)(2q-p)=$

(a) G^2 (b) $4G$
(c) $2G^2$ (d) none of these

Solution (a)

45. If the H.M. of two numbers is to their G.M. as 12 : 13, then the numbers are in the ratio

(a) 4 : 9 (b) 9 : 4
(c) 2 : 9 (d) 9 : 2

Solution (a), (b)

i.e., the numbers are in the ratio 4 : 9 or 9 : 4.

46. If a, b, c are in A.P. and a, mb, c are in G.P, then a, m^2b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Solution (c)

47. Sum to infinity terms of the series

$$\frac{1}{5}+\frac{1}{7^2}+\frac{1}{5^3}+\frac{1}{7^4}+\dots \text{ is}$$

(a) $\frac{9}{48}$ (b) $\frac{5}{24}$
(c) $\frac{11}{48}$ (d) none of these

Solution (c)

48. The value of $1\cdot\dot{2}\dot{3}$ is

(a) $\frac{61}{495}$ (b) $\frac{41}{495}$
(c) $\frac{51}{495}$ (d) none of these

Solution (a)

49. The value of $(0.2)^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots\infty\right)}$ is

(a) 1 (b) 2
(c) $\frac{1}{2}$ (d) 4

Solution (d)

50. $i-2-3i+4$ ... to 100 terms =

(a) $50(1-i)$ (b) $25i$
(c) $25(1+i)$ (d) $100(1-i)$

Solution (a)

51. The sum of first n terms of the series
$1\cdot 1!+2\cdot 2!+3\cdot 3!+4\cdot 4!+\dots$ is

(a) $(n+1)!-1$
(b) $n!-1$
(c) $(n-1)!-1$
(d) none of these

Solution (a)

52. If a, b, c are digits, then the rational number represented by $0\cdot cababab$ ... is

(a) $\frac{99c+ab}{990}$ (b) $\frac{99c+10a+b}{99}$
(c) $\frac{99c+10a+b}{990}$ (d) none of these

Solution (c)

53. If $\Sigma n=210$, then $\Sigma n^2=$

(a) 2870 (b) 2160
(c) 2970 (d) none of these

Solution (a)

54. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1=$

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)}{2}$
(c) $\left(\frac{n(n+1)}{2}\right)^2$ (d) $\frac{n(n+1)(n+2)}{6}$

Solution (d)

55. Let $a=1\cdot 2\cdot 3\cdot 4\cdot 5$. Then

(a) $5^5\geq a$
(b) $3^5\geq 5!$
(c) $5^5\geq 6a$
(d) none of these

Solution (a), (b)

56. If $T_n=n\times(n!)$, then $\sum_{n=1}^{20}T_n$ is equal to

(a) $(21)!-1$ (b) $(20)!-1$
(c) $(21)!+1$ (d) none of these

Solution (a)

57. The sum of the products of the $2n$ numbers
$\pm 1, \pm 2, \pm 3, \dots, \pm 2n$ taking two at a time is

(a) $-\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$
(c) $-\frac{n(n+1)(2n+1)}{6}$ (d) none of these

Solution (c)

58. $\cos x=b$. For what b do the roots of the equation form an A.P.?

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$
(c) -1 (d) none of these

Solution (c)

59. If the sum of the first n terms of an A.P. is $Pn + Qn^2$ where P and Q are constants, then common difference of A.P. will be

(a) $P - Q$ (b) $P + Q$
(c) $2Q$ (d) $2P$

Solution (c)

60. The largest value of the positive integer k for which $n^k + 1$ divides $1 + n + n^2 + \ldots + n^{127}$ is divisible by

(a) 8 (b) 16
(c) 32 (d) 64

Solution (d)

Problems for Practice

Level of Difficulty I

1. If $\log_3 2$, $\log_3 (2^x - 5)$ and $\log_3 \left(2^x - \frac{7}{2}\right)$ are in A.P., then the value of x is

(a) 3 (b) 2 (c) 4 (d) 5

2. In an A.P. of 81 terms, the 41th term is 10. Then the sum of series is

(a) 10×41 (b) $\frac{10 \times 41}{2}$
(c) 10×81 (d) 41×81

3. If $\log 2$, $\log (2^x - 1)$ and $\log (2^x + 3)$ are in A.P. then $x =$

(a) $\frac{5}{2}$ (b) $\log_2 5$ (c) $\log_3 2$ (d) $\frac{3}{2}$

4. If the sum of the first n terms of a series be $5n^2 + 2n$, then its second term is

(a) 7 (b) 17 (c) 24 (d) 42

5. If the pth term of an A.P. is q and the qth term is p, then its $(p + q)$th term is

(a) 0 (b) $p - q$ (c) $p + q$ (d) none of these

6. If m times the mth term of an A.P. is equal to n times the nth term, then its $(m + n)$th term is

(a) 1 (b) -1 (c) 0 (d) none of these

7. The number of terms in the series 20, $19\frac{1}{3}$, $18\frac{2}{3}$, ... of which the sum is 300, is

(a) 25 (b) 36
(c) 31 (d) none of these

8. If the nth term of a series is $\frac{3 + n}{4}$, then the sum of 105 terms is

(a) 1470 (b) 1360 (c) 1530 (d) none of these

9. If $5^{1+x} + 5^{1-x}$, $\frac{a}{2}$ and $25^x + 25^{-x}$ are three consecutive terms of an A.P., then the values of a are given by

(a) $a \geq 12$ (b) $a > 12$
(c) $a < 12$ (d) $a \leq 12$

10. If x, y, z are in A.P., then

$(x + 2y - z)(2y + z - x)(z + x - y) =$

(a) $4xyz$ (b) $2xyz$
(c) xyz (d) none of these

11. If the A.M. between two numbers is 34 and their G.M. is 16, then the two numbers are

(a) 64 and 8 (b) 64 and 4
(c) 8 and 4 (d) none of these

12. If x, y and z are 4th, 10th and 16th terms of a G.P., then x, y and z are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

13. If the first term of an infinite G. P. is 1 and each term is twice the sum of the succeeding terms, then the common ratio is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) none of these

14. If a, b, c are in G.P., then $\log a^n$, $\log b^n$, $\log c^n$ are in

(a) A.P. (b) G.P. (c) H.P (d) none of these

15. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and a, b, c are in G.P., then x, y, z are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

16. If A_1, A_2 are the two A.M.s between two numbers a and b and G_1, G_2 be two G.M.s between same two numbers, then $\frac{A_1 + A_2}{G_1 \cdot G_2} =$

(a) $\frac{a+b}{ab}$ (b) $\frac{a+b}{2ab}$
(c) $\frac{2ab}{a+b}$ (d) $\frac{ab}{a+b}$

17. If a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P., then a, c, e are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

18. If a, b, c, d, e, f are in A.P., then $e - c$ is equal to

(a) $2(c - a)$ (b) $2(d - c)$
(c) $2(f - d)$ (d) $d - c$

19. If a, b, c are in H.P. then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$, are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

20. The H.M. of two numbers is 4. If their A.M. A and G.M. G satisfy the relation $2A + G^2 = 27$, then the numbers are

(a) 2, 6 (b) 3, 6
(c) 1, 3 (d) none of these

Level of Difficulty II

21. If x, y, z are in G.P. and $a^x = b^y = c^z$, then
(a) $\log_c b = \log_a c$
(b) $\log_a c = \log_b a$
(c) $\log_a b = \log_c b$
(d) $\log_b a = \log_c b$

22. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then
(a) $a = b = c$
(b) $2b = 3a + c$
(c) $b^2 = \sqrt{\dfrac{ac}{8}}$
(d) none of these

23. If a, b, c are in H.P, then $\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)A =$
(a) $\dfrac{4}{ac}-\dfrac{3}{b^2}$ (b) $\dfrac{4}{ac}+\dfrac{3}{b^2}$
(c) $\dfrac{2}{ac}-\dfrac{3}{b^2}$ (d) none of these

24. Sum to infinity of the series

$$\frac{2}{3}-\frac{5}{6}+\frac{2}{3}-\frac{11}{24}+\ldots \text{ is}$$

(a) $\dfrac{4}{9}$ (b) $\dfrac{1}{3}$
(c) $\dfrac{2}{9}$ (d) none of these

25. If $S_n = 1+\dfrac{1}{2}+\dfrac{1}{2^2}+\ldots+\dfrac{1}{2^{n-1}}$, then the least value of n such that $2 - S_n < \dfrac{1}{100}$ is
(a) 6 (b) 8 (c) 10 (d) none of these

26. If the 7th term of an H.P. is $\dfrac{1}{10}$ and the 12th term is $\dfrac{1}{25}$, then the 20th term is
(a) $\dfrac{1}{37}$ (b) $\dfrac{1}{41}$
(c) $\dfrac{1}{45}$ (d) $\dfrac{1}{49}$

27. The A.M., G.M. and H.M. of two numbers are x, y and z respectively. Then, which of the following is true?
(a) $z < x < y$ (b) $x < y < z$
(c) $y < x < z$ (d) $z < y < x$

28. The sum of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+\ldots$ is
(a) $242\,(\sqrt{3}\ - 1)$ (b) $\dfrac{121}{\sqrt{3}-1}$
(c) $243\,(\sqrt{3}\ + 1)$ (d) $121\,(\sqrt{6}+\sqrt{2}\,)$

29. If p, q, r are in A.P., a is G.M. between p and q and b is G.M. between q and r, then a^2, q^2, b^2 are in
(a) G.P. (b) A.P. (c) H.P. (d) none of these

30. If the A.M. of the roots of a quadratic equation is $\dfrac{8}{5}$ and A.M. of their reciprocals is $\dfrac{8}{5}$, then the quadratic equation is
(a) $5x^2 - 8x + 7 = 0$
(b) $5x^2 - 16x + 7 = 0$
(c) $7x^2 - 16x + 5 = 0$
(d) $7x^2 + 16x + 5 = 0$.

31. If a, b, c are in G.P. and p, q are respectively A.M. between a, b and b, c then
(a) $2 = \dfrac{a}{p}+\dfrac{c}{q}$ (b) $2 = \dfrac{a}{p}-\dfrac{c}{q}$
(c) $\dfrac{2}{b}=\dfrac{1}{q}-\dfrac{1}{p}$ (d) $\dfrac{2}{b}=\dfrac{1}{p}-\dfrac{1}{q}$

32. If a, b, c are in A.P., a, mb, c are in G.P. then a, m^2b, c are in
(a) H.P. (b) G.P.
(c) A.P. (d) none of these

33. The product $(32)\,(32)^{\frac{1}{6}}\,(32)^{\frac{1}{36}}\ldots\infty$ is equal to
(a) 64 (b) 16 (c) 32 (d) 0

34. If the first and the $(2n - 1)$th terms of an A.P., G.P. and H.P. are equal and their nth terms are a, b, c respectively, then
(a) $a = b = c$
(b) $a \geq b \geq c$
(c) $a + c = b$
(d) $ac - b = 0$

35. If 1, $\log_3 (3^{1-x} + 2)$, $\log_3 (4.3^x - 1)$ are in A.P. then x equal
(a) $\log_3 4$ (b) $1 - \log_3 4$
(c) $1 - \log_4 3$ (d) $\log_4 3$

36. The value of $2^{1/4}$, $4^{1/8}$, $8^{1/6}$, $+ \ldots\infty$ is
(a) 1 (b) 2 (c) 3/2 (d) 4

37. Fifth term of an GP is 2, then the product of its 9 terms is
(a) 256 (b) 512
(c) 1024 (d) none of these

38. Let T, be the rth term of an A.P. whose first term is a and common difference is d. If for some positive integers $m,n,m \neq n$, $T_m = \dfrac{1}{n}$ and $T_n = \dfrac{1}{m}$, then a – d equals
(a) 1/mn (b) 1 (c) 0 (d) $\dfrac{1}{m}+\dfrac{1}{n}$

39. The sum of first n terms of the series$1^2 + 2$, $2^2 + 3^2 + 2$, $4^2 + 5^2 + 2$, $6^2 + \ldots$ is. $\dfrac{n(n+1)^2}{2}$ when n is even. When n is odd the sum is
(a) $\dfrac{n(n+1)^2}{4}$ (b) $\dfrac{n(n+1)^2}{2}$
(c) $\dfrac{3n(n+1)^2}{2}$ (d) $\left[\dfrac{n(n+1)}{2}\right]^2$

40. In a G.P., $(m + n)$th term is 9 and $(m - n)$th term is 4, then mth term is
(a) 16 (b) $\dfrac{1}{6}$
(c) 6 (d) none of these

ANSWERS

1. (a)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (a),(b)	**8.** (a)	**9.** (a)	**10.** (a)
11. (b)	**12.** (b)	**13.** (a)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (b)
21. (d)	**22.** (a)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (d)	**27.** (d)	**28.** (d)	**29.** (b)	**30.** (b)
31. (a)	**32.** (a)	**33.** (a)	**34.** (b),(d)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (c)

Quadratic Equations

20

BRIEF REVIEW OF THE CONCEPTS

QUADRATIC EQUATION

An algebraic expression of the form: $ax^2 + bx + c$, where $a(\neq 0)$, b, $c \in R$ is called a quadratic expression.

An equation of the form: $ax^2 + bx + c = 0$, where $a\ (\neq 0)$, b, $c \in R$ is called a quadratic equation.

The numbers a, b, c are called the coefficients of the quadratic equation and the expression $b^2 - 4ac$ is called its discriminant. Discriminant of a quadratic equation is usually denoted by D or Δ.

Roots of the Quadratic Equation

A root of the quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{...(1)}$$

is a number α (real or complex) such that $a\alpha^2 + b\alpha + c = 0$. The roots of the quadratic equation (1) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Nature of Roots of the Quadratic Equation

1. If $D < 0$, then roots α, β are imaginary
2. If $D > 0$, then roots α, β are real and distinct
3. If $D = 0$, then roots α, β are real and equal.

Key Points to Remember

1. Roots are rational $\Leftrightarrow D$ is a perfect square
2. Roots are irrational $\Leftrightarrow D$ is positive but not a perfect square.
3. If $a + b + c = 0$, then 1 is a root of the equation
$$ax^2 + bx + c = 0$$
4. If a and c are of opposite sign, the roots must be of opposite sign
5. If the roots are equal in magnitude but opposite in sign, then $b = 0$, $ac > 0$.
6. If the roots are reciprocal of each other, then $c = a$.
7. If $ax^2 + bx + c = 0$ is satisfied by more than two values, it is an identity and $a = b = c = 0$ and vice-versa.
8. If $ax^2 + bx + c = 0$, where a, b, $c \in R$, has one root $p + iq$, then the other root will be $p - iq$. Hence the imaginary roots occur in conjugate pair.
9. If $ax^2 + bx + c = 0$, where a, b, c are rational, has one root $p + \sqrt{q}$ then the other root will be $p - \sqrt{q}$. Hence irrational roots occur in conjugate pair if the coefficients are rational.
10. The quadratic equation whose roots are reciprocals of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$ (i.e., the coefficients are written in reverse order).
11. If $a = 1$, b, $c \in Z$ and the roots are rational numbers, then these roots must be integers.

Sum and Product of the Roots

If α and β are roots of $ax^2 + bx + c = 0$, then

$$\text{Sum of roots} = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Formation of Equation with Given Roots

If α and β are roots of $f(x) = ax^2 + bx + c = 0$, then

$$f(x) = (x - \alpha)(x - \beta) = 0$$
$$= x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

i.e., $x^2 -$ (sum of the roots)x + (product of the roots) = 0.

COMMON ROOTS

One root common If α is a common root of the equations

$$a_1x^2 + b_1x + c_1 = 0 \qquad \text{...(1)}$$

and

$$a_2x^2 + b_2x + c_2 = 0 \qquad \text{...(2)}$$

then we have

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \text{ and } a_2\alpha^2 + b_2\alpha + c_2 = 0$$

These give

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1}$$

$$= \frac{1}{a_1b_2 - a_2b_1} \left(a_1b_2 - a_2b_1 \neq 0 \right).$$

Thus, the required condition for one common root is $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$ and the value of the common root is

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ or } \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}.$$

Both roots common If the equations (1) and (2) have both roots common, then these equations will be identical. Thus the required condition for both roots common is:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Key Points to Remember

1. To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of x so obtained is the required common root.
2. If two quadratic equations with real coefficients have an imaginary root common, then both roots will be common and the two equations will be identical. The required condition is:
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$
3. If two quadratic equations have an irrational root common, then both roots will be common and the two equations will be identical. The required condition is:
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$
4. If α is a repeated root of the quadratic equation
$$f(x) = ax^2 + bx + c = 0,$$
then α is also a root of the equation $f'(x) = 0$.
5. If α is repeated common root of two quadratic equations $f(x) = 0$ and $\phi(x) = 0$, then α is also a common root of the equations $f'(x) = 0$ and $\phi'(x) = 0$.

SYMMETRIC FUNCTION OF THE ROOTS

A function of α and β is said to be a symmetric function if it remains unchanged when α and β are interchanged.

For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 - \beta^2 + 3\alpha\beta$ is not a symmetric function of α and β.

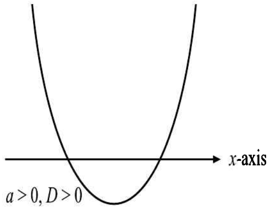
In order to find the value of a symmetric function of α and β, express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. The following results may be useful.

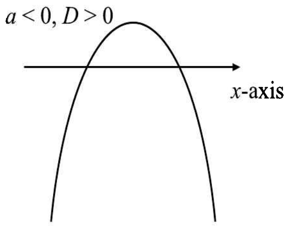
1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
3. $\alpha^4 + \beta^4 = (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2)$
4. $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$
5. $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$
6. $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$
7. $\alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$
8. $\alpha^4 - \beta^4 = (\alpha + \beta)(\alpha - \beta)(\alpha^2 + \beta^2)$.

GRAPH OF A QUADRATIC EXPRESSION

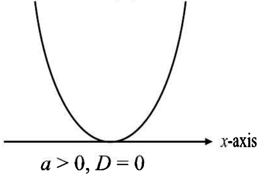
We have, y or $f(x) = ax^2 + bx + c$ where $a, b, c \in R, a \neq 0$.

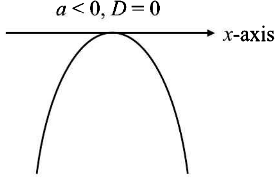
1. The shape of the curve $y = f(x)$ is a parabola
2. The axis of the parabola is y-axis (incase $b = 0$) or parallel to y-axis.
3. If $a > 0$, then the parabola opens upwards.
4. If $a < 0$, then the parabola opens downwards
5. For $D > 0$, parabola cuts x-axis in two distinct points



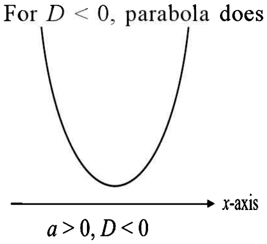


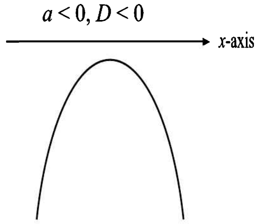
6. For $D = 0$, parabola touches x-axis in one point.





7. For $D < 0$, parabola does not cut x-axis.





GREATEST AND LEAST VALUES OF A QUADRATIC EXPRESSION

1. If $a > 0$, then the quadratic expression $y = ax^2 + bx + c$ has no greatest value but it has least value
$$\frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}.$$
2. If $a < 0$, then the quadratic expression $y = ax^2 + bx + c$ has no least value but it has greatest value
$$\frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}.$$

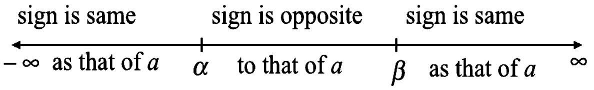
SIGN OF QUADRATIC EXPRESSION

If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then for $x = \alpha$ and $x = \beta$, the value of the expression $ax^2 + bx + c$ is equal to zero. For other real values of x, the expression

$$ax^2 + bx + c > 0 \text{ or } < 0.$$

The sign of $ax^2 + bx + c$, $x \in R$ is determined by the following rule:

1. If α, β ($\alpha < \beta$) are real and unequal (i.e., $D > 0$) roots of the quadratic equation, then the sign of the expression
$ax^2 + bx + c$, $x \in R$ is determined as follows:



2. If α, β are real and equal (i.e., $D = 0$) roots of the quadratic equation, then the sign of the expression
$ax^2 + bx + c$, $x \in R$ is as follows:
$$ax^2 + bx + c \geq 0 \text{ if } a > 0$$
and $$ax^2 + bx + c \leq 0 \text{ if } a < 0$$
3. If α and β are imaginary (i.e., $D < 0$), then the expression
$$ax^2 + bx + c > 0 \text{ if } a > 0$$
and $$ax^2 + bx + c < 0 \text{ if } a < 0.$$

Key Points to Remember

1. Let $f(x) = 0$ be a polynomial equation. Let p and q be two real numbers.
   (a) If $f(p)\cdot f(q) < 0$, then the equation $f(x) = 0$ has odd number of real roots between p and q.
   (b) If $f(p)\cdot f(q) > 0$, then the equation $f(x) = 0$ has either no real root or even number of real roots between p and q.
   (c) If $f(p) = f(q)$, then the equation $f'(x) = 0$ has at least one real root between p and q (This is due to **Rolle's Theorem**)
2. (a) If the coefficients of the polynomial equation $f(x) = 0$ have p changes of signs, then the equation $f(x) = 0$ will have atmost p, positive roots.
   (b) If the coefficients of the polynomial equation $f(-x) = 0$ have q changes of signs, then the equation $f(x) = 0$ will have atmost q, negative roots.
   (c) The polynomial equation $f(x) = 0$ will have atmost $p + q$ real roots where p and q are the changes of signs of coefficients in $f(x)$ and $f(-x)$. (This is due to **Descarte's Rule of signs**)

   For example, consider

   $f(x) = 2x^5 - 6x^4 + 7x^3 - 8x^2 + 5x + 3$

   $+ \quad - \quad + \quad - \quad + \quad +$

   Then, $f(-x) = -2x^5 - 6x^4 - 7x^3 - 8x^2 - 5x + 3$

   $- \quad - \quad - \quad - \quad - \quad +$

   Clearly, $f(x)$ has 4 changes of signs and $f(-x)$ has only one change of sign, Therefore, the equation

   $f(x) = 2x^5 - 6x^4 + 7x^3 - 8x^2 + 5x + 3 = 0$ has atmost four positive roots and one negative root. Also, the equation has atmost $(4 + 1) = 5$ real roots.
3. (a) A polynomial equation $f(x) = 0$ has exactly one root equal to α if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.
   (b) A polynomial equation $f(x) = 0$ has exactly two roots equal to α if $f(\alpha) = 0$, $f'(\alpha) = 0$ and $f''(\alpha) \neq 0$.
   (c) In general, a polynomial equation $f(x) = 0$ has exactly n roots equal to α if

   $f(\alpha) = f'(\alpha) = f''(\alpha) = \dots$

   $= f^{n-1}(\alpha) = 0$ and $f^n(\alpha) \neq 0$.

RELATION BETWEEN ROOTS AND COEFFICIENTS OF A POLYNOMIAL EQUATION

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$, $a_0, a_1, a_2, \dots, a_n \in R$, $a_0 \neq 0$ be a polynomial equation of degree n, having n roots $\alpha_1, \alpha_2, \dots \alpha_n$. Then,

1. Sum of all roots

$$\sigma_1 = \alpha_1 + \alpha_2 + \dots + \alpha_n$$
$$= \Sigma\alpha_1 = -\frac{a_1}{a_0} = (-1)^1\frac{a_1}{a_0}$$

2. Sum of the product of two roots

$$\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \dots$$
$$= \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0} = (-1)^2\frac{a_2}{a_0}$$

3. Sum of the product of three roots

$$\sigma_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \dots$$
$$= \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$
$$= (-1)^3\frac{a_r}{a_0} \text{ and so on.}$$

In general, $\sigma_r = \Sigma\alpha_1\alpha_2 \dots \alpha_r = (-1)^r\dfrac{a_r}{a_0}$.

Particular Cases

Quadratic Equation If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Cubic Equation If α, β, γ are roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\sigma_1 = \alpha + \beta + \gamma = -\frac{b}{a}$$
$$\sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$
$$\sigma_3 = \alpha\beta\gamma = -\frac{d}{a}$$

Biquadratic Equation If α, β, γ, δ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\sigma_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$
$$\sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$
$$\sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$
$$\sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a}.$$

RATIONAL ALGEBRAIC EXPRESSION

An expression of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$, is known as a rational algebraic expression.

Sign Scheme for a Rational Algebraic Expression in *x*

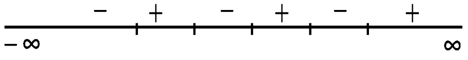
Step 1: Factorise the numerator and denominator of the given rational expression into linear factors.

Step 2: Find the real values of x by equating all the factors to zero.

Step 3: If n distinct real values of x are obtained then the entire line will be divided into $(n + 1)$ parts.

Step 4: Plot all these points on the number line in order.

Step 5: Start with '+' sign from extreme right and change the sign alternatively in other parts.



Note: If the rational expression in x occurs under modulus sign, then first of all remove the modulus sign and then proceed.

In order to remove the modulus sign, the following results may be useful:

(a) $|x| = k \Leftrightarrow x = \pm k$

(b) $|x| < k \Leftrightarrow = -k < x < k$

(c) $|x| > k \Leftrightarrow = x < -k$ or $x > k$.

To find the values of a Rational Expression in x, where x is Real

Key Points to Remember

1. Put the given rational expression equal to y and form the quadratic equation in x.
2. Find the discriminant D of the quadratic equation obtained in step 1.
3. Since x is real, therefore, put $D \geq 0$. We get an inequation in y.
4. Solve the above inequation for y. The values of y so obtained determine the set of values attained by the given rational expression.

Note: The general quadratic expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ in x and y may be resolved into two linear rational factors if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Solved Problems

1. If $x = \sqrt{7+4\sqrt{3}}$, then $x + \dfrac{1}{x} =$

(a) 4 (b) 6

(c) 3 (d) 2

Solution (a) We have, $x = \sqrt{7+4\sqrt{3}}$

$$\Rightarrow \frac{1}{x} = \sqrt{7+4\sqrt{3}}$$

$$= \frac{\sqrt{7-4\sqrt{3}}}{\sqrt{7+4\sqrt{3}}\,\sqrt{7-4\sqrt{3}}} = \sqrt{7-4\sqrt{3}}$$

$$\Rightarrow x + \frac{1}{x} = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$$

$$= (\sqrt{3}+2) + (2-\sqrt{3}) = \mathbf{4}.$$

2. If the roots of the equation

$(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$ are real, then a^2, bd, c^2 are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

Solution (b) Since the roots of the given equation are real, therefore the discriminant ≥ 0

$$\Rightarrow 4(bc + ad)^2 - 4(a^2 + b^2)(c^2 + d^2) \geq 0$$

$$\Rightarrow b^2c^2 + a^2d^2 + 2abcd - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 \geq 0$$

$$\Rightarrow (ac - bd)^2 \leq 0.$$

But $(ac - bd)^2$ cannot be negative as it is a square of real number

$\therefore$ $ac - bd = 0$; or $b^2d^2 = a^2c^2$.

Hence a^2, bd, c^2 are in **G.P.**

3. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, then $\dfrac{(r+1)^2}{r} =$

(a) $\dfrac{a^2}{bc}$ (b) $\dfrac{b^2}{ca}$

(c) $\dfrac{c^2}{ab}$ (d) none of these

Solution (b) Given equation is $ax^2 + bx + c = 0$ ...(1)

Let the root of equation (1) be α and $r\alpha$, then

$$\alpha + r\alpha = \frac{-b}{a} \quad \text{...(2)}$$

and $$r\alpha^2 = \frac{c}{a} \quad \text{...(3)}$$

From (2), $$\alpha = -\frac{b}{a(r+1)} \quad \text{...(4)}$$

Putting the value of α in (3), we get

$$\frac{rb^2}{a^2(r+1)^2} = \frac{c}{a} \text{ or, } \frac{b^2}{ac} = \frac{(r+1)^2}{r}$$

4. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(a) 4 (b) 3

(c) 1 (d) 2

Solution (a) We have, $|x|^2 - 3|x| + 2 = 0$

$\Rightarrow$ $(|x| - 1)(|x| - 2) = 0$

$\Rightarrow$ either $|x| = 1$ or $|x| = 2 \Rightarrow x = \pm 1$, $x = \pm 2$.

$\therefore$ Number of real solutions is **4.**

5. If α, β are irrational roots of $ax^2 + bx + c = 0$ ($a, b, c \in Q$), then

(a) $\alpha = \beta$

(b) $\alpha\beta = 1$

(c) α and β are conjugate roots

(d) $\alpha^2 + \beta^2 = 1$

Solution (c) Since α and β are roots of the equation $ax^2 + bx + c = 0$

$$\therefore \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

[$\because$ roots are irrational]

$\therefore$ $b^2 - 4ac > 0$ and not a perfect square

$\therefore$ **α and β are conjugate roots.**

6. The equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}} = 1 \text{ has}$$

(a) no solution (b) one solution

(c) two solutions (d) more than two solutions

Solution (d) Put $\sqrt{x-1} = t \Rightarrow x - 1 = t^2$ or $x = t^2 + 1$, The given equation reduces to

$$\sqrt{t^2+1+3-4t}+\sqrt{t^2+1+8-6t} = 1 \text{ where } t \geq 0.$$

$\Rightarrow |t - 2| + |t - 3| = 1$, where $t \geq 0$. This equation will be satisfied if $2 \leq t \leq 3$.

Therefore, $2 \leq \sqrt{x-1} \leq 3$ or $5 \leq x \leq 10$.

$\therefore$ The given equation is satisfied for all values of x lying in [5, 10].

7. In a quadratic equation with leading coefficient 1, a student reads the coefficient 16 of x wrongly as 19 and obtain the roots as – 15 and – 4. The correct roots are

(a) 6, 10 (b) – 6, –10

(c) – 7, – 9 (d) none of these

Solution (b) Since coefficient of x = 16,

$\therefore$ sum of roots = – 16

Since constant term = (– 15)(– 4) = 60,

$\therefore$ correct answer is **– 6, – 10.**

8. The equation $125^x + 45^x = 2.27^x$ has

(a) no solution (b) one solution

(c) two solutions (c) more than two solutions

Solution (b) The given equation can be written as

$$(5/3)^{3x} + (5/3)^x = 2$$

Putting $(5/3)^x = t$, the equation becomes

$$t^3 + t - 2 = 0 \quad \Rightarrow t^3 - 1 + (t - 1) = 0$$

$$\Rightarrow \quad (t - 1)(t^2 + t + 1) + (t - 1) = 0$$

$$\Rightarrow \quad (t - 1)(t^2 + t + 2) = 0 \Rightarrow t = 1 \text{ or } t^2 + t + 2 = 0$$

But $t^2 + t + 2 = 0$ does not have real solutions. Therefore, $t = 1 \Rightarrow (5/3)^x = 1 \Rightarrow x = 0$.

9. The value of p for which the quadratic equation $x^2 - px + p + 3 = 0$ has reciprocal roots is

(a) 1 (b) – 1

(c) 2 (d) – 2

Solution (d) Let the roots of the given equation be α and $\frac{1}{\alpha}$

$$\text{Then } \alpha\cdot\frac{1}{\alpha} = p + 3; \Rightarrow p = -\mathbf{2}.$$

10. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the reciprocals of their squares, then bc^2, ca^2 and ab^2 are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

Solution (a) Let α and β be the roots of the given equation;

then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Given $\alpha + \beta = \frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$,

$$\Rightarrow -\frac{b}{a} = \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{b^2-2ca}{c^2}$$

$$\Rightarrow 2ca^2 = bc^2 + ab^2$$

Hence, bc^2, ca^2 and ab^2 are in **A. P.**

11. If the equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ have a common root, then $a : b : c$ =

(a) 2 : 4 : 5 (b) 1 : 3 : 4

(c) 1 : 2 : 3 (d) none of these

Solution (c) For the equation $x^2 + 2x + 3 = 0$,

Discriminant = $(2)^2 - 4\cdot1\cdot3 < 0$.

$\therefore$ roots of $x^2 + 2x + 3 = 0$ are imaginary. Since the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ are given to have a common root, therefore both roots will be common. Hence both the equations are identical.

$$\therefore \quad \frac{a}{1}=\frac{b}{2}=\frac{c}{3} \qquad \text{i.e., } a : b : c = \mathbf{1 : 2 : 3.}$$

12. The roots of the equation $2^{x+2}\cdot 3^{\frac{3x}{x-1}} = 9$ are given by

(a) $\log_2\left(\frac{2}{3}\right)$, – 2 (b) 3, – 3

(c) – 2, $1 - \frac{\log 3}{\log 2}$ (d) $1 - \log_2 3$, 2

Solution (c) We have, $2^{x+2}.3^{3x/(x-1)} = 9 = 3^2$

$$\Rightarrow \quad (x + 2) \log 2 + \frac{3x}{x-1} \log 3 = 2 \log 3$$

$$\Rightarrow \quad (x + 2) \log 2 + \left(\frac{3x}{x-1} - 2\right) \log 3 = 0$$

$$\Rightarrow \quad (x + 2) \left(\log 2 + \frac{1}{x-1}\log 3\right) = 0$$

$$\Rightarrow \quad x = -\mathbf{2} \text{ or } x = \mathbf{1} - \frac{\log 3}{\log 2}.$$

13. In copying a quadratic equation of the form $x^2 + px + q = 0$, a student wrote the coefficient of x incorrectly and the roots were found to be 3 and 10; another student wrote the same equation but he wrote the constant term incorrectly and thus he found the roots to be 4 and 7. The roots of the correct equation are

(a) 5, 6 (b) 4, 6

(c) 4, 5 (d) none of these

Solution (a) In case of the first student, product of the roots = 3 × 10 = q. So the correct value of q is 30.

In case of the second student,

sum of the roots = 4 + 7

= – p. So the correct value of p is –11.

$\therefore$ The correct equation is $x^4 - 11x + 30 = 0$

or $(x - 5)(x - 6) = 0$; $\therefore x = 5, 6$.

$\therefore$ Roots of the correct equation are **5, 6.**

14. The sum of the real roots of the equation

$$|x - 2|^2 + |x - 2| - 2 = 0 \text{ is}$$

(a) 2 (b) 6

(c) 4 (d) 8

Solution (c) Put $|x-2| = t$.

The given equation becomes

$$t^2 + t - 2 = 0 \text{ or } (t+2)(t-1) = 0$$

Since $t + 2 = |x-2| + 2 > 0$

$\therefore$ we get $t - 1 = 0 \Rightarrow |x-2| = 1 \Rightarrow x - 2 = \pm 1$

$\Rightarrow$ $x = 3, 1$. Thus, the sum of roots is **4.**

15. If α, β are the roots of $x^2 - 2px + q = 0$ and γ, δ are roots of $x^2 - 2rx + s = 0$ and $\alpha, \beta, \gamma, \delta$ are in A.P., then

(a) $p - q = r^2 - s^2$ (b) $s - q = r^2 - p^2$

(c) $r - s = p^2 - q^2$ (d) none of these

Solution (b) We have, $\alpha + \beta = 2p$;

$\alpha\beta = q$, $\gamma + \delta = 2r$ and $\gamma\delta = s$.

$\because$ $\alpha, \beta, \gamma, \delta$ are in A.P.

$\therefore$ $\beta - \alpha = \delta - \gamma \Rightarrow (\beta - \alpha)^2 = (\delta - \gamma)^2$

$\Rightarrow$ $(\beta + \alpha)^2 - 4\beta\alpha = (\delta + \gamma)^2 - 4\delta\gamma$

$\Rightarrow$ $4p^2 - 4q = 4r^2 - 4s$; or $\boldsymbol{s - q = r^2 - p^2}$.

16. If $a \le 0$, then the root of the equation

$x^2 - 2a\,|x - a| - 3a^2 = 0$ is

(a) $(1-\sqrt{2})a$ (b) $(-1+\sqrt{6})a$

(c) $(1+\sqrt{2})a$ (d) $-(1+\sqrt{6})a$

Solution (a), (b) If $x - a < 0$, $|x - a| = -(x - a)$

$\therefore$ equation becomes $x^2 + 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$\Rightarrow$ $x = -(1+\sqrt{6})\,a,\ (-1+\sqrt{6})\,a$

$\because$ $x < a \le 0$

$\therefore$ $x = \boldsymbol{(-1+\sqrt{6})\,a}$

If $x - a \ge 0$, $|x - a| = x - a$

$\therefore$ the equation becomes $x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$\Rightarrow$ $x = (1+\sqrt{2})\,a,\ (1-\sqrt{2})\,a$

$\because$ $x \ge a$ and $a \le 0$

$\therefore$ $x = \boldsymbol{(1-\sqrt{2})\,a}$.

17. If $f(x) = x - [x]$, $x\,(\ne 0) \in R$, where $[x]$ is the greatest integer less than or equal to x, then the number of solutions of $f(x) + f\left(\dfrac{1}{x}\right) = 1$ are

(a) 0 (b) 1

(c) infinite (d) 2

Solution (c) We have, $f(x) + f\left(\dfrac{1}{x}\right) = 1$

$$\Rightarrow \quad x - [x] + \frac{1}{x} - \left[\frac{1}{x}\right] = 1$$

$$\Rightarrow \quad x + \frac{1}{x} - 1 = [x] + \left[\frac{1}{x}\right]$$

$$\Rightarrow \quad \frac{x^2 + 1 - x}{x} = \text{(integer) } k \text{ (say)}$$

$\Rightarrow$ $x^2 - (k+1)x + 1 = 0$

Since x is real, so $(k+1)^2 - 4 \ge 0$

$\Rightarrow$ $k^2 + 2k - 3 \ge 0 \Rightarrow (k+3)(k-1) \ge 0$

$\Rightarrow$ $k \le -3$ or $k \ge 1$.

Therefore, number of solutions is **infinite.**

18. If $(\log_5 x)^2 + \log_5 x < 2$, then x belongs to the interval

(a) $\left(\dfrac{1}{25}, 5\right)$ (b) $\left(\dfrac{1}{5}, \dfrac{1}{\sqrt{5}}\right)$

(c) $(1, \infty)$ (d) none of these

Solution (a) We have, $(\log_5 x)^2 + \log_5 x < 2$

Put $\log_5 x = a$ then $a^2 + a < 2$

$\Rightarrow$ $a^2 + a - 2 < 0 \Rightarrow (a+2)(a-1) < 0$

$\Rightarrow$ $-2 < a < 1$ or $-2 < \log_5 x < 1$

$\therefore$ $5^{-2} < x < 5$ *i.e.,* $\boldsymbol{\dfrac{1}{25} < x < 5}$.

19. The number of solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

(a) 0 (b) 1

(c) 2 (d) infinite

Solution (a) Put $5^x = y$. Then the given equation becomes

$$\sin(e^x) = y + \frac{1}{y} = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 + 2 \quad [\because 5^x > 0]$$

$\Rightarrow \sin(e^x) \ge 2$. which is not possible for any real value of x.

Hence the given equation has no real solution.

20. The greatest negative integer satisfying

$x^2 - 4x - 77 < 0$ and $x^2 > 4$ is

(a) -4 (b) -6

(c) -7 (d) none of these

Solution (c) We have, $x^2 - 4x - 77 < 0$ and $x^2 - 4 > 0$

$\Rightarrow$ $(x+7)(x-11) < 0$ and $(x-2)(x+2) > 0$

$\Rightarrow$ $-7 < x < 11$ and $x < -2$ or $x > 2$.

$\therefore$ $-7 < x < -2$.

Hence, the greatest negative integer is **– 7.**

21. If $x = 2 + 2^{2/3} + 2^{1/3}$ then the value of $x^3 - 6x^2 + 6x$ is

(a) 3 (b) 2

(c) 1 (d) none of these

Solution (b) We have, $x - 2 = 2^{2/3} + 2^{1/3}$. Cube both sides, we get

$$(x-2)^3 = 2^2 + 2 + 3\cdot 2^{2/3}\cdot 2^{1/3}(x-2)$$
$$= 6 + 6(x-2)$$

or $x^3 - 6x^2 + 12x - 8 = -6 + 6x$.

$\therefore$ $x^3 - 6x^2 + 6x = \mathbf{2}$.

22. The roots of the equation $3^{2x+1} + 3^2 = 3^{x+3} + 3^x$ are

(a) $1, -2$ (b) $1, 2$

(c) $-1, 2$ (d) $-1, -2$

Solution (c) We have, $3^{2x+1} + 3^2 = 3^{x+3} + 3^x$

$\Rightarrow$ $3^{2x}\cdot 3^1 + 9 = 3^x\cdot 3^3 + 3^x$

$\Rightarrow$ $3a^2 + 9 = 27a + a$, where $a = 3^x$

$\Rightarrow$ $3a^2 - 28a + 9 = 0$ or, $(3a-1)(a-9) = 0$

$\therefore$ $a = \dfrac{1}{3}, 9$

Now $a = \frac{1}{3} \Rightarrow 3^x = \frac{1}{3} = 3^{-1} \Rightarrow x = -1$

and $a = 9 \Rightarrow 3^x = 9 = 3^2 \Rightarrow x = 2$

Hence $x = \mathbf{-1, 2}$.

23. The values of a, for which the quadratic equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ possesses roots of opposite sign, are

(a) $1 < a < 2$ (b) $a \in (2, \infty)$

(c) $1 < a < 3$ (d) none of these

Solution (a) Roots are of opposite sign if

(*a*) roots are real and distinct (*b*) product is negative.

So $D = 4(a^2 + 1)^2 - 12(a^2 - 3a + 2) > 0$

and product of roots $= \frac{a^2 - 3a + 2}{3} < 0$

$\Rightarrow a^2 - 3a + 2 = (a - 1)(a - 2) < 0$

$\therefore 1 < a < 2.$

Clearly for these values of a, $D > 0$.

Hence $\mathbf{1 < a < 2}$.

24. Solution of the equation

$$\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1 \text{ is}$$

(a) $x \in [2, 3]$ (b) $x \in [2, 5]$

(c) $x \in (5, 10)$ (d) $x \in [5, 10]$

Solution (d) Let $\sqrt{x - 1} = t$

then $x = t^2 + 1, t \geq 0.$

The given equation reduces to

$$\sqrt{t^2 + 4 - 4t} + \sqrt{t^2 + 9 - 6t} = 1$$

$\Rightarrow |t - 2| + |t - 3| = 1$

$\therefore 2 \leq t \leq 3 \Rightarrow 4 \leq t^2 \leq 9$

$\Rightarrow 4 \leq x - 1 \leq 9 \Rightarrow \mathbf{5 \leq x \leq 10}.$

$\therefore$ Solution of the original equation is $x \in [5, 10]$.

25. If α and β are the roots of the equation $ax^2 - 2bx + c = 0$, then $\alpha^3\beta^3 + \alpha^2\beta^3 + \alpha^3\beta^2 =$

(a) $\frac{c^2}{a^3}(c - 2b)$ (b) $\frac{c^2}{a^3}(c + 2b)$

(c) $\frac{bc^2}{a^3}$ (d) none of these

Solution (b) Since α, β are roots of the equation

$ax^2 - 2bx + c = 0$

$\therefore \alpha + \beta = \frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Now, $\alpha^3\beta^3 + \alpha^2\beta^3 + \alpha^3\beta^2 = (\alpha\beta)^3 + \alpha^2\beta^2(\beta + \alpha)$

$$= \left(\frac{c}{a}\right)^3 + \frac{c^2}{a^2}\left(\frac{2b}{a}\right) = \frac{c^3 + 2bc^2}{a^3} = \frac{c^2(c + 2b)}{a^3}.$$

26. Root of the equation $3^{x-1} + 3^{1-x} = 2$ is

(a) 1 (b) 2

(c) 0 (d) -1

Solution (a) We have, $3^{x-1} + 3^{1-x} = 2$

$\Rightarrow 3^x \cdot 3^{-1} + \frac{3^1}{3^x} = 2 \Rightarrow \frac{y}{3} + \frac{3}{y} = 2$, where $y = 3^x$

$\Rightarrow y^2 - 6y + 9 = 0 \Rightarrow (y - 3)^2 = 0 \quad \therefore \quad y = 3$

$\Rightarrow 3^x = 3^1 \Rightarrow \mathbf{x = 1}.$

27. The real values of x for which $3^{72}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^{\sqrt{x}} > 1$, are

(a) $x \in [0, 64]$ (b) $x \in (0, 64)$

(c) $x \in [0, 64)$ (d) none of these

Solution (c) The given inequation is valid only

when $x \geq 0$ ...(1)

The given inequation can be written in the form

$$3^{72 - x - \sqrt{x}} > 1$$

$\Rightarrow 72 - x - \sqrt{x} > 0 \qquad (\because 3 > 1)$

$\Rightarrow x + \sqrt{x} - 72 < 0$

$\Rightarrow (\sqrt{x} + 9)(\sqrt{x} - 8) < 0$

But $\sqrt{x} + 9 > 0$ for all $x \geq 0$

$\therefore \sqrt{x} - 8 < 0 \Rightarrow \sqrt{x} < 8$

$\therefore \mathbf{0 \leq x < 64}$ {from (1)}.

28. If α, β are the roots of the equation $x^2 + x\sqrt{\alpha} + \beta = 0$ then the values of α and β are

(a) $\alpha = 1, \beta = -1$ (b) $\alpha = 1, \beta = -2$

(c) $\alpha = 2, \beta = 1$ (d) $\alpha = 2, \beta = -2$

Solution (b) Since α, β are roots of the equation

$x^2 + x\sqrt{\alpha} + \beta = 0$

$\therefore \alpha + \beta = -\sqrt{\alpha}$ and $\alpha\beta = \beta$

$\Rightarrow \alpha = 1 \therefore \beta = -1 - \sqrt{\alpha} = -1 - 1 = -2.$

Thus, $\alpha = \mathbf{1}$ and $\beta = \mathbf{-2}$.

29. For real values of x, the expression $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lies between

(a) $-\frac{1}{7}$ and 7 (b) $\frac{1}{7}$ and 7

(c) $\frac{1}{3}$ and 3 (d) none of these

Solution (b) Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow yx^2 + 3xy + 4y = x^2 - 3x + 4$

$\Rightarrow (y - 1)x^2 + 3(y + 1)x + 4(y - 1) = 0$

Since x is real, $\therefore$ discriminant ≥ 0

$\Rightarrow 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow 9(y^2 + 2y + 1) - 16(y^2 - 2y + 1) \geq 0$

$\Rightarrow -7y^2 + 50y - 7 \geq 0 \Rightarrow 7y^2 - 50y + 7 \leq 0$

$\Rightarrow (y - 7)\left(y - \frac{1}{7}\right) \leq 0 \quad \Rightarrow \mathbf{\frac{1}{7} \leq y \leq 7}.$

30. If x is real, then the maximum value of $3 - 6x - 8x^2$ is

(a) $\frac{17}{8}$ (b) $\frac{33}{8}$

(c) $\frac{21}{8}$ (d) none of these

Solution (b) Let $y = 3 - 6x - 8x^2$ then $8x^2 + 6x + y - 3 = 0$.

Since x is real, $\therefore\ 6^2 - 4\cdot 8(y - 3) \geq 0$,

or $36 - 32y + 96 \geq 0$ or $32y \leq 132$

$\therefore\quad y \leq \frac{132}{32}$ or $y \leq \frac{33}{8}$.

Hence maximum value of $y = \mathbf{\frac{33}{8}}$.

31. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{19} and β^7 is

(a) $x^2 - x + 1 = 0$ (b) $x^2 + x + 1 = 0$

(c) $x^2 + x + 3 = 0$ (d) $x^2 - x + 3 = 0$

Solution (b) We have, $x^2 + x + 1 = 0$

$\Rightarrow\quad (x - \omega)(x - \omega^2) = 0 \Rightarrow x = \omega, \omega^2$

$\therefore\quad \alpha = \omega$ and $\beta = \omega^2$

$\therefore\quad \alpha^{19} = \omega^{19} = \omega$ and $\beta^7 = \omega^{14} = \omega^2$.

$\therefore\quad$ Required equation is $\mathbf{x^2 + x + 1 = 0}$.

32. The number of real solutions of the equation $27^{1/x} + 12^{1/x} = 2\cdot 8^{1/x}$ is

(a) one (b) two

(c) infinite (d) zero

Solution (d) The given equation can be written as

$$\left(\frac{3}{2}\right)^{3/x} + \left(\frac{3}{2}\right)^{1/x} = 2.$$

Put $\left(\frac{3}{2}\right)^{1/x} = t$, then the equation becomes $t^3 + t - 2 = 0$

$\Rightarrow\quad (t - 1)(t^2 + t + 2) = 0$.

But $t^2 + t + 2 = 0$ has no real roots, $\therefore\ t = 1$

$\Rightarrow\quad \left(\frac{3}{2}\right)^{1/x} = 1 \Rightarrow \frac{1}{x} = 0$

which is not possible for any value of x.

33. The values of a which make the expression $x^2 - ax + 1 - 2a^2$ always positive for real values of x, are

(a) $-\frac{2}{3} < a < \frac{2}{3}$ (b) $-\frac{2}{3} \leq a \leq \frac{2}{3}$

(c) $-\frac{2}{3}\ \ a < 1$ (d) $0 < a < \frac{2}{3}$

Solution (a) Since the coefficient of x^2 is 1 which is positive, $\therefore$ the given expression is positive for all real values of x if $D < 0$.

$\Rightarrow\quad (-a)^2 - 4(1 - 2a^2) < 0 \Rightarrow 9a^2 - 4 < 0$

$\Rightarrow\quad (3a + 2)(3a - 2) < 0 \Rightarrow -\frac{2}{3} < a < \frac{2}{3}$.

34. For all real values of x, $\left|\frac{12x}{4x^2+9}\right|$

(a) ≤ 1 (b) ≤ 2

(c) > 1 (d) > 2

Solution (a) Let $\frac{12x}{4x^2+9} = y$,

Now, $4yx^2 - 12x + 9y = 0$.

As x is real, $D = 144 - 4\cdot 4y\cdot 9y \geq 0 \Rightarrow 1 - y^2 \geq 0$

$\Rightarrow\quad y^2 \leq 1$; $\therefore\ |y| \leq 1$. Hence $\mathbf{\left|\frac{12x}{4x^2+9}\right| \leq 1}$.

35. If $x^2 - 3x + 2$, be one of the factors of the expression $x^4 - px^2 + q$, then

(a) $p = 4, q = 5$ (b) $p = 5, q = 4$

(c) $p = -5, q = -4$ (d) none of these

Solution (b) Since $x^2 - 3x + 2 = 0$ is one of the factors of the expression $x^4 - px^2 + q$, therefore, on dividing the expression by factor, remainder = 0 i.e., on dividing $x^4 - px^2 + q$ by $x^2 - 3x + 2$, the remainder

$$(15 - 3p)x + (2p + q - 14) = 0.$$

On comparing both sides, we get

$15 - 3p = 0$ or $\mathbf{p = 5}$ and $2p + q - 14 = 0$ or $\mathbf{q = 4}$.

36. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$ are

(a) 2 (b) 3

(c) 4 (d) 1

Solution (c) The given equation can be written as

$|x|^2 - 3|x| + 2 = 0 \Rightarrow (|x| - 1)(|x| - 2) = 0$

$\Rightarrow\quad |x| = 1$ or $|x| = 2 \Rightarrow x = \pm 1$ or $x = \pm 2$

$\therefore\quad$ **Four** roots are possible.

37. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$ (b) $2 \leq a \leq 3$

(b) $3 \leq a \leq 4$ (d) $a > 4$

Solution (a) Since the roots of the given equation are real

$\therefore\quad B^2 - 4AC \geq 0 \Rightarrow 4a^2 - 4(a^2 + a - 3) \geq 0$

$\Rightarrow\quad -a + 3 \geq 0$ or $a \leq 3$. ...(1)

Since the root is less than 3, so $f(3) > 0$

$\Rightarrow\quad 3^2 - 2a(3) + a^2 + a - 3 > 0$

$\Rightarrow\quad a^2 - 5a + 6 > 0$ or $(a - 2)(a - 3) > 0$

$\Rightarrow\quad a < 2$ or $a > 3$ ...(2)

From (1) and (2), we have $\mathbf{a < 2}$.

38. The roots of the equation $x^2 + ax + b = 0$ where $a = 2\sqrt{b}$ and $b > 3$ are

(a) imaginary (b) equal

(c) unequal (d) none of these

Solution (b) $D = B^2 - 4AC = a^2 - 4b = 0$ $[\because a = 2\sqrt{b}]$

$\therefore\quad$ Roots are **equal.**

39. The equation $\frac{a}{x-a} + \frac{b}{x-b} = 1$ has roots equal in magnitude but opposite in sign, then value of $a + b$ is

(a) 1 (b) -1

(c) 0 (d) none of these

Solution (c) The given equation can be written as

$$x^2 - 2(a + b)x + 3ab = 0$$

The roots are equal in magnitude but opposite in sign, hence their sum,

$2(a + b) = 0$ or $a + b = \mathbf{0}$.

40. The value of k for which the number 3 lies between the roots of the equation $x^2 + (1 - 2k)x + (k^2 - k - 2) = 0$ is given by

(a) $2 < k < 5$ (b) $k < 2$

(c) $2 < k < 3$ (d) $k > 5$

Solution (a) Let $f(x) = x^2 + (1 - 2k)x + k^2 - k - 2$

The number 3 lies between the roots of the given equation, if $f(3) < 0$

Now, $f(3) = 9 + (1 - 2k)3 + k^2 - k - 2$

$= 10 - 7k + k^2 = k^2 - 7k + 10$

Hence $f(3) < 0 \Rightarrow k^2 - 7k + 10 < 0$

$\Rightarrow (k - 2)(k - 5) < 0 \Rightarrow \mathbf{2 < k < 5}$.

41. The minimum value of $|x|\left|x+\frac{1}{2}\right| + |x-3|\left|x-\frac{5}{2}\right|$ is

(a) 2 (b) 4

(c) 6 (d) 0

Solution (c) The minimum value occurs at $x = 0$, because minimum value of $|x| = 0$.

$\therefore$ Minimum value

$$= |0|\left|0+\frac{1}{2}\right| + |0-3| + \left|0-\frac{5}{2}\right|$$

$$= \frac{1}{2} + 3 + \frac{5}{3} = \mathbf{6}.$$

42. If $a < b$, then the solution of $x^2 + (a + b)x + ab < 0$, is given by

(a) $x < -b$ or $x < -a$ (b) $a < x < c$

(b) $x < a$ or $x > b$ (d) $-b < x < -a$

Solution (d) $x^2 + (a + b)x + ab < 0$

$\Rightarrow (x + a)(x + b) < 0$

$\Rightarrow x + a < 0, x + b > 0$ or $x + a > 0, x + b < 0$

$\Rightarrow x < -a, x > -b$ or $x > -a, x < -b$

$\Rightarrow -b < x < -a$ or $-a < x < -b$

Since $a < b \therefore -a > -b$

Hence $\mathbf{-b < x < -a}$.

43. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $\left(\frac{\alpha^2}{\beta}\right)^{\frac{1}{3}} + \left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{3}}$ is

(a) $\frac{1}{3}$ (b) $\frac{7}{2}$

(c) 4 (d) $\frac{1}{4}$

Solution (d)

44. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is

(a) 1 (b) 0

(c) 2 (d) none of these

Solution (a)

45. The conditions that the equation $ax^2 + bx + c = 0$ has both the roots positive is that

(a) a and b are of the same sign

(b) a, b and c are of the same sign

(c) a and c are of the same sign opposite to that of b

(d) b and c are of the same sign opposite to that of a

Solution (c)

46. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the quadratic equation $ax^2 + bx + c = 0$ has

(a) purely imaginary roots

(b) only one root

(c) real roots

(d) complex roots

Solution (c)

47. The number of roots of the equation $\frac{(x+2)(x-5)}{(x-3)(x+6)} = \frac{(x-2)}{(x+4)}$ is

(a) 0 (b) 1

(c) 2 (d) 3

Solution (b)

48. If α, β are the roots of the equation $x^2 + x\sqrt{\alpha} + \beta = 0$, then the values of α and β are

(a) $\alpha = 2$ and $\beta = -2$ (b) $\alpha = 2$ and $\beta = 1$

(c) $\alpha = 1$ and $\beta = -2$ (d) $\alpha = 1$ and $\beta = -1$

Solution (c)

49. The number of negative integral solutions of

$$x^2 \cdot 2^{x+1} + 2^{|x-3|+2} = x^2 \cdot 2^{(|x-3|+4)} + 2^{x-1}$$ is

(a) 4 (b) 2

(c) 1 (d) 0

Solution (d)

50. If α and β ($\alpha < \beta$), are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$

(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Solution (b)

51. The number of solution of $|[x] - 2x| = 4$, where $[x]$ is the greatest integer $\le x$, is

(a) 2 (b) 4

(c) 1 (d) infinite

Solution (b)

52. The common roots of the equations $x^3 + 2x^2 + 2x + 1 = 0$ and $1 + x^{130} + x^{1988} = 0$ are (where ω is a none real cube root of unity)

(a) ω (b) ω^2

(c) -1 (d) ω, ω^2

Solution (a), (b), (d)

53. The number of solutions (s) of the equation $\sqrt{3x^2+6x+7} + \sqrt{5x^2+10x+14} \le 4 - 2x - x^2$ is

(a) one (b) two

(c) four (d) infinite

Solution (a)

54. If $\log_2 x + \log_x 2 = \frac{10}{3} = \log_2 y + \log_y 2$ and $x \ne y$, then $x + y =$

(a) 2 (b) $\frac{65}{8}$

(c) $\frac{1}{2}$ (d) none of these

Solution (d)

55. $\{x \in R : |x - 2| = x^2\} =$

(a) $-1, 2$ (b) $1, 2$

(c) $-1, -2$ (d) $1, -2$

Solution (d)

Problems for Practice

Level of Difficulty I

1. If p and q are the roots of the equation $x^3 + px + q = 0$, then
(a) $p = 1$ (b) $p = 1$ or 0
(c) $p = -2$ (d) $p = -2$ or 0

2. If α, β are roots of the equation $x^2 - p(x + 1) - c = 0$, then $(\alpha + 1)(\beta + 1) =$
(a) c (b) $c - 1$ (c) $1 - c$ (d) none of these

3. It α, β are the roots of the equation

$$x^2 - (1 + n^2)\,x + \frac{1}{2}(1 + n^2 + n^4) = 0, \text{then } \alpha^2 + \beta^2 =$$

(a) $n^2 + 2$ (b) $-n^2$
(c) n^2 (d) $2n^2$

4. If the equation $5x^2 + 13x + k = 0$ has roots α and $\frac{1}{\alpha}$, then k is equal to
(a) 1 (b) 13 (c) -5 (d) 5

5. Roots of the equation $\frac{2x+31}{9} + \frac{x^2+7}{x^2-7} = \frac{2x+47}{9}$ are
(a) ± 3 (b) $\pm\sqrt{3}$ (c) ± 5 (d) $\pm\sqrt{5}$

6. If one root of a quadratic equation is $\frac{-i+\sqrt{3}}{2}$, then the sum of the roots is
(a) $-i$ (b) $\sqrt{3}/2$ (c) $+i$ (d) $\sqrt{3}$

7. The values of k for which the quadratic equation $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are
(a) $\{-11, -3\}$ (b) $\{5, 7\}$
(c) $\{5, -7\}$ (d) none of these

8. If $a + b + c = 0$ and a, b, c are rational, then the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are
(a) rational (b) irrational
(c) imaginary (d) equal

9. The value of m for which the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots, is
(a) 0 (b) 1 (c) 2 (d) 3

10. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite sign is
(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(0, \infty)$

11. If the roots of the equation $x^2 + a^2 = 8x + 6a$ are real, then
(a) $a \in (-\infty, -2]$ (b) $a \in [-2, 8]$
(c) $a \in [8, \infty)$ (d) none of these

12. If the roots of the equation $9x^2 + 4ax + 4 = 0$ are imaginary, then
(a) $a \in (-3, 3)$
(b) $a \in [-3, 3]$
(c) $a \in (-\infty, -3] \cup [3, \infty)$
(d) none of these

13. The value of m for which the roots of the equation $x^2 + (m - 2)x + m + 2 = 0$ are in the ratio 2 : 3, is
(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{26}{3}$ (d) $-\frac{26}{3}$

14. The value of k so that the equations $x^2 - x - 12 = 0$ and $kx^2 + 10x + 3 = 0$ may have one root in common, is
(a) $\frac{43}{16}$ (b) 3 (c) -3 (d) $\frac{-43}{16}$

15. For the equation $|x^2| + |x| - 6 = 0$, the roots are
(a) real and equal (b) real with sum 0
(c) real with sum 1 (d) real with product 0

16. In the equation $x^2 + px + q = 0$, the coefficient of x was incorrectly written as 17 instead of 13, then the roots were found to be -2 and -15. The correct equation is
(a) $x^2 + 13x + 26 = 0$ (b) $x^2 + 13x - 30 = 0$
(c) $x^2 + 13x + 30 = 0$ (d) none of these

17. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, the value of q is
(a) 49/4 (b) 4/49 (c) 4 (d) none of these

18. The quadratic equation with rational coefficients whose one root is $\frac{2+\sqrt{3}}{2-\sqrt{3}}$, is
(a) $x^2 - 14x + 1 = 0$ (b) $x^2 + 14x + 1 = 0$
(c) $x^2 - 14x - 1 = 0$ (d) none of these

19. If a and b are rational and α, β be the roots of $x^2 + 2ax + b = 0$, then the equation with rational coefficients one of whose roots is $\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$ is
(a) $x^2 + 4ax - 2b = 0$ (b) $x^2 + 4ax + 2b = 0$
(c) $x^2 - 4ax + 2b = 0$ (d) $x^2 - 4ax - 2b = 0$

20. If x is real, then the expression $\frac{x^2+34x-71}{x^2+2x-7}$
(a) lies between 4 and 7
(b) lies between 5 and 9
(c) has no value between 4 and 7
(d) has no value between 5 and 9

Level of Difficulty II

21. If α, β are roots of the equation $x^2 + px + p^2 + q = 0$, then the value of $\alpha^2 + \alpha\beta + \beta^2 + q$ is
(a) 0 (b) 1 (c) q (d) $2q$

22. The value of k for which the expression $x^2 + (2 - k)x + k - \frac{3}{4}$ will be a perfect square, is
(a) 1 (b) 6 (c) 7 (d) none of these

23. The roots of an equation $x^3 - 9x^2 + 14x + 24 = 0$ are in the ratio 3 : 2. The roots will be
(a) 6, 4, -1 (b) 6, 4, 1
(c) -6, 4, 1 (d) -6, -4, 1

24. Given that, for all real x, the expression $\frac{x^2-2x+4}{x^2+2x+4}$ lies between $\frac{1}{3}$ and 3. The values between which the expression $\frac{9\cdot 3^{2x}+6\cdot 3^x+4}{9\cdot 3^{2x}-6\cdot 3^x+4}$ lies are

(a) 0 and 2 (b) – 1 and 1
(c) – 2 and 0 (d) $\frac{1}{3}$ and 3.

25. The value of k ($k > 0$) for which the equations $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$, both will have real rootsis
(a) 16 (b) – 64 (c) – 16 (d) 8

26. If $x^2 - x + 1 = 0$, then value of x^{3n} is
(a) 0 (b) – 1 (c) 1 (d) – 1, 1

27. If the roots of the equation $px^2 + qx + r = 0$ and $qx^2 + 2\sqrt{pr}\,x + q = 0$ are simultaneously real, then
(a) $\frac{p}{q} = \frac{q}{r}$ (b) $q = r \neq 0$
(c) $2q = \pm\sqrt{pr}$ (d) $p = q, r \neq 0$

28. If the roots of $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, then $a + c =$
(a) b^2 (b) b (c) $2b$ (d) $3b$

29. The values of p and q ($p \neq 0, q \neq 0$) for which p, q are the roots of the equation $x^2 + px + q = 0$ are
(a) $p = 1, q = -2$ (b) $p = -1, q = -2$
(c) $p = -1, q = 2$ (d) $p = 1, q = 2$

30. If α and β are the roots of $x^2 + px + q = 0$ and α^4 and β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always
(a) two real roots (b) two positive roots
(c) two negative roots
(d) one positive and one negative root

31. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$ then a, b, c and d.
(a) are in A.P.(b) are in G.P.
(c) are in H.P.(d) satisfy $ab = cd$

32. The sum of all real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is
(a) 2 (b) 4 (c) 1 (d) none of these

33. If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation having α/β and β/α as its roots is
(a) $3x^2 + 19x + 3 = 0$
(b) $3x^2 - 19 + 3 = 0$
(c) $3x^2 - 19x - 3 = 0$
(d) $x^2 - 16x + 1 = 0$

34. The number of real roots of $3^{2x^2-7x+7} = 9$ is
(a) zero (b) 2 (c) 1 (d) 4

35. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in
(a) Arithmetic progression
(b) Geometric progression
(c) Harmonic progression
(d) Arithmetico-Geometric progression

36. The number of the real solutions of the equation $x^2 - 3|x| + 2 = 0$ is
(a) 2 (b) 4 (c) 1 (d) 3

37. The value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is
(a) $\frac{2}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

38. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation
(a) $x^2 + 18x - 16 = 0$
(b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x + 16 = 0$
(d) $x^2 - 18x - 16 = 0$

39. If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its roots are
(a) 0, –1 (b) –1, 1 (c) 0.1 (d) –1, 2

40. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of q is
(a) 3 (b) 12 (c) 49/4 (d) 4.

ANSWERS

1. (b)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (a)	**9.** (a),(d)	**10.** (b)
11. (b)	**12.** (a)	**13.** (b),(c)	**14.** (b),(d)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (d)
21. (a)	**22.** (a),(c)	**23.** (a)	**24.** (d)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (a)
31. (b)	**32.** (b)	**33.** (b)	**34.** (b)	**35.** (c)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (c)

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Permutations and Combinations

BRIEF REVIEW OF THE CONCEPTS

FACTORIAL NOTATION

We often come across products of the form $1 \cdot 2$, $1 \cdot 2 \cdot 3$, $1 \cdot 2 \cdot 3 \cdot 4, \ldots$

Instead of writing all the factors of such a product in full, it is convenient to use a special notation. We write

$$\begin{aligned} 1\,! &= 1, \\ 2\,! &= 1 \cdot 2, \\ 3\,! &= 1 \cdot 2 \cdot 3, \\ &\ldots\ldots\ldots\ldots \\ n! &= 1 \cdot 2 \cdot 3 \ldots n. \end{aligned}$$

"$n!$" denotes the product of the first n natural numbers. We read '$n!$' as 'n factorial'. $n!$ is also written as '$\lfloor\!\underline{n}$' and read as 'factorial n'. It is easy to see that

$1! = 1$, $2! = 1 \cdot 2 = 2$, $3\,! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4. = 24$, and so on.

***Notes*:**

- We know that

$n\,! = n\,(n-1)\,(n-2)\,(n-3) \ldots 3 \cdot 2 \cdot 1$

$= n\,(n-1)! = n\,(n-1)\,(n-2)!$

$= n\,(n-1)\,(n-2)\,(n-3)!$ and so on.

Thus, if $m,\ n \in N$ and $m > n$, then $m\,!$ can be expressed in terms of $n!$

For example $8! = 8 \cdot 7 \cdot 6!$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!$

Also $m\,! = n\,!$ if and only if $m = n$

- Putting $n = 1$ in $n! = n\,(n-1)!$, we have

$$\begin{aligned} 1! &= 1 \cdot 0! \\ \therefore \quad 0! &= 1. \end{aligned}$$

- The factorial is defined only for whole numbers

  We do not define the factorial of proper fractions.

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication Principle If an operation can be performed in 'm' different ways; following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in $m \times n$ different ways.

Addition Principle If an operation can be performed in 'm' different ways and another operation, which is independent of the first operation, can be performed in 'n' different ways. Then either of the two operations can be performed in $(m + n)$ ways.

***Note*:** The above two principles can be extended for any finite number of operations.

PERMUTATION

Each of the different arrangements which can be made by taking some or all of given number of things or objects at a time is called a *Permutation*.

***Note*:** Permutation of things means arrangement of things. The word arrangement is used if order of things is taken into account. Thus, if order of different things changes, then their arrangement also changes.

Notations Let r and n be positive integers such that $1 \le r \le n$. Then, the number of permutations of n different things, taken r at a time, is denoted by the symbol ${}^nP_r$ or $P\,(n, r)$.

Key Points to Remember

1. ${}^nP_r = \dfrac{n!}{(n-r)!} = n(n-1)\,(n-2) \ldots \{n-(r-1)\},\ 0 \le r \le n.$
2. Number of Permutations of n different things taken all at a time is: ${}^nP_n = n!$.
3. The number of permutations of n things, taken all at a time, out of which p are alike and are of one type, q are alike and are of second type and rest are all different is $\dfrac{n!}{p!\,q!}$.
4. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.
5. **Permutations under Restrictions**
   - **(a)** Number of Permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is:
     $$r \cdot {}^{n-1}P_{r-1}.$$
   - **(b)** Number of permutations of n different things, taken r at a time, when s particular things are to be always included in each arrangement, is
     $$s!\ (r-(s-1)) \cdot {}^{n-s}P_{r-s}.$$
   - **(c)** Number of permutations of n different things, taken r at a time, when a particular thing is never taken in each arrangement, is
     $${}^{n-1}P_r.$$

(d) Number of permutations of n different things, taken all at a time, when m specified things always come together, is $m! \times (n - m + 1)!$

(e) Number of permutations of n different things, taken all at a time, when m specified things never come together, is $n! - m! \times (n - m + 1)!$.

6. **Circular Permutations**

(a) Number of circular arrangements (permutations) of n different things is: $(n - 1)!$

(b) Number of circular arrangements (permutations) of n different things when clockwise and anticlockwise arrangements are not different, i.e., when observation can be made from both sides is: $\frac{1}{2}(n-1)!$.

(c) Number of circular permutations of n different things, taken r at a time, when clockwise, and anticlockwise orders are taken as different, is $= \frac{{}^nP_r}{r}$.

(d) Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are not different, is $= \frac{{}^nP_r}{2r}$.

COMBINATION

Each of the different groups or selections which can be made by taking some or all of a number of things (irrespective of order) is called a combination.

Note: Combination of things means selection of things. Obviously, in selection of things order of things has no importance. Thus, with the change of order of things selection of things does not change.

Notations The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$. Thus,

$$^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!} \qquad (0 \le r \le n)$$

$$= \frac{n\,(n-1)\,(n-2)\ldots.(n-r+1)}{r\,(r-1)\,(r-2)\ldots 3\cdot 2\cdot 1}$$

If $r > n$, then ${}^nC_r = 0$.

Key Results on Combinations

1. ${}^nC_r = {}^nC_{n-r}$
2. ${}^nC_0 = {}^nC_n = 1,\ {}^nC_1 = n$
3. If ${}^nC_x = {}^nC_y$ then either $x = y$ or $y = n - x$, i.e., $x + y = n$.
4. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
5. $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$
6. $n \cdot {}^{n-1}C_{r-1} = = (n - r + 1)\ {}^nC_{r-1}$
7. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
8. If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.
9. If n is odd then the greatest value of ${}^nC_r$ is

$${}^nC_{\frac{n+1}{2}} \text{ or } {}^nC_{\frac{n-1}{2}}.$$

10. $${}^nC_r = \frac{r \text{ decreasing numbers starting with } n}{r \text{ increasing numbers starting with } 1}$$

$$= \frac{n\,(n-1)\,(n-2)\ldots.(n-r+1)}{1\cdot 2\cdot 3\ldots..r}.$$

11. ${}^nP_r = r!\ {}^nC_r = n(n - 1)(n - 2)\ldots..(n - r + 1)$.
12. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.
13. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.
14. Number of combinations of n different things taken r at a time
    - **(a)** when p particular things are always included $= {}^{n-p}C_{r-p}$
    - **(b)** when p particular things are never included $= {}^{n-p}C_r$
    - **(c)** when p particular things are not together in any selection $= {}^nC_r - {}^{n-p}C_{r-p}$.
15. **(a)** Number of selections of r consecutive things out of n things in a row $= n - r + 1$.

    (b) Number of selections of r consecutive things out of n things along a circle

$$\begin{cases} n, \text{ when } r < n \\ 1, \text{ when } r = n \end{cases}.$$

16. **(a)** Number of selections of zero or more things out of n different things

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

(b) Number of combinations of n different things selecting at least one of them is

$${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1.$$

(c) Number of selections of r things ($r \le n$) out of n identical things is 1.

(d) Number of selections of zero or more things out of n identical things $= n + 1$.

(e) Number of selections of one or more things out of n identical things $= n$.

17. **(a)** Number of ways of dividing $m + n$ different things in two groups containing m and n things respectively ($m \neq n$):

$${}^{m+n}C_m = \frac{(m+n)!}{m!\,n!}.$$

(b) Number of ways of dividing $m + n + p$ different things in three groups containing m, n and p things respectively ($m \neq n \neq p$):

$$\frac{(m+n+p)!}{m!\,n!\,p!}.$$

NUMBER OF RECTANGLES AND SQUARES

1. Number of rectangles of any size in a square of size $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.
2. Number of rectangles of any size in a rectangle of size $n \times p$ ($n < p$) is $\frac{np}{4}(n + 1)(p + 1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

Key Results on Combinations

1. If n distinct points are given in the plane such that no three of which are collinear, then the number of line segments formed = ${}^nC_2$

   If m of these points are collinear ($m \geq 3$), then the number of line segments is $({}^nC_2 - {}^mC_2) + 1$.
2. The number of diagonals in an n-sided closed polygon = ${}^nC_2 - n$.
3. If n distinct points are given in the plane such that no three of which are collinear, then the number of triangles formed = ${}^nC_3$.

   If m of these points are collinear ($m \geq 3$), then the number of triangles formed = ${}^nC_3 - {}^mC_3$.
4. If n distinct points are given on the circumference of a circle, then

   (a) Number of st. lines = ${}^nC_2$

   (b) Number of triangles = ${}^nC_3$

   (c) Number of quadrilaterals = ${}^nC_4$ and so on
5. The sum of the digits in the unit place of all numbers formed with the help of $a_1, a_2, ..., a_n$ taken all at a time is = $(n-1)!\,(a_1 + a_2 + ... + a_n)$

   (repetition of digits not allowed)
6. The sum of all n digit numbers that can be formed using the digits $a_1, a_2, ..., a_n$ is

   $= (n-1)!\,(a_1 + a_2 + ... + a_n)\,\dfrac{(10^n - 1)}{9}$.

Solved Problems

1. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$, then

(a) $n > 5$
(b) $n > 6$
(c) $n > 7$
(d) none of these

Solution (c) We have, ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$

$$\Rightarrow \quad {}^nC_4 > {}^nC_3 \Rightarrow \frac{n!}{(n-4)!\,4!} > \frac{n!}{(n-3)!\,3!}$$

$$\Rightarrow \quad \frac{1}{4\cdot 3!\,(n-4)!} > \frac{1}{3!\,(n-3)(n-4)!}$$

$$\Rightarrow \quad \frac{1}{4} > \frac{1}{n-3}$$

$\Rightarrow n - 3 > 4 \Rightarrow \mathbf{n > 7.}$

2. ${}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + ... + n\cdot{}^nP_n =$

(a) ${}^{n+1}P_{n+1}$ (b) ${}^{n+1}P_{n+1} - 1$
(c) ${}^{n+1}P_{n+1} - 2$ (d) none of these

Solution (b) rth term = $r\cdot{}^rP_r = r\cdot r!$
$= (r+1) - 1)\,r!$.

$= (r+1)! - r!$

Putting $r = 1, 2, 3, ... n$ and adding, we get

$${}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + ... + n\cdot{}^nP_n$$
$$= (2! - 1!) + (3! - 2!) + (4! - 3!) + ... + ((n+1)! - n!)$$
$$= (n+1)! - 1! = \mathbf{{}^{n+1}P_{n+1} - 1.}$$

3. ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3 =$

(a) ${}^{52}C_4$
(b) ${}^{51}C_4$
(c) ${}^{52}C_3$
(d) none of these

Solution (a) We have,

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3 = {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$
$$= ({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 = {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_3$$
$$= {}^{51}C_4 + {}^{51}C_3 = \mathbf{{}^{52}C_4.}$$

4. ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 =$

(a) 8
(b) 0
(c) 6
(d) none of these

Solution (b) We have,

$${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = ({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7)$$
$$= {}^{16}C_9 - {}^{16}C_7 \quad (\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$$
$$= {}^{16}C_9 - {}^{16}C_9 \quad (\because {}^nC_r = {}^nC_{n-r})$$
$$= \mathbf{0.}$$

5. The number of positive terms in the sequence

$$x_n = \frac{195}{4\,{}^nP_n} - \frac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}},\ n \in N \text{ is}$$

(a) 2
(b) 3
(c) 4
(d) none of these

Solution (c) We have,

$$x_n = \frac{145}{4\cdot{}^nP_n} - \frac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}}$$
$$= \frac{195}{4\cdot n!} - \frac{(n+3)(n+2)(n+1)}{(n+1)!}$$
$$= -\ \frac{195}{4\cdot n!} - \frac{(n+3)(n+2)}{n!}$$

$$= \frac{195 - 4n^2 - 20n - 24}{4 \cdot n!}$$

$$= \frac{171 - 4n^2 - 20n}{4 \cdot n!}$$

$\because$ x_n is positive.

$$\therefore \quad \frac{171 - 4n^2 - 20n}{4 \cdot n!} > 0 \Rightarrow 4n^2 + 20n - 171 < 0$$

which is true for $n = 1, 2, 3, 4$.

Hence, the given sequence has **4** positive terms.

6. A man has got seven friends. The number of ways in which he can invite one or more of his friends to dinner, is

(a) 116 (b) 128

(c) 127 (d) none of these

Solution (c) Number of friends = 7.

The men can invite one friend, two friends ...., or seven friends.

$\therefore$ Required number of ways

$$= {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$$

$$= \frac{7}{1} + \frac{7 \times 6}{1 \times 2} + \frac{7 \times 6 \times 5}{1 \times 2 \times 3} + \frac{7 \times 6 \times 5}{1 \times 2 \times 3} + \frac{7 \times 6}{1 \times 2} + \frac{7}{1} + 1$$

$= 7 + 21 + 35 + 35 + 21 + 7 + 1 =$ **127.**

7. The number of numbers that can be formed with the help of the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy odd places is

(a) 24 (b) 18

(c) 12 (d) 30

Solution (b) The 4 odd digits 1, 3, 3, 1 can be arranged in the 4 odd places in $\frac{4!}{2!2!} = 6$ ways.

and even digits 2, 4, 2 can be arranged in the three even places in $\frac{3!}{2!} = 3$ ways.

Hence the required number of ways = 6 × 3 = **18.**

8. The chief minister of 11 states of India meet to discuss the language problem. The number of ways they can seat themselves at a round table so that the Punjab and Madras Chief Ministers sit together is

(a) 725760

(b) 725748

(c) 725778

(d) none of these

Solution (a) Treat the Panjab and Madras Chief Ministers as one (*P*, *M*) + 9 others.

$\therefore$ We have to arrange 10 persons round a table. This can be done in (10 – 1)! = 9! ways. Corresponding to each of these 9! ways, the Panjab and Madras Chief Ministers can interchange their places in 2! ways. Associating the two operations, total number of ways

$= 9! \times 2!$

$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (2 \times 1)$

= **725760.**

9. The total number of 8 digits numbers which have all different digits is

(a) 3265920

(b) 3265860

(c) 3268620

(d) none of these

Solution (a) The number is to be of 9 digits, the first place can be filled in 9 ways only (as 0 cannot be in the left-most position). Having filled up the first place, the remaining 8 places can be filled up by the remaining 9 digits in

$${}^9P_8 = \frac{9!}{(9-8)!} \text{ ways} = 9! \text{ ways.}$$

Hence the required no. of numbers

$= 9 \times 9! = 9 \times (9 \times 8 \times 7 \times 6!)$

$= 8! \times 56 \times 6! = 8! \times 56 \times 720 =$ **3265920.**

10. The sum of the digits in the unit place of all the numbers formed with the help of 3, 4, 5, 6 taken all at a time is

(a) 432 (b) 108

(c) 36 (d) 18

Solution (b) The total number of numbers that can be formed with the digits 3, 4, 5, 6 taken all at a time

$= P(4, 4) = 4! = 24.$

Each of the digits 3, 4, 5, 6 occurs in 3!

$= 3 \times 2 = 6$ times in unit's place.

$\therefore$ Sum of the digits in the unit's place of all the numbers

$= (3 + 4 + 5 + 6) \times 6 = 18 \times 6 =$ **108.**

11. The number of ways in which a committee of 5 can be chosen from 10 candidates so as to exclude the youngest if it includes the oldest, is

(a) 196

(b) 178

(c) 202

(d) none of these

Solution (a) There are two different ways of forming the committee

(i) oldest may be included

(ii) oldest may be excluded

(i) If oldest is included, then youngest has to be excluded and we are to select 4 candidates out of 8. This can be done in

$${}^8C_4 = \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70 \text{ ways.}$$

(ii) If oldest is excluded, then we are to select 5 candidates from 9 which can be done in

$${}^9C_5 = \frac{9!}{5!4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126 \text{ ways}$$

Hence the total number of ways in which committee can be formed

$= 126 + 70 =$ **196.**

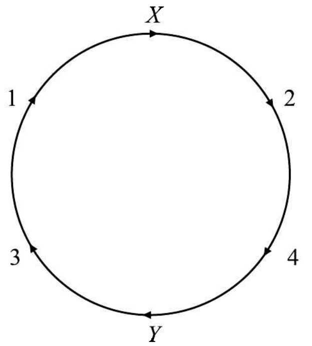
12. Three boys and three girls are to be seated around a table, in a circle. Among them, the boy *X* does not want any girl neighbour and the girl *Y* does not want any boy neighbour. The number of such arrangements possible is

(a) 4 (b) 6

(c) 8 (d) none of these

Solution (a) As shown in figure, 3, 1, 2 and *X* are the three boys and 3, 4 and *Y* are three girls, Boy *X* will have neighbours as boys 1 and 2 and the girl Y will have neighbors as girls 3 and 4.]

1 and 2 can be arranged in *P* (2, 2) ways



$= 2! = 2 \times 1 = 2$ ways.

Also 3 and 4 can be arranged in $P(2, 2)$ ways

$= 2! = 2 \times 1 = 2$ ways.

Hence required no. of permutations

$= 2 \times 2 =$ **4.**

13. There are 10 lamps in a hall. Each one of them can be switched on independently. The number of ways in which the hall can be illuminated is

(a) 10^2 (b) 18

(c) 2^{10} (d) 1023

Solution (d) Each bulb has two choices, either switched on or off

∴ Required number of ways

$= 2^{10} - 1 = 1024 - 1 =$ **1023**

[Since in one way when all are switched off, the hall will not be illuminated].

14. The number of ways in which a committee of 3 ladies and 4 gentlemen can be appointed from a meeting consisting of 8 ladies and 7 gentlemen, if Mrs *X* refuses to serve in a committee if Mr. *Y* is a member is

(a) 1960 (b) 1540

(c) 3240 (d) none of these

Solution (b) 3 ladies out of 8 can be selected in ${}^8C_3$ ways and 4 gentlemen out of 7 in ${}^7C_4$ ways.

Now each way of selecting 3 ladies is associated with each way of selecting 4 gentlemen.

Hence the required number of ways

$= {}^8C_3 \times {}^7C_4 = 56 \times 35 = 1960.$

We now find the no. of committees of 3 ladies and 4 gentlemen in which both Mrs. X and Mr. Y are members. In this case, we can select 2 other ladies from the remaining 7 in ${}^7C_2$ ways and 3 other gentlemen from the remaining 6 in ${}^6C_3$ ways.

∴ The no. of ways in which both Mrs. *X* and Mr. *Y* are always included $= {}^7C_2 \times {}^8C_3 = 21 \times 20 = 420.$

Hence the required no. of committes in which Mrs. *X* and Mr. *Y* do not serve together $= 1960 - 420 =$ **1540.**

15. A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. The total number of signals that can be made is

(a) 473 (b) 1023

(c) 1173 (d) none of these

Solution (b) Each arm can be set in 4 ways.

∴ Five arms can be set in $4 \times 4 \times 4 \times 4 \times 4$ ways.

But this includes the way when all the arms are in the position of rest, when no signal is sent.

Hence required number of signals

$= 4^5 - 1 = 1024 - 1 =$ **1023.**

16. The number of ways in which the letters of the word BALLOON can be arranged so that two L's do not come together, is

(a) 700 (b) 800

(c) 900 (d) none of these

Solution (c) There are in all seven letters in the word BALLOON in which *L* occurs 2 times and *O* occurs 2 times.

∴ The number of arrangements of the seven letters of the word $= \dfrac{7!}{2! \times 2!} = 1260.$

If two L's always come together, taking them as one letter, we have to arrange 6 letters in which *O* occurs 2 times.

∴ The no. of arrangements in which the two L's come together

$$= \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360.$$

Hence the required no. of ways in which the two L's do not come together $= 1260 - 360 =$ **900.**

17. The number of divisors of 9600 including 1 and 9600 are

(a) 60 (b) 58

(c) 48 (d) 46

Solution (c) $\because 9600 = 2^7 \times 3 \times 5^2$

∴ No. of divisors $= (7 + 1) \times (1 + 1) \times (2 + 1)$

$= 8 \times 2 \times 3 =$ **48.**

18. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

(a) 6 (b) 18

(c) 12 (d) 9

Solution (b) Required number of parallelograms

$$= {}^4C_2 \times {}^3C_2 = \frac{4!}{2!2!} \times \frac{3!}{2!1!} = \frac{4 \times 3}{2 \times 1} \times \frac{3}{1} = \mathbf{18.}$$

19. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups each containing 6 questions. He is not permitted to attempt more than 5 questions from each group. The number of ways in which he can choose the 7 questions is

(a) 780 (b) 640

(c) 820 (d) none of these

Solution (a) A candidate can attempt 5 questions from group 1 and 2 from group II or 4 from group 1 and 3 from group II or 3 from group I and 4 from group III or 2 from group I and 5 from group II. This can be done in

${}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5.$

$= 6 \times 15 + 15 \times 20 + 20 \times 15 + 15 \times 6$

$= 90 + 300 + 300 + 90 =$ **780.**

20. The number of words that can be formed from the letters *a, b, c, d, e, f,* taken 3 at a time, each word containing at least one vowel is

(a) 96 (b) 84

(c) 106 (d) none of these

Solution (a) The total number of words

$= ({}^2C_1 \times {}^4C_2 + {}^2C_2 \times {}^4C_1)\, 3!$

$= (12 + 4) \times 6 =$ **96.**

21. If each of 10 points on a straight line be joined to each of 10 points on a parallel line then the total number of triangles that can be formed with the given points as vertices, is

(a) 860 (b) 900

(c) 920 (d) none of these

Solution (b) A triangle is formed for each selection of 2 points from one line and 1 point from the other line.

∴ The number of triangles

$$= {}^{10}C_2 \times {}^{10}C_1 + {}^{10}C_1 \times {}^{10}C_2$$

$$= \frac{10\times9}{2}\times10+10\times\frac{10\times9}{2} = \mathbf{900.}$$

22. In an examination a candidate has to pass in each of the papers to be successful. It the total number of ways to fail is 63, how many papers are there in the examination?

(a) 6 (b) 8

(c) 14 (d) none of these

Solution (a) Let the number of papers be n.

∴ Total number of ways to fail or pass

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n.$$

But there is only one way to pass, i.e., when he fails in none.

∴ Total number of ways to fail = $2^n - 1$.

∴ From question, $2^n - 1 = 63$; ∴ $2^n = 64 = 2^6$

∴ $n = \mathbf{6.}$

23. The total number of selections from 4 boys and 3 girls if each selection has to contain at least one boy is

(a) 106 (b) 120

(c) 240 (d) none of these

Solution (b) Number of selections of at least one boy from 4 boys

$$= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 - 1.$$

Number of selections of any number of girls from 3 girls

$$= {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3.$$

∴ Required number of selections of at least one boy from 4 boys and 3 girls = $(2^4 - 1)2^3 = 15 \times 8 = \mathbf{120.}$

24. The number of ways in which a mixed doubles game in tennis can be arranged from 5 married couples, if no husband and wife play in the same game, is

(a) 46 (b) 54

(c) 60 (d) none of these

Solution (c) Let the sides of the game be A and B. Given 5 married couples, i.e., 5 husbands and 5 wives. Now, 2 husbands for two sides A and B can be selected out of $5 = {}^5C_2 = 10$ ways.

After choosing the two husbands their wives are to be excluded (since no husband and wife play in the same game). So we are to choose 2 wives out of remaining $5 - 2 = 3$ wives, i.e., ${}^3C_2 = 3$ ways.

Again two wives can interchange their sides A and B in $2! = 2$ ways.

By the principle of multiplication, the required number of ways = $10 \times 3 \times 2 = \mathbf{60.}$

25. The number of seven letter words that can be formed by using the letters of the word SUCCESS so that the two C are together but no two S are together is

(a) 24 (b) 36

(c) 54 (d) none of these

Solution (a) Considering CC as single object, U, CC, E can be arranged in 3! ways

$$\times\ U \times CC \times E\ \times$$

Now the three S are to be placed in the four available places.

Hence required no. of ways = $3! \cdot {}^4C_3 = \mathbf{24.}$

26. If 7 points out of 12 are in the same straight line, then the number of triangles formed is

(a) 19 (b) 185

(c) 201 (d) none of these

Solution (b) Number of triangles formed = ${}^{12}C_3 - {}^7C_3$

$$= \frac{12.11.10}{1.2.3} - \frac{7.6.5}{1.2.3} = 220 - 35 = \mathbf{185.}$$

27. In a class tournament where the participants were to play one game with another, two class players fell ill, having played 3 games each. If the total number of games played is 84, the number of participants at the beginning was

(a) 22 (b) 15

(c) 17 (d) none of these

Solution (b) Suppose the two players did not play at all so that the remaining $(n - 2)$ players played ${}^{n-2}C_2$ matches. Since these two players played 3 matches each, hence the total number of matches is

$${}^{n-1}C_2 + 3 + 3 = 84 \text{ (given)}$$

or $\dfrac{(n-2)(n-3)}{1.2} = 78$ or $n^2 - 5n + 6 = 156$

or $n^2 - 5n - 150 = 0$ or $(n - 15)(n + 10) = 0$

∴ $n = \mathbf{15}$ $(n \neq -10)$.

28. In an examination a minimum is to be secured in each of 5 subjects for a pass. The number of ways in which a student can fail is

(a) 31

(b) 36

(c) 42

(d) none of these

Solution (a) The student will fail if he fails in one or more subjects.

Now the student can fail in one or more subjects out of 5 subjects in

$${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= {}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 - {}^5C_0$$

$$= 2^5 - 1 = 31 \text{ ways.}$$

∴ Required number = **31.**

29. Out of 18 points in a plane no three are in the same straight line except five points which are collinear. The number of straight lines that can be formed joining them is

(a) 143 (b) 144

(c) 153 (d) none of these

Solution (b) The number of st. lines

$$= {}^{18}C_2 - ({}^5C_2 - 1) = \mathbf{144.}$$

30. A polygon has 44 diagonals. The number of its sides are

(a) 13 (b) 12

(c) 11 (d) 10

Solution (c) Let n be the number of sides.

∴ Number of diagonals = ${}^nC_2 - n = 44$ (Given)

$$\Rightarrow \frac{n(n-1)}{2} - n = 44 \Rightarrow n^2 - n - 2n = 88$$

$\Rightarrow \quad n^2 - 3n = 88 \Rightarrow (n - 11)(n + 8) = 0$

$\Rightarrow \quad n = \mathbf{11} \qquad (\because n \neq -8)$

31. In a steamer there are stalls for 12 animals and there are cows, horses and calves (not less than 12 of each) ready to be shipped. The total number of ways in which the ship load can be made is

(a) $^{12}C_3$ (b) $^{12}P_3$
(c) 3^{12} (d) 12^3

Solution (c) Each stall can be occupied by a cow, a horse or a calf i.e., in 3 ways.

∴ The required number of ways = $\mathbf{3^{12}}$.

32. Given five line segments of length 2, 3, 4, 5, 6 units. Then the number of triangles that can be formed by joining these lines is

(a) $^5C_3 - 3$ (b) $^5C_3 - 1$
(c) $^5C_3$ (d) $^5C_3 - 2$

Solution (a) We know that in any trinagle the sum of two sides is always greater than the third side.

∴ The triangle will not be formed if we select segments of lengths (2, 3, 5), (2, 3, 6) or (2, 4, 6).

Hence no. of triangles formed = $\mathbf{^5C_3 - 3}$.

33. The number of words which can be made out of the letters of the word "MOBILE" when consonants always occupy odd places, is

(a) 20 (b) 36
(c) 30 (d) 720

Solution (b) There are three odd places in the words formed by the letters of MOBILE. At these 3 places 3 consonants can be placed in 3! ways and the remaining 3 letters at the remaining three places can be placed in 3! ways.

∴ Number of words formed = 3! 3! = **36.**

34. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n–digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is

(a) 6 (b) 7
(c) 8 (d) 9

Solution (b) Distinct n numbers which can be formed using digits 2, 5 and 7 are 3^n. We have to find n so that

$$3^n \geq 900 \Rightarrow 3^{n-2} \geq 100 \Rightarrow n - 2 \geq 5 \Rightarrow n \geq 7.$$

So, the least value of n is **7.**

35. There are 5 roads leading to a town from a village. The number of different ways in which a villager can go to the town and return back, is

(a) 25 (b) 20
(c) 10 (d) 5

Solution (a) No. of ways to go to a town = 5.

No. of ways to return back = 5.

Total no. of ways in which villager can go to town & return back are = 5 × 5 = **25.**

36. The least positive integral value of x which satisfies the inequality $^{10}C_{x-1} > 2.\ ^{10}C_x$ is

(a) 7 (b) 8
(c) 9 (d) 10

Solution (b) $10 \geq x - 1 \Rightarrow x \leq 11$ and $10 \geq x$

$\therefore \quad x \leq 10$

$$\because \quad ^{10}C_{x-1} > 2 \cdot {}^{10}C_x \quad \Rightarrow 1 > 2 \cdot \frac{^{10}C_x}{^{10}C_{x-1}}$$

$$\Rightarrow 1 > 2 \cdot \frac{10 - x + 1}{x}$$

$$\Rightarrow x > 22 - 2x\ ; \Rightarrow x > \frac{22}{3}$$

$$\Rightarrow x > 7\frac{1}{3} \quad \therefore x = \mathbf{8}.$$

37. There are 10 points in a plane of which no three points are collinear and 4 points are concyclic. The number of different circles that can be drawn through at least 3 of these points is

(a) 116 (b) 120
(c) 117 (d) none of these

Solution (c) The required no. of circles

$$= (^{10}C_3 - {}^4C_3) + 1 = \mathbf{117}.$$

38. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. The number of choices available to him is

(a) 140 (b) 196
(c) 280 (d) 346

Solution (b) The number of choices available to him is

$$= {}^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5$$

$$= \frac{5!}{4!1!} \times \frac{8!}{6! \times 2!} + \frac{5!}{5!0!} \times \frac{8!}{5!3!}$$

$$= 5 \times \frac{8 \times 7}{2} + 1 \times \frac{8 \times 7 \times 6}{3 \times 2}$$

$$= 5 \times 4 \times 7 + 8 \times 7 = 140 + 56 = 196$$

39. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(a) 6! ×5! (b) 30
(c) 5! × 4! (d) 7! × 5!

Solution (a) The number of ways to sit men = 5! and the number of ways to sit women = $^6C_5 \times 5!$

Total number of ways = $5! \times {}^6C_5 \times 5!$

$= 5! \times 6 \times 5! = 6!\ 5!$

40. If ^nC_r denotes the number of combinations of n things taken r at a time, then the expression

$$^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r \text{ equals}$$

(a) ^{n+2}C (b) $^{n+2}C_{r+1}$
(c) $^{n+1}C_r$ (d) $^{n+1}C_{r+1}$

Solution (b) $^nC_{r+1} + {}^nC_{r-1} + 2.{}^nC_r$

$$= {}^nC_{r+1} + {}^nC_r + {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_{r+1} + {}^{n+1}C_r$$

$$= {}^{n+2}C_{r+1}.$$

Problems for Practice

Level of Difficulty I

1. The number of positive integers satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \leq 100$ is
(a) nine (b) eight (c) five (d) none of these

2. ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} =$
(a) ${}^{n+1}P_r$ (b) ${}^{n}P_r$
(c) $r \cdot {}^{n}P_r$ (d) none of these

3. If ${}^{n}P_r = {}^{n}P_{r+1}$ and ${}^{n}C_r = {}^{n}C_{r-1}$, then the values of n and r are
(a) 4, 3 (b) 3, 2 (c) 4, 2 (d) none of these

4. The positive integer r, such that ${}^{15}C_{3r} = {}^{15}C_{r+3}$, is equal to
(a) 3 (b) 4 (c) 5 (d) none of these

5. The number of ways in which 6 different beads can be string into a necklace is
(a) 60 (b) 48 (c) 72 (d) none of these

6. The number of ways in which n books can be arranged on a shelf so that two particular books shall not be together is
(a) $(n - 2)\,(n - 1)!$ (b) $(n - 1)\,n!$
(c) $(n - 2)\,n!$ (d) none of these

7. The number of ways in which the letters of the word "STRANGE" can be arranged so that the vowels may appear in the odd places, is
(a) 1440 (b) 1470 (c) 1370 (d) none of these

8. If there are 12 persons in a party, and if each of them shakes hands with each other, then number of handshakes happen in the party is
(a) 66 (b) 48 (c) 72 (d) none of these

9. The number of ways in which 7 people can be arranged at a round table so that 2 particular persons may be together, is
(a) 132 (b) 148 (c) 240 (d) none of these

10. In an examination there are three multiple choice questions and each question has 4 choices. Number of ways in which a student can fail to get all answers correct is
(a) 11 (b) 12 (c) 27 (d) 63

11. The number of diagonals that can be drawn by joining the vertices of an octagon is
(a) 28 (b) 48 (c) 20 (d) none of these

12. The number of ways in which a team of eleven players can be selected from 22 players including 2 of them and excluding 4 of them is
(a) ${}^{16}C_{11}$ (b) ${}^{16}C_{5}$ (c) ${}^{16}C_{9}$ (d) ${}^{20}C_{9}$

13. The number of diagonals in a polygon of n sides is
(a) $\dfrac{n(n-3)}{2}$ (b) $\dfrac{n\,(n-1)}{2}$
(c) $\dfrac{(n-1)\,(n-2)}{2}$ (d) none of these

14. A boat is to be manned by eight men of whom 2 can only row on bow side and 1 can only row on stroke side; the number of ways in which the crew can be arranged is
(a) 4360 (b) 5760 (c) 5930 (d) none of these

15. The number of ways in which 10 boys and 10 girls can be seated in a row so that boys and girls alternate is
(a) 3628800 (b) 3628730
(c) 3628860 (d) none of these

16. The number of lines drawn through 6 points lying on a circle is
(a) 12 (b) 15 (c) 24 (d) 30

17. The maximum number of points of intersection of 8 straight lines is
(a) 56 (b) 28 (c) 16 (d) 8

18. ${}^{n-2}C_r + 2.\, {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$ equals
(a) ${}^{n+1}C_r$ (b) ${}^{n}C_r$
(c) ${}^{n}C_{r+1}$ (d) ${}^{n-1}C_r$

19. How many ways are there to arrange the letters in the word GARDEN with the vowels in alphabetical order?
(a) 360 (b) 240 (c) 120 (d) 480

20. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is
(a) 38 (b) 21 (c) 5 (d) ${}^{8}C_3$

Level of Difficulty II

21. The number of ways in which 10 examination papers can be arranged so that the best and the worst papers never come together, is
(a) 9.9! (b) 8.9! (c) 4.9! (d) none of these

22. Four boys picked up 30 mangoes. The number of ways in which they can divide them if all mangoes be identical, is
(a) 5456 (b) 3456 (c) 5462 (d) none of these

23. A parallelogiam is cut by two sets of m lines parallel to its sides. The number of parallelogram then formed is
(a) $({}^{m}C_2)^2$ (b) $({}^{m+1}C_2)^2$
(c) $({}^{m+2}C_2)^2$ (d) none of these

24. The least positive integral value of n which satisfies the inequality ${}^{10}C_{n-1} > 2 \cdot {}^{10}C_n$ is
(a) 7 (b) 8 (c) 9 (d) 10

25. The maximum number of points into which 4 circles and 4 straight lines intersect is
(a) 26 (b) 50 (c) 56 (d) 72

26. There are n seats round a table numbered 1, 2, 3, ...n. The number of ways in which $m(\leq n)$ persons can take seats is
(a) ${}^{n}P_m$ (b) ${}^{n}C_m \times (m - 1)!$
(c) $\dfrac{1}{2}.{}^{n}P_m$ (d) ${}^{n-1}P_m$

27. The value of

$$P\ (n,\ 1) + \frac{P(n,2)}{2!} + \frac{P(n,3)}{3!} + \ldots + \frac{P(n,n)}{n!}$$

is equal to
(a) $2^n + 1$ (b) 2^{n-1}
(c) $2^n - 1$ (d) 2^n

28. On a railway route there are 15 stations. The number of tickets required in order that it may be possible to book a passenger from every station to every other is

(a) $\frac{15!}{13!\,2!}$ (b) $\frac{15!}{13!}$

(c) 15! (d) $\frac{15!}{2!}$

29. The number of straight lines that can be drawn out of 10 points of which 7 are collinear

(a) 23 (b) 21 (c) 25 (d) 24

30. In a cricket championship there are 36 matches. The number of teams, if each plays one match with other are

(a) 9 (b) 10 (c) 8 (d) none of these

31. A five digit number divisible by 3 is to be formed using the numerals 0, 1, 3, 4 and 5 without repetition. The total number of ways this can be done is

(a) 216 (b) 600 (c) 240 (d) 3125

32. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is

(a) 26 (b) 35

(c) 34 (d) none of these

33. Number of ways in which the letters of the word ARRANGE can be arranged such that both R do not come together is

(a) 360 (b) 900 (c) 1260 (d) 1620

34. Two straight lines intersect at a point O. Points $A_1, A_2, ..., A_n$ are taken on one line and points $B_1, B_2, ..., B_n$ on the other. If the point O is not to be used, the number of triangles that can be drawn using these points as vertices, is

(a) $n(n-1)$ (b) $n(n-1)^2$

(c) $n^2(n-1)$ (d) $n^2(n-1)^2$

35. How many numbers greater than 24000 can be formed by using digits 1, 2, 3, 4, 5 when no digit is repeated, is

(a) 36 (b) 60 (c) 84 (d) 120

36. Out of 6 and 4 girls, a group of 7 is to be formed. In how many ways can this be done, if the group is to have a majority of boys?

(a) 120 (b) 90 (c) 100 (d) 180

37. The number of ways in which the six faces of a cube be painted with six different colours is

(a) 6 (b) 6! (c) ${}^6C_2$ (d) none of these

38. The number of numbers of 9 different nonzero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle is

(a) 2 (4!) (b) $(4!)^2$

(c) 8! (d) none of these

39. In a certain test, these are n questions. In this test 2^{n-i} students gave wrong answers to at least i questions, where $i = 1, 2, 3, ..., n$. If the total number of wrong answers given is 2047, then n is equal to

(a) 10 (b) 11 (c) 12 (d) 13

ANSWERS

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (d)
11. (c)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (b)	**19.** (a)	**20.** (b)
21. (b)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (d)	**30.** (a)
31. (a)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (c)	**37.** (d)	**38.** (b)	**39.** (b)	

Binomial Theorem 22

BRIEF REVIEW OF THE CONCEPTS

BINOMIAL EXPRESSION

An algebraic expression consisting of only two terms is called a binomial expression. For example, expressions such as:

$$x + a, \quad 4x + 3y, \quad 2x - \frac{4}{y}$$

are all binomial expressions.

BINOMIAL THEOREM

This theorem gives a formula by which any power of a binomial expression can be expanded. It was first given by Sir Isaac Newton.

Binomial Theorem for Positive Integral Index

If x and y are real numbers, then for all $n \in N$,

$$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n\, x^0 y^n \quad \ldots(1)$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **binomial coefficients**.

Special Cases

1. Replacing y by $-y$, in (1) we get
$$(x - y)^n = {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 \ldots + (-1)^n\, {}^nC_n\, x^0 y^n \quad \ldots(2)$$
2. Replacing x by 1 and y by x, we get
$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n$$

Notes:

- The coefficient of (r + 1)th term in the expansion of (1 + x)n is nCr.
- The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

3. Replacing x by 1 and y by $-x$, we get
$$(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - \ldots + (-1)^n\, {}^nC_n\, x^n.$$
4. Adding (1) and (2), we get
$$(x + y)^n + (x - y)^n = 2\,[x^n + {}^nC_2\, x^{n-2} y^2 + {}^nC_4\, x^{n-4} y^4 + \ldots]$$
$$= 2 \text{ (sum of terms at odd places).}$$
The last term is ${}^nC_n\, y^n$ or ${}^nC_{n-1}\, xy^{n-1}$ according as n is even or odd respectively.
5. Subtracting (2) from (1), we get
$$(x + y)^n - (x - y)^n = 2\,[{}^nC_1\, x^{n-1} y + {}^nC_3\, x^{n-3} y^3 + \ldots]$$
$$= 2 \text{ (sum of terms at even places)}$$
The last term is ${}^nC_{n-1}\, xy^{n-1}$ or ${}^nC_n\, y^n$ according as n is even or odd respectively.

Notes:

- The positive integer n is called the index of the binomial.
- Number of terms in the expansion of $(x + y)^n$ is $n + 1$ i.e., one more than the index n.
- In the expansion of $(x + y)^n$, the power of x goes on decreasing by 1 and that of y goes on increasing by 1 so that the sum of powers of x and y in any term is n.
- The binomial coefficients of the terms equidistant from the beginning and the end are equal.
- Students are advised to remember the following values:

$${}^nC_0 = 1, {}^nC_1 = n, {}^nC_2 = \frac{n(n-1)}{2!},$$

$${}^nC_3 = \frac{n(n-1)(n-2)}{3!},$$

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \text{ and } {}^nC_n = 1.$$

Also, ${}^nC_r = {}^nC_{n-r}$ for $1 \le r \le n$.

Pascal's Triangle

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ or simply $C_0, C_1, C_2 \ldots, C_n$ are called *binomial coefficients* and they can be evaluated with the help of *Pascal's triangle* as below:

Exponent of Binomial	Coefficients of successive terms in Binomial Theorem
$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1

I. Each row starts and ends in 1.
II. The coefficients of an expansion are obtained from the coefficients of the previous expansion.
III. Each coeffcient is the sum of the two coefficients which lie on either side of it in the previous row.

General Term in the Expansion of $(x + y)^n$

In the binomial expansion of $(x + y)^n$, the $(r + 1)$th term from the beginning is usually called the *general term* and it is denoted by T_{r+1}, i.e.,

$$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$$

Greatest Term (Numerically) in the Expansion of $(1 + x)^n$

Method 1

1. Let T_r (the r^{th} term) be the greatest term.
2. Find T_{r-1}, T_r, T_{r+1} from the given expansion.
3. Put $\dfrac{T_r}{T_{r+1}} \geq 1$ and $\dfrac{T_r}{T_{r-1}} \geq 1$. This will give an inequality from where value or values of r can be obtained.
4. Then, find the rth term T_r which is the greatest term.

Method 2

1. Find the value of $k = \dfrac{(n+1)|x|}{1+|x|}$
2. If k is an integer, then T_k and T_{k+1} are equal and both are greatest term.
3. If k is not an integer, then $T_{[k]+1}$ is the greatest term, where $[k]$ is the greatest integral part of k.

Note: To find the greatest term in the expansion of $(x + y)^n$, write $(x + y)^n = x_n \left(1+\dfrac{y}{x}\right)^n$ and then find the greatest term in $\left(1+\dfrac{y}{x}\right)^n$.

MIDDLE TERM IN THE BINOMIAL EXPANSION

The middle term in the binomial expansion of $(x + y)^n$ depends upon the value of n.

1. If n is even, then there is only one middle term i.e., $\left(\dfrac{n}{2}+1\right)$th term.
2. If n is odd, then there are two middle terms i.e., $\left(\dfrac{n+1}{2}\right)$th and $\left(\dfrac{n+3}{2}\right)$ th terms.

*p*th Term from the End in the Binomial Expansion of $(x + y)^n$

pth term from the end in the expansion of $(x + y)^n$ is $(n - p + 2)$th term from the beginning.

Properties of Binomial Coefficients

In the binomial expansion of $(1 + x)^n$, the coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_n$ are denoted by C_0, C_1, C_2, ... C_n respectively.

1. If n is even, then greatest coefficient = ${}^nC_{n/2}$
2. If n is odd, then greatest coefficient is ${}^nC_{(n-1)/2}$ or ${}^nC_{(n+1)/2}$.
3. $C_0 + C_1 + C_2 + ... + C_n = 2^n$
4. $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$
5. $C_0 - C_1 + C_2 - C_3 + C_4 - ... + (-1)^n C_n = 0$
6. $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \dfrac{(2n)!}{(n!)^2} = {}^{2n}C_n$
7. $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \begin{cases} 0, \text{ if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nc_{n/2}, \text{ if } n \text{ is even} \end{cases}$
8. $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1}\, C_n = {}^{2n}C_{n-1}$
9. $C_0C_r + C_1C_{r+1} + ... C_{n-r}C_n = {}^{2n}C_{n-r}$ or ${}^{2n}C_{n+r}$

Properties of ${}^nC_r$

If $0 < r < n$, n, $r \in N$, then

1. $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$
2. $\dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$
3. ${}^nC_r = \dfrac{n}{r}\, {}^{n-1}C_{r-1}$
4. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
5. $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$
6. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
7. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$
8. ${}^nC_r = {}^nC_{n-r}$.

Key Points to Remember

1. The number of terms in the expansion of $(x + y + z)^n$, where n is a positive integer, is $1/2(n + 1)(n + 2)$.
2. In the expansion of $(x_1 + x_2 + ... + x_k)^n$, the sum of all the coefficients is obtained by putting all the variables x_i equal to 1 and it is equal to k^n.

BINOMIAL THEOREM FOR ANY INDEX

If n is a rational number and x is real number such that $|x| < 1$, then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$
$$+ \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r + ... \text{ to } \infty \quad ...(1)$$

Observations

1. In the above expansion, the first term must be unity. In the expansion of $(a + x)^n$, where n is either a negative integer or a fraction, we proceed as follows:

$$(a + x)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(1+\frac{x}{a}\right)^n$$
$$= a^n\left[1+n\cdot\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+...\right]$$

and the expansion is valid when $\left|\dfrac{x}{a}\right| < 1$ i.e., $|x| < |a|$.

2. There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.
3. If x is so small that its square and higher powers may be neglected, then approximate value of $(1 + x)^n = 1 + nx$.

General Term in the Expansion of $(1 + x)^n$

The $(r + 1)^{th}$ term in the expansion of $(1 + x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} x^r$$

Some Important Deductions from $(1 + x)^n$

1. Replacing n by $-n$ in (1), we get

$$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3$$
$$+ ... + (-1)^r \frac{n(n+1)(n+2)...(n+r-1)}{r!} x^r + ...$$
$$T_{r+1} = (-1)^r \frac{n(n+1)(n+2)...(n+r-1)}{r!} x^r$$

(General term).

2. Replacing x by $-x$ in (1), we get

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3$$
$$+ ... + (-1)^r \frac{n(n-1)(n-2)...(n-r+1)}{r!} x^r + ...$$
$$T_{r+1} = (-1)^r \frac{n(n-1)(n-2)...(n-r+1)}{r!} x^r$$

(General term).

3. Replacing x by $-x$ and n by $-n$ in (1), we get

$$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3$$
$$+ ... + \frac{n(n+1)(n+2)...(n+r-1)}{r!} x^r + ...$$
$$T_{r+1} = \frac{n(n+1)(n+2)...(n+r-1)}{r!} x^r \text{ (General term).}$$

Some Useful Expansions

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$

2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$

3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1) x^r + ...$

4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1) x^r + ...$

5. $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r \dfrac{(r+2)(r+1)}{2} x^r + ...$

6. $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \dfrac{(r+1)(r+2)}{2} x^r + ...$

Solved Problems

1. The largest coefficient in the expansion of $(1 + x)^{24}$ is

(a) ${}^{24}C_{13}$ (b) ${}^{24}C_{11}$

(c) ${}^{24}C_{24}$ (d) ${}^{24}C_{12}$

Solution (d) We know that if n is even, then ${}^nC_r$ is greatest for

$$r = \frac{n}{2}.$$

$\therefore$ ${}^{24}C_r$ is greatest for $r = \mathbf{12}$.

2. The sum of coefficients in the expansion of $(1 - x)^{10}$ is

(a) 1024 (b) 0

(c) 1 (d) 10^2

Solution (b) We have,

$$(1 - x)^{10} = {}^{10}C_0 + {}^{10}C_1 (-x) + {}^{10}C_2 (-x)^2 + {}^{10}C_3 (-x)^3 + ... + {}^{10}C_{10} (-x)^{10}.$$

$\therefore$ Sum of the coefficients

$$= {}^{10}C_0 - {}^{10}C_1 + {}^{10}C_2 - {}^{10}C_3 + ... + {}^{10}C_{10} = \mathbf{0}.$$

[Putting $x = 1$ on both sides]

3. The sum of the coefficients in the expansion of

$$(1 + 5x - 7x^3)^{3165} \text{ is}$$

(a) 1 (b) 2^{3165}

(c) 2^{3164} (d) -1

Solution (d) Putting $x = 1$ in $(1 + 5x - 7x^3)^{3165}$, the required sum of coefficients $= (1 + 5 - 7)^{3165} = (-1)^{3165} = \mathbf{-1}$.

4. The 8th term of $\left(3x + \dfrac{2}{3x^2}\right)^{12}$, when expanded in ascending power of x, is

(a) $\dfrac{228096}{x^3}$ (b) $\dfrac{228096}{x^9}$

(c) $\dfrac{328179}{x^9}$ (d) none of these

Solution (a) When $\left(3x + \dfrac{2}{3x^2}\right)^{12}$ is expanded, the power of x goes on decreasing as the term proceed. Hence, it is expanded in descending powers of x. So $\left(3x + \dfrac{2}{3x^2}\right)^{12}$, when expanded, will be in ascending powers of x.

$$\text{Now, } t_8 \text{ in } \left(3x + \frac{2}{3x^2}\right)^{12} = {}^{12}C_7 \left(\frac{2}{3x^2}\right)^{12-7} \cdot (3x)^7$$
$$= \frac{12!}{7!\,5!} \cdot \left(\frac{2}{3x^2}\right)^5 \cdot (3x)^7$$
$$= \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} \cdot \frac{2^5 \cdot 3^2}{x^3}$$
$$= \mathbf{\frac{228096}{x^3}}.$$

5. If A is the sum of the odd terms and B the sum of even terms in the expansion of $(x + a)^n$, then $A^2 - B^2 =$

(a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$

(c) $2\,(x^2 - a^2)^n$ (d) none of these

Solution (b) We have,

$$\begin{aligned}(x+a)^n &= {}^nC_0\,x^n + {}^nC_1\,x^{n-1}\,a^1 + {}^nC_2\,x^{n-2}\,a^2 \\ &\qquad + {}^nC_3\,x^{n-3}a^3 + \ldots + {}^nC_n\,x^n \\ &= ({}^nC_0\,x^n + {}^nC_2\,x^{n-2}\,a^2 + \ldots) \\ &\qquad + ({}^nC_1\,x^{n-1}\,a^1 + {}^nC_3\,x^{n-3}\,a^3 + \ldots) \\ &= A + B\end{aligned}$$

$$\begin{aligned}(x-a)^n &= {}^nC_0\,x^n - {}^nC_1\,x^{n-1}\,a^1 + {}^nC_2\,x^{n-2}a^2 \\ &\qquad - {}^nC_3\,x^{n-3}\,a^3 + \ldots + {}^nC_n\,(-1)^n\,a^n \\ &= ({}^nC_0\,x^n + {}^nC_2\,x^{n-2}\,a^2 + \ldots) \\ &\qquad - ({}^nC_1\,x^{n-1}\,a^1 + {}^nC_3\,x^{n-3}\,a^3 + \ldots) \\ &= A - B\end{aligned}$$

$$\therefore \quad A^2 - B^2 = (A+B)(A-B) = (x+a)^n\,(x-a)^n = \mathbf{(x^2 - a^2)^n}.$$

6. The term independent of x in $(1+x)^m\left(1+\dfrac{1}{x}\right)^n$ is

(a) ${}^{m+n}C_m$ (b) ${}^{m+n}C_n$
(c) ${}^{m+n}C_{m-n}$ (d) none of these

Solution (b) We have,

$$(1+x)^m\left(1+\frac{1}{x}\right)^n = (1+x)^m\left(1+\frac{1}{x}\right)^n$$

$$= \frac{(1+x)^{m+n}}{x^n} = x^{-n}\,(1+x)^{m+n}$$

∴ Required term independent of x = coefficient of x^0 in

$x^{-n}\,(1+x)^{m+n}$ = coefficient of x^n in $(1+x)^{m+n}$

$= \mathbf{{}^{m+n}C_n}$.

7. The greatest term (numerically) in the expansion of $(2+3x)^9$, when $x = \dfrac{3}{2}$, is

(a) $\dfrac{5\times 3^{11}}{2}$ (b) $\dfrac{5\times 3^{13}}{2}$

(c) $\dfrac{7\times 3^{13}}{2}$ (d) none of these

Solution (c) We have,

$$(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9 = 2^9\left(1+\frac{9}{4}\right)^9 \qquad \left(\because x = \frac{3}{2}\right)$$

$$\therefore \quad m = \left|\frac{x(n+1)}{(x+1)}\right|$$

$$= \left|\frac{\left(\frac{9}{4}\right)(9+1)}{\left(\frac{9}{4}\right)+1}\right| = \frac{90}{13} = 6\,\frac{12}{13} \neq \text{Integer}$$

The greatest term in the expansion is

$$T_{[m]+1} = T_{6+1} = T_7$$

Hence the greatest term $= 2^9 \cdot T_7$

$$= 2^9 \cdot T_{6+1} = 2^9 \cdot {}^9C_6\left(\frac{9}{4}\right)^6$$

$$= 2^9 \cdot {}^9C_3\left(\frac{9}{4}\right)^6 = 2^9 \cdot \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{3^{12}}{2^{12}} = \mathbf{\frac{7\times 3^{13}}{2}}.$$

8. The greatest term (numerically) in the expansion of $(3-5x)^{11}$ when $x = \dfrac{1}{5}$ is

(a) 55×3^9 (b) 46×3^9
(c) 55×3^6 (d) none of these

Solution (a) We have,

$$(3-5x)^{11} = 3^{11}\left(1-\frac{5x}{3}\right)^{11} = 3^{11}\left(1-\frac{1}{3}\right)^{11} \qquad \left(\because x = \frac{1}{5}\right)$$

$$\therefore \quad m = \frac{|x|\,(n+1)}{(|x|+1)} \qquad \left(-\frac{1}{3} < 0\right)$$

$$= \frac{\left(\left|-\frac{1}{3}\right|\right)(11+1)}{\left(\left|-\frac{1}{3}\right|+1\right)} = 3$$

The greatest terms in the expansion are T_3 and T_4

∴ Greatest term (when $r = 2$) $= 3^{11}\,|T_{2+1}|$

$$= 3^{11}\left|{}^{11}C_2\left(-\frac{1}{3}\right)^2\right| = 3^{11}\left|\frac{11\cdot 10}{1\cdot 2}\times\frac{1}{9}\right| = 55\times 3^9$$

and greatest term (when $r = 3$) $= 3^{11}\,|T_{3+1}|$

$$= 3^{11}\left|{}^{11}C_3\left(-\frac{1}{3}\right)^3\right| = 3^{11}\left|\frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}\times -\frac{1}{27}\right| = \mathbf{55\times 3^9}$$

From above we see that the values of both greatest terms are equal.

9. The coefficient of x^{53} in the expansion

$$\sum_{m=0}^{100} {}^{100}C_m(x-3)^{100-m}\cdot 2^m \text{ is}$$

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$
(c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

Solution (c) We have, $\displaystyle\sum_{m=0}^{100} {}^{100}C_m(x-3)^{100-m}\cdot 2^m$

$$\begin{aligned}&= (x-3)^{100} + {}^{100}C_1\,(x-3)^{99}\cdot 2^1 \\ &\qquad + {}^{100}C_2\,(x-3)^{98}\cdot 2^2 + \ldots + {}^{100}C_{100}\,2^{100} \\ &= [(x-3)+2]^{100} = (x-1)^{100} = (1-x)^{100}\end{aligned}$$

∴ coefficient of $x^{54} = {}^{100}C_{53}\,(-1)^{53} = \mathbf{-\,{}^{100}C_{53}}$.

10. The coefficient of x^3 in the expansion of $(1-x+x^2)^6$ is

(a) 50 (b) – 50
(c) 68 (d) none of these

Solution (b) $(1-x+x^2)^6 = \{1-x\,(1-x)\}^6$

$$\begin{aligned}&= {}^6C_0 - {}^6C_1\,x\,(1-x) + {}^6C_2\,x^2\,(1-x)^2 - {}^6C_3\,x^3\,(1-x)^3 \\ &\qquad + \ldots \text{ to 7 terms} \\ &= {}^6C_0 - {}^6C_1\,x\,(1-x) + {}^6C_2\,x^2\,(1-2x+x^2) \\ &\qquad - {}^6C_3\,x^3\,(1-3x+3x^2-x^3) + \ldots \text{ to 7 terms}\end{aligned}$$

∴ Coefficient of $x^3 = -2\cdot{}^6C_2 - {}^6C_3$, (collecting coefficients of x^3 from each term)

$$= -2\cdot\mathbf{\frac{6!}{2!4!}} - \mathbf{\frac{6!}{3!3!}} = \mathbf{-\,50.}$$

11. The greatest coefficient in the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$ is

(a) $\dfrac{1\cdot 3\cdot 5\ldots(2n-1)\cdot 2^n}{n!}$ (b) $\dfrac{2n!}{(n!)^2}$

(c) $\dfrac{n!}{\left(\left(\frac{n}{2}\right)!\right)^2}$ (d) none of these

Solution (a), (b) Since the middle term has greatest coefficient,

$\therefore$ greatest coefficient = coefficient of the middle term

$$= {}^{2n}C_n = \frac{(2n)!}{n!n!}$$

$$= \frac{2n(2n-1)(2n-2)(2n-3)...4\cdot3\cdot2\cdot1}{n!n!}$$

$$= \frac{[(2n-1)(2n-3)...3\cdot1]\,[2n(2n-2)(2n-4)...4\cdot2]}{n!n!}$$

$$= \frac{[1\cdot3\cdot5...(2n-1)]2^n[n(n-1)(n-2)...2\cdot1]}{n!n!}$$

$$= \frac{1\cdot3\cdot5...(2n-1)2^n n!}{n!n!} = \frac{\mathbf{1\cdot3\cdot5...(2n-1)2^n}}{\mathbf{n!}}.$$

12. The value of x in the expression $(x+x^{\log_{10} x})^5$, if the third term in the expansion is 10,00,000, is

(a) 10^{-1} (b) 10^1

(c) $10^{-5/2}$ (d) $10^{5/2}$

Solution (b), (c) Put $\log_{10} x = z$

Then, given expression $= (x + x^z)^5$.

Now, $T_3 = {}^5C_2\, x^3\, (x^z)^2 = 10x^{3+2z} = 10^6$

$\therefore \quad x^{3+2z} = 10^5$.

Taking log, we get

$(3 + 2z) \log_{10} x = 5 \log_{10} 10$

$\Rightarrow \quad (3 + 2z)\, z = 5 \quad$ or $\quad 2z^2 + 3z - 5 = 0$

$\Rightarrow \quad (z - 1)(2z + 5) = 0 \Rightarrow z = 1, -\dfrac{5}{2}$

$\therefore \quad \log_{10} x = 1$ or $-\dfrac{5}{2}$ $\quad\therefore \quad x = \mathbf{10^1}$ or $\mathbf{10^{-5/2}}$.

13. If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then α is equal to

(a) 3 (b) 2

(c) 1 (d) -2

Solution (c) We have, sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$.

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

$\Rightarrow \quad (\alpha - 1)^{35} = (1 - \alpha)^{35}$ [Putting $x = y = 1$]

$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35} \Rightarrow 2(\alpha - 1)^{35} = 0$

$\Rightarrow \quad \alpha - 1 = 0. \quad \therefore \quad \alpha = \mathbf{1}$.

14. If $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + ... + a_{12}x^{12}$, then $a_2 + a_4 + a_6 + ... + a_{12} =$

(a) 21 (b) 11

(c) 31 (d) none of these

Solution (c) Given

$(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + ... + a_{12}x^{12}$

Putting $x = 1$, we get

$0 = 1 + a_1 + a_2 + ... + a_{12}$ ...(1)

Putting $x = -1$, we get

$64 = 1 - a_1 + a_2 - ... + a_{12}$ ...(2)

Adding (1) and (2), we get

$64 = 2(1 + a_2 + a_4 + ...)$

$\therefore \quad a_2 + a_4 + a_6 + ... + a_{12} = \mathbf{31}$.

15. If 7^{103} is divided by 25, then the remainder is

(a) 20 (b) 16

(c) 18 (d) 15

Solution (c) We have, $7^{103} = 7(49)^{51} = 7(50 - 1)^{51}$

$= 7(50^{51} - {}^{51}C_1\, 50^{50} + {}^{51}C_2\, 50^{49} - ... - 1)$

$= 7(50^{51} - {}^{51}C_1\, 50^{50} + {}^{51}C_2\, 50^{49} - ...) - 7 + 18 - 18$

$= 7(50^{51} - {}^{51}C_1\, 50^{50} + {}^{51}C_2\, 50^{49} - ...) - 25 + 18$

$= k + 18$ (say) $\quad \because k$ is divisible by 25,

$\therefore$ remainder is **18.**

16. The sum of rational terms in the expansion of $(\sqrt{2} + 3^{1/15})^{10}$ is

(a) 31 (b) 41

(c) 51 (d) none of these

Solution (b) $(r + 1)^{\text{th}}$ term in the given expansion is given by

$$t_{r+1} = 10C_r\, 2^{\frac{10-r}{2}}\, 3^{\frac{r}{5}}, \text{ where } r = 0, 1, 2, ..., 10$$

For rational terms

r = a multiple of 5 = 0, 5, 10 ...(1)

$10 - r$ = a multiple of 2 = 0, 2, 4, 6, 8, 10 ...(2)

From (1) and (2) possible values of r are : 0 and 10

$\therefore$ sum of rational terms

$= t_1 + t_{11} = {}^{10}C_0\, (\sqrt{2})^{10}\, (3^{1/5})^0 + {}^{10}C_{10}\, (\sqrt{2})^0\, (3^{1/5})^{10}$

$= 2^5 + 3^2 = 32 + 9 = \mathbf{41}$.

17. In the expansion of $(x + a)^n$ if the sum of odd terms be P and the sum of even terms be Q, then $4PQ =$

(a) $(x + a)^n - (x - a)^n$ (b) $(x + a)^n + (x - a)^n$

(c) $(x + a)^{2n} - (x - a)^{2n}$ (d) none of these

Solution (c) We have,

$$(x + a)^n = x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + {}^nC_3\, x^{n-3} a^3 + ...$$

$$= (x^n + {}^nC_2\, x^{n-2} a^2 + ...) + ({}^nC_1\, x^{n-1} a + {}^nC_3\, x^{n-3} a^3 + ...)$$

$$= P + Q$$

$\therefore (x - a)^n = P - Q$, as the terms are alternatively positive and negative.

$\therefore \quad 4PQ = (P + Q)^2 - (P - Q)^2 = \mathbf{(x + a)^{2n} - (x - a)^{2n}}$.

18. The value of is

$$\frac{(18^3 + 7^3 + 3\cdot18\cdot7\cdot25)}{3^6 + 6\cdot243\cdot2 + 15\cdot81\cdot4 + 20\cdot27\cdot8 + 15\cdot9\cdot16 + 6\cdot3\cdot32 + 64}$$

(a) 0 (b) 1

(c) 2 (d) none of these

Solution (b) The numerator is of the form

$a^3 + b^3 + 3ab(a + b) = (a + b)^3$.

where $\quad a = 18$ and $b = 7$

$\therefore$ Numerator $= (18 + 7)^3 = 25^3$.

For denominator, $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243$

${}^6C_1 = 6,\ {}^6C_2 = 15,\ {}^6C_3 = 20$.

${}^6C_4 = {}^6C_2 = 15,\ {}^6C_5 = {}^6C_1 = 6,\ {}^6C_6 = 1$

$\therefore$ denominator $= 3^6 + {}^6C_1\, 3^5 \cdot 2^1 + {}^6C_2\, 3^4 \cdot 2^2 + {}^6C_3\, 3^3 \cdot 2^3 + {}^6C_4\, 3^2 \cdot 2^4 + {}^6C_5\, 3 \cdot 2^5 + {}^6C_6\, 2^6$.

This is clearly the expansion of

$(3 + 2)^6 = 5^6 = (25)^3$

$$\therefore \quad \frac{\text{Numerator}}{\text{Denominator}} = \frac{(25)^3}{(25)^3} = \mathbf{1}.$$

19. If $C_0, C_1, C_2, \ldots, C_n$ are the coefficients of the expansion of $(1 + x)^n$, then the value of $\sum_0^n \frac{C_k}{k+1}$ is

(a) 0 (b) $\frac{2^n - 1}{n}$

(c) $\frac{2^{n+1} - 1}{n+1}$ (d) none of these

Solution (c) Here, $t_{r+1} = \frac{{}^nC_r}{r+1} = \frac{1}{r+1} \cdot {}^nC_r$

$$= \frac{1}{n+1} \cdot {}^{n+1}C_{r+1}$$

Putting $r = 0, 1, 2, \ldots n$ and adding we get, $\sum_0^n \frac{C_k}{k+1}$

$$= \frac{1}{n+1} \{{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1}\}$$

$$= \frac{1}{n+1} \{2^{n+1} - {}^{n+1}C_0\} = \boldsymbol{\frac{2^{n+1} - 1}{n+1}}.$$

20. Larger of $99^{50} + 100^{50}$ and 101^{50} is

(a) 101^{50} (b) $99^{50} + 100^{50}$

(c) both are equal (d) none of these

Solution (a) We have,

$$101^{50} = (100 + 1)^{50}$$

$$= 100^{50} + 50 \cdot 100^{49} + \frac{50 \cdot 49}{1 \cdot 2} \cdot 100^{48} + \ldots$$

and $99^{50} = (100 - 1)^{50}$

$$= 100^{50} - 50 \cdot 100^{49} + \frac{50 \cdot 49}{1 \cdot 2} \cdot 100^{48} - \ldots$$

Subtracting, we get

$$101^{50} - 99^{50} = 2\left[50 \cdot 100^{49} + \frac{50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3} \times 100^{47} + \ldots\right]$$

$$= 100^{50} + 2 \cdot \frac{50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3} \cdot 100^{47} + \ldots > 100^{50}$$

Hence $\mathbf{101^{50}} > 99^{50} + 100^{50}$.

21. For all $n \in N$, $2^{4n} - 15n - 1$ is divisible by

(a) 225 (b) 125

(c) 325 (d) none of these

Solution (a) We have, $2^{4n} = (2^4)^n = (16)^n = (1 + 15)^n$

$\therefore\ 2^{4n} = 1 + {}^nC_1 \cdot 15 + {}^nC_2\, 15^2 + {}^nC_3\, 15^3 + \ldots$

$\Rightarrow\ 2^{4n} - 1 - 15n = 15^2\,[{}^nC_2 + {}^nC_3 \cdot 15 + \ldots]$

$= 225\,K$, where K is an integer.

Hence $2^{4n} - 15n - 1$ is divisible by 225.

22. When 5^{99} is divided by 13, the remainder is

(a) 8 (b) 9

(c) 10 (d) none of these

Solution (a) We have,

$5^{99} = 5^3 \cdot 5^{96} = (125)\,(625)^{24}$

$= [13 \times 9 + 8]\,(1 + 48 \times 13)^{24}$

$= (13 \times 9 + 8)\,[1 + {}^{24}C_1 \times (48 \times 13)$

$+ {}^{24}C_2\,(48 \times 13)^2 + \ldots + (48 \times 13)^{24}]$

$= 8$ + terms containing powers of 13.

Hence remainder = **8.**

23. The last digit of the number $(32)^{32}$ is

(a) 4 (b) 6

(c) 8 (d) none of these

Solution (b) $(32)^{32} = (2 + 3 \times 10)^{32}$

$= 2^{32} + 10k$, where $k \in N$

Therefore, last digits in $(32)^{32}$ = last digit in $(2)^{32}$

But $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$

$\therefore\ 2^{32} = (2^5)^6 \cdot 2^2 = (32)^6 \cdot 4 = (2 + 30)^6 \cdot 4$

$= (2^6 + 10r)\,4,\ r \in N$

Last digit in 2^{32} = last digit in $(2)^6 \cdot 4$ = last digit in $4 \times 4 = \mathbf{6}$

$\therefore$ Last digit in $(32)^{32} = 6$.

24. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then

$$2C_0 + 2^2 \cdot \frac{C_1}{2} + 2^3 \cdot \frac{C_2}{3} + \ldots + 2^{n+1} \frac{C_n}{n+1} =$$

(a) $\frac{3^{n+1} - 1}{n+1}$ (b) $\frac{3^n - 1}{n}$

(c) $\frac{3^{n+2} - 1}{n+2}$ (d) none of these

Solution (a) We have,

$$t_{r+1} = 2^{r+1} \frac{{}^nC_r}{r+1} = 2^{r+1} \cdot \frac{1}{n+1} \cdot {}^{n+1}C_{r+1}$$

Putting $r = 0, 1, 2, \ldots, n$ and adding, we get the required sum

$$= \frac{1}{n+1} \{2 \cdot {}^{n+1}C_1 + 2^2 \cdot {}^{n+1}C_2 + \ldots + 2^{n+1} \cdot {}^{n+1}C_{n+1}\}$$

$$= \frac{1}{n+1} \{(1 + 2)^{n+1} - {}^{n+1}C_0\} = \boldsymbol{\frac{3^{n+1} - 1}{n+1}}.$$

25. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then for n even, $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$ is equal to

(a) 0 (b) $(-1)^{n/2}\,{}^nC_{n/2}$

(c) ${}^nC_{n/2}$ (d) none of these

Solution (b) We have,

$(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - \ldots + (-1)^n\, {}^nC_n\, x^n$,

and $(x + 1)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \ldots + {}^nC_n$.

The given series is the coefficient of x^n in the product of R.H.S. of the above two.

$\therefore$ Sum of the series = coefficient of x^n in $(1 - x)^n \cdot (x + 1)^n$

= coefficient of x^n in $(1 - x^2)^n$ = coefficient of x^n in

$[{}^nC_0 + {}^nC_1\,(-x^2) + {}^nC_2\,(-x^2)^2 + \ldots + {}^nC_n\,(-x^2)^n]$

Since n is even, let $n = 2m$. Then

sum = coefficient of x^{2m} in

$[{}^{2m}C_0 + {}^{2m}C_1\,(-x^2) + {}^{2m}C_2\,(-x^2)^2 + \ldots + {}^{2m}C_{2m}\,(-x^2)^{2m}]$

$= {}^{2m}C_m\,(-1)^m = \boldsymbol{{}^nC_{n/2}\,(-1)^{n/2}}.$

26. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then

$$\sum\sum_{0 \le i \le j \le n} (C_i + C_j) =$$

(a) $(n - 1) \cdot {}^{2n}C_n + 2^{2n}$

(b) $n \cdot {}^{2n}C_n + 2^{2n}$

(c) $(n + 1) \cdot {}^{2n}C_n + 2^{2n}$

(d) none of these

Solution (a) $\sum\sum_{0 \le i \le j \le n} (C_i + C_j)^2$ $\quad i = 0, 1, 2, \ldots, (n - 1)$

$j = 1, 2, 3, \ldots, n$

and $i < j$

$$= n\,(C_0^2 + C_1^2 + \ldots + C_n^2) + 2\,\Sigma\,\Sigma\, C_i\, C_j \quad 0 \le i < j \le n$$
$$= n \cdot {}^{2n}C_n + [(C_0 + C_1 + \ldots + C_n)^2 - (C_0^2 + C_1^2 + \ldots + C_n^2)]$$
$$= n \cdot {}^{2n}C_n + (2^n)^2 - {}^{2n}C_n = \mathbf{(n-1) \cdot {}^{2n}C_n + 2^{2n}}.$$

27. The greatest cofficient in the expansion of $(x + y + z + w)^{15}$ is

(a) $\frac{15!}{3!(4!)^3}$ (b) $\frac{15!}{(3!)^3 4!}$

(c) $\frac{15!}{2!(4!)^2}$ (d) none of these

Solution (a) The greatest coefficient is

$$= \frac{n!}{(q!)^{k-r}[(q+1)!]^r} \quad [\text{Here } n = 15, q = 3, r = 3, k = 4]$$

28. The coefficient of x^n in the expansion of $(1 - 2x + 3x^2 - 4x^3 + \ldots \text{ to } \infty)^{-n}$ is

(a) $\frac{(2n)!}{n!(n-1)!}$ (b) $\frac{(2n)!}{[(n-1)!]^2}$

(c) $\frac{(2n)!}{(n!)^2}$ (d) none of these

Solution (c) We have, $(1 - 2x + 3x^2 - 4x^3 + \ldots \text{ to } \infty)^{-n}$
$$= [(1 + x)^{-2}]^{-n} = (1 + x)^{2n}.$$

$\therefore$ Coefficient of $x^n = {}^{2n}C_n = \mathbf{\frac{(2n)!}{(n!)^2}}$.

29. The values of a and b so that the coefficient of x^n in the expansion of $\frac{a+bx}{(1-x)^2}$ may be $2n + 1$, are

(a) 1, 1 (b) 1, 2

(c) 2, 3 (d) none of these

Solution (a) We have, $\frac{a+bx}{(1-x)^2} = (a + bx)(1 - x)^{-2}$

$$= (a + bx)[1 + 2x + 3x^2 + \ldots + (n + 1)x^n + \ldots]$$

$\therefore$ Coefficient of x^n in $\frac{a+bx}{(1-x)^2} = a(n + 1) + bn$

But $a(n + 1) + bn = 2n + 1$. Equating coefficient of n and constant term, we get $a + b = 2$ and $a = 1$. So $a = 1, b = 1$

$\therefore$ When $a = 1$, $b = 1$, coefficient of $x^n = 2n + 1$.

30. If x is nearly equal to 1, then the approximate value of $\frac{mx^m - nx^n}{m - n}$ is

(a) x^m (b) x^n

(c) x^{m+n} (d) none of these

Solution (c) Since x is very nearly equal to 1, let $x = 1 + h$, where h is very nearly equal to zero, so that h^2 and higher powers of h may be neglected.

Now, $\frac{mx^m - nx^n}{m - n} = \frac{m(1+h)^m - n(1+h)^n}{m - n}$

$$= \frac{m(1+mh) - n(1+nh)}{m - n}$$

[Neglecting h^2 and higher powers]

$$= \frac{(m-n)[1+(m+n)h]}{m-n}$$
$$= 1 + (m + n)h = (1 + h)^{m+n}$$
$$= \mathbf{x^{m+n}}.$$

31. The number of non zero terms in the expansion of $(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$ is

(a) 9 (b) 0

(c) 5 (d) 10

Solution (c) In the expansion of

$$(1 + 3\sqrt{2}\,x)^9 + (1 - 3\sqrt{2}\,x)^9$$

2nd, 4th, 6th, 8th and 10th terms get cancelled.

$\therefore$ Number of non-zero terms in

$$2\,[{}^9C_0 + {}^9C_2\,(3\sqrt{2})^2 + \ldots + {}^9C_8\,(3\sqrt{2})^8] \text{ is } \mathbf{5}.$$

32. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(a) 5 (b) 6

(c) 7 (d) 8

Solution (c) $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$
$$= 2\,[{}^5C_0\,x^5 + {}^5C_2\,x^3(x^3 - 1) + {}^5C_4\,x(x^3 - 1)^2]$$
$$= 2\,[x^5 + 10x^3(x^3 - 1) + 5x(x^3 - 1)^2]$$
$$= 5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x$$

which is polynomial of degree **7**.

33. The value of x, for which the 6th term in the expansion of

$$\left[2^{\log_2\sqrt{(9^{x-1}+7)}} + \frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$$ is 84, is equal to

(a) 4 (b) 3

(c) 2 (d) 1

Solution (c), (d) The given expression

$$= \left[\sqrt{9^{x-1}+7} + \frac{1}{(3^{x-1}+1)^{1/5}}\right]^7$$

Given, $T_6 = 84$

$$\Rightarrow \quad {}^7C_5\,(\sqrt{9^{x-1}+7})^{7-5}\left(\frac{1}{(3^{x-1}+1)^{1/5}}\right)^5 = 84$$

$$\Rightarrow \quad {}^7C_5\,(9^{x-1} + 7) \cdot \frac{1}{(3^{x-1}+1)} = 84$$

$$\Rightarrow \quad 9^{x-1} + 7 = 4\,(3^{x-1} + 1)$$
$$\Rightarrow \quad 3^{2x} - 12 \cdot 3^x + 27 = 0$$
$$\Rightarrow \quad (3^x - 3)(3^x - 9) = 0$$
$$\Rightarrow \quad 3^x = 3, 9 \Rightarrow x = \mathbf{1, 2}.$$

34. If $\frac{1}{1-2x+x^2} = 1 + a_1 x + a_2 x^2 + \ldots$ then the value of a_r is

(a) $2r$ (b) $r + 1$

(c) r (d) $r - 1$

Solution (b) $\frac{1}{1-2x+x^2} = (1 - x)^{-2}$

$$= 1 + 2x + 3x^2 + \ldots = 1 + a_1\,x + a_2\,x^2 + \ldots$$
$$\Rightarrow \quad a_r = \mathbf{r + 1}.$$

35. If the second term in the expansion $\left(\sqrt[13]{a} + \frac{a}{\sqrt{a^{-1}}}\right)^n$ is $14\,a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is

(a) 8 (b) 12

(c) 4 (d) none of these

Solution (c) Given : $T_2 = 14a^{5/2}$

$$\Rightarrow \quad {}^nC_1\,(a^{1/13})^{n-1}\cdot\left(\frac{a}{a^{-1/2}}\right)^1 = 14a^{5/2}$$

$$\Rightarrow \quad n\cdot a^{(n-1)/13}\cdot a^{3/2} = 14a^{5/2}$$

$$\Rightarrow \quad n\cdot a^{(n-1)/13} = 14a \Rightarrow n\cdot a^{(n-14)/13} = 14$$

$$\Rightarrow \quad n = 14$$

$$\therefore \quad \frac{{}^nC_3}{{}^nC_2} = \frac{{}^{14}C_3}{{}^{14}C_2} = \mathbf{4.}$$

36. The number of irrational terms in the expansion of $(4^{1/5} + 7^{1/10})^{45}$ is

(a) 40 (b) 5

(c) 41 (d) none of these

Solution (c) Total number of terms in the expansion of $(4^{1/5} + 7^{1/10})^{45}$ is $45 + 1$ i.e., 46.

The general term in the expansion is

$$T_{r+1} = {}^{45}C_r\cdot 4^{\frac{45-r}{5}}\cdot 7^{\frac{r}{10}}$$

T_{r+1} is rational if $r = 0, 10, 20, 30, 40$.

$\therefore$ Number of rational terms = 5.

$\therefore$ Number of irrational terms = $46 - 5 = \mathbf{41}$.

37. The coefficient of x in the expansion of $[\sqrt{1+x^2} - x]^{-1}$ in ascending powers of x, when $|x| < 1$, is

(a) 0 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) 1

Solution (d) We have,

$$(\sqrt{1+x^2} - x)^{-1} = \frac{1}{(\sqrt{1+x^2}-x)}\times\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}$$

$$= \frac{(1+x^2)^{1/2}+x}{1+x^2-x^2} = (1+x^2)^{1/2} + x$$

$\therefore$ coefficient of x in the expansion of $(\sqrt{1+x^2} - x)^{-1}$

= coefficient of x in the exp. of $[(1 + x^2)^{1/2} + x]$

= **1.** ($\because$ coefficient of x in the exp. of $\sqrt{1+x^2}$ is 0)

38. The middle term in the expansion of $(1 + 3x + 3x^2 + x^3)^6$ is

(a) 4th (b) 3rd

(c) 10th (d) none of these

Solution (c) $(1 + 3x + 3x^2 + x^3)^6 = [(1 + x)^3]^6 = (1 + x)^{18}$.

$\therefore$ Middle term = $\left(\frac{18}{2}+1\right)$th term i.e., **10th** term.

39. If A and B are coefficients of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then

(a) $A = B$ (b) $2A = B$

(c) $A = 2B$ (d) none of these

Solution (c) Coefficient of x^n in $(1 + x)^{2n}$ is

$$A = {}^{2n}C_n \qquad \text{...(1)}$$

Coefficient of x^n in $(1 + x)^{2n-1}$ is

$$B = {}^{2n-1}C_n \qquad \text{...(2)}$$

From (1) and (2), we get

$$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{2\cdot{}^{2n-1}C_n}{{}^{2n-1}C_n} = 2 \quad \therefore\ \boldsymbol{A = 2B.}$$

40. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then the value of $C_1 - 2C_2 + 3C_3 + \ldots + (-1)^{n-1}nC_n$ is

(a) 1 (b) 0

(c) -1 (b) none of these

Solution (b) We have,

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = C_1 + 2xC_2 + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

Putting $x = -1$, we get

$$\mathbf{0} = C_1 - 2C_2 + 3C_3 - 4C_4 \ldots + (-1)^n nC_n.$$

41. If $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + a_3x^3 + \ldots$ then the value of $a_2 + a_4 + a_6 + \ldots + a_{12}$ will be

(a) 32 (b) 31

(c) 64 (d) 1024

Solution (b) We have, $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \ldots$

Putting $x = 1$, we get

$$0 = 1 + a_1 + a_2 + a_3 + \ldots + a_{12} \qquad \text{...(1)}$$

Putting $x = -1$, we get

$$64 = 1 - a_1 + a_2 - a_3 + \ldots + a_{12} \qquad \text{...(2)}$$

On adding, we get

$$64 = 2 + 2a_2 + 2a_4 + 2a_6 + \ldots + 2a_{12}$$

$$\Rightarrow \quad a_2 + a_4 + a_6 + \ldots + a_{12} = \mathbf{31.}$$

42. If in the expansion of $(1 + x)^{2n}$, coefficients of $(p + 1)$th term and $(p + 3)$th term are same, then

(a) $p = n$ (b) $p = n + 3$

(c) $p = n + 1$ (d) $p = n - 1$

Solution (d) Given, ${}^{2n}C_p = {}^{2n}C_{p+2}$ $\Rightarrow 2n = p + p + 2$

$\Rightarrow n = p + 1$

$\Rightarrow \boldsymbol{p = n - 1.}$

43. If x is so small that its second and higher power can be neglected, then coefficient of x in the expansion of $(1 + 4x)^{-5/4}\cdot(1 + 2x)^{1/2}$ is

(a) 0 (b) 2

(c) 4 (d) -4

Solution (d) We have,

$$(1+4x)^{-5/4}\cdot(1+2x)^{1/2} = \left(1-\frac{5}{4}\cdot 4x\right)\left(1+\frac{1}{2}(2x)\right)$$

$= (1 - 5x)(1 + x)$ $\therefore$ coefficient of $x = 1 - 5 = \mathbf{-4.}$

44. If $A = {}^{2n}C_0\cdot{}^{2n}C_1 + {}^{2n}C_1\,{}^{2n-1}C_1 + {}^{2n}C_2\,{}^{2n-2}C_1 + \ldots$, then A is

(a) 0 (b) 2^n

(c) $n2^{2n}$ (d) 1

Solution (c) A = coeff. of x in $[{}^{2n}C_0(1 + x)^{2n} + {}^{2n}C_1(1 + x)^{2n-1} + \ldots)]$

= coeff. of x in $(1 + (1 + x))^{2n}$

= coeff. of x in $(2 + x)^{2n}$

= coeff. of x in $2^{2n}\left(1+\frac{x}{2}\right)^{2n}$ = $\boldsymbol{n\cdot 2^{2n}}$.

45. The greatest integer which divides the number $101^{100} - 1$ is

(a) 100 (b) 1000

(c) 10000 (d) 100000

Solution (c) By Binomial theorem

$$(1+x)^n = \left[1+nx+\frac{n(n-1)}{2}\cdot x^2 \dots x^n\right]$$

or
$$(1+x)^n - 1 = nx + \frac{n(n-1)}{2}\, x^2 \dots x^n$$

If $x = n$, $(1+n)^n - 1 = n^2 + \frac{n(n-1)}{2}\, n^2 \dots n^n$

$$(1+n)^n - 1 = n^2\left[1+\frac{n(n-1)}{2}\dots n^{n-2}\right]$$

Put $n = 100$,

$$(1+100)^{100} - 1 = (100)^2\left[1+\frac{100(100-1)}{2}\dots 100^{98}\right]$$

$$(101)^{100} - 1 = (100)^2\left[1+\frac{100\times 99}{2}\dots 100^{98}\right]$$

Clearly $(101)^{100} - 1$ is divisible by $(100)^2$ = **10000.**

46. If $s_n = \sum_{r=0}^{n}\frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n}\frac{r}{{}^nC_r}$ then $\frac{t_n}{s_n}$ is

(a) $n - 1$ (b) $\frac{1}{2}n - 1$

(c) $\frac{1}{2}n$ (d) $\frac{2n-1}{2}$

Solution (c) $t_n = \sum_{r=0}^{n}\frac{r}{{}^nC_r} = \sum_{r=0}^{n}\frac{n-(n-r)}{{}^nC_{n-r}}$

[∴ ${}^nC_r = {}^nC_{n-r}$]

$$= n\sum_{r=0}^{n}\frac{1}{{}^nC_{n-r}} - \sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}$$

$$= n\sum_{r=0}^{n}\frac{1}{{}^nC_r} - \left[\frac{n}{{}^nC_n}+\frac{n-1}{{}^nC_{n-1}}+\dots+\frac{1}{{}^nC_1}+0\right]$$

$$= n\,.\,S_n - \sum_{r=0}^{n}\frac{r}{{}^nC_r} = nS_n - t_n$$

$$\therefore\quad 2t_n = n\,.\,S_n \Rightarrow \frac{t_n}{S_n} = \frac{n}{2}.$$

47. The coefficient of x^n in expansion of $(1 + x)(1 - x)^n$ is

(a) $(-1)^{n-1}(n-1)^2$ (b) $(-1)^n(1-n)$

(c) $(n-1)$ (d) $(-1)^{n-1}n.$

Solution (b) The coeff. of x^n in $(1 + x)(1 - x)^n$

$= \text{coeff of } x^n \text{ in } (1-x)^n + \text{coeff. of } x^n \text{ in } x(1-x)^n$

$= (-1)^n.\ {}^nC_n + \text{coeff. of } x^{n-1} \text{ in } (1-x)^n$

$= (-1)^n + (-1)^{n-1}\,{}^nC_{n-1} = (-1)^{n-1}(-1+n)$

$= (n-1)(-1)^{n-1} = (1-n)(-1)^n.$

48. The coefficient of the middile term in the binomial expansion in powers of x of $(1 + \alpha)^4$ and of $(1 - \alpha)^6$ is the same if α equals

(a) $-3/10$ (b) $10/3$

(c) $-5/3$ (d) 4

Solution (a)

49. If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is

(a) 7th dterm (b) 5th term

(c) 8th term (d) 6th term

Solution (c)
$$(1+x)^{27/5} = 1 + \frac{27}{5}x + \frac{27}{5}\left(\frac{27}{5}-1\right)\frac{x^2}{2!}$$
$$+ \frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\frac{x^3}{3!}$$
$$+ \frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\left(\frac{27}{5}-3\right)\frac{x^4}{4!}$$
$$+ \frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\left(\frac{27}{5}-3\right)\left(\frac{27}{5}-4\right)\frac{x^5}{5!}$$
$$+ \frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\left(\frac{27}{5}-3\right)\left(\frac{27}{5}-4\right)\left(\frac{27}{5}-5\right)\frac{x^6}{6!}$$
$$+ \frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\left(\frac{27}{5}-3\right)\left(\frac{27}{5}-4\right)\left(\frac{27}{5}-5\right)$$
$$\times\left(\frac{27}{5}-6\right)\frac{x^7}{7!} + \dots$$

Here $\frac{27}{5} - 6$ is negative i.e., 8th term is negative in the expansion of $(1 + x)^{27/5}$.

50. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is

(a) 32 (b) 33

(c) 34 (d) 35

Solution (b)

Problems for Practice

Level of Difficulty I

1. If the 21st and 22nd terms in the expansion of $(1 - x)^{44}$ are equal, then $x =$

(a) $\frac{8}{7}$ (b) $-\frac{7}{8}$

(c) $\frac{7}{8}$ (d) $-\frac{8}{7}$

2. The sum of the binomial coefficients in the expansion of $(1 + x)^7$ is

(a) 256 (b) 64

(c) 128 (d) 0

3. In the expansion of $(3x + 2)^4$, the coefficient of middle term is

(a) 36 (b) 216

(c) 54 (d) 81

4. Constant term in the expansion of $\left(x-\dfrac{1}{x}\right)^{10}$ is

(a) 152 (b) − 152
(c) − 252 (d) 252

5. The term independent of x in $\left(x^2-\dfrac{1}{x}\right)^{9}$ is

(a) 1 (b) 49
(c) − 1 (d) none of these

6. The coefficient of x^4 in $\left(\dfrac{x}{2}-\dfrac{3}{x^2}\right)^{10}$ is

(a) $\dfrac{450}{263}$ (b) $\dfrac{405}{256}$
(c) $\dfrac{504}{259}$ (d) none of these

7. The term independent of x in $\left(\sqrt{x}-\dfrac{2}{x}\right)^{18}$ is

(a) ${}^{18}C_6\, 2^6$ (b) ${}^{18}C_6\, 2^{12}$
(c) ${}^{18}C_{18}\, 2^{18}$ (d) none of these

8. The coefficient of x^3 in $\left(\sqrt{x^5}+\dfrac{3}{\sqrt{x^3}}\right)^{6}$ is

(a) 0 (b) 120 (c) 420 (d) 540

9. The coefficient of x^{32} and x^{-16} in the expansion of

$$\left(x^4-\frac{1}{x^3}\right)^{15} \text{ are}$$

(a) 1365, 240 (b) 1365, 0
(c) 240, 0 (d) none of these

10. The sum of the coefficients in the expansion of $(6a-5b)^n$, where n is a positive integer, is

(a) 1 (b) − 1 (c) 2^n (d) 2^{n-1}

11. In the expansion of $\left(3x-\dfrac{1}{x^2}\right)^{10}$, the 5th term from the end is

(a) $\dfrac{16486}{x^8}$ (b) $\dfrac{17010}{x^8}$
(c) $\dfrac{13486}{x^8}$ (d) none of these

12. The 13th term of $\left(9x-\dfrac{1}{3\sqrt{x}}\right)^{18}$ is

(a) 17682 (b) 18564
(c) $18564\,x^6$ (d) none of these

13. The $(n+1)^{\text{th}}$ term from the end in the expansion of $\left(2x-\dfrac{1}{x}\right)^{3n}$ is

(a) $\dfrac{3n!}{2n!n!}2^n\cdot x^{-n}$ (b) $\dfrac{3n!}{2n!n!}2^{2n}\cdot x^{-2n}$
(c) $\dfrac{3n!}{2n!n!}2^n\cdot x^{n}$ (d) none of these

14. The 7th term in $\left(\dfrac{1}{y}+y^2\right)^{10}$, when expanded in descending power of y, is

(a) $\dfrac{210}{y^2}$ (b) $\dfrac{y^2}{210}$
(c) $210y^2$ (d) none of these

15. The coefficient of x^{30} in the expansion of $(1+3x+3x^2+x^3)^{15}$ is

(a) ${}^{45}C_{15}$ (b) ${}^{45}C_{25}$
(c) ${}^{45}C_{30}$ (d) ${}^{15}C_{11}$

16. If $(1+x)^{15}=C_0+C_1x+C_2x^2+\ \dots+C_{15}x^{15}$, then ${}^{15}C_0^2-{}^{15}C_1^2+{}^{15}C_2^2-{}^{15}C_3^2+\dots-{}^{15}C_{15}^2$ is equal to

(a) 0 (b) 1
(c) − 1 (d) none of these

17. $\dfrac{C_1}{C_0}+2\dfrac{C_2}{C_1}+3\dfrac{C_3}{C_2}+\dots n\dfrac{C_n}{C_{n-1}}=$

(a) $\dfrac{n(n-1)}{2}$ (b) $\dfrac{n(n+1)}{2}$
(c) $\dfrac{(n+1)(n+2)}{2}$ (d) none of these

18. If the binomial expansion of $(a+bx)^{-2}$ is $\dfrac{1}{4}-3x+\dots$, then the value of a and b are

(a) 2, 12 (b) 2, 10
(c) 1, 12 (d) none of these

19. If $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to $a+bx$ for all small values of x, then

(a) $a=1,\ b=-\dfrac{35}{24}$

(b) $a=-\dfrac{35}{24},\ b=1$

(c) $a=1,\ b=\dfrac{35}{24}$

(c) none of these

20. ${}^7C_0+{}^7C_1+{}^7C_2+{}^7C_3+{}^7C_4+{}^7C_5=$

(a) 128 (b) 121
(c) 120 (d) 129

Level of Difficulty II

21. If the r^{th} term in the expansion of $\left(\dfrac{x}{3}-\dfrac{2}{x^2}\right)^{10}$ contains x^4, then r is equal to

(a) 2 (b) 3 (c) 4 (d) 5

22. The coefficient of x^n in the expansion of $(1+x)^{2n}$ and the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ are in the ratio

(a) 4 : 1 (b) 3 : 1
(c) 2 : 1 (d) 1 : 1

23. The coefficient of x^5 in the expansion of $(1+x^2)^5\,(1+x)^4$ is

(a) 40 (b) 50
(c) –50 (d) 60

24. The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ is

(a) 990 (b) 605 (c) 810 (d) none of these

25. If the coefficients of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P., then

(a) $2n^2 - 9n + 7 = 0$ (b) $2n^2 + 5n + 7 = 0$

(c) $n^2 - 9n + 7 = 0$ (d) none of these

26. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 =$

(a) $\frac{(2n)!}{(n!)^2}$ (b) $\frac{(2n)!}{(n-1)!(n+1)!}$

(c) $\frac{(2n)!}{(n-2)!(n+2)!}$ (d) none of these

27. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1} C_n =$

(a) $\frac{(2n)!}{(n!)^2}$ (b) $\frac{(2n)!}{(n-1)!(n+1)!}$

(c) $\frac{(2n)!}{(n-2)!(n+2)!}$ (d) none of these

28. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2} C_n =$

(a) $\frac{(2n)!}{(n!)^2}$ (b) $\frac{(2n)!}{(n-1)!(n+1)!}$

(c) $\frac{(2n)!}{(n-2)!(n+2)!}$ (d) none of these

29. The total number of terms in the expansion of $(a + b + c + d)^n$, $n \in N$ is

(a) $\frac{n(n+1)(n+2)}{6}$

(b) $\frac{n(n+1)(n+2)(n+3)}{6}$

(c) $\frac{(n+1)(n+2)(n+3)}{6}$

(d) none of these

30. If x is so small that its square and higher powers may be neglected, then the value of $\frac{(8+3x)^{2/3}}{(2+3x)(4-5x)^{1/2}}$ is

(a) $1 - \frac{3}{2}x$ (b) $1 + \frac{5}{8}x$

(c) $1 - \frac{5}{8}x$ (d) none of these

31. The expansion of $(8 - 3x)^{3/2}$ in terms of powers of x is valid only if

(a) $x > \frac{8}{3}$ (b) $|x| < \frac{8}{3}$

(c) $x < \frac{3}{8}$ (d) $x < \frac{8}{3}$

32. The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is

(a) $2^{19} - \frac{1}{2} \cdot {}^{20}C_{10}$

(b) $2^{19} + \frac{1}{2} \cdot {}^{20}C_{10}$

(c) 2^{19}

(d) 2^{20}

33. Coefficient of x^r in the expansion of $(1 - 2x)^{-1/2}$ is

(a) $\frac{2r!}{(r!)^2 \cdot 2^{2r}}$ (b) $\frac{2r!}{2^r(r+1)!(r-1)!}$

(c) $\frac{2r!}{(r!)^2}$ (d) $\frac{2r!}{(r!)^2 \cdot 2^r}$

34. $1 + \frac{2nx}{1+x} + \frac{n(n+1)}{2!} + ...\infty$ equals

(a) $\left(\frac{1+x}{1-x}\right)^n$ (b) $\left(\frac{1-x}{1+x}\right)^n$

(c) $\left(\frac{1-x}{1+x}\right)^n$ (d) none of these

35. ${}^{n+1}C_2 + 2\,[{}^2C_2 + {}^3C_2 + {}^4C_2 + ... + {}^nC_2] =$

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{n(n-1)(2n-1)}{6}$ (d) none of these

36. If the coefficients of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal, then $r =$

(a) 12 (b) 10 (c) 8 (d) 6

37. In the expansion of $(1 + x + x^3 + x^4)^{10}$, then coefficient of x^4 is

(a) ${}^{40}C_4$ (b) ${}^{10}C_4$ (c) 210 (d) 310

38. The term independent of x in $\left[\sqrt{\left(\frac{x}{3}\right)} + \sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is

(a) none (b) ${}^{10}C_1$ (c) $\frac{5}{12}$ (d) 1

39. If the absolute term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then k is equal to

(a) ± 2 (b) ± 3 (c) ± 1 (d) none of these

40. In the expansion of $\left(x^3 - \frac{3}{x^2}\right)^{10}$, the term independent of x is

(a) 150000 (b) 153090

(c) 153180 (d) 150090

ANSWERS

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (d)	**9.** (b)	**10.** (a)
11. (b)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (c)
21. (b)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (c)
31. (b)	**32.** (b)	**33.** (d)	**34.** (a)	**35.** (a)	**36.** (d)	**37.** (d)	**38.** (a)	**39.** (d)	**40.** (b)

Exponential and Logarithmic Series

23

BRIEF REVIEW OF THE CONCEPTS

EXPONENTIAL SERIES

If x is any real number, then

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty.$$

is known as exponential series and e^x is called **exponential function.**

EXPONENTIAL THEOREM

For $a > 0$, $a^x = e^{x \log_e a} =$

$$1 + \frac{x \log_e a}{1!} + \frac{x^2 (\log_e a)^2}{2!} + \frac{x^3 (\log_e a)^3}{3!} + \ldots\infty,$$

for all real values of x.

Deductions from Exponential Series

1. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\infty$
2. $e^x + e^{-x} = 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\infty\right]$
3. $e^x - e^{-x} = 2\left[\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\infty\right]$
4. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\infty$
5. $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots\infty$
6. $e + e^{-1} = 2\left[1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\infty\right]$
7. $e - e^{-1} = 2\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\infty\right]$

Graph of Exponential Function

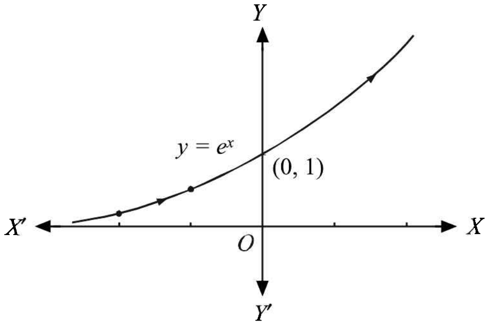
Let $y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$

For all $x \in R$, y has a unique and definite value. Since $e > 2$, e^x is positive for all x. As x increase from 0 to ∞, y increases from 1 to ∞. As x decreases from 0 to $-\infty$, y decreases from 1 to 0.

The approximate values of y for some values of x are as follows:

x	–2	–1	0	1	2	3
y	0.1	0.4	1	2.7	7.3	19.7

The graph is as shown below:



The value of e The value of e lies between 2 and 3 i.e., $2 < e < 3$.

The value of e, correct upto ten places of decimals, is 2.7182818284.

Key Results to Remember

1. $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\infty = \sum_{n=0}^{\infty} \frac{1}{n!}$

   [Since $0! = 1$ and $\frac{1}{(n-m)!}$ when $n < m = 0$

   $\therefore \quad \sum_{n=0}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{(n-k)!} = e$]

2. $e - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\infty = \sum_{n=1}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}$
3. $e - 2 = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots\infty$

$$= \sum_{n=2}^{\infty}\frac{1}{n!}=\sum_{n=0}^{\infty}\frac{1}{(n+2)!}=\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\ .$$

4. $$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}=\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}$$

5. $$\frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots$$

$$= \sum_{n=1}^{\infty}\frac{1}{(2n-1)!}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!} =$$

LOGARITHMIC SERIES

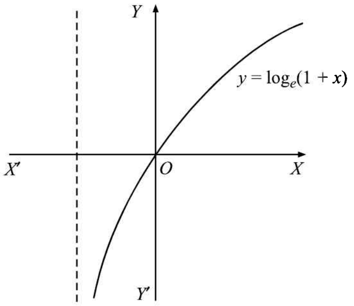
If $|x| < 1$, then

$$\log_e(1 + x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$$

is known as logarithmic series. ...(A)

Remarks:

1. It can be shown that (A) is valid even when $x = 1$ (but not for $x = -1$).
2. $\log_e x$ is also written as $In\ x$.
3. The graph of the function $y = \log_e(1 + x)$ is as shown in figure below.



1. $\log_e(1 - x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\infty$
2. $\log_e\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\infty\right) =$
3. $\log_e 2 = 1 - \frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\infty$

DIFFERENCES BETWEEN THE EXPONENTIAL AND LOGARITHMIC SERIES

The main differences between the exponential series and the logarithmic series are:

1. In the expansion for e^x, all the terms are positive but in the expansion for $\log_e(1 + x)$, the terms are alternatively positive and negative.
2. In the exponential series, the denominators of the terms involve factorials, which do not occur in logarithmic series.
3. The exponential series is valid for all values of x but the logarithmic series is valid for $-1 < x \le 1$.

Solved Problems

1. The coefficient of x^n in the expansion of $\frac{e^{7x}+e^{x}}{e^{3x}}$ is:

(a) $\frac{4^{n-1}+(-2)^n}{n!}$ (b) $\frac{4^{n-1}+2^n}{n!}$

(c) $\frac{4^{n-1}+(-2)^{n-1}}{n!}$ (d) $\frac{4^{n}+(-2)^n}{n!}$

Solution (d) Given expression = $\frac{e^{7x}+e^{x}}{e^{3x}}$

$$= \frac{e^{7x}+e^{x}}{e^{3x}} = e^{4x} + e^{-2x}.$$

Coefficient of x^n in expansion of $e^{4x} = \frac{4^n}{n!}$.

Coefficient of x^n in expansion of $e^{-2x} = \frac{(-2)^n}{n!}$.

∴ Coefficient of x^n in expansion of given expression

$$= \frac{4^n+(-2)^n}{n!}.$$

2. The value of $\sqrt{e}$ will be

(a) 1.648 (b) 1.547

(c) 1.444 (d) 1.348

Solution (a) We have, $e^x = 1 + x + \frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$$\therefore \quad \sqrt{e} = e^{1/2} = 1 + \frac{1}{2}+\frac{1}{8}+\frac{1}{48}+\ldots$$

$= 1 + .5 + .125 + .0208 + \ldots =$ **1.648** nearly.

3. The coefficient of x^{12} in the expansion of e^{2x} is

(a) $\frac{2^{12}}{12!}$ (b) $\frac{2^{11}}{12!}$

(c) $\frac{2^{10}}{12!}$ (d) none of these

Solution (a) We have

$$e^{2x} = 1 + \frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\ldots+\frac{(2x)^n}{n!}+\ldots$$

$$= 1 + \frac{2x}{1!}+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots+\frac{2^nx^n}{n!}+\ldots$$

$\therefore$ The coefficient of x^n in the expansion of

$$e^{2x} = \frac{2^n}{n!}.$$

$\therefore$ Coefficient of x^{12} in the expansion of $e^{2x} = \mathbf{\frac{2^{12}}{12!}}$.

4. The coefficient of x^n in the expansion of $\dfrac{a-bx}{e^x}$ is

(a) $\dfrac{(-1)^n}{n!}(a-bn)$ (b) $\dfrac{(-1)^n}{n!}(a+bn)$

(c) $\dfrac{(-1)^n}{n!}(b+an)$ (d) none of these

Solution (b) We have, $\dfrac{a-bx}{e^x} = (a - 6x)e^{-x}$

$$= (a - bx)\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right.$$
$$\left.+(-1)^{n-1}\frac{x^{n-1}}{(n-1)!}+\frac{(-1)^n x^n}{n!}+\ldots\right].$$

$$\therefore \text{ Coefficient of } x^n = a\cdot\frac{(-1)^n}{n!}-b\,\frac{(-1)^{n-1}}{(n-1)!}$$

$$= (-1)^n\cdot\frac{a}{n!}+\frac{(-1)^n b}{(n-1)!}=\frac{(-1)^n}{n!} = \boldsymbol{]\ (a + bn).}$$

5. The coefficient of x^n in the series

$\dfrac{1+x}{1!}+\dfrac{(1+x)^2}{2!}+\dfrac{(1+x)^3}{3!}+\ldots$ is

(a) $\dfrac{2e}{n!}$ (b) $\dfrac{4e}{n!}$

(c) $\dfrac{e}{n!}$ (d) none of these

Solution (c) We have, $\dfrac{1+x}{1!}+\dfrac{(1+x)^2}{2!}+\dfrac{(1+x)^3}{3!}+\ldots$

$$= \left[1+\frac{1+x}{1!}+\frac{(1+x)^2}{2!}+\frac{(1+x)^3}{3!}+\ldots\right] - 1$$

$$= e^{1+x} - 1 = e.\ e^x - 1$$

$$= e\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\right)-1.$$

$$\therefore \text{ Coefficient of } x^n = \mathbf{\frac{e}{n!}}.$$

6. The sum of the series $\dfrac{2}{1!}+\dfrac{4}{3!}+\dfrac{6}{5!}+\ldots\infty$ is

(a) e (b) $2e$

(c) $3e$ (d) $4e$

Solution (a) The given series $= \dfrac{2}{1!}+\dfrac{4}{3!}+\dfrac{6}{5!}+\ldots$

$$= \frac{1+1}{1!}+\frac{3+1}{3!}+\frac{5+1}{5!}+\ldots$$

$$= 1 + \frac{1}{1!}+\frac{3}{3!}+\frac{1}{3!}+\frac{5}{5!}+\frac{1}{5!}+\ldots$$

$$= 1 + \frac{1}{1!}+\frac{3}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots = e^1 = \boldsymbol{e.}$$

7. The sum of the series $1 + \dfrac{3}{2!}+\dfrac{5}{4!}+\dfrac{7}{6!}+\ldots\infty$ is

(a) e (b) $2e$

(c) $3e$ (b) none of these

Solution (a) We have, $1 + \dfrac{3}{2!}+\dfrac{5}{4!}+\dfrac{7}{6!}+\ldots\infty$

$$= 1 + \frac{2+1}{2!}+\frac{4+1}{4!}+\frac{6+1}{6!}+\ldots\infty$$

$$= 1 + \left(\frac{2}{2!}+\frac{1}{2!}\right)+\left(\frac{4}{4!}+\frac{1}{4!}\right)+\left(\frac{6}{6!}+\frac{1}{6!}\right)+\ldots\infty$$

$$= 1 + \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\ldots\infty = \boldsymbol{e.}$$

8. The value of

$$\left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\right)^2-\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\right)^2 \text{ is}$$

(a) 2 (b) -2

(c) 1 (d) -1.

Solution (c) We have $\dfrac{e+e^{-1}}{2} = 1 + \dfrac{1}{2!}+\dfrac{1}{4!}+\ldots$

and $\dfrac{e-e^{-1}}{2} = 1 + \dfrac{1}{3!}+\dfrac{1}{5!}+\ldots$

$$\therefore \quad \left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\right)^2-\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\right)^2$$

$$= \left(\frac{e+e^{-1}}{2}\right)^2-\left(\frac{e-e^{-1}}{2}\right)^2$$

$$= \frac{e^2+e^{-2}+2}{4}-\frac{e^2+e^{-2}-2}{4}$$

$$\frac{e^2+e^{-2}+2-e^2-e^{-2}+2}{4}=\frac{2+2}{4}$$

$$= \frac{2+2}{4}$$

$$= \frac{4}{4} = \mathbf{1.}$$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{C(n,0)+C(n,1)+\ldots+C(n,n)}{P(n,n)}$ is equal to

(a) e^2 (b) $e^2 + 1$

(c) $e^2 - 1$ (d) none of these

Solution (c) $\because$ $C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n$ and $P(n, n) = n!$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{C(n,0)+C(n,1)+\ldots+C(n,n)}{P(n,n)}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{n!} = \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots \infty$$

$$= \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots\right) - 1 = \mathbf{e^2 - 1.}$$

10. The sum of the series

$$\frac{1}{1\cdot2} + \frac{1\cdot3}{1\cdot2\cdot3\cdot4} + \frac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \ldots \text{ is}$$

(a) $\sqrt{e}$ (b) $\sqrt{e} - 1$

(c) $\sqrt{e} - 2$ (d) none of these

Solution (b) The general term of the given series is

$$T_n = \frac{1\cdot3\cdot5\ldots(2n-1)}{(2n)!} = \frac{1\cdot2\cdot3\cdot4\cdot5\ldots(2n-1)2n}{(2n)!\{2\cdot4\cdot6\ldots2n\}} =$$

$$= \frac{(2n)!}{(2n)!\,2^n n!} = \frac{1}{2^n n!} = \frac{(1/2)^n}{n!}$$

Hence sum of the series $= \sum\limits_{n=1}^{\infty} T_n = \sum\limits_{n=1}^{\infty} \frac{(1/2)^n}{n!}$

$$= \left[\frac{(1/2)^1}{1!} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \ldots\right]$$

$$= e^{1/2} - 1 = \mathbf{\sqrt{e} - 1.}$$

11. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$, then x is equal to

(a) $\log(1 + y)$ (b) e^y

(c) $y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ (d) $e^{-y} - 1$

Solution (c) Since $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$\Rightarrow \quad y = \log_e(1 + x)$$

$$\therefore \quad e^y = 1 + x \quad \text{or} \quad 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots = 1 + x$$

$$\therefore \quad x = \mathbf{y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots \infty.}$$

12. The sum of the series $1 + \frac{1}{2!} + \frac{1\cdot3}{4!} + \frac{1\cdot3\cdot5}{6!} + \ldots$ is

(a) $\sqrt{e}$ (b) $e^{3/2}$

(c) $e^{-1/2}$ (d) e

Solution (a) The nth term of the given series is

$$T_n = \frac{1\cdot3\cdot5\ldots(2n-3)}{(2n-2)!}$$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\ldots(2n-3)(2n-2)}{(2n-2)!\cdot2\cdot4\cdot6\ldots(2n-2)}, \text{ for } n \geq 2$$

$$\Rightarrow T_n = \frac{(2n-2)!}{(2n-2)!\cdot2^{n-1}\cdot(n-1)!} = \frac{\left(\frac{1}{2}\right)^{n-1}}{(n-1)!} \quad \ldots(1)$$

Clearly T_1 = Ist term of the series = 1

From (1), we have

$$T_2 = \frac{\frac{1}{2}}{1!},\ T_3 = \frac{\left(\frac{1}{2}\right)^2}{2!},\ T_4 = \frac{\left(\frac{1}{2}\right)^3}{3!}, \text{ and so on}$$

Thus, the given series become

$$1 + \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \ldots = e^{1/2} = \sqrt{e}.$$

13. The series $3\log2 + \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3$ ... is equal to

(a) $\log 3$ (b) $\log 5$

(c) $\log 10$ (d) none of these

Solution (c) The given series

$$= 3\log2 + \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 \ldots$$

$$= \log 2^3 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) \qquad \left[\text{where } x = \frac{1}{4}\right]$$

$$= \log 8 + \log(1 + x) = \log 8 + \log\left(1 + \frac{1}{4}\right)$$

$$= \log 8 + \log\frac{5}{4} = \log\left(8 \times \frac{5}{4}\right) = \mathbf{\log 10.}$$

14. If α, β are the roots of the equation $x^2 - px + q = 0$, then $(\alpha + \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 + \frac{\alpha^3 + \beta^3}{3}x^3$ ... is equal to

(a) $\log(1 + px + qx^2)$ (b) $\log(1 - px + qx^2)$

(c) $\log(1 + px - qx^2)$ (d) none of these

Solution (a) Since α, β are the roots of $x^2 - px + q = 0$

$\therefore \quad \alpha + \beta = p$ and $\alpha\beta = q$.

Now, the given series

$$= (\alpha + \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 + \frac{\alpha^3 + \beta^3}{3}x^3 - \ldots$$

$$= \left[\alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \ldots\right] + \left[\beta x - \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \ldots\right]$$

$$= \log(1 + \alpha x) + \log(1 + \beta x)$$

$$= \log[(1 + \alpha x)(1 + \beta x)] = \log[1 + (\alpha + \beta)x + \alpha\beta x^2]$$

$$= \mathbf{\log(1 + px + qx^2)} \quad [\because \alpha + \beta = p \text{ and } \alpha\beta = q].$$

15. $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$ will be equal to

(a) $2e^{-2}$ (b) e^{-2}

(c) e^{-1} (d) $2e^{-1}$

Solution (c) The given series is $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$

Here $T_n = \frac{(2n)}{(2n+1)!} = \frac{(2n+1) - 1}{(2n+1)!}$

$$= \frac{1}{(2n)!} - \frac{1}{(2n+1)!}$$

$$\therefore \quad S = T_1 + T_2 + T_3 + \dots = \sum_{n=1}^{\infty} T_n$$

$$\Rightarrow \quad S = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots = \mathbf{e^{-1}}.$$

16. The product of the following series $\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots\right)$ is

(a) 1 (b) e^{-2}

(c) e^2 (d) -1

Solution (a) We have,

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots\right)$$

$$= e \cdot e^{-1} = e^0 = \mathbf{1}.$$

17. The sum of the series $1 + \frac{3}{2!} + \frac{7}{3!} + \frac{15}{4!} + \dots \infty$ is

(a) $e(e+1)$ (b) $e(1-e)$

(c) $e(e-1)$ (d) $3e$

Solution (c) The general term of the given series is

$$T_n = \frac{2^n - 1}{n!} \Rightarrow S_n = \sum_{n=1}^{\infty} T_n = \sum_{n=1}^{\infty}\left[\frac{2^n}{n!} - \frac{1}{n!}\right]$$

$$= (e^2 - 1) - (e - 1) = e^2 - e = \mathbf{e(e-1)}.$$

18. The value of $(1+3)\log_e 3 + \frac{(1+3^2)}{2!}(\log_e 3)^2 + \frac{(1+3^3)}{3!}(\log_e 3)^3 + \dots \infty$ is

(a) 18 (b) 28

(c) 36 (d) none of these

Solution (b) We have,

$$(1+3)\log_e 3 + \frac{(1+3^2)}{2!}(\log_e 3)^2 + \frac{(1+3^3)}{3!}(\log_e 3)^3 + \dots$$

$$= \left[1 + \log_e 3 + \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} + \dots\right]$$

$$+ \left[1 + 3\log_e 3 + \frac{(3\log_e 3)^2}{2!} + \frac{(3\log_e 3)^3}{3!} + \dots\right] - 2$$

$$= e^{\log_e 3} + e^{3\log_e 3} - 2$$

$$= e^{\log_e 3} + e^{\log_e 3^3} - 2 = 3 + 3^3 - 2 = \mathbf{28}.$$

19. The value of $1 + \frac{1+a}{2!} + \frac{1+a+a^2}{3!} + \dots \infty$ is

(a) $\frac{e - e^a}{a-1}$ (b) $\frac{e^a - e}{a-1}$

(c) $\frac{e^{a-1} - e}{a-1}$ (d) none of these

Solution (b) The general term of the given series is

$$T_n = \frac{1 + a + a^2 + \dots + a^{n-1}}{n!} = \frac{1\cdot(a^n - 1)/(a-1)}{n!}$$

(Series in the numerator is G.P with first term as 1 and common ratio a)

$$= \frac{1}{(a-1)}\left[\frac{a^n}{n!} - \frac{1}{n!}\right].$$

Putting $n = 1, 2, 3, \dots$ and adding, we get the given series

$$= \frac{1}{(a-1)}\left[\left(\frac{a}{1!} + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right)\right]$$

$$= \frac{1}{(a-1)}[(e^a - 1) - (e^1 - 1)] = \mathbf{\frac{e^a - e}{a - 1}}.$$

20. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, then x is

(a) $e^y - 1$ (b) $\log(1+y)$

(c) $e^y + 1$ (d) e^y

Solution (a) We have, $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

$$= \log_e(1+x)$$

$$\therefore \quad e^y = 1 + x \Rightarrow \mathbf{x = e^y - 1}.$$

21. If $y = -\left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + \dots\right)$, then

(a) $x^3 = 1 - e^y$ (b) $x = \log(1+y)$

(b) $x^3 = e^y$ (d) $x = 1 + e^y$

Solution (a) We have, $y = \log_e(1 - x^3)$

$$\therefore \quad e^y = 1 - x^3 \Rightarrow \mathbf{x^3 = 1 - e^y}.$$

22. $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \dots$ is

(a) $-\frac{x}{1+x} + \log(1+x)$

(b) $\frac{x}{1+x} + \log(1+x)$

(c) $\frac{x}{1-x} + \log(1-x)$

(d) none of these

Solution (a) The general term of the given series is

$$T_n = \frac{n}{n+1}x^{n+1} = \frac{\overline{n+1}-1}{n+1}x^{n+1}$$

$$= \left(1 - \frac{1}{n+1}\right)x^{n+1} = x^{n+1} - \frac{x^{n+1}}{n+1}$$

Putting $n = 1, 2, 3, \dots$

$$\therefore \quad S = (x^2 + x^3 + x^4 + \dots) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\dots\right)$$

$$= x + \frac{x^2}{1-x} + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\dots\right)$$

$$= \frac{x - x^2 + x^2}{1 - x} + \log(1 - x) = \frac{\mathrm{x}}{\mathbf{1} - \mathrm{x}} + \log(\mathbf{1} - \mathrm{x}).$$

23. The sum of these series $\sum_{n=0}^{\infty} \frac{n^2 - n + 1}{n!}$ is

(a) $2e$ (b) $\frac{3}{2}e$

(c) e (d) $3e$

Solution (a) We have,

$$\sum_{n=0}^{\infty} \frac{n^2 - n + 1}{n!} = \sum_{n=1}^{\infty} \frac{n(n-1)}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{n-1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-2!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= \left[\frac{1}{0!} + \frac{1}{1!} + \ldots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right]$$

$= e + e =$ **$2e$**.

24. $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots \infty$ is equal to

(a) $e + 1$ (b) $e - 1$

(c) e^{-1} (d) e

Solution (d) The general term of the given series is

$$T_n = \frac{2n}{(2n-1)!} = \frac{2n - 1 + 1}{(2n-1)!}$$

$$= \frac{1}{(2n-2)!} + \frac{1}{(2n-1)!}$$

$$\therefore S_n = \sum_{n=1}^{\infty} T_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = e.$$

25. The value of $(x + y)(x - y) + \frac{1}{2!}(x + y)(x - y)(x^2 + y^2)$

$+ \frac{1}{3!}(x + y)(x - y)(x^4 + y^4 + x^2y^2) + \ldots \infty$ is

(a) $e^{x^2} + e^{y^2}$ (b) $e^{x^2 - y^2}$

(c) $e^{x^2} - e^{y^2}$ (d) none of these

Solution (c) The given expression

$$= x^2 - y^2 + \frac{1}{2!}(x^4 - y^4) + \frac{1}{3!}(x^6 - y^6) + \ldots$$

$$= \left(x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots\right) - \left(y^2 + \frac{(y^2)^2}{2!} + \frac{(y^2)^3}{3!} + \ldots\right)$$

$$= \left(1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots.\right) - \left(1 + y^2 + \frac{(y^2)^2}{2!} + \frac{(y^2)^3}{3!} + \ldots.\right)$$

$= e^{x^2} - e^{y^2}$.

26. If $S = \dfrac{y - 1 - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 \ldots}{a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 \ldots}$, then S is equal to

(a) $\log_a y$ (b) $\log_y a$

(c) $\log_e a$ (d) $\log_e y$

Solution (a) $S = \dfrac{\log[1 + (y-1)]}{\log[1 + (a-1)]} = \dfrac{\log y}{\log a}$ **$\log_a y$.**

27. The value of $1 - \log 2 + \frac{(\log 2)^2}{2!} - \frac{(\log 2)^3}{3!} + \ldots$ is

(a) 2 (b) $\frac{1}{2}$

(c) $\log 3$ (d) none of these

Solution (b) We have,

$$1 - \frac{\log 2}{1!} + \frac{(\log 2)^2}{2!} - \frac{(\log 2)^3}{3!} + \ldots = e^{-\log 2} = \mathbf{\frac{1}{2}}.$$

28. The sum of the series $\frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \ldots$ is

(a) $\log\left(\frac{e}{2}\right)$ (b) $\log\left(\frac{2}{3}\right)$

(c) $\frac{e}{2}$ (d) $\frac{2}{e}$

Solution (a) Given series $= \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} \ldots$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \ldots = 1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \ldots$$

$$= 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$$

$$= 1 - \log 2 = \log e - \log 2 = \mathbf{\log \frac{e}{2}}.$$

29. $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots =$

(a) $3e$ (b) $5e$

(c) $6e$ (d) $7e$

Solution (b) The general term of the given series is

$$T_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{n^2 - 1}{(n-1)!} + \frac{1}{(n-1)!}$$

$$= \frac{n+1}{(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{n-2}{(n-2)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

$$\therefore S_n = \sum_{n=1}^{\infty} T_n$$

$$= \sum_{n=1}^{\infty}\frac{1}{(n-3)!}+\sum_{n=1}^{\infty}\frac{3}{(n-2)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$$

$$= e + 3e + e = \mathbf{5e}.$$

30. $\sum_{n=0}^{\infty}\frac{(\log_e x)^n}{n!}$ is equal to

(a) $\log_e x$ (b) $\log x$

(c) x (d) none of these

Solution (c) $\sum_{n=0}^{\infty}\frac{(\log_e x)^n}{n!} = e^{\log_e x} = \boldsymbol{x}.$

31. $e^{(x-1)-\frac{1}{2}(x-1)^2+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\dots}$ equal to

(a) $\log(x-1)$ (b) $\log x$

(c) x (d) none of these

Solution (c) We have $e^{(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3}$

$$= e^{\log(1+x-1)} = \boldsymbol{x}.$$

32. The sum of the series

$$\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}\dots \text{ upto } \infty$$

is equal to

(a) $2\log_e 2$ (b) $\log_e 2 - 1$

(c) $\log_e 2$ (d) $\log_e\left(\frac{4}{e}\right)$

Solution (d) $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}\dots$

$$= \left(1-\frac{1}{2}\right)-\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= 1 - 2.\frac{1}{2} + 2.\frac{1}{3} - 2.\frac{1}{4} + \dots$$

$$= 2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\dots\right] - 1 = 2\log(1+1) - 1$$

$$= 2\log 2 - \log e = \log 4 - \log e = \mathbf{\log\frac{4}{e}}.$$

33. The sum of series $\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots$ is

(a) $\frac{(e^2-1)}{2e}$ (b) $\frac{(e-1)^2}{2e}$

(c) $\frac{(e^2-1)}{2}$ (d) $\frac{(e^2-1)}{e}$

Solution (b) $e = 1 + \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$ to ∞

$$e^{-1} = 1 - \frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots \text{ to } \infty$$

$$\therefore\quad e + e^{-1} = 2 + 2\left[\frac{1}{2!}+\frac{1}{4!}+\dots\right]$$

$$\therefore\quad \frac{1}{2!}+\frac{1}{4!}+\dots = \frac{1}{2}\left(\frac{e^2+1}{e}-2\right) = \mathbf{\frac{(e-1)^2}{2e}}.$$

Problems for Practice

1. The coefficient of x^n in the expansion of $\left[1+\frac{a+bx}{1!}+\frac{(a+bx)^2}{2!}+\frac{(a+bx)^3}{3!}+\dots\right]$ is

(a) $\frac{e^a b^n}{n!}$ (b) $\frac{e^a \cdot b^n}{(n-1)!}$

(c) $\frac{e^b \cdot a^n}{n!}$ (d) none of these

2. If $\frac{e^x}{1-x} = B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1} + B_nx^n + \dots$, then the value of $B_n - B_{n-1}$ is

(a) $\frac{1}{(n-2)!}$ (b) $\frac{1}{(n-1)!}$

(c) $\frac{1}{n!}$ (d) none of these

3. If n is a multiple of 3, then the coefficient of x^n in the expansion of $\log(1+x+x^2)$ ($|x|<1$), is

(a) $\frac{-2}{n}$ (b) $\frac{-1}{n}$ (c) $\frac{1}{n}$ (d) $\frac{2}{n}$

4. The sum of the series $1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\dots\infty$ is

(a) $\log 3$ (b) $2\log 3$ (c) $\log\frac{3}{2}$ (d) none of these

5. The sum of the series

$2\left[\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\dots\right]$ is

(a) $\log\frac{n}{n+1}$ (b) $\log\frac{n+1}{n}$

(c) $\log n$ (d) none of these

6. $\log 2 + 2\left[\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5^2}+\frac{1}{5}\cdot\frac{1}{5^3}+\dots\infty\right] =$

(a) $\log 3$ (b) $2\log 3$

(c) $3\log 3$ (d) none of these

7. $\frac{1}{n!}+\frac{1}{2!(n-2)!}+\frac{1}{4!(n-4)!}+\ldots\infty$ is

(a) $\frac{2^{n-1}}{n!}$ (b) $\frac{2^{n}}{(n+1)!}$

(c) $\frac{2^{n}}{n!}$ (d) $\frac{2^{n-2}}{(n-1)!}$

8. Coefficient of x^4 in the expansion of $\frac{1-3x-x^2}{e^x}$ is

(a) $\frac{5}{24}$ (b) $\frac{4}{25}$ (c) $\frac{24}{25}$ (d) $\frac{25}{24}$

9. $\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots$ equals

(a) e^{-1} (b) $\log 2$ (c) $\log e$ (d) e

10. The sum of the series

$$\log_4 2 - \log_8 2 + \log_{16} 2 - \ldots \infty \text{ is}$$

(a) e^2

(b) $\log_e 2 + 1$

(c) $\log_e 2 - 1$

(d) $1 - \log_e 2$

ANSWERS

1. (a) **2.** (c) **3.** (a) **4.** (a) **5.** (b) **6.** (a) **7.** (a) **8.** (d) **9.** (a) **10.** (d)

Matrices

24

BRIEF REVIEW OF THE CONCEPTS

MATRIX

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ) or || ||. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In compact form the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The numbers a_{11}, a_{12}, ... etc. are known as elements of the matrix A, a_{ij} belongs to the ith row and jth column and is called the (i, j)th element of the matrix $A = [a_{ij}]$.

For example, $A = \begin{bmatrix} 3 & 7 & 2 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements:

$$a_{11} = 3,\ a_{12} = 7,\ a_{13} = 2,\ a_{21} = 0,\ a_{22} = -1,\ a_{23} = 9.$$

TYPES OF MATRICES

Row Matrix A matrix having only one row is called a row matrix or a row vector. For example, $A = [2 \quad 3 \quad -4 \quad 1]$ is a row matrix of order 1×4.

Column Matrix A matrix having only one column is called a column matrix or a column vector.

For example, $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1.

Zero Matrix or Null Matrix A matrix each of whose elements is zero, is called a zero matrix or a null matrix.

For example, the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, are null matrices of order 2×2 and 2×3 respectively.

Square Matrix A matrix in which number of rows is equal to the number of columns, say n, is called a square matrix of order n.

For example, the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 4 \\ 3 & 4 & 2 \\ 5 & 3 & 6 \end{bmatrix}$ are square matrices of order 2 and 3 respectively.

Diagonal Matrix A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if all the elements except those in the leading diagonal are zero, *i.e.*, $a_{ij} = 0$ for $i \neq j$. In other words

$$A = \text{diag. } [a_{11} \quad a_{22} \quad a_{33} \quad \ldots \quad a_{nn}].$$

For example, the matrix $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is a diagonal matrix, and is denoted by $A = \text{diag. } [6 \quad 4 \quad -2]$.

Scalar Matrix A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is known as scalar matrix.

For example, the matrices $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

Unit Matrix A square matrix in which every non-diagonal element is zero and every diagonal element is 1, is called a unit matrix or an identity matrix. Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{cases}.$$

A unit matrix of order n is denoted by I_n or I. For example,

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2 and 3 respectively.

Sub Matrix A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a sub-matrix of the given matrix. For example,

$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ is a sub-matrix of the matrix $\begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & -1 \\ 6 & 3 & 5 \end{bmatrix}$.

Equality of Matrices Two matrices A and B are said to be equal if they are of same order and all the corresponding elements are equal. It is written as $A = B$. For example,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1+2 & 1+3 \\ 2+3 & 1 & 0 \end{bmatrix}$$

are equal matrices, whereas

$$C = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

are not equal, because their orders are not same.

ALGEBRA OF MATRICES

Addition of Matrices Let A and B be two matrices each of order $m \times n$. Then the sum matrix $A + B$ is defined only if matrices A and B are of same order. The new matrix, say $C = A + B$ is of order $m \times n$ and is obtained by adding the corresponding elements of A and B. Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of same order then the sum $A + B$ is defined to be the matrix of order $m \times n$ such that $A + B = C = [a_{ij} + b_{ij}]$ for all i and j.

For example, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$

then $C = A + B = \begin{bmatrix} 2+4 & 3-2 & 4+3 \\ 1+1 & 0+1 & 5+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 7 \\ 2 & 1 & 9 \end{bmatrix} =$, whereas the addition of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

is not defined since the two matrices are not of same order. If A is any matrix, the negative of A, denoted by $-A$, is the matrix obtained by replacing each entry in A by its negative. For example, if

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ -6 & 0 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -2 & 1 \\ -5 & -4 \\ 6 & 0 \end{bmatrix}.$$

Properties of Addition of Matrices

1. *Matrix addition is commutative*. If A and B are two matrices of the same order, then $A + B = B + A$.
2. *Matrix addition is associative*. If A, B and C are three matrices of the same order, then $(A + B) + C = A + (B + C)$.
3. *Existence of additive identity*. If O is the zero matrix of the same order as that of the matrix A, then $A + O = A = O + A$.
4. *Existence of additive inverse*. If A is any matrix, then $A + (-A) = O = (-A) + A$.
5. *Cancellation laws hold good in case of addition of matrices*. If A, B, C are matrices of the same order, then

   $A + B = A + C \Rightarrow B = C$ (left cancellation law)

   and $B + A = C + A \Rightarrow B = C$ (right cancellation law)

Note: The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Subtraction of Matrices

Let A and B be two matrices of the same order. Then by $A - B$, we mean $A + (-B)$. In other words, to find $A - B$ we subtract each element of B from the corresponding element of A.

For example, if $A = \begin{bmatrix} 2 & 3 \\ 6 & 1 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 \\ 2 & 5 \\ 6 & 3 \end{bmatrix}$

then $A - B = \begin{bmatrix} 2+3 & 3-4 \\ 6-2 & 1-5 \\ 7-6 & -2-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & -4 \\ 1 & -5 \end{bmatrix}$.

Mulitiplication of a Matrix by a Scalar

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar. Then the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus, if $A = [a_{ij}]_{m \times n}$, then $kA = [ka_{ij}]_{m \times n}$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -6 & 8 \\ 0 & 2 & 5 \end{bmatrix} \text{ then}$$

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & -18 & 24 \\ 0 & 6 & 15 \end{bmatrix} \text{ and } \frac{1}{2}A = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & -3 & 4 \\ 0 & 1 & \frac{5}{2} \end{bmatrix}.$$

Properties of Scalar Multiplication

1. If A and B are two matrices of the same order and k be a scalar, then

   $$k(A + B) = kA + kB.$$
2. If k_1 and k_2 are two scalars and A is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.
3. If k_1 and k_2 are two scalars and A is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$.
4. If A is any matrix, then $1A = A$.

Multiplication of Matrices

Two matrices A and B can be multiplied only if the number of columns in A (pre-multiplier) is same as the number of rows in B (first multiplier). For example, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{i,k} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \dots + a_{in} b_{nk}$$

$$= [a_{i1} \; a_{i2} \; \cdots \; a_{in}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

$= (i^{\text{th}}$ row of $A)$ $(k^{\text{th}}$ column of $B)$.

$(AB)_{ik} =$ Sum of the product of elements of i^{th} row of A with the corresponding elements of k^{th} column of B.

Remarks

1. If A and B are square matrices of the same order, say n, then both the products AB and BA are defined and each is a sqaure matrix of order n.
2. In the matrix product AB, the matrix A is called pre-multiplier (prefactor) and B is called post-multiplier (post-factor).

3. The rule of multiplication of matrices is row column wise (or → ↓ wise), viz. the first row of AB is obtained by multiplying the first row of A with first, second, third, ... columns of B respectively. Similarly second row of A with first, second, third, ... columns of B respectively and so on.

$$\text{Let } A = \begin{bmatrix} 1 & -4 \\ 5 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 4 & 1 & 6 \\ 2 & 7 & 3 & 8 \end{bmatrix}$$

be two matrices.

Since the number of columns in A are equal to the number of rows in B, the product AB is defined. As order of matrix A is 3×2 and B is 2×4, the product AB will be of order 3×4.

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}.$$

The entry c_{11} is obtained by summing the products of each entry in row 1 of A by the corresponding entry in column 1 of B, i.e.,

$$c_{11} = (1)\,(-2) + (-4)\,(2) = -10.$$

Similarly, for c_{21}, we use the entries in row 2 of A and those in column 1 of B:

$$c_{21} = (5)\,(-2) + (3)\,(2) = -4.$$

Also,
$$\begin{aligned} c_{12} &= (1)\,(4) + (-4)\,(7) = -24 \\ c_{13} &= (1)\,(1) + (-4)\,(3) = -11 \\ c_{14} &= (1)\,(6) + (-4)\,(8) = -26 \\ c_{22} &= (5)\,(4) + (3)\,(7) = 41 \\ c_{23} &= (5)\,(1) + (3)\,(3) = 14 \\ c_{24} &= (5)\,(6) + (3)\,(8) = 54 \\ c_{31} &= (0)\,(-2) + (2)\,(2) = 4 \\ c_{32} &= (0)\,(4) + (2)\,(7) = 14 \\ c_{33} &= (0)\,(1) + (2)\,(3) = 6 \\ c_{34} &= (0)\,(6) + (2)\,(8) = 16. \end{aligned}$$

$$\text{Thus, } AB = \begin{bmatrix} -10 & -24 & -11 & -26 \\ -4 & 41 & 14 & 54 \\ 4 & 14 & 6 & 16 \end{bmatrix}$$

The product BA is not defined since the number of columns of B is not equal to the number of rows of A. This shows that matrix multiplication is not commutative. That is, for any two matrices A and B, it is usually the case that $AB \neq BA$ (even if both products are defined).

Properties of Matrix Multiplication

1. *Multiplication is distributive over matrix addition.* If A, B, C are $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then
$$A(B + C) = AB + AC.$$
2. *Multiplication is associative.* If A, B, C are matrices of order $m \times n$, $n \times p$ and $p \times r$ respectively, then
$$(AB)C = A(BC)$$
3. *Multiplicative identity.* If A is an $m \times n$ matrix and I_n the identity matrix of order $n \times n$ and I_m the identity matrix of order $m \times m$, then
$$I_m A = A \text{ and } AI_n = A.$$
In particular if A is a square matrix of order n, then
$$AI_n = I_n A = A.$$
4. $AB = 0$ (null matrix) does not necessarily imply that $A = 0$ or $B = 0$ or both $= 0$. For example, if
$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq 0 \text{ and}$$
$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq 0, \text{ then } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
5. If A is a square matrix of order n, then A^2 is defined as AA. In general $A^m = AA \ldots A$ (m times), where m is any positive integer.
6. If I be a unit matrix, then $I = I^2 = I^3 = \ldots = I^n$.

TRANSPOSE OF A MATRIX

Let A be an $m \times n$ matrix. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A, and is denoted by A' or A'. Thus,

1. if order of A is $m \times n$, then, the order of A' is $n \times m$.
2. (i, j)th element of A = (j, i)th element of A'.

$$\text{For example, if } A = \begin{bmatrix} 2 & -3 & -1 \\ 4 & 2 & 3 \end{bmatrix}, \text{ then}$$

$$A' = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}_{3\times 2}.$$

Properties of the Transpose of a Matrix

1. Let A and B be two matrices of order $m \times n$, then $(A \pm B)' = A' \pm B'$.
2. Let A be a matrix of order $m \times n$ and k be a scalar, then $(kA)' = kA'$.
3. Let A and B be two matrices of order $m \times n$ and $n \times p$ respectively. Then, $(AB)' = B'A'$.
4. The double transpose of any matrix is the original matrix. For exampe, if A is any marix, then $(A')' = A$.

SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A' = A$. That is, the matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric provided $a_{ij} = a_{ji}$ for all i and j.

$$\text{For example, } A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix} \text{ is symmetric, since}$$

$$A' = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix} = A.$$

SKEW SYMMETRIC MATRIX

A square matrix A is said to be skew symmetric, if $A' = -A$. That is, the matrix $A = [a_{ij}]_{n \times n}$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

$$\text{For example, } A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix} \text{ is skew symmetric, since}$$

$$A' = \begin{bmatrix} 0 & -5 & -7 \\ 5 & 0 & -3 \\ 7 & 3 & 0 \end{bmatrix} = -A$$

Remark: Elements of main diagonal of a skew-symmetric matrix are all zero, because by definition,

$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \text{ or } a_{ii} = 0 \text{ for all values of } i.$$

Properties of Symmetric and Skew-Symmetric Matrices

1. If A is a square matrix, then
   (a) $A + A'$ is symmetric
   (b) $A - A'$ is skew-symmetric.
2. If A and B are two symmetric (or skew-symmetric) matrices of the same order, then so is $A + B$.
3. If A is symmetric (or skew-symmetric) matrix and k is a scalar, then kA is also symmetric (or skew-symmetric).
4. If A and B are symmetric matrices of the same order, then the product AB is symmetric if and only if $AB = BA$.
5. Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.
6. The matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.
7. All positive integral powers of a symmetric matrix are symmetric.
8. All positive integral powers of a symmetric matrix are symmetric.
9. All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.
10. If A and B are symmetric matrices of the same order, then
    (a) $AB - BA$ is a skew-symmetric matrix.
    (b) $AB + BA$ is a symmetric matrix.

SINGULAR MATRIX

A square matrix A is said to be singular matrix if determinant of A denoted by det A or $|A|$ is zero, i.e., $|A| = 0$, otherwise, it is a non-singular matrix.

For example, the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{bmatrix}$

is singular as $\begin{vmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{vmatrix} = 0.$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in the determinant $|A|$. Then the adjoint of A, denoted by *adj A*, is defined as the transpose of the cofactor matrix.

The adjoint of a square matrix A is obtained on replacing each (i, j)th element of A by the cofactor of the (j, i)th element in $|A|$.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 3 & 2 \end{bmatrix}$, then we have

$C_{11} = -3$ $\quad C_{12} = 6$ $\quad C_{13} = -3$
$C_{21} = 5$ $\quad C_{22} = -10$ $\quad C_{23} = 5$
$C_{31} = 2$ $\quad C_{32} = -4$ $\quad C_{33} = 2$

Thus, adj $A = \begin{bmatrix} -3 & 5 & 2 \\ 6 & -10 & -4 \\ -3 & 5 & 2 \end{bmatrix}.$

Properties of the Adjoint of a Matrix

1. If A is a square matrix of order n, then
   $$A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A,$$
   where I_n is a square matrix of order n.
2. If A is a square matrix of order n, then adj $(A') = (\text{adj } A)'$.
3. If A and B are two square matrices of the same order, then adj $(AB) = (\text{adj } B)\,(\text{adj } A)$.
4. adj (adj A) $= |A|^{n-2}A$, where A is a non-singular matrix.
5. $|\text{adj (adj } A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix.
6. Adjoint of a diagonal matrix is a diagonal matrix.

INVERSE OF A SQUARE MATRIX

Let A be any n-rowed square matrix. Then a square matrix B, such that $AB = BA = I$, is called inverse of A. The inverse of A denoted by A^{-1} is determined by the formula:

$$A^{-1} = \frac{1}{|A|} (\text{adj } A).$$

It may be noted that

$$AA^{-1} = A^{-1}A = I.$$

Properties of the Inverse of a Matrix

1. A square matrix is invertible if and only if it is non-singular.
2. The inverse of the inverse is the original matrix itself, i.e., $(A^{-1})^{-1} = A$.
3. The inverse of the transpose of a matrix is the transpose of its inverse, i.e., $(A')^{-1} = (A^{-1})'$.
4. If A and B are two invertieble matrices of the same order, then AB is also invertible and moreover
   $(AB)^{-1} = B^{-1}A^{-1}$.
5. Let A, B, C be square matrices of the same order n. If A is a non-singular matrix, then
   (a) $AB = AC \Rightarrow B = C$ (Left cancellation law)
   (b) $BA = CA \Rightarrow B = C$ (Right cancellation law)
   Note that these cancellation laws hold only if the matrix A is non-singular.
6. If A is a non-singular matrix such that A is symmetric then A^{-1} is also symmetric.
7. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$.

ELEMENTARY TRANSFORMATIONS

Any one of the following operations on a matrix is called an elementary transformation.

1. Interchanging any two rows (or columns). This transformation is indicated by $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$
2. Mulitplication of the elements of any row (or column) by a non-zero scalar quantity and indicated as $R_i \leftrightarrow kR_i (C_i \leftrightarrow kC_i)$
3. Addition of constant multiple of the elements of any row to the corresponding element of any other row, indicated as $R_i \to R_i + kR_j$.

Two matrices are said to be **equivalent** if one is obtained from the other by elementary transformations. The symbol ≈ is used for **equivalence**.

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots \quad \vdots \quad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n.$$

We can express these equations as a single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & A & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad B$$

Let $|A| \neq 0$, so that A^{-1} exists uniquely. Pre-multiplying both sides of $AX = B$ by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \quad \text{or} \quad (A^{-1}A)X = A^{-1}B$$
$$\text{or} \qquad IX = A^{-1}B \quad \text{or} \quad X = A^{-1}B.$$

Hence $X = A^{-1}B$ is the unique solution of $AX = B$, $|A| \neq 0$.

Criterion of Consistency

Let $AX = B$ be a system of n linear equations in n variables.

1. If $|A| \neq 0$, then the system of equations in consistent and has a unique solution given by $X = A^{-1}B$.
2. If $|A| = 0$ and (adj $A)B = 0$, then the system of equations is consistent and has infinitely many solutions.
3. If $|A| = 0$ and (adj $A)B \neq 0$, then the system of equations is inconsistent, i.e., it has no solution.

Homogeneous Equations

The system of equations $AX = B$ is said to be homogeneous if the constants $b_1, b_2 \ldots b_n$ are all zero. That is, if the matrix B is a zero matrix and the system is of the form

$$AX = O$$

where O is a null matrix of order $n \times 1$.

1. If $|A| \neq 0$, then its only solution $X = 0$, is called the trivial solution.
2. If $|A| = 0$, then $AX = O$ have a non-trivial solution. It will have infinitely many solutions.

Solved Problems

1. The inverse of a symmetric matrix is

(a) diagonal matrix

(b) symmetric matrix

(c) skew-symmetric matrix

(d) none of these

Solution (b) Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then

$$A_{11} = bc - f^2, A_{22} = ac - g^2, A_{33} = ab - h^2$$
$$A_{12} = -(hc - gf) = gf - hc = A_{21},$$
$$A_{13} = hf - bg = A_{31}$$
$$A_{23} = -(af - hg) = gh - af = A_{32}$$

Since $A_{12} = A_{21}, A_{13} = A_{31}, A_{23} = A_{32}$

$\therefore$ A^{-1} is **symmetric.**

2. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ then the value of $|\text{adj } A|$ is

(a) a^{27} (b) a^9

(c) a^6 (d) a^2

Solution (c) Cofactor matrix = $\begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$$\therefore \quad \text{adj } A = (\text{cofactor Matrix})' = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$$

$$\therefore \quad |\text{adj } A| = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} = \boldsymbol{a^6}.$$

3. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$, then A^{4n} equals

(a) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Solution (c) We have,

$$A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore \quad A^{4n} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{n \text{ times}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. If $AB = A$ and $BA = B$, then B^2 is equal to

(a) B (b) A

(c) 1 (d) 0

Solution (a) Since $BA = B$, $\therefore$ $(BA)\,B = BB = B^2$

$\Rightarrow \quad B\,(AB) = B^2 \Rightarrow BA = B^2 \qquad (\because AB = A)$

$\Rightarrow \quad \boldsymbol{B = B^2}. \qquad (\because BA = B)$

5. If the matrix AB is zero, then

(a) It is not necessary that either $A = 0$ or $B = 0$

(b) $A = 0$ or $B = 0$

(c) $A = 0$ and $B = 0$

(d) all the above statements are wrong

Solution (a) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $A \neq 0$, $B \neq 0$ but $AB = 0$.

6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that $A = BX$ then X is equal to

(a) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (d) none of these

Solution (a) Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Since $A = BX$,

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

$\therefore \quad a = 1,\ b = 2,\ 2c = 3,\ 2d = -5$

i.e., $\quad c = \frac{3}{2},\ d = -\frac{5}{2}$.

$$\therefore \quad X = \begin{bmatrix} 1 & 2 \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}.$$

7. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then det. (adj (adj A)) is

(a) $(14)^4$ (b) $(14)^3$

(c) $(14)^2$ (d) $(14)^1$

Solution (a) We know that adj (adj A) $= |A|^{n-2} A$ if $|A| \neq 0$, provided order of A is n.

$\therefore \quad$ adj (adj A) $= |A|\, A$ (as $n = 3$)

$\therefore \quad$ det (adj (adj A)) $= |A|^3$ det $A = |A|^4$.

$$\text{But } |A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 14.$$

$\therefore \quad$ det (adj (adj A)) $= \mathbf{(14)^4}$.

8. For any 2×2 matrix A, if A (adj A) $= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ is equal to

(a) 20 (b) 100

(c) 10 (d) 0

Solution (c) Since A (adj A) $= |A|\, I$,

$$\therefore \quad \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$$

$\therefore \quad |A| = \mathbf{10}$.

9. If A is a square matrix of order n, then det (adj A) =

(a) $(\det A)^{n-1}$ (b) $(\det A)^{n-2}$

(c) $(\det A)^n$ (d) none of these

Solution (a) We know that

$$|A \cdot \text{adj } A| = \begin{vmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |A| \end{vmatrix}$$

$$= |A|^n$$

$\Rightarrow \quad |A|\,|\text{adj } A| = |A|^n \ (\because |AB| = |A|\,|B|)$

$\Rightarrow \quad |\text{adj } A| = \mathbf{|A|^{n-1}}$.

10. If the system of equations $\lambda x + 2y - 2z = 1$, $4x + 2\lambda y - z = 2$, $6x + 6y + \lambda z = 3$ has a unique solution, then

(a) $\lambda \neq 1$ (b) $\lambda \neq 2$

(c) $\lambda \neq 3$ (d) none of these

Solution (b) The given system of equations can be written in a single matrix equation:

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The given system of equations will have a unique solution if and only if the coefficient matrix is non-singular i.e., if

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \neq 0.$$

$\Rightarrow$ $\lambda (2\lambda^2 + 6) - 2 (4\lambda + 6) - 2 (24 - 12\lambda) \neq 0$

or $\lambda^3 + 11\lambda - 30 \neq 0.$

or $(\lambda - 2) (\lambda^2 + 2\lambda + 15) \neq 0$ or $\boldsymbol{\lambda \neq 2}$

(only real root).

11. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$

(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) none of these

Solution (d) We have,

$$X^2 = X \cdot X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

For $n = 2$, matrices in (a), (b) and (c) do not match with

$$\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}.$$

12. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2 =$

(a) $2AB$ (b) $2BA$

(c) $A + B$ (d) AB

Solution (c) We have,

$A^2 + B^2 = AA + BB = A (BA) + B (AB)$

($\because$ $AB = B$ and $BA = A$)

$= (AB) A + (BA) B$

$= BA + AB = \boldsymbol{A + B}$ ($\because$ $AB = B$ and $BA = A$)

13. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

(a) $k \neq 0$ (b) $-1 < k < 1$

(c) $-2 < k < 2$ (d) $k = 0$

Solution (a) The given system of equations has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow \boldsymbol{k \neq 0.}$$

14. If A is a square matrix, then

(a) AA' is symmetric (b) AA' is skew-symmetric

(c) $A'A$ is symmetric (d) $A'A$ is skew symmetric

Solution (a), (c) We have,

$(AA')' = (A')' \cdot A''$ [$\because$ $(AB)' = B'A'$]

$= A'A'$ [$\because$ $(A')' = A$]

$\Rightarrow$ AA' is **symmetric.**

Also, $(A'A)' = A' \cdot (A')'$ [$\because$ $(AB)' = B' \cdot A'$]

$= A' \cdot A$ [$\because$ $(A')' = A$]

$\Rightarrow$ $A'A$ is **symmetric.**

15. Which of the following is correct

(a) $B'AB$ is symmetric if A is symmetric

(b) $B'AB$ is skew-symmetric if A is symmetric

(c) $B'AB$ is symmetric if A is skew-symmetric

(d) $B'AB$ is skew-symmetric if A is skew-symmetric

Solution (a), (d) Let A be a symmetric matrix.

Then $A' = A$.

Now, $(B'AB)' = B'A' (B')'$ [$\because$ $(AB)' = B'A'$]

$= B'A'B$ [$\because$ $(B')' = B$]

$= B'AB$ [$\because$ $A' = A$]

$\Rightarrow$ $\boldsymbol{B'AB}$ **is a symmetric matrix.**

Now, let A be a skew-symmetric matrix.

Then $A' = -A$.

$\therefore$ $(B'AB)' = B'A' (B')'$ [$\because$ $(AB)' = B'A'$]

$= B'A'B$ [$\because$ $(B')' = B$]

$= B' (-A) B$ [$\because$ $A' = -A$]

$= -B'AB$

$\therefore$ $\boldsymbol{B'AB}$ **is a skew-symmetric matrix.**

16. Let A and B be two non-null square matrices. If the product AB is a null matrix, then

(a) A is singular (b) B is singular

(c) A is non-singular (d) B is non-singular

Solution (a), (b) Let B be non-singular, then B^{-1} exists.

Now, $AB = O$ (given)

$\Rightarrow$ $(AB) B^{-1} = OB^{-1}$

(post-multiplying both sides by B^{-1})

$\Rightarrow$ $A (BB^{-1}) = O$ (by associativity)

$\Rightarrow$ $AI_n = O$ ($\because$ $BB^{-1} = I_n$)

$\Rightarrow$ $A = O$.

But A is a non-null matrix. Hence $\boldsymbol{B}$ **is a singular matrix.** Similarly it can be shown that $\boldsymbol{A}$ **is a singular matrix.**

17. If A is symmetric as well as skew symmetric matrix, then A is

(a) diagonal (b) null

(c) triangular (d) none of these

Solution (b) Let $A = [a_{ij}]$

Since A is skew-symmetric,

$\therefore$ $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ $(i \neq j)$.

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j.

$\therefore$ $a_{ij} = 0$ for all $i \neq j$.

Hence $a_{ij} = 0$ for all i and j i.e., A is a **null matrix.**

18. If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined. Then B is of the type

(a) 3×4 (b) 3×3

(c) 4×4 (d) 4×3

Solution (a) Let the order of B be $m \times n$.

Since A is 3×4 matrix, $\therefore$ A' is 4×3 matrix.

Since $A'B$ is defined, $\therefore$ number of columns of A' must be equal to number of rows of B, $\therefore$ $m = 3$.

Also, since BA' is defined, $\therefore$ number of columns of B must be equal to number of rows of A', $\therefore$ $n = 4$.

$\therefore$ B is **3 × 4** matrix.

19. If A and B are square matrices of same order such that $(A + B)^2 = A^2 + B^2 + 2AB$, then

(a) $AB = BA$ (b) $A = B$

(c) $A = B'$ (d) $A = -B$

Solution (a) We have, $(A + B)^2 = A^2 + B^2 + 2AB$

$\Rightarrow$ $(A + B)(A + B) = A^2 + B^2 + 2AB$

$\Rightarrow$ $A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB$

$\Rightarrow$ $\mathbf{BA = AB}$.

20. If the system of linear equations

$x + 2ay + az = 0$

$x + 3by + bz = 0$

$x + 4cy + cz = 0$

has a non-zero solution, then a,b,c

(a) are in A.P. (b) are in G.P.

(c) are in H.P.` (d) satisfy $a+2b+3c = 0$

Solution (c) The system of linear equations has a non-zero solution, then

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\begin{vmatrix} 1 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0$$

$\Rightarrow$ $(3b - 2a)(c - a) - (4c - 2a)(b - a) = 0$

$\Rightarrow$ $3bc - 3ba - 2ac + 2a^2 = 4bc - 2ab - 4ac + 2a^2$

$\Rightarrow$ $4ac - 2ab = 2ab - 3bc + 3ab$

$\Rightarrow$ $2ac = bc + ab$

Dividing by abc, we get

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

i.e., a, b, c, are in HP.

21. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a) $\alpha = a^2 + b^2$, $\beta = ab$

(b) $\alpha = a^2 + b^2$, $\beta = 2ab$

(c) $\alpha = a^2 + b^2$, $\beta = a^2 - b^2$

(d) $\alpha = 2ab$, $\beta = a^2 + b^2$

Solution (b) $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2+b^2 & ab+ba \\ ba+ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

$\Rightarrow$ $\alpha = a^2 + b^2$, $\beta = 2ab$

22. Let $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The only correct statement about the matrix A is

(a) A^{-1} does not exist

(b) $A = (-1)I$, where I is a unit matrix

(c) A is a zero matrix (*d*) $A^2 = I$

Solution (d) (i) Clearly A is not a zero matrix

(ii) $(-1)I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \neq A$

(iii) $|A| = 1 \neq 0 \Rightarrow A^{-1}$ exist.

(iv) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

23. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and $(10)\ B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$. If B is the inverse of matrix A, then α is

(a) 2 (b) –1

(c) –2 (d) 5

Solution (d) Given, $\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix} = 10A^{-1}$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} = 10I$$

$$= \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

$\Rightarrow$ $-5 + \alpha = 0 \Rightarrow \alpha = 5$ (Equating the element of 2nd row and first column).

Problems for Practice

Level of Difficulty I

1. If $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$, then A^{-1} equals

(a) $-\frac{1}{8}\begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$ (b) $-\frac{1}{8}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

(c) $\frac{1}{8}\begin{bmatrix} -1 & -3 \\ -2 & 2 \end{bmatrix}$ (d) none of these

2. The matrix product $\begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ equals

(a) $\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$ (d) none of these

3. The matrix $\begin{bmatrix} 0 & 5 & -7 \\ -5 & 0 & 11 \\ 7 & -11 & 11 \end{bmatrix}$ is known as

(a) skew-symmetric matrix

(b) symmetric matrix

(c) diagonal matrix

(d) upper triangular matrix

4. The value of x for which the matrix product

$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$

equals an identity matrix is

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

5. If $[1 \;\; x \;\; 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$, then x is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) -1

6. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, then AB is equal to

(a) B (b) A (c) 0 (d) I

7. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^2 + 2A$ equals

(a) $4A$ (b) $3A$ (c) $2A$ (d) A

8. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$ then

(a) $AB = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$

(b) $AB = [-2 \;\; -1 \;\; 4]$

(c) $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(d) AB does not exist

9. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then A^2 is equal to

(a) $\begin{bmatrix} \sin 2\alpha & \cos 2\alpha \\ \cos 2\alpha & -\sin 2\alpha \end{bmatrix}$

(b) $\begin{bmatrix} \cos 2\alpha & -\sin\alpha \\ -\sin\alpha & \cos 2\alpha \end{bmatrix}$

(c) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

10. If $A = (a_{ij})_{2\times 2}$, where $a_{ij} = i + j$, then A is equal to

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

11. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then A (adj A) is equal to

(a) $\begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ (d) none of the above

12. If a 3 × 3 matrix A has its inverse equal to A, then A^2 is equal to

(a) $\begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}$ (b) $\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}$

(c) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

13. If $[1\ \ x\ \ 1]\begin{bmatrix}1&3&2\\0&5&1\\0&3&2\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix} = 0$, then x is equal to

(a) 1 (b) -1

(c) $\dfrac{-9\pm\sqrt{53}}{2}$ (d) none of these

14. If $A = \begin{bmatrix}1&0\\-1&7\end{bmatrix}$ and $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, then the value of k so that $A^2 = 8A + kI$ is

(a) $k = 7$ (b) $k = -7$ (c) $k = 0$ (d) none of these

15. If $A = \begin{bmatrix}3&-5\\-4&2\end{bmatrix}$, then $A^2 - 5A - 14I$ is equal to

(a) $\begin{bmatrix}2&-3\\1&0\end{bmatrix}$ (b) $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

(c) $\begin{bmatrix}1&-3\\-1&0\end{bmatrix}$ (d) none of these

Level of Difficulty II

16. If $U = [2\ \ -3\ \ 4]$, $V = \begin{bmatrix}3\\2\\1\end{bmatrix}$, $X = [0\ \ 2\ \ 3]$ and

$$Y = \begin{bmatrix}2\\2\\4\end{bmatrix} \text{ then } UV + XY =$$

(a) 20 (b) $[-20]$ (c) -20 (d) $[20]$

17. Which of the following is correct?

(a) skew symmetric matrix of even order is always singular

(b) skew symmetric matrix of odd order is non-singular

(c) skew symmetric matrix of odd order is singular

(d) none of the above

18. The equations $x + 2y + 3z = 1$, $x - y + 4z = 0$, $2x + y + 7z = 1$ have

(a) only two solutions

(b) only one solutions

(c) no solution

(d) infinitely many solutions

19. Let A be an invertible matrix, which of the following is not true?

(a) $(A')^{-1} = (A^{-1})'$ (b) $A^{-1} = |A|^{-1}$

(c) $(A^2)^{-1} = (A^{-1})^2$ (d) none of these

20. If A is a singular matrix, then adj A is

(a) non-singular (b) singular

(c) symmetric (d) not defined

21. If B is a non-singular matrix and A is a square matrix, then det $(B^{-1}AB)$ is equal to

(a) det (A^{-1}) (b) det (B^{-1})

(c) det (A) (d) det (B)

22. If A is a square matrix, then AA' is a

(a) skew-symmetric matrix

(b) symmetric matrix

(c) diagonal matrix

(d) none of these

23. If A, B are two $n \times n$ non-singular matrices, then

(a) AB is non-singular

(b) AB is singular

(c) $(AB)^{-1} = A^{-1}B^{-1}$

(d) $(AB)^{-1}$ does not exist

24. If A is a square matrix such that $A^2 = I$, then A^{-1} is equal to

(a) $A + I$ (b) A

(c) O (d) $2A$

25. If A and B are two matrices such that $A + B$ and AB are both defined, then

(a) A, B are square matrices of same order

(b) number of columns of A = number of rows of B

(c) A and B can be any matrices

(d) none of these

26. The multiplicative inverse of $A = \begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$ is

(a) $\begin{bmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{bmatrix}$

(b) $\begin{bmatrix}-\cos\theta&-\sin\theta\\\sin\theta&-\cos\theta\end{bmatrix}$

(c) $\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$

(d) $\begin{bmatrix}-\cos\theta&-\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$

27. If A and B are any 2×2 matrices, then det $(A + B) = 0$ implies

(a) det $A = 0$ and det $B = 0$

(b) det A + det $B = 0$

(c) det $A = 0$ or det $B = 0$

(d) none of these

28. If $A = \begin{bmatrix}3&1&-1\\0&1&2\end{bmatrix}$, then AA' is

(a) symmetric matrix

(b) skew-symmetric matrix

(c) orthogonal matrix

(d) none of these

29. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then $A^2 =$

(a) A (b) $-A$

(c) $2A$ (d) $-2A$

30. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to

(a) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (b) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$

(c) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (d) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

ANSWERS

1. (a)	**2.** (a)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (d)
11. (b)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (b)
21. (c)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (a)

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Determinants

BRIEF REVIEW OF THE CONCEPTS

DETERMINANTS

A determinant is a pure number associated with a square matrix. Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

there is associated an expression, called the *determinant of A, denoted by det A* or $|A|$, written as

$$|A| = \det A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

A matrix is an arrangement of numbers and it has no fixed value but a determinant is a number and it has a fixed value. A determinant having n rows and n columns is called a determinant of order n.

Determinant of a Square Matrix of Order 1

Let $A = [a_{11}]$ be a 1×1 matrix, then the determinant of A is the number a_{11} itself i.e., $|a_{11}| = a_{11}$.

Determinant of a Square Matrix of Order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

i.e., the determinant of a 2×2 matrix is obtained by taking the product of the entries on the main diagonal and subtracting from it the product of the entries in the other diagonal.

Determinant of a Square Matrix of Order 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then the expression $a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{32}\, a_{21} - a_{11}\, a_{23}\, a_{32} - a_{22}\, a_{13}\, a_{31} - a_{12}\, a_{21}\, a_{33}$ is defined as the determinant of A i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}\,(a_{21}\, a_{33} - a_{31}\, a_{23}) + a_{13}\,(a_{32}\, a_{21} - a_{22}\, a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The above expansion of $|A|$ is known as the expansion along the first row.

Working Rule to Expand a Determinant of Order 3

- Write the elements of the first row with alternatively positive and negative sign, the first element always has positive sign before it.
- Multiply each signed element by the determinant of second order obtained after deleting the row and the column in which that element occurs.

Notes:

- The value of determinant is same when expanded by any row or any column.
- The above method of expansion is general and is valid for determinant of any order.
- If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.
- Always expand a determinant along a row or column with maximum number of zeros.
- If each element above or below the main diagonal of a determinant is zero, then the value of the determinant is the product of elements along the main diagonal.

Sarrus Diagram for Expansion of a Determinant of Order 3

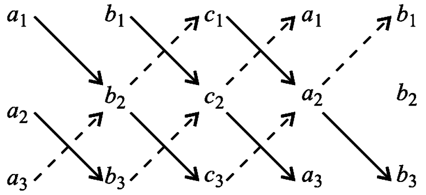
The following diagram called sarrus diagram, enables us to write the value of the determinant of order 3 very conveniently.

Working Rule

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3.

Write the elements as shown in the diagram below.

Multiply the elements joined by arrows. Assign the positive sign to an expression if it is formed by a downward arrow and negative sign to an expression if it is formed by an upward arrow. Note that the first two columns are repeated in the above table to complete the process. The value of the given determinant



$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1.$$

Remark: Sarrus rule does not work for determinants of order greater than 3.

MINORS AND COFACTORS

Minor

The minor of any element a_{ij} in $|A|$ is a determinant of the sub-matrix obtained from A, by deleting the row and the column containing a_{ij}. It is denoted by M_{ij}. Consider the determinant of a square matrix A of order 3. That is,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactor

The cofactor of any element a_{ij}, denoted by C_{ij}, is defined as $(-1)^{i+j}M_{ij}$ where $i+j$ is the sum of the row number i and column number j in which the element a_{ij} lies. That is,

$$C_{ij} = (-1)^{i+j} \times \text{minor of the element } a_{ij}$$
$$= (-1)^{i+j} M_{ij}$$

and $$C_{ij} = \begin{bmatrix} \mathrm{M}_{ij} & \text{if } i+j \text{ is even} \\ -\mathrm{M}_{ij} & \text{if } i+j \text{ is odd} \end{bmatrix}$$

Consider the determinant of a square matrix A of order 3, i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
$$= (a_{22}a_{33} - a_{32}a_{23}).$$

Cofactor of a_{12} is

$$C_{12} = (-1)^{1+2}M_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$= -(a_{21}a_{33} - a_{31}\,a_{23}).$$

and so on.

***Note*:** The sum of products of elements of a row (or column) with their corresponding cofactors = det A.

For example, in the above case (ii)

$$|A| = (-1)C_{11} + (2)C_{12} + (0)C_{13}$$
$$= -1 \times 7 + 2 \times 1 + 0 \times 5 = -5.$$

PROPERTIES OF DETERMINANTS

Properties of determinants of order three only are stated below. However these properties hold for determinants of any order. These properties help a good deal in the evaluation of determinants.

1. The value of the determinant remains unchanged if rows are changed into columns and columns are changed into rows, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

2. If two adjacent rows (columns) of a determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

3. If two rows or columns of a determinant are identical then its value is zero, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

4. If each element of a row or column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant, i.e.,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

5. If any two rows or columns of a determinant are proportional, then its value is zero, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_1 & c_2 & c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

6. If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of the two or more determinants, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & a_{33}+ka_{23} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

7. If each element of a row (column) of a determinant is multiplied by a constant k and then added to the corresponding elements of some other row (column), then the value of the determinant remains the same, i.e.,

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & a_{33}+ka_{23} \end{vmatrix}.$$

8. If each element of a row (column) of a determinant is zero, then its value is zero.
9. If the elements of a determinant that involve x are polynomials in x, and if the determinant is equal to zero when a is substituted for x, the $x - a$ is a factor of given determinant.

Note: In general, if r rows (or r columns) become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

10. The sum of the products of the elements of any row (or column) of a determinant with the corresponding co-factors is equal to the value of determinant, i.e., if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta \text{ and so on.}$$

11. The sum of the products of elements of any row (or column) of a determinant with the co-factors of the corresponding elements of any other row (or column) is zero, i.e., if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0 \text{ and so on.}$$

Note: In $\Delta = |a_{ij}|$ is a determinant of order n, then the value of the determinant $|A_{ij}|$, where A_{ij} is the cofactor of a_{ij} is Δ^{n-1}.

$$\text{For example, } \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2.$$

EVALUATION OF DETERMINANTS USING ELEMENTARY OPERATIONS

To evaluate determinants of higher order, we should always try to introduce zeros at the maximum number of places in a particular row (column) by using the properties of the determinant. We denote the rows of the determinant by R_1, R_2, R_3, ... and columns by C_1, C_2, C_3, ...

We shall use the following notations to evaluate a determinant.

1. The operation of interchanging the ith row and jth row will be denoted by $R_i \leftrightarrow R_j$.
2. The operation of multiplying each element of the ith row by a number k will be denoted by $R_i \to kR_i$.
3. The operation of adding to each element of the ith row, k times the corresponding elements of the jth row ($j \neq i$) will be denoted by $R_i \to R_i + kR_j$.

Similar notations are used for operations on columns replacing R by C.

SOLUTION OF LINEAR EQUATIONS BY DETERMINANTS

1. Cramer's Rule: Solution of system of linear equations in two unknowns The solution of the system of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is given by $x = \dfrac{D_1}{D}$ and $y = \dfrac{D_2}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, provided $D \neq 0$.

2. Cramer's Rule: Solution of system of linear equations in three unknowns The solution of the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by

$$x = \frac{D_1}{D}, y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}, \text{ where}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}; \quad D_3 = ;$$

provided $D \neq 0$.

Conditions for consistency The following cases may arise:

1. If $D \neq 0$, then the system is consistent and has a unique solution, which is given by Cramer's rule:

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}.$$

2. If $D = 0$ and atleast one of the determinants D_1, D_2, D_3 is non-zero, the given system is inconsistent, i.e., it has no solution.
3. If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system is consistent and dependent, and has infinitely many solutions.

Homogeneous and Non-homogeneous System

If $D_1 = D_2 = D_3 = 0$, then the system is said to be, *homogeneous*, otherwise it is called non-homogeneous.

If the system of equations is homogeneous, then $D_1 = D_2 = D_3 = 0$ (value of the determinant is zero, if one column has all elements = 0). Thus,

1. if $D \neq 0$, the system has only trivial solution ($x = y = z = 0$), and
2. if $D = 0$, the system has infinitely many solutions.

Solved Problems

1. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ if

(a) x, y, z are in A.P (b) x, y, z are in G.P.

(c) x, y, z are in H.P (d) xy, yz, zx are in A.P.

Solution (b) The given determinant

$$= \begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ -(xp^2+2yp+z) & 0 & 0 \end{vmatrix}$$

[Applying $R_3 \to R_3 - pR_1 - R_2$]

$= -(xp^2 + 2yp + z)(xz - y^2)$

$= 0$ if **x, y, z are in G. P.**

2. If $\alpha = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$, then

(a) $\alpha = -1$ (b) $\alpha = 1$

(c) $\alpha = 0$ (d) none of these

Solution (c) Operate $R_1 \to R_1 + R_2 + R_3$, we get

$$\alpha = \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \mathbf{0}.$$

3. If $a \neq b \neq c$, one value of x which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ is given by

(a) $x = 0$ (b) $x = c$

(c) $x = b$ (d) $x = a$

Solution (a) For $x = 0$, the determinant reduces to the determinant of a skew symmetric matrix of odd order which is always zero. Hence $\mathbf{x = 0}$ is the solution of the given equation.

4. If T_p, T_q, T_r are pth, qth, rth terms of an A.P, then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) $p+q+r$ (b) 0

(c) -1 (d) 1

Solution (b) We have, $T_p = a + (p-1)d$, $T_q = a + (q-1)d$, $T_r = a + (r-1)d$, where a is the first term and d is the common difference.

$\therefore$ The given determinant

$$= \begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \text{ [Applying } R_1 \to R_1 - (R_2 - R_3)d \text{]}$$

$$= a \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{0}.$$

5. If $a_i, b_i, c_i \in R$ (i = 1, 2, 3) and $x \in R$ and $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = 0$, then

(a) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 4$

(b) $x = \pm 1$ (c) $x = 2$

(d) none of these

Solution (b) Clearly $\mathbf{x = \pm 1}$ satisfies the given equation.

6. If $a + b + c = 0$, one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

(a) $x = 0$ (b) $x = 1$

(c) $x = 2$ (d) $x = a^2 + b^2 + c^2$

Solution (a) Operate $C_1 \to C_1 + C_2 + C_3$, we get

$$(a+b+c-x) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

i.e., $$x \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$\therefore$ $\mathbf{x = 0}$ is one of the roots of

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

7. The roots of the equation $\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0$ are independent of

(a) α (b) β

(c) γ (d) all of these

Solution (a) We have, $\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-\beta & \alpha-x & 0 \\ 0 & x-\gamma & 0 \\ \beta & \gamma & 1 \end{vmatrix} = 0$$

[Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$]

$\Rightarrow$ $(x - \beta)(x - \gamma) = 0$ [Expanding along C_3]

Clearly, the roots are independent of α.

8. If $f(x) = \begin{vmatrix} \cos^2 x & \cos x\cdot\sin x & -\sin x \\ \cos x\sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, then for all x

(a) $f(x) = 0$ (b) $f(x) = 1$
(c) $f(x) = 2$ (d) none of these

Solution (b) We have,

$$f(x) = \begin{vmatrix} \cos^2 x & \cos x\cdot\sin x & -\sin x \\ \cos x\sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

[Applying $C_1 \to C_1 - \sin x \cdot C_3$ and $C_2 \to C_2 + \cos x \,.\, C_3$]

$$= \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ 0 & -\cos x & \sin^2 x \end{vmatrix}$$

[Applying $R_3 \to R_3 - \sin x\cdot R_1$]

$= \sin^2 x + \cos^2 x = \mathbf{1}$ for all x. [Expanding along C_1]

9. The value of the determinant

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix} \text{ is}$$

(a) independent of n (b) independent of a
(c) independent of x (d) none of these

Solution (a) We have,

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

$$= \begin{vmatrix} a^2 - 2a\cos x + 1 & a & 1 \\ 0 & \cos(n+1)x & \cos(n+2)x \\ 0 & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2\cos x C_2$]

$= (a^2 - 2a\cos x + 1)\sin x$ [Expanding along C_1], which is **independent of n.**

10. For all values of A, B, C and P, Q, R, the value of the determinant

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} \text{ is}$$

(a) 1 (b) 0
(c) 2 (d) none of these

Solution (b) We have,

$$= \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$= 0.0 = \mathbf{0}$.

11. If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim\limits_{x\to 0} \dfrac{f'(x)}{x} = \lim\limits_{x\to 0}$ is

(a) 2 (b) – 2
(c) 1 (d) – 1

Solution (b) We have, $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$$= \begin{vmatrix} \cos x - \tan x & 0 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

[Applying $R_1 \to R_1 - R_3$]

$= (\cos x - \tan x)(x^2 - 2x^2)$

$= -x^2(\cos x - \tan x)$ [Expanding along R_1]

$\therefore \quad f'(x) = -2x(\cos x - \tan x) - x^2(-\sin x - \sec^2 x)$

$\therefore \quad \lim\limits_{x\to 0}\left[\dfrac{f'(x)}{x}\right] - 2(\cos x - \tan x) + x(\sin x + \sec^2 x)$

$= -2 \times 1 = \mathbf{-2}$.

12. If $a \neq b \neq c$, one value of x which satisfies the equation

$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0 \text{ is given by}$$

(a) $x = a$ (b) $x = b$
(c) $x = c$ (d) $x = 0$

Solution (d) Let $\Delta = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$.

On putting $x = a$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix} = (a+c)(a+b)(a-c)$$

Clearly $\Delta \neq 0$ on expansion along second column, so that $x = a$ does not satisfy the equation $\Delta = 0$. Similarly $x = b$ and $x = c$ also do not satisfy. Now, put $x = 0$, we get

$$\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0.$$

Hence, $\mathbf{x = 0}$ satisfies the equation $\Delta = 0$.

13. If a, b, c are positive and not all equal, then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) non-negative (b) non-positive
(c) negative (d) positive

Solution (c) We have,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a + b + c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

[Applying $R_1 \to R_1 + R_2 + R_3$ and taking $(a + b + c)$ common from R_1]

$$= (a + b + c)\begin{vmatrix} 1 & 0 & 0 \\ b & c - b & a - b \\ c & a - c & b - c \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$= (a + b + c)\,[-(b - c)^2 - (a - c)(a - b)]$

[Expanding along R_1]

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$

$= -\dfrac{1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$,

which is clearly **negative** [∵ a, b, c are + *ve* and not all equal]

14. If a, b, c are all different and the equations $ax + a^2y + (a^3 + 1) = 0$, $bx + b^2y + (b^3 + 1) = 0$, $cx + c^2y + (c^3 + 1) = 0$ are consistent, then

(a) $abc = -1$ (b) $abc = 1$

(c) $abc = -2$ (d) $abc = 2$

Solution (a) For consistency of three equations in two unknowns, we must have $D = 0$

i.e., $$\begin{vmatrix} a & a^2 & a^3 + 1 \\ b & b^2 & b^3 + 1 \\ c & c^2 & c^3 + 1 \end{vmatrix} = 0$$

$\Rightarrow$ $1 + abc = 0$ (See Q.No. 70)

$\therefore$ $\mathbf{abc = -1}$.

15. The value of k, for which the system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non-trivial solution over the set of rationals, is

(a) $-\dfrac{33}{2}$ (b) $\dfrac{33}{2}$

(c) 11 (d) none of these

Solution (b) For the given system to have a non-trivial solution, we must have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 2 & 3 - 2k & -10 \end{vmatrix} = 0$$

[Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$]

$\Rightarrow$ $20k + 11(3 - 2k) = 0 \Rightarrow k = \mathbf{\dfrac{33}{2}}$.

16. In a third order determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consists of sum of four terms. Then it can be decomposed into n determinants, where n has the value.

(a) 1 (b) 9

(c) 16 (d) 24

Solution (d) $n = 2 \times 3 \times 4 = \mathbf{24}$.

17. If the value of a third order determinant is 11, then the value of the determinant formed by its cofactors will be

(a) 11 (b) 121

(c) 1331 (d) 14641

Solution (b)

18. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\dfrac{d}{dx}\Delta_1 = 3\Delta_2$

(c) $\dfrac{d}{dx}\Delta_1 = 3\Delta_2^2$ (d) $\Delta_1 = 3(\Delta_2)^{3/2}$

Solution (b)

19. The system of linear equation $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution, if

(a) $k \neq 0$ (b) $-1 < k < 1$

(b) $-2 < k < 2$ (c) $k = 0$

Solution (a)

20. If $U_n = \begin{vmatrix} 1 & k & k \\ 2n & k^2 + k + 1 & k^2 + k \\ 2n - 1 & k^2 & k^2 + k + 1 \end{vmatrix}$

and $\sum\limits_{n=1}^{K} U_n = 72$ then $k =$

(a) 8 (b) 9

(c) 6 (d) none of these

Solution (a)

21. If $f(x) = \begin{vmatrix} x^3 & \cos^2 x & 2^{x^4} \\ \tan^5 x & 1 & \sec 2x \\ \sin^3 x & x^4 & 5 \end{vmatrix}$ then

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx =$$

(a) 2 (b) -2

(c) 0 (d) none of these

Solution (c)

22. The value of the determinant

$$\begin{vmatrix} 1 + \alpha & 1 + \alpha z & 1 + \alpha z^2 \\ 1 + \beta & 1 + \beta z & 1 + \beta z^2 \\ 1 + \gamma & 1 + \gamma z & 1 + \gamma z^2 \end{vmatrix} \text{ is}$$

(a) $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$

(b) 0.

(c) $\alpha\beta\gamma$

(d) none of these

Solution (b)

23. If [ ] denotes the greatest integer less than or equal to the real number under consideration, and $-1 \le x < 0$; $0 \le y < 1$; $1 \le z < 2$, then the value of the determinant

$$\begin{vmatrix} [x] + 1 & [y] & [z] \\ [x] & [y] + 1 & [z] \\ [x] & [y] & [z] + 1 \end{vmatrix} \text{ is}$$

(a) $[z]$ (b) $[y]$
(c) $[x]$ (d) none of these

Solution (a)

24. If l, m, n are the pth, qth and rth term of a G.P. all positive, then

$$\begin{vmatrix} \log l & p & l \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} \text{ equals}$$

(a) 3 (b) 2
(c) 1 (d) zero

Solution (d)

25. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and a, b, c are distinct, then the product abc equals

(a) 2 (b) -1
(c) 1 (d) 0

Solution

26. If $a_1, a_2, a_3, \ldots a_n, \ldots$ are in G.P., then the value of the determinant $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is

(a) 2 (b) 1
(c) 0 (d) -2

Solution (c)

27. If $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$, $a \neq b \neq c$, then

(a) $x = 0$ if $b(a+c) \le ac$

(b) $x = \pm\sqrt{b(a+c) - ac}$ if $b(a+c) > ac$

(c) $x = 0, \pm\sqrt{b(a+c) - ac}$ if $b(a+c) > ac$

(d) none of these

Solution (a), (c)

28. If $\omega\,(\neq 1)$ is a cube root of unity, then

$$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix} \text{ equals}$$

(a) 0 (b) 1
(c) i (d) ω

Solution (a)

29. If $f(x)$

$$= \begin{vmatrix} 1 & 2(x-1) & 3(x-1)(x-2) \\ (x-1) & (x-1)(x-2) & (x-1)(x-2)(x-3) \\ x & (x-1)x & (x-1)(x-2)x \end{vmatrix},$$

then $f(41) =$

(a) 41 (b) -41
(c) 0 (d) none of these

Solution (c)

30. Let α, β, γ be the cube roots of unity, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 1 (b) -1
(c) i (d) 0

Solution (d)

31. If $f(x) = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$, then $f(3x) - f(x) =$

(a) $3x\lambda^2$ (b) $6x\lambda^2$
(c) $x\lambda^2$ (d) none of these

Solution (b)

32. Let $D_k = \begin{vmatrix} \alpha & \beta & \gamma \\ 2.3^k & 16.9^k & 26.27^k \\ (3^{10}-1) & 2(9^{10}-1) & (27^{10}-1) \end{vmatrix}$ then the value of $\sum\limits_{k=1}^{10} D_k$ is

(a) $2(\alpha + \beta + \gamma)$

(b) $\alpha\beta + \alpha\gamma + \beta\gamma$

(c) $\alpha\,\beta\,\gamma$

(d) 0

Solution (d)

33. The value of the determinant

$$\begin{vmatrix} \sin\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta \\ \cos\alpha & -\sin\alpha & 0 \end{vmatrix} \text{ is}$$

(a) is independent of α

(b) independent of β

(c) independent of α and β

(d) none of these

Solution (b)

34. If $b^2 - ac < 0$ and $a > 0$ then the value of the determinant

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} \text{ is}$$

(a) positive (b) negative
(c) zero (d) $b^2 + ac$

Solution (b)

35. If α, β and γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) q (b) 0
(c) p (d) $p^2 - 2q$

Solution (b)

36. If

$$f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix},$$

then $f(100)$ is equal to

(a) 0 (b) 1

(c) 100 (d) – 100

Solution (a)

37. If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, then maximum value of Δ is

(a) 1 (b) 9

(c) 16 (d) none of these

Solution (d)

38. The system of linear equations $ax + by = 0$, $cx + dy = 0$ has a non-trivial solution if

(a) $ad - bc = 0$ (b) $ac + bd = 0$

(c) $ad - bc < 0$ (d) $ad - bc > 0$

Solution (a) System of given equations has a non-trivial solution if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ i.e., if $\boldsymbol{ad - bc = 0}$.

39. The value of the determinant

$$\begin{vmatrix} 1 & (5^{2x}-5^{-2x})^2 & (5^{2x}+5^{-2x})^2 \\ 1 & (6^{2x}-6^{-2x})^2 & (6^{2x}+6^{-2x})^2 \\ 1 & (7^{2x}-7^{-2x})^2 & (7^{2x}+7^{-2x})^2 \end{vmatrix} \text{ is}$$

(a) 210^{2x} (b) 210^{-2x}

(c) 0 (d) none of these

Solution (c)

40. The value of n for which the determinant

$$\begin{vmatrix} {}^8C_3 & {}^9C_5 & {}^{10}C_7 \\ {}^8C_4 & {}^9C_6 & {}^{10}C_8 \\ {}^9C_n & {}^{10}C_{n+2} & {}^{11}C_{n+4} \end{vmatrix} \text{ becomes zero is}$$

(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) none of these

Solution (c)

Problems for Practice

1. The value of the determinant

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} \text{ is}$$

(a) 1 (b) 0 (c) – 1 (d) none of these

2. The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) none of these

3. If $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 + Kabc$, then K is equal to

(a) 3 (b) – 3 (c) 4 (d) – 4

4. If $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$= (a^3 + b^3 + c^3 + kabc)^2$, then k is equal to

(a) 2 (b) – 2 (c) 3 (d) – 3

5. The three roots of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ are

(a) – 9, 2, 7 (b) 9, – 2, 7

(c) 9, 2, – 7 (d) None of these

6. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$ where x, y, z are not all zero, then

(a) $a^2 + b^2 + c^2 + 2abc = 0$

(b) $a^2 + b^2 + c^2 - 2abc = 1$

(c) $a^2 + b^2 + c^2 + 2abc = 1$

(d) none of these

7. The value of the determinant

$$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} =$$

(a) 0 (b) $(p - q)(q - r)(r - p)$

(c) pqr (d) $3pqr$

8. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to 0, if

(a) a, b, c are in A.P.

(b) a, b, c are in G.P

(c) a, b, c are in H.P.

(d) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

9. If $\begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix}$

$$= ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

be an identity in x, where a, b, c, d, e, f are independent of x, then the value of f is

(a) 0 (b) 15 (c) 17 (d) none of these

10. If the system of equations

$x + ay + a^2 = 0,\ x + by + b^2 = 0,\ x + cy + c^2 = 0$ is consistent then the equations.

$x + bcy + (b + c)z = 0$

$x + cay + (c + a)z = 0$

$x + aby + (a + b)z = 0$ have

(a) a unique solution

(b) infinite number of solutions

(c) no solution

(d) none of these

11. If $\begin{vmatrix} \alpha & -\beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{vmatrix} = 0$ then

(a) α/β is one of the cube roots of unity

(b) α is one of the cube roots of unity

(c) β is one of the cube roots of unity

(d) none of these

12. If α, β and γ are the roots of the equation $x^3 + ax + b = 0$, then the value of the determinant

$$\begin{vmatrix} \alpha-\beta-\gamma & 2\alpha & 2\alpha \\ 2\beta & \beta-\gamma-\alpha & 2\beta \\ 2\gamma & 2\gamma & \gamma-\alpha-\beta \end{vmatrix} \text{ is}$$

(a) 0 (b) a (c) $-b$ (d) none of these

13. If $1, \omega, \omega^2$ are the cube roots of unity, then

$D = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ has the value

(a) 0 (b) ω (c) ω^2 (d) 1

14. If the determinant

$$\begin{vmatrix} a & b & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2a\alpha+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0, \text{ then}$$

(a) a, b, c are in H.P

(b) α is root of $4ax^2 + 12bx + 9c = 0$ or a, b, c are in G.P.

(c) a, b, c are in G.P. only

(d) a, b, c are in A.P.

15. If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$, then for all θ,

(a) $f(\theta) = 1$ (b) $f(\theta) = 2$

(c) $f(\theta) = 3$ (d) none of these

16. One root of the equation

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0 \text{ is}$$

(a) $\frac{2}{3}$ (b) $\frac{8}{3}$ (c) $\frac{16}{3}$ (d) $\frac{1}{3}$

17. The value of λ for which the equations $x + y - 3 = 0$, $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$, $x - (1 + \lambda)y + (2 + \lambda) = 0$ are consistent is

(a) 1 (b) 5/3 (c) – 5/3 (d) none of these

18. For positive numbers x, y, z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) none of these

19. If $D_1 = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$ and $D_2 = \begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ x & y & z \end{vmatrix}$, then

(a) $D_1 = D_2$ (b) $D_1 = -D_2$

(c) $D_1 = -2D_2$ (d) $D_2 = 2D_1$

20. If $f(x) = \begin{vmatrix} x^{n-1} & \cos x & \dfrac{1}{x+3} \\ 0 & \cos\dfrac{n\pi}{2} & \dfrac{(-1)^n n!}{3^{n+1}} \\ \alpha & \alpha^3 & \alpha^5 \end{vmatrix}$ then

$$\frac{d^n}{dx^n}[f(x)]_{x=0} =$$

(a) 1 (b) –1 (c) 0 (d) none of these

ANSWERS

1. (b) **2.** (a) **3.** (b) **4.** (d) **5.** (a) **6.** (c) **7.** (a) **8.** (b),(d) **9.** (c) **10.** (b)

11. (a) **12.** (a) **13.** (a) **14.** (b) **15.** (a) **16.** (a) **17.** (a),(c) **18.** (a) **19.** (a) **20.** (c)

Set Theory

26

BRIEF REVIEW OF THE CONCEPTS

SET

A *set* is any collection of objects such that given an object, it is possible to determine whether that object belongs to the given collection or not.

For example, the collection of all students of Delhi University, is a set, whereas, collection of all good books on mathematics, is not a set, since a mathematics book considered good by one person might be considered bad or average by another.

Notations The sets are usually denoted by capital letters A, B, C, etc. and the members or elements of the set are denoted by lower-case letters a, b, c etc. If x is a member of the set A, we write $x \in A$ (read as 'x belongs to A') and if x is not a member of the set A, we write $x \notin A$ (read as 'x does not belong to A'). If x and y both belong to A, we write $x, y \in A$.

REPRESENTATION OF A SET

Usually, sets are represented in the following two ways:

1. Roster form or Tabular form
2. Set Builder form or Rule Method

Roster Form

In this from, we list all the member of the set within braces (curly brackets) and separate these by commas.

For example, the set A of all odd natural numbers less than 10 in the roster form is written as:

$$A = \{1, 3, 5, 7, 9\}$$

Note:

- In roster form, every element of the set is listed only once.
- The order in which the elements are listed is immaterial For example, each of the following sets denotes the same set $\{1, 2, 3\}$, $\{3, 2, 1\}$, $\{1, 3, 2\}$.

Set-Builder Form

In this form, we write a variable (say x) representing any member of the set followed by a property satisfied by each member of the set.

For example, the set A of all prime numbers less than 10 in the set-builder form is written as

$$A = \{x \mid x \text{ is a prime number less than } 10\}$$

The symbol '|' stands for the words 'such that'. Sometimes, we use the symbol ':' in place of the symbol '|'.

TYPES OF SETS

Empty Set or Null Set

A set which has no element is called the *null set* or *empty set*. It is denoted by the symbol Φ.

For example, each of the following is a null set:

1. The set of all real numbers whose square is –1.
2. The set of all rational numbers whose square is 2.
3. The set of all those integers that are both even and odd.

A set consisting of atleast one element is called a non-empty set.

Singleton Set

A set having only one element is called *singleton set*.

For example, $\{0\}$ is a singleton set, whose only member is 0.

Finite and Infinite Set

A set which has finite number of elements is called a *finite set*. Otherwise, it is called an *in finite set*.

For example, the set of all days in a week is a finite set whereas, the set of all integers, denoted by $\{ \ldots, -2, -1, 0, 1, 2, \ldots\}$ or $\{x \mid x \text{ is an integer}\}$, is an infinite set.

An empty set Φ which has no element, is a finite set.

The number of distinct elements in a finite set A is called the *cardinal number of the set* A and it is denoted by $n(A)$.

Equal Sets

Two sets A and B are said to be *equal*, written as $A = B$, if every element of A is in B and every element of B is in A.

Equivalent Sets

Two finite sets A and B are said to be *equivalent*, if $n(A) = n(B)$.

Clearly, equal sets are equivalent but equivalent sets need not be equal.

For example, the sets $A = \{4, 5, 3, 2\}$ and $B = \{1, 6, 8, 9\}$ are equivalent but are not equal.

Subset

Let A and B be two sets. If every elements of A is an element of B, then A is called a *subset* of B and we write $A \subseteq B$ or $B \supseteq A$ (read as 'A is contained in B' or B contains A'). B is called *superset* of A.

Notes:

- If $A \subseteq B$ and $A \neq B$, we write $A \subset B$ or $B \supset A$ (read as : A is a proper subset of B or B is a proper superset of A).
- Every set is a subset and a superset of itself.
- If A is not a subset of B, we write $A \not\subseteq B$.
- The empty set is the subset of every set.
- If A is a set with $n(A) = m$, then the number of subsets of A are 2^m and the number of proper subsets of A are 2^m-1.

  For example, let $A = \{3, 4\}$, then the subsets of A are ϕ, $\{3\}$, $\{4\}$, $\{3, 4\}$. Here, $n(A) = 2$ and number of subsets of $A = 2^2 = 4$.

  Also, $\{3\} \subset \{3, 4\}$ and $\{2, 3\} \not\subseteq \{3, 4\}$

Power Set

The set of all subsets of a given set A is called the *power set* of A and is denoted by $P(A)$.

For example, if $A = \{1, 2, 3\}$, then

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\} \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Clearly, if A has n elements, then its power set $P(A)$ contains exactly 2^n elements.

OPERATIONS ON SETS

Union of Two Sets

The union of two sets A and B, written as $A \cup B$ (read as 'A union B'), is the set consisting of all the elements which are either in A or in B or in both. Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Clearly, $x \in A \cup B \Rightarrow x \in A$ or $x \in B$, and

$x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

For example, if $A = \{a, b, c\ d\}$ and $B = \{c, d, e, f\}$, then $A \cup B = \{a, b, c, d, e, f\}$

Intersection of Two Sets

The intersection of two sets A and B, written as $A \cap B$ (read as 'A intersection B') is the set consisting of all the common elements of A and B. Thus,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Clearly, $x \in A \cap B \Rightarrow x \in A$ and $x \in B$, and

$x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.

For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, then $A \cap B = \{c, d\}$.

Disjoint Sets

Two sets A and B are said to be *disjoint*, if $A \cap B = \phi$, i.e., A and B have no element in common.

For example, if $A = \{1, 2, 5\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \phi$, so A and B are disjoint sets.

Difference of Two Sets

If A and B are two sets, then their difference $A - B$ is defined as:

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$.

For example , if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A - B = \{2, 4\}$ and $B - A = \{7, 9\}$.

Important Results

1. $A - B \neq B - A$
2. The sets $A - B$, $B - A$ and $A \cap B$ are disjoint sets
3. $A - B \subseteq A$ and $B - A \subseteq B$
4. $A - \phi = A$ and $A - A = \phi$

Symmetric Difference of Two Sets

The symmetric difference of two sets A and B, denoted by $A \Delta B$, is defined as

$$A \Delta B = (A - B) \cup (B - A).$$

For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$ then $A \Delta B = (A - B) \cup (B - A) = \{2, 4\} \cup \{7, 9\} = \{2, 4, 7, 9\}$.

Complement of a Set

If U is a universal set and A is a subset of U, then the complement of A is the set which contains those elements of U, which are not contained in A and is denoted by A' or A^c. Thus,

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

For example, if $U = \{1, 2, 3, 4, ...\}$ and $A = \{2, 4, 6, 8, ...\}$, then, $A' = \{1, 3, 5, 7, ...\}$

Important Results

(a) $U' = \phi$ (b) $\phi' = U$

(c) $A \cup A' = U$ (d) $A \cap A' = \phi$.

ALGEBRA OF SETS

1. **Idempotent Laws** For any set A, we have
   (a) $A \cup A = A$ (b) $A \cap A = A$
2. **Identity Laws** For any set A, we have
   (a) $A \cup \phi = A$ (b) $A \cap \phi = \phi$
   (c) $A \cup U = U$ (d) $A \cap U = A$
3. **Commutative Laws** For any two sets A and B, we have
   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$
4. **Associative Laws** For any three sets A, B and C, we have
   (a) $A \cup (B \cup C) = (A \cup B) \cup C$
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$
5. **Distributive Laws** For any three sets A, B and C, we have
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. For any two sets A and B, we have
   (a) $P(A) \cap P(B) = P(A \cap B)$
   (b) $P(A) \cup P(B) \subseteq P(A \cup B)$, where $P(A)$ is the power set of A.
7. If A is any set, we have $(A')' = A$.
8. **Demorgan's Laws** For any three sets A, B and C, we have
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$.

Key Results on Operations on Sets

1. $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$, $A \cap B \subseteq B$
2. $A - B = A \cap B'$
3. $(A - B) \cup B = A \cup B$
4. $(A - B) \cap B = \phi$
5. $A \subseteq B \Leftrightarrow B' \subseteq A'$
6. $A - B = B' - A'$
7. $(A \cup B) \cap (A \cup B') = A$
8. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$
9. $A - (A - B) = A \cap B$

10. $A - B = B - A \Leftrightarrow A = B$
11. $A \cup B = A \cap B \Leftrightarrow A = B$
12. $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Some Basis Results about Cardinal Number

If A, B and C are finite sets and U be the finite universal set, then

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. $n(A \cup B) = n(A) + n(B)$, where A and B are disjoint non-empty sets
4. $n(A \cap B') = n(A) - n(A \cap B)$
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$
9. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$ $- n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
10. If $A_1, A_2, A_3, \ldots A_n$ are disjoint sets, then $$n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_n)$$
11. $n(A \,\Delta\, B)$ = number of elements which belong to exactly one of A or B

CARTESIAN PRODUCT OF TWO SETS

If A and B are any two non-empty sets, then *cartesian product* of A and B is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Notes:

- If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.
- $A \times B \neq B \times A$
- If A has n elements and B has m elements then $A \times B$ has mn elements.
- If $A_1, A_2, \ldots, A_p$ are p non-empty sets, then their cartesian product, is defined as $\prod_{i=1}^{p} A_i = \{(a_1, a_2, a_3, \ldots a_p);\ a_i \in A_i \text{ for all } i\}$

Key Results on Operations on Sets

If A, B, C are three sets, then

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$, where S and T are two sets.
5. If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$
6. If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$
7. If $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$
8. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
9. If A and B are two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.
10. $A \times B = B \times A$ if and only if $A = B$
11. $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
12. $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

RELATIONS

Let A, B be any two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

If R is a relation from A to B and if $(a, b) \in R$, then we write $a\,R\,b$ and say that 'a is related to b' and if $(a, b) \notin R$, then we write $a \not R\, b$ and say that a is not related to b.

Key Results on Operations on Sets

1. Every subset of $A \times A$ is said to be a relation on A.
2. If A has m elements and B has n elements, then $A \times B$ has mn elements and total number of different relations from A to B is 2^{mn}.
3. Let R be a relation from A to B, *i.e.*, $R \subseteq A \times B$, then

   Domain of $R = \{a : a \in A, (a, b) \in R \text{ for a some } b \in B\}$

   Range of $R = \{b : b \in B, (a, b) \in R \text{ for some } a \in A\}$

   For example, let $A = \{1, 3, 4, 5, 7\}$, $B = \{2, 4, 6, 8\}$ and R be the relation 'is one less than' from A to B, then

   $$R = \{(1, 2), (3, 4), (5, 6), (7, 8)\}.$$

   Here, domain of $R = \{1, 3, 5, 7\}$ and range of $R = \{2, 4, 6, 8\}$.

Identity Relation

R is an *identity relation* if $(a, b) \in R$ iff $a = b$, $a \in A$, $b \in A$. In other words, every element of A is related to only itself.

Universal Relation

Let A be any set and R be the set $A \times A$, then R is called the *universal relation* in A.

Void Relation

ϕ is called void relation in a set.

Inverse Relation

Let $R \subseteq A \times B$ be a relation from A to B. Then $R^{-1} \subseteq B \times A$ is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Thus, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall a \in A, b \in B.$

Notes:

- dom (R^{-1}) = range (R) and range (R^{-1}) = dom (R)
- $(R^{-1})^{-1} = R$. For example, if $R = \{(1, 2), (3, 4), (5, 6)\}$

  then $R^{-1} = \{(2, 1), (4, 3), (6, 5)\}$

  and $(R^{-1})^{-1} = \{(1, 2), (3, 4), (5, 6) = R.$

  dom $(R) = \{1, 3, 5\}$, range $(R) = \{2, 4, 6\}$

  and dom $(R^{-1}) = \{2, 4, 6\}$, range $(R^{-1}) = \{1, 3, 5\}$

  So, dom (R^{-1}) = range (R) and range (R^{-1}) = dom (R).

TYPES OF RELATIONS ON A SET

Let A be a non-empty set, then a relation R on A is said to be:

1. **Reflexive** If $a\,R\,a, \forall a \in A$, i.e., if $(a, a) \in R, \forall a \in A$
2. **Symmetric** If $a\,R\,b \Rightarrow b\,R\,a, \forall a, b \in A$, i.e., if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
3. **Anti-Symmetric** If $a\,R\,b$ and $b\,R\,a \Rightarrow a = b, \forall a, b \in A$
4. **Transitive** If $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,C, \forall a, b, c \in A$ i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

EQUIVALENCE RELATION

A relation R on a non-empty set A is called an *equivalence relation* if and only if it is

1. reflexive **2.** symmetric and **3.** transitive.

That is, R satisfies following properties:

1. $a\ R\ a, \forall\ a \in A$
2. $a\ R\ b \Rightarrow b\ R\ a, \forall\ a, b \in A$ and
3. $a\ R\ b, b\ R\ c \Rightarrow a\ R\ c, \forall\ a, b, c \in A$.

For example, let I be the set of all integers, m be a positive integer. Then the relation, R on I is defined by

$$R = \{(x, y) : x, y \in I, x - y \text{ is divisible by } m\}.$$

Consider any $x, y, z \in I$.

1. Since $x - x = 0 = 0.m \quad \Rightarrow x - x$ is divisible by m
$\Rightarrow (x, x) \in R \Rightarrow R$ is reflexive.

2. Let $(x, y) \in R \quad \Rightarrow x - y$ is divisible by m
$\Rightarrow x - y = mq$, for some $q \in I$
$\Rightarrow y - x = m\ (-q)$
$\Rightarrow y - x$ is divisible by m
$\Rightarrow (y, x) \in R$
Thus, $(x, y) \in R \Rightarrow (y, x) \in R \Rightarrow R$ is symmetric.

3. Let $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow x - y$ is divisible by m and $y - z$ is divisible by m
$\Rightarrow x - y = mq$ and $y - z = mq'$ for some $q, q' \in I$
$\Rightarrow (x - y) + (y - z) = m\ (q + q')$
$\Rightarrow x - z = m\ (q + q'), q + q' \in I$
$\Rightarrow (x, z) \in R$
Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, so R is transitive.

Hence the relation R is reflexive, symmetric and transitive and it is also an equivalence relation.

Note: It is important to note that every identity relation is reflexive but every reflexive relation need not be an identity relation. Also, identity relation is reflexive, symmetric and transitive.

CONGRUENCE MODULO *m*

Let m be a positive integer and $x, y \in I$, then x is said to be congruent to y modulo m, written as $x \equiv y \pmod m$, iff $x - y$ is divisible by m.

For example, $155 \equiv 7 \pmod 4$ as

$$\frac{155 - 7}{4} = \frac{148}{4} = 37 \text{ (integer)}.$$

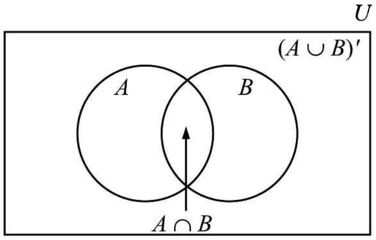
Solved Problems

1. If $n\ (U) = 60$, $n\ (A) = 35$, $n\ (B) = 24$ and $n\ (A \cup B)' = 10$ then $n\ (A \cap B)$ is

(a) 9 (b) 8
(c) 6 (d) none of these

Solution (a) We have, $n(A \cup B) = n(U) - n(A \cup B)'$
$= 60 - 10 = 50$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



$\Rightarrow 50 = 35 + 24 - n(A \cap B) \Rightarrow n(A \cap B) = 59 - 50 = \mathbf{9}$.

2. If $X = \{8^n - 7n - 1/n \in N\}$ and $Y = \{49\ (n - 1)/n \in N\}$, then

(a) $X \subset Y$ (b) $Y \subset X$
(c) $X = Y$ (d) none of these

Solution (a) We have, $8^n - 7n - 1$

$$= (7 + 1)^n - 7n - 1 = ({}^nC_2 7^2 + {}^nC_3 7^3 + \ldots + {}^nC_n 7^n)$$
$$= 49({}^nC_2 + {}^nC_3 7 + \ldots + {}^nC_n 7^{n-2}) \text{ for } n \geq 2$$

For $n = 1$, $8^n - 7n - 1 = 0$

Thus, $8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$ and 0 for $n = 1$. Hence X consists of all positive integral multiple of 49 of the form $49K_n$. where $K_n = {}^nC_2 + {}^nC_2 7 + \ldots + {}^nC_n 7^{n-2}$ together with zero. Also Y consists of all positive integral multiples of 49 including zero. Therefore, $\boldsymbol{X \subset Y}$.

Hence (a) is the correct answer.

3. If X and Y are two sets, then $X \cap (Y \cup X)'$ equals

(a) X (b) Y
(c) ϕ (d) none of these

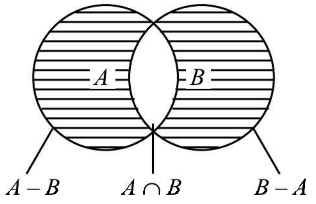
Solution (c) We have,

$$X \cap (Y \cup X)' = X \cap (Y' \cap X') \cap Y' = (X \cap X') \cap Y'$$
$$\phi \cap Y' = \phi.$$

4. If A, B and C are non-empty subsets of a set, then $(A - B) \cup (B - A)$ equals

(a) $(A \cap B) \cup (A \cup B)$
(b) $(A \cup B) - (A \cap B)$
(c) $A - (A \cap B)$
(d) $(A \cup B) - B$

Solution (b) $(A - B) \cup (B - A) = \boldsymbol{(A \cup B) - (A \cap B)}$.



5. Let A and B two non-empty subsets of a set X such that A is not a subset of B then

(a) A is subset of the complement of B

(c) B is a subset of A

(c) A and B are disjoint

(d) A and the complement of B are non-disjoint

Solution (d) Since $A \not\subseteq B$, $\exists\ x \in A$ such that $x \notin B$

Then $x \in B'$. $\therefore$ $\mathbf{A \cap B' \neq \phi}$.

6. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is

(a) $p + q + 1$ (b) pq

(c) p^2 (d) $p + q$

Solution (b) As A has p elements and B has q elements so, $A \times B$ has $\boldsymbol{pq}$ elements.

7. If the sets A and B are defined as

$$A = \{(x, y) : y = \frac{1}{x},\ 0 \neq x \in R\}$$

$$B = \{(x, y) : y = -x,\ x \in R\}, \text{ then}$$

(a) $A \cap B = A$ (b) $A \cap B = B$

(c) $A \cap B = \phi$ (d) none of these

Solution (c) $\mathbf{A \cap B = \phi}$

$$[\because y = \frac{1}{x},\ y = -x \text{ meet when } -x = \frac{1}{x}$$

$$\Rightarrow \quad x^2 = -1 \text{ which does not give any real value of } x]$$

8. Two finite sets have $\boldsymbol{m}$ and $\boldsymbol{n}$ elements, then total number of subsets of the first set is 56 more that the total number of subsets of the second. The values of m and n are

(a) 7, 6 (b) 6, 3

(d) 5, 1 (d) 8, 7

Solution (b) Since the two finite sets have m and n elements, so number of subsets of these sets well be 2^m and 2^n respectively. According to the question

$$3^m - 2^n = 56$$

putting $m = 6, n = 3$ we get

$$2^6 - 2^3 = 56 \text{ or } 64 - 8 = 56.$$

9. Let $U = R$ (the set of all real numbers) If $A = \{x: x \in R,\ 0 < x < 2\}$, $B = \{x: x \in R,\ 1 < x \leq 3\}$, then

(a) $A \cup B = \{x: x \in R \text{ and } 0 < x \leq 3\}$

(b) $A \cap B = \{x: x \in R \text{ and } 1 < x < 2\}$

(c) $A - B = \{x: x \in R \text{ and } 0 < x \leq 1\}$

(d) all of these.

Solution (d) We have

$$A' = R - A = \{x: \in R \text{ and } x \notin A\}$$

$$= \{x: (x \in R \text{ and } x \geq 2) \text{ or } (x \in R \text{ and } x \leq 0)\}$$

$$= \{x: x \in R \text{ and } x \geq 2\} \cup \{x: x \in R \text{ and } x \leq 0\}$$

Similarly, $B' = \{x: x \in R \text{ and } x \leq 1\} \cup \{x: x \in R \text{ and } x > 3\}$.

$$\therefore \quad A \cup B = \{x: x \in R \text{ and } 0 < x \leq 3\},$$

$$A \cap B = \{x: x \in R \text{ and } 1 < x < 2\}$$

$$A - B = \{x: x \in R \text{ and } 0 < x \leq 1\}$$

10. If $Y \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then

(a) The smallest set of Y is {3, 5, 9}

(b) The smallest set of Y is {2, 3, 5, 9}

(c) The largest set of Y is {1, 2, 3, 4, 9}

(d) The largest set of Y is {2, 3, 4, 9}

Solution (a), (c) Since the set on the right hand side has 5 elements,

$\therefore$ smallest set of Y has three elements and largest set of Y has five elements,

$\therefore$ smallest set of Y is **{3, 5, 9}**

and largest of Y is **{1, 2, 3, 4, 9}**.

11. If A has 3 elements and B has 6 elements, then the minimum number of elements in the set $A \cup B$ is

(a) 6 (b) 3

(c) ϕ (d) none of these

Solution (a) Clearly the number of elements in $A \cup B$ will be minimum when $A \subset B$. Hence the minimum number of elements in $A \cup B$ is the same as the number of elements in B, that is, **6**.

12. If $aN = \{ax: x \in N\}$, then $3N \cap 7N =$

(a) $3N$ (b) $7N$

(c) N (d) $21\ N$

Solution (d) We have $3N = \{3x: x \in N\} = \{3, 6, 9, 12, ...\}$

and $7N = \{7x: x \in N\} = \{7, 14, 21, 28, 35, 42...\}$

Hence $3N \cap 7N = \{21, 42, 63, ...\}$

$= \{21x: x \in N\} = \mathbf{21N}$.

13. Suppose $A_1, A_2, ... A_{30}$ are thirty sets, each with five elements and $B_1, B_2, ..., B_n$ are n sets each with three elements. Let

$$\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$$

If each elements of S belongs to exactly ten of the A_i's and exactly nine of the B_j's then $n =$

(a) 45 (b) 35

(c) 40 (d) none of these

Solution (a) Given A's are thirty sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \qquad ...(1)$$

If the $\boldsymbol{m}$ distinct elements in S and each element of S belongs to exactly 10 of the A_i's, we have

$$\sum_{i=1}^{30} n(A_i) = 10\text{m} \qquad ...(2)$$

$\therefore$ From (1) and (2), we get $10\boldsymbol{m} = 150$

$$\therefore \quad m = 15 \qquad ...(3)$$

Similarly $\sum_{j=1}^{30} n(B_j) = 3n$ and $\sum_{j=1}^{30} n(B_j) = 9m$

$$\therefore \quad 3n = 9m$$

$$\Rightarrow \quad n = \frac{9m}{3} = 3m = 3 \times 15 = 45 \qquad \text{[from (3)]}$$

Hence, $n = \mathbf{45}$.

14. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, ..., 18\}$ and N is the universal set, then $A' \cup ((A \cup B) \cap B')$ is

(a) A (b) N

(c) B (d) none of these

Solution (b) We have, $(A \cup B) \cap B' = A$

$((A \cup B) \cap B') \cup A' = A \cup A' = N$.

Hence (b) is the correct answer.

15. If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)'$ equals

(a) X (b) Y
(c) ϕ (d) none of these

Solution (c) $X \cap (X \cup Y)' = X \cap (X' \cap Y')$

[$\because$ By De-Morgan's Law $(A \cup B)' = (A' \cap B')$

$= (X \cap X') \cap Y' = \phi \cap Y' = \phi$.

16. If $n(U) = 700$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to

(a) 400 (b) 240
(c) 300 (d) none of these

Solution (c) $n(A' \cap B') = (A \cup B)'$

$= n(U) - n(A \cup B)$

$= 700 - n(A \cup B)$

Now $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 200 + 300 - 100 = 400$.

Hence, $n(A' \cap B') = 700 - 400 =$ **300.**

17. In a committee 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. The number of persons speaking at least one of these two languages is

(a) 60 (b) 40
(c) 38 (d) none of these

Solution (a) Let S be the set of people who speak Spanish and F be the set of people who speak French.

$\therefore$ $n(S) = 20$, $n(F) = 50$, $n(S \cap F) = 10$,

we have to find $n(S \cup F)$.

Using $n(S \cup F) = n(S) + n(F) - n(S \cap F)$

$\Rightarrow$ $n(S \cup F) = 20 + 50 - 10 =$ **60.**

18. In a group of 70 people, 37 like coffee, 52 like tea and each person like atleast one of the two drinks. The number of persons liking both coffee and tea is

(a) 16 (b) 13
(c) 19 (d) none of these

Solution (c) Let A = set of people who like coffee.

and B = set of people who like tea.

Then, $A \cup B$ = set of people who like at least one of the two drinks.

and $A \cap B$ = set of people who like both the drinks.

Here, $n(A) = 37$, $n(B) = 52$, $n(A \cup B) = 70$

Using the result

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

we have, $70 = 37 + 52 - n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 89 - 70 = 19$

$\therefore$ **19** people like both coffee and tea.

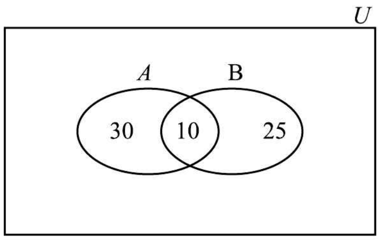
19. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. The number of persons liking tennis only and not cricket is

(a) 21 (b) 25
(c) 15 (d) none of these

Solution (b) Let A be the set of people who like cricket and B the set of people who like tennis.

Then $n(A \cup B) = 65$

$n(A) = 40$, $n(A \cap B) = 10$



$\because$ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$65 = 40 + n(B) - 10$

$n(B) = 65 - 40 + 10 = 35$

Number of people who like only tennis

$= n(B) - n(A \cap B) = 35 - 10 = 25$

$\therefore$ Number of people who like tennis only and not cricket = **25.**

20. In a group of 1000 people, there are 750 people who can speak Hindi and 400 who can speak English. Then number of persons who can speak Hindi only is

(a) 300 (b) 400
(c) 600 (d) none of these

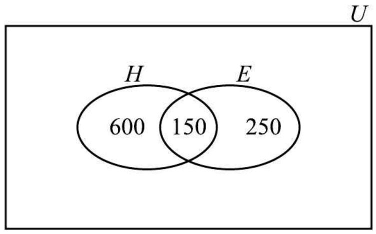
Solution (c) Here $n(H \cup E) = 1000$, $n(H) = 750$,

$n(E) = 400$

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$1000 = 750 + 400 - n(H \cap E)$

$\Rightarrow$ $n(H \cap E) = 1150 - 1000 = 150$.



Number of people who can speak Hindi only

$= n(H \cap E') = n(H) - n(H \cap E)$

$= 750 - 150 =$ **600.**

21. If $f : R \rightarrow R$, defined by $f(x) = x^2 + 1$, then the values of $f^{-1}(17)$ and $f^{-1}(-3)$ respectively are

(a) ϕ, $\{4, -4\}$ (b) $\{3, -3\}$, ϕ
(c) ϕ, $\{3, -3\}$ (d) $\{4, -4\}$, ϕ

Solution (d) Let $y = x^2 + 1$. Then for $y = 17$, we have $x = \pm 4$ and for $y = -3$, x becomes imaginary that is, there is no value of x.

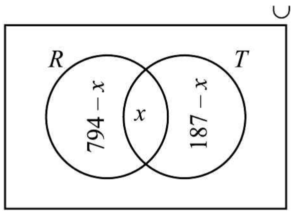
Hence $f(17) = \{-4, 4\}$ and $f^{-1}(-3) = \phi$

Hence (d) is the correct answer.

22. In a statistical investigation of 1,003 families of Calcutta, it was found that 63 families had neither a radio nor a TV, 794 families had a radio and 187 had a TV. The number of families in that group having both a radio and a TV is

(a) 36 (b) 41
(c) 32 (d) none of these

Solution (b) Let R be the set of families having a radio and T be, the set of families having a TV, then



$n(R \cup T)$ = The no of families having at least one of radio and TV

$= 1003 - 63 = 940$

$n(R) = 794$ and $n(T) = 187$.

Let x families had both a radio and a TV i.e.,

$$n(R \cap T) = x.$$

The no. of families who have only Radio = $794 - x$ and the no. of families who have only T.V.= $187 - x$

From Venn diagram, $794 - x + x - 187 - x = 940$

$\Rightarrow \quad 981 - x = 940$ or $x = 981 - 940 = 41$

Hence, the required no. of families having both a radio and a TV = **41.**

23. In a city, three daily newspapers A, B, C are published. 42% of the people in that city read A, 51% read B and 68% read C. 30% read A and B; 28% read B and C; 36% read A and C; 8% do not read any of the three newspapers. The percentage of persons who read all the three papers is

(a) 25% (b) 18%

(c) 20% (d) none of these

Solution (a) Let the no. of persons in the city be 100.

Then we have

$n(A) = 42$, $n(B) = 51$, $n(C) = 68$;

$n(A \cap B) = 30$, $n(B \cap C) = 28$, $n(A \cap C) = 36$

$n(A \cup B \cup C) = 100 - 8 = 92$

Using $n(A \cup B \cup C) = n(A)\ n(B) - n(A \cap B)$

$$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Substituting the above values, we have

$$92 = 42 + 51 + 68 - 30 - 28 - 36 + n(A \cap B \cap C)$$

$\Rightarrow \quad n(A \cap B \cap C) = 92 - 161 + 94$

$\Rightarrow \quad n(A \cap B \cap C) = 92 - 67 = 25$

Hence, **25%** of the people read all the three papers.

24. Let n be a fixed positive integer. Let a relations R be defined on I (the set of all integers) as follows: $a\ R\ b$ iff $n/(a - b)$, that is iff $a - b$ is divisible by n. Then, the relation R is

(a) reflexive only (b) symmetric only

(c) transitive only (d) an equivalence relation.

Solution (d) R is reflexive since for any integer a we have $a - a = 0$ and 0 is divisible by n. Hence $a\ R\ a \ \forall\ a \in I$.

R is symmetric, let $a\ R\ b$. Then by definition of R, $a - b = nk$ where $k \in I$. Hence $b - a = (-k)\ n$ where $-k \in I$ and so $b\ R\ a$. Thus we have shown that

$$a\ R\ b \Rightarrow b\ R\ a.$$

R is transitive, let $a\ R\ b$ and $b\ R\ c$. Then by definition of R, we have $a - b = k_1 n$ and $b - c = k_2 n$, where $k_1, k_2 \in I$. It then follows that

$$a - c = (a - b) + (b - c) = k_1 n + k_2 n = (k_1 + k_2) n$$

where $\quad k_1 + k_2 \in I.$

25. Let R be a relation defined as $a\ R\ b$ iff $|a - b| > 0$. Then ,the relation R is

(a) reflexive (b) symmetric

(c) transitive (d) none of these

Solution (b) R is not reflexive since $|a - a| = 0$ and so $|a - a| \not> 0$.

Thus $a \not R\ a$ for any real number a.

R is symmetric since if $|a - b| > 0$, then

$|b - a| = |a - b| > 0.$

Thus $a\ R\ b \Rightarrow b\ R\ a$

R is not transitive. For example, consider the numbers 3, 7, 3. Then we have $3\ R\ 7$ since $|3 - 7| = 4 > 0$ and $7\ R\ 3$ since $|7 - 3| = 4 > 0$.

But $3 \not R\ 3$ since $|3 - 3| = 0$ so that $|3 - 3| \not> 0$.

26. Let R be a relation defined as $a\ R\ b$ iff $1 + ab > 0$. Then, the relation R is

(a) reflexive (b) symmetric

(c) transitive (d) none of these

Solution (a), (b)

27. Let R be a relation defined as $a\ R\ b$ if $|a| \le b$. Then, relation R is

(a) reflexive (b) symmetric

(c) transitive (d) none of these

Solution (c)

28. N is the set of natural numbers. The relation R is defined on $N \times N$ as follows $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$. Then, R is

(a) reflexive only (b) symmetric only

(c) transitive only (d) an equivalence relation

Solution (d)

29. In a class 100 students, 55 students have passed in Mathematics and 67 students have passed in Physics. Then the number of students who have passed in Physics only is

(a) 33 (b) 22

(c) 45 (d) 10

Solution (b)

30. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey; 24 played all the three games. The number of boys who did not play any game is

(a) 160 (b) 240

(c) 216 (d) 128

Solution (b)

31. Let $S = \{1, 2, 3, 4, 5\}$ and let $A = S \times S$. Define the relation R on A as follows "$(a, b)\ R\ (c, d)$ if and only if $ad = cd$. Then, R is

(a) reflexive only (b) symmetric only

(c) transitive only (d) equivalence relation

Solution (d)

32. The relation 'less than' in the set of natural numbers is

(a) only symmetric (b) only transitive

(c) only reflexive (d) equivalence relation

Solution (b)

33. If R and R' are symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is

(a) reflexive (b) symmetric
(c) transitive (d) none of these

Solution (b)

34. In a certain town 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Consider the following statements in this regard:

1. 10% families own both a car and a phone.
2. 35% families own either a car or a phone.
3. 40,000 families live in the town.

Which of the above statements are correct?

(a) 1 and 2 (b) 1 and 3
(c) 2 and 3 (d) 1, 2 and 3

Solution (c)

35. If R is a relation '<' from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$ i.e., $(a, b) \in R$ iff $a < b$, then $R \text{ o } R^{-1}$ is

(a) $\{(1, 2), (1, 5), (2, 3) (2, 5) (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)$
(d) $\{(3, 3), (3, 4), (4, 5)\}$

Solution (c)

Problems for Practice

Level of Difficulty I

1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6\}$ and $C = \{3, 4, 6, 7\}$, then

(a) $A - (B \cap C) = \{1, 3, 4\}$
(b) $A - (B \cap C) = \{1, 2, 4\}$
(c) $A - (B \cup C) = \{2, 3\}$
(d) $A - (B \cup C) = \{1\}$.

2. If the sets A and B are given by $A = \{1, 2, 3, 4, \}$, $B = \{2, 4, 6, 8, 10\}$ and the universal set

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then

(a) $(A \cup B)' = \{5, 7, 9\}$
(b) $(A \cap B)' = \{1, 3, 5, 6, 7\}$
(c) $(A \cap B)' = \{1, 3, 5, 6, 7, 8, 9, 10\}$
(d) none of these

3. Let $A = \{2, 3, 4\}$ and $X = \{0, 1, 2, 3, 4\}$, then which of the following statements is correct?

(a) $\{0\} \in A'$ in X (b) $\phi \in A'$ w.r.t. X
(c) $\{0\} \subset A'$ w.r.t X (d) $0 \subset A'$ w.r.t X.

4. $A = \{x : x \neq x\}$ represents

(a) $\{x\}$ (b) $\{1\}$ (c) $\{\}$ (d) $\{0\}$

5. The number of non-empty subsets of the set $\{1, 2, 3, 4\}$ is

(a) 15 (b) 14 (c) 16 (d) 17

6. X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$, $n(X \cup Y) = 38$ then $n(X \cap Y)$ is

(a) 4 (b) 2 (c) 6 (d) none of these

7. If A and B are two sets such that A has 12 elements, B has 17 elements and $A \cup B$ has 21 elements, then number of elements in $A \cap B$ are

(a) 6 (b) 4 (c) 8 (d) none of these

8. If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements and Y has 15 elements; then number of elements in $X \cap Y$ are

(a) 5 (b) 8 (c) 6 (d) none of these

9. If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, then number of elements $S \cup T$ has

(a) 42 (b) 50 (c) 48 (d) none of these

10. Given the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $[A \cup (B \cap C)]$ is

(a) $\{1, 2, 3, 4, 5, 6\}$ (b) $\{1, 2, 4, 5\}$
(c) $\{1, 2, 3, 4\}$ (d) $\{3\}$

11. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, $C = \{a, b, d, e,\}$, then $A \cap (B \cup C)$

(a) (c) (b) $\{a, b, c\}$
(c) $\{b, c, d\}$ (d) $\{a, b, d, e\}$.

12. Let $A = \{x : x$ is a multiple of $3\}$ and $B = \{x : x$ is a multiple of $5\}$. Then $A \cap B$ is given by

(a) $\{3, 6, 9...\}$ (b) $\{5, 10, 15, 20, ...\}$
(c) $\{15, 30, 45,...\}$ (d) none of these

13. Which of the following is the empty set?

(a) $\{x / x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x / x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x / x$ is a real number and $x^2 - 9 = 0\}$
(d) $\{x / x$ is a real number and $x^2 = x + 2\}$

14. If $n(U) = 700$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 100$, then $n(A' \cap B') =$

(a) 300 (b) 350 (c) 400 (d) none of these

15. The set of intelligent students in a class is

(a) a null set
(b) a singleton set
(c) a finite set
(d) not a well defined collection.

Level of Difficulty II

16. If P, Q and R are subsets of a set A, then $R \times (P^c \cup Q^c)^c$ equals

(a) $(R \times P) \cap (R \times Q)$ (b) $(R \times Q) \cap (R \times P)$
(c) $(R \times P) \cup (R \times Q)$ (d) None of these

17. The set $A = \{x: x \in R,\ x^2 = 16. \text{ and } 2x = 6\}$ equals

(a) ϕ (b) $\{14, 3, 4\}$ (c) $\{3\}$ (d) $\{4\}$

18. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to

(a) B^c (b) A^c (c) B (d) A

19. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49\ (n - 1): n \in N\}$, then

(a) $X \subseteq Y$ (b) $Y \subseteq X$

(c) $X = Y$ (d) none of these

20. If sets A and B are defined as

$A = \{(x, y): y = e^x, x \in R\}$

$B = \{(x, y): y = x, x \in R\}$ then

(a) $B \subset A$ (b) $A \subset B$

(c) $A \cap B = \phi$ (d) $A \cup B = A$

21. The solution of $3x^2 - 12x = 0$ when

(a) $x \in N$ is $\{4\}$

(b) $x \in I$ is $\{0, 4\}$

(c) $x \in S = \{a + ib: b \neq 0, a, b \in R\}$ is ϕ

(d) all of these

22. X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, then number of elements Y has

(a) 58 (b) 40 (c) 30 (d) none of these

23. If R be a relation $<$ from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$ i.e., $(a, b) \in R$ iff $a < b$, then RoR^{-1} is

(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

(b) $\{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$

(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$

(d) $\{(3, 3), (3, 4), (4, 5)$

24. (*i*) Let R be the relation on the set R of all real numbers defined by setting $a\ R\ b$ iff $|a - b| \leq \frac{1}{2}$. Then R is

(a) reflexive and symmetric but not transitive

(b) symmetric and transitive but not reflexive

(c) transitive but neither reflexive nor symmetric

(d) none of these

25. If $aN = \{ax : x \in N\}$ and $bN \cap cN = dN$, where $b, c \in N$ are relatively prime, then

(a) $c = bd$ (b) $b = cd$

(c) $d = bc$ (d) none of these

26. Set A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$?

(a) 18 (b) 9 (c) 6 (d) 3

27. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of second set. The value of m and n are

(a) 7, 6 (b) 6, 3 (c) 5, 1 (d) 8, 7

28. Let R be a relation defined on the set of natural numbers N as

$R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$. Then

(a) Domain of $R = \{1, 2, 3, ..., 19, 20\}$

(b) Range of $R = \{39, 37, 35, 9, 7, 5, 3, 1\}$

(c) R is reflexive

(d) R is symmetric.

29. Let $A = \{x: x \in R, |x| < 1\}$,

$B = \{x: x \in R, |x - 1| \geq 1\}$

and $A \cup B = R - D$, then the set D is

(a) $\{x: 1 < x \leq 2\}$

(b) $\{x: 1 \leq x < 2\}$

(c) $\{x: 1 \leq x \leq 2\}$

(d) none of these

30. With reference to a universal set, the inclusion of a subset in another, is relation which is

(a) Symmetric only (b) equivalence

(c) reflexive only (d) none of these

ANSWERS

1. (b),(d)	**2.** (a),(c)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (c)
11. (b)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (d)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (a)	**20.** (c)
21. (d)	**22.** (c)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (a),(b)	**29.** (b)	**30.** (c)

27 Vector Algebra

BRIEF REVIEW OF THE CONCEPTS

SCALARS AND VECTORS

Scalar quantity: A quantity which has only magnitude and no direction is called a *scalar quantity* or simply a *scalar*.

Examples of scalar are mass, temperature, volume, work and so on. To specify a scalar two things are needed.

1. a unit in terms of which it is measured
2. a real number (+ ve, –ve or zero)

Vector quantity: A quantity which has magnitude as well as direction is called a *vector quantity* or simply a *vector*.

Examples of vectors are displacement, velocity, acceleration, force and so on. To specify a vector, three things are needed

1. a unit in terms of which it is measured
2. a real number (+ve, –ve or zero)
3. a particular direction

REPRESENTATION OF VECTORS

The best way to represent a vector is with the help of a directed line segment. Suppose A and B are two points, then by the vector $\overrightarrow{AB}$, we mean a quantity whose magnitude is the length AB and whose direction is from A to B.

A and B are called the end points of the vector $\overrightarrow{AB}$. In particular A is called the initial point and B is called the terminal point.

Sometimes a vector $\overrightarrow{AB}$ is expressed by a single letter **a** (which is always written in bold type, to distinguish it from a scalar). Sometimes, however, we write the vector **a** as $\vec{a}$ **or** $\bar{a}$**.**

Modulus (or magnitude) of a Vector

The positive real number which is the measure of the length of the vector, is called the *modulus, length, magnitude, absolute value* or *norm* of the vector.

The modulus of a vector **a** or **OA** is usually denoted by $|\mathbf{a}|$ or $|\overrightarrow{OA}|$ or by the corresponding letter '*a*' (not in bold-faced type), i.e.,

$$|\overrightarrow{OA}| = OA \text{ and } |\mathbf{a}| = a.$$

Multiplication of a Vector by a Scalar

The product of a scalar m and a vector **a**, is defined as a vector $m\mathbf{a}$ or $\mathbf{a}m$ whose magnitude is the product of the magnitudes of m and **a** and whose direction is that of **a** or opposite to **a** accordingly as m is positive or negative.

TYPES OF VECTORS

Equal Vectors Two vectors **a** and **b** are equal when they have (1) the same magnitude and (2) the same direction. Symbolically such vectors are written as: $\mathbf{a} = \mathbf{b}$.

Unit Vectors A vector whose magnitude is unity is called a *unit vector*. The unit vector having the same direction as that of given vector **a** is usually denoted by the symbol $\hat{\mathbf{a}}$ (read as '**a** cap'), i.e.,

A vector = Modulus of vector × Unit vector in its direction

or $$\mathbf{a} = |\mathbf{a}|\ \hat{\mathbf{a}}$$

Also, Unit vector in a direction

$$= \frac{\text{vector in that direction}}{\text{modulus of vector}}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Zero or Null Vector A vector whose magnitude is zero, is called a *zero vector*. For such a vector, initial and terminal points are coincident so that its direction is indeterminate. A zero vector is denoted by the bold-faced symbol **O** or $\vec{O}$.

Collinear (or Parallel Vectors) The vectors which are parallel to the same straight line are called collinear vectors.

Vectors which are not parallel to the same line are called *non-collinear vectors.*

Like and Unlike Vectors Collinear vectors having the same direction are called *like* vectors and those having the opposite directions are called *unlike* vectors.

Remark: If two vectors **a** and **b** are collinear, then there exists a scalar m such that $\mathbf{b} = m\mathbf{a}$, m being positive or negative according as **a** and **b** are like or unlike vectors.

Conversely, if $\mathbf{b} = m\mathbf{a}$ be given, then **a** and **b** must be collinear (or parallel) vectors such that $|\mathbf{b}| = |m|\ |\mathbf{a}|$.

Reciprocal Vector Let $|\mathbf{a}|$ be the modulus of the given vector **a**. Then a vector whose direction is that of **a** but modulus is $1/|\mathbf{a}|$ (reciprocal of the modulus of **a**) is called the reciprocal of **a** and is written as $\mathbf{a}^{-1}$. Thus,

$$\mathbf{a}^{-1} = \hat{\mathbf{a}} = \frac{|\mathbf{a}|}{|\mathbf{a}|^2}\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|^2}.$$

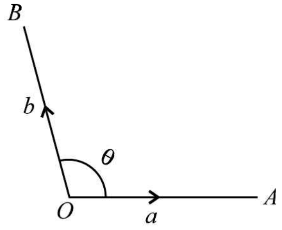
Coplanar and Non-coplanar Vectors Three or more vectors are said to be *coplanar* when they are parallel to the same plane. Otherwise they are said to be *non-coplanar vectors*.

Co-initial Vectors The vectors which have the same initial point are called *co-initial vectors*.

Negative of a Vector A vector having the same modulus as that of a given vector **a** and the direction opposite to that of **a**, is called the negative of **a** and is denoted by – **a**. Clearly, if **OA** = **a**, then **AO** = – **a**, and therefore, **OA** = – **AO**.

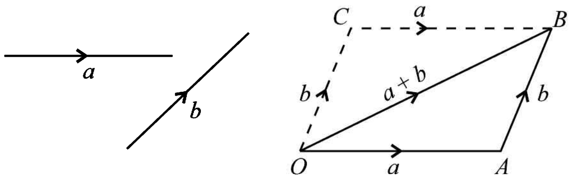
ANGLE BETWEEN TWO VECTORS

The angle between two vectors **a** and **b** represented by **OA** and **OB**, is defined as the angle *AOB* which does not exceed π. This is also known as the *inclination of given vectors* **a** and **b**. If the angle *AOB* be θ, then $0 \le \theta \le \pi$.



If $\theta = \frac{\pi}{2}$, then vectors are said to be *perpendicular* or *orthogonal* and if $\theta = 0$ or π, then vectors are said to be *parallel* or *coincident*.

ADDITION (SUM OR RESULTANT) OF TWO VECTORS



Let **a**, **b** be two vectors. Take any point *O* and draw the vectors **OA** = **a** and **AB** = **b** such that the terminal point of the vector **a** is the initial point of vector **b**. Join *OB*. Then the vector **OB** is defined as the sum of **a** and **b** and is written as

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{a} + \mathbf{b} \quad \text{...(1)}$$

This method of addition of vectors is known as the **triangle law of addition**.

Completing the parallelogram *OABC*. Since

$$\mathbf{AB} = \mathbf{OC} = \mathbf{b},$$

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{OA} + \mathbf{OC} \quad \text{...(2)}$$

That is, the sum of two co-initial vectors is the vector represented by the diagonal of the parallelogram formed with the component vectors as adjacent sides.

This method of addition of vectors is known as the **parallelogram law of addition**.

Remark: From Equation (1), – **BO** = **OA** + **AB** or **OA** + **AB** + **BO** = **O**, showing that *the sum of vectors determined by the sides of a triangle, taken in order, is zero.*

Properties of Vector Addition

1. **Vector addition is commutative** For any two vectors **a** and **b**, we have
$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$
2. **Vector addition is associative** For any three vectors **a**, **b** and **c**, we have
$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$
3. **Existence of additive identity** For every vector **a**, we have
$$\mathbf{a} + \mathbf{O} = \mathbf{a} = \mathbf{O} + \mathbf{a},$$
where **O** is the null vector.
4. **Existence of additive inverse** For a given vector **a**, there exists a vector –**a** such that
$$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}.$$
The vector – **a** is called the additive inverse of **a**.

Properties of Multiplication of Vector by a Scalar

1. If $m = O$, then $m\mathbf{a} = o$
2. If m and n be two scalars, then
$$m(n\mathbf{a}) = mn\mathbf{a} = n(m\mathbf{a})$$
3. If m and n be two scalars, then
$$(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}.$$
4. If **a**, **b** are any two vectors and m be any scalar, then
$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$

SUBTRACTION (OR DIFFERENCE) OF TWO VECTORS

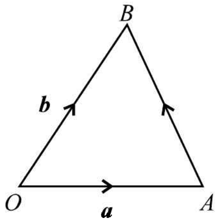
If **a** and **b** are any two given vectors, then subtraction of **b** from **a** is defined as the additon of – **b** to **a**, i.e.,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Hence to subtract a vector **b** from **a**, we should reverse the direction of **b** and add to **a**.

POSITION VECTOR OF A POINT

The position vector of a point A relative to a point *O* is the vector **OA** which specifies uniquely the position of *A* relative to *O*. The point *O* is called the *origin* or *origin of reference*.

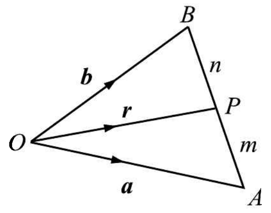


The Expression of a Vector in Terms of the Position Vectors of its End Points Let **AB** be a given vector. Let the position vectors of *A* and *B* with respect to origin *O* be **a** and **b** respectively so that **OA** = **a** and **OB** = **b**. Then

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \text{p.v. of } B - \text{p.v. of } A.$$

SECTION FORMULA

Internal Division The position vector of the point *P* which divides internally two given points *A* and *B* whose position vectors are **a** and **b** in a given ratio *m* : *n*, is



$$\mathbf{OP} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}.$$

Notes:

- If P is the mid-point of AB, then it divides AB in the ratio 1 : 1. Thus position vector of P is given by

$$\mathbf{OP} = \frac{\mathbf{a} + \mathbf{b}}{2}.$$

- Since $\mathbf{OP} = \mathbf{r} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n} = \frac{m}{m+n}\mathbf{b} + \frac{n}{m+n}\mathbf{a}$

$$= \lambda\mathbf{a} + \mu\mathbf{b},$$

where $\lambda = \frac{n}{m+n}$ and $\mu = \frac{m}{m+n}$.

Thus, p.v. of any point P on AB can always be taken as $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}$, where $\lambda + \mu = 1$.

- Since $\mathbf{OP} = \mathbf{r} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}$

$$\Rightarrow (m + n)\,\mathbf{r} = m\mathbf{b} + n\mathbf{a}$$

$$\Rightarrow n \cdot \mathbf{OA} + m \cdot \mathbf{OB} = (n + m)\,\mathbf{OP},$$

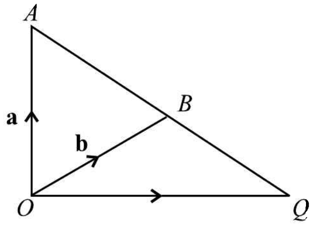
where P is a point on AB dividing it in the ratio $m : n$.

In particular, if P is the mid-point of AB, then

$$\mathbf{OA} + \mathbf{OB} = 2\mathbf{OP}.$$

That is, twice the position vector of the middle point is equal to the sum of the vectors of the ends.

External Division The position vector of the point Q, which divides externally the join of two given points A and B whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ in the given ratio $m : n$, is

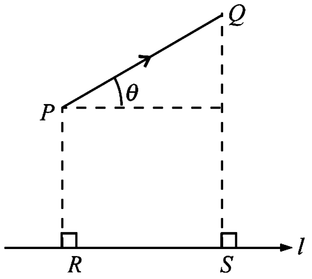


$$\mathbf{OQ} = \frac{m\mathbf{b} - n\mathbf{a}}{m - n}$$

COMPONENT OF A VECTOR

The component of a vector **PQ** on a line l is RS, where R and S are the feet of perpendiculars from P and Q on the line l.

The vector component of **PQ** on l will be denoted by **RS**.



1. If θ is the angle between **PQ** and **RS**, then the component of **PQ** on $l = PQ\cos\theta = |\mathbf{PQ}|\cos\theta$ and the vector component of **PQ** on $l = \mathbf{PQ}\cos\theta$.
2. If $\mathbf{r}$ is the position vector of a point P, having coordinates (x, y, z) then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along x, y and z axes respectively.
3. $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$ are the vector components of $\mathbf{r}$ on x, y and z axes repectively.
4. If a point P in space has coordinates (x, y, z), then its *pv* $\mathbf{r}$ is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

LINEAR COMBINATION

A vector $\mathbf{r}$ is said to be a linear combination of the given vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ ..., and so on if there exist a system of scalars $x, y, z, \ldots$, and so on such that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$$

LINEARLY DEPENDENT AND INDEPENDENT SYSTEM OF VECTORS

The system of n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, \ldots, x_n$ not all zero such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$$

The same system of vectors is said to be linearly independent, if all scalers are zero, i.e., $x_1 = x_2 = \ldots = x_n = 0$.

Remark: When the system of n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is linearly dependent (or independent), then n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are said to be linearly dependent (or independent).

COLLINEARITY OF THREE POINTS

The necessary and sufficient condition for three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to be collinear is that there exist three scalars x, y, z, not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0, \text{ where } x + y + z = 0.$$

Test of Collinearity of Two Vectors To prove that two vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, find a scalar m such that one of the vectors is m times the other. In case no such scalar m exists, then the two vectors will be non-collinear vectors.

Test of Collinearity of Three Points

Method 1: To prove that three points A, B, C are collinear, find the vectors **AB** and **AC** and show that there exists a scalar m such that $\mathbf{AB} = m\mathbf{AC}$.

If no such scalar m exists, then the points are not collinear.

Method 2: To prove that three points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively are collinear, find three scalars x, y, z (not all zero) such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0, \text{ where } x + y + z = 0.$$

If no such scalars x, y, z exist, then the points are not collinear.

COPLANARITY OF FOUR POINTS

The necessary and sufficient condition for four points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to be coplanar is that there exist scalars x, y, z and w, not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + w\mathbf{d} = 0$$

where $\quad x + y + z + w = 0.$

Test of Coplanarity of Three Vectors To prove that three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, express one of these vectors as the linear combination of the other two such as $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$.

Now, compare the coefficients from the two sides and find the values of x and y. If real values of scalars x and y exist, then the vectors are coplanar otherwise non-coplanar.

Test of Coplanarity of Four Points

Method 1: To prove that four points A, B, C and D are coplanar, find the vectors **AB**, **AC** and **AD** and show that these three vectors are coplanar.

Method 2: To prove that four points A, B, C and D with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively are coplanar, find four scalars x, y, z, w (not all zero) such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + w\mathbf{d} = 0 \text{ where } x + y + z + w = 0.$$

If no such scalars x, y, z, w exist, then the points are non-coplanar.

SOME RESULTS ON LINEARLY DEPENDENT AND INDEPENDENT VECTORS

1. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors, then these are linearly independent and conversely if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent, then they are non-coplanar.
2. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ are three linearly dependent vectors, then
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$
3. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-coplanar vectors. Then, vectors $x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}$, $x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$ and $x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}$ will be coplanar if
$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0.$$
4. Two non-zero, non-collinear vectors are linearly independent.
5. Any two collinear vectors are linearly dependent.
6. Any three non-coplanar vectors are linearly independent.
7. Any three coplanar vectors are linearly dependent.
8. Any four vectors in 3-dimensional space are linearly dependent.

PRODUCT OF TWO VECTORS

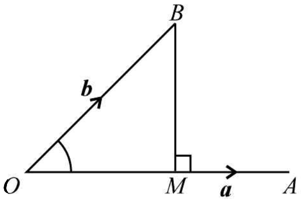
Between two vectors, two distinct kinds of products are defined. One being a pure number is called the *scalar product* while the other being a vector quantity is called the *vector product*.

SCALAR PRODUCT OF TWO VECTORS

The scalar product or dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as: $|\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where θ is the angle between them such that $0 \le \theta \le \pi$. It is denoted by placing a dot between the vectors $\mathbf{a}$ and $\mathbf{b}$. Thus,

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta.$$

If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{O}$, we define $\mathbf{a}\cdot\mathbf{b} = \mathbf{O}$.



Key Results on Operations on Sets

1. Scalar product is commutative. For any two vectors $\mathbf{a}$ and $\mathbf{b}$ we have
$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}.$$
2. If m is any scalar and $\mathbf{a}$, $\mathbf{b}$ be any two vectors, then
$$(m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(m\mathbf{b})$$
3. Scalar product is distributive with respect to vector addition, i.e., for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have
$$\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.$$
4. Magnitude of a vector as a scalar product: For any vector $\mathbf{a}$
$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a^2.$$
5. Scalar product of two perpendicular vectors is zero, i.e., if $\mathbf{a}$ and $\mathbf{b}$ are two perpendicular vectors, then $\mathbf{a}\cdot\mathbf{b} = 0$.
However, if $\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow$ Either $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a} \perp \mathbf{b}$.
6. Scalar product of mutually orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:
$$\mathbf{i}\cdot\mathbf{i} = 1 = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k}$$
and $\quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$
7. Scalar porduct of two vectors in terms of components: If
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$
then $\quad \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.
8. Angle between two vectors in terms of the components of the given vectors.
If θ is the angle between two vectors
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$
then $\quad \cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
9. Components of a vector $\mathbf{b}$ along and perpendicular to vector $\mathbf{a}$
$$\text{Component of } \mathbf{b} \text{ along } \mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$
$$\text{Component of } \mathbf{b} \text{ perpendicular to } \mathbf{a} = \mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$
10. Any vector $\mathbf{r}$ can be expressed as:
$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{i})\mathbf{i} + (\mathbf{r}\cdot\mathbf{j})\mathbf{j} + (\mathbf{r}\cdot\mathbf{k})\mathbf{k}.$$

SOME USEFUL IDENTITIES

Since scalar product satisfies commutative and distributive laws, we have

1. $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}\cdot\mathbf{b}$
2. $(\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$
3. $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$.

WORK DONE BY A FORCE

Work done by a force F in displacing a particle from A to B is defined by

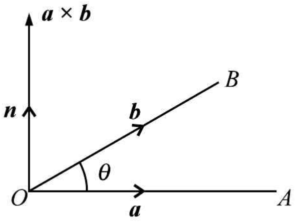
$$W = \boldsymbol{F}\cdot\boldsymbol{AB}.$$

Note: If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force

VECTOR PRODUCT OF TWO VECTORS

The vector product or cross product of two vectors **a** and **b** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \mathbf{n}$$



1. $|\mathbf{a}|\ |\mathbf{b}|\ \sin\theta$ is the modulus of $\mathbf{a}\times\mathbf{b}$, θ being the angle between the directions of **a** and **b** and $0 \le \theta \le \pi$;
2. direction of $\mathbf{a}\times\mathbf{b}$ is that of the unit vector **n** which is perpendicular to both **a** and **b** such that **a**, **b** and **n** form a right handed system.

Key Points to Remember

1. By right handed system we mean that as the first vector **a** is turned towards the second vector **b** through an angle θ, **n** will point in the direction in which a right handed screw would advance if turned in a similar manner.
2. If either **a** or **b** is **O**, we have $\mathbf{a}\cdot\mathbf{b} = O$.
3. $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$
4. A unit vector perpendicular to the plane of two given vectors **a** and **b** is given as $\mathbf{n} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$.
5. $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane of **a** and **b**.

Key Results on Operations on Sets

1. Vector product is not commutative. For any two vectors **a** and **b**

   $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$.

   In fact, $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$.
2. Vector product is associative with respect to a scalar. If m and n be any scalars and **a**, **b** any vectors, then

   $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = (\mathbf{a} \times \mathbf{b})m;$

   $(m\mathbf{a}) \times (n\mathbf{b}) = (n\mathbf{a}) \times (m\mathbf{b}) = (mn\mathbf{a}) \times \mathbf{b}$

   $= \mathbf{a} \times (mn\mathbf{b}) = mn(\mathbf{a} \times \mathbf{b})$.
3. Vector product is distributive with respect to addition. For any three vectors **a**, **b** and **c**

   $$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
4. If two vectors **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
5. Vector product of mutually orthogonal unit vectors **i**, **j**, **k**:

   $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = O$

   and $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\,\mathbf{j} \times \mathbf{i}$,

   $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\,\mathbf{i} \times \mathbf{k}$.
6. Vector product in terms of components. Let

   $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

   and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

   $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

   $$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
7. Angle between two vectors: If θ is the angle between two vectors **a** and **b**, then $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$.

Geometrical Interpretation of Cross Product

1. The area of a parallelogram with adjacent sides **a** and **b** is given by $|\mathbf{a}\times\mathbf{b}|$.
2. The area of a triangle with adjacent sides **a** and **b** is given by $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
3. The area of a triangle ABC is

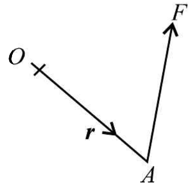
   $$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| \text{ or } \frac{1}{2}|\mathbf{BC}\times\mathbf{BA}| \text{ or } \frac{1}{2}|\mathbf{CB}\times\mathbf{CA}|.$$
4. The area of a parallelogram with diagonals **a** and **b** is given by $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
5. The area of a quadrilateral $ABCD$ is given by $\frac{1}{2}|\mathbf{AC}\times\mathbf{BD}|$, where AC and BD are its diagonals.
6. Vector area of a ΔABC, when **a**, **b**, **c** are the position vectors of A, B, C respectively is given by

   $$\Delta ABC = \frac{1}{2}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}).$$

MOMENT OF A FORCE ABOUT A POINT

The vector moment or torque **M** of a force **F** acting at a point A about the point O is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \mathbf{OA} \times \mathbf{F}$$



where **r** = **OA** is the position vector of the point A with respect to the point O.

Note: The algebraic sum of the moments of a system of forces about any point is equal to the moment of their resultant about the same point.

TRIPLE PRODUCTS

Scalar Triple Product

The scalar product of two vectors, one of which is itself the vector product of two vectors is a scalar quantity called a *scalar triple product*.

Thus if $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then $(\vec{a}\times\vec{b})\cdot\vec{c}$ is the scalar triple product of the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. Sometimes this scalar triple product is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$ or $[\vec{a},\vec{b},\vec{c}]$. Since the scalar triple product involves both the signs of 'cross' and 'dot', it is also called the *mixed product*.

Geometrical interpretation of scalar Triple product Geometrically, scalar triple product (**a** × **b**) × **c** represents volume of the parallelopiped with **a**, **b**, **c** as coterminous edges.

Notes:

- Since the ordered sets $(\vec{c},\ \vec{a},\ \vec{b})(\vec{a},\vec{b},\vec{c}),(\vec{b},\ \vec{c},\ \vec{a}),(\vec{c},\ \vec{a},\ \vec{b})$ are all right-handed or all left-handed, we easily observe that $(\vec{a}\times\vec{b})\cdot\vec{c}\ =(\vec{b}\times\vec{c})\cdot\vec{a}\ =(\vec{c}\times\vec{a})\cdot\vec{b}$.
- Since the dot product is commutative,

  $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot(\vec{a}\times\vec{b})$ ...(1)

  $(\vec{b}\times\vec{c})\cdot\vec{a}\ =\vec{a}\cdot(\vec{b}\times\vec{c})$ ...(2)

  Further $(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}$, [from Remark 1]

  Hence $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$ using (2)

  Thus, the scalar triple product is not altered if the dot and cross are interchanged
- Since the vector product is anti-commutative,

  $(\vec{a}\times\vec{b})\quad=-\vec{b}\times\vec{a}$

  $\therefore\ (\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{b}\times\vec{a})\cdot\vec{c}$.

  i.e., $[\vec{a}\,\vec{b}\,\vec{c}]\ =-[\vec{b}\,\vec{a}\,\vec{c}]$.

  Thus, if two factors of a scalar triple product are interchanged, its sign is changed.

Properties of Scalar Triple Product

1. If $\vec{i},\vec{j},\vec{k}$ constitute an orthonormal right-handed triad of vectors, then $[\vec{i}\ \vec{j}\ \vec{k}]$ = 1.

   We have $[\vec{i}\ \vec{j}\ \vec{k}]$ = $(\vec{i}\times\vec{j})\cdot\vec{k}$ = $\vec{k}\cdot\vec{k}$ = 1.
2. The value of a scalar triple product, if two of its vectors are equal, is zero. If $\vec{a}$,$\vec{a}$,$\vec{b}$ are three vectors, then

   $[\vec{a}\ \vec{a}\ \vec{b}]\ =(\vec{a}\times\vec{a})\cdot\vec{b}\ =\ \vec{0}\cdot\vec{b}\ =0.\ [\because\ \vec{a}\times\vec{a}=0]$
3. The value of a scalar triple product, if two of its vectors are parallel, is zero. If $[\vec{a}\ \vec{b}\ \vec{c}]$ are three vectors such that $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{b}$ =t $\vec{a}$ for some scalar t.

   $\therefore\ (\vec{a}\times\vec{b})\cdot\vec{c}\ [\vec{a}\ \vec{b}\ \vec{c}]\ =\ [\vec{a}\ \vec{b}\ \vec{c}]\ =\ (\vec{a}\times t\vec{a})\cdot\vec{c}$

   $=\ t[\vec{a}\times\vec{a}]\cdot\vec{c}\ =\ t(\vec{0}\times\vec{c})\ [\because\ \vec{a}\times\vec{a}=0]$

   $=\ t(0)\ =\ 0.$
4. The necessary and sufficient condition that three non-parallel and non-zero vectors $\vec{a}\ \vec{b}\ \vec{c}$ to be coplanar is that = 0.

Expression for Scalar Triple Product as a Determinant

To express $[\vec{a}\ \vec{b}\ \vec{c}]$ in terms of rectangular components of $\vec{a},\vec{b},\vec{c}$, let us take

$\vec{a}=\ a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}\ =\ b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$,

$\vec{c}=\ c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$. Then

$$\vec{b}\times\vec{c}=(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\times(c_1\vec{i}+c_2\vec{j}+c_3\vec{k})$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\ (b_2c_3-b_3c_2)\vec{i}-(b_1c_3-b_3c_1)\vec{j}+(b_1c_2-b_2c_1)\vec{k}\ .$$

Therefore, we have

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\qquad(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot[(b_2c_3-b_3c_2)\vec{i}$$

$$-(b_1c_3-b_3c_1)\vec{j}\ +(b_1c_2-(b_2c_1)\vec{k}]$$

$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)$$

$$+\ a_3\ (b_1c_2-b_2c_1)$$

$$\left[\because\ \vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1,\text{while}\ \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0\right]$$

Hence $$[\vec{a}\ \vec{b}\ \vec{c}]\ =\ \vec{a}\cdot(\vec{b}\times\vec{c})\ =\ \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

VECTOR TRIPLE PRODUCT

The vector product of two vectors, one of which is itself the vector product of two vector, is a vector quantity called a *vector triple product*.

Thus if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, the products of the form $\vec{a}\times(\vec{b}\times\vec{c}),(\vec{a}\times\vec{b})\times\vec{c}$ etc., are vector triple products. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,

we have $\vec{a}\times(\vec{b}\times\vec{c})\ =\ (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Notes:

- $(\vec{a}\times\vec{b})\times\vec{c}=-[\vec{c}\times(\vec{a}\times\vec{b})]\ [\because\ \vec{a}\times\vec{b}=-\vec{b}\times\vec{a}]$

  $=-[(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}]$

  $=\ (\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}$
- Vector triple product is not associative. We have, $\vec{a}\times(\vec{b}\times\vec{c})$ $=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

  Also, by Cor. 1., $(\vec{a}\times\vec{b})\times\vec{c}\ =(\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}$.

  Hence, $\vec{a}\times(\vec{b}\times\vec{c})\ \neq(\vec{a}\times\vec{b})\times\vec{c}$.

Solved Problems

1. If **a** and **b** are two perpendicular vectors, then
(a) $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2$
(b) $(\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2$
(c) $(\mathbf{a} + \mathbf{b})^2 = (\mathbf{a} - \mathbf{b})^2$
(d) all of the above are true

Solution (d) All the results are true as $\mathbf{a} \cdot \mathbf{b} = 0$.

2. Let $\mathbf{a} = -\frac{1}{2}$, $\mathbf{b} = \mathbf{j}$. The value of **c**, for which **a**, **b**, **c** form a right handed system is
(a) $y\mathbf{j}$ (b) $-z\mathbf{i} + x\mathbf{k}$
(c) 0 (d) $z\mathbf{i} - x\mathbf{k}$

Solution (d) **a**, **b**, **c** form a right handed system,
$\therefore$ $\mathbf{c} = \mathbf{a} \times \mathbf{b} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \mathbf{j} = x\mathbf{k} - z\mathbf{i}$.

3. The vectors $2\mathbf{i} - m\mathbf{j} + 3m\mathbf{k}$ and $(1 + m)\mathbf{i} - 2m\mathbf{j} + \mathbf{k}$ include an acute angle for
(a) $m = \frac{-1}{2}$ (b) $m \in \left[-2, \frac{-1}{2}\right]$
(c) all values of m (d) $m < -2$ or $m > \frac{-1}{2}$

Solution (d) The given vectors include an acute angle if
$(2\mathbf{i} - m\mathbf{j} + 3m\mathbf{k}) \cdot [(1 + m)\mathbf{i} - 2m\mathbf{j} + \mathbf{k}] > 0$
$\Rightarrow$ $2(1 + m) + 2m^2 + 3m > 0$
$\Rightarrow$ $2m^2 + 5m + 2 > 0 \Rightarrow (2m + 1)(m + 2) > 0$
$\Rightarrow$ $m < -2$ or $m > -\frac{1}{2}$.

4. If $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = 144$ and $|\mathbf{a}| = 4$ then $|\mathbf{b}| =$
(a) 16 (b) 8
(c) 3 (d) 12

Solution (c) We know that $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$
$\Rightarrow$ $144 = 16|\mathbf{b}|^2 \Rightarrow |\mathbf{b}| = \mathbf{3}$.

5. The value of $(\mathbf{a} - \mathbf{b}) \cdot [(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})]$ is:
(a) $2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $3[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(c) 0 (d) none of these

Solution (c) The given value
$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{c} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$
$= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})$
$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + 0 + 0 - 0 - \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$
$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = 0$
[Since $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$].

6. If the sum of two unit vectors is a unit vector, then the angle between them is equal to
(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/2$ (d) $2\pi/3$

Solution (d) Let $\hat{a} + \hat{b} = \hat{c}$
$\Rightarrow$ $(\hat{a} + \hat{b})^2 = (\hat{c})^2 \Rightarrow (\hat{a})^2 + (\hat{b})^2 + 2\hat{a} \cdot \hat{b} = (\hat{c})^2$
$\Rightarrow$ $\hat{a} \cdot \hat{b} = -1/2 \Rightarrow \cos\theta = -1/2 \Rightarrow \theta = \mathbf{2\pi/3}$.

7. If three points A, B, C whose position vector are respectively $\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$, $5\mathbf{i} - 2\mathbf{k}$ and $11\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ are collinear, then the ratio in which B divides AC is
(a) 1 : 2 (b) 2 : 3
(c) 2 : 1 (d) none of these

Solution (d) Let $B(5\mathbf{i} - 2\mathbf{k})$, divides AC where A $(\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})$ and $C(11\mathbf{i} + 3\mathbf{j} + 7\mathbf{k})$ in the ratio $\lambda : 1$ at P
$\therefore$ p.v. of P is

$$\frac{(11\lambda + 1)\,\mathbf{i} + (3\lambda - 2)\,\mathbf{j} + (7\lambda - 8)\,\mathbf{k}}{\lambda + 1}$$

i.e., $$\frac{11\lambda + 1}{\lambda + 1}\mathbf{i} + \frac{3\lambda - 2}{\lambda + 1}\mathbf{j} + \frac{7\lambda - 8}{\lambda + 1}\mathbf{k}$$

If P coincides with B, then

$$\frac{11\lambda + 1}{\lambda + 1} = 5,\ \frac{3\lambda - 2}{\lambda + 1} = 0,\ \frac{7\lambda - 8}{\lambda + 1} = -2$$

$\Rightarrow$ $\lambda = \frac{2}{3}$

$\therefore$ Required ratio is **2 : 3.**

8. The values of x for which the angle between $\mathbf{a} = 2x^2\,\mathbf{i} + 4x\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$ is obtuse and the angle between **b** and the **k**-axis is acute and less then $\frac{\pi}{6}$ are
(a) $x > \frac{1}{2}$ or $x < 0$ (b) $0 < x < \frac{1}{2}$
(b) $\frac{1}{2} < x < 15$ (d) form a null set

Solution (b) Since the angle between **a** and **b** is obtuse
$\therefore$ $\mathbf{a} \cdot \mathbf{b} < 0$
$\therefore$ $14x^2 - 8x + x < 0 \Rightarrow 14x^2 - 7x < 0$
i.e., $7x(2x - 1) < 0$
$\Rightarrow$ $x > 0, x < \frac{1}{2}$ or $x < 0, x > \frac{1}{2}$
$\Rightarrow$ $0 < x, x < \frac{1}{2}$ or $\frac{1}{2} < x < 0$ (not possible)
$\Rightarrow$ $\mathbf{0 < x < \frac{1}{2}}$.

9. Vectors **a** and **b** are inclined at an angle $\theta = 120°$. If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, then $[(\mathbf{a} + 3\mathbf{b}) \times (3\mathbf{a} - \mathbf{b})]^2$ is equal to
(a) 300 (b) 325
(c) 275 (d) 225

Solution (a) We have, $[(\mathbf{a} + 3\mathbf{b}) \times (3\mathbf{a} - \mathbf{b})]^2$
$= [10(\mathbf{b} \times \mathbf{a})]^2 = 100\,|\mathbf{b} \times \mathbf{a}|^2 = 100[\,|\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2]$
$= 100[(1)(4) - ((1)(2)\cos 120°)^2] = 100[4 - 1] = \mathbf{300}$.

10. Given the two vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$, the unit vector coplanar with the two vectors and perpendicular to the first is
(a) $\frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$ (b) $\frac{2\mathbf{i} + 2\mathbf{j}}{2\sqrt{2}}$
(c) $\frac{\mathbf{i} + \mathbf{j}}{2\sqrt{2}}$ (d) none of these

Solution (c) The vector coplanar with the given two vectors is $\lambda(\mathbf{i}-\mathbf{j})+\mu(\mathbf{i}+2\mathbf{j})$

$$= (\lambda+\mu)\mathbf{i}+(-\lambda+2\mu)\mathbf{j}$$

This is $\perp$ to $\mathbf{i}-\mathbf{j}$

$\therefore \quad (\lambda+\mu)\cdot 1+(-\lambda+2\mu)(-1)=0$

$\Rightarrow \quad \lambda+\mu+\lambda-2\mu=0 \Rightarrow 2\lambda-\mu=0 \Rightarrow 2\lambda=\mu$

$\therefore$ vector is $3\lambda\,\mathbf{i}+3\lambda\mathbf{j}=3\lambda(\mathbf{i}+\mathbf{j})$.

Since this is a unit vector,

$\therefore \quad 3\lambda\sqrt{1+1}=1$

$\therefore \quad \lambda=\dfrac{1}{3\sqrt{2}}$,

$\therefore$ required vector $=\dfrac{1}{\sqrt{2}}$ **$(\mathbf{i}+\mathbf{j})$.**

11. The vectors $2\mathbf{i}+3\mathbf{j}$, $5\mathbf{i}+6\mathbf{j}$ and $8\mathbf{i}+\lambda\mathbf{j}$ have their initial points at (1, 1). The value of λ so that the vectors terminate on one straight line is

(a) 9 (b) 6

(c) 3 (d) 0

Solution (a) Since the vector $2\mathbf{i}+3\mathbf{j}$ and $5\mathbf{i}+6\mathbf{j}$ have (1, 1) as initial point. $\therefore$ their terminal points are (3, 4) and (6, 7) respectively. The equation of the line jointing these two points is $y-4=\dfrac{7-4}{6-3}(x-3)$ i.e., $x-y+1=0$.

The terminal point of $8\mathbf{i}+\lambda\mathbf{j}$ is $(9\lambda+1)$. Since the vectors terminate on the same st. line. $\therefore$ $(9, \lambda+1)$ lies on $x-y+1=0$ and

$\therefore \quad 9-(\lambda-1)+1=0 \Rightarrow \lambda=\mathbf{9}.$

12. The unit vector parallel to the resultant of the vectors $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ is

(a) $\dfrac{1}{\sqrt{37}}(6\mathbf{i}+\mathbf{k})$ (b) $\dfrac{1}{\sqrt{37}}(6\mathbf{i}+\mathbf{j})$

(c) $\dfrac{1}{\sqrt{37}}(6\mathbf{i}+\mathbf{j})$ (d) none of these

Solution (a) Let $\mathbf{R}$ be the resultant of the given vectors. Then,

$\mathbf{R}=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})+(4\mathbf{i}-3\mathbf{j}+2\mathbf{k})=6\mathbf{i}+0\mathbf{j}+\mathbf{k}$.

The required unit vector parallel to $\mathbf{R}$

$$=\frac{\mathbf{R}}{|\mathbf{R}|}=\frac{6\mathbf{i}+0\mathbf{j}+\mathbf{k}}{\sqrt{(6^2+0^2+1^2)}}=\frac{6\mathbf{i}+0\mathbf{j}+\mathbf{k}}{\sqrt{(37)}}$$

$$=\frac{1}{\sqrt{(37)}}\ \mathbf{(6i+k)}.$$

13. If $\mathbf{a}=(2,1,-1)$, $\mathbf{b}=(1,-1,0)$ and $\mathbf{c}=(5,-1,1)$, then the unit vector parallel to $\mathbf{a}+\mathbf{b}-\mathbf{c}$, but in the opposite direction is

(a) $\dfrac{-1}{3}(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$ (b) $\dfrac{1}{3}(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$

(c) $\dfrac{1}{3}(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$ (d) none of these

Solution (b) Here $\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{b}=\mathbf{i}+\mathbf{j}+0\cdot\mathbf{k}$, $\mathbf{c}=5\mathbf{i}-\mathbf{j}+\mathbf{k}$.

$\therefore \quad \mathbf{a}+\mathbf{b}-\mathbf{c}=2\mathbf{i}+\mathbf{j}-\mathbf{k}+\mathbf{i}-\mathbf{j}-(5\mathbf{i}-\mathbf{j}+\mathbf{k})$

$$=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}.$$

$\therefore \quad |\mathbf{a}+\mathbf{b}-\mathbf{c}|=\sqrt{\{(-2)^2+1^2+(-2)^2\}}=3.$

$\therefore$ Unit vector in the direction of $\mathbf{a}+\mathbf{b}-\mathbf{c}$

$$=\frac{\mathbf{a}+\mathbf{b}-\mathbf{c}}{|\mathbf{a}+\mathbf{b}-\mathbf{c}|}=\frac{-2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{3}=-\frac{1}{3}(2\mathbf{i}-\mathbf{j}+2\mathbf{k}).$$

So the desired unit vector $=-\left\{-\dfrac{1}{3}(2\mathbf{i}-\mathbf{j}+2\mathbf{k})\right\}$

$$=\frac{1}{3}\ \mathbf{(2i-j+2k)}.$$

14. If $|\mathbf{a}|=3$, $|\mathbf{b}|=1$, $|\mathbf{c}|=4$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=$

(a) 12 (b) – 12

(c) 13 (d) – 13

Solution (d) We have, $(\mathbf{a}+\mathbf{b}+\mathbf{c})^2=\mathbf{0}$

$\Rightarrow \quad |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{a}\cdot\mathbf{b}+2\mathbf{b}\cdot\mathbf{c}+2\mathbf{c}\cdot\mathbf{a}=0$

$\Rightarrow \quad 9+1+16+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-\dfrac{26}{2}=\mathbf{-13}.$

15. If $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ and $|\mathbf{a}|=3$, $|\mathbf{b}|=5$, $|\mathbf{c}|=7$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{6}$

Solution (b) Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$

$\therefore \quad \mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$

$\therefore \quad \cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{(3)(5)}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{15}$ ...(1)

Now $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$

$\therefore \quad \mathbf{a}+\mathbf{b}=-\mathbf{c}$

$\Rightarrow \quad |\mathbf{a}+\mathbf{b}|=|-\mathbf{c}|=|\mathbf{c}| \Rightarrow |\mathbf{a}+\mathbf{b}|^2=|\mathbf{c}|^2$

$\Rightarrow \quad (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=(7)^2 \Rightarrow |\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2=49$

$\Rightarrow \quad (3)^2+2\mathbf{a}\cdot\mathbf{b}+(5)^2=49$

$\Rightarrow \quad 2\mathbf{a}\cdot\mathbf{b}=49-9-25=15 \Rightarrow \mathbf{a}\cdot\mathbf{b}=\dfrac{15}{2}.$

$\therefore$ From (1), we get

$$\cos\theta=\frac{15}{2\times 15}=\frac{1}{2}=\cos 60^\circ \Rightarrow \theta=\mathbf{60^\circ}.$$

16. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and θ is the angle between them, then $|\mathbf{a}-\mathbf{b}|=$

(a) $\left|\sin\dfrac{\theta}{2}\right|$ (b) $2\left|\sin\dfrac{\theta}{2}\right|$

(c) $4\left|\sin\dfrac{\theta}{2}\right|$ (d) None of these

Solution (b) We have,

$(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2+\mathbf{b}^2-2\mathbf{a}\cdot\mathbf{b}$

$$=1+1-2(1)(1)\cos\theta=2-2\cos\theta$$

$\Rightarrow \quad (\mathbf{a}-\mathbf{b})^2=2(1-\cos\theta)$

$\Rightarrow \quad |(\mathbf{a} - \mathbf{b})^2| = 2(2\sin^2\theta/2) = 4\sin^2\theta/2.$

$\Rightarrow \quad |\mathbf{a} - \mathbf{b}| = \mathbf{2}\,|\mathbf{sin}\,\theta/\mathbf{2}|.$

17. If the sum of two unit vectors, is a unit vector then the magnitude of their difference is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) 1 (d) none of these

Solution (b) Let **a** and **b** be two unit vectors such that their sum is also a unit vector

$|\mathbf{a}| = 1, |\mathbf{b}| = 1, |\mathbf{a} + \mathbf{b}| = 1.$

Now $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2$

$= (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b})\cdot(\mathbf{a} - \mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b})$

$= (|\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2) + (|\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2)$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{a}|^2 + |\mathbf{b}|^2$

$= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = 2(1)^2 + 2(1)^2 = 2 + 2 = 4.$

$\therefore \quad \mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 4$

But $|\mathbf{a} + \mathbf{b}| = 1,$

$\therefore \quad (1)^2 + |\mathbf{a} - \mathbf{b}|^2 = 4$

$\Rightarrow \quad |\mathbf{a} - \mathbf{b}|^2 = 4 - 1 = 3$

$\therefore \quad |\mathbf{a} - \mathbf{b}| = \boldsymbol{\sqrt{3}}.$

18. If $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, then a unit vector along the vector $\mathbf{a} \times \mathbf{b}$ is

(a) $\dfrac{-2\mathbf{j} + \mathbf{k}}{\sqrt{5}}$ (b) $\dfrac{-\mathbf{j} - 2\mathbf{k}}{\sqrt{5}}$

(c) $\dfrac{-2\mathbf{j} - \mathbf{k}}{\sqrt{5}}$ (d) none of these

Solution (c) Given $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$; $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & -2 & 4 \end{vmatrix}$$

$= \mathbf{i}(-4 + 4) - \mathbf{j}(12 + 2) + \mathbf{k}(-6 - 1) = -14\mathbf{j} - 7\mathbf{k}$

A unit vector along $\mathbf{a} \times \mathbf{b} = \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

$$= \frac{-14\mathbf{j} - 7\mathbf{k}}{\sqrt{(14)^2 + (7)^2}} = \frac{-14\mathbf{j} - 7\mathbf{k}}{\sqrt{245}} = \frac{-14\mathbf{j} - 7\mathbf{k}}{\sqrt{49 \times 5}}$$

$$= \frac{-14\mathbf{j} - 7\mathbf{k}}{7\sqrt{5}} = \frac{-2\mathbf{j} - \mathbf{k}}{\sqrt{5}}.$$

19. A vector whose length is 3 and which is perpendicular to each of the vectors

$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ is

(a) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (b) $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(c) $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ (d) none of these

Solution (c) A vector which is ⊥ to **a** as well as **b** is given by

$\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

$\mathbf{a} \times \mathbf{b} = (3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \times (6\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$= (-2 + 20)\mathbf{i} - (-6 + 24)\mathbf{j} + (15 - 6)\mathbf{k}$

$= 8\mathbf{i} - 18\mathbf{j} + 9\mathbf{k} = 9(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$

$\therefore$ A unit vector ⊥ to both **a** and **b** is

$$\eta = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{9(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})}{9\sqrt{4 + 4 + 1}} = \frac{2\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{3}$$

Hence the required vector

$$= 3(\eta) = 3\left(\frac{2\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{3}\right) = \mathbf{2i - 2j + k}.$$

20. The moment of the couple formed by the forces $5\mathbf{i} + \mathbf{k}$ and $-5\mathbf{i} - \mathbf{k}$ acting at the points (9, − 1, 2) and (3, − 2, 1) respectively, is

(a) $\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ (b) $\mathbf{i} - \mathbf{j} - 5\mathbf{k}$

(b) $\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ (d) none of these

Solution (b) Here $\mathbf{F} = 5\mathbf{i} + \mathbf{k}$, $-\mathbf{F} = -5\mathbf{i} - \mathbf{k}$.

The points of application of **F** and − **F** are

$A \equiv (9, -1, 2)$ and $B \equiv (3, -2, 1)$ respectively.

$\therefore \quad \mathbf{OA} = 9\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{OB} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\mathbf{BA} = \mathbf{OA} - \mathbf{OB} = (9\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\mathrm{BA} = 6\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\therefore$ Moment of couple $= \mathbf{BA} \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 1 \\ 5 & 0 & 1 \end{vmatrix}$$

$= \mathbf{i}(1 - 0) - \mathbf{j}(6 - 5) + \mathbf{k}(0 - 5) = \mathbf{i - j - 5k}.$

21. The torque about the point $3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ of a force $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ through the point $5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, is

(a) $\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$ (b) $\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$

(c) $\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$ (d) none of these

Solution (a) We have, $\mathbf{F} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Let *O* be the point $3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and *P* be the point $5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

$\therefore \quad \mathbf{OP} = (5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) - (3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} = \mathbf{r}$ (say)

Torque (vector moment) of **F** about *O*

$$= \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$= \mathbf{i}(3 - 2) - \mathbf{j}(2 - 4) + \mathbf{k}(4 - 12) = \mathbf{i + 2j - 8k}.$

22. If **i**, **j**, **k** are the unit vectors and mutually perpendicular, then $[\mathbf{i} - \mathbf{j} \;\; \mathbf{j} - \mathbf{k} \;\; \mathbf{k} - \mathbf{i}] =$

(a) 0 (b) 1

(c) − 1 (d) none of these

Solution (a) $(\mathbf{i} - \mathbf{j}) \cdot \{(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})\}$

$= (\mathbf{i} - \mathbf{j}) \cdot (\mathbf{j} \times \mathbf{k} - \mathbf{j} \times \mathbf{i} - \mathbf{k} \times \mathbf{k} + \mathbf{k} \times \mathbf{i})$

$= (\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + \mathbf{k} - 0 + \mathbf{j})$

$= \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{k} + \mathbf{i} \cdot \mathbf{j} - \mathbf{j} \cdot \mathbf{i} - \mathbf{j} \cdot \mathbf{k} - \mathbf{j} \cdot \mathbf{j}$

$= \mathbf{i}^2 + 0 + 0 - 0 - 0 - \mathbf{j}^2 = \mathbf{i}^2 - \mathbf{j}^2 = 1 - 1 = \mathbf{0}.$

23. For any three vectors **a**, **b**, **c**, $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$ is equal to

(a) 0 (b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(c) $2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (d) none of these

Solution (c) We have, $(\mathbf{a}+\mathbf{b})\cdot[(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})]$

$= (\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a})]$

$= (\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}]$ $[\because \mathbf{c}\times\mathbf{c}=0]$

$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{a})+\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})+\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$
$+\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{b}\cdot(\mathbf{b}\times\mathbf{a})+\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+0+0+0+0+\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

[scalar triple product is zero if two vectors are equal]

$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$

[scalar triple product is unchanged if vectors are cyclically permuted]

$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$= 2\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Hence, $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}] = \mathbf{2}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

24. If **i**, **j**, **k** are the unit vectors and mutually orthogonal, then $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})+\mathbf{j}\times(\mathbf{k}\times\mathbf{i})+\mathbf{k}\times(\mathbf{i}\times\mathbf{j}) =$

(a) **O** (b) 1

(c) – 1 (d) none of these

Solution (a) We have, $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})+\mathbf{j}\times(\mathbf{k}\times\mathbf{i})+\mathbf{k}\times(\mathbf{i}\times\mathbf{j})$

$= (\mathbf{i}\cdot\mathbf{k})\mathbf{j}-(\mathbf{i}\cdot\mathbf{j})\mathbf{k}+(\mathbf{j}\cdot\mathbf{i})\mathbf{k}-(\mathbf{j}\cdot\mathbf{k})\mathbf{i}+(\mathbf{k}\cdot\mathbf{j})\mathbf{i}-(\mathbf{k}\cdot\mathbf{i})\mathbf{j}$

$= \mathbf{0}$ $[\because \mathbf{i}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}, \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i} \text{ and } \mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}]$.

25. If the angle between the vectors $(x, 3, -7)$ and $(x, -x, 4)$ is acute, the interval in which x lies is

(a) $(-4, 7)$ (b) $[-4, 7]$

(c) $R-(-4, 7)$ (d) $R-[-4, 7]$

Solution (c) Let $\mathbf{a} = (x, 3, -7)$ and $\mathbf{b} = (x, -x, 4)$.

$\because$ since the agle between is acute

$\therefore$ $\mathbf{a}\cdot\mathbf{b} > 0 \Rightarrow x^2-3x-28 > 0 \Rightarrow (x-7)(x+4) > 0$

$\Rightarrow$ x does not belong to $(-4, 7)$

$\Rightarrow$ $x \in R-(-4, 7)$.

Hence the interval in which x lies $\boldsymbol{R-(-4, 7)}$.

26. The value of c so that for all real x the vectors $cx\mathbf{i}-6\mathbf{j}+3\mathbf{k}$ and $x\mathbf{i}+2\mathbf{j}+2cx\mathbf{k}$ make an obtuse angle are

(a) $c < 0$ (b) $0 < c < -\frac{4}{3}$

(c) $\frac{-4}{3} < c < 0$ (d) $c > 0$

Solution (c) $\because$ $cx\mathbf{i}-6\mathbf{j}+3\mathbf{k}$ and $x\mathbf{i}+2\mathbf{j}+cx\mathbf{k}$ make an obtuse angle

$\therefore$ $(cx\mathbf{i}-6\mathbf{j}+3\mathbf{k})\cdot(x\mathbf{i}+2\mathbf{j}+cx\mathbf{k}) < 0$

$\Rightarrow$ $cx^2+6cx-12 < 0$

$\Rightarrow$ (Coefficient of x^2), i.e., $c < 0$ and discriminant < 0

$\Rightarrow$ $36c^2+48c < 0 \Rightarrow c(3c+4) < 0$

$\Rightarrow$ c should lie between $-\frac{4}{3}$ and 0.

Hence $\boldsymbol{-\frac{4}{3} < c < 0}$.

27. If $\mathbf{u} = \mathbf{a}-\mathbf{b}$, $\mathbf{v} = \mathbf{a}+\mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}| = 2$, then $|\mathbf{u}\times\mathbf{v}| =$

(a) $2\sqrt{[16-(\mathbf{a}\cdot\mathbf{b})^2]}$ (b) $\sqrt{[16-(\mathbf{a}\cdot\mathbf{b})^2]}$

(c) $2\sqrt{[4-(\mathbf{a}\cdot\mathbf{b})^2]}$ (d) $\sqrt{[4-(\mathbf{a}\cdot\mathbf{b})^2]}$

Solution (a) $\mathbf{u}\times\mathbf{v} = (\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\times\mathbf{b}$

$\therefore$ $|\mathbf{u}\times\mathbf{v}| = 2|\mathbf{a}\times\mathbf{b}|$

$= 2\sqrt{[a^2b^2\sin^2\theta]} = 2\sqrt{(a^2b^2-a^2b^2\cos^2\theta)}$

$= \mathbf{2}\sqrt{[16-(\mathbf{a}\cdot\mathbf{b})^2]}$ $[\because |\mathbf{a}| = |\mathbf{b}| = 2]$

28. If the unit vectors **a** and **b** are inclined at angle 2θ $(0 \le \theta \le \pi)$ and $|\mathbf{a}-\mathbf{b}| < 1$, then θ lies in the interval

(a) $\left[0, \frac{\pi}{6}\right)$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

(c) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$ (d) none of these

Solution (a) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos 2\theta$

$\Rightarrow$ $\mathbf{a}\cdot\mathbf{b} = (1)(1)\cos 2\theta = \cos 2\theta$.

$|\mathbf{a}-\mathbf{b}| < 1$

$\Rightarrow$ $\mathbf{a}^2+\mathbf{b}^2-2\mathbf{a}\cdot\mathbf{b} < 1 \Rightarrow 1+1-2\cos 2\theta < 1$

$\Rightarrow$ $2(1-\cos 2\theta) < 1 \Rightarrow 2(2\sin^2\theta) < 1$

$\Rightarrow$ $\sin^2\theta < \frac{1}{4} \Rightarrow \theta$ lies in $\left[\mathbf{0}, \frac{\pi}{\mathbf{6}}\right)$.

29. The value of $\frac{[(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}\cdot\mathbf{b})^2]}{2a^2b^2} =$

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) $\frac{1}{4}$

Solution (b) $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = a^2b^2\sin^2\theta$

$(\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta = a^2b^2\cos^2\theta$

$$\therefore \quad \frac{[(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}\cdot\mathbf{b})^2]}{2a^2b^2} = \frac{a^2b^2\sin^2\theta+a^2b^2\cos^2\theta}{2a^2b^2}$$

$$= \frac{a^2b^2(\sin^2\theta+\cos^2\theta)}{2a^2b^2} = \frac{\mathbf{1}}{\mathbf{2}}.$$

30. If **a**, **b**, **c** are three vectors such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$ and **a**, **b**, **c** are perpendicular to $\mathbf{b}+\mathbf{c}$, $\mathbf{c}+\mathbf{a}$, $\mathbf{a}+\mathbf{b}$ respectively, then $|\mathbf{a}+\mathbf{b}+\mathbf{c}| =$

(a) $6\sqrt{2}$ (b) $4\sqrt{2}$

(c) $3\sqrt{2}$ (d) $5\sqrt{2}$

Solution (d) $\because$ $\mathbf{a} \perp (\mathbf{b}+\mathbf{c})$

$\therefore$ $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = 0$ ...(1)

Similarly $\mathbf{b} \perp (\mathbf{c}+\mathbf{a}) \Rightarrow \mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a} = 0$ ...(2)

and $\mathbf{c} \perp (\mathbf{a}+\mathbf{b}) = 0 \Rightarrow \mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b} = 0$ ...(3)

Adding (1), (2), (3), we get

$2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}) = 0$.

Now, $(\mathbf{a}+\mathbf{b}+\mathbf{c})^2 = \mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$

$= |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+0$

$= (3)^2+(4)^2+(5)^2 = 50$.

Hence, $|\mathbf{a}+\mathbf{b}+\mathbf{c}| = \mathbf{5}\sqrt{\mathbf{2}}$.

31. If $A = \mathbf{i}+\lambda\mathbf{j}+\mathbf{k}$, $B = \mathbf{i}+\mathbf{j}+\mathbf{k}$ then for $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}|+|\mathbf{B}|$ to be true, the value of $\lambda =$

(a) – 1 (b) 2

(c) – 2 (d) 1

Solution (d) We have, $|\mathbf{A} + \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$

$\Rightarrow$ $\mathbf{A} + \mathbf{B}$, $\mathbf{A}$, $\mathbf{B}$ are collinear.

$\Rightarrow$ $\mathbf{i} + \lambda\mathbf{j} + \mathbf{k} = t(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow t = 1, t = \lambda$

Hence $\lambda = \mathbf{1}$.

32. For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2 =$

(a) a^2 (b) $3a^2$

(c) $4a^2$ (d) $2a^2$

Solution (d) Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\therefore$ $\mathbf{a} \times \mathbf{i} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \mathbf{i} = -y\mathbf{k} + z\mathbf{j}$

$\therefore$ $(\mathbf{a} \times \mathbf{i})^2 = (-y\mathbf{k} + z\mathbf{j})^2 = \left(\sqrt{(-y)^2 + z^2}\right)^2$

$\Rightarrow$ $(\mathbf{a} \times \mathbf{i})^2 = y^2 + z^2$.

Similarly $(\mathbf{a} \times \mathbf{j})^2 = z^2 + x^2$ and $(\mathbf{a} \times \mathbf{k})^2 = x^2 + y^2$.

Hence, $(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2$

$= y^2 + z^2 + z^2 + x^2 + x^2 + y^2$

$= 2(x^2 + y^2 + z^2) = \mathbf{2a^2}$

$[\because a = |\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}$,

$\therefore a^2 = x^2 + y^2 + z^2]$.

33. If $\mathbf{a} = (2, 1, 1)$, $\mathbf{b} = (1, 0,3)$, $\mathbf{c} = (2, 1, 3)$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, then $(x, y, z) =$

(a) $(0, -8, 5)$ (b) $(8, 0, -5)$

(c) $(0, 8, -5)$ (d) $(8, -5, 0)$

Solution (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\therefore$ $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

$\Rightarrow$ $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

$\Rightarrow$ $x = 0, y = \mathbf{a}\cdot\mathbf{c}$ and $z = -(\mathbf{a}\cdot\mathbf{b}) \Rightarrow x = 0$

$y = (2\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 4 + 1 + 3 = 8$

$z = -[2(1) + 1(0) + 1(3)] = -5$

Hence $(x, y, z) = \mathbf{(0, 8, -5)}$.

34. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is an orthogonal system of vectors, $\mathbf{a}$ is a vector and $\mathbf{a} \times \mathbf{i} + 2\mathbf{a} - 5\mathbf{j} = 0$ then $\mathbf{a} =$

(a) $2\mathbf{j} + \mathbf{k}$ (b) $2\mathbf{j} - \mathbf{k}$

(c) $2\mathbf{i} + \mathbf{j}$ (d) $2\mathbf{j} + \mathbf{j}$

Solution (a) Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Then $\mathbf{a} \times \mathbf{i} + 2\mathbf{a} - 5\mathbf{j} = 0$

$\Rightarrow$ $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \mathbf{i} + \mathbf{2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - 5\mathbf{j} = 0$

$\Rightarrow$ $-y\mathbf{k} + z\mathbf{j} + 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - 5\mathbf{j} = 0$

$\Rightarrow$ $(2x)\mathbf{i} + (z + 2y - 5)\mathbf{j} + (-y + 2z)\mathbf{k} = 0$

$\Rightarrow$ $x = 0, z + 2y - 5 = 0, -y + 2z = 0$

$\Rightarrow$ $x = 0, y = 2, z = 1.$

$\therefore$ $\mathbf{a} = (0)\mathbf{i} + 2\mathbf{j} + (1)\mathbf{k} = \mathbf{2j + k}$.

35. The volume of the tetrahedron whose vertices are the points $0(0, 0, 0)$, $A(1, -1, 1)$, $B(\lambda, 0, 1)$ and $C(0, 1, \lambda)$ is $\dfrac{5}{6}$ cubic units, if the value of λ is

(a) -3 (b) 3

(c) -2 (d) 2

Solution (a), (d) Volume $= \dfrac{1}{6}[\mathbf{OA\ OB\ OC}]$

$$\therefore \quad \frac{5}{6} = \frac{1}{6}\begin{vmatrix} 1 & -1 & 1 \\ \lambda & 0 & 1 \\ 0 & 1 & \lambda \end{vmatrix}$$

$\Rightarrow$ $\lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3) = 0 \Rightarrow \lambda = \mathbf{2, -3}$.

36. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ then the value of $\dfrac{\text{the projection of } \mathbf{a} \text{ on } \mathbf{b}}{\text{the projection of } \mathbf{b} \text{ on } \mathbf{a}}$ is equal to

(a) $\dfrac{7}{3}$ (b) 2

(c) 4 (d) none of these

Solution (a) We have,

$$\frac{\text{the projection of } \mathbf{a} \text{ on } \mathbf{b}}{\text{the projection of } \mathbf{b} \text{ on } \mathbf{a}} = \frac{\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}{\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}} = \frac{|\mathbf{a}|}{|\mathbf{b}|}$$

$$= \frac{\sqrt{4 + 9 + 36}}{\sqrt{4 + 4 + 1}} = \frac{\mathbf{7}}{\mathbf{3}}.$$

37. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ be three vectors. A vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ whose projection on $\mathbf{a}$ is of magnitude $\sqrt{\dfrac{2}{3}}$ is

(a) $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

(c) $-2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (d) $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

Solution (a), (c) Any vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ is

$\mathbf{r} = \mathbf{b} + \lambda\mathbf{c} = (1 + 2\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

$= (1 + \lambda)\mathbf{i} + (2 + \lambda)\mathbf{j} + (-1 - 2\lambda)\mathbf{k}$

Projection of $\mathbf{r}$ on $\mathbf{a}$ is $= \dfrac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{a}|}$

$$= \frac{2(1 + \lambda) - (2 + \lambda) - (1 + 2\lambda)}{\sqrt{4 + 1 + 1}} = \frac{-\lambda - 1}{\sqrt{6}}$$

$\therefore$ $\dfrac{-\lambda - 1}{\sqrt{6}} = \pm\sqrt{\dfrac{2}{3}} \Rightarrow -\lambda - 1 = \pm 2; \lambda = -3$ or 1

Hence $\mathbf{r} = -\mathbf{2i - j + 5k}$ or $\mathbf{r} = \mathbf{2i + 3j - 3k}$.

38. If $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$, then $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$ is equal to

(a) a non-zero vector (b) 1

(c) -1 (d) $|\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$

Solution (d) We have, $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

$\Rightarrow$ $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular

$\therefore$ $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = |\mathbf{a}|\,|(\mathbf{b} \times \mathbf{c})| \cos 0$

$= |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}| \sin\dfrac{\pi}{2} = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$.

39. If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and $|\mathbf{a} + \mathbf{b}| = 5$, then $|\mathbf{a} - \mathbf{b}| =$

(a) 8 (b) 5

(c) 4 (d) 3

Solution (b) We know that

$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{bb}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$

$\therefore$ $25 + |\mathbf{a} - \mathbf{b}|^2 = 2(9 + 16)$

$\Rightarrow$ $|\mathbf{a} - \mathbf{b}|^2 = 50 - 25 = 25 \Rightarrow |\mathbf{a} - \mathbf{b}| = \mathbf{5}$.

40. If $\mathbf{a}\cdot\mathbf{i} = 4$, then $(\mathbf{a}\cdot\mathbf{j}) \times (2\mathbf{j} - 3\mathbf{k}) =$

(a) 12 (b) 2

(c) 0 (d) – 12

Solution (d) We have, $(\mathbf{a}\cdot\mathbf{j}) \times (2\mathbf{j} - 3\mathbf{k})$

$$= \mathbf{a}\cdot(\mathbf{j}\times(2\mathbf{j}-3\mathbf{k}) = \mathbf{a}\cdot(-3(\mathbf{j}\times\mathbf{k}) = \mathbf{a}\cdot(-3(\mathbf{j}\times\mathbf{k})$$

$$= -3(a\cdot\mathbf{i}) \qquad [\because \mathbf{j}\times\mathbf{k} = \mathbf{i}]$$

$$= -3(4) = -\mathbf{12}.$$

41. **a** and **b** are two unit vectors and θ is the angle between them. Then **a** + **b** is a unit vector if

(a) $\theta = \pi/3$ (b) $\theta = \pi/4$

(c) $\theta = \pi/2$ (d) $\theta = 2\pi/3$

Solution (d) **a** + **b** is unit vector if $|(\mathbf{a}+\mathbf{b})|^2 = 1$

or $(\mathbf{a}+\mathbf{b})^2 = 1$

or $\mathbf{a}^2 + \mathbf{b}^2 + 2|\mathbf{a}|\,|\mathbf{b}|\cos\theta = 1$

or $1 + 1 + 2\cdot1\cdot1\cdot\cos\theta = 1$

$\therefore$ $\cos\theta = -\dfrac{1}{2} = -\cos(\pi/3).$

$= \cos(\pi - \pi/3) = \cos(2\pi/3);\ \therefore\ \theta = \mathbf{2\pi/3}.$

42. θ is the angle between two vectors **a** and **b** then $\mathbf{a}\cdot\mathbf{b} \geq 0$ only if

(a) $0 \leq \theta \leq \pi$

(b) $\pi/2 \leq \theta \leq \pi$

(c) $0 \leq \theta \leq \pi/2$

(d) $0 < \theta < \pi/2$

Solution (c) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \geq 0$ if $\cos\theta \geq 0$; i.e., $0 \leq \theta \leq \dfrac{\pi}{2}$.

43. If **a** and **b** are unit vectors and α is the angle between them, then the value of α such that **a** + **b** is a unit vector, is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$

(c) $\dfrac{2\pi}{3}$ (d) none of these

Solution (c) Given $|\mathbf{a}| = 1 = |\mathbf{b}|$.

$\therefore$ $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\alpha = 1\cdot1\cdot\cos\alpha = \cos\alpha$

$\because$ **a** + **b** is a unit vector

$\therefore$ $|\mathbf{a}+\mathbf{b}| = 1$ or $|\mathbf{a}+\mathbf{b}|^2 = 1$

or $(\mathbf{a}+\mathbf{b})^2 = 1$ or $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{2}\,\mathbf{a}\cdot\mathbf{b} = 1$

or $1 + 1 + 2\cos\alpha = 1$

or $\cos\alpha = -1/2\ \therefore\ \alpha = \mathbf{120^\circ}.$

44. Consider a tetrahedron with faces F_1, F_2, F_3, F_4. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the vectors whose magnitudes are respectively equal to areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to their faces in outward direction. Then $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4|$ equals

(a) 1 (2) 4

(c) 0 (d) none of these

Solution (c) We have,

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{a}\times\mathbf{b}),\ \mathbf{v}_2 = \frac{1}{2}(\mathbf{b}\times\mathbf{c})$$

$$\mathbf{v}_3 = \frac{1}{3}(\mathbf{c}\times\mathbf{a}) \text{ and } \mathbf{v}_4 = \frac{1}{2}\{(\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})\}$$

$\therefore$ $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$

$\therefore$ $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4| = \mathbf{0}.$

45. Let **a**, **b** and **c** be three non-zero and non-coplanar vectors and **p**, **q** and **r** be three vectors given by

$\mathbf{p} = \mathbf{a} + \mathbf{b} - 2\mathbf{c}$, $\mathbf{q} = 3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ and $\mathbf{r} = \mathbf{a} - 4\mathbf{b} + 2\mathbf{c}$. If the volume of the parallelopiped determined by **a**, **b** and **c** is V_1 and the volume of the parallelopiped determined by **p**, **q** and **r** is V_2 then $V_2 : V_1 =$

(a) 3 : 1 (b) 7 : 1

(c) 11 : 1 (d) 15 : 1

Solution (d) Given $\mathbf{p} = \mathbf{a} + \mathbf{b} - 2\mathbf{c}$

$\mathbf{q} = 3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$

and $\mathbf{r} = \mathbf{a} - 4\mathbf{b} + 2\mathbf{c}$

Also, given $V_1 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ ...(1)

and $$V_2 = [\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 1 & -4 & 2 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$\Rightarrow$ $V_2 = 15[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ ...(2)

From (1) and (2), $V_2 : V_1 = \mathbf{15 : 1}.$

46. If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mu\mathbf{j} + 5\mathbf{k}$ are coplanar, then μ is a root of the equation

(a) $x^2 + 3x = 4$ (b) $x^2 + 2x = 6$

(c) $x^2 + 3x = 6$ (d) none of these

Solution (a) Since **a**, **b**, **c** are coplanar,

$$\therefore\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \mu & 5 \end{vmatrix} = 0$$

$\Rightarrow$ $\mu = -4$

which is a root of the equation $\mathbf{x^2 + 3x = 4}$.

47. $p\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\left|\sqrt{q}\,\mathbf{i} + 5\mathbf{k}\right|$ are two vectors, where $p, q \geq 0$ are two scalars, then the length of the vectors is equal for

(a) all values of (p, q)

(b) only finite number of value of (p, q)

(c) infinite number of values of (p, q)

(d) no value of (p, q)

Solution (c) Given $|p\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}| = \left|\sqrt{q}\,\mathbf{i} + 5\mathbf{k}\right|$

$\Rightarrow$ $\sqrt{(p^2 + 25)} = \sqrt{(q + 25)}$

$\Rightarrow$ $p^2 + 25 = q + 25 \Rightarrow p^2 = q,\ p, q \geq 0.$

Hence there are infinite values of p, q satisfying $p^2 = q$, $p, q \geq 0$.

48. If **a**, **b**, **c** are three unit vectors such that $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 1$ and $\mathbf{a} \perp \mathbf{b}$. If **c** makes angles α, β with **a**, **b** respectively, then $\cos\alpha + \cos\beta$ is equal to

(a) $\dfrac{3}{2}$ (b) 1

(c) – 1 (d) none of these

Solution (c) Since $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 1\ \therefore\ |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = 1$

$\Rightarrow$ $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2|\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}| = 1$

$\Rightarrow$ $1 + 1 + 1 + 2(0 + 1\cdot1\cos\beta + 1\cdot1\cos\alpha) = 1$

$\Rightarrow$ $2(\cos\alpha + \cos\beta) = -2$

$\Rightarrow$ $\cos\alpha + \cos\beta = -\mathbf{1}.$

49. For a non-zero vector **a**, the set of real numbers satisfying the inequality $|(5-x)\mathbf{a}| < |2\mathbf{a}|$ consists of all x such that

(a) $0 < x < 3$ (b) $3 < x < 7$

(c) $-7 < x < -3$ (d) $-7 < x < 3$

Solution (b) We have, $|(5-x)\mathbf{a}|^2 < |2\mathbf{a}|^2$.

$\Rightarrow$ $(5-x)^2 < 4 \Rightarrow 5 - x < 2$ or $5 - x > -2$

$\Rightarrow$ $5 - 2 < x$ or $-x > -7$

$\Rightarrow$ $3 < x$ or $x < 7 \Rightarrow \mathbf{3 < x < 7}$.

50. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1,2,1)$, $B(2, 1, 3)$ and $C(-1, 1,2)$. Then the angle between the faces OAB and ABC will be

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{71}{31}\right)$

(c) 30° (d) 90°

Solution (a) Vector $\perp$ to face OAB

$$= \mathbf{OA} \times \mathbf{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \quad \text{...(1)}$$

Vector $\perp$ to the face ABC

$$= \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 3\mathbf{k} \quad \text{...(2)}$$

Since the angle between the faces = angle between their normals

$$\therefore \quad \cos\theta = \left|\frac{5+5+9}{\sqrt{35}\,\sqrt{35}}\right| = \frac{19}{35}$$

$$\therefore \quad \theta = \mathbf{\cos^{-1}}\left(\frac{19}{35}\right).$$

51. If P, Q, R are three points with respective position vectors **i** + **j**, **i** – **j** and a**i** + b**j** + c**k**. The points P, Q, R are collinear if

(a) $a = b = c = 1$ (b) $a = b = c = 0$

(c) $a = 1, b, c \in R$ (d) $a = 1, c = 0, b \in R$

Solution (d) Since P, Q, R are collinear

$\therefore$ $\mathbf{PQ} = t\mathbf{QR}$ [where t is a non-zero scalar]

$\Rightarrow$ $(\mathbf{i} - \mathbf{j}) - (\mathbf{i} + \mathbf{j}) = t[a\mathbf{i} + b\mathbf{j} + c\mathbf{k} - \mathbf{i} + \mathbf{j}]$

$\Rightarrow$ $-2\mathbf{j} = t[(a-1)\mathbf{i} + (b+1)\mathbf{j} + c\mathbf{k}]$

for some non-zero scalar t.

$\Rightarrow$ $a - 1 = 0$, $t(b + 1) = -2$, $ct = 0$

$\Rightarrow$ $\mathbf{a = 1, c = 0, b \in R}$.

52. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$, then the vectors

$(\mathbf{a}\cdot\mathbf{i})\mathbf{i} + (\mathbf{a}\cdot\mathbf{j})\mathbf{j} + (\mathbf{a}\cdot\mathbf{k})\mathbf{k}$,

$(\mathbf{b}\cdot\mathbf{i})\mathbf{i} + (\mathbf{b}\cdot\mathbf{j})\mathbf{j} + (\mathbf{b}\cdot\mathbf{k})\,\mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(a) are mutually perpendicular

(b) are coplanar

(c) form a parallelopiped of volume 6 units

(d) form a parallelopiped of volume 3 units

Solution (a), (c) Let $\mathbf{A} = (\mathbf{a}\cdot\mathbf{i})\mathbf{i} + (\mathbf{a}\cdot\mathbf{j})\mathbf{j} + (\mathbf{a}\cdot\mathbf{k})\mathbf{k}$

$= (1)\mathbf{i} + (1)\mathbf{j} + (1)\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{B} = (\mathbf{b}\cdot\mathbf{i})\mathbf{i} + (\mathbf{b}\cdot\mathbf{j})\mathbf{j} + (\mathbf{b}\cdot\mathbf{k})\mathbf{k}$

$= (1)\mathbf{i} + (-1)\mathbf{j} + 0\mathbf{k} = \mathbf{i} - \mathbf{j}$ and

$\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Then, $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbf{A} = \mathbf{O}$

$\therefore$ **A**, **B** and **C** are mutually perpendicular

Also, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 6$, $\therefore$ **A**, **B**, **C**, form a parallelopiped of volume 6 units.

53. For non-zero vectors **a**, **b**, **c**, $|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ holds if and only if

(a) $\mathbf{a}\cdot\mathbf{b} = 0$, $\mathbf{b}\cdot\mathbf{c} = 0$ (b) $\mathbf{b}\cdot\mathbf{c} = 0$, $\mathbf{c}\cdot\mathbf{a} = 0$

(c) $\mathbf{c}\cdot\mathbf{a} = 0$, $\mathbf{a}\cdot\mathbf{b} = 0$ (d) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

Solution (d) $|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$

$\Rightarrow$ $|ab \sin\theta\ \mathbf{n}\cdot\mathbf{c}| = abc$

$\Rightarrow$ $|(ab \sin\theta)\ 1c \cos\phi| = abc$

$\Rightarrow$ $|\sin\theta|\,|\cos\phi| = 1 \Rightarrow \theta = \frac{\pi}{2}$ and $\phi = 0$

$\Rightarrow$ **a** is $\perp$ to **b** and **c** $\parallel$ **n**

$\Rightarrow$ **a** is $\perp$ to **b** and **c** is $\perp$ to both **a** and **b**

$\Rightarrow$ **a**, **b**, **c** are mutually perpendicular

$\therefore$ $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = \mathbf{0}$.

54. The vectors $2\mathbf{i} - m\mathbf{j} + 3m\mathbf{k}$ and $(1 + m)\mathbf{i} - 2m\mathbf{j} + \mathbf{k}$ include an acute angle for

(a) all value of m (b) $m < -2$ or $m > \frac{-1}{2}$

(c) $m = \frac{-1}{2}$ (d) $m \in \left[-2, \frac{-1}{2}\right]$

Solution (b) The given vectors include an acute angle if

$(2\mathbf{i} - m\mathbf{j} + 3m\mathbf{k})\cdot[(1 + m)\mathbf{i} - 2m\mathbf{j} + \mathbf{k}] > 0$

$\Rightarrow$ $2(1 + m) + 2m^2 + 3m > 0 \Rightarrow 2m^2 + 5m + 2 > 0$

$\Rightarrow$ $(2m + 1)(m + 2) > 0 \Rightarrow \boldsymbol{m < -2}$ **or** $\boldsymbol{m > -\frac{1}{2}}$.

55. If **A**, **B**, **C** are three non-coplanar vectors, then

$$\frac{\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}}{\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}} + \frac{\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}}{\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}} =$$

(a) 0 (b) 1

(c) – 1 (d) none of these

Solution (a) The given expression

$$= \frac{[\mathbf{A\ B\ C}]}{[\mathbf{C\ A\ B}]} + \frac{[\mathbf{B\ A\ C}]}{[\mathbf{C\ A\ B}]} = \frac{[\mathbf{A\ B\ C}] - [\mathbf{A\ B\ C}]}{[\mathbf{A\ B\ C}]} = \mathbf{0}.$$

56. The position vectors of three points A, B and C are **i** + **j**, **i** – **j** and l**i** + m**j** + n**k** respectively. The vectors **OA**, **OB**, **OC** are coplanar if

(a) $l = m = n = 1$

(b) $l = 1$, m and n are any scalars

(c) $n = 0$, l, m are scalar

(d) $m = 0$, $n = 1$, l is any scalar

Solution (c) Let O be the origin and A, B, C the given points. Then A, B, C are coplanar if

$$\mathbf{OA}\cdot(\mathbf{OB} \times \mathbf{OC}) = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ l & m & n \end{vmatrix} = 0$$

$\Rightarrow$ $-2n = 0 \Rightarrow \boldsymbol{n = 0}$. $\therefore$ $\boldsymbol{l, m}$ **can be any scalars.**

57. $[\mathbf{a\ b\ a} \times \mathbf{b}]$ is equal to

(a) $|\mathbf{a} \times \mathbf{b}|$ (b) $|\mathbf{a} \times \mathbf{b}|^2$

(c) $|\mathbf{a}\cdot\mathbf{b}|$ (d) $|\mathbf{a}|\,|\mathbf{b}|$

Solution (b) We have,

$[\mathbf{a}\ \mathbf{b}\ \mathbf{a}\times\mathbf{b}] = [\mathbf{a}\times\mathbf{b}\ \mathbf{a}\ \mathbf{b}\ ] = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = |\mathbf{a}\times\mathbf{b}|^2$.

58. If $\mathbf{x}\times\mathbf{b} = \mathbf{c}\times\mathbf{b}$ and $\mathbf{x}\perp\mathbf{a}$ then $\mathbf{x}$ is equal to

(a) $\dfrac{\mathbf{b}\times(\mathbf{a}\times\mathbf{c})}{\mathbf{b}\cdot\mathbf{c}}$ (b) $\dfrac{(\mathbf{b}\times\mathbf{c})\times\mathbf{a}}{\mathbf{b}\cdot\mathbf{a}}$

(c) $\dfrac{\mathbf{a}\times(\mathbf{c}\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{b}}$ (d) none of these

Solution (b), (c) We have, $\mathbf{x}\times\mathbf{b} = \mathbf{c}\times\mathbf{b}$

$\Rightarrow\ \mathbf{a}\times(\mathbf{x}\times\mathbf{b}) = \mathbf{a}\times(\mathbf{c}\times\mathbf{b})$

$\Rightarrow\ (\mathbf{a}\cdot\mathbf{b})\mathbf{x} - (\mathbf{a}\cdot\mathbf{x})\mathbf{b} = \mathbf{a}\times(\mathbf{c}\times\mathbf{b})$

$\Rightarrow\ \mathbf{x} = \dfrac{\mathbf{a}\times(\mathbf{c}\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{b}}$ $(\because\ \mathbf{a}\perp\mathbf{x})$.

Also, $\mathbf{a}\times(\mathbf{c}\times\mathbf{b}) = -(\mathbf{c}\times\mathbf{b})\times\mathbf{a} = (\mathbf{b}\times\mathbf{c})\times\mathbf{a}$

$\therefore\ \mathbf{x} = \dfrac{(\mathbf{b}\times\mathbf{c})\times\mathbf{a}}{\mathbf{b}\cdot\mathbf{a}}$.

59. If **a**, **c**, **d** are non-coplanar vectors then

$\mathbf{d}\cdot\{\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\}$ is equal to

(a) $(\mathbf{b}\cdot\mathbf{d})\,[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$ (b) $(\mathbf{a}\cdot\mathbf{d})\,[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$

(c) $(\mathbf{c}\cdot\mathbf{d})\,[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$ (d) none of these

Solution (a) We have, $\mathbf{d}\cdot\{\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\}$

$= \mathbf{d}\cdot(\mathbf{a}\times((\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}\ ))$

$= \mathbf{d}\cdot((\mathbf{b}\cdot\mathbf{d})\ (\mathbf{a}\times\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\ (\mathbf{a}\times\mathbf{d}))$

$= (\mathbf{b}\cdot\mathbf{d})\mathbf{d}\cdot(\mathbf{a}\times\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})\ \mathbf{d}\cdot(\mathbf{a}\times\mathbf{d})$

$= (\mathbf{b}\cdot\mathbf{d})\ [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$ $[\because\ \mathbf{d}\cdot(\mathbf{a}\times\mathbf{d}) = 0]$.

60. Let **a**, **b**, **c** be three vectors such that $\mathbf{a}\neq 0$ and $\mathbf{a}\times\mathbf{b} = 2\mathbf{a}\times\mathbf{c}$, $|\mathbf{a}| = |\mathbf{c}| = 1$, $|\mathbf{b}| = 4$ and $|\mathbf{b}\times\mathbf{c}| = \sqrt{15}$. If $\mathbf{b} - 2\mathbf{c} = \lambda\,\mathbf{a}$, then λ equals

(a) 1 (b) – 1

(c) 2 (d) – 4

Solution (d) If the angle between **b** and **c** is α

then $|\mathbf{b}\times\mathbf{c}| = \sqrt{15}$

$\Rightarrow\ |\mathbf{b}|\,|\mathbf{c}|\sin\alpha = \sqrt{15}\ \Rightarrow \sin\alpha = \dfrac{\sqrt{15}}{4}$

$\therefore\ \cos\alpha = \dfrac{1}{4}$.

Now, $\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$

$\Rightarrow\ (\mathbf{b} - 2\mathbf{c})^2 = \lambda^2\,(\mathbf{a})^2 \Rightarrow (\mathbf{b})^2 + 4(\mathbf{c})^2 - 4\mathbf{b}\cdot\mathbf{c} = \lambda(\mathbf{a})^2$

$\Rightarrow\ 16 + 4 - 4\ \{|\mathbf{b}|\ |\mathbf{c}|\cos\alpha\} = \lambda^2$

$\therefore\ \lambda^2 = 16 \Rightarrow \lambda = \pm\ \mathbf{4}.$

Problems for Practice

Level of Difficulty II

1. If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, then

$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$ is given by

(a) $20\mathbf{i} + 22\mathbf{j} + 6\mathbf{k}$ (b) $6\mathbf{i} + 20\mathbf{j} + 22\mathbf{k}$

(c) $-(20\mathbf{i} + 6\mathbf{j} + 22\mathbf{k})$ (d) $20\mathbf{i} + 6\mathbf{j} - 22\mathbf{k}$

2. If the vectors $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $x\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ are coplanar, then x =

(a) 8/5 (b) 5/8 (c) 0 (d) 1

3. The area of the parallelogram whose diagonals are given by the vectors $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ is

(a) $5\sqrt{3}$ (b) 4 (c) $10\sqrt{3}$ (d) 8

4. The vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + (m+1)\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + m\mathbf{k}$, $\mathbf{c} = \mathbf{j} - \mathbf{j} + m\mathbf{k}$ are coplanar for

(a) $m = 2$ (b) $m = \dfrac{1}{2}$

(c) $m = -\dfrac{1}{2}$ (d) no value of m

5. A constant force $F = 2\,\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is acting on a particle such that the particle is displaced from the point $A\,(1, 2, 3)$ to the point $B\,(3, 4, 5)$. The work done by the force is

(a) $2\sqrt{51}$ (b) $\sqrt{17}$

(c) 3 (d) 2

6. The vector $\dfrac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ is

(a) a unit vector

(b) makes an angle $\dfrac{\pi}{3}$ with the vector $(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$

(c) parallel to the vector $\left(-\mathbf{i} + \mathbf{j} - \dfrac{1}{2}\mathbf{k}\right)$

(d) perpendicular to the vector $(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

7. If $|a + b| = |a - b|$ then angle between a and b is

(a) 0° (b) 45° (c) 60° (d) 90°

8. The value of a such that the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + a\mathbf{j} + 5\mathbf{k}$ are coplanar, is

(a) 4 (b) – 4 (c) 2 (d) – 2

9. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, then the angle between the vectors $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-\mathbf{b})$ is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) none of these

10. If the scalar projection of $\mathbf{a} = \lambda\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ is 4 units then λ is equal to

(a) 5 (b) 3 (c) 1 (d) – 5

11. If $\mathbf{a} = 4\mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = 3\mathbf{j} + 4\mathbf{k}$,

then the vector form of component of **a** along **b** is

(a) $\dfrac{18}{10\sqrt{3}}(3\mathbf{i} + 4\mathbf{k})$ (b) $\dfrac{18}{25}(3\mathbf{j} + 4\mathbf{k})$

(c) $\dfrac{18}{\sqrt{3}}(3\mathbf{i} + 4\mathbf{k})$ (d) $3\mathbf{j} + 4\mathbf{k}$

12. Constant forces $P = 2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ and $Q = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ act on a particle. The work done, when the particle is displaced from a point A with position vector $4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ to a point B with position vector $6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, is given by

(a) 15units (b) 22units
(c) 18units (d) none of these

13. If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{k}$ and $\mathbf{c} = 2\mathbf{j} - \mathbf{k}$, then the area of the parallelogram having diagonals $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$ is

(a) $\sqrt{21}$ (b) $\frac{1}{2}\sqrt{21}$
(c) $\sqrt{23}$ (d) $\frac{1}{2}\sqrt{23}$

14. If the volume of a parallelopiped whose edges are represented by $-12\mathbf{i} + \lambda\mathbf{k}$, $3\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 15\mathbf{k}$ is 546, then the value of λ is

(a) 3 (b) – 3 (c) – 179 (d) 179

15. If **i**, **j**, **k** are the unit vectors and mutually orthogonal, then for any vector **a**,

$\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) =$

(a) **O** (b) **a** (c) 2**a** (d) none of these

16. If **a** and **b** are position vectors of *A* and *B* respectively, then the position vector of a point *C* in *AB* produced such that **AC** = 3**AB** is

(a) $3\mathbf{a} - \mathbf{b}$ (b) $3\mathbf{b} - \mathbf{a}$
(c) $3\mathbf{a} - 2\mathbf{b}$ (d) $3\mathbf{b} - 2\mathbf{a}$

17. If **a**, **b**, **c** are unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ then the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} =$

(a) 1 (b) 2 (c) $\frac{-3}{2}$ (d) none of these

18. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $|\mathbf{a} \times \mathbf{b}| = 8$, then $\mathbf{a}\cdot\mathbf{b}$ is equal to

(a) 2 (b) 4 (c) 6 (d) 8

19. If force represented by $5\mathbf{i} + \mathbf{k}$ is acting through the point $9\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, then the moment about the point $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is

(a) $3\mathbf{i} + \mathbf{j} + 15\mathbf{k}$ (b) $3\mathbf{i} - \mathbf{j} + 15\mathbf{k}$
(c) $-3\mathbf{i} - \mathbf{j} + 15\mathbf{k}$ (d) $3\mathbf{i} + \mathbf{j} - 15\mathbf{k}$

20. Let **p** and **q** be the position vectors of *P* and *Q* respectively with respect to *O* and $|\mathbf{p}| = p$, $|\mathbf{q}| = q$. The points *R* and *S* divide *PQ* internally and externally in the ratio 2 : 3 respectively.

It **OR** and **OS** are perpendicular, then
(a) $9p^2 = 4q^2$ (b) $4p^2 = 9q^2$
(c) $9p = 4q$ (d) $4p = 9q$

Level of Difficulty II

21. Given the vectors $\mathbf{a}(2, 1, -1)$, $\mathbf{b}(1, 2, 1)$, $\mathbf{c}(2, -1, 3)$ and $\mathbf{d}(3, -1, 2)$. The projection of the vector $\mathbf{a} + \mathbf{c}$ on the vector $(\mathbf{b} - \mathbf{d}) \times \mathbf{c}$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{6}$ (d) none of these

22. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If **c** is a vector such that $\mathbf{a}\cdot\mathbf{c} = |\mathbf{c}|$, $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and **c** is 30° then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$ is equal to

(a) 2/3 (b) 3/2 (c) 2 (d) 3

23. If **a**, **b**, **c** are three non coplanar, non-zero vectors then $(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b} \times \mathbf{c} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \times \mathbf{a} + (\mathbf{a}\cdot\mathbf{c})\,\mathbf{a} \times \mathbf{b}$ is equal to

(a) $[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{a}$ (b) $[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{b}$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}$ (d) none of these

24. If $\mathbf{n}_2$, $\mathbf{n}_2$ are two unit vectors and θ is the angle between them, then $\cos\frac{\theta}{2} =$

(a) $\frac{1}{2}|\mathbf{n}_1 + \mathbf{n}_2|$ (b) $\frac{1}{2}|\mathbf{n}_1 - \mathbf{n}_2|$
(c) $\frac{1}{2}(\mathbf{n}_1 + \mathbf{n}_2)$ (d) $\frac{|\mathbf{n}_1 \times \mathbf{n}_2|}{2|\mathbf{n}_1||\mathbf{n}_2|}$

25. $\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$ is equal to

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ (b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3$ (d) none of these

26. If **a**, **b**, **c** and **p**, **q**, **r** are reciprocal system of vectors, then $\mathbf{a} \times \mathbf{p} + \mathbf{b} \times \mathbf{q} + \mathbf{c} \times \mathbf{r}$ equals

(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $(\mathbf{p} + \mathbf{q} + \mathbf{r})$
(c) **0** (d) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

27. Given two vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, the unit vector coplanar with the two vectors and perpendicular to first is

(a) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) $\frac{1}{\sqrt{5}}(2\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(c) $\pm\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) none of these

28. The vector $\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{\mathbf{i}} + (4\mathbf{x} - 2)\,\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The value of *x* is

(a) $-\frac{2}{3}, 2$ (b) $\frac{1}{3}, 2$

(c) $\left(\frac{2}{3}, 0\right)$ (d) (2, 7)

29. A parallelopiped is formed by planes drawn through the points (2, 3, 5) and (5, 9, 7), parallel to the coordinate planes. The length of a diagonal of the parallelopiped is

(a) 7 (d) $\sqrt{38}$ (c) $\sqrt{155}$ (d) none of these

30. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the sides *BC*, *CA* and *AB* respectively, of a triangle *ABC*, then

(a) $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} + \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} + \vec{\mathbf{c}}\cdot\vec{\mathbf{a}} = 0$

(b) $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \vec{\mathbf{c}} \times \vec{\mathbf{a}}$

(c) $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} = \vec{\mathbf{c}}\cdot\vec{\mathbf{a}} = 0$

(d) none of these

31. A tetrahedron has vertices at *O* (0, 0, 0), A (1, 2, 1), *B* (2, 1, 3) and *C* (– 1, 1, 2). Then the angle between the faces *OAB* and *ABC* will be

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$

(c) 30° (d) 90°

32. $\vec{a}$, $\vec{b}$, $\vec{c}$ are 3 vectors, such that $\vec{a} + \vec{b} + \vec{c} = \vec{\mathbf{0}}$, $|\vec{\mathbf{a}}| = 1$, $|\vec{\mathbf{b}}| = 2$, $|\vec{\mathbf{c}}| = 3$, then $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} + \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} + \vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$ is equal to

(a) 0 (b) – 7 (c) 7 (d) 1

33. If $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$ and $\vec{\mathbf{w}}$ are three non-coplanar vectors, then

$(\vec{\mathbf{u}} + \vec{\mathbf{v}} - \vec{\mathbf{w}}).\ (\vec{\mathbf{u}} - \vec{\mathbf{v}}) \times (\vec{\mathbf{v}} - \vec{\mathbf{w}})$ equals

(a) 0 (b) $\vec{u}.\,\vec{v}\times\vec{w}$

(c) $\vec{u}.\,\vec{w}\times\vec{v}$ (d) $3\vec{u}\times\vec{v}\times\vec{w}$

34. Consider points *A*, *B*, *C* and *D* with position vectors $7\hat{i}-4\hat{j}+7\hat{k}$, $\hat{i}-6\hat{j}+10\hat{k}$, $-\hat{i}-3\hat{j}+4\hat{k}$ and $5\hat{i}-\hat{j}+5\hat{k}$ respectively. Then *ABCD* is a
(a) square
(b) rhombus
(c) rectangle
(d) none of these

35. The vectors **AB** = 3**i** + 4**k** and **AC** = 5**i** – 2**j** + 4**k** are the sides of a triangle *ABC*. The length of the median through *A* is
(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{288}$

36. A particle acted on by constant forces 4**i** + **j** – 3**k** and 3**i** + **j** – **k** is displaced from the point **i** + 2**j** + 3**k** to the point 5**i** + 4**j** + **k**. The total work done by the forces is
(a) 20 units (b) 30 units
(c) 40 units (d) 50 units

37. Let **u** = **i** + **j** , **v** = **i** – **j** and **w** = **i** + 2**j** + 3**k**. if **n** is a unit vector such that **u.n** = 0 and **v.n** = 0, then | **w.n** | is equal to
(a) 0 (b) 1 (c) 2 (d) 3

38. Let **a**, **b** and **c** be three non-zero vectors such that no two of these are colinear. If the vector **a** + 2**b** is collinear with **c** and **b** + 3**c** is collinear with **a** (λ being some non-zero scalar) then **a** + 2**b** + 6**c** equals
(a) λ**c** (b) λ**b** (c) λ**a** (d) 0

39. A particle is acted upon by constant forces 4**i** + **j** – 3**k** and 3**i** + **j** – **k** which displace it from a point **i** + 2**j** + 3**k** to the point 5**i** + 4**j** + **k**. The work done in standard units by the forces is given by
(a) 25 (b) 30 (c) 40 (d) 15

40. If **a**, **b**, **c** are non-coplanar vectors and λ is a real number, then the vectors **a** + 2**b** + 3**c**, **b** + 4**c** and (2λ – 1)**c** are non-coplanar for
(a) all except two values of λ
(b) all except one value of λ
(c) all value of λ
(d) no value of λ

41. Let **u**, **v**, **w** be such that | **u** | = 1, | **v** | = 2, | **w** |, = 3. If the projection of **v** along u is equal to that of **w** along **u** and **v**, **w** are perpendicular to each other then | **u** – **v** + **w** | equals
(a) $\sqrt{14}$ (b) $\sqrt{7}$ (c) 2 (d) 14

42. Let **a**, **b** and **c** be non-zero vectors such that (**a** × **b**) × **c** = $\frac{1}{3}$ | **b** | | **c** | **a**. If θ is the acute angle between the vectors **b** and **c**, then sin θ equals
(a) $\frac{2}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2\sqrt{2}}{3}$

43. If **a**, **b** and **c** are unit vectors then
$|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2$ does not exceed
(a) 4 (b) 9 (c) 8 (d) 6

44. Let **a** = **i** – **j**, **b** = **j** – **k**, **c** = **k** – **i**. If $\vec{d}$ is a unit vector such that **a**·**d** = 0 = [**b c d**], then **d** equals
(a) $\pm\frac{\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{6}}$ (b) $\pm\frac{\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{3}}$
(c) $\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ (d) ±**k**

45. If **A**, **B** and **C** are three non-coplanar vectors, then (**A** + **B** + **C**)·[(**A** + **B**) × (**A** + **C**)] equals
(a) 0 (b) [**A**, **B**, **C**]
(c) 2 [**A**, **B**, **C**] (d) – [**A**, **B**, **C**]

ANSWERS

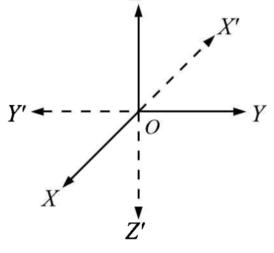
1. (c)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (a,c,d)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (a)
11. (b)	**12.** (a)	**13.** (b)	**14.** (b),(d)	**15.** (c)	**16.** (d)	**17.** (c)	**18.** (c)	**19.** (c)	**20.** (a)
21. (c)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (a)	**29.** (a)	**30.** (b)
31. (a)	**32.** (b)	**33.** (b)	**34.** (d)	**35.** (c)	**36.** (c)	**37.** (d)	**38.** (d)	**39.** (c)	**40.** (a)
41. (a)	**42.** (a)	**43.** (b)	**44.** (a)	**45.** (d)					

28 Three–Dimensional Geometry

BRIEF REVIEW OF THE CONCEPTS

ORIGIN

Let $X'OX$, $Y'OY$, $Z'OZ$ be three mutually perpendicular lines which intersect at O. Then O is called the *origin*.



COORDINATE AXES

$X'OX$ is called the x-axis, $Y'OY$ the y-*axis*, $Z'OZ$ the z-axis and taken together these are called the *coordinate axes*.

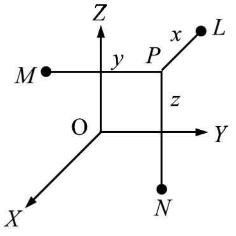
COORDINATE PLANES

1. XOY is called the xy-plane.
2. YOZ is called the yz-plane.
3. ZOX is called the zx-plane.

These three, taken together, are called the *coordinate planes*.

COORDINATES

Let P be any point in space. Draw PL, PM, PN perpendiculars to the xy, yz and xz planes, then



1. LP is called the x-coordinate of P.
2. MP is called the y-coordinate of P.
3. NP is called the z-coordinate of P.

These three, taken together, are known as coordinates of P.

DISTANCE FORMULA

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

SECTION FORMULA

1. The coordinates of the point dividing the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ internally are
$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right).$$
2. The coordinates of the point dividing the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ externally are
$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right).$$
3. The coordinates of the mid-point of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

CENTROID OF A TRIANGLE

The coordinates of the centroid of the triangle ABC, whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

CENTROID OF A TETRAHEDRON

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

DIRECTION COSINES

If a vector makes angles α, β, γ with the positive directions of x-axis, y-axis and z-axis respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called its *direction cosines*.

The direction cosines are generally denoted as l, m, n i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

The angles α, β, γ are known as *direction angles*.

DIRECTION RATIOS

Three numbers a, b, c proportional to the direction cosines l, m, n of a vector are known as the *direction ratios* of the vector.

Thus a, b, c are the direction ratios of a vector, provided

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}.$$

Key Results on Direction Cosines and Direction Ratios

If $P(x, y, z)$ is a point in space such that $\mathbf{r} = OP$ has direction cosines l, m, n, then

(a) $x = l|\mathbf{r}|$, $y = m|\mathbf{r}|$, $z = n|\mathbf{r}|$

(b) $l|\mathbf{r}|$, $m|\mathbf{r}|$, $n|\mathbf{r}|$ are projections of $\mathbf{r}$ on OX, OY, OZ respectively.

(c) $\mathbf{r} = |\mathbf{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$ and $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$.

(d) $l^2 + m^2 + n^2 = 1$.

(e) If $\mathbf{r} = a\hat{i} + b\hat{j} + c\hat{k}$, then

(a) a, b, c are the direction ratios of $\mathbf{r}$.

(b) Direction cosines of $\mathbf{r}$ are given by

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\quad m = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

(f) Direction ratios of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, and its direction cosines are

$$\frac{x_2-x_1}{|\mathbf{PQ}|}, \frac{y_2-y_1}{|\mathbf{PQ}|}, \frac{z_2-z_1}{|\mathbf{PQ}|}$$

(g) The direction cosines of

(a) $\vec{OX}$ are (1, 0, 0)

(b) $\vec{OY}$ are (0, 1, 0)

(c) $\vec{OZ}$ are (0, 0, 1)

Note: Direction cosines of a line are unique but direction ratios of a line are not unique and can be infinite.

ANGLE BETWEEN TWO VECTORS

If θ is the angle between two vectors whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2, then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$ and

$$\sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}$$

Key Points to Remember

1. If $l_1l_2 + m_1m_2 + n_1n_2 = 0$, then two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are orthogonal.
2. If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ then two vectors are parallel.

ANGLE IN TERMS OF DIRECTION RATIOS

If θ is the angle between two vectors having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}.$$

The two vectors are orthogonal when

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

and parallel when

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

PROJECTION OF THE JOIN OF TWO POINTS ON A LINE

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then the length of projection of PQ on a line whose direction cosines are l, m, n is given by

$$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n.$$

VECTOR EQUATION OF A LINE THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

The vector equation of a straight line passing through a given point with position vector $\mathbf{a}$ and parallel to a given vector $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$$

where λ is a scalar.

Notes:

- The position vector of any point on the line is taken as $\mathbf{a} + \lambda\,\mathbf{b}$.
- $\mathbf{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore, $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Cartesian Form The equation of a straight line with direction ratios a, b, c and passing through a fixed point (x_1, y_1, z_1) is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

Key Points to Remember

1. The equation of a line whose direction cosines are l, m, n and which passes through the point (x_1, y_1, z_1) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{c}$$

2. The coordinates of any point on the line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \text{ are given by}$$

$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ is a real number.

3. Equation of x-axis:

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0} \text{ or } y = 0, z = 0$$

Equation of y-axis:

$$\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0} \text{ or } x = 0, z = 0$$

Equation of z-axis:

$$\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1} \text{ or } x = 0, y = 0.$$

VECTOR EQUATION OF A LINE PASSING THROUGH TWO POINTS

The vector equation of a line passing through two points with position vectors **a** and **b** is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

Cartesian Form The equation of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

ANGLE BETWEEN TWO LINES

Vector Form The angle between the two lines

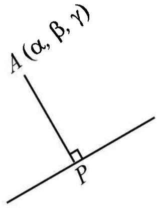
$$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1 \text{ and } \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$$

is given by $\cos\theta = \dfrac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$

PERPENDICULAR FROM A POINT TO A LINE

Let the equation of the line be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r \text{ (say)}$$



and $A(\alpha, \beta, \gamma)$ be the given point. Then,

1. the coordinates of the foot of the perpendicular from A on the given line are
   $P(lr + a, mr + b, nr + c)$
2. length of perpendicular (AP) is
   $$\sqrt{(lr + a - \alpha)^2 + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$$
3. equation of the perpendicular is given by
   $$\frac{x-\alpha}{lr + a - \alpha} = \frac{y - \beta}{mr + b - \beta} = \frac{z - \gamma}{nr + c - \gamma}$$
   where $r = (\alpha - a)l + (\beta - b)m + (\gamma - c)n.$

Length of the perpendicular from a point $A(\mathbf{r}_1)$ upon the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is given by

$$= \frac{|(\mathbf{a} - \mathbf{r}_1) \times \mathbf{b}|}{|\mathbf{b}|}.$$

SKEW LINES

Two straight lines which are not parallel and which do not intersect, are known as skew lines. Clearly, two skew lines are never coplanar.

Shortest Distance between Two Skew Lines

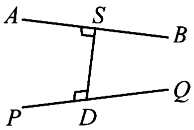
Let AB and PQ be the skew lines. Let a line SD be perpendicular to AB and PQ. Then, the length of line SD is called the shortest distance between them.

Shortest Distance

Shortest distance between the lines

$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}.$$



Notes:

- Shortest distance between two parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is
  $$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}.$$
- Two lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\,\mathbf{b}_2$ will intersect provided $d = 0$, i.e., when
  $$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0.$$

THE PLANE

Plane It is a surface such that if any two points on it are taken, then every point of the line joining them lies on it.

General Equation of a Plane The general equation of a plane is

$$ax + by + cz + d = 0$$

where a, b, c, are not all zero.

Remarks:

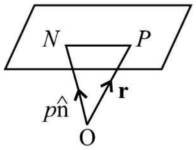
1. a, b, c are the direction ratios of the normal to the plane $ax + by + cz + d = 0$.
2. Equation of yz-plane is $x = 0$
3. Equation of zx-plane is $y = 0$
4. Equation of xy-plane is $z = 0$
5. Equation of any plane parallel to xy-plane is $z = c$. Similarly, planes parallel to yz-plane and zx-plane are $x = c$ and $y = c$ respectively.

Equation of a Plane in Normal Form

Vector Form If $\hat{n}$ be a unit vector normal to a given plane and p be the length of perpendicular from the origin to the plane, then the equation of the plane is given by

$$\mathbf{r} \cdot \hat{n} = p.$$

Cartesian Form If l, m, n be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is $lx + my + nz = p$.



Transformation of General Form to Normal Form To reduce the general equation $ax + by + cz + d = 0$ to normal form, we follow the following working rule:

1. Write the terms containing x, y and z on left hand side and the constant term on the right hand side.
2. If the constant term on the right hand side is not positive, make it positive by multiplying both sides by -1.
3. Divide each term by $\sqrt{a^2 + b^2 + c^2}$, we get

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}} + \frac{by}{\pm\sqrt{a^2+b^2+c^2}} + \frac{cz}{\pm\sqrt{a^2+b^2+c^2}} = \frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

where + sign is to be taken if $d > 0$ and – sign is to be taken if $d < 0$.

Equation of a Plane Passing through a Given Point

Vector Form The vector equation of a plane through a given point $\mathbf{r}_1$ and perpendicular to $\hat{n}$ is

$$(\mathbf{r} - \mathbf{r}_1)\cdot \hat{n} = 0.$$

Cartesian Form The equation of a plane passing through a given point $A(x_1, y_1, z_1)$ and normal to the plane having direction ratios a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

Equation of a Plane in Intercept Form

If a plane makes intercepts of lengths a, b, c with x-axis, y-axis and z-axis respectively, then the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Equation of a Plane Passing through Three Points

Vector Form The equation of a plane passing through three points having position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ is $[\,\mathbf{r} - \mathbf{r}_1 \mathbf{r} - \mathbf{r}_2\ \mathbf{r} - \mathbf{r}_3] = 0.$

Cartesian Form The equation of a plane passing through three given points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0\cdot$$

Working Rules

- In order of find the equation of a plane passing through three given points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, instead of using the above formula, we proceed as follows:

Step 1: The equation of any plane passing through the point $A(x_1, y_1, z_1)$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$ ....(1)

Step 2: Since the plane (1) is to pass through the points B and C also, so the coordinates of B and C must satisfy the equation (1). This will give two linear homogeneous equations in a, b, and c. Solving these equations, we get the proportionate values of a, b and c.

Step 3: Since the R.H.S of (1) is zero, so putting these proportionate values of a, b, c in (1), we get the equation of the required plane.

- **Coplanarity of four given points:** Suppose we are to show whether the four given points A, B, C and D are coplanar or not. First we find the equation of the plane passing through the three points A, B and C. If the coordinates of D satisfy this equation, the four given points A, B, and D are coplanar otherwise not.

Equation of a Plane through Two Given Points and Parallel to a Given Vector

Vector Form The equation of a plane through two given points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and parallel to a given vector $\mathbf{m}$ is

$$(\mathbf{r} - \mathbf{r}_1)\cdot[(\mathbf{r}_2 - \mathbf{r}_1)\times \mathbf{m}] = 0$$

Cartesian Form The equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and parallel to a line having direction ratios a, b, c is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ a & b & c \end{vmatrix} = 0\cdot$$

Equation of a Plane Passing through a Given Point and Parallel to Two Given Vectors

Vectors Form The equation of a plane passing through a point having position vector $\mathbf{a}$ and parallel to two given vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \text{ where } \lambda \text{ and } \mu \text{ are scalars}$$

or $$(\mathbf{r} - \mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) = 0 \text{ or } \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).$$

Cartesian Form The equation of a plane passing through a point (x_1, y_1, z_1) and parallel to two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0\,.$$

Planes Parallel to a Given Plane

Cartesian Form Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant to be detemined by the given condition.

Vector Form The equation of a plane parallel to the plane $\mathbf{r}.\mathbf{n} = d_1$ is $\mathbf{r}.\mathbf{n} = d_2$, where d_2 is a constant to be determined by the given condition.

Angle between Two Planes

Angle between two planes is the angle between their normals.

Vector Form The angle between the planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_1}{|\mathbf{n}_1||\mathbf{n}_2|}\,.$$

Cartesian Form The angle between the planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \text{ is given by}$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note:
- If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the planes are parallel to each other.

Angle between a Line and a Plane

The angle between a line and a plane is the angle between the line and the normal to the plane.

Vector Form If θ is the angle between the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r}.\mathbf{n} = d$, then

$$\sin\theta = \frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}|\,|\mathbf{n}|}.$$

Cartesian Form If θ is the angle between the line

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and the plane}$$

$a_2x + b_2y + c_2z + d = 0$, then

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Note: If the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to the plane $a_2x + b_2y + c_2z + d = 0$,
then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Planes Bisecting the Angles between Two Planes

Cartesian Form The equations of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad ..(1)$$

Vector Form The equations of the planes bisecting the angles between the planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ are

$$\frac{|\mathbf{r}.\mathbf{n}_1 - \mathbf{d}_1|}{|\mathbf{n}_1|} = \frac{|\mathbf{r}.\mathbf{n}_2 - \mathbf{d}_2|}{|\mathbf{n}_2|}$$

or $$\mathbf{r}\cdot(\hat{\mathbf{n}}_1 \pm \hat{\mathbf{n}}_2) = \frac{d_1}{|\mathbf{n}_1|} \pm \frac{d_2}{|\mathbf{n}_2|}$$

Bisector of the Angle Containing the Origin After making the constant term in both the equations positive, the positive sign in (1) gives the bisector of the angle which contains the origin.

Bisector of Acute/Obtuse Angle

1. Write the equations of the given planes such that their constant terms are positive.
2. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then origin lies in obtuse angle and hence positive sign in (1) gives the bisector of the obtuse angle.
3. If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then origin lies in acute angle and hence positive sign in (1) gives the bisector of the acute angle.

Distance of a Point from a Plane

Vector Form The length of the perpendicular from a point having position vector **a** to the plane $\mathbf{r}.\mathbf{n} = d$ is given by

$$p = \frac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}.$$

Cartesian Form The length of the perpendicular from a point P(x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}.$$

Distance between Two Parallel Planes

Vector Form The distance between two parallel planes $\mathbf{r}.\mathbf{n} = d_1$ and $\mathbf{r}.\mathbf{n} = d_2$ is given by

$$p = \frac{|d_1 - d_2|}{|\mathbf{n}|}.$$

Cartesian Form The distance between two parallel planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ is given by

$$p = \frac{|d_1 - d_2|}{\sqrt{a_1^2 + b_1^1 + c_1^2}}\frac{x-x_1}{l}.$$

Planes Passing through the Intersection of Two Planes

Vector Form The equation of a plane passing through the intersection of the planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ is

$$(\mathbf{r}.\mathbf{n}_1 - d_1) + k(\mathbf{r}.\mathbf{n}_2 - d_2) = 0$$

or $$\mathbf{r}.(\mathbf{n}_1 + k\mathbf{n}_2) = d_1 + kd_2,$$

where k is an arbitrary constant.

Cartesian Form The equation of a plane passing through the intersection of planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$, where k is an arbitrary constant.

Two Sides of a Plane

The two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same side or the opposite sides of the plane $ax + by + cz + d = 0$ according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ have the same sign or the opposite signs.

Condition for a Line to Lie in a Plane

Vector Form If the line $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$ lies in the plane $\mathbf{r}\cdot\mathbf{n} = d$, then

$$\mathbf{b}\cdot\mathbf{n} = 0 \quad\text{and}\quad \mathbf{a}\cdot\mathbf{n} = d.$$

Cartesian Form If the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane $ax + by + cz + d = 0$, then

1. $ax_1 + by_1 + cz_1 + d = 0$ and
2. $al + bm + cn = 0$.

Condition for the Two Lines to be Intersecting (Coplanar) and the Equation of the Plane Containing Them

Vector Form If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ are intersecting (coplanar), then

$$[\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$$

and the equation of the plane containing the two lines is

$$[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2]$$

or $$[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$$

Cartesian Form If the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ are intersecting (coplanar)}$$

then $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing the two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or
$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

SPHERE

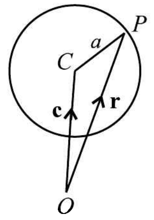
A *sphere* is the locus of a point which remains at a constant distance from a fixed point. The constant distance is called the *radius* and the fixed point is called the *centre* of the sphere.

Equation of a Sphere

Vector Equation The vector equation of a sphere of radius a and centre having position vector
$\mathbf{c}$ is $|\mathbf{r}-\mathbf{c}| = a$

Note: The vector equation of a sphere of radius a with centre at the origin, is $|\mathbf{r}| = a$.

Cartesian Equation The equation of a sphere with centre (a, b, c) and radius k is given by



$$(x-a)^2+(y-b)^2+(z-c)^2=k^2$$

Note: The equation of a sphere with centre at origin and radius k is

$$x^2+y^2+z^2=k^2.$$

General Equation of a Sphere

The general equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

represents a sphere with centre $(-u, -v, -w)$ and radius equal to $\sqrt{u^2+v^2+w^2-d}$.

Equation of a Sphere through Four Points

Equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

or

1. Assume the equation of the sphere to be
   $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ...(1)
2. Put the coordinates of four given points in Equation (1) to obtain four equations in u, v, w and d.
3. Solve the four equations obtained in Step (2) to get the values of u, v, w, and d.
4. Put the values of u, v, w and d obtained in Step (3) in Equation (1) to obtain the required equation of sphere.

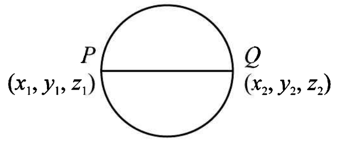
Equation of a Sphere, the Extremities of Diameter Being given

Cartesian Form The equation of a sphere described on the join of two points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

as diameter is given by

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$



Vector Form The vector equation of a sphere, described on the join of two points P and Q, having position vectors $\mathbf{a}$ and $\mathbf{b}$, as diameter, is given by

$$(\mathbf{r}-\mathbf{a}) \cdot (\mathbf{r}-\mathbf{b}) = 0.$$

or $$|\mathbf{r}^2| - \mathbf{r}\cdot(\mathbf{a}-\mathbf{b}) + \mathbf{a}\cdot\mathbf{b} = 0$$

or $$|\mathbf{r}-\mathbf{a}|^2 + |\mathbf{r}-\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2 .$$

Condition of Tangency

Vector Form Condition for the plane $\mathbf{r}\cdot\mathbf{n} = d$ to touch the sphere $|\mathbf{r}-\mathbf{c}| = a$ is

$$\frac{|\mathbf{c}.\mathbf{n}-d|}{|\mathbf{n}|} = a.$$

Cartesian Form Condition for the plane $lx+my+nz=p$ to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $(ul+vm+wn+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

Solved Problems

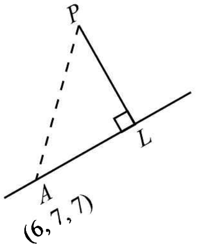
1. The perpendicular distance of $P(1, 2, 3)$ from the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(a) 7 (b) 5

(c) 0 (d) none of these

Solution (a) The point A (6, 7, 7) is on the line. Let the perpendicular from P meet the line in L. Then

$$AP^2 = (6-1)^2+(7-2)^2+(7-3)^2 = 66.$$



Also AL = projection of AP on line

$$\left(\text{actual d.c.'s} \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$$

$$= (6-1)\cdot\frac{3}{\sqrt{17}} + (7-2)\cdot\frac{2}{\sqrt{17}} + (7-3)\cdot\frac{-2}{\sqrt{17}} = \sqrt{17}.$$

$\therefore$ $\perp$ distance d of P from the line is given by

$d^2 = AP^2 - AL^2 = 66 - 17 = 49$, so that d = **7.**

2. The points (5, 2, 4), (6, –1, 2) and (8, –7, k) are collinear if k is equal to

(a) –1 (b) 3

(c) 2 (d) –2

Solution (d) Let $A(5, 2, 4)$, $B(6, -1, 2)$, and $C(8, -7, k)$ be the given points.

Direction ratios of AB are

$<6-5, -1-2, 2-4>$, i.e., $<1, -3, -2>$

Direction ratios of BC are

$<8-6, -7+1, k-2>$, i.e., $<2, -6, k-2>$

Since A, B, C are collinear

$$\therefore \quad \frac{2}{1} = \frac{-6}{-3} = \frac{k-2}{-2}$$

$\therefore$ $k - 2 = -4 \Rightarrow k = -4 + 2 = \mathbf{-2}$.

3. The number of straight lines that are equally inclined to three dimensional coordinate axes, is

(a) 2 (b) 4

(c) 6 (d) 8

Solution (b) If α, β, γ are the angles made by the line with x, y, and z-axis respectively, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Given $\alpha = \beta = \gamma$, $\therefore$ $3\cos^2\alpha = 1$

or $$\cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Possible direction cosines are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.

Different sets of Dc's are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and $$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus four lines are equally inclined to axes.

4. Perpendicular distance of the point (3, 4, 5) from the y-axis, is

(a) $\sqrt{34}$ (b) $\sqrt{41}$

(c) 4 (d) 5

Solution (a) Distance of (α, β, γ) from y-axis is given by

$$d = \sqrt{\alpha^2 + \gamma^2}$$

$\therefore$ Distance (d) of (3, 4, 5) from y-axis is

$$d = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \mathbf{\sqrt{34}}.$$

5. $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ are three points forming a triangle. If AD, the bisector of $\angle BAC$ meets BC in D, then coordinates of D are

(a) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$

(c) $\left(\frac{19}{8}, \frac{57}{16}, -\frac{17}{16}\right)$ (d) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

Solution (d) $\because$ AD is the bisector of $\angle BAC$.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC}$$

$$\text{Now} \quad AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2}$$

$$= \sqrt{4+1+4} = \sqrt{9} = 3$$

$$AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2}$$

$$= \sqrt{144+16+9} = 13$$

$$\therefore \quad \frac{BD}{DC} = \frac{3}{13} \Rightarrow D \text{ divides } BC \text{ in the ratio } 3:13.$$

$\therefore$ The co-ordinates of D are

$$\left[\frac{3(-9)+13(5)}{3+13}, \frac{3(6)+13(3)}{3+13}, \frac{3(-3)+13(2)}{3+13}\right]$$

$$= \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)\left(\mathbf{\frac{19}{8}, \frac{57}{16}, \frac{17}{16}}\right).$$

6. The ratio in which the plane $x - 2y + 3z = 17$ divides the line joining the points (2, –4, 7) and (3, – 5, 8) is

(a) 1 : 5 (b) 5 : 1

(c) 3 : 10 (d) 10 : 3

Solution (c) Let the required radio be $k:1$

The co-ordinates of the point which divides the join of (–2, 4, 7) and (3, –5, 8) in the ratio $k:1$ are

$$\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right)$$

Since this point lies on the plane

$$x - 2y + 3z - 17 = 0$$

$$\therefore \quad \left(\frac{3k-2}{k+1}\right) - 2\left(\frac{-5k+4}{k+1}\right) + 3\left(\frac{8k+7}{k+1}\right) - 17 = 0$$

$\Rightarrow$ $(3k - 2) - 2(-5k + 4) + 3(8k + 7) = 17k - 17$

$\Rightarrow$ $3k + 10k + 24k - 17k = 17 + 2 + 8 - 21$

$\Rightarrow$ $37k - 17k = 6 \Rightarrow 20k = 6$; $k = \frac{6}{20} = \mathbf{\frac{3}{10}}$.

7. If a line OP makes with x-axis an angle of measure 120° and with y-axis of measure 60°, then the angle made by the line with z-axis is

(a) 45° (b) 135°

(c) 60° (d) none of these

Solution (a), (b) We have, $\alpha = 120°$ and $\beta = 60°$

$\therefore \quad \cos\alpha = \cos 120° = -\dfrac{1}{2}$

and $\quad \cos\beta = \cos 60° = \dfrac{1}{2}$

But $\cos^2\alpha + \cos^2\beta + \cos^2\lambda = 1$

$\therefore \quad \left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 + \cos^2\lambda = 1$

$\cos^2\lambda = 1 - \dfrac{1}{4} - \dfrac{1}{4} = \dfrac{1}{2}$

$\therefore \quad \cos^2\lambda = \pm\dfrac{1}{\sqrt{2}} \quad \therefore \quad \lambda = \mathbf{45°}$ or $\mathbf{135°}$.

8. A vector **r** is inclined at equal angles to *OX*, *OY* and *OZ*. If the magnitude of **r** is 6 units, then **r** is equal to

(a) $\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (b) $-\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

(c) $2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (d) $-2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

Solution (c), (d) Let **r** be inclined at an angle α to each axis, then

$l = m = n = \cos\alpha$

Since $l^2 + m^2 + n^2 = 1 \Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \quad \cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

If α is acute: $l = m = n = \dfrac{1}{\sqrt{3}} : |\mathbf{r}| = 6$

$\mathbf{r} = |\mathbf{r}|\ (l\hat{i} + m\hat{j} + n\hat{k})$

$= 6\left(\dfrac{1}{\sqrt{3}}\hat{i} + \dfrac{1}{\sqrt{3}}\hat{j} + \dfrac{1}{\sqrt{3}}\hat{k}\right)$

$= \mathbf{2\sqrt{3}}\ (\hat{i} + \hat{j} + \hat{k})$.

If α is obtuse $l = m = n = -\dfrac{1}{\sqrt{3}} : |\mathbf{r}| = 6$

$\mathbf{r} = |\mathbf{r}|\ (l\hat{i} + m\hat{j} + n\hat{k})$

$= 6\left(-\dfrac{1}{\sqrt{3}}\hat{i} - \dfrac{1}{\sqrt{3}}\hat{j} - \dfrac{1}{\sqrt{3}}\hat{k}\right)$

$= \mathbf{-2v}\ (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$.

9. If the projections of a line on the axes are 6, 2, 3, then its direction cosines are

(a) $\dfrac{6}{7}, -\dfrac{2}{7}, \dfrac{3}{7}$ (b) $\dfrac{6}{7}, \dfrac{2}{7}, -\dfrac{3}{7}$

(c) $-\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$ (d) $\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$

Solution (d) Suppose the line is drawn through the origin and its length be *r*.

$\therefore$ If *l*, *m*, *n* are its direction cosines, then

$lr = 6,\ mr = 2$ and $nr = 3$

$\therefore \quad l^2r^2 = 36,\ m^2r^2 = 4$ and $n^2r^2 = 9$.

On adding these, we get

$l^2r^2 + m^2r^2 + n^2r^2 = 36 + 4 + 9$

$\Rightarrow \quad r^2(l^2 + m^2 + n^2) = 49$

$\Rightarrow \quad r^2 = 49 \Rightarrow r = 7 \qquad (\because l^2 + m^2 + n^2 = 1)$

$\therefore \quad \boldsymbol{l = \dfrac{6}{7},\ m = \dfrac{2}{7},\ n = \dfrac{3}{7}}$.

10. The direction cosines of a line equally inclined to the axes are

(a) $\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}$ (b) $-\dfrac{1}{3}, -\dfrac{1}{3}, -\dfrac{1}{3}$

(c) $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$ (d) $-\dfrac{1}{\sqrt{3}} -\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}$

Solution (c), (d) If a line makes angles α, β, γ with the axes, we have $\alpha = \beta = \gamma$.

$\therefore \quad \cos\alpha = \cos\beta = \cos\gamma \quad \Rightarrow l = m = n$.

$\because \quad l^2 + m^2 + n^2 = 1$,

$\therefore \quad l^2 + l^2 + l^2 = 1 \Rightarrow 3l^2 = 1$,

$\therefore \quad l^2 = \dfrac{1}{3}$ or $l = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ The d.c's of the line are $\left(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}\right)$.

11. The direction cosines of any normal to the *xy*-plane are

(a) 1, 0, 0 (b) 0, 1, 0

(c) 1, 1, 0 (d) 0, 0, 1

Solution (d) The equation of *xy* plane is $z = 0$.

$\therefore$ direction cosines of its normal are **0, 0, 1.**

12. If **r** is a vector of magnitude 21 and has d.r's Rs 2, –3, 6, then **r** is equal to

(a) $6\hat{i} - 9\hat{j} + 18\hat{k}$ (b) $6\hat{i} + 9\hat{j} + 18\hat{k}$

(c) $6\hat{i} + 9\hat{j} - 18\hat{k}$ (d) $6\hat{i} + 9\hat{j} - 18\hat{k}$

Solution (a) d.r's Rs of **r** are 2, –3, 6. Therefore its d.c's are

$l = \dfrac{2}{7},\ m = \dfrac{-3}{7},\ n = \dfrac{6}{7}$

$\therefore \quad \mathbf{r} = |\mathbf{r}|\ (l\hat{i} + m\hat{j} + n\hat{k}) = 21\left(\dfrac{2}{7}\hat{i} - \dfrac{3}{7}\hat{j} + \dfrac{6}{7}\hat{k}\right)$

$= 6\hat{i} - 9\hat{j} + 18\hat{k}$.

13. The equation of the plane containing the points *A* (1, 0, 1) and *B* (3, 1, 2) and parallel to the line joining the origin to the point *C* (1, – 1, 2) is

(a) $x + y + z = 0$ (b) $x + y - z = 0$

(c) $x - y + z = 0$ (d) $x - y - z = 0$

Solution (d) Direction ratios of *OC* are [1, –1, 2].

Since the plane passes through (1, 0, 1), let its equation be

$$a(x - 1) + b(y - 0) + c(z - 1) = 0 \qquad \text{...(1)}$$

Its normal is ⊥ to OC.

$\therefore \quad 1.a + (-1)\,b + 2c = 0$,

i.e., $\quad a - b + 2c = 0 \qquad$ ...(2)

Also since (1) passes through (3, 1, 2), we have $2a+b+c=0$. ....(3)

From (2) and (3),

$$\frac{a}{-1-2}=\frac{b}{4-1}=\frac{c}{1+2}, \text{ i.e., } \frac{a}{-1}=\frac{b}{1}=\frac{c}{1}$$

Thus the plane is

$$(x-1)-1(y-0)-(z-1)=0$$

or $\mathbf{x-y-z=0}$.

14. If the axes are rectangular, the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=17$ is

(a) 1 (b) 2

(c) 3 (d) none of these

Solution (c) Any point on the line is $(r+3.\ 2r+4,\ 2r+5)$.

It lies on the plane $x+y+z=17$,

$\therefore$ $(r+3)+(2r+4)+(2r+5)=17$,

i.e., $r=1$.

Thus the point of intersection of the plane and the line is (4,6, 7).

Required distance = distance between (3, 4, 5) and (4, 6, 7)

$$=\sqrt{\{(4-3)^2+(6-4)^2+(7-5)^2\}}$$

$$=\sqrt{(1+4+4)}=\mathbf{3}.$$

15. The distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{-6}$ is

(a) 1 (b) 2

(c) 4 (d) none of these

Solution (b) Equation of the line through (1, –2, 3) parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{-6}$ is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-1}{-6}=r \text{ (say)}. \quad ...(1)$$

Then any point on (1) is $(2r+1,\ 3r-2,\ -6r+3)$.

If this point lies on the plane $x-y+z=5$, then

$$(2r+1)-(3r-2)+(-6r+3)=5$$

$\Rightarrow$ $-7r+6=5$, i.e., $r=\frac{1}{7}$.

Hence the point is $\left(\frac{9}{7},-\frac{11}{7},\frac{15}{7}\right)$.

Distance between (1, –2, 3) and $\left(\frac{9}{7},-\frac{11}{7},\frac{15}{7}\right)$

$$=\sqrt{\left(\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\right)}=\sqrt{\left(\frac{49}{49}\right)}=\mathbf{1}.$$

16. The planes $3x-y+z+1=0$, $5x+y+3z=0$ intersect in the line PQ. The equation of the plane through the point (2, 1, 4) and perpendicular to PQ is

(a) $x+y-2z=5$ (b) $x+y-2z=-5$

(c) $x+y+2z=5$ (d) $x+y+2z=-5$

Solution (b) Let $[l, m, n]$ be the direction-cosines of PQ, then

$$3l-m+n=0$$

and $5l+m+3n=0$,

$$\therefore \quad \frac{l}{-3-1}=\frac{m}{5-9}=\frac{n}{3+5} \text{ i.e., } \frac{l}{1}=\frac{m}{1}=\frac{n}{-2}.$$

Now a plane ⊥ to PQ will have l, m, n as the coefficients of x, y and z.

Hence the plane ⊥ to PQ is $x+y-2z=\lambda$.

It passes through (2, 1, 4);

$\therefore$ $2+1-2.4=\lambda$ i.e., $\lambda=-5$.

Hence the required plane is

$$\mathbf{x+y-2z=-5.}$$

17. The equation of the plane through the line $x+y+z+3=0=2x-y+3z+1$ and parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is

(a) $x-5y+3z=7$ (b) $x-5y+3z=-7$

(c) $x+5y+3z=7$ (d) $x+5y+3z=-7$

Solution (a) Any plane through the given line

$$2x-y+3z+1+\lambda(x+y+z+3)=0$$

(From $S+\lambda S'=0$)

If this plane is parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, then the normal to the plane is also perpendicular to the above line or

$$(2+\lambda)1+(\lambda-1)2+(3+\lambda)3=0.$$

(From $l_1l_2+m_1m_2+n_1n_2=0$)

This gives $\lambda=-\frac{3}{2}$ and the required plane is

$$\mathbf{x-5y+3z-7=0.}$$

18. The equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point (0, 7, –7) is

(a) $x+y+z=2$ (b) $x+y+z=3$

(c) $x+y+z=0$ (d) none of these

Solution (b) $a(x+1)+b(y-3)+c(z+2)=0$, ...(1)

where $-3a+2b+c=0$. ...(2)

If the plane through (0, 7, –7), then

$$a+4b-5c=0 \quad ...(3)$$

From (2) and (3), $\frac{a}{-10-4}=\frac{b}{1-15}=\frac{c}{-12-2}$,

i.e., $\frac{a}{1}=\frac{b}{1}=\frac{c}{1}$.

Hence the plane (1) becomes

$(x+1)+(y-3)+(z+2)=0$, *i.e.*, $\mathbf{x+y+z=0}$.

19. The equation of the plane passing through the straight line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and perpendicular to the plane $x+2y+z=12$ is

(a) $9x + 2y - 5z + 4 = 0$

(b) $9x - 2y - 5z + 4 = 0$

(c) $9x + 2y + 5z + 4 = 0$

(d) none of these

Solution (b) Any plane through the given line is

$$a(x - 1) + b(y + 1) + c(z - 3) = 0, \quad ...(1)$$

where $$2a - b + 4c = 0 \quad ...(2)$$

If this plane is perpendicular to $x + 2y + z = 12$, then their normals are also perpendicular to each other.

$$\therefore \quad a + 2b + c = 0 \quad ...(3)$$

From (2) and (3), $\dfrac{a}{-1-8} = \dfrac{b}{4-2} = \dfrac{c}{4+1}$,

i.e., $\dfrac{a}{-9} = \dfrac{b}{2} = \dfrac{c}{5}$.

$\therefore$ plane (1) becomes

$$-9(x - 1) + 2(y + 1) + 5(z - 3) = 0.$$

i.e., $\mathbf{9x - 2y - 5z + 4 = 0.}$

20. The lines $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} \times \mathbf{c})$ and $\mathbf{r} = \mathbf{b} + \mu(\mathbf{c} \times \mathbf{a})$ will intersect if

(a) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ (b) $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$

(c) $\mathbf{b} \times \mathbf{a} = \mathbf{c} \times \mathbf{a}$ (d) none of these

Solution (b) The lines $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} \times \mathbf{c})$ and $\mathbf{r} = \mathbf{b} + \mu\ (c \times \mathbf{a})$ pass through points **a** and **b** respectively and are parallel to vector $\mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ respectively. Therefore, they intersect if $\mathbf{a} - \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ are coplanar and so

$$(\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})\} = 0$$

$$\Rightarrow \quad (\mathbf{a} - \mathbf{b}) \cdot ([\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\ \mathbf{c} - [\mathbf{b}\ \mathbf{c}\ \mathbf{c}]\ \mathbf{a}) = 0$$

$$\Rightarrow \quad ((\mathbf{a} - \mathbf{b}) \cdot \mathbf{c})\ [\ \mathbf{b}\ \mathbf{c}\ \mathbf{a}] = 0$$

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} = 0 \Rightarrow \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}.$$

21. The equation of the plane containing the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\,\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is

(a) $\mathbf{r}\ .(\mathbf{a}_1 - \mathbf{a}_2) \times \mathbf{b} = [\mathbf{a}_1\mathbf{a}_2\mathbf{b}]$

(b) $\mathbf{r}\ .(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} = [\mathbf{a}_1\mathbf{a}_2\mathbf{b}]$

(c) $\mathbf{r}\ .(\mathbf{a}_1 + \mathbf{a}_2) \times \mathbf{b} = [\mathbf{a}_2\mathbf{a}_1\mathbf{b}]$

(d) none of these

Solution (b) The required plane passes through the points having position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ and is parallel to the vector **b**. Therefore, if **r** is the position vector of any variable point on the plane, then the vectors $\mathbf{r} - \mathbf{a}_1$, $\mathbf{a}_2 - \mathbf{a}_1$ and **b** are coplanar.

$$\therefore \quad (\mathbf{r} - \mathbf{a}_1) \cdot ((\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}) = 0$$

$$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} - \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{b}) = 0$$

$$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}].$$

22. A straight line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ meets the plane $\mathbf{r}\ .\ \mathbf{n} = 0$ in P. The position vector of P is

(a) $\mathbf{a} + \dfrac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ (b) $\mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$

(c) $\mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ (d) none of these

Solution (c) A straight line $r = a + \lambda b$ meets the plane $r \cdot n = 0$ in P for which λ is given by

$$(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{n} = 0 \Rightarrow \lambda = -\frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

Thus, the position vector of P is

$\mathbf{r} = \mathbf{a} - \dfrac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\ \mathbf{b}$ [Putting the value of λ in $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$]

23. Lines $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$ lie on a plane if

(a) $\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{O}$ (b) $\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{O}$

(c) $(\mathbf{a}_2 - \mathbf{a}_1) . (\mathbf{b}_1 \times \mathbf{b}_2) = \mathbf{O}$

(d) none of these

Solution (c) Lines lie in a plane if

$$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$$

$\because$ $\mathbf{b}_1 \times \mathbf{b}_2$ is a vector $\perp$ to $\mathbf{b}_1$, $\mathbf{b}_2$.

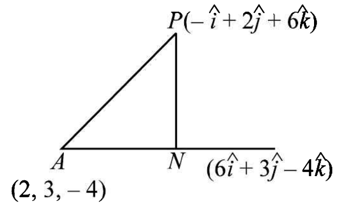
24. The distance from the point $-\hat{i} + 2\hat{j} + 6\hat{k}$ to the straight line through the point $(2, 3, -4)$ and parallel to the vector $6\hat{i} + 3\hat{j} - 4\hat{k}$ is

(a) 7 (b) 10

(c) 9 (d) none of these

Solution (a) We have $\mathbf{AP} = -3\hat{i} - \hat{j} + 10\hat{k}$

$$\therefore \quad |\mathbf{AP}| = \sqrt{9 + 1 + 100} = \sqrt{110}$$



AN = Projection of **AP** on $6\hat{i} + 3\hat{j} - 4\hat{k}$

$$= \left|\frac{\mathbf{AP} \cdot (6\hat{i} + 3\hat{j} - 4\hat{k})}{|6\hat{i} + 3\hat{j} - 4\hat{k}|}\right| = \left|\frac{-8 - 3 - 40}{\sqrt{61}}\right| = \sqrt{61}$$

$$\therefore \quad PN = \sqrt{AP^2 - AN^2} = \sqrt{110 - 61} = 7.$$

25. The equation of the plane containing the line $\mathbf{r} = \hat{i} + \hat{j} + t(2\hat{i} + \hat{j} + 4\hat{k})$, is

(a) $\mathbf{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$

(b) $\mathbf{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 6$

(c) $\mathbf{r} \cdot (-\hat{i} - 2\hat{j} + \hat{k}) = 3$

(d) none of these

Solution (a) The position vector of any point on the given line is

$$\hat{i} + \hat{j} + t\ (2\hat{i} + \hat{j} + 4\hat{k}) = (1 + 2t)\ \hat{i} + (1 + t)\ \hat{j} + 4t\,\hat{k}$$

This lies on $\mathbf{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$

if $$(1 + 2t) \cdot 1 + (1 + t) \cdot 2 + 4t(-1) = 3$$

i.e., if $1 + 2t + 2 + 2t - 4t = 3$. i.e., if $3 = 3$ which is true.

Hence the plane $\mathbf{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the given line.

26. The line of intersection of the planes $\mathbf{r} \cdot (3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\mathbf{r} \cdot (\hat{i} + 4\hat{j} - 2\hat{k}) = 2$ is parallel to the vector

(a) $-2\hat{i} + 7\hat{j} + 13\hat{k}$ (b) $2\hat{i} + 7\hat{j} - 13\hat{k}$

(c) $-2\hat{i} - 7\hat{j} + 13\hat{k}$ (d) $2\hat{i} + 7\hat{j} + 13\hat{k}$

Solution (a) The line of intersection of the planes

$\mathbf{r} \cdot 3\hat{i} - \hat{j} + \hat{k} = 1$ and

$\mathbf{r}\cdot \hat{i}+4\hat{j}-2\hat{k}=2$ is $\perp$ to each of the normal vectors

$\mathbf{n}_1=(3\hat{i}-\hat{j}+\hat{k})$ and $\mathbf{n}_2=(\hat{i}+4\hat{j}-2\hat{k})$

$\therefore$ It is parallel to the vector

$$\mathbf{n}_1\times\mathbf{n}_2 = (3\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+4\hat{j}-2\hat{k})$$

$$= -2\hat{i}+7\hat{j}+13\hat{k}\,.$$

27. The distance between the planes $\mathbf{r}\bullet(2\hat{i}-\hat{j}+3\hat{k})=4$ and $\mathbf{r}\cdot(6\hat{i}-3\hat{j}+9\hat{k})+13=0$ is

(a) $\dfrac{5}{3\sqrt{14}}$ (b) $\dfrac{10}{3\sqrt{14}}$

(c) $\dfrac{25}{3\sqrt{14}}$ (d) none of these

Solution (c) The distance between the parallel planes

$$\mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k})=4$$

and $$\mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k})=-\frac{13}{3}$$

$$\left[\begin{array}{l}\because \mathbf{r}\cdot(6\hat{i}-3\hat{j}+9\hat{k})+13=0\\ \text{so } \mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k})=-\dfrac{13}{3}\end{array}\right]$$

is $$\frac{\left|4-\left(\dfrac{-13}{3}\right)\right|}{\sqrt{2^2+(-1)^2+3^2}}\quad\left[\because \text{Required distance}=\frac{|d-k|}{|\mathbf{n}|}\right]$$

$$=\frac{\left|4+\dfrac{13}{3}\right|}{\sqrt{4+1+9}}=\frac{\dfrac{25}{3}}{\sqrt{14}}=\mathbf{\frac{25}{3\sqrt{14}}}.$$

28. The radius of the circular section of the sphere $|\mathbf{r}|=5$ by the plane $\mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k})=3\sqrt{3}$ is

(a) 16 (b) 8

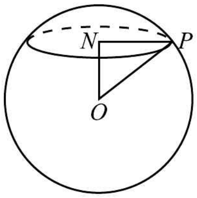
(c) 4 (d) none of these

Solution (c) The sphere $|\mathbf{r}|=5$ has centre at the origin and radius 5.

Distance of the plane

$\mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k})=3\sqrt{3}$ from the origin.

$$=\frac{3\sqrt{3}}{|\hat{i}+\hat{j}+\hat{k}|}=\frac{3\sqrt{3}}{\sqrt{1^2+1^2+1^2}}=3.$$



Thus, according of the figure

$OP=5$, $ON=3$.

$\therefore$ $NP^2=OP^2-ON^2=(5)^2-(3)^2=16$

$\therefore$ $NP=4$

Hence, the radius of the circular section $=NP=\mathbf{4}$.

29. The equation of the sphere whose centre has the position vector $\mathbf{r}\cdot(2\hat{i}-2\hat{j}-\hat{k})$ and which touches the plane $|\mathbf{r}-(3\hat{i}+6\hat{j}-4\hat{k})|=4$ $=10$ is

(a) $|\mathbf{r}-(3\hat{i}+6\hat{j}-4\hat{k})|=4$

(b) $|\mathbf{r}-(3\hat{i}+6\hat{j}+4\hat{k})|=4$

(c) $|\mathbf{r}-(3\hat{i}+6\hat{j}-4\hat{k})|=2$

(d) none of these

Solution (a) The centre of the sphere has the position vector $3\hat{i}+6\hat{j}-4\hat{k}$.

Radius = The distance of the centre whose position vector $\mathbf{a}=3\hat{i}+6\hat{j}-4\hat{k}$ from the plane

$$\mathbf{r}\cdot 3\hat{i}+6\hat{j}-4\hat{k}=10$$

$$=\frac{|\mathbf{a}\cdot\mathbf{n}-d|}{|\mathbf{n}|}\;\frac{|\mathbf{a}\cdot\mathbf{n}-d|}{|\mathbf{n}|}$$

$$=\frac{|(3\hat{i}+6\hat{j}-4\hat{k})\cdot(2\hat{i}-2\hat{j}-\hat{k})-10|}{\sqrt{2^2+(-2)^2+(-1)^2}}$$

$$=\frac{|6-12+4-10|}{\sqrt{4+4+1}}=\frac{|-12|}{3}=\frac{12}{3}=4.$$

Centre = (3, 6, −4); Radius = 4.

Required equation of the sphere is

$$|\mathbf{r}-(3\hat{i}+6\hat{j}-4\hat{k})|=\mathbf{4}.$$

30. The equation $|\mathbf{r}|^2-\mathbf{r}\,.(2\hat{i}+4\hat{j}-2\hat{k})-10=0$ represents a

(a) circle (b) plane

(c) sphere of radius 4 (d) sphere of radius 3

(e) none of these

Solution (c) Since the equation $|\mathbf{r}|^2-2(\mathbf{r}\cdot\mathbf{a})+\lambda=0$ represents a sphere of radius $\sqrt{|\mathbf{a}|^2-\lambda}$, therefore,

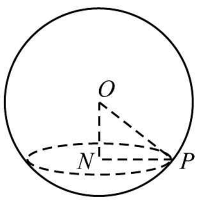
$|\mathbf{r}|^2-\mathbf{r}\cdot(2\hat{i}+4\hat{j}-2\hat{k})-10=0$ represents a sphere of radius $=\sqrt{|\hat{i}+2\hat{j}-\hat{k}|^2+10}=\sqrt{6+10}=\mathbf{4}$.

31. The radius of the circle $x^2+y^2+z^2=49$, $2x+3y-z-5\sqrt{14}=0$ is

(a) $\sqrt{6}$ (b) $2\sqrt{6}$

(c) $4\sqrt{6}$ (d) none of these

Solution (b) The sphere $x^2+y^2+z^2=49$ has centre at the origin (0, 0, 0) and radius 7.



Distance of the plane $2x+3y-z-5\sqrt{14}=0$ from the origin.

$$=\frac{\left|2(0)+3(0)-(0)-5\sqrt{14}\right|}{\sqrt{2^2+3^2+(-1)^2}}$$

$$=\frac{\left|-5\sqrt{14}\right|}{\sqrt{14}}A=\frac{5\sqrt{14}}{\sqrt{14}}=5.$$

Thus according to Figure.

$OP = 7,\ ON = 5$

$NP^2 = OP^2 - ON^2 = (7)^2 - (5)^2 = 49 - 25 = 24$

$\therefore \quad NP = 2\sqrt{6}$.

Hence the radius of the circle $= NP = \mathbf{2\sqrt{6}}$ **.**

32. If the direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$ then

(a) $0 < c < 1$ (b) $c > 2$

(c) $c > 0$ (d) $c = \pm\sqrt{3}$

Solution (d) Since d.c's of line are $< \frac{1}{c}, \frac{1}{c}, \frac{1}{c} >$

$$\therefore \quad \frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}.$$

33. The line of intersection of the planes $\mathbf{r}\cdot(3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\mathbf{r}\cdot(\hat{i} + 4\hat{j} - 2\hat{k}) = 2$ is parallel to the vector

(a) $2\hat{i} + 7\hat{j} + 13\hat{k}$ (b) $-2\hat{i} - 7\hat{j} + 13\hat{k}$

(c) $2\hat{i} + 7\hat{j} + 13\hat{k}$ (d) $-2\hat{i} + 7\hat{j} + 13\hat{k}$

Solution (d) The line of intersection of the planes

$\mathbf{r}\cdot(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and

$\mathbf{r}\cdot(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is $\perp$ to each of the normal vectors

$\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\therefore$ it is parallel to the vector $\mathbf{n}_1 \times \mathbf{n}_2$

$= (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

$\mathbf{= -2i + 7j + 13k.}$

34. A plane which passes through the point (3, 2, 0) and the line $\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$ is

(a) $x - y + z = 1$ (b) $x + y + z = 5$

(c) $x + 2y - z = 1$ (d) $2x - y + z = 5$

Solution (a) Any plane pasing through (3, 2, 0)

$a(x - 3) + b(y - 2) + c(z - 0) = 0$ ...(1)

It passes through (4, 7, 4)

$a + 5b + 4c = 0$

Normal to plane (i) is perpendicular to given line

$a + 5b + 4c = 0$ ...(2)

$\frac{a}{1} = \frac{b}{-1} = \frac{c}{1} = k$

so $a = k,\ b = -k,\ c = k$

Putting the value of a, b, c in Equation (1)

we get, $x - y + z = 1$

35. The equation of the plane containing the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{y-y_1}{m}$ is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where

(a) $ax_1 + by_1 + cz_1 = 0$

(b) $al + bm + cn = 0$

(c) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

(d) $lx_1 + my_1 + nz_1 = 0$

Solution (b) If the given plane contains the given line then the normal to the plane, must be perpendicular to the line and the condition for the same is

$\boldsymbol{al + bm + cn = 0}$

36. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is

(a) 1 (b) 2

(c) 3 (d) 4

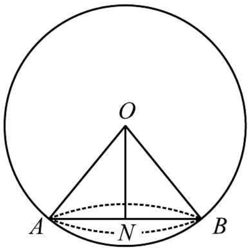
Solution (c) Centre of the given sphere is (–1, 1, 2)

and radius$= \sqrt{(-1)^2 + (1)^2 + (2)^2 + 19}$

$= \sqrt{25} = 5$

Length of the perpendicular from centre O on the given plane is

$$ON = \frac{-1\times1 + 1\times2 + 2\times2 + 7}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{12}{3} = 4$$



In ΔOBN, $OB^2 = ON^2 + NB^2$

$5^2 = 4^2 + NB$

$NB^2 = 25 - 16$

$NB^2 = 9$

$NB = 3.$

37. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if

(a) $k = 0$ or -1 (b) $k = 1$ or -1

(c) $k = 0$ or -3 (d) $k = 3$ or -3

Solution (c) The lines

$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ ...(1)

$\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ ...(2)

are coplanar

$$\therefore \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1 + 2k) + 1(1 + k^2) - (2 - k) = 0$

$\Rightarrow \quad 2k + 1 + 1 + k^2 - 2 + k = 0 \Rightarrow k = 0$ or -3.

38. The two lines $x = ay + b,\ z = cy + d$ and $x = a'y + b',\ z = c'y + d'$ will be perpendicular, if and only if

(a) $aa' + bb' + cc' + 1 = 0$

(b) $aa' + bb' + cc' = 0$

(c) $(a + a')(b + b') + (c + c') = 0$

(d) $aa' + cc' + 1 = 0$

Solution (d) Equation of the line

$x = ay + b,\ z = cy + d$

can be written as

$$\frac{x-b}{a} = y,\ \frac{z-d}{c} = y$$

$$\Rightarrow \quad \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \qquad \text{...(1)}$$

Also, $x = a'y + b'$, $z = c'y + d'$ can be written as

$$\frac{x-b'}{a'} = y,\ \frac{z-d'}{c'} = y$$

$$\Rightarrow \quad \frac{x-b'}{a} = \frac{y-0}{1} = \frac{z-d'}{c'} \qquad \text{...(2)}$$

Lines (1) and (2) will be perpendicular to each other if

$l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow \quad aa' + 1 \times 1 + c \times c' = 0$

$aa' + cc' + 1 = 0$

39. The shortest distance from the plane $12x + 4y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 555$ is

(a) 26 (b) $11\frac{4}{13}$

(c) 13 (d) 39

Solution (c) Centre of the given sphere is (–2, 1, 3)

Length of the perpendicular from centre on the plane is

$$= \left|\frac{-2\times12+1\times4+3\times3-327}{\sqrt{144+16+9}}\right|$$

$$= \left|\frac{-11-327}{13}\right| = 26$$

Shortest distance between the plane and sphere

= 26 – radius of sphere

$= 26 - \sqrt{4+1+9+155} = 26 - 13 = 13.$

40. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a', b', c' from the origin, then

(a) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2} = 0$

(b) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}-\frac{1}{c'^2} = 0$

(c) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2} = 0$

(d) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}1 = 0$

Solution (d) Consider OX, OY, OZ and Ox, Oy, Oz are two system of rectangular axes.

Equation of the plane corresponding to OX, OY, OZ as axes is

$$\frac{X}{a}+\frac{Y}{b}+\frac{Z}{c} = 1 \qquad \text{...(1)}$$

Similarly, equation of the plane corresponding to Ox, Oy, Oz as axes is

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'} = 1 \qquad \text{...(2)}$$

Length of perpendicular from origin to (1) and (2) must be same

i.e,
$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}1 = \frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2} = 0.$$

41. A line makes the same angle θ, with each of the x and z-axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(a) 3/5 (b) 1/5

(c) 3/2 (d) 2/5

Solution (a) Here $l = \cos\theta$, $m = \cos\beta$, $n = \cos\theta$

Now $l^2 + m^2 + n^2 = 1 \Rightarrow \quad 2\cos^2\theta + \cos^2\beta = 1$

$\Rightarrow 2\cos^2\theta = \sin^2\beta \qquad$ ...(1)

Given : $\sin^2\beta = 3\sin^2\theta$

$\Rightarrow \quad 2\cos^2\theta = 3\sin^2\theta \Rightarrow 5\cos^2\theta = 3.$

42. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is

(a) 7/2 (b) 5/2

(c) 3/2 (d) 9/2

Solution (a) Given planes are $4x + 2y + 4z - 16 = 0$ ...(1)

and $4x + 2y + 4z + 5 = 0 \qquad$ ...(2)

Distance between planes (1) and (2) is

$$= \left|\frac{-16-5}{\sqrt{16+4+16}}\right| = \frac{21}{6} = \frac{7}{2}.$$

43. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The co-ordinates of each of the points of intersection are given by

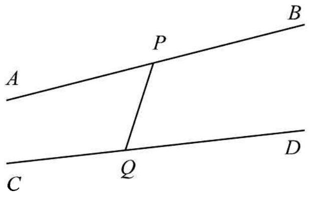
(a) $(3a, 2a, 3a), (a, a, 2a)$

(b) $(3a, 2a, 3a), (a, a, a)$

(c) $(3a, 3a, 3a), (a, a, a)$

(d) $(2a, 3a, 3a), (2a, a, a)$

Solution (b) Given AB : $\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1}$



$$CD : \frac{x+a}{2} = \frac{y}{1} = \frac{z}{1}$$

Let $p \cong (r, r - a, r)$ and $Q = (2\lambda - a, \lambda, \lambda)$

Direction ratios of PQ are $r - 2\lambda + a, r - \lambda - a, r - \lambda$

According to question direction ratios of PQ are (2, 1, 2)

$$\therefore \quad \underset{\text{(i)}}{\frac{r-2\lambda+a}{2}} = \underset{\text{(ii)}}{\frac{r-\lambda-a}{1}} = \underset{\text{(iii)}}{\frac{r-\lambda}{2}}$$

(ii) and (iii) $\Rightarrow r - a = 2a$ ...(1)

(i) and (iii) $\Rightarrow \lambda = a \quad \therefore \quad r = 3a, \lambda = a$

$\therefore \quad p \equiv (3a, 2a, 3a)$ and $Q \equiv (a, a, a)$.

44. If the straight lines $x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}$, $y = 1 + t$, $z = 2 - t$, with parameters s and t respectively, are co-planar, then λ equals

(a) $-1/2$ (b) -1

(c) -2 (d) 0

Solution (c) Given lines are

$x = 1 + s, y = -3 - \lambda s, z = 1 + \lambda s$

i.e., $\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda}$ ...(1)

and $x = \frac{t}{2}$, y $= 1 + t, z = 2 - t$

i.e., $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{-2}$ ...(2)

Lines (1) and (2) are coplanar,

$$\Rightarrow \begin{vmatrix} 1-0 & -3-1 & 1-2 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix}1 = 0$$

$\Rightarrow -5\lambda - 10 = 0 \Rightarrow \lambda = -2.$

45. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the spheres and the plane

(a) $x - y - 2z = 1$ (b) $x - 2y - z = 1$

(c) $x - y - z = 1$ (d) $2x - y - z = 1$

Solution (d) Equation of plane of intersection of the two spheres is

$$S_1 - S_2 = 0 \Rightarrow x^2 + y^2 + z^2 + 7x - 2y - z - 13 - (x^2 + y^2 + z^2 - 3x + 3y + 4z - 8) = 0$$

$\Rightarrow 10x - 5y - 5z - 5 = 0 \Rightarrow 2x - y - z = 1.$

Problems for Practice

Level of Difficulty I

1. The value of k so that $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ may be perpendicular is given by

(a) $-\frac{7}{10}$ (b) $-\frac{10}{7}$ (c) -10 (d) $\frac{10}{7}$

2. The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

(a) $\frac{10}{6\sqrt{5}}$ (b) $\frac{4}{5\sqrt{2}}$ (c) $\frac{\sqrt{2}}{10}$ (d) $\frac{2\sqrt{3}}{5}$

3. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$, is

(a) 0° (b) 30° (c) 45° (d) 90°

4. The equation of the plane through (2, 3, 4) and parallel to the plane $x + 2y + 4z = 5$ is:

(a) $x + 2y + 4z = 10$ (b) $x + 2y + 4z = 3$

(c) $x + y + 2z = 2$ (d) $x + 2y + 4z = 24$

5. The angle between the straight lines

$\frac{x+1}{2} = \frac{y-2}{5} = \frac{z+3}{4}$ and $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-3}{-3}$ is:

(a) 45° (b) 30° (c) 60° (d) 90°

6. Equation of the line passing through (1, 1, 1) and parallel to the plane $2x + 3y + z + 5 = 0$ is

(a) $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{1}$

(b) $\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z-1}{-1}$

(c) $\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-1}{1}$

(d) $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{1}$

7. The distance between the line

$\mathbf{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and

the plane $\mathbf{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ is

(a) $\frac{10}{9}$ (b) $\frac{10}{3\sqrt{3}}$ (c) $\frac{10}{3}$ (d) none of these

8. The angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane $x + y + 4 = 0$, is:

(a) 0° (b) 30° (c) 45° (c) 90°

9. The equation of straight line passing through the point (a, b, c) and parallel to z-axis, is

(a) $\frac{x-a}{1} = \frac{y-b}{1} = \frac{z-c}{0}1$

(b) $\frac{x-a}{0} = \frac{y-b}{1} = \frac{z-c}{1}$

(c) $\frac{x-a}{1} = \frac{y-b}{0} = \frac{z-c}{0}$

(d) $\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$

10. The ratio in which the line joining the points (a, b, c) and $(-a, -c, -b)$ is divided by the xy-plane is

(a) $a : b$ (b) $b : c$ (c) $c : a$ (d) $c : b$

11. The acute angle between the lines whose direction cosines are proportional to (2, 3, –6) and (3, –4, 5), is

(a) $\dfrac{36}{7\sqrt{48}}$ (b) $\dfrac{36}{7\sqrt{50}}$

(c) $\dfrac{18}{7\sqrt{50}}$ (d) none of these

12. The ratio in which the line joining (1, 2, 3) and (–3, 4, –5) is divided by *xy*-plane is

(a) 5 : 3 (b) 3 : 5 (c) 2 : 3 (d) none of these

13. The acute angle between two lines whose direction ratios are 2, 3, 6 and 1, 2, 2 is

(a) $\cos^{-1}\left(\dfrac{20}{21}\right)$ (b) $\cos^{-1}\left(\dfrac{18}{21}\right)$

(c) $\cos^{-1}\left(\dfrac{17}{21}\right)$ (d) none of these

14. The direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to (1, –1, 2) and (2, 1, –1), are

(a) $\dfrac{1}{\sqrt{35}}, -\dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}$

(b) $-\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}$

(c) $\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}$

(d) none of these

15. The projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3, is

(a) $\dfrac{22}{7}$ (b) $\dfrac{15}{7}$ (c) $\dfrac{9}{7}$ (d) none of these

16. If the line through the points (4, 1, 2) and (5, *k*, 0) is parallel to the line through the points (2, 1, 1) and (3, 3, –1), then *k* is equal to

(a) 3 (b) 2 (c) 1 (d) none of these

17. If a line makes α, β, γ angles with the positive directions of the axes, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

(a) 1 (b) 2 (c) 0 (d) none of these

18. The equation of the plane that passes through (2, –3, 1) and is perpendicular to the line joining the points (3, 4, –1) and (2, –1, 5) is

(a) $x + 5y - 6z + 19 = 0$
(b) $x + 5y + 6z + 19 = 0$
(c) $x - 5y + 6z + 19 = 0$
(d) none of these

19. The equation of the plane through the points (2, 3, 1) and (4, –5, 3) and parallel to *x*-axis is

(a) $x - z - 1 = 0$ (b) $4x + y - 11 = 0$
(c) $y + 4z - 7 = 0$ (d) none of these

20. The equation of the plane perpendicular to the *yz*-plane and passing through the points (1, –2, 4) and (3, –4, 5) is

(a) $y + 2z = 5$
(b) $2y + z = 5$
(c) $y + 2z = 6$
(d) $2y + z = 6$

Level of Difficulty II

21. The equation of the plane through (1, 1, 1) and passing through the line of intersection of the planes

$x + 2y - z + 1 = 0$ and $3x - y - 4z + 3 = 0$ is

(a) $8x + 5y - 11z + 8 = 0$
(b) $8x + 5y + 11z + 8 = 0$
(c) $8x - 5y - 11z + 8 = 0$
(d) none of these

22. The equation of the plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$, is

(a) $2x + 17y + 9z = 0$ (b) $2x - 17y + 9z = 0$
(c) $2x + 17y - 9z = 0$ (d) none of these

23. The equation of the plane through the point (–1, 3, 2) and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$, is

(a) $2x - 4y + 3z + 8 = 0$
(b) $2x + 4y + 3z + 8 = 0$
(c) $2x + 4y - 3z + 8 = 0$
(d) none of these

24. The equation of the plane through the points (1, 1, 0), (–2, 2, –1) and (1, 2, 1) is

(a) $2x + 3y + 3z = 5$
(b) $2x - 3y + 3z = 5$
(c) $2x + 3y - 3z = 3$
(d) $2x + 3y - 3z = 5$

25. The coordinates of the point of intersection of the line $\dfrac{x+1}{1} = \dfrac{y+3}{3} = \dfrac{z-2}{-2}$ with the plane $3x + 4y + 5z = 25$ are

(a) (5, 15, 10) (b) (5, 15, –10)
(c) (5, –15, 10) (d) none of these

26. The equation of the plane through the points (2, –1, 0), (3, – 4, 5) and parallel to the line $2x = 3y = 4z$ is

(a) $29(x - 2) + 27(y + 1) - 22z = 0$
(b) $29(x - 2) - 27(y + 1) - 22z = 0$
(c) $29(x - 2) + 27(y + 1) + 22z = 0$
(d) none of these

27. The angle between the planes $\mathbf{r}.(3\hat{i} + \hat{j} - \hat{k}) = 1$ and $\mathbf{r}.(\hat{i} + 4\hat{j} - 2\hat{k}) = 2$ is

(a) $\cos^{-1}\left(\dfrac{9}{\sqrt{231}}\right)$ (b) $\cos^{-1}\left(\dfrac{4}{\sqrt{231}}\right)$

(c) $\cos^{-1}\left(\dfrac{11}{\sqrt{231}}\right)$ (d) none of these

28. If the planes $\mathbf{r}\cdot(2\hat{i} + \lambda\hat{j} - 3\hat{k}) = 0$ and $\mathbf{r}\cdot(\lambda\hat{i} + 3\hat{j} + \hat{k}) = 5$ are perpendicular, then λ is equal to

(a) 2 (b) –2 (c) 3 (d) –3

29. If the planes $\mathbf{r}\bullet(2\hat{i} - \lambda\hat{j} + \hat{k}) = 3$ and $\mathbf{r}\cdot(4\hat{i} + \hat{j} - \mu\hat{k}) = 5$ are parallel, then the values of λ and μ are

(a) $\frac{1}{2}, -2$ (b) $-\frac{1}{2}, 2$

(c) $-\frac{1}{2}, -2$ (d) $\frac{1}{2}, 2$

30. The ratio in which the plane $\mathbf{r}.(\hat{i}-2\hat{j}+2\hat{k})=17$ divides the line joining the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ is

(a) 3 : 5 (b) 1 : 10 (c) 3 : 10 (d) 1 : 5

31. A straight line which makes an angle of 60° with each of y and z-axis, inclines with x-axis at an angle

(a) 30° (b) 60° (c) 75° (d) 45°

32. The position vector of the centre of the circle $|\mathbf{r}| = 5$, $\mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 3\sqrt{3}$ is

(a) $\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$ (b) $\hat{i}+\hat{j}+\hat{k}$

(c) $3(\hat{i}+\hat{j}+\hat{k})$ (d) none of the above

33. Equation of the sphere with centre (1, –1, 1) and radius equal to that of sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y - 6z = 1$ is

(a) $x^2 + y^2 + z^2 - 2x + 2y - 2z + 1 = 0$

(b) $x^2 + y^2 + z^2 + 2x - 2y + 2z + 1 = 0$

(c) $x^2 + y^2 + z^2 - 2x + 2y - 2z - 1 = 0$

(d) none of the above

34. If a straight line makes an angle of $\frac{\pi}{4}$ with each of x-axis and y-axis, then the angle made by the line with z-axis is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none of these

35. A plane meets the coordinate axes at A, B, C such that the centroid of the ΔABC is the point (a, b, c). The equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = k$, where $k =$

(a) 1 (b) 2 (c) 3 (d) none of these

36. The position vectors of points A and B are $\hat{i}-\hat{j}+3\hat{k}$ and $3\hat{i}+3\hat{j}+3\hat{k}$ respectively. The equation of a plane is $\mathbf{r}.(5\hat{i}+2\hat{j}-7\hat{k}) + 9 = 0$. The points A and B

(a) lie on the plane

(b) are on the same side of the plane

(c) are on the opposite side of the plane

(d) none of these

37. The equation of the plane which contains the origin and the line of intersection of the planes $\mathbf{r}.\mathbf{a} = p$ and $\mathbf{r}.\mathbf{b} = q$ is

(a) $\mathbf{r}\cdot(p\mathbf{a} - q\mathbf{b}) = 0$

(b) $\mathbf{r}\cdot(p\mathbf{a} + q\mathbf{b}) = 0$

(c) $\mathbf{r}\cdot(q\mathbf{a} + p\mathbf{b}) = 0$

(d) $\mathbf{r}\cdot(q\mathbf{a} - p\mathbf{b}) = 0$

38. Chord AB is a diameter of the sphere $|\mathbf{r}-2\hat{i}-\hat{j}+6\hat{k}| = \sqrt{18}$. If the coordinates of A are (3, 2, –2), then the coordinates B are

(a) (1, 0, 10) (b) (1, 0, – 10)

(c) (– 1, 0, 10) (d) none of these

39. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is the sphere

(a) $x^2 + y^2 + z^2 - ax - by - cz = 0$

(b) $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0$

(c) $x^2 + y^2 + z^2 - 4ax - 4by - 4cz = 0$

(d) none of these

40. A sphere of constant radius k passes through origin and meets axes in A, B, C. The centroid of the triangle ABC lies on the sphere

(a) $9(x^2 + y^2 + z^2) = 4k^2$

(b) $3(x^2 + y^2 + z^2) = 4k^2$

(c) $x^2 + y^2 + z^2 = 4k^2$

(d) none of these

41. Radius of the circle $\mathbf{r}^2 + \mathbf{r}\bullet(2\hat{i}-2\hat{j}-4\hat{k}) - 19 = 0$

$\mathbf{r}\bullet(\hat{i}-2\hat{j}+4\hat{k}) + 8 = 0$

(a) 5 (b) 4 (c) 3 (d) 2

42. Gives the line $L : \frac{x-1}{3}=\frac{y+1}{2}=\frac{z-3}{-1}$ and the plane $\pi : x - 2y = 0$. Of the following assertions, the only one that is always true is

(a) L is $\perp$ to π

(b) L lies in π

(c) L is parallel to π

(d) none of these

43. Lines $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$ lie in a plane if

(a) $\mathbf{b}_1 \times \mathbf{b}_2 = 0$

(b) $(\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{b}_1 \times \mathbf{b}_2) = 0$

(c) $\mathbf{a}_1 \times \mathbf{a}_2 = 0$

(d) none of these

44. The coordinates of a point which is equidistant from the points (o, o, o), (a, o, o), (o, b, o) and (o, o, c) are given by

(a) $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ (b) $\left(\frac{-a}{2},\frac{-b}{2},\frac{c}{2}\right)$

(c) $\left(\frac{a}{2},\frac{-b}{2},\frac{-c}{2}\right)$ (d) $\left(\frac{-a}{2},\frac{b}{2},\frac{-c}{2}\right)$

45. If a plane passes through the point (1, 1, 1) and is perpendicular to the line $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$, then its perpendicular distance from the origin is

(a) 3/4 (b) 4/3 (c) 7/5 (d) 1

ANSWERS

1. (b) **2.** (c) **3.** (d) **4.** (d) **5.** (d) **6.** (b) **7.** (b) **8.** (c) **9.** (d) **10.** (d)
11. (b) **12.** (b) **13.** (a) **14.** (b) **15.** (a) **16.** (a) **17.** (b) **18.** (a) **19.** (c) **20.** (c)
21. (c) **22.** (b) **23.** (a) **24.** (d) **25.** (b) **26.** (b) **27.** (a) **28.** (d) **29.** (c) **30.** (c)
31. (d) **32.** (a) **33.** (c) **34.** (c) **35.** (c) **36.** (c) **37.** (d) **38.** (b) **39.** (a) **40.** (a)
41. (b) **42.** (b) **43.** (b) **44.** (a) **45.** (c)

Probability 29

BRIEF REVIEW OF THE CONCEPTS

- **Die:** A die is a small cube used in games of chance. On its six faces dots are marked as

  . .. ... : : : . : ⋮ ⋮

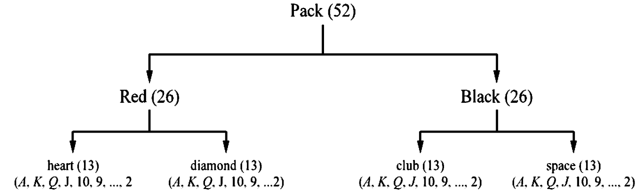
  Plural of die is dice. The outcome of throwing (or tossing) a die is the number of dots on its uppermost face. An ace on a die means one dot.

- **Cards:** A pack (or deck) of playing cards has 52 cards, divided into four suits:

  (i) Spades हुकम (♠) (ii) Clubs चिड़ा (♣)
  (iii) Hearts पान (♥) (iv) Diamonds ईंट (♦)

  Each suit has 13 cards, nine cards numbered 2 to 10, an Ace (इक्का), a King (बादशाह), Queen (बेगम) and a Jack or Knave (गुलाम). Spades and Clubs are black-faced cards while Hearts and Diamonds are red-faced cards. The Aces, Kings, Queens and Jacks are called *face cards* and other cards are called *number cards*. The Kings, Queens and Jacks are called *court cards*.

Description of normal pack of Cards



- The number of combinations of n objects taken r at a time ($r \leq n$) is denoted by $C(n, r)$ or ${}^nC_r$ and is defined as

$$ {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{1 \cdot 2 \cdot 3 \ldots r}. $$

- **Random Experiment or Trial:** The performance of an experiment is called a *trial*. An experiment is characterised by the property that its observations under a given set of circumstances do not always lead to the same observed outcome but rather to the different outcomes. If in an experiment all the possible outcomes are known in advance and none of the outcomes can be predicted with certainity, then such an experiment is called a *random experiment*.

  For example, tossing a coin or throwing a die are random experiments.

- **Event:** The possible outcomes of a trial are called *events*. Events are generally denoted by capital letters A, B, C and so on.

- **Sample Space:** The set of all possible outcomes of an experiment is called a *sample space*. We generally denote it by S.

- **Equally Likely Events:** Events are said to be *equally likely* if there is no reason to expect any one in preference to other. Thus, equally likely events mean outcome is as likely to occur as any other outcome.

- **Simple and Compound Events**

  In the case of *simple events* we consider the probability of happening or non-happening of single events.

- **Exhaustive Events:** It is the total number of all possible outcomes of any trial.

- **Algebra of Events**

  If A and B are two events associated with sample space S, then

1. $A \cup B$ is the event that either A or B or both occur.
2. $A \cap B$ is the event that A and B both occur simultaneously.
3. $\bar{A}$ is the event that A does not occur.
4. $\bar{A} \cap \bar{B}$ is an event of non-occurrence of both A and B, i.e., none of the events A and B occurs.

- **Mutually Exclusive Events**

  Two events are said to be mutually exclusive if occurrence of one precludes or rules out the happening of other.

Mutually Exclusive and Exhaustive Events

Events E_1, E_2, ..., E_n are called mutually exclusive and exhaustive if $E_1 \cup E_2 \cup \ldots \cup E_n = S$, i.e., $\bigcup_{i=1}^{n} E_i = S$ and $E_i \cap E_j = \phi$ for all $i \neq j$.

For example, in a single throw of a die, let A be the event of getting an even number and B be the event of getting an odd number, then

$$ A = \{2, 4, 6\}, B = \{1, 3, 5\} $$

$$ A \cap B = \phi, A \cup B = \{1, 2, 3, 4, 5, 6\} = S $$

$\therefore$ A and B are mutually exclusive and exhaustive events.

PROBABILITY OF AN EVENT

The probability of an event is defined in the following two ways:

1. Mathematical (or *a priori*) definition
2. Statistical (or empirical) definition.

Mathematical Definition of Probability: Probability of an event A, denoted as $P(A)$, is defined as

$$P(A) = \frac{\text{Number of cases favourable to } A}{\text{Number of possible outcomes}}$$

Thus, if an event A can happen in m ways and fails (does not happen) in n ways and each of $m + n$ ways is equally likely to occur then the probability of happening of the event A (also called success of A) is given by

$$P(A) = \frac{m}{m+n}$$

and the probability of non-occurrence of the event A (also called its failure) is given by

$$P(\text{not } A) \text{ or } P(\bar{A}) = \frac{n}{m+n}$$

If the probability of the happening of a certain event is denoted by p and that of not happening by q, then

$$p + q = \frac{m}{m+n} + \frac{n}{m+n} = 1.$$

Here, p, q are non-negative and cannot exceed unity, i.e., $0 \le p \le 1$ and $0 \le q \le 1$.

When $p = 1$, then the event is certain to occur.

When $p = 0$, then the event is impossible. For example, the probability of throwing eight with a single die is zero.

Probability as defined above is sometimes called **Priori Probability,** i.e., it is determined before hand, that is, before the actual trials are made.

Odds of an Event

Suppose, there are m outcomes favourable to a certain event and n outcomes unfavourable to the event in a sample space, then

odds in favour of the event

$$= \frac{\text{Number of favourable outcomes}}{\text{Number of unfavourable outcomes}} = \frac{m}{n}$$

and odds against the event

$$= \frac{\text{Number of unfavourable outcomes}}{\text{Number of favourable outcome}} = \frac{n}{m}.$$

If odds in favour of an event A are $a : b$, then the probability of happening of event $A = P(A) = \dfrac{a}{a+b}$ and probability of not happening of event $A = P(\bar{A}) = \dfrac{b}{a+b}$.

If odds against happening of an event A are $a : b$, then probability of happening of event $A = P(A) = \dfrac{b}{a+b}$ and probability of not happening of event A

$$= P(\bar{A}) = \frac{a}{a+b}.$$

Key Results on Probability

1. In a random experiment, if S is the sample space and E is an event, then

   (a) $P(E) \ge 0$ **(b)** $P(\phi) = 0$ **(c)** $P(S) = 1$.

Remarks: It follows from above results that

- **(i)** probability of occurrence of an event is always non-negative;
- **(ii)** probability of occurrence of an impossible event is 0;
- **(iii)** probability of occurrence of a sure event is 1.

2. If E and F are mutually exclusive events, then

   (i) $P(E \cap F) = 0$ and

   (ii) $P(E \cup F) = P(E) + P(F)$.

Notes:

- For mutually exclusive events E and F, we have
  $$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F).$$
- If $E_1, E_2, \ldots, E_k$ are mutually exclusive events, then
  $$P(E_1 \cup E_1 \cup, \ldots, \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k).$$
- If E and F are two mutually exclusive and exhaustive events, then $P(E) + P(F) = 1$.
- Let E be any event and $\bar{E}$ be its complementary event, then $P(\bar{E}) = 1 - P(E)$.
- For any two events E and F,
  $$P(E - F) = P(E) - P(E \cap F).$$
- **(Addition Theorem).** For any two events E and F,
  $$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
- We may express the above result as
  $$P(E \text{ or } F) = P(E) + (F) - P(E \text{ and } F)$$
- If E_1 and E_2 be two events such that $E_1 \subseteq E_2$, then $P(E_1) \le P(E_2)$.
- If E is an event associated with a random experiment, then $0 \le P(E) \le 1$.
- For any three events E, F, G,
  $$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G)$$

INDEPENDENT EVENTS

Two event A and B are said to be independent if the occurrence (or non-occurrence) of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Remarks If A and B are independent events, then

1. A and $\bar{B}$ are independent events.
2. $\bar{A}$ and B are independent events.
3. $\bar{A}$ and $\bar{B}$ are independent events.
4. $P(A \cup B) = 1 - P(\bar{A}) \cdot P(\bar{B})$.
5. $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$.
6. $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$.
7. $P(A) = P(A \cap B) + P(A \cap \bar{B})$.

8. $P(B) = P(B \cap A) + P(B \cap \bar{A})$.

Mutual Independence and Pairwise Independence

Three events A, B, C are said to be mutually independent if

$$P(A \cap B) = P(A)\cdot P(B),\ P(A \cap C) = P(A)\cdot P(C),\ P(B \cap C) = P(B)\cdot P(C) \text{ and } P(A \cap B \cap C) = P(A)\cdot P(B)\cdot P(C).$$

These events would be pairwise independent if,

$$P(A \cap B) = P(A)\cdot P(B),\ P(B \cap C) = P(B)\cdot P(C)$$

and, $P(A \cap C) = P(A)\cdot P(C)$.

Thus, mutually independent events are pairwise independent but the converse may not be true.

CONDITIONAL PROBABILITY

Let A and B be the two events. If the event A happens when the event B has already happened, then the probability of happening of the event A is called the **conditional probability**. For example, if a bag contains 5 white and 6 red balls, and a white ball is drawn, then the probability of drawing a white or red ball when the first ball drawn is not replaced, is the conditional probability. In case the first ball drawn is replaced and mixed in the contents of the bag, the probability of a drawing the second ball in such a case is independent of the probability of drawing the first ball. In such a case the probability is not conditional.

Probability of A when B has already happened is denoted by $P(A/B)$.

MULTIPLICATION THEOREM ON PROBABILITY

If A and B are any two events, then

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{or} \quad P(A \cap B) = P(B)\cdot P(A/B).$$

Key Results to Remember

1. $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$, or $P(A \cap B) = P(A)\cdot P(B/A)$.
2. If A and B are independent events, then
$$P(B/A) = P(B).$$
3. If A and B are two events such that $B \neq \phi$, then
$$P(A/B) + P(\bar{A}/B) = 1.$$
4. If A and B are two events such that $A \neq \phi$, then
$$P(B) = P(A)\cdot P(B/A) + P(\bar{A})\cdot P(B/\bar{A})$$

LAW OF TOTAL PROBABILITY

Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... E_n, then

$$P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E) + \ldots + P(E_n)\cdot P(A/E_n)$$

BAYE'S THEOREM

Consider any event 'A' of the sample space 'S'. This event would have occurred due to the different causes (or due to the occurrence of any of the events $E_1, E_2, \ldots E_n$).

Now, let us assume that event A is found to have occurred and we have to find the probability that it has occurred due to the occurrence of cause, say, E_i. That means, we are interested in finding $P(E_i/A)$. These types of problems are solved with the help of Baye's theorem.

Statement: Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P(E_i/A) = \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)},\ i = 1, 2, \ldots, n.$$

The probability $P(E_i)$ and $P(E_i/A)$ are known as priori and posteriori probabilities, respectively.

Note: $P(A/_{Ei})$ gives the contribution of E_i in the occurrence of A.

PROBABILITY DISTRIBUTION

Random Variable: A random variable is a real valued function whose domain is the sample space of a random experiment.

A random variable is usually denoted by the capital letters X, Y, Z,..., and so on.

1. **Discrete Random Variable:** A random variable which can take only finite or countably infinite number of values is called a discrete random variable.
2. **Continuous Random Variable:** A random variable which can take any value between two given limits is called a continuous random variable.

Probability Distribution of a Random Variable

If the values of a random variable together with the corresponding probabilites are given, then this description is called a probability distribution of the random variable.

For example, if two coins are tossed, then the probability of 0, 1 and 2 heads occurred is given by,

$$P(X = 0) = P(TT) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$P(X = 1) = P(HT) + P(TH) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Thus, the probability distribution of number of heads, when two coins are tossed is given as:

X	:	0	1	2
$P(X)$	:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

BINOMIAL DISTRIBUTION

Consider a random experiment and an event E associated with it. Let p = Probability of occurrence of event E in one trial. and $q = 1 - p$ = probability of non-occurrence of event E in one trial.

If X denotes the number of successes in n trials of the random experiment, then $P(X = r)$ = Probability of r successes = ${}^nC_r\, p^r\, q^{n-r}$.

Remarks

1. Probability of at most '*r*' successes in *n* trials

$$= \sum_{n=0}^{r} {}^{n}C_r p^r q^{n-r}$$

2. Probability of at least '*r*' successes in *n* trials

$$= \sum_{n=r}^{n} {}^{n}C_4 p^r q^{n-r}$$

3. Probability of having first success at the r^{th} trial $= p \cdot q^{r-1.}$

Important Results

1. The mean, variance and standard deviation of a binomial distribution are np, npq and $\sqrt{npq}$ respectively.
2. Mode of binomial distribution is that value of r for which $P(X = r)$ is maximum. In other words,

$$(n + 1)p - 1 \le r \le (n + 1)p.$$

MATHEMATICAL EXPECTATION

Let X be discrete random variable which assumes the values x_1, $x_2,\ldots x_n$ with corresponding probabilities $p_1, p_2,\ldots p_n$. Then, the expected value of X, denoted by $E(X)$ is defined as

$$E(X) = \sum_{i=1}^{n} x_i\, p_i \text{, where } \sum_{i=1}^{n} p_i = 1.$$

Solved Problems

1. A and B throw a dice. The probability that A's throw is not greater than B's is

(a) $\frac{5}{12}$ (b) $\frac{7}{12}$

(c) $\frac{1}{6}$ (d) $\frac{1}{2}$

Solution (b) If B throws 1, then A can throw only 1, if B throws 2, then A can throw 1 and 2 and so on

∴ the required probability

$$= \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{2}{6} + \frac{1}{6}\cdot\frac{3}{6} + \frac{1}{6}\cdot\frac{4}{6} + \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{6}{6}$$

$$= \frac{1}{6\times6}\,(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{6\times7}{6\times6\times2} = \mathbf{\frac{7}{12}}.$$

2. A committee of five is to be chosen from a group of 9 people. The probability that a certain married couple will either serve together or not at all is

(a) $\frac{2}{3}$ (b) $\frac{4}{9}$

(c) $\frac{1}{2}$ (d) $\frac{5}{9}$

Solution (b) The total number of ways in which 5 person can be chosen out of 9 person is ${}^9c_5 = 126$ The couple serves the committee in ${}^7C_3 \times {}^2C_2 = 35$ ways

The couple does not serve the committee in

${}^7C_5 - {}^7C_2 = 21$ways

Since the couple will either be together or not at all

∴ favourable number of cases = 35 + 21 = 56

∴ required porbability $= \frac{56}{126} = \mathbf{\frac{4}{9}}$.

3. A five digit number is formed by the digits 1, 2, 3, 4, 5, 6 and 8. The probability that the number has even digit at both ends is

(a) $\frac{3}{7}$ (b) $\frac{4}{7}$

(c) $\frac{2}{7}$ (d) none of these

Solution (b) Total number of 5 digit numbers obtained by the digits 1, 2, 3, 4, 5, 6 and 8

$= {}^7P_5 = 7\cdot6\cdot5\cdot4\cdot3 = 2520$

There are 4 even digits (2, 4, 6 and 8)

∴ 2 even digits can be selected in ${}^4C_2 = 6$ways

∴ the two ends can be filled in 2 × 6 = 12 ways

Remaining 3 places from remaining 6 digits can be filled in ${}^6P_3 = 6\cdot5\cdot4 = 120$ ways

Hence the required probability $= \frac{12\times120}{2520} = \mathbf{\frac{4}{7}}$.

4. If $\frac{2-3p}{2}, \frac{1+4p}{3}, \frac{1+p}{6}$ are the probability of three mutually exclusive and exhaustive events, then the set of all values of p is,

(a) $(0, 1)$ (b) $\left(-\frac{1}{4}, \frac{1}{3}\right)$

(c) $\left(0, \frac{1}{3}\right)$ (d) $(0, 2)$

Solution (b) Since $\frac{1-3p}{2} > 0 \Rightarrow 1 - 3p > 0$

$\Rightarrow -3p > -1 \Rightarrow p < 1/3$ ...(1)

Since $\frac{1+4p}{3} > 0 \Rightarrow 1 + 4p > 0$

$\Rightarrow 4p > -1 \Rightarrow p > -1/4$ ...(2)

Since $\frac{1+p}{6} > 0 \Rightarrow 1 + p > 0 \Rightarrow p > -1$ ...(3)

So from (1), (2) and (3), we have the set of values of p as $\left(-\mathbf{\frac{1}{4}}, \mathbf{\frac{1}{3}}\right)$.

5. The letters of the word 'ARTICLE' are arranged at random. The probability that the vowels may occupy the even places, is

(a) $\frac{1}{35}$ (b) $\frac{4}{35}$

(c) $\frac{7}{35}$ (d) none of these

Solution (a) The word 'article' has seven alphabet.

They can be arranged among themselves in 7! ways.

In this word 'article' there are 3 vowels viz. *a*, *i* and *e* and four consonants. These three vowels have to be placed in three even places 2nd, 4th and 6th. The three vowels can occupy these 3 places in 3! ways and the 4 consonants can occupy the remaining 4 places in 4! ways.

Thus the number of ways favourable to the event are 3! × 4!.

$\therefore$ Required probability $= \frac{3! \times 4!}{7!} = \frac{3 \times 2 \times 1}{7 \times 6 \times 5} = \mathbf{\frac{1}{35}}$.

6. A coin and a dice is thrown. If A denotes the event head and even face and B denotes the event tail and multiple of 3, then

(a) $P(A) = \frac{1}{4}$ (b) $P(B) = \frac{1}{5}$

(c) $P(A) = \frac{1}{3}$ (d) $P(B) = \frac{1}{6}$

Solution (a), (d) $S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

$A = \{(H, 2), (H, 4), (H, 6)\}$

$B = \{(T, 3), (T, 6)\}$.

$n(A) = 3, n(B) = 2, n(S) = 12$

$\therefore \quad P(A) = \frac{3}{12} = \mathbf{\frac{1}{4}}$ and $P(B) = \frac{2}{12} = \mathbf{\frac{1}{6}}$.

7. There are 100 students in a collage class of which 36 are boys studying statistics and 13 girls not studying statistics. If there are 55 girls in all, the probability that a boy picked up at random is not studying, is

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$

(c) $\frac{1}{5}$ (d) none of these

Solution (c) Number of students = 100

Number of girls = 55

$\therefore$ Number of boys = 100 – 55 = 45

Out of 45 boys 36 boys are studying Statistics.

$\therefore$ Number of boys not studying Statistics = 45 – 36 = 9

$\therefore$ Probability that a boy picked up at random is not studyng Statistics $= \frac{9}{45} = \mathbf{\frac{1}{5}}$.

8. A room has 3 lamps. From a collection of 10 light bulbs of which 6 are not good, a person selects 3 at random and puts them in a socket, the probability that he will have light, is

(a) $\frac{5}{6}$ (b) $\frac{1}{6}$

(c) $\frac{1}{2}$ (d) none of these

Solution (a) There are 10 bulbs in all and of these 3 can be selected in

$${}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways}$$

$\therefore \quad n(S) = 120$

Let A be the event of getting light. Then $\bar{A}$ denotes the event of having no light. Now 6 bulbs are not good. Hence the number of ways in which all not good bulbs are chosen are

$${}^{6}C_3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$$

Thus $n(\bar{A}) = 20$

$\therefore \quad P(\bar{A}) = \frac{n(\bar{A})}{n(S)} = \frac{20}{120} = \frac{1}{6}$

$\therefore \quad P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{6} = \mathbf{\frac{5}{6}}$.

9. A, B, C are three mutually exclusive and exhaustive events associated with a random experiment. If $P(B) = \frac{3}{2} P(A)$ and $P(C) = \frac{1}{2} P(B)$, then $P(A) =$

(a) $\frac{4}{13}$ (b) $\frac{6}{13}$

(c) $\frac{8}{13}$ (d) none of these

Solution (a) Let $P(A) = p$, then $P(B) = \frac{3}{2} P(A) = \frac{3}{2} p$ and

$$P(C) = \frac{1}{2} P(B) = \frac{1}{2} \times \frac{3}{2} p = \frac{3}{4}$$

$\because$ A, B, C are mutually exclusive and exhaustive events

$\therefore \quad (A \cup B \cup C) = S$

$\therefore \quad P(A \cup B \cup C) = P(S)$

$\Rightarrow \quad P(A) + P(B) + P(C) = 1$

$\Rightarrow \quad p + \frac{3}{2} p + \frac{3}{4} p = 1$

$\Rightarrow \quad \frac{13}{4} p = 1 \qquad \therefore p = \mathbf{\frac{4}{13}}$.

10. Two dice are thrown. The Probability that the number appeared have a sum 8 if it is known that the second dice always exhibits 4, is

(a) $\frac{5}{6}$ (b) $\frac{1}{6}$

(c) $\frac{2}{3}$ (d) none of these

Solution (a) Let A: Second dice always exhibits 4

$= \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)\}$

$\therefore \quad n(A) = 6$

B: Sum of the numbers on both dice is 8 = $\{(4, 4)\}$

Similarly, $A \cap B = A \cap \{(4, 4)\} = \{(4, 4)\}$

$\therefore \quad n(A \cap B) = 1$

$\therefore \quad P(B|A) = \frac{n(A \cap B)}{n(A)} = \mathbf{\frac{1}{6}}$.

11. If the probabilities that A and B will die within a year are p and q respectively then the probability that only one of them will be alive at the end of the year is

(a) $p + q$ (b) $p + q - 2pq$

(c) $p + q - pq$ (d) $p + q + pq$

Solution (b) P(only one of A and B will die in a year)

$= P(\bar{A}B) + P(A\bar{B}) = P(\bar{A})\,P(B) + P(A)\,P(\bar{B})$

[Since A and B are independent]

$= (1 - p)\,q + p(1 - q) = \mathbf{p + q - 2pq}.$

12. You are given a box with 20 cards in it. 10 of these cards have the letter I printed on them. The other ten have the letter T printed on them. If you pick up 3 cards at random and keep them in the same order, the probability of making the word IIT is

(a) $\frac{9}{80}$ (b) $\frac{1}{8}$

(c) $\frac{4}{27}$ (d) $\frac{5}{38}$

Solution (d) Required probability

$= P(\text{I in Ist draw}) \times P(\text{I in 2nd draw}) \times P(T \text{ in 3rd draw})$

$$= \frac{{}^{10}C_1}{{}^{20}C_1} \times \frac{{}^{9}C_1}{{}^{19}C_1} \times \frac{{}^{10}C_1}{{}^{18}C_1} = \frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} = \mathbf{\frac{5}{38}}.$$

13. A pack of cards contains 4 aces, 4 kings, 4 queens and 4 jacks. Two cards are drawn at random. The probability that atleast one of them is an ace is

(a) $\frac{1}{5}$ (b) $\frac{3}{16}$

(c) $\frac{9}{20}$ (d) $\frac{1}{9}$

Solution (c) $P(\text{both are aces}) = \frac{4}{16} \times \frac{3}{15} = \frac{1}{20}$

$P(\text{one is ace}) = \frac{1}{4} \cdot \frac{12}{15} + \frac{12}{16} \cdot \frac{4}{15} = \frac{2}{5}$

$\therefore$ $P(\text{at least one is ace}) = \frac{1}{20} + \frac{2}{5} = \mathbf{\frac{9}{20}}.$

14. A man is known to speak truth 3 out of 4 times. He throws a dice and reports that it is six. The probability that it is actually six is

(a) $\frac{3}{8}$ (b) $\frac{1}{5}$

(c) $\frac{3}{5}$ (d) none of these

Solution (a) Let A denotes the event that a six occurs and B the event that the man reports that it is a six.

Then the probability that it is actually a six is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now $P(A \cap B) = \frac{1}{6} \times \frac{3}{4} = \frac{3}{24}$

$P(B) = P(A \cap B) + P(\bar{A} \cap B) = \frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4} = \frac{8}{24}$

Hence $P(A/B) = \frac{3/24}{8/24} = \mathbf{\frac{3}{8}}.$

15. Ram and Shyam throw with one dice for a prize of Rs 88 which is to be won by the player who throws 1 first. If Ram starts, then mathematical expectation for shyam is

(a) Rs 32 (b) Rs 40

(c) Rs 48 (d) none of these

Solution (b) Probability of winning of Shyam

$$= \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \ldots = \frac{5}{36}\left[1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \ldots\right]$$

$$= \frac{5}{36} \cdot \frac{1}{1 - 25/36} = \frac{5}{36} \times \frac{36}{11} = \frac{5}{11}$$

Mathematical expectation for Shyam = Rs $\frac{5}{11}$ × 88 = **Rs 40.**

16. An unbiased die is tossed until a number greater than 4 appears. The probability that an even number of tosses is needed is

(a) $\frac{1}{2}$ (b) $\frac{2}{5}$

(c) $\frac{1}{5}$ (d) $\frac{2}{3}$

Solution (b) p = probability of successs $(S) = \frac{2}{6} = \frac{1}{3}$

q = probability of failure $(F) = 1 - \frac{1}{3} = \frac{2}{3}$

Probability that the success occurs in even number of tosses

$= P(FS) + P(FFFS) + P(FFFFFS) + \ldots$

$$= qp + q^3p + q^5p + \ldots = \frac{qp}{1 - q^2}$$

$$= \frac{2/3 \times 1/3}{1 - (2/3)^2} = \frac{2/9}{1 - 4/9} = \frac{2}{9} \times \frac{9}{5} = \mathbf{\frac{2}{5}}.$$

17. Sixteen players $S_1, S_2, \ldots S_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assuming that all the palyers are to equal strength. The probability that the players S_1 is among the eight winners is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) none of these

Solution (a) Since all player are of equal strength, with whom S_1 is paired, will have no bearing on the probability of his winning.

The number of ways of choosing 8 winners out of 16: ${}^{16}C_8$.

The number of ways of choosing S_1 and 7 other winners out of 15 is ${}^{15}C_7$.

$\therefore$ Probability that S_1 will win $= \frac{{}^{15}C_7}{{}^{16}C_8}$

$$= \frac{15!}{7!\,8!} \cdot \frac{8!\,8!}{16!} = \mathbf{\frac{1}{2}}.$$

18. These are four balls of different colours and four boxes of colours, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is

(a) $\frac{5}{8}$ (b) $\frac{3}{8}$

(c) $\frac{1}{8}$ (d) none of these

Solution (b) The exhaustive cases are 4! = 24

The favourable cases are $4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

$= 12 - 4 + 1 = 9$

$\therefore$ the required probability $= \dfrac{9}{24} = \mathbf{\dfrac{3}{8}}$.

19. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probabiity that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is,

(a) $\dfrac{16}{81}$ (b) $\dfrac{1}{81}$

(c) $\dfrac{80}{81}$ (d) $\dfrac{65}{81}$

Solution (a) Probability that the outcome of a single throw of a die is any one of 2, 3, 4 and 5 is equal to $\dfrac{4}{6} = \dfrac{2}{3}$.

If the die is rolled four times the required probability is equal to $\left(\dfrac{2}{3}\right)^4 = \mathbf{\dfrac{16}{81}}$.

20. The probability of India winning a test match against West Indies is $\dfrac{1}{2}$. Assuming independence from match to match the probability that in a 5 match series India's second win occurs in the third test is

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{2}{3}$

Solution (b) Let the win in materials I to V be denoted by A_1, A_2, A_3, A_4, A_5

Required probability $= P(A_1\ A_2\ A_3) + P(A_1\ A_2\ A_3)$

$$= P(A_1)\ P(A_2)\ P(A_3) + P(A_1)\ P(A_2)\ P(A_3)$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \mathbf{\frac{1}{4}}.$$

21. An unbaised die is tossed until a number greater than 4 appears. The probability that an even number of tosses is needed is

(a) 1/2 (b) 2/5

(c) 1/5 (d) 2/3

Solution (b) We have $P(s) = P[5, 6] = 2/6 = 1/3$.

$\Rightarrow$ $P(f) = 2/3$.

P(an even number of tosses is needed)

$$= P(FS \text{ or } FFFS \text{ or } FFFFFFS \text{ or } ...)$$

$$= P(F)\ P(S) + P(F)^3\ P(S) + P(F)^5\ P(S) + ...$$

$$= \frac{P(F)\,P(S)}{1 - P(F)^2} = \frac{2/9}{1 - 4/9} = \mathbf{\frac{2}{5}}.$$

22. There are four machines and it is known that exactly two of them are faulty. They are tested one by one, in a random order till both the faulty machines are identifed. Then the probability that only two tests are needed is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{6}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$

Solution (b) The probability that only two tests are needed = (probability that the first machine tested is faulty) × (probability that the second machine tested is faulty given the first machine tested is faulty)

$$= \frac{2}{4} \times \frac{1}{3} = \mathbf{\frac{1}{6}}.$$

23. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black ball will be drawn is

(a) $\dfrac{13}{32}$ (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{32}$ (d) $\dfrac{3}{16}$

Solution (a) P(2 white and 1 black)

$$= P(W_1\ W_2\ B_3 \text{ or } W_1\ B_2\ W_3 \text{ or } B_1\ W_2\ W_3)$$

$$= P(W_1)\ P(W_2)\ P(B_3) + P(W_1)\ P(B_2)\ P(W_3) + P(B_1)\ P(W_2)\ P(W_3)$$

$$= \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{32}(9 + 3 + 1) = \mathbf{\frac{13}{32}}.$$

24. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently, equals

(a) $\dfrac{1}{2}$ (b) $\dfrac{7}{15}$

(c) $\dfrac{2}{15}$ (d) $\dfrac{1}{3}$

Solution (b) The number of ways of placing 3 black ball is ${}^{10}C_3$. The number of ways in which two black balls are not together is equal to the number of ways of choosing 3 places marked with X out of eight places

$$XWXWXWXWXWXWXWX$$

This can be done in ${}^8C_3$ ways. Thus, probability of the required event $= \dfrac{{}^8C_3}{{}^{10}C_3} = \dfrac{8 \times 7 \times 6}{10 \times 9 \times 8} = \mathbf{\dfrac{7}{15}}$.

25. n (≥ 3) persons are sitting in a row. Two of them are selected at random. The probability that they are not together is

(a) $1 - \dfrac{1}{n}$ (b) $1 - \dfrac{2}{n}$

(c) $\dfrac{2}{n+1}$ (d) none of these

Solution (b) The total number of ways of choosing 2 persons out of n is

$${}^nC_2 = \frac{n(n-1)}{2}$$

$\therefore$ The number of ways in which two chosen persons are together is $(n - 1)$

$\therefore$ The number of favourable ways $= {}^nC_2 - (n - 1)$

$$= \frac{n(n-1)}{2} - (n - 1) = \frac{(n-1)(n-2)}{2}$$

$$\therefore \text{ required probability } = \frac{(n-1)(n-2)/2}{n(n-1)/2}$$

$$= \frac{n-2}{n} = \mathbf{1 - \frac{2}{n}}.$$

26. The mean and variance of a binomial vaniate X are 2 and 1 respectively, then the probability that X takes a value greater than 1 is

(a) $\frac{15}{16}$ (b) $\frac{7}{8}$

(c) $\frac{4}{5}$ (d) $\frac{2}{3}$

Solution (a) We have $E(X) = np = 2$ and var $(X) = npq = 1$

$$\therefore \quad q = \frac{1}{2} \text{ and } \therefore p = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \quad n\left(\frac{1}{2}\right) = 2 \Rightarrow n = 4$$

Now $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^4c_0\left(\frac{1}{2}\right)^4 = \frac{1}{16} = \mathbf{\frac{15}{16}}.$$

27. Four digit number are formed using each of the digits 1, 2,...,8 only once. One number from these is picked up at random. The probability that the selected number contains unity is

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) none of these

Solution (c) Required probability = $\frac{{}^7C_3}{{}^8C_4} = \frac{4}{8} = \mathbf{\frac{1}{2}}.$

28. Two fair dice are tossed. Let x be the event that the first dice shows an even number and y be the event that the second die shows an odd number. The two events x and y are

(a) independent and mutually exclusive

(b) dependent

(c) mutually exclusive (d) none of these

Solution (d) The two events are independent but not mutually exclusive.

29. n biscuits are distributed among N boys at random. The probability that particular boy gets r ($< n$) biscuits is

(a) ${}^nC_r\left(\frac{1}{N}\right)^r\left(\frac{N-1}{N}\right)^{n-r}$ (b) ${}^nC_r\left(\frac{1}{N}\right)^r$

(c) ${}^nC_r$ (d) $\frac{r}{n}$

Solution (a) It is a case of Bernoullian trials with number of trials n and probability, = $\frac{1}{N}$ (a success in one trial)

$$\therefore \quad P(r \text{ successes}) = {}^nc_r\, q^{n-1} p^r = {}^nc_r\,(1-p)^{n-r} p^r$$

$$= {}^nc_r\left(1-\frac{1}{N}\right)^{n-r}\left(\frac{1}{N}\right)^r$$

$$= {}^n\mathbf{C}_r\left(\frac{\mathbf{1}}{\mathbf{N}}\right)^r\left(\frac{\mathbf{N-1}}{\mathbf{N}}\right)^{n-r}.$$

30. The binomial distrubtion whose mean is 10 and $S.D.$ is $2\sqrt{2}$ is

(a) $\left(\frac{4}{5}+\frac{1}{5}\right)^{50}$ (b) $\left(\frac{4}{5}+\frac{1}{5}\right)^{1/50}$

(c) $\left(\frac{4}{5}+\frac{5}{1}\right)^{50}$ (d) none of these

Solution (a) Mean = np = 10, S. D = $\sqrt{npq} = 2\sqrt{2}$ ⇒ $npq = 8$

$$\therefore \quad q = \frac{8}{10} = \frac{4}{5}$$

$$\therefore \quad p = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\therefore \quad n \times \frac{1}{5} = 10 \quad \text{i.e., } n = 50$$

$$\therefore \quad \text{Binomial distribution} = (q+p)^n = \left(\mathbf{\frac{4}{5}+\frac{1}{5}}\right)^{50}.$$

31. Four positive integers are taken at random and are multiplied together. Then the probability that the product ends in an odd digit other than 5 is

(a) $\frac{3}{5}$ (b) $\frac{609}{625}$

(c) $\frac{16}{625}$ (d) $\frac{2}{5}$

Solution (c) In any number the last digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, We want that the last digit in the product is an odd digit other than 5 *i.e.,* it is any one of the digits 1, 3, 7, 9. This means that the product is not divisible 2 or 5. The probability that a number is divisible by 2 or 5 is $\frac{6}{10}$

[∵ in that case the last digit can be one of 0, 2, 4, 5, 6, 8]

∴ The probability that the number is not divisible by 2 or 5 = $1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}$

In order that the product is not divisible by 2 or 5 none of the constituent numbers should be divisible by 2 or 5 and its probability = $\left(\frac{2}{5}\right)^4 = \mathbf{\frac{16}{625}}.$

32. If A and B are two independent events such that $P(A') = \frac{7}{10}$, $P(B') = \alpha$ and $P(A \cup B) = \frac{8}{10}$, then α is

(a) $\frac{2}{7}$ (b) $\frac{5}{7}$

(c) 1 (d) none of these

Solution (a) Since A, B are independent

$$\therefore \quad P(A' \cap B') = P(A')\,P(B')$$

$$\Rightarrow \quad 1 - P(A \cup B) = P(A')\,P(B')$$

$$\Rightarrow \quad 1 - \frac{8}{10} = \left(\frac{7}{10}\right)a \Rightarrow \alpha = \mathbf{\frac{2}{7}}.$$

33. For two events A and B if $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$, then

(a) A is subevent of B
(b) A and B are mutually exclusive
(c) A and B are independent and $P\left(\frac{A'}{B}\right) = \frac{3}{4}$
(d) none of the above

Solution (c) $P(A) = P(A/B) = P(A \cap B)/P(B)$

$\Rightarrow \quad P(A \cap B) = P(A)\,P(B)$

Thus, A and B are independent

Also, $P(A'/B) \;=\; \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)}$

$= 1 - P(A/B) = 1 - \dfrac{1}{4} = \mathbf{\dfrac{3}{4}}.$

34. A random Variable X has the following proability distribution:

X	: 0	1	2	3	4	5	6	7	8
$P(X = x)$	: a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

then the value of a is:

(a) $\frac{7}{81}$ (b) $\frac{5}{81}$
(c) $\frac{2}{81}$ (d) $\frac{1}{81}$

Solution (d) We must have

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\therefore \quad \frac{9}{2}(a + 17a) = 1$ [$\because$ sum of the probability = 1]

or $\quad \dfrac{9 \times 18a}{2} = 1 \quad \therefore \quad a = \dfrac{2}{9 \times 18} = \mathbf{\dfrac{1}{81}}.$

35. In tossing 10 coins the proability of getting exactly 5 heads is

(a) $\frac{193}{256}$ (b) $\frac{9}{128}$
(c) $\frac{1}{2}$ (d) $\frac{63}{256}$

Solution (d) Probability for a head $= \frac{1}{2}$ *i.e.*, $p = \frac{1}{2}$

$\therefore \quad q = \frac{1}{2}$ in a toss.

The probability that out of 10 coins, r coins show head

$= {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}$

$\therefore$ required probability $= {}^{10}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5$

$= \dfrac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5} \cdot \left(\dfrac{1}{2}\right)^{10} = \dfrac{63}{2^8} = \mathbf{\dfrac{63}{256}}.$

36. For a poisson distribution whose mean is λ, the standard deviation will be

(a) λ^2 (b) $\frac{1}{\lambda}$
(c) $\sqrt{\lambda}$ (d) λ.

Solution (c) S. D = $\sqrt{\text{mean}}$ for P. D.

37. A person draws a card from a pack of 52 playing cards, replaces it and shuffles the pack. He continues doing this until be draws a spade, the chance that he will fail in the first two draws is

(a) $\frac{1}{16}$ (b) $\frac{9}{16}$
(c) $\frac{9}{64}$ (d) $\frac{1}{64}$

Solution (b) The required probability

$= \dfrac{39}{52} \times \dfrac{39}{52} = \dfrac{3}{4} \times \dfrac{3}{4} = \mathbf{\dfrac{9}{16}}.$

38. If A and B are such that $P(A) > 0$ and $P(B) \neq 1$, then $P(\bar{A}/\bar{B})$ is equal to

(a) $\dfrac{1 - P(A \cup B)}{P(\bar{B})}$ (b) $\dfrac{P(A)}{P(\bar{B})}$
(c) $1 - P(A/B)$ (d) $1 - P(\bar{A}/B)$.

Solution (a) $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$

$= \dfrac{P(\overline{A \cap B})}{P(\bar{B})} = \mathbf{\dfrac{1 - P(A \cup B)}{P(\bar{B})}}.$

39. 6 ordinary dice are rolled. The probability that at least half of them will show at least 3 is

(a) $41 \times \frac{2^4}{3^6}$ (b) $\frac{2^4}{3^6}$
(c) $20 \times \frac{2^4}{3^6}$ (d) none of these

Solution (a) The probability of getting at least 3 in a throw

$= \dfrac{4}{6} = \dfrac{2}{3}.$

$\therefore$ the required probability

$= {}^6C_3 \cdot \left(\frac{2}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^3 + {}^6C_4 \cdot \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)$

$= 41 \, \dfrac{2^4}{3^6}.$

40. If the mean of a binomial distribution is 25, then its standard deviation lies in the interval given below:

(a) $[0, 5)$ (b) $(0, 5]$
(c) $[0, 25)$ (d) $(0, 25]$

Solution (a) We have, $np = 25$. Now, $0 \le p < 1$ and $0 \le q \le 1$

$\Rightarrow \quad 0 \le n\,pq \le np \Rightarrow 0 \le \sqrt{npq} \le \sqrt{np}$

$\Rightarrow \quad 0 \le$ S.D. ≤ 5. But $p \neq 0$, therefore $0 \le$ S.D. < 5

$\Rightarrow \quad$ S.D. $\in$ **[0, 5).**

41. The value of C for which $P(X = k) = Ck^2$ can serve as the probability function of a random variable X that takes value 0, 1, 2, 3, 4 is

(a) 1/30 (b) 1/10
(c) 1/3 (d) 1/15

Solution (a) $\sum\limits_{k=0}^{4} P(X=k) = 1 \Rightarrow \sum\limits_{k=0}^{4} Ck^2 = 1$

$\Rightarrow \quad C(1^2 + 2^2 + 3^2 + 4^2) = 1 \Rightarrow C = \mathbf{\frac{1}{30}}$.

42. Three squares of a chess board are chosen at random, the probability that two are of one colour and one of another is

(a) $\frac{16}{21}$ (b) $\frac{8}{21}$

(c) $\frac{32}{12}$ (d) none of these

Solution (a) Three squares can be chosen out of 64 squares in ${}^{64}C_3$ ways. Two squares of one colour and one another colour can be chosen in two mutually exclusive ways:

(i) two white and one black,

and (ii) two black and one white.

Thus, the favourable number of cases

$= {}^{32}C_2 \times {}^{32}C_1 + {}^{32}C_1 \times {}^{32}C_2$

Hence, the required probability

$= \frac{2\ {}^{32}C_1 \cdot {}^{32}C_2}{{}^{64}C_3} = \mathbf{\frac{16}{21}}$.

43. The mean and variance of a binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than 1 is

(a) $\frac{2}{3}$ (b) $\frac{4}{5}$

(c) $\frac{7}{8}$ (d) $\frac{11}{16}$

Solution (d) We have $E(X) = np = 2$ $\text{var}(X) = npq = 1$

$\therefore \quad q = \frac{1}{2}$ and $\therefore \quad p = 1 - \frac{1}{2} = \frac{1}{2}$.

$\therefore \quad n\left(\frac{1}{2}\right) = 2 \Rightarrow n = 4$.

Now $P(X > 1) = 1 - P(X \leq 0)$

$= 1 - \left[{}^4C_0\left(\frac{1}{2}\right)^4 + {}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\right]$

$= 1 - \left(\frac{1}{2}\right)^4 (1 + 4) = 1 - \frac{5}{16} = \mathbf{\frac{11}{16}}$.

44. $n(\geq 3)$ persons are sitting in a row. Two of them are selected at random. The probability that they are not together is

(a) $1 - \frac{2}{n}$ (b) $\frac{2}{n+1}$

(c) $1 - \frac{1}{n}$ (d) none of these

Solution (a) The total number of ways of choosing 2 persons out of n in ${}^nC_2 = \frac{n(n-1)}{2}$.

The no. of ways in which two chosen persons are together is $(n - 1)$

$\therefore$ the no. of favourable ways $= {}^nC_2 - (n - 1)$

$= \frac{n(n-1)}{2} - (n - 1) = \frac{n(n-1)(n-2)}{2}$

$\therefore \quad$ required probability $= \frac{(n-1)(n-2)/2}{n(n-1)/2}$

$= \frac{n-2}{n} = \mathbf{1 - \frac{2}{n}}$.

45. For a B.D., the parameters n and p are 16 and $\frac{1}{2}$ respectively. The its S.D. σ is equal to

(a) 2 (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) 4.

Solution (a) S.D $\sigma = \sqrt{npq} = \sqrt{16 \times \frac{1}{2} \times \frac{1}{2}} = \sqrt{4} = \mathbf{2}$.

46. The mean and variance of a binomial variable X are 2 and 1 respectively. The probability that X takes values greater than 1, is

(a) $\frac{5}{16}$ (b) $\frac{9}{16}$

(c) $\frac{11}{16}$ (d) none of these

Solution (c) Given: mean $np = 2$ ...(1)

and Variance $npq = 1$ ...(2)

Dividing (2) by (1), then $q = \frac{1}{2}$

$\therefore \quad p = 1 - q = \frac{1}{2}$

From (1), $n \times \frac{1}{2} = 2$, $\therefore \ n = 4$

The binomial distribution is $\left(\frac{1}{2} + \frac{1}{2}\right)^4$

Now, $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$

$= {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4$

$= \frac{6+4+1}{16} = \mathbf{\frac{11}{16}}$.

47. A man takes a step forward with probability 0.4 and backward with probability 0.6. The probability that at the end of eleven steps he is one step away from the starting point, is

(a) 0.37 (b) 0.32

(c) 0.54 (d) none of these

Solution (a) Since the man is one step away from the starting point, $\therefore$ either

(i) man has taken 6 steps forward and 5 steps backward.

or (ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure.

$\therefore \quad p$ = Probability of success = 0.4

and $\quad q$ = Probability of failure = 0.6

$\therefore$ Required probability $= P(X = 6 \text{ or } X = 5)$

$= {}^{11}C_6\, p^6 q^5 + {}^{11}C_5\, p^5 q^6 = {}^{11}C_5\, (p^6 q^5 + p^5 q^6)$

$= \frac{11 \cdot 10 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \{(0 \cdot 4)^6 (0 \cdot 6)^5 + (0 \cdot 4)^5 (0 \cdot 6)^6\}$

$$= \frac{11.10.9.8.7}{1.2.3.4.5}\ (0.24)^5 = 0\cdot37.$$

Hence the required probability = **0.37.**

48. A student is given a true-false exam with 10 questions. If he gets 8 or more correct answers he passes the exam. Given that he guesses at the answer to each question, the probability that he passes the exam, is

(a) $\frac{6}{128}$ (b) $\frac{9}{128}$

(c) $\frac{7}{128}$ (d) none of these

Solution (c) n = total no. of ways = 2^{10} = 1024.

Since each answer can be true or false.

and m = favourable number of ways

$= {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} = 45 + 10 + 1 = 56.$

since to pass the exam, he must give 8 or 9 or 10 true answers,

Hence, $p = \frac{m}{n} = \frac{56}{1024} = \mathbf{\frac{7}{128}}$.

49. There are three events A, B, C one of which must, and only one can happen. If the odds are 8 to 3 against A, 5 to 2 against B, then the odds against C are

(a) 32 : 34 (b) 29 : 34

(c) 43 : 34 (d) none of these

Solution (c) Since odds against A are as 8: 3, the probability of A's occurring,

$$P(A) = \frac{3}{8+3} = \frac{3}{11}.$$

Similarly, $P(B) = \frac{2}{5+2} = \frac{2}{7}$.

Since the events A, B, C are mutually exclusive and totally exhaustive, the sum of their probabilities must be unity, that is,

$$P(A) + P(B) + P(C) = 1$$

or $$\frac{3}{11} + \frac{2}{7} + P(C) = 1$$

$$\therefore \quad P(C) = 1 - \frac{3}{11} - \frac{2}{7} = \frac{34}{77}$$

Hence the odds against C are 77 – 34 : 34 i.e., **43 : 34.**

50. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose that each of them independently and with equal probability, can leave the cabin at any floor beginning with the first. The probability of all five persons leaving at different floors, is

(a) $\frac{{}^8P_5}{7^4}$ (b) $\frac{{}^9P_5}{7^6}$

(c) $\frac{{}^7P_5}{7^5}$ (d) none of these

Solution (c) Besides the ground floor, there are seven floors. The total number of ways in which each of the five persons can leave cabin at any of the 7 floors = 7^5.

And the favourable number of ways, that is, the number of ways in which 5 persons leave at different floors is ${}^7P_5$.

$\therefore$ The required probability = $\mathbf{{}^7P_5/7^5}$.

Problems for Practice

Level of Difficulty I

1. Three identical dice are rolled. The probability that the same number will appear on each of them is

(a) $\frac{1}{18}$ (b) $\frac{3}{28}$ (c) $\frac{1}{36}$ (d) $\frac{1}{6}$

2. Two events A and B have probability 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14. Then the probability that neither A nor B occurs is

(a) 0.25 (b) 0.11 (c) 0.39 (d) none of these

3. $P(A) = \frac{1}{4}$, $P(\bar{B}) = \frac{1}{2}$ and $P(A \cup B) = \frac{5}{9}$, then $P(A/B)$ is

(a) $\frac{7}{36}$ (b) $\frac{7}{9}$ (c) $\frac{7}{18}$ (d) $\frac{7}{72}$

4. A bag contains 5 brown and 4 white socks. A man pulls out two socks. The probability that they are of the same colour is

(a) $\frac{4}{9}$ (b) $\frac{5}{18}$ (c) $\frac{5}{108}$ (d) $\frac{1}{6}$

5. The probability of the occurence of a multiple of 3 on one die and a multiple of 2 on the other die if both are thrown together is

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{11}{16}$ (d) $\frac{1}{4}$

6. The probability of a student getting I, II and III division in an examination are respectively $\frac{1}{10}, \frac{3}{5}$ and $\frac{1}{4}$. The probability that the student fails in the examination is

(a) $\frac{197}{200}$ (b) $\frac{27}{100}$ (c) $\frac{83}{100}$ (d) none of these

7. Two dice are thrown together. The probability that the sum of the numbers on the two faces is neither 9 nor 11 is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) none of these

8. In a single throw of three dice, the probability of getting a total of atleast 5 is

(a) $\frac{1}{3}$ (b) $\frac{53}{54}$ (c) $\frac{1}{54}$ (d) none of these

9. The letters of word 'SOCIETY' are placed at random in a row. The probability that three vowels come together, is

(a) $\frac{6}{7}$ (b) $\frac{1}{7}$ (c) $\frac{3}{7}$ (d) none of these

10. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, the probability that it is rusted or is a bolt, is

(a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{5}{8}$ (d) none of these

11. An integer is chosen at random from first two hundred digits. The probability that the ineger chosen is divisible by 6 or 8 is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) none of these

12. 100 Students appeared for two examinations, 60 passed the first, 50 passed the second and 30 passed both. The probability that a student selected at random has failed in both examinations, is

(a) 0.4 (b) 0.2 (c) 0.3 (d) none of these

13. The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and the probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting atleast one contract is $\frac{4}{5}$, then the probability that he will get both is

(a) $\frac{14}{45}$ (b) $\frac{7}{45}$ (c) $\frac{13}{45}$ (d) none of these

14. A card is drawn drom a pack of 52 cards. The Probability of getting a king or a heart or a red card is

(a) $\frac{9}{13}$ (b) $\frac{7}{13}$ (c) $\frac{5}{13}$ (d) none of these

15 A class consists of 100 students, 25 of them are girls and 75 boys, 20 of them are rich and remaining poor, 40 of them are fair complexioned. The Probability of selecting a fair complexioned rich girl is

(a) 0.02 (b) 0.04 (c) 0.05 (d) 0.08

16. *A* speaks truth in 60% of the cases and *B* in 90% of the cases. The percentage of cases they are likely to contradict each other in stating the same fact is

(a) 36% (b) 48% (c) 42% (d) none of these

17. If $P(A) = 0.65$, $P(B) = 0.15$, $P(\bar{A}) + P(\bar{B}) =$

(a) 1.5 (b) 1.2 (c) .8 (d) none of these

18. In a class of 125 students 70 passed in Mathematics, 55 in Statistics and 30 in both. The probability that a student selected at random from the class, has passed in only one subject is

(a) $\frac{13}{25}$ (b) $\frac{3}{25}$ (c) $\frac{17}{25}$ (d) $\frac{8}{25}$.

19. Two events *A* and *B* have probabilities 0.25 and 0.50 respectively. The probability that both *A* and *B* occur simultaneously is 0.14. Then the probability that neither *A* nor *B* occurs is

(a) 0.28 (b) 0.39 (c) 0.61 (d) 0.72

20. In a certain town, 40% of the people have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. If a person selected at random from the town, has brown hair, the probability that he also has brown eyes is

(a) $\frac{1}{5}$ (b) $\frac{3}{8}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Level of Difficulty II

21. A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. The probability that the number formed is divisible by 4, is

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) none of these

22. A committee consists of 9 experts taken from three institutions *A*, *B* and *C*, of which 2 are from *A*, 3 from *B* and 4 from *C*. If three experts resign, then the probability that they belong to different institutions is:

(a) $\frac{1}{729}$ (b) $\frac{1}{24}$ (c) $\frac{1}{21}$ (d) $\frac{2}{7}$

23. A number is chosen form each of two sets

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{8, 7, 6, 5, 4, 3, 2, 1\}$.

If p_1 is the probability that the sum of the numbers is 9 and p_2 is the probability that sum of the numbers is 7, then $p_1 + p_2$ is equal to

(a) $\frac{5}{52}$ (b) $\frac{3}{16}$ (c) $\frac{9}{32}$ (d) $\frac{7}{32}$

24. The probability of getting the sum as a prime number when two dice are thrown together, is

(a) $\frac{1}{2}$ (b) $\frac{7}{12}$ (c) $\frac{5}{12}$ (d) none of these

25. One of the two events must happen. Given that the chance of one is two-third of the other, the odds in favour of the other are

(a) 3 : 5 (b) 2 : 5 (c) 3 : 2 (d) none of these

26. A bag contains 3 red and 7 black balls. Two balls are selected at rendom without replacement. If the second selected ball is given to be red, the probability that the first selected ball is also red, is

(a) $\frac{4}{9}$ (b) $\frac{2}{9}$ (c) $\frac{2}{9}$ (d) none of these

27. A bag contains 10 mangoes out of which 4 are rotten, two mangoes are taken out together. If one of them is found to be good, the probability that other is also good is

(a) $\frac{1}{3}$ (b) $\frac{8}{15}$ (c) $\frac{5}{18}$ (d) $\frac{2}{3}$

28. Numbers are selected at random, one at a time, from the two-digit numbers 00, 01, 02,...,99 with replacement. An even *E* occurs if and only if the product of the two digits of selected number is 18. If four numbers are selected, the probability that the event *E* occurs at least 3 times, is

(a) $\frac{99}{(25)^4}$ (b) $\frac{86}{(25)^4}$

(c) $\frac{74}{(25)^4}$ (d) $\frac{97}{(25)^4}$

29. A fair coin is tossed repeatedly. If the tail appears on first four tosses, then the probability of head appearing on the fifth toss equals

(a) $\frac{1}{2}$ (b) $\frac{1}{32}$ (c) $\frac{31}{32}$ (d) $\frac{1}{5}$

30. Out of 40 consecutive natural numbers, two are chosen at random. Probability that the sum of the numbers is odd, is

(a) $\frac{14}{29}$ (b) $\frac{20}{39}$ (c) $\frac{1}{2}$ (d) none of these

31. Cards are drawn from a pack of 52 cards one by one. The probability that exactly 10 cards will be drawn before the first ace is

(a) $\frac{451}{884}$ (b) $\frac{241}{1456}$ (c) $\frac{164}{4165}$ (d) none of these

32. Let A, B, C be three equally and collectively exhaustive events. If the odds are $3:3$ against A, $5:2$ against B, then the odds against C are

(a) $50:31$ (b) $77:34$
(c) $20:17$ (d) $43:34$

33. If $P(A \cap B) = \frac{1}{2}$, $P(\overline{A} \cap B) = \frac{1}{2}$ and $2P(A) = P(B) = p$, then the value of p is equal to

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

34. A and B are any two mutally exclusive events, then

(a) $P(A) < P(B)$ (b) $P(A) > P(\bar{B})$
(c) $P(A) \le P(\bar{B})$ (d) none of these

35. From a box containing 20 tickets of value 1 to 20, four tickets are drawn one by one. After each draw, the ticket is replaced. The proability that the largest value of tickets drawn is 15 is

(a) $\left(\frac{3}{4}\right)^4$ (b) $\frac{27}{320}$ (c) $\frac{27}{1280}$ (d) none of these

36. A man alternately tosses a coin and throws a die beginning with the coin. The probability that he gets a head in the coin before he gets a 5 or 6 in the dice is

(a) 3/4 (b) 1/2 (c) 1/3 (d) none of these

37. A certain player, say X, is known to win with probability 0.3 if the track is fast and 0.4 if the track is slow. For Monday, there is a 0.7 probability of a fast track and 0.3 probability of a slow track. The probability that player X will win a Monday, is

(a) 0.22 (b) 0.11 (c) 0.33 (d) none of these

38. It is $8:5$ against a person who is now 40 years old living till he is 70 and $4:3$ against a person now 50 living till he is 80. The probability that at least one of these persons will be alive 30 years hence, is

(a) $\frac{63}{91}$ (b) $\frac{48}{91}$ (c) $\frac{59}{91}$ (d) none of these

39. An MBA applies for a job in two firms X and Y. The probability of his being selected in firim X is 0.7 and being rejected at Y is 0.5. The probability of at least one of his applications being rejected is 0.6. The probability that he will be selected in one of the firms, is

(a) 0.6 (b) 0.4 (c) 0.8 (d) none of these

40. The probability that a leap year selected at random will contain either 53 Thursdays or 53 Fridays, is

(a) $\frac{3}{7}$ (b) $\frac{2}{7}$ (c) $\frac{5}{7}$ (d) none of these

ANSWERS

1. (c)	**2.** (c)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (c)
11. (b)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (b)
21. (a)	**22.** (d)	**23.** (d)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (b)
31. (c)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (c)	**40.** (a)

30 Measures of Central Tendency and Dispersion

BRIEF REVIEW OF THE CONCEPTS

MEASURES OF CENTRAL TENDENCY

For a given data, a single value of the variable which describes its characteristics is identified. This single value is known as the *average*. An average value generally lies in the central part of the distribution and therefore, such values are called the *measures of central tendency*.

The commonly used measures of central tendency are:

1. Arithmetic Mean.
2. Geometric Mean.
3. Harmonic Mean.
4. Median.
5. Mode.

ARITHMETIC MEAN

Mean of Unclassified Data

Let $x_1, x_2, ..., x_n$ be n observations, then their arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x$$

Mean of Grouped Data

Let $x_1, x_2, x_3, ..., x_n$ be n observations and let $f_1, f_2, ..., f_n$ be their corresponding frequencies, then their arithmetic mean is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

Short Cut Method For the given data, we suitably choose a term, usually the middle term and call it the assumed mean, to be denoted by A.

We find the deviation, $d_i = (x_i - A)$ for each term. Then the arithmetic mean is given by

$$\bar{x} = A + \frac{\sum f_id_i}{\sum f_i}.$$

Step Deviation Method When the class intervals in a grouped data are equal, then the calculations can be simplified further by taking out the common factor from the deviations. This common factor is equal to the width of the class interval. In such cases, the deviation $d_i = x_i - A$, of variates x_i from the assumed mean A are divided by the common factor. The $A.M$ is then obtained by the following formula:

$$\bar{x} = A + \frac{\sum f_id_i}{N} \times h;\ N = \sum f_i.$$

where A = assumed mean,

$d_i = \frac{x_i - A}{h}$ = the deviation of any variate from A,

h = the width of the class - interval.

Weighted Arithmetic Mean

If $w_1, w_2, w_3, ..., w_n$ are the weights assigned to the values $x_1, x_2, x_3, ..., x_n$ respectively, then the weighted average is defined as :

$$\text{Weighted } A.M. = \frac{w_1x_1 + w_2x_2 + ... + w_n\,x_n}{w_1 + w_2 + ... + w_n}$$

Combined Mean

If we are given the A. M. of two data sets and their sizes, then the combined A. M. of two data sets can be obtained by the formula:

$$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}}{n_1 + n_2}$$

where, $\bar{x}_{12}$ = Combined mean of the two data sets 1 and 2

$\bar{x}_1$ = Mean of the first data

$\bar{x}_2$ = Mean of the second data

n_1 = Size of the first data

n_2 = Size of the second data.

Important Results

1. In a statistical data, the sum of the deviations of individual values from $A.M.$ is always zero, i.e.,

$$\sum_{i=1}^{n} f_i\,(x_i - \bar{x}) = 0,$$

where f_i is the frequency of $x_i (1 \le i \le n)$

2. In a statistical data, the sum of squares of the deviations of individual values from A. M. is least, i.e.,

$$\sum_{i=1}^{n} f_i\,(x_i - \bar{x})^2 \text{ is least.}$$

3. If each of the n given observations is doubled, then their mean is doubled.

4. If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$, then the mean of $ax_1, ax_2, \ldots, ax_n$ where a is any number different from zero, is $a\bar{x}$.

GEOMETRIC MEAN

If $x_1, x_2, x_3, \ldots, x_n$ are n observations, none of them being zero, then their geometric mean is defined as

$$G.M. = (x_1.x_2.x_3 \ldots x_n)^{\frac{1}{n}}$$

$$G.M. = \text{antilog}\left(\frac{\log x_1 + \log x_2 + \ldots\ldots + \log x_n}{n}\right)$$

In the case of a grouped data, geometric mean of n observations $x_1, x_2, \ldots\ldots, x_n$, is given by

$$G.M. = \left(x_1^{f_1}.x_2^{f_2}\ldots\ldots x_n^{f_n}\right)^{1/N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

or $$G.M. = \text{anti-log}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{N}\right)$$

***Note*:** In the case of continuous or grouped frequency distribution, the values of the variate x are taken to be the values corresponding to the mid-points of the class intervals.

HARMONIC MEAN

The harmonic mean of n observation $x_1, x_2, \ldots, x_n$ is defined as:

$$H.M. = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots + \frac{1}{x_n}}$$

If $x_1, x_2, x_3, \ldots, x_n$ are n observations which occur with frequencies $f_1, f_2, \ldots, f_n$ respectively, then, their $H.M.$ is given by

$$H.M. = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

RELATION BETWEEN A. M, G. M. AND H. M.

The arithmetic mean ($A.M.$), geometric mean ($G.M.$) and harmonic mean (H.M.) for a given set of observations are related as under:

$$A.M \geq G.M. \geq H.M.$$

Equality sign holds only when all the observations are equal.

MEDIAN

Median is the middle most or the central value of the variate in a set of observations, when the observations are arranged either in ascending or in descending order of their magnitudes. It divides the arranged series in two equal parts.

Calculation of Median

1. **Median of an Individual Series** Let n be the number of observations.
   (i) Arrange the data in ascending or descending order.
   (ii) (a) If n is odd, then
   Median = value of the $\frac{1}{2}(n + 1)$ th observation
   (b) If n is even, then
   Median = mean of the $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observation.
2. **Median of a Discrete Series**
   (i) Arrange the values of the variate in ascending or descending order.
   (ii) Prepare a cumulative frequency table.
   (iii) (a) If n is odd, then
   Median = size of the $\left(\frac{n+1}{2}\right)$th term.
   (b) If n is even, then
   Median = size of the $\left(\frac{\left(\frac{n}{2}\right)+\left(\frac{n}{2}+1\right)}{2}\right)$th term.
3. **Median of a Continuous Series**
   (i) Prepare the cumulative frequency table.
   (ii) Find the median class, i.e., the class in which the $\left(\frac{n}{2}\right)$ th observation lies.
   (iii) The median value is given by the formula

$$\text{Median} = l + \left(\frac{\left(\frac{n}{2}\right) - c_f}{f}\right) \times h, \text{ where}$$

l = lower limit of the median class
n = total frequency
f = frequency of the median class
h = width of the median class
c_f = cumulative frequency of the class preceding the median class.

QUARTILES, DECILES AND PERCENTILES

Quartile Just as the median divides a set of observations (when arranged in ascending or descending order of magnitudes), into two equal parts, similarly *Quartile* divides the observations into four equal parts. The value of the item midway, between the first item and the median is known as *first or lower quartile* and is denoted be Q_1. The value of the item midway between the last item and the median is known *Third or Upper Quartile* and is denoted Q_3. The median is known as the *Second Quartile* and is denoted by Q_2. The methods for finding the values of Q_1 and Q_3 are similar to that of the median. In the case of ungrouped data, when arranged in ascending or descending order of magnitudes Q_1, Q_3 can be obtained as follows:

$$Q_1 = \frac{n+1}{4}\text{th item, } Q_3 = \frac{3(n+1)}{4}\text{th item.}$$

For a frequency distribution, Q_1 and Q_3 are given by

$$Q_1 = l + \frac{[(n/4) - C_f]}{f} \times h,$$

$$Q_3 = l + \frac{[(3n/4) - C_f]}{f} \times h,$$

where l = lower limit of the class in which a particular quartile lies,

f = frequency of the class-interval in which a particular quartile lies,

i = class-interval of the class in which a particular quartile lies,

c_f = cumulatively frequency of the class preceding the class in which the particular quartile lies.

In general, $Q_i = l + \dfrac{[(nh/4) - c_f]}{f} \times h$, $i = 1, 2, 3, 4$.

Decile The value of the variable which divides the series, when arranged in ascending or descending order, into 10 equal parts is called decile. There are 9 deciles denoted by $D_1, D_2 \ldots D_9$. When the series is ungrouped the deciles are calculated as follows:

$$D_i = \frac{n \times h}{10}, \; i = 1, 2, \ldots, 9.$$

When the data is classified or grouped,

$$D_i = l + \frac{[(nh/10) - c_f]}{f} \times h$$

where symbols have their usual meaning.

Percentile The value of the variable which divides the series, when arranged in ascending or descending order, into 100 equal parts is called *percentile*. There are 99 percentiles denoted by $P_1, P_2, P_3, P_4, \ldots, P_{99}$ respectively. When the series is ungrouped the percentiles are calculated by the following formula:

$$P_i = \frac{n \times h}{100}, \; h = 1, 2, \ldots, 99.$$

When the data is classified or grouped, the percentiles are calculated by the formula

$$P_i = l + \frac{[(nh/100) - c_f]}{f} \times h, \; i = 1, 2, \ldots., 99,$$

where symbols have their usual meanings.

MODE

Mode is that value in a series which occurs most frequently.

In a frequency distribution, mode is that variate which has the maximum frequency.

Computation of Mode

1. **Mode of Individual Series** In the case of individual series, the value which is repeated maximum number of times is the mode of the series.
2. **Mode of Discrete Series** In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.
3. **Mode of Continuous Series**
   (i) Find the modal class, i.e., the class which has maximum frequency. The modal class can be determined either by inspection or with the help of grouping table.
   (ii) The mode is given by the formula

$$\text{Mode} = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h,$$

where

l = the lower limit of the modal class

h = the width of the modal class

f_{m-1} = the frequency of the class preceding modal class

f_m = the frequency of the modal class

f_{m+1} = the frequency of the class succeeding modal class.

In case, the modal value lies in a class other than the one containing maximum frequency, we take the help of the following formula;

$$\text{Mode} = l + \frac{f_{m+1}}{f_{m-1} + f_{m+1}} \times h,$$

where symbols have usual meaning.

A distribution in which mean, median and mode coincide is called a symmetrical distribution. If the distribution is moderately skewed, then mode can be calculated as follows:

Mode = 3 median − 2 Mean.

MEASURES OF DISPERSION

The degree to which numerical values in the set of values tend to spread about an average value is called the dispersion or variation.

The commonly used measures of dispersion are:

1. Range
2. Quartile Deviation or Semi-interquartile range
3. Mean Deviation
4. Standard Deviation.

Range It is the difference between the greatest and the smallest observations of the distribution.

If L is the largest and S is the smallest observation in a distribution, then its Range = $L - S$. Also,

$$\text{Coefficient of range} = \frac{L - S}{L + S}.$$

Quartile deviation Quartile deviation or semi-interquartile range is given by

$$Q.D. = \frac{1}{2}(Q_3 - Q_1)$$

$$\text{Coefficient of } Q.D. = \frac{Q_3 - Q_1}{Q_3 + Q_1}.$$

Mean deviation For a frequency distribution, the mean deviation from an average (median, or arithmetic mean) is given by

$$M.D. = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$$

$$\text{Coefficient of } M.D. = \frac{\text{Mean deviation}}{\text{Corresponding average}}.$$

Standard deviation The *standard deviation* of a statistical data is defined as the positive square root of the squared deviations of observations from the *A.M.* of the series under consideration.

1. Standard deviation (also denoted by σ) for ungrouped set of observations is given by

$$\sigma = \frac{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}}{n}$$

2. Standard deviation for frequency distribution is given by,

$$S.D. = \frac{\sqrt{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}}{N}$$

where f_i is the frequency of $x_i (1 \le i \le n)$.

When the values of the variable are given in the form of classes, then their respective mid-points are taken as the values of the variable.

Standard Deviation of n Natural Numbers:

$$\sigma = \left(\frac{1}{12}(n^2-1)\right)^{1/2}.$$

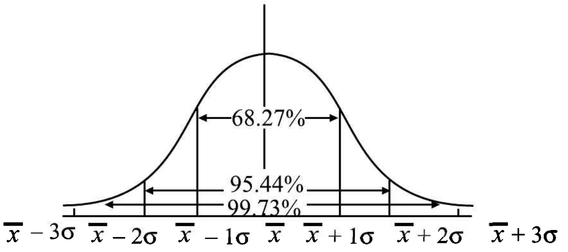
Standard deviation shows the limits of variability by which the individual observation in a distribution will vary from the mean. For a symmetrical distribution with mean $\bar{x}$, the following area relationship holds good:

$\bar{x} \pm \sigma$ covers 68.27 % observations.

$\bar{x} \pm 2\sigma$ covers 95.45 % observations.

$\bar{x} \pm 3\sigma$ covers 99.73 % observations.

These limits are illustrated by the following curve known as *Normal Curve*.



Empirical relationships If the data is moderately non-symmetrical, then the following empirical relationships hold:

Mean deviation $= \frac{4}{5}\sigma$

Semi-Inter-quartile range $= \frac{2}{3}\sigma$.

Probable error of standard deviation $= \frac{2}{3}\sigma =$ Semi-inter-quartile range.

Quartile deviation $= \frac{5}{6}$ *M.D.*

From these relationships, we have

4 *S.D.* = 5 *M.D.* = 6 *Q.D.*

Coefficient of *S.D.* (*C.V.*) For comparing two or more series for variability, the relative measure, called coefficient of variation (*C.V.*) is used. This measure is defined as:

$$C.V. = \frac{\sigma}{\bar{x}} \times 100.$$

The coefficient of variation is also represented as percentage.

The square of *S.D.* is called the variance of the distribution and is denoted by σ^2.

Computation of Standard Deviation

1. **Direct Method**

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

2. **Short-cut Method**

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}\text{, for ungrouped data}$$

where A is assumed mean and $d = x - A$.

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}\text{, for grouped data}$$

where $N = \sum f$.

3. **Step-Deviation Method**

$$\sigma = h\sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2}\ ;\ d' = \frac{x-A}{h}$$

Combined Standard Deviation

Let A_1 and A_2 be two series having n_1 and n_2 observations respectively. Let their *A.M.* be $\bar{x}_1$ and $\bar{x}_2$, and standard deviations be σ_1 and σ_2. Then the combined standard deviation σ or σ_{12} of A_1 and A_2 is given by

$$\sigma_{12} \text{ or } \sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1+n_2}}$$

$$= \sqrt{\frac{n_1(\sigma_1^2+d_1^2) + n_2(\sigma_2^2+d_2^2)}{n_1+n_2}}$$

where $d_1 = (\bar{x}_1 - \bar{x}_{12}),\ d_2 = (\bar{x}_2 - \bar{x}_{12})$,

and $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$ is the combined mean.

Solved Problems

1. A person purchases one kg of tomatoes from each of the 4 places at the rate of 1 kg, 2 kg, 3 kg, 4 kg per rupee respectively. On the average he has purchased x kg of tomatoes per rupee, then the value of x is

(a) 2 (b) 2.5

(c) 1.92 (d) none of these

Solution (c) Since we are given rate per rupee, harmonic mean will give the correct answer

$$H.M. = \frac{4}{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = \frac{4\times 12}{12+6+4+3} = \frac{48}{25}$$

= **1.92** kg per rupee.

2. The mean height of 15 students is 154 cm. It is discovered later on that while calculating the mean the reading 175 cm. was wrongly read as 145 cm. The correct mean height is

(a) 145 cm. (b) 170 cm.

(c) 156 cm. (d) none of these

Solution (c) Total height of 15 students = $\Sigma x = 154 \times 15$

= 2310 cm.

It was found that 175 cm. was wrongly read as 145

Correct sum = 2310 – 145 + 175 = 2340 cm.

Correct mean $= \frac{2340}{15} =$ **156 cm.**

3. Mean of 25 observations was found to be 78.4. But later on it was found that 96 was misread as 69. The correct mean is

(a) 79.48 (b) 76.54

(c) 81.32 (d) 78.4

Solution (a) We know that the mean is given by

$$\bar{x} = \frac{\sum x}{n} \text{ or } \sum x = n\bar{x}$$

Here $\bar{x}$ = 78.4, n = 25

$\therefore$ Σx = 25 × 78.4 = 1960

But this Σx is incorrect as 96 was misread as 69.

Correct Σx = 1960 – 69 + 96 = 1987

$\therefore$ Correct mean = $\frac{1987}{25}$ = **79.48.**

4. The weighted mean of the first n natural numbers if their weights are the same as the numbers, is

(a) $\frac{n+1}{3}$ (b) $\frac{2n+1}{3}$

(c) $\frac{2n-1}{3}$ (d) none of these

Solution (b) Here the numbers are 1, 2, 3, ... , n and their weights also are repectively 1, 2, 3, ... , n.

$$\text{So weighted } A_w = \frac{\sum wx}{\sum w}$$

$$= \frac{1.1 + 2.2 + 3.3 + \ldots\ldots + n.n}{1 + 2 + 3 + \ldots\ldots + n}$$

$$= \frac{1^2 + 2^2 + 3^2 + \ldots.. + n^2}{1 + 2 + 3 + \ldots.. + n}$$

$$= \frac{n(n+1)(2n+1)}{6.\frac{n(n+1)}{2}} = \mathbf{\frac{2n+1}{3}}.$$

5. If the frequencies of first four numbers out of 1, 2, 4, 6, 8 are 2, 3, 3, 2 respectively, then the frequency of 8 if their $A.M.$ is 5, is

(a) 4 (b) 5

(c) 6 (d) none of these

Solution (c) Here mean A = 5.

Let the frequency of 8 be x. Then by the formula

$$A = \frac{\sum xf}{\sum f}$$

$$5 = \frac{1.2 + 2.3 + 4.3 + 6.2 + 8.x}{2 + 3 + 3 + 2 + x} = \frac{32 + 8x}{10 + x}$$

or 18 = $3x$; $\therefore$ x = **6.**

6. The mean weight of 120 students in the second year class of a college is 56 kg. If the mean weights of the boys and that of the girls in the class are 60 kg and 50 kg respectively, then the number of boys and girls separately in the class are

(a) 72, 64 (b) 38, 64

(c) 72, 48 (d) none of these

Solution (c) We know that the combined mean

$$A = \frac{n_1A_1 + n_2A_2}{n_1 + n_2} \quad \ldots(1)$$

Here A_1 = mean weight of boys = 60 kg.

A_2 = mean weight of girls = 50 kg.

A = combined mean = 56 kg.

and $n_1 + n_2$ = 120. ...(2)

So, from (1) and (2),

$$56 = \frac{n_1.60 + n_2.50}{120};$$

$\therefore$ 56 × 120 = n_1.60 + (120 – n_1) 50 ;

$\therefore$ 120 (56 – 50) = 10n_1 ;

$\therefore$ n_1 **= 72,** n_2 **= 48.**

Thus, the number of boys = 72 and the number of girls = 48.

7. The mean of 10 numbers is 12.5; the mean of the first six is 15 and the last five is 10. The sixth number is

(a) 15 (b) 12

(c) 18 (d) none of these

Solution (a) Let the mean of the last four be A_2. Then by the formula for combined mean,

$$12.5 = \frac{6 \times 15 + 4 \times A_2}{6 + 4}; \text{ or } 125 = 90 + 4A_2;$$

$\therefore$ $A_2 = \frac{35}{4}$.

Let the sixth number = x; then taking the sixth number as a collection, the combined mean of this collection and the collection of the last four is 10, by question.

$\therefore$ By definition of combined mean

$$10 = \frac{1 \times x + 4 \times \frac{35}{4}}{1 + 4}; \therefore 50 = x + 35; \therefore x = \mathbf{15.}$$

$\therefore$ Sixth number = 15.

8. If the mean of the set of number x_1, x_2, ..., x_n is $\bar{x}$, then the mean of the numbers $x_i + 2i$, $1 \le i \le n$ is

(a) $\bar{x} + 2n$ (b) $\bar{x} + n + 1$

(c) $\bar{x} + 2$ (d) $\bar{x} + n$.

Solution (b) $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \sum_{i=1}^{n} x_i = n\bar{x}$

$$\therefore \frac{\sum_{i=1}^{n}(x_i + 2i)}{n} = \frac{\sum_{i=1}^{n} x_i + 2(1 + 2 + \ldots. + n)}{n}$$

$$= \frac{n\bar{x} + 2\frac{n(n+1)}{2}}{n} = \bar{x} + \mathbf{(n + 1).}$$

9. The AM of n observations is M. If the sum of n – 4 observations is a, then the mean of remaining 4 observations is

(a) $\frac{nM - a}{4}$ (b) $\frac{nM + a}{2}$

(c) $\frac{nM - a}{4}$ (d) $nM + a$.

Solution (a) Let the mean of the remaining 4 observations be $\bar{X}_1$. Then, $M = \frac{a + 4\bar{X}_1}{(n-4) + 4} \Rightarrow \bar{X}_1 = \mathbf{\frac{nM - a}{4}}$.

10. The weighted mean of first n natural numbers whose weights are equal to the squares of corresponding numbers is

(a) $\dfrac{n+1}{2}$ (b) $\dfrac{3n(n+1)}{2(2n+1)}$

(c) $\dfrac{(n+1)(2n+1)}{6}$ (d) $\dfrac{n(n+1)}{2}$.

Solution (b) Weighted Mean = $\dfrac{1.1^2 + 2.2^2 + \ldots\ldots + n.n^2}{1^2 + 2^2 + \ldots\ldots. + n^2}$

$$= \frac{\Sigma n^3}{\Sigma n^2} = \frac{\frac{n(n+1)}{2}\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}} = \boldsymbol{\frac{3n(n+1)}{2(2n+1)}}.$$

11. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70, 80 were wrongly read as 40, 20, 50 respectively. The correct mean is

(a) 48 (b) $82\frac{1}{2}$

(c) 50 (d) 80

Solution (c) Sum of 100 items = 49 × 100 = 4900

Sum of items added = 60 + 70 + 80 = 210

Sum of items replaced = 40 + 20 + 50 = 110

New Sum = 4900 – 110 + 210 = 5000

$\therefore$ New mean = $\dfrac{5000}{100}$ = **50.**

12. The geometric mean of numbers 7, 7^2, 7^3, ..., 7^n is:

(a) $7^{7/n}$ (b) $7^{n/7}$

(c) $7^{(n-1)/2}$ (d) $7^{(n+1)/2}$

Solution (d) The G.M. of numbers 7, 7^2, 7^3, ..., 7^n

$$(7.7^2.7^3\ldots\ldots..7^n)^{1/n} = (7^{\,1+2+3+\ldots+n})^{1/n}$$

$$= \left(7^{\frac{n(n+1)}{2}}\right)^{1/n} = 7^{\frac{(n+1)}{2}}.$$

13. The mean of n items is $\bar{X}$. If the first item is increased by 1, second by 2 and so on, then the new mean is

(a) $\bar{X} + n$ (b) $\bar{X} + \dfrac{n}{2}$

(c) $\bar{X} + \dfrac{n+1}{2}$ (d) none of these

Solution (c) Let $x_1, x_2, \ldots x_n$ be n items.

Then $\bar{X} = \dfrac{1}{n}\Sigma x_i$.

Let $y_1 = x_1 + 1, y_2 = x_2 + 2,$
$y_3 = x_3 + 3, \ldots.., y_n = x_n + n.$

Then the mean of the new series is

$$\frac{1}{n}\Sigma y_i = \frac{1}{n}\Sigma(x_i + i)$$

$$= \frac{1}{n}\Sigma x_i + \frac{1}{n}(1 + 2 + 3 + \ldots.. + n)$$

$$= \bar{X} + \frac{1}{n}.\frac{n(n+1)}{2} = \bar{X} + \frac{\text{n}+\mathbf{1}}{\mathbf{2}}.$$

14. A sample of 35 observations has the mean 80 and *S.D.* as 4. A second sample of 65 observations from the same population has mean 70 and *S.D.* 3. The s.d. of the combined sample is

(a) 5.85 (b) 5.58

(c) 3.42 (d) none of these

Solution (a) Here $n_1 = 35$, $\bar{x}_1 = 80$, $\sigma_1 = 4$,

$n_2 = 65$, $\bar{x}_2 = 70$, $\sigma_2 = 3$.

$$\therefore \quad \bar{x}_{12} = \frac{35\times 80 + 65\times 70}{35 + 65} = 73.5.$$

$$\sigma_{12} = \sqrt{\left[\frac{35(16 + 42\times 25) + 65(9 + 12\times 25)}{100}\right]}$$

$$= \sqrt{34.21} = \mathbf{5.85}$$

15. The means of five observations is 4 and their variance is 5.2. If three of these observations are 1, 2, and 6, then the other two are:

(a) 2 and 9 (b) 3 and 8

(c) 4 and 7 (d) 5 and 6.

Solution (c) $\bar{x}_1 = 4$, $N = 5$ and $\dfrac{\Sigma(x - \bar{x})^2}{N} = 5.2$

$\Rightarrow \quad \Sigma(x - \bar{x})^2 = (5.2)\ 5$

$\therefore \quad \Sigma(x - \bar{x})^2 = 26$

$\therefore \quad (1 - 4)^2 + (2 - 4)^2 + (6 - 4)^2 + (\alpha - 4)^2 + (\beta - 4)^2 = 26$

where α, β are the other two observatios.

$\therefore \quad 9 + 4 + 4 + (\alpha - 4)^2 + (\beta - 4)^2 = 26$

$\therefore \quad (\alpha - 4)^2 + (\beta - 4)^4 = 9$

Also, $\dfrac{1 + 2 + 6 + \alpha + \beta}{5} = 4$

$\therefore \quad \alpha + \beta = 20 - 9 = 11.$

Clearly 4, 7 only satisfy the above equation in α, β.

Hence reqd. numbers are **4, 7.**

16. If M.D. is 12, the value of S.D. will be

(a) 15 (b) 12

(c) 24 (d) none of these

Solution (a) *M.D.* = $\dfrac{4}{5}$ (*S.D.*)

$\therefore$ *S.D.* = $\dfrac{5}{4}$ (*M.D.*) = $\dfrac{5}{4}$ (12) = **15.**

17. If the mean of numbers 27, 31, 89, 107, 156 is 82, then the mean of 130, 126, 68, 50, 1 is

(a) 80 (b) 82

(c) 157 (d) 75

Solution (d) Mean = $\dfrac{130 + 126 + 68 + 50 + 1}{5}$

$= \dfrac{375}{5} = \mathbf{75.}$

18. The average of n numbers $x_1, x_2, x_3, \ldots.. x_n$ is M. If x_n is replaced by x', then the new average is

(a) $\dfrac{M - x_n + x'}{n}$ (b) $\dfrac{(n-1)M + x'}{n}$

(c) $\dfrac{nM - x_n + x'}{n}$ (d) $M - x_n + x'$.

Solution (c) $M = \dfrac{x_1 + x_2 + \ldots\ldots + x_n}{n}$

$\therefore \quad nM = x_1 + x_2 + \ldots\ldots. + x_n$

$$\Rightarrow \quad \frac{nM - x_n + x'}{n} = \frac{x_1 + x_2 + \ldots\ldots + x_{n-1} + x'}{n}$$

$\therefore$ New average $= \dfrac{\boldsymbol{nM - x_n + x'}}{\boldsymbol{n}}$.

19. For a symmetrical distribution $Q_1 = 20$ and $Q_3 = 40$. The value of 50 th percentile is

(a) 20 (b) 30
(c) 40 (d) none of these

Solution (b) The value of 50th percentile

$$= \frac{Q_3 + Q_1}{2} = \frac{40 + 20}{2} = \mathbf{30.}$$

20. The mean deviation of the numbers 3, 4, 5, 6, 7 is

(a) 25 (b) 5
(c) 1.2 (d) 0

Solution (c) Mean $\bar{x} = \dfrac{3+4+5+6+7}{5} = 5, N = 5$

x	$\lvert x - \bar{x} \rvert$
3	2
4	1
5	0
6	1
7	2

$\therefore \quad \Sigma |x - \bar{x}| = 6 \;\therefore$ Mean deviation from the mean

$$= \frac{\Sigma |x - \bar{x}|}{N} = \frac{6}{5} = \mathbf{1.2.}$$

21. In a class of 100 students there are 70 boys whose average marks in a subject are 75. If the average marks of the complete class is 72, then what is the average of the girls?

(a) 73 (b) 65
(c) 68 (d) 74

Solution (b) Total students = 100

Average marks = 72

Total marks = 72 × 100 = 7200

Total marks of boys in a subject = 70 × 75 = 5250.

$\therefore$ Total marks of girls in a subject

= 7200 − 5250 = 1950

$\therefore$ Average marks of girls $= \dfrac{1950}{30} = 65$

22. The median of a set of 9 distinct observations is 20.5. If each of the largest 4 observations in the set is increased by 2, then the median of the new set

(a) is increased by 2
(b) is decreased by 2
(c) is two times the original median
(d) remains the same as that of the original set

Solution (d) Remains the same as that of the original set.

23. In an experiment with 15 observations on x, the following results were available

$$\Sigma x^2 = 2830, \Sigma x = 170$$

One observation that was 20 was found to be wrong and was replaced by the correct value 30. Then the corrected variance is

(a) 78.00 (b) 188.66
(c) 177.33 (d) 8.33

Solution (a) Given $N = 15$

$\Sigma x = 2830, \Sigma x = 170$

One observation 20 was replaced by 30, then

$\Sigma x^2 = 2830 - 400 + 900 = 3330$

$\Sigma x = 170 - 20 + 30 = 180$

Variance $\sigma^2 = \dfrac{\Sigma x^2}{N} - \left(\dfrac{\Sigma x}{N}\right)^2$

$$= \frac{3330 - 15 \times 144}{15} = \frac{3330 - 2160}{15}$$

$$= \frac{1170}{15} = 78.0$$

24. Consider the following statements

(1) Mode can be computed from histogram
(2) Median is not independent of change of scale
(3) Variance is independent of change of origin and scale.

Which of these is/are correct?

(a) only (1) and (2) (b) only (2)
(c) only (1) (d) (1), (2) and (3)

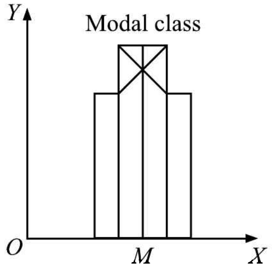
Solution (a) Mode can be computed from histogram

Median is not independent of change of scale.

Median of 1, 2, 3, is 2

Meidan of 2, 4, 6 is 4

Variance depends on change of scale



If $\quad d = \dfrac{x - a}{h}$,

then $x = a + hd$ and $\bar{x} = a + h\bar{d}$

variance $= \dfrac{h^2 f\,(d - \bar{d})^2}{N}$, where $N = \Sigma f$.

25. In a series of 2n observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the ovservations is 2, then $|a|$ equals

(a) 2 (b) $\sqrt{2}$
(c) $\dfrac{1}{n}$ (d) $\dfrac{\sqrt{2}}{n}$

Solution (a) Here $x_1 = x_2 = \ldots = x_n = a$

$x_{n-1} = x_{n+2} = \ldots = x_{2n} = -a$

$\therefore \quad$ mean $\bar{x} = 0$

$$2 = \text{S.D.} = \sqrt{\frac{\Sigma (x - \bar{x})^2}{2n}} = \sqrt{\frac{\Sigma x^2}{2n} - \bar{x}^2}$$

$$= \sqrt{\frac{\Sigma x^2}{2n}} = \sqrt{\frac{2na^2}{2n}} = |a|$$

$\therefore \quad |a| = \mathbf{2.}$

26. In a family, there are 8 men, 7 women and 5 children whose mean ages separately are respectively 24, 20 and 6 years. The mean age of the family is

(a) 17.1 years (b) 18.1 years
(c) 19.1 years (d) none of these

Solution (b) Here we have three collections for which $A_1 = 24$, $n_1 = 8$; $A_2 = 20$, $n_2 = 7$ and $A_3 = 6$, $n_3 = 5$. Their combined mean is the required mean.

By the formula $A = \dfrac{n_1A_1 + n_2A_2 + n_3A_3}{n_1 + n_2 + n_3}$

$$\therefore \quad A = \frac{8\times24 + 7\times20 + 5\times6}{8 + 7 + 3}$$

$$= \frac{192 + 140 + 30}{20} = \frac{362}{20} = 18.1.$$

∴ The mean age of the family = **18.1 years.**

27. The mean of 100 items is 50 and their *S.D.* is 4. the sum of all the items and also the sum of the squares of the items is

(a) 5000, 251600 (b) 4000, 251600
(c) 5000, 261600 (d) none of these

Solution (a) Here $n = 100$, $A = 50$, $\sigma = 4$.

Now, $A = \dfrac{\sum x}{n}$; ∴ $\sum x = nA = 100 \times 50 =$ **5,000.**

Again, from the formula,

$$\sigma^2 + A^2 = \frac{\sum x^2}{n}, \text{ we get } \sum x^2 = n(\sigma^2 + A^2).$$

$$\therefore \quad \sum x^2 = n(\sigma^2 + A^2) = 100(16 + 2500) = \mathbf{2,51,600.}$$

28. If the *A.M.* of two numbers is 10 and their *G.M.* is 8, then their *H.M.* and the two numbers are

(a) 6.4, 16, 4 (b) 4.4, 16, 4
(c) 6.4, 12, 4 (d) none of these

Solution (a) We are given that *A.M.* = 10, *G.M.* = 8.

Also we know that (*A.M.*) (*H.M.*) = $(G.M.)^2$
or (10) (*H.M.*) = $8^2 = 64$

or $H.M. = \dfrac{64}{10} =$ **6.4.**

Let the two numbers be *a* and *b*.

Now $A.M. = \dfrac{a+b}{2} = 10$ or $a + b = 20$.

$G.M. = \sqrt{ab} = 8$ or $ab = 64$

But $(a - b)^2 = (a + b)^2 - 4ab = 20^2 - 4 \times 64 = 144.$

∴ $(a - b) = \sqrt{144} = 12.$

Again $a + b = 20$ and $a - b = 12$ implies that
$2a = 32$ or $a = 16$ and $b = a - 12 = 16 - 12 = 4.$

Hence the numbers are **16, 4.**

29. The coefficient of variation of two series are 58% and 69%. If their standard deviations are 21.2 and 15.6, then their *A.M*s are

(a) 36.6, 22.6
(b) 34.8, 22.6
(c) 36.6, 24.4
(d) none of these

Solution (a)

30. If $n = 10$, $\bar{x} = 12$, $\sum x^2 = 1530$, then the coefficient of variation is

(a) 36% (b) 41%
(c) 25% (d) none of these

Solution (c)

31. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}$, $\alpha - 3$, $\alpha - 2$, $\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$, $\alpha + 5 (\alpha > 0)$, then the median is

(a) $\alpha - \frac{5}{4}$ (b) $\alpha - \frac{1}{2}$
(c) $\alpha - 2$ (d) $\alpha + \frac{5}{4}$

Solution (a)

32. The mean deviation from the mean for the set of observations −1, 0, 4 is

(a) less than 3 (b) less than 4
(c) greater than 2.5 (d) greater than 4.9

Solution (a), (b)

∴ Mean Deviation = $\frac{1}{3}$ [| – 1 – 1 | + |0 – 1| + |4 – 1|] = 2

33. If the S.D of a set of observations is 4 and if each observation is divided by 4, the S.D of the new set of observations will be

(a) 4 (b) 3
(c) 2 (d) 1

Solution (d) We know that if $y = x/h$ when $\sigma y = \sigma_x/|h|$.

∴ The s.d. of new set of observations will be 4/4 = **1.**

34. The quartile deviation for the data

x :	2	3	4	5	6	
f :	3	4	8	4	1	is

(a) 0 (b) $\frac{1}{4}$
(c) $\frac{1}{2}$ (d) 1

Solution (d)

35. If 25% of the items are less than 20 and 25% are more then 40, the quartile deviation is

(a) 20 (b) 30
(c) 40 (d) 10

Solution (d)

36. The sum of squares of deviations for 10 observations taken from mean 50 is 250. The coefficient of variation is

(a) 10% (b) 40%
(c) 50% (d) none of these

Solution (a)

37. The number of observations in a group is 40. If the average of first 10 is 4.5 and that of the remaining 30 is 3.5, then the average of the whole group is

(a) $\frac{15}{4}$ (b) $\frac{1}{5}$
(c) 8 (d) 4

Solution (a)

38. A firm of readymade garments make both men's and women's shirts. Its profit average is 6% of sales. Its profits in men's shirts average 8% of sales and women's shirts comprise 60% of output. The average profit per sales rupee in women's shirts is

(a) 0.0466 (b) 0.0166
(c) 0.0666 (d) none of these

Solution (a)

39. Suppose values taken by a variable X are such that $a \le x_i \le b$ where x_i denotes the value of X in the ith case for $i = 1, 2, ... n$ then

(a) $(b - a)^2 \ge \text{var}(X)$ (b) $\frac{a^2}{4} \le \text{var}(X)$
(c) $a^2 \le \text{var}(X) \le b^2$ (d) $a \le \text{var}(X) \le b$

Solution (a)

40. The *A.M.* of a set of 50 numbers is 38. If two numbers of the set, namely 55 and 45 are discarded, the *A.M.* of the remaining set of numbers is

(a) 36 (b) 36.5
(c) 37.5 (d) 38.5

Solution (c)

Problems for Practice

1. Harmonic mean of 2, 4 and 5 is

(a) 4.21 (b) 3.16 (c) 2.98 (d) none of these

2. Geometric mean of 3, 9 and 27 is

(a) 18 (b) 6 (c) 9 (d) none of these

3. The number of runs scored by 11 players of a cricket team of a school are 5, 19, 42, 11, 50, 30, 21, 0, 52, 36, 27. The median is

(a) 21 (b) 27 (c) 30 (d) none of these

4. If the value of mode and mean is 60 and 66 respectively, then the value of median is

(a) 60 (b) 64 (c) 68 (d) none of these

5. The median of the items 6, 10, 4, 3, 9, 11, 22, 18 is

(a) 9 (b) 10 (c) 9.5 (d) 11

6. The variance of first n natural numbers is

(a) $\frac{n^2+1}{12}$ (b) $\frac{n^2-1}{12}$

(c) $\frac{(n+1)(2n+1)}{6}$ (d) none of these

7. Mean deviation of the series a, $a + d$, $a + 2d$, $a + 2nd$ from its mean is

(a) $\frac{(n+1)\,d}{(2n+1)}$ (b) $\frac{nd}{2n+1}$

(c) $\frac{n\,(n+1)\,d}{(2n+1)}$ (d) $\frac{(2n+1)\,d}{n\,(n+1)}$

8. In any discrete series (when all values are not same), the relationship between *M.D.* about mean and *S.D.* is

(a) *M.D.* = *S.D.* (b) *M.D.* ≥ *S.D.*
(c) *M.D.* < *S.D.* (d) *M.D.* ≤ *S.D*

9. If μ is the mean of a distribution, then $\sum f_i (y_i - \mu)$ is equal to

(a) M.D. (b) S.D. (c) 0 (d) none of these

10. The *S.D.* of scores 1, 2, 3, 4, 5 is:

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

11. The mean of the squares of first n natural numbers is

(a) $\frac{1}{2} n^2$ (b) $\frac{1}{8} n(n + 1)$

(c) $\frac{1}{6} n(2n + 1)$ (d) $\frac{1}{6} (n + 1)(2n + 1)$.

12. The arithmetic mean of first n odd natural numbers is

(a) n (b) $(n + 1)/2$
(c) $(n - 1)$ (d) none of these

13. The arithmetic mean of the series 1, 2, 2^2, ... 2^{n-1} is

(a) $2^n/n$ (b) $(2^n - 1)/n$
(c) $(2^n + 1)/n$ (d) none of these

14. The arithmetic mean of the series ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ ..., ${}^nC_n$ is:

(a) $2^n/(n+1)$ (b) $2^n/n$
(c) $(2^{n-1}/(n+1)$(d) none of these

15. The variance of the data 2, 4, 6, 8, 10 is

(a) 6 (b) 7 (c) 8 (d) none of these

ANSWERS

1. (b) **2.** (c) **3.** (b) **4.** (b) **5.** (c) **6.** (b) **7.** (c) **8.** (d) **9.** (c) **10.** (a)
11. (d) **12.** (a) **13.** (b) **14.** (a) **15.** (c)

Mathematical Induction

31

BRIEF REVIEW OF THE CONCEPTS

MATHEMATICAL INDUCTION

In drawing mathematical or scientific conclusions, there are two basic processes of reasoning that are commonly used. These are deduction and induction. Deduction is the process of reasoning from general to particular and induction is the process of reasoning from particular to general. In this unit, we plan to study induction. Induction begins by observations and from observations we arrive at some tentative conclusions, called conjectures. A conjecture may be true or false. The principle of mathematical induction helps us in proving some of these conjectures which are true.

A Notation for Statements

Consider the statements:

1. $n(n + 1)$ is divisible by 2.
2. $2^{3n} - 1$ is divisible by 7.
3. $2^n > n$, etc.

All these statements are concerned with n, which takes values 1, 2, 3,… Such statements are usually denoted by $p(n)$. By giving particular values to n, we get particular statement.

For example, if the statement "$3^{2n} - 1$ is divisible by 8" is denoted by $p(n)$, then $p(4)$ is the statement "$3^{2.4} - 1$ is divisible by 8".

The Principle of Mathematical Induction

This principle states: If $p(n)$ is a statement involving natural number n, then

(i) if $p(1)$ is true, and

(ii) if $p(k + 1)$ is true whenever $p(k)$ is true, then $p(n)$ is true for all natural numbers n.

Thus, in order to prove a statement $p(n)$ to be true for all natural numbers, we have the following working rule:

Working Rule

1. Prove that $p(1)$ is true; i.e., $p(n)$ is true for $n = 1$.
2. Assume $p(k)$ to be true; i.e., $p(n)$ is true for $n = k$.
3. Prove that $p(k + 1)$ is also true; i.e., $p(n)$ is also true for $n = k + 1$.

Note: It is important to note that for the proof by mathematical induction both the conditions (i) and (ii) as stated above must be fulfilled. The result obtained may be fallacious if only one of these conditions is satisfied. Even if we prove a certain statement for a large number of values of n, say $n = 1, 2, \ldots, 100$, we cannot say that the statement is true for all values of n unless we also establish the condition (ii).

Solved Problems

1. If $n \in N$, then $7^{2n} + 3^{3n-3}. 3^{n-1}$ is always divisible by

(a) 25 (b) 35

(c) 45 (d) None of these

Solution Putting $n = 1$ is $7^{2n} + 2^{3n-3} . 3^{n-1}$, we get

$$7^{2\cdot 1} + 2^{3\cdot 1-3}.3^{1-1} = 7^2 2^0.3^0 = 49 + 1 = 50 \quad \ldots..(1)$$

Also, for $n = 2$

$$7^{2\cdot 2} + 2^{3\cdot 2-3}.3^{2-1} = 2401 + 24 = 2425 \quad \ldots..(2)$$

From (1) and (2), it is always divisible by 25.

2. For every natural number n, $(n^2 - 1)$ is divisible by

(a) 4

(b) 6

(c) 10

(d) None of these

Solution We have, n, $(n^2 - 1) = (n - 1)\, n(n + 1)$.

It is product of three consecutive natural numbers, so by Lagrange's theorem it is divisible by 3! i.e. 6.

3. For a positive integer n,

Let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n - 1}$. Then

(a) $a\ (100) \le 100$ (b) $a\ (100) > 100$

(c) $a\ (200) \le 100$ (d) $a\ (200) > 100$

Solution It can be proved by induction that $\frac{n}{2} > a(n) \le n$.

$\therefore \frac{200}{2} < a(200) \Rightarrow a(200) > 100$ and $a(100) \le 100$.

4. Let $p(n)$ denote the statement that $n^2 + n$ is odd. It is seen that $p(n) \Rightarrow p(n + 1)$, Pn is true for all

(a) $n > 1$ (b) n

(c) $n > 2$ (d) None of these

Solution Since the square of any odd number is always odd and sum of two odd numbers is always even, so for no 'n' this statement is true.

5. If $n \in N$, then $10^n + 3\ (4^{n+2}) + 5$ is divisible by

(a) 7 (b) 5

(c) 9 (d) 17

Solution For n =, $10^2 + 3(4^4) + 5 = 100 + 768 + 5 = 873$, which is divisible by 9.

6. The value of the natural numbers n such that the inequality $2^n > 2n + 1$ is valid, for

(a) $n \ge 3$ (b) $n < 3$

(c) no n (d) any n

Solution Check through options, the condition $2^n > 2n + 1$ is valid for $n \ge 3$.

7. If $p(n) = 2 + 4 + 6 + \ldots + 2n$, $n \in N$, then $p(k) = k\ (k + 1) + 2 \Rightarrow p(k + 1) = (k + 1)\ (k + 2) + 2$ for all K $\in N$. So, we can conclude that $p(n)\ n\ (n + 1) + 2$ for

(a) all $n \in N$ (b) $n > 1$

(c) $n > 2$ (d) nothing can be said

8. $x(x^{n-1} - n\, a^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$ for

(a) $n > 1$ (b) $n > 2$

(c) all $n \in N$ (d) None of these

Solution Check the options. The condition is satisfied for all $n \in N$.

9. For natural number n, $2^n(n - 1)! < n^n$, if

(a) $n < 2$ (b) $n > 2$

(c) $n \ge 2$ (d) for no n

Solution Check the options. The condition is satisfied for $n > 2$.

10. If n is a natural number then $\left(\frac{n+1}{2}\right)^n \ge n!$ is true when

(a) $n > 1$ (b) $n \ge 1$

(c) $n > 2$ (d) $n \ge 2$

Solution Check the options, the condition is true for $n \ge 1$.

Problems for Practice

1. For each natural number n, $3^n > n^3$ for

(a) $n > 2$ (b) $n \ge 3$

(c) $n \ge 4$ (d) $n < 4$

2. For $n \in N$, $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7}{15}n$ is

(a) an integer

(b) a natural number

(c) a positive fraction

(d) None of these

3. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then for $n \in N$, A^n is equal to

(a) $\begin{bmatrix} \cos^n\theta & \sin^n\theta \\ -\sin^n\theta & \cos^n\theta \end{bmatrix}$

(b) $\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

(c) $\begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$

(d) $\begin{bmatrix} n\cos\theta & n\sin\theta \\ -n\sin\theta & n\cos\theta \end{bmatrix}$

4. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots$ to n terms =

(a) $\frac{1}{5n - 1}$ (b) $\frac{1}{3n - 1}$

(c) $\frac{n}{3n + 1}$ (d) $\frac{n}{5n - 1}$

5. $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$ if n is

(a) a positive integer

(b) an even positive integer

(c) an odd positive integer

(d) None of these

6. For every positive integer n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is

(a) an integer

(b) a rational number

(c) an odd integer

(d) a negative real number

7. $10^n + 3\cdot 4^{n+2} + k$ is divisible by 9 for $n \in N$. Then, the least positive integral value of k is

(a) 1 (b) 3

(c) 5 (d) 7

8. The sum of the cubes of three consecutive natural numbers is divisible by

(a) 2 (b) 4

(c) 6 (d) 9

9. For all $n \in N$, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{4}}$

(a) $> \sqrt{n}$ (b) $< \sqrt{n}$

(c) $\leq \sqrt{n}$ (d) $\geq \sqrt{n}$

10. The statement $p(n)$: $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$ is

(a) true for all $n > 1$ (b) not true for any n

(c) true for all $n \in N$ (d) None of these

ANSWERS

1. (c) **2.** (b) **3.** (b) **4.** (c) **5.** (a) **6.** (a) **7.** (c) **8.** (d) **9.** (d) **10.** (c)

32 Mathematical Reasoning

BRIEF REVIEW OF THE CONCEPTS

MATHEMATICAL REASONING

In mathematical language, there are two kinds of reasoning-inductive and deductive. We have already discussed the inductive reasoning in the context of mathematical induction. In this chapter, we shall discuss some fundamentals of deductive reasoning.

Statement

A sentence is called mathematically acceptable statement if it is either true or false but not both.

For example, consider the following sentences:

(i) 8 is less than 6

This sentence is false because 8 is greater than 6. Hence, it is a statement.

(ii) The sun is a star

It is a scientifically established fact that sun is a star and therefore, this sentence is always true. Hence, it is a statement.

(iii) Mathematics is fun

This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others, it may not be. This means that this sentence is not always true. Hence, it is not a statement.

Negation of a statement

The denial of a statement is called negation of the statement. If p is a statement, then the negation of p is also a statement and is denoted by ~p and read and 'not p'

For example, If

p: Diagonals of a rectangle are equal, Then

~p: Diagonals of a rectangle are not equal.

This may also be written as

~p: It if false that diagonals of a rectangle are equal.

We may also write it as

~p: There is at least one rectangle whose diagonals are not equal.

Compound statement

A compound statement is a statement which is made up of two or more statements. Each statement is called a compound statement.

The connecting word 'AND'

We can connect the two statement by the word 'AND'

For example,

P: All rational numbers are real

q: All real numbers are complex

p and q: All rational numbers are real and all real numbers are complex.

Truth Value of p and q

The compound statement with 'and' is true if all its components of statements are true otherwise it is false, i.e, it is false when

(i) p is true and q is false

(ii) p is false and q is true

(iii) p is false and q is false

For example, consider the following compound statements:

(i) p and q: A line is straight and extends indefinitely in both directions.

The components of statements are

p: A line is straight

q: A line extends indefinitely in both directions.

Both these statements are true, therefore, the compound statement is true.

(ii) P and q: 0 is less than every positive integer and every negative integer.

The components of statements are

p: 0 is less than every positive integer.

q: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

(iii) p and q: All living things have two legs and two eyes.

The components of statements are

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statements is false.

CAUTION

A Statement with ‘And’ is not always a compound statement.

For example, consider the statement:

A mixture of alcohol and water can be separated by chemical methods.

This statement cannot be considered as a compound statement with ‘And’. Here, the word ‘And’ refers to two things—alcohol and water.

The Connecting Word ‘OR’

The statement p, q may be connected by the connecting word ‘OR’, i.e., p or q.

For example,

p: Cold drink is available at dinner.

q: Coffee is available at dinner.

p or q: Cold drink or coffee is available at dinner.

Exclusive ‘OR’

In a statement p or q, if exactly one of the two alternatives occurs, the connecting of word ‘OR’ is exclusive.

For example, consider the statement

q or q: An ice cream or pepsi is available with a thali in a restaurant.

This means that a person who does not want ice cream can have a pepsi along with thali or one does not want pepsi can have an ice cream along with thali. A person cannot have both ice cream and pepsi. This is called exclusive ‘OR’.

Inclusive ‘OR’

In a statement p or q, if at least one of the two alternatives occurs, the connecting of word “OR” is inclusive.

For example, consider the statement:

p or q: A student who has taken physics or mathematics can apply for M.Tech programme.

This means that the students who have taken physics or mathematics or both can apply for the M.Tech programme. In this case, we are using inclusive ‘OR’

Truth Value of p or q

When p and q statements both are false, then p or q is also false, otherwise it is true.

Thus, p or q is true when

(i) p is true, q is false.

(ii) p is false, q is true.

(iii) p and q both are true.

Quantifiers

Quantifiers are phrases like, “There exists” and “For all”.

“There Exists”

There exists, is used of at least one.

For example, consider the statement

p: There exists, a quadrilateral whose all sides are equal.

The statements is equivalent to

There is al least one quadrilateral whose all sides are equal.

“For All”

The quantifier “for all” or “for every” can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

For example, consider the statement

p: For every prime number p, $\sqrt{p}$ is an irrational number.

This means that if S denotes the set of all prime numbers, then for all the members p of the set S, $\sqrt{P}$ is an irrational number.

Note: The words ‘And’ and ‘Or’ are called connectives and ‘There exists’ and ‘For all’ are called quantifiers.

Implications

The statements with “If–then”, “only if” and “if and only if” are known as implications.

“If p then q”

The statement “if p then q” says that in the event if p is true then q must be true.

For example, consider the following statement:

If a number is a multiple of 4 then it is a multiple of 2.

Here, p: A number is a multiple of 4.

q: The number is a multiple of 2.

When p is true i.e., a number is a multiple of 4, then q is true i.e., the number is a multiple of 2.

Note: “If p then q” is the same as the following:

1. p implies q: It is denoted by p $\Rightarrow$ q. The symbol $\Rightarrow$ stands for implies.
   This says that a number is a multiple of 4 implies that it is a multiple of 2.
2. p is a sufficient condition for q.
   This says that knowing that a number is a multiple of 4 is sufficient to conclude that it is a multiple of 2.
3. p only if q.
   This says that a number is a multiple of 4 only if it is a multiple of 2.
4. q is a necessary condition for p.
   This says that when a number is a multiple of 4, it is necessary a multiple of 2.
5. ~q implies ~p.
   This says that if a number is not a multiple of 2, then it is not a multiple of 4.

Truth Value of “If p then q”.

Truth value of the statement “if p then q” is false when p is true and q is false otherwise it is true i.e., it is true when

(i) p is true, q is true

(ii) p is false, q is true

(iii) p is false, q is false

Note: There methods are adopted to test the truth value of this statement:

1. Assuming that p is true, prove that q must be true.
2. Assuming that q is false, prove that p must be false.
3. Assuming that p is true and q is false, obtain a contradiction.

Converse Statement

If p and q are two statements, then the converse of the implication "if p then q" is "if q then p".

For example,

If a number x is odd, then x^2 is also odd.

Its converse is : If x^2 is odd then x is also odd.

Contrapositive statement

If p and q are two statements, then the contrapositive of the implication "if p then q" is "if ~q then ~p".

For example,

If a number is divisible by 4, then it is divisible by2.

Its contrapositive statement is

If a number is not divisible by 2, then it is not divisible by4.

"If And Only If" Implication

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called "if and only if" implication and is denoted by $p \Leftrightarrow q$.

For example, consider the following two statements

p: If a rectangle is a square, then all its four sides are equal.

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

Then, the compound statement using "if and only if" is

A rectangle is a square if and only if all its four sides are equal.

Truth Value of "If and Only if"

The statement with "if and only if" is true when

(i) p is true, q is true (ii) p is false, q is false

The statement with "if and only if" is false when

(i) p is true, q is false (ii) p is false, q is true

Problems for Practice

1. Which of the following is not a statement?
   (a) Give me a glass of water.
   (b) Asia is a continent.
   (c) The earth revolves round the sun.
   (d) The number 6 has two prime factors 2, 3.
2. Which of the following is not a statement?
   (a) Every set is a finite set.
   (b) 8 is less then 6.
   (c) Where are you going?
   (d) The sum of interior angles of a triangle is 180 degrees.
3. Which of the following is not a statement?
   (a) Roses are red.
   (b) New Delhi is in India.
   (c) Every square is a rectangle.
   (d) Alas! I have failed.
4. Which of the following is a statement?
   (a) Open the door.
   (b) Do your homework.
   (c) Switch on the fan.
   (d) Two plus two is four.
5. Which of the following is a statement?
   (a) May you live long!
   (b) May God bless you!
   (c) The sun is a star.
   (d) Hurrah! We have won the match.
6. Which of the following is not a statement?
   (a) Please do me a favour.
   (b) 2 is an even integer.
   (c) $2 + 1 = 3$.
   (d) The number 17 is prime.
7. Negation of "Paris is in France and London is in England" is
   (a) Paris is in England and London is in France.
   (b) Paris is not in France or London is not in England.
   (c) Paris is in England or London is in France.
   (d) None of these
8. Negation of "$2 + 3 = 5$ and $8 < 10$" is
   (a) $2 + 3 \neq 5$ and $8 < 10$
   (b) $2 + 3 = 5$ and $8 \nless 10$
   (c) $2 + 3 \neq 5$ or $8 \nless 10$
   (d) None of these
9. Negation of the conditional: "If it rains, I shall go to school" is
   (a) It rains and I shall go to school.
   (b) It rain and I shall not go to school.
   (c) It does not rain and I shall go to school.
   (d) None of these
10. Negation of "Manu is in class X or Anu is in class XII" is
    (a) Manu is not in class X but Anu is in class XII.
    (b) Manu is not is class X but Anu is not in class XII.
    (c) Either Manu is not in class X or Anu is not is class XII.
    (d) None of these
11. Truth value of the statement "if p then q" is false when
    (a) p is true, q is true (b) p is true, q is false
    (c) p is false, q is true (d) p is false, q is false
12. Truth value of the statement "p or q" is false, when
    (a) p is true, q is false (b) p is false, q is true
    (c) p and q both are true (d) p and q both are false
13. Truth value of the statement with "if and only if " is false, when
    (a) p is true, q is true (b) p is false, q is false
    (c) p is true, q is false (d) p is false, q is true

ANSWERS

1. (a) **2.** (c) **3.** (d) **4.** (d) **5.** (c) **6.** (a) **7.** (b) **8.** (c) **9.** (b) **10.** (d)
11. (b) **12.** (d) **13.** (d)

PART C CHEMISTRY

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Basic Concepts of Chemistry

1

BRIEF REVIEW OF THE CONCEPTS

BRIEF REVIEW OF THE CONCEPTS

Measurement is a comparison of the magnitude of the observed property with a **unit** of the property.

A **unit** is a fixed magnitude of the property which is accepted by scientists as a standard. The system of units used these days is called as the **international system of units** or commonly known as SI units. There are seven base units in this system (Table 1.1) and units for other physical quantities are derived from these. (Table 1.2). Table 1.3 describes equivalence between the S.I. units and other commonly used units for some physical quantities.

Table 1.1 Base Physical Quantities and Their SI Units

Quantity (Symbol)	*Name of unit*	*Unit symbol*
Length (l)	meter (or metre)	m
Mass (m)	kilogram	kg
Time (t)	second	s
Electric current (I)	Ampere	A
Temperature (T)	Kelvin	K
Luminous intensity (l_v)	Candela	Cd
Amount of substance (n)	mole	mol

Table 1.2 Derived Physical Quantities and Their SI units

Quantity (symbol)	*Name of unit*	*Derived unit*
Area (A)	square meter	m^2
Volume (V)	cubic meter	m^3
Density (ρ)	kilogram per cubic meter	kg/m^3
Velocity (u)	meter per second	m/s
Force, (F)	Newton (N)	$kg\ m/s^2$
Pressure (p)	Pascal (Pa)	$kg/m\ s^2$
Energy (E)	Joule (J)	$kg\ m^2/s^2$
Quantity of electricity	Coulomb (C)	As
Electromotive force	volt (V)	$kg\ m^2/As^3$

Table 1.3 Equivalence Between Units

Length

1 m = 3.281 ft

1 cm = 0.3937 inch

1 nm = 10 Å = 10^{-9}m

Volume

1L = 1 dm^3 = $10^{-3}m^3$

1 cm^3 = 1 mL

Mass

1 kg = 2.205 lb

1 metric ton = 10^3kg

Energy

1 J = 1 kg m^2/s^2

1 J = 0.239 cal

1 erg = 10^{-7} J

1 cal = 4.184 J

1 eV = 1.6022×10^{-19} J

Force

1 dyne = 1 g cm/s^2

1N = 1 kg m/s^2

Pressure

1 Pa = $1N/m^2$ = $1kg/ms^2$

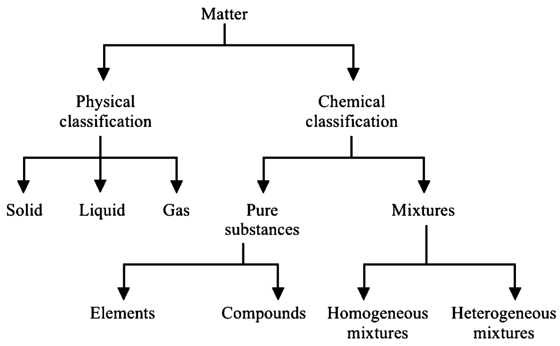
1 atm = 101, 325 Pa or 1.01325×10^5 Pa

1 atm = 760 Torr = 760 mm Hg

1 bar = 1×10^5 Pa

1 bar = 0. 987 atm

Matter is the physical material of the universe that has mass and occupies space. Matter may be classified on the basis of physical and chemical characteristics as shown below.



The law of Conservation of Mass

This law was proposed by Lavoisier in 1744. It states that the total mass of products of a chemical reaction is the same as that of the reactants present before the reaction. An alternate statement of the law is that matter is neither created nor destroyed as a result of any chemical change.

The Law of Constant Composition

The law was proposed by Proust in 1799. It states that a chemical compound has a constant composition and contains same elements combined together in the same proportion by mass.

According to the law, the composition of any compound is the same irrespective of its source. For example, magnesium oxide whether prepared in the laboratory, or procured from the market contains 61% magnesium and 39% oxygen. This law is also known as the *Law of Definite or Constant Proportions*.

The Law of Multiple Proportions

This law was proposed by Dalton in 1805. It states that when two elements A and B combine to form two or more compounds, then different masses of B which combine with a fixed mass of A bear a simple ratio to one another.

The Law of Reciprocal Proportions

This law was proposed by Richter in 1794. It states that if an element A combines separately with other elements B and C, the masses of B and C which combine with a fixed mass of A are the masses of B and C which combine with each other or simple multiples of those masses.

Gay-Lussac's Law of Combining Volumes

This law was proposed by Gay Lussac, a French chemist in 1809. It gives a relation between volumes of gaseous reactants and products measured under identical conditions of temperature and pressure. It states that volumes of gaseous reactants and products bear a simple ratio to one another when measured under identical conditions of temperature and pressure.

Consider the formation of water vapour from hydrogen and oxygen gases. It is found that the ratio of volumes of hydrogen, and oxygen which combine and that of water vapours formed is 2 : 1 : 2. That is, 2 volumes of hydrogen and 1 volume of oxygen combine to form 2 volumes of water vapours, provided the volumes are measured at the same temperature and pressure. This law is nothing but the Law of Definite Proportions stated in terms of volume.

Dalton's Atomic Theory

The laws of chemical combination led Dalton to propose his atomic theory in 1808. The main features of *Dalton's atomic theory* are:

1. Matter is made up of extremely small particles called **atoms.**
2. The atoms can neither be created nor destroyed. They are indivisible particles.
3. The atoms of a particular element are similar in all respects and have the same mass. The atoms of an element remain unchanged in all chemical processes.
4. The atoms of different elements differ from each other and have different masses.
5. In a chemical reaction between two or more elements, the atoms of these elements combine to form *compound atoms* (now called molecules). The atoms combine in a simple but fixed ratio by numbers.

Later developments in the field of chemistry brought about some changes in Dalton's atomic theory, though the basic concept of atom remains unchanged.

Atomic Theory and Laws of Chemical Combination

The reason of success of Dalton's atomic theory and its immediate acceptance was that the laws of chemical combination could be explained by it. *The Law of Conservation of Mass* follows from the postulate that atoms are indestructible. In a chemical reaction, atoms only rearrange themselves to from molecules of the products. Therefore, total mass remains unchanged during a chemical reaction. *The Law of Constant Composition* follows from the fact that atoms combine in a fixed ratio to form a molecule of the compound. Hence, whatever be the source of the compounds, the ratio of atoms of different elements and hence the ratio of their masses is fixed in it. *The Law of Multiple Proportion* also follows from the atomic molecular theory. Suppose two elements A and B combine to form two compounds A_2B and AB. In the case of A_2B, 1 atom of B combines with 2 atoms of A while in AB, 1 atom of B combines with 1 atom of A. The ratio of number of atoms of A that combine with 1 atom of B in these compounds is 2 : 1. Same will be that ratio of masses of A that combine with a fixed mass of B to from these compounds.

Atomic Mass Unit (μ): An atomic mass unit is defined as exactly one-twelfth (1/12) of the mass of ^{12}C atom. The earlier symbol used was amu.

Atomic Mass or the relative atomic mass of an atom is the ratio of its mass to one-twelfth (1/12) of the mass of one ^{12}C atom. Relative atomic masses are pure numbers only since they represent the ratio of masses of atoms

Mole: A mole is the amount of a substance that contains as many entities (atoms, molecules or ions) as there are atoms in exactly 0.012 kg (or 12 g) of ^{12}C isotope. One mole has 6.022×10^{23} entities. This number is also known as Avogadro's number (N_A)

Molar Mass: For an element, the molar mass is the mass in grams numerically equal to its atomic mass (in atomic mass units). For a compound, the molar mass is the sum of the molar masses of all the atoms that have combined in one molecule or one formula unit.

$$\text{Number of moles} = \frac{\text{Mass is grams}}{\text{Molar mass}}$$

or $$\text{Mass in grams} = \text{Number of moles} \times \text{Molar mass}$$

Chemical Equations and Stoichiometry: A chemical equation represents an actual chemical change in terms of the symbols and the formulae of the reactants and products. In a balanced chemical equation, the number of atoms of each kind are same in the reactants and products. The calculation of quantitative relationships in balanced chemical equations is called stoichiometry. Various relationships, viz., mole, mass or volume can be used to make different types of calculations. Most of the stoichiometric problems can be solved using the following four steps.

(i) Write the balanced chemical equation.

(ii) Convert the known quantity (in grams) to number of moles.

(iii) Use a mole ratio from the balanced chemical equation to find the unknown in terms of number of moles.

(Choose that mole ratio which includes both, the substance, the mass of which is given and the substances the mass of which is required)

(iv) Convert from number of moles to the unknown quantity that is required.

Limiting Reagent: Sometimes only a limited amount of one of the reagents needed for a chemical reaction is available or there is an excess of one of the reactants. *When one of two reactants is present in less than the stoichiometric amount required by the other reactant, that reactant is known as the limiting reagent.* In such cases the maximum amount of product that can be formed is determined by the amount of the reactant that is used up first.

Solved Problems

1. The number of moles in 4.68 mg of silicon are

(a) 1.67×10^{-1} mol (b) 1.67×10^{-4} mol

(c) 1.67×10^{2} mol (d) 4.68 mol

Solution (b) No of moles = $\dfrac{\text{Mass in grams}}{\text{Molar mass}}$

4.68 mg of Si = 4.68×10^{-3} g of Si

and Molar mass of Si is 28.08 g mol^{-1}

No of moles of Si = $\dfrac{4.68 \times 10^{-3}}{28.08\,g\,mol^{-1}}$ = 1.67×10^{-4} mol

2. The number of carbon atoms present in 0.35 mol of glucose ($C_6H_{12}O_6$) are

(a) 3.61×10^{24} (b) 3.61×10^{23}

(c) 2.11×10^{23} (d) 1.26×10^{24}

Solution (d) 1 mol of $C_6H_{12}O_6$ contains 6 N_A atoms of carbon so 0.35 mol will contain $6 \times 0.35 \times N_A$ atoms of carbon

$= 6 \times 0.35 \times 6.022 \times 10^{23} = 1.26 \times 10^{24}$ carbon atoms

3. It was found that the black oxide of copper and the red oxide of copper contain 79.9% and 88.8% copper respectively. This data is in accordance to law of

(a) equivalent proportions
(b) constant proportions
(c) multiple proportions
(d) conservation of matter

Solution (c) In black oxide 79.9 g copper is combined with (100 – 79.9) g of oxygen = 20.1 g oxygen.

In the red oxide 88.8 g copper is combined with (100 – 88.8)g of oxygen = 11.2 g oxygen

So in red oxide 79.9 g of copper will combine $\dfrac{11.2 \times 79.9}{88.8}$ g of oxygen = 10.08 g oxygen

Thus the weights of oxygen (20.1 g and 10.08 g) which combine with 79.9 g of copper are in the ratio of 2 : 1, which is a simple whole number ratio. Hence the law of multiple proportion is established.

4. How many moles of H_2O are formed when 3.3 moles of O_2 are brunt in 2 moles of H_2?

(a) 2 (b) 4

(c) 6.6 (d) 3.3

Solution (a) From the mole relationship we know that 1 mole O_2 combines with 2 moles of H_2 to form 2 moles of water. Thus, for complete conversion of O_2 into water the moles of H_2 required are

Moles of H_2 required = (3.3 mol O_2) × $\left(\dfrac{2\text{ mol }H_2}{1\text{ mol }O_2}\right)$

= 6.6 mol H_2

Since the moles of $H_2(g)$ are less than 6.6 moles, required for complete conversion of 3.3 moles O_2 into water, the entire amount of O_2 will not burn. The moles of O_2 those combine with 2 moles of H_2 are

Moles of O_2 burnt = (2 mol H_2) × $\left(\dfrac{1\text{ mol }O_2}{2\text{ mol }H_2}\right)$ = 1 mol O_2

The number of moles of H_2O formed can be calculated as

Moles of H_2O formed = 2 × moles of O_2 burnt
= 2 mol H_2O

5. How much $BaCl_2$ is required to react with 10.00 g $Fe_2(SO_4)_3$? The formula masses are 399.88 for $Fe_2(SO_4)_3$ and 208.24 for $BaCl_2$.

(a) 10.00 g (b) 5.21 g
(c) 31.24 g (d) 15.62 g

Solution (d) The balanced chemical equation for the reaction is

$$\underset{1\text{ mole}}{Fe_2(SO_4)_3} + \underset{3\text{ moles}}{3BaCl_2} \longrightarrow \underset{3\text{ mole}}{3BaSO_4} + \underset{2\text{ moles}}{2FeCl_3}$$

From the mole relationship

Moles of $BaCl_2$ = 3 × moles of $Fe_2(SO_4)_3$

Amount of $BaCl_2$ required = $\dfrac{(10.00\,g\,Fe_2(SO_4)_3)}{(399.88\,g\,Fe_2(SO_4)_3 mol^{-1})}$

× 3 × (208.24 g $BaCl_2$ mol^{-1})

= 15.62 g $BaCl_2$

6. How many molecules of acetylene C_2H_2 would be produced in the reaction of 10.00 g calcium carbide with water?

$$CaC_2 + 2H_2O \longrightarrow C_2H_2 + Ca(OH)_2$$

The molar mass of CaC_2 is 64 g mol^{-1}.

(a) 6.02×10^{24} (b) 6.02×10^{23}

(c) 9.4×10^{22} (d) 9.4×10^{23}

Solution (c) From the mole relationship.

Moles of C_2H_2 produced = moles of CaC_2 consumed

$= \left(\dfrac{10.00\,g\,CaC_2}{64\,g\,CaC_2\,mol^{-1}}\right)$

Number of molecules of C_2H_2 formed

= (moles of C_2H_2) × (6.02×10^{23} molecules mol^{-1})

$= \dfrac{10.0}{64} \times 6.02 \times 10^{23}$ molecules

= 9.4×10^{22} molecules.

Problems for Practice

1. The density of water is 1g cm^{-3}. What will be its values in S.I. Units?

(a) 1 kg m^{-3} (b) 100 kg m^{-3}

(c) 100 kg L^{-1} (d) 1000 kg m^{-3}

2. What is the volume of 5 cubic meters in cubic centimeters?

(a) 5×10^3 (b) 5×10^6

(c) 5×10^9 (d) 5×10^{10}

3. The S.I. units of pressure will be

(a) Atmosphere (b) mm Hg

(c) Kg $m^{-1}s^{-2}$ (d) kg m^2s^{-2}

4. S.I. units of density will be

(a) g cm^{-3} (b) kg cm^{-3}

(c) kg m^{-3} (d) g m^{-3}

5. Correct S.I. units of intensity of light will be

(a) Ampere (b) Coulomb

(c) Candela (d) Hertz

6. Which of the following is not a unit of pressure?

(a) Pascal (b) Newton

(c) Atmosphere (d) mm Hg

7. Which of the following is a heterogeneous mixture?

(a) Wood (b) Brass

(c) Air (d) Sugar dissolved in water

8. Which of the following is not conserved in a chemical reaction?

(a) Number of atoms (b) Total mass

(c) Number of moles (d) Net charge

9. One sample of atmospheric air is found to have 0.03% carbon dioxide and another sample 0.04%. This evidence shows that

(a) the law of constant composition is not always true

(b) the law of multiple proportions is true

(c) air is a compound

(d) air is a mixture

10. Which of the following is the best example of law of conservation of mass?

(a) 12 g of carbon combines with 32 g of oxygen to form 44 g of CO_2.

(b) When 12g of carbon is heated in a vacuum there is no change in mass.

(c) A sample of air increases in volume when heated at constant pressure but its mass remains unaltered.

(d) The weight of a piece of platinum is the same before and after heating in air

11. Which of the following pairs of substances illustrates the law of multiple proportions?

(a) Sodium chloride and sodium bromide

(b) Water and heavy water

(c) Carbon monoxide and carbon dioxide

(d) Magnesium hydroxide and magnesium oxide

12. The law of multiple proportions is illustrated by the two compounds.

(a) Sodium chloride and sodium bromide

(b) Ordinary water and heavy water

(c) Caustic soda and caustic potash

(d) Sulphur dioxide and sulphur trioxide

13. After a chemical reaction, the total mass of reactants and products

(a) is always increased

(b) is always decreased

(c) is not changed

(d) is always less or more

14. Carbon and oxygen combine to form two oxides, carbon monoxide and carbon dioxide in which the ratio of the weights of carbon and oxygen is respectively 12 : 16 and 12 : 32. These figures illustrate the

(a) law of multiple proportions

(b) law of reciprocal proportions

(c) law of conservation of mass

(d) law of constant proportions

15. Zinc sulphate contains 22.65% of zinc and 43.9% of water of crystallization. If the law of constant proportions is true, then the weight of zinc required to produce 20 g of the crystals will be

(a) 45.3 g (b) 4.53 g

(c) 0.453 g (d) 453 g

16. A sample of calcium carbonate ($CaCO_3$) has the following percentage composition:

Ca = 40%; C = 12%; O = 48%

If the law of constant proportions is true, then the weight of calcium in 4 g of a sample of calcium carbonate from another source will be

(a) 0.016 g (b) 0.16 g

(c) 1.6 g (d) 16 g

17. A sample of pure carbon dioxide, irrespective of its source contains 27.27% carbon and 72.73% oxygen. The data supports

(a) law of constant composition

(b) law of conservation of mass

(c) law of reciprocal proportions

(d) law of multiple proportions

18. The law of conservation of mass is valid for all the following except

(a) All chemical reactions

(b) Nuclear reactions

(c) Endothermic reactions

(d) Exothermic reaction

19. Different proportions of oxygen in the various oxides of nitrogen, prove the law of

(a) equivalent proportions

(b) multiple proportions

(c) constant proportions

(d) conservation of matter

20. One part of an element A combines with two parts of another element B. Six parts of the element C combine four parts of the element B. If A and C combine together the ratio of their weights will be governed by

(a) law of definite proportions
(b) law of multiple proportions
(c) law of reciprocal proportions
(d) law of conservation of mass

21. Two samples of lead oxide were separately reduced to metallic lead by heating in a current of hydrogen. The weight of lead from one oxide was half the weight of lead obtained from the other oxide. The data illustrates
(a) law of reciprocal proportions
(b) law of constant proportions
(c) law of multiple proportions
(d) law of equivalent proportions

22. The percentages of copper and oxygen in samples of CuO obtained by different methods were found to be the same. This illustrates the law of
(a) constant proportions
(b) conservation of mass
(c) multiple proportions
(d) reciprocal proportions

23. Chemical equation is balanced according to the law of
(a) multiple proportions
(b) reciprocal proportions
(c) definite proportions
(d) conservation of mass

24. Number of carbon atoms present in 22g CO_2 is
(a) 6.02×10^{23} (b) 3.01×10^{23}
(c) 3.01×10^{-23} (d) 6.02×10^{-23}

25. Volume occupied by 14g of nitrogen at NTP is.
(a) 22.4 L (b) 11.2 L
(c) 44.8 L (d) 14 L

26. Sucrose reacts with oxygen to produce CO_2 and H_2O according to the equation

$$C_{12} H_{22} O_{11} + 12O_2 \longrightarrow 12CO_2 + 11H_2O$$

The amount of CO_2 produced per g of sucrose is
(a) 1.54 g (b) 3.08 g
(c) ≈ 5g (d) ≈ 7.5 g

27. Avogadro's number represents the number of molecules present in:
(a) 1cc of a gas at SPT
(b) 11.2 L of a gas at SPT
(c) 1 L of a gas at STP
(d) 22.4 L of a gas at STP

28. 5.6 L of gas at STP weighs equal to 8 g. The vapour density of the gas is
(a) 32 (b) 16
(c) 64 (d) 8

29. The number of moles of ammonia required to produce 2.5 moles of $[Cu(NH_3)_4]SO_4$ according to the equation

$$CuSO_4 + 4NH_4 \longrightarrow [Cu(NH_3)_4]\ SO_4$$ is

(a) 1 (b) 10
(c) 5 (d) 15

30. The one that contains the same number of atoms as present in 6 g of C is
(a) 24 g of Mg (b) 12 g of Mg
(c) 12 g of Ca (d) 36 g of Ca

31. 1 atom of an element weighs 1.792×10^{-22}g. The atomic weight of elements is
(a) 108 (b) 17.92
(c) 1.192 (d) 64

32. How many grams of H_2SO_4 are present in 0.05 litres of 0.5 M solution?
(a) 4.90 g (b) 9.80 g
(c) 2.45 g (d) 3.98 g

33. Hydrogen combines with oxygen to form H_2O in which 16 g of oxygen combine with 2 g of hydrogen. Hydrogen also combines with carbon to form CH_4 in which 2 g of hydrogen combine with 6 g of carbon. If carbon and oxygen combine together then they will do so in the ratio of
(a) 6 : 16 or 12 : 32 (b) 6 : 18
(c) 1 : 2 (d) 12 : 24

34. In compound A, 1.00 g nitrogen combines with 0.57 g oxygen. In compound B, 2.00 g nitrogen combines with 2.24 g oxygen. In compound C, 3.00 g nitrogen combines with 5.11 g oxygen. These results obey the following law:
(a) Law of constant proportions
(b) Law of multiple proportions
(c) Law of reciprocal proportions
(d) Dalton's law of partial pressure

35. The total number of electrons in 18 g of H_2O are
(a) 18 (b) 10
(c) 6.023×10^{23} (d) 6.023×10^{24}

36. Which of the following has the largest number of atoms?
(a) 0.5g atom of Cu (b) 10^{23} atoms of Cu
(c) 0.623g of Cu (d) 3.175g of Cu

37. The mass of Ca which contains the same number of atoms as are present in 3.2g of S.
(a) 2 g (b) 4 g
(c) 8 g (d) 4.4 g

38. The number of atoms in 52 amu of He are
(a) 13 atoms (b) $13 \times 6.023 \times 10^{23}$ atoms
(c) 52 atoms (d) $52 \times 6.023 \times 10^{23}$ atoms

39. The number of gram atoms in 3.2g Ca (atomic mass 40)
(a) 3.2 (b) 0.08
(c) 0.8 (d) 1.008

40. Calculate the number of gram-atoms in a sample of lead (atomic mass = 207) weighing 100 grams.
(a) 0.0483 (b) 0.24
(c) 4.8 (d) 0.48

41. The volume of a drop of water is 0.05 ml, and the density of water is 1.0g/mL. How many water molecules are there in a drop of water?
(a) 1.7×10^{21} (b) 5.4×10^{23}
(c) 2.2×10^{26} (d) 1.7×10^{20}

42. What is the mass of 0.5 moles of ozone (O_3) molecules ?
(a) 8 g (b) 16 g
(c) 24 g (d) 48 g

43. How many moles of Al_2O_3 will be formed when a mixture of 5.4g of Al and 3.2g of O_2 is heated?
(a) $\frac{1}{10}$ (b) $\frac{1}{15}$

(c) $\frac{1}{5}$ (d) $\frac{1}{20}$

44. Volume of oxygen required at NTP to completely burn 1 kg of coal (100% carbon):

(a) 22400 L (b) 22.4×10^3 L

(c) 1.86×10^3 L (d) 1000 L

45. When 0.1g of hydrogen is burnt in oxygen the number of water molecules produced is equal to

(a) 3.01×10^{23} (b) 6.02×10^{23}

(c) 3.01×10^{22} (d) 6.02×10^{22}

46. Given the following reaction

$Na_2CO_3(aq) + CaBr_2(aq) \longrightarrow CaCO_3(s) + 2NaBr(aq)$. The weight of Na_2CO_3 that combines with 50 g of $CaBr_2$ is

(a) 5.3 g (b) 13.2 g

(c) 26.5 g (d) 53.0 g

47. If 6.02×10^{23} molecules of nitrogen react completely with hydrogen according to the equation

$$N_2(g) + 3H_2(g) \longrightarrow 2NH_3(g)$$

the number of molecules of ammonia formed is

(a) 6.02×10^{23} (b) $3 \times 6.02 \times 10^{23}$

(c) $2 \times 6.02 \times 10^{23}$ (d) $\frac{1}{2} \times 6.02 \times 10^{23}$

48. Calculate the mass in grams of $KClO_3$ necessary to produce 67.2 litres of oxygen at S.T.P. according to the reaction

$$2\ KClO_3(s) \longrightarrow 2\ KCl\ (s) + 3O_2(g)$$

(a) 96 (b) 2

(c) 122.5 (d) 245

49. How many litres of ammonia will be formed when 2 litres of N_2 and 2 litres of H_2 are allowed to react?

(a) 0.665 (b) 1.0

(c) 4.00 (d) 1.33

50. Which of the following is the standard for atomic mass?

(a) ${}_{8}^{16}0$ (b) ${}_{1}^{1}H$

(c) ${}_{6}^{14}C$ (d) ${}^{12}_{6}C$

51. How many grams of oxygen is required for complete reaction with 27 g of aluminium (atomic mass = 27)?

(a) 8 g (b) 10 g

(c) 24 g (d) 48 g

52. In SO_2 and SO_3, the ratio of the masses of oxygen which combine with a fixed mass of sulphur is 2 : 3. This is an example of law of

(a) constant proportions

(b) multiple proportions

(c) reciprocal proportions

(d) Gay-Lussac law

53. Which of the following is the correct relation?

(a) 1 L = 1 m^3 (b) 1 L = 1 dm^3

(c) 10 L = 1 dm^3 (d) 1 L = 10 dm^3

54. The set of numerical coefficients that balances the chemical equation

$$K_2CrO_4 + HCl \longrightarrow K_2Cr_2O_7 + KCl + H_2O$$

(a) 2, 2, 2, 1, 1 (b) 1, 1, 2, 1, 2

(c) 2, 2, 1, 2, 1 (d) 2, 1, 2, 1, 2

55. 1.5 mol of O_2 combine with Mg to form oxide MgO. The mass of Mg used up is

(a) 24 g (b) 36 g

(c) 48 g (d) 72 g

56. Assuming full decomposition, the volume of CO_2 released at STP on heating 9.85 g of $BaCO_3$ (Atomic mass of Ba = 137) will be

(a) 0.84 L (b) 0.24 L

(c) 4.06 L (d) 1.12 L

57. The number of molecules in 16g of methane is

(a) 3.0×10^{23} (b) 6.02×10^{23}

(c) $\frac{16}{602} \times 10^{23}$ (d) $16 \times 6.02 \times 10^{23}$

58. The largest number of molecules is in

(a) 34 g water (b) 28 g CO_2

(c) 48 g (d) 54 g N_2O_5

59. The number of oxygen atoms in 4.4g of CO_2 is approximately

(a) 6×10^{22} (b) 1.2×10^{23}

(c) 6×10^{23} (d) 12×10^{23}

60. 2.76 g of silver carbonate (atomic mass of Ag = 108) on being heated strongly yields a residue weighing

(a) 2.16 g (b) 2.48 g

(c) 2.32 g (d) 2.64 g

ANSWERS

1. (d)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (a)
11. (c)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (c)
21. (c)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (b)
31. (a)	**32.** (c)	**33.** (a)	**34.** (b)	**35.** (d)	**36.** (a)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (d)
41. (a)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (c)	**46.** (c)	**47.** (c)	**48.** (d)	**49.** (d)	**50.** (d)
51. (c)	**52.** (b)	**53.** (b)	**54.** (c)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (a)	**59.** (b)	**60.** (c)

States of Matter–I (Gaseous and Liquid State)

2

BRIEF REVIEW OF THE CONCEPTS

Matter exists in three different states viz., solid, liquid and gas.

Solids have a definite shape and volume.

Liquids have a definite volume, but no definite shape. They acquire the shape of the container.

Gases have neither definite shape nor volume. They completely fill up container of any size and shape.

Temperature measures the degree of hotness of a body. The SI unit of temperature is kelvin (K).

Volume is expressed in m^3 or cm^3 or litres

$$1 \text{ Litre } (1 \text{ L}) = 1 \text{ dm}^3 = 10^3 \text{ cm}^3 = 10^{-3} \text{ m}^3$$

Pressure of a gas is measured with the help of an instrument called *manometer*. It in defined as force per unit area. SI unit of pressure is Pascal.

$$1 \text{ Pa} = 1 \text{ Nm}^{-2} = 1 \text{ Kg m}^{-1}\text{s}^{-2}$$

Usually, the pressure is expressed in terms of *bar* or *atm*. These are related as

$$1 \text{ bar} = 10^5 \text{ Pa} = 0.9862 \text{ atm.}$$

$$1 \text{ atm} = 1.01325 \text{ bar}$$

$$= 760 \text{ mm Hg} = 760 \text{ torr}$$

Boyle's Law is stated as follows:

At constant temperature, the volume of a fixed mass of gas is inversely proportional to the pressure. i.e., as the pressure (p) increases the volume (V) decreases.

Mathematically, $p \propto \frac{1}{V}$ or p= constant × $\frac{1}{V}$

or

$$pV = \text{constant}$$

A plot between pressure and volume at constant temperature is called *isotherm*.

Charles's Law At constant pressure the volume of a fixed mass of gas is directly proportional to its temperature (Kelvin scale)

That is as the temperature (T) increases, the volume of the gas increases.

Mathematically, $V \propto T$ or $V = \text{constant} \times T$

$$\text{or } \frac{V}{T} = \text{constant}$$

Charles' Law can also be stated as the volume of a fixed mass of a gas at constant pressure increases or decreases by 1/273 of its volume at 0°C for 1°C rise or fall in temperature.

$$V_t = V_0 + \frac{V_0}{273} \times t \quad \text{(at constant pressure)}$$

where V_t and V_0 are the volumes at temperature t°C and 0°C, respectively

A plot of V versus t(°C) at constant pressure for a fixed mass of a gas is straight line and is called an **isobar.**

Variation of Pressure with Temperature For a given mass of a gas, at constant volume the pressure of the gas is directly proportional to the temperature (Kelvin scale) of the gas.

Mathematically, $p \propto T$

For a fixed mass of gas at constant volume. It can also be written as

$$\frac{p_1}{T_1} = \frac{p_2}{T_2} \quad \text{(at constant volume).}$$

This generalization is also known as **Amonton's Law.**

Ideal Gas Equation Combining Boyle's law and Charles's law gives: For a given mass of a gas

$$\frac{pV}{T} = \text{constant}$$

$$\text{or} \quad \frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

For n moles of a gas, ideal gas equation is

$$pV = nRT$$

Nature of Gas Constant *R* Ideal gas equation is

$$pV = nRT$$

$$\text{or} \quad R = \frac{pV}{nT} = \frac{(\text{Force/Area}) \times \text{volume}}{\text{moles} \times \text{Degrees(K)}}$$

$$= \frac{\text{Force} \times \text{Length}}{\text{moles} \times \text{Degrees(K)}} = \frac{\text{work}}{\text{moles} \times \text{Degrees (K)}}$$

So R is identified as work per degree for mole

Value of R in different units can be calculated by substituting the values of p, V, n and T

(i) **Value of R** (in joules)

Let n be 1 mol and T be 273.15 K

$p = 10^5$ Pa, $V = 22.7 \times 10^{-3}\text{m}^3$

Substituting the values in equation, gives

$$R = \frac{10^5\text{Pa} \times 22.7 \times 10^{-3}\ \text{m}^3}{1\ \text{mol} \times 273.15\text{K}}$$

$$= \frac{10^5\text{Nm}^{-2} \times 22.7 \times 10^{-3}\text{m}^3}{1\ \text{mol} \times 273.15\text{K}}$$

$$= 8.314\ \text{J K}^{-1}\ \text{mol}^{-1}$$

(ii) Value of R in dm³, bar and Kelvin

$$R = \frac{1\ \text{bar} \times 22.7\text{dm}^3}{1\ \text{mol} \times 273.15\text{K}}$$

$$= 0.083\ \text{bar dm}^3\text{K}^{-1}\text{mol}^{-1} = 0.083\ \text{bar LK}^{-1}\text{mol}^{-1}$$

Note: The value of $R = 0.0821\ \text{L atm K}^{-1}\ \text{mol}^{-1}$ when pressure is taken in atmosphere units.

Avogadro Law Equal volumes of all gases, under similar conditions of temperature and pressure, contain equal number of molecules.

Mathematically, $V \propto n$

Partial Pressure of a gas in a mixture of gases is the pressure exerted by the individual gas if it alone occupies the whole space.

Dalton's Law of Partial Pressure The law states, "The total pressure exerted by a mixture of nonreacting gases is equal to the sum of partial pressure of each gas present in the mixture".

Mathematically,

$$P_{\text{Total}} = p_1 + p_2 + p_3 + \ldots$$

(p_1, p_2 … p_3 are the partial pressures of gas 1, 2 and 3 respectively).

Diffusion is the process of inter-mixing of gases irrespective of force of gravity and without the help of external agency. Due to this process, scent smell can be detected at a distance when a bottle of scent is opened.

Effusion is the process of passage of gases through a small aperture under pressure.

Graham's Law of Diffusion It states that the rate of diffusion or effusion of a gas is inversely proportional to the square root of its density or molar mass provided temperature and pressure remain constant.

$$\text{Rate of diffusion/effusion} \propto \sqrt{\frac{1}{d}} \quad \text{or Rate} \propto \sqrt{\frac{1}{M}}$$

If two gases having their densities d_1 and d_2 (or molar masses M_1 and M_2) and rates of diffusion r_1 and r_2, respectively be compared. Then Graham's Law can be put as

$$\frac{r_1}{r_2} = \sqrt{\frac{d_2}{d_1}} = \sqrt{\frac{M_2}{M_1}}$$

This law may be modified for simplicity as shown below

(i) **Comparison of time taken for the diffusion of the same volume of two gases.**

$$\frac{r_1}{r_2} = \frac{v_1/t_1}{v_2/t_2} = \sqrt{\frac{d_2}{d_1}} = \sqrt{\frac{M_2}{M_1}}$$

when we compare for same volume i.e. $v_1 = v_2 = v$

$$\frac{r_1}{r_2} = \frac{v}{t_1} \times \frac{t_2}{v} = \frac{t_2}{t_1} = \sqrt{\frac{d_2}{d_1}} = \sqrt{\frac{M_2}{M_1}}$$

Thus, *the time taken for the diffusion of the same volume of two gases under similar conditions of temperature and pressure, are directly proportional to the square roots of their densities or molar masses.*

(ii) **Comparison of the volumes of two gases that diffuse in the same time.**

In this case $t_1 = t_2 = t$

so $\frac{r_1}{r_2} = \frac{v_1}{t} \times \frac{t}{v_2} = \frac{v_1}{v_2} = \sqrt{\frac{d_2}{d_1}} = \sqrt{\frac{M_2}{M_1}}$

Thus, *the volume of two gases that diffuse in the same time are inversely proportional to the square root of their densities.*

Most Probable Velocity (u_{mp}) of a gas is the velocity possessed by maximum number of molecules of the gas at a given temperature.

$$u_{\text{mp}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2PV}{M}}$$

Root Mean square velocity (u_{rms} or $\sqrt{\overline{u^2}}$) is defined as the square root of the mean of the squares of the velocities of different molecules of the gas at particular conditions.

$$u_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3PV}{M}}$$

Average Velocity (u_{av}) is the arithmetic mean of the velocities of different molecules of the gas at a particular temperature.

$$u_{\text{av}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8PV}{\pi M}}$$

The three types of velocities are related as

$u_{\text{mp}} : u_{\text{av}} : u_{\text{rms}} : : 1 : 1.128 : 1.224$

Average Kinetic Energy, E_k. The average kinetic energy of a gas is proportional to its temperature (Kelvin).

Kinetic energy of one mole of a gas at temperature, T is given as $E_k = 3/2\ RT$, where R is gas constant.

Kinetic gas equation is

$$pV = \frac{1}{3} mn\ \overline{u^2} = RT.$$

Ideal Gas A gas which obeys gas, laws, for example, Boyle's Law, Charle's Law, or the general gas equation $PV = nRT$ for all values of temperature and pressure is known as an ideal gas.

Real gases actually deviate from ideal behavior because of the following two wrong assumptions of kinetic theory.

(i) The actual volume of the gas molecules is negligible as compared to the total volume of the gas.

(ii) The gas molecules exert no force of attraction or repulsion on one another.

van der Waals' equation for real gases

For 1 mol of gas

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

and for n moles of gas

$$\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

Here, a and b are van der Waals' constant.

Critical Temperature (T_C) A gas must be cooled to a certain minimum temperature so that it can be liquefied. This temperature is called critical temperature, T_C. No gas can be liquefied above its critical temperature.

Critical Pressure (p_c) is the minimum pressure that must be applied to liquefy a gas at its critical temperature, T_c.

Critical Volume, (V_c) is the volume of 1 mol of a gas at its critical temperature, T_c and critical pressure, p_c.

Boyle's Temperature. The temperature at which a real gas shows ideal behavior for a considerable range of pressure.

$$\text{Compressibility Factor } Z = \frac{PV}{RT}.$$

$$Z = 1 \text{ for ideal gas.}$$

When value of Z is more than one, it indicates that gas is less compressible and when value of Z is less than one, it indicates that the gas is more compressible.

Vapour Pressure of a liquid is the pressure exerted by its vapours in equilibrium with the liquid at a particular temperature. Vapour pressure of a liquid increases with increase in temperature.

Boiling Point of a liquid is the temperature at which its vapour pressure becomes equal to the external pressure.

Surface Tension (γ): It is defined as the force acting per unit length perpendicular to the line drawn on the surface of the liquid. It can also be defined as the energy required to increases the surface area of a liquid by one unit. It is expressed in SI units as Nm^{-1} or as Jm^{-2}.

Viscosity The resistance of a liquid to flow is viscosity—the opposite of fluidity.

Coefficient of Viscosity, (η) may be defined as the force per unit area required to maintain unit difference of velocity between two layers in the liquid which are unit distance apart. S.I. unit is kg $m^{-1}s^{-1}$. C.G.S. unit, **poise,** is also in common use.

Solved Problems

1. A weather balloon has a volume of 160 L when filled with hydrogen at a pressure of 1.0 atm. What will be its volume when it rises to a height where atmospheric pressure is 0.80 atm? Assuming that the temperature remains constant.

(a) 160 L (b) 200 L
(c) 152 L (d) 168 L

Solution (b) This is solved by using Boyle's law.

$$P_1V_1 = P_2V_2$$

where P_1 = 1 atm

P_2 = 0.80 atm

V_1 = 160 L

V_2 = ?

Substituting the values

$$V_2 = \frac{P_1V_1}{P_2} = \frac{1\times160}{0.80} = 200 \text{ L}$$

2. What will be the volume of 600 cm^3 of a gas when it is cooled from 27°C to 7°C?

(a) 560 cm^3 (b) 155.6 cm^3
(c) 400 cm^3 (d) 643 cm^3

Solution (a) Applying Charle's Law

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

where V_1 = 600 cm^3

T_1 = 273 + 27 = 300 K

T_2 = 273 + 7 = 280 K

Substituting the values

$$V_2 = \frac{V_1T_2}{T_1} = \frac{600\times280}{300} = 560 \text{ cm}^3$$

3. The molecular mass of oxygen and sulphur dioxide are 32 and 64 respectively. If 1 L of oxygen at 15°C and 740 mm pressure contains N molecules, the number of molecules in 2 L of SO_2 under the same conditions of temperature and pressure will be

(a) N/2 (b) N
(c) 2N (d) 4N

Solution (c) This is the simple application of Gaylussac's Law that "under identical conditions of temperature and pressure, the same volume of all gases contain same number of molecules". Since in this case the conditions are identical so the no. of molecules of SO_2 will be double i.e., 2N to the no of molecule of O_2 (given N) because the volume of SO_2 is double. Molecular mass has no role in this case.

4. 250 cm^3 of oxygen at 680 mm pressure and 500 cm^3 of nitrogen at 720 mm pressure were put together in a one litre flask at the same temperature. What will be the final pressure of the mixture?

(a) 700 mm (b) 350 mm
(c) 360 mm (d) 530 mm

Solution (d) First the partial pressure due to two gases in flask is to be calculated.

For oxygen $P_1V_1 = P_2V_2$

where P_1 = 680 mm

P_2 = ?

V_1 = 250 cm^3

V_2 = 1000 cm^3

$$P_2 = \frac{P_1V_1}{V_2} = \frac{680\times250}{1000} = 170 \text{ mm}$$

For nitrogen $P_1'V_1' = P_2'V_2'$

P_1' = 720 mm

P_2' = ?

V_1' = 500 cm^3

V_2' = 1000 cm^3

$$P_2' = \frac{720\times500}{1000} = 360 \text{ mm}$$

So total pressure of the mixture = 170 + 360 = 530 mm

5. The root-mean square velocity of molecules of a given gas at 27°C and 1 atm pressure is 450 ms^{-1}. If the temperature is raised to 327°C and pressure reduced to half atmosphere, the root-mean square velocity of the molecule is

(a) 630 ms^{-1} (b) 1566 ms^{-1}

(c) 37 ms^{-1} (d) 74 ms^{-1}

Solution (a) For a gas $u_{rms} = \sqrt{\frac{3RT}{M}}$

Ratio of root mean square velocities at two different temperatures will be

$$\frac{u_{rms1}}{u_{rms2}} = \frac{\sqrt{\frac{3RT_1}{M}}}{\sqrt{\frac{3RT_2}{M}}} = \sqrt{\frac{T_1}{T_2}}$$

Substituting the values of

$$T_1 = 27 + 273 = 300K$$

$$T_2 = 327 + 273 = 600K$$

and $u_{rms1} = 450\ ms^{-1}$

we obtain

$$u_{rms2} = \sqrt{\frac{600}{300}} \times 450$$

$$= 630\ ms^{-1}$$

Problems for Practice

1. Which of the following statements about gases is not true?

(a) Diffuse rapidly (b) Do not mix well

(c) Posses mass (d) Highly compressible

2. The density of neon will be highest at

(a) STP (b) 0°C and 2 atm

(c) 273°C and 1 atm (d) 273°C and 2 atm

3. If 10g of a gas at atmospheric pressure is cooled from 273°C to 0°C keeping the volume constant, its pressure would become

(a) 1/2 atm (b) 1/273 atm

(c) 2 atm (d) 273 atm

4. "One gram mole of a gas at N.T.P. occupies 22.4 litres." This fact was derived from

(a) Dalton's theory

(b) Avogadro's hypothesis

(c) Berzelius hypothesis

(d) Law of gaseous volume

5. A gas with a volume of 20 cm^3 at 1 atmosphere expands to 50 cm^3 at constant temperature. The final pressure of the gas will be

(a) $\frac{50 \times 1}{20}$ atmosphere

(b) $\frac{20 \times 1}{50}$ atmosphere

(c) $\frac{50 \times 20}{1}$ atmosphere

(d) $\frac{1}{50 \times 20}$ atmosphere

6. At a certain temperature the volume of gas sample is 120 mL at one atmosphere pressure. Keeping the temperature same the pressure is raised to five atmospheres. The volume of the gas would change to

(a) 24 mL (b) 115 mL

(c) 125 mL (d) 600 mL

7. Avogadro's number, N_A means

(a) number of protons in nucleus of an atom

(b) number of atoms in one gram atom of an element

(c) sum of the number of protons and the neutrons in the nucleus of an atom

(d) number of protons or electrons in one gram atom of sodium

8. 40 mL of a gas is collected at 25°C. If the temperature is raised to 50°C and the pressure remains constant, the new volume of the gas expressed in millilitres is equal to

(a) 40 × 323/298 (b) 40 × 298/323

(c) 40 × 25/50 (d) 40 × 50/25

9. He and Ar are monoatomic gases and their atomic masses are 4 and 40 respectively. Under similar conditions He will diffuse through semi-permeable membrane

(a) 3.16 times as fast as Ar

(b) 7.32 times as fast as Ar

(c) 1.58 times as fast as Ar

(d) 10 times as fast as Ar

10. The ratio of RMS velocity to average velocity of a gas molecule is

(a) 1.086 : 1 (b) 1 : 1.86

(c) 2 : 1.086 (d) 1.086 : 2

11. At 27°C a gas was compressed to half of its volume. To what temperature must it be now heated so that it occupies just its original volume. Pressure remains constant?

(a) 54°C (b) 327°C

(c) 600°C (d) 327 K

12. If 4 g of oxygen diffuses through a very narrow hole, how much hydrogen would have diffused under identical conditions

(a) 16 g (b) 1 g

(c) 1/4 g (d) 64 g

13. Since the atomic masses of carbon, nitrogen and oxygen are 12, 14 and 16 respectively, among the following pairs of gases, the pair that will diffuse at the same rate is

(a) carbon dioxide and nitrous oxide
(b) carbon dioxide and nitrogen dioxide
(c) carbon dioxide and carbon monoxide
(d) nitrous oxide and nitrogen dioxide

14. 273 mL of a gas at STP was taken to 27°C and 600 mm pressure. The final volume of the gas would be
(a) 273 mL (b) 300 mL
(c) 380 mL (d) 546 mL

15. 10.0g of oxygen, assumed ideal, is introduced into an evacuated vessel of 5 litre capacity at 47°C. What is the pressure of the gas in the container?
($R = 0.0821$ litre atm K^{-1} mol^{-1})
(a) 5.25 atm (b) 3.28 atm
(c) 1.64 atm (d) 0.24 atm

16. The rate of diffusion of hydrogen is about
(a) ½ that of He
(b) 1.4 times that of He
(c) Twice that of He
(d) Four times that of He

17. For a gas deviation from ideal gas behaviour is maximum at
(a) 0ºC and atmospheric pressure
(b) 100°C and 2 atm
(c) –100°C and 5 atm
(d) 500ºC and 1 atm

18. The root mean square velocity of hydrogen molecules at any temperature is
(a) 8 times that of oxygen molecules
(b) 4 times that of oxygen molecules
(c) 16 times that of oxygen molecules
(d) same as that of oxygen molecules

19. The relative rate of diffusion of a gas (molecular weight = 128) as compared to oxygen is
(a) 2 times (b) 1/4
(c) 1/8 (d) 1/2

20. Weight of 112 mL of oxygen at NTP on liquefication, would be
(a) 0.32 g (b) 0.64 g
(c) 0.16 g (d) 0.96 g

21. A weather balloon filled with hydrogen at 1 atm and 27°C has volume equal to 12000 litres. On ascending it reaches a place where temperature is –23°C and pressure is 0.5 atm. The volume of the balloon is
(a) 24000 litres (b) 20000 litres
(c) 10000 litres (d) 12000 litres

22. In a closed flask of 5 litre, 1.0 gram H_2 is heated from 300K to 600K. Which statement is false?
(a) pressure of the gas increases
(b) the energy of molecules of the gas increases
(c) the rate of collision increases
(d) the number of moles of the gas increases.

23. The van der Waal's equation explains the behaviour of
(a) Ideal gases (b) Real gases
(c) Vapours (d) Non-Real gases

24. Rate of diffusion of a gas is
(a) Directly proportional to its density
(b) Directly proportional to its molecular mass
(c) Directly proportional to the square of its molecular mass
(d) Inversely proportional to the square root of its molecular mass

25. If two moles of an ideal gas at 546 K occupies a volume of 44.8 litres, the pressure must be
(a) 2 atm (b) 3 atm
(c) 4 atm (d) 1 atm

26. Some of the properties of system remain unchanged, i.e., conserved even when the system undergoes a change. In a chemical change
(a) molecules are conserved
(b) atoms are conserved
(c) moles are conserved
(d) none of these is conserved

27. Four litres of CO_2 is kept at 27°C. What will be the volume if the temperature is lowered to 150 K at the same pressure?
(a) 0.5 litre (b) 1 litre
(c) 2 litres (d) 4 litres

28. A real gas approaches the behaviour of an ideal gas at
(a) 15 atm and 200 K
(b) 1 atm and 273 K
(c) 0.5 atm and 500 K
(d) 15 atm and 500 K

29. Vander waal's constant 'a' has the dimensions of
(a) Mol L^{-1} (b) Atm L^2 mol^{-2}
(c) Litre mol^{-1} (d) Atm L mol^{-2}

30. In the Haber process, gaseous nitrogen and hydrogen react to yield ammonia whose volume (STP) relative to the total volume of reactants (STP) would be
(a) one-fourth (b) one-half
(c) the same (d) three-fourth

31. The ratio amongst most probable velocity : mean velocity : root mean square velocity is given by
(a) 1 : 2 : 3 (b) $1 : \sqrt{2} : \sqrt{3}$
(c) $\sqrt{2} : \sqrt{3} : \sqrt{8/\pi}$ (d) $\sqrt{2} : \sqrt{8/\pi} : \sqrt{3}$

32. A bottle of dry ammonia and a bottle of dry hydrogen chloride connected through a long tube are opened simultaneously at both ends. The white ammonium chloride ring first formed will be
(a) At the centre of the tube
(b) Near the hydrogen chloride bottle
(c) Near the ammonia bottle
(d) Throughout the length of the tube

33. The rate of diffusion of methane at a given temperature is twice that of a gas *X*. The molecular mass of *X* is
(a) 64.0 (b) 32.0
(c) 4.0 (d) 8.0

34. The rates of diffusion of two gases *X* and *Y* are in the ratios 1 : 5 and those of *Y* and *Z* in the ratio of 1 : 6. The ratio of rates of diffusion of *Z* w.r.t. *X* is
(a) 5/6 (b) 1/30
(c) 6/5 (d) 30

35. The rates of diffusion of gases *A* and *B* of molecular masses 100 and 81 respectively are in the ratio of
(a) 9 : 10 (b) 10 : 9
(c) 100 : 18 (d) 81 : 100

36. At what temperature the RMS velocity of oxygen will be same as that of methane at 27°C?

(a) 54°C (b) 327 K
(c) 600 K (d) 573 K

37. A flask of methane (CH_4) was weighed. Methane was then pushed out and the flask again weighed when filled with oxygen at the same temperature and pressure. The mass of oxygen would be

(a) the same as the methane
(b) half of the methane
(c) double to that of methane
(d) negligible in comparison to that of methane

38. Non-reacting gases have a tendency to mix with each other. This property is known as

(a) diffusion (b) fusion
(c) mixing (d) none of these

39. Two litres of a gas are maintained at 25°C and two atmospheric pressure. If the pressure is doubled and absolute temperature is halved, the gas will now occupy

(a) 2 litres (b) 4 litres
(c) 0.5 litre (d) 1 litre

40. A closed vessel contains equal number of oxygen and hydrogen molecules at a total pressure of 740 mm. If oxygen is removed from the system, the pressure will

(a) become 1/9th of 740 mm
(b) become half of 740 mm
(c) remain unchanged
(d) become double of 740 mm

41. A sample of gas at 1.2 atm and 27°C is heated at constant pressure to 57°C. Its final volume is found to be 4.75 litres. What was its original volume?

(a) 4.32 litres (b) 5.02 litres
(c) 4.22 litres (d) None of these

42. If the density of a certain gas at 30°C and 768 torr is 1.35 kg/m^3 its density at STP would be

(a) 1.48 kg/m^3 (b) 1.58 kg/m^3
(c) 1.25 kg/m^3 (d) 1.4 kg/m^3

43. A sample of O_2 gas is collected over water at 23°C at a barometric pressure of 751 mm Hg (vapour pressure of water at 23°C is 21 mm Hg). The partial pressure of O_2 gas in the sample collected is

(a) 21 mm Hg (b) 751 mm Hg
(c) 0.96 atm (d) 1.02 atm

44. Equal weights of two gases of molecular weight 4 and 40 are mixed. The pressure of the mixture is 1.1 atm. The partial pressure of lighter gas in this mixture is

(a) 0.55 atm (b) 0.11 atm
(c) 1 atm (d) 0.1 atm

45. Vander Waal found what two assumptions made by kinetic theory of gases were wrong, one of them is that gas molecules are

(a) Very large
(b) Compressible
(c) Point particles without significant volume
(d) Spherical

46. The ratio of rates of diffusion of SO_2, O_2 and CH_4 is

(a) $1 : \sqrt{2} : 2$ (b) $1 : 2 : 4$
(c) $2 : \sqrt{2} : 2$ (d) $1 : 2 : 2\sqrt{2}$

47. Equal masses of methane and oxygen are mixed in an empty container at 25°C. The fraction of the total pressure exerted by oxygen is :

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{2}{3}$ (d) $\frac{1}{3}\times\frac{273}{298}$

48. At constant volume, for a fixed number of moles of a gas the pressure of the gas increases with rise of temperature due to

(a) increase in average molecular speed
(b) increased rate of collisions amongst molecules
(c) increase in molecular attraction
(d) decreases in mean free path

49. According to kinetic theory of gases there are

(a) Intermolecular attractions
(b) Molecules have considerable volume
(c) No intermolecular attraction
(d) The velocity of molecules decreases after each collision

50. If the pressure and absolute temperature of 2 litres of CO_2 are doubled, the new volume of carbon dioxide will be

(a) 2 litres (b) 4 litres
(c) 5 litres (d) 1 litre

51. Pressure remaining the same the volume of a given mass of an ideal gas increases for every degree rise in temperature by a definite fraction of its volume at

(a) zero degree celcius
(b) its critical temperature
(c) absolute zero
(d) its Boyle temperature

52. Gases often deviate from the ideal gas behaviour because their molecules

(a) possess negligible volume
(b) are polyatomic
(c) have forces of attraction between them
(d) are not attracted to one another

53. The critical temperature of a substance is defined as

(a) the temperature above which the substance decomposes
(b) the temperature above which a substance can exist only as a gas
(c) melting point of the substance
(d) boiling point of the substance

54. The vapour density of a gas is 11.2, the volume occupied by 11.2 g of this gas at NTP is

(a) 1 L (b) 11.2 L
(c) 22.4 L (d) 2 L

55. If a vessel containing hydrogen chloride at a pressure *P* is connected with another vessel of the same volume containing ammonia at a pressure *P* and the connecting tube opened so that they can mix. A white solid is formed and the gas pressure

(a) is equal to the pressure *P*
(b) will be $P/P = 1$
(c) will be doubled i.e., $2P$
(d) drops to zero

56. At what temperature will the volume of a gas at 0°C double itself, pressure remaining constant?

(a) 373°C (b) 273K

(c) 546°C (d) 546K

57. The stopcock connecting two bulbs of volume 5 litres and 10 litres containing an ideal gas at 9 atm and 6 atm respectively is opened. What is the final pressure in the two bulbs if the temperature remains constant?

(a) 15 atm (b) 7 atm

(c) 12 atm (d) 21 atm

58. In vander Waals equation of state for a non-ideal gas, the term that accounts for intermolecular forces is

(a) $V - 1$ (b) RT

(c) $\left[P+\frac{a}{V^2}\right]$ (d) $(RT)^{-1}$

59. Two sealed containers of the same capacity and at the same temperature are filled with 44 g of hydrogen gas in one and 44 g of carbon dioxide in the other. If the pressure of carbon dioxide in the second container is 1 atm then that of hydrogen in the first container would be

(a) 1 atm (b) 10 atm

(c) 22 atm (d) 44 atm

60. Real gases will approach the behaviour of ideal gas at

(a) low temperature and low pressure

(b) low temperature and high pressure

(c) high temperature and high pressure

(d) high temperature and low pressure

61. A pressure cooker reduces cooking time because

(a) the heat is more evenly distributed

(b) the higher pressure tenderizes the food

(c) the boiling point of the water inside is elevated

(d) a large flame is used

62. The kinetic theory of gases predicts that total kinetic energy of gaseous assembly depends on

(a) pressure of the gas

(b) temperature of the gas

(c) volume of the gas

(d) pressure, temperature and volume of the gas

63. Two gases, oxygen and nitrogen, having pressures P_1 and P_2 kept in two vessels of equal volume at constant temperature T are pumped into a third vessel of the same volume at the same temperature T. The observed pressure P of the gas mixture is $P = P_1 + P_2$. This relationships is known as

(a) Boyle's law (b) Charle's law

(c) Avogadro's law (d) Dalton's law

64. The partial pressure of hydrogen in a flask containing 2g H_2 and 32 g SO_2 is

(a) 1/16th of total pressure

(b) 1/9th of total pressure

(c) 2/3rd of total pressure

(d) 1/8th of total pressure

65. The total kinetic energy in joules of the molecules in 8g of methane at 27°C is

(a) 3741.30 J (b) 935.3 J

(c) 1870.65 J (d) 700 J

66. If only pressure of a fixed quantity of a gas is increased 4 times, the r.m.s. velocity will be

(a) 4 times (b) 2 times

(c) Same (d) $\frac{1}{2}$ time

67. The compressibility factor 'Z' for the gas is given by

(a) $Z = PV_{obs}$ (b) $Z = \frac{PV_{obs}}{nRT}$

(c) $Z = nRT$ (d) $z = PV.nRT$

68. The critical temperature of a gas is related to van der Waal's constants as

(a) $Tc = 3b$ (b) $Tc = \frac{a}{27b^2}$

(c) $Tc = \frac{8a}{27bR}$ (d) $Tc = \frac{27bR}{8a}$

69. Boyle's temperature T_b is equal to

(a) $\frac{a}{b}$ (b) $\frac{a}{bR}$

(c) $\frac{2a}{bR}$ (d) $\frac{a}{2bR}$

70. Boyle's temperature and inversion temperature are related as

(a) $T_i = T_b$ (b) $2T = Tb$

(c) $T_i = 2T_b$ (d) $T_i = \sqrt{T_b}$

ANSWERS

1. (b)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (a)	**10.** (a)
11. (b)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (d)	**20.** (c)
21. (b)	**22.** (d)	**23.** (b)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (b)
41. (a)	**42.** (a)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (a)	**47.** (b)	**48.** (a)	**49.** (c)	**50.** (a)
51. (a)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (c)	**59.** (c)	**60.** (d)
61. (c)	**62.** (b)	**63.** (d)	**64.** (c)	**65.** (c)	**66.** (c)	**67.** (b)	**68.** (c)	**69.** (b)	**70.** (c)

3

States of Matter–II (Solid State)

BRIEF REVIEW OF THE CONCEPTS

Solids Macroscopically, solids are substances which have a definite shape and volume. They are characterised by rigidity, incompressibility, slow diffusion and mechanical strength.

In the solid state of matter, the inter particle forces of attractions are very strong. The particles are very closely placed and occupy more or less fixed positions relative to one another. Hence, they possess very low kinetic energy.

An amorphous solid is a substance in which atoms, molecules, or ions assume a random and non-repetitive three-dimensional arrangement.

Crystalline solid is a substance in which the atoms, molecules or ions assume a characteristic, regular, and repetitive three-dimensional arrangement.

Crystalline solid is characterized mainly by three characteristics

(i) Sharp melting point,
(ii) Definite heat of fusion and
(iii) Definite crystal lattice.

Types of Solids Solids are classified on the basis of their structural units such as ions atoms or molecules and the nature of forces which hold these structural units together in a crystal.

Ionic Solids They are made up of anions and cations which are held together by strong electrostatic forces.

Molecular Solids consist of neutral molecules held together by relatively weak dipole-dipole or van der Waals' forces.

Covalent Solids In these solids, the neighbouring atoms are held by covalent bonds.

Metallic Solids In the structural units these are positive metal ions (same radii) dispersed in sea of electrons. There exists strong electrostatic attraction between metal ions and mobile valence electrons.

Isomorphism is the phenomenon by virtue of which different chemical substances exhibit the same crystalline form.

Polymorphism is the phenomenon by virtue of which a substance crystallises in more than one form. In case of elements this phenomenon is called as **allotropy.**

Crystal lattice It is a regular pattern of points, which describes the three-dimensional arrangement of particles (ions, atoms or molecules) in a crystal structure.

Lattice points The points representing the atoms, molecules or ions in a unit cell are known as lattice points or lattice sites.

Unit cell It is the smallest portion of the space lattice, which can generate the complete crystal by repeating its own dimensions in various directions or *repetition of unit cell in three dimensions gives rise to a space lattice*. Depending upon the length of edges (a, b, c) and angles (α, β, γ), seven type of simple crystal systems have been observed. These are summarized in table 3.1

Table 3.1 Different crystal systems and characteristics of their unit cells

	System	*Axial dimensions of the unit cell*	*Axial angles*	*Examples*
1.	Cubic	$a = b = c$	$\alpha = \beta = \gamma = 90°$	NaCl, KCl, Cu, ZnS
2.	Tetragonal	$a = b \neq c$	$\alpha = \beta = \gamma = 90$	SnO_2, TiO_2, $CaSO_4$, white Sn
3.	Orthorhombic	$a \neq b \neq c$	$\alpha = \beta = \gamma = 90°$	Rhombic S, KNO_3, $BaSO_4$, K_2SO_4
4.	Hexagonal	$a = b \neq c$	$\alpha = \beta = 90°$; $\gamma = 120°$	Quartz, ZnO, graphite, Ice
5.	Trigonal or Rhombohedral	$a = b = c$	$\alpha = \beta = \gamma \neq 90°$	Calcite, Magnesite
6.	Monoclinic	$a \neq b \neq c$	$\alpha = \gamma = 90°$; $\beta \neq 90°$	Monoclinic S, Gypsum
7.	Triclinic	$a \neq b \neq c$	$\alpha \neq \beta \neq \gamma \neq 90°$	$CuSO_4 . 5H_2O$, $K_2Cr_2O_7$, H_3BO_3

Bragg's Law gives a relation between the distance between the layers of particles in a crystal (*d*), the wave length of the *X*-ray used (λ) and the angle of diffraction (2θ). The relation is

$$n\lambda = 2d \sin \theta$$

where *n* is an integer, 1, 2, 3, ... and is called order of diffraction.

Hexagonal Close Packing (HCP) or ABAB A ... type: It is a three dimensional packing of identical spheres in such a way that the third layer is exactly identical to the first layer. In this packing 74% of the available space is occupied by the spheres.

Cubic Close Packing (CCP) or Face Centred Cubic (fcc) or ABCABCA ... type: In this type of arrangement the fourth layer is identical to the first layer. In this also 74% of the available space is occupied by the spheres.

Void: It is the vacant space present among the constituent particles when they are packed to give a crystal structure. Two types of voids are formed in close packed structures, which are called tetrahedral voids and octahedral voids.

In fcc or ccp, a unit cell contains 8 tetrahedral voids and 4 octahedral voids.

Coordination number: The number of nearest neighbours of an atom, ion or molecule in the crystal structure to which this atom, ion or molecule is touching is called its coordination number. Coordination number in an octahedral void is 6 while in a tetrahedral void it is 4.

Density of a cubic crystal is given by the formula

$$\rho = \frac{M \times Z}{a^3 \times N_A}$$

where ρ = density
M = molar mass of the substance or atomic mass in the case of metals
Z = number of formula units per unit cell or no. of atoms per unit cell
a = edge length of cubic cell
and N_A = Avogadro's number

Distance (*d*), radius of atom (*r*) and edge length (*a*) for the different cubic unit cells is given in the table 3.2

Compounds of *AB* type show three different types of cubic structural variations, which are described in the table 3.3.

Table 3.2

Unit Cell	*Relation between d and a*	*Relation between r (radius of atoms) and a (edge length)*	*No of particles per unit cell*
Simple	$d = a$	$r = a/2$	1
Body centred	$d = \frac{\sqrt{3}}{2}a = 0.866a$	$r = \frac{\sqrt{3}}{4}a$	2
Face centred	$d = \frac{1}{\sqrt{2}}a = 0.707a$	$r = \frac{1}{2\sqrt{2}}a$	4

Table 3.3

	Structure	*Description*	*Coordination number*	*No. of units of AB per unit cell*	*Example*
1.	NaCl type (Rock salt type)	Cl^- form fcc and Na^+ occupy all the octahedral voids	$Na^+ = 6$ $Cl^- = 6$	4	NaCl, AgCl, MgO
2.	CsCl type	Cl^- from a simple cubic unit cell with Cs^+ occupying the body centre	$Cs^+ = 8$ $Cl^- = 8$	1	CsCl, TlCl, CsBr
3.	ZnS type	S^{-2} form fcc unit cell and Zn^{+2} occupy alternate tetrahedral voids	$Zn^{2+} = 4$ $S^{2-} = 4$	4	ZnS, CuCl, ZnO

Diamagnetic substances are weakly repelled by the magnetic field and do not have any unpaired electron. e.g., NaCl, $MgCl_2$.

Paramagnetic substances are attracted by the magnetic field and have unpaired electrons e.g., TiO, CuO.

Ferromagnetic substances are attracted by the magnetic field and permanent magnetism is due to the spontaneous and permanent alignment of magnetic moments in the same direction as

↑↑↑↑↑ e.g., Fe, Ni, Co.

Antiferromagnetic substances are those which are expected to possess paramagnetism or ferromagnetism on the basis of unpaired electrons but actually possess zero net magnetic moment because the small magnetic moments align in compensatory way as shown

↑↓↑↓↑↓ e.g., Cr_2O_3, MnO, Mn_2O_3.

Ferrimagnetic substances are those which are expected to possess large magnetism on the basis of unpaired electrons but actually have small net magnetic moment because the small magnetic moments align more in one direction and less in the opposite direction.

↑↑↓↑↑↓↑↑ e.g., Fe_3O_4.

Piezo electricity is the electricity produced on applying mechanical stress on some polar crystals.

Pyro electricity is the electricity produced on heating some polar crystals.

Photovoltaic are substances which can convert solar energy into electrical energy.

Defects in Crystals The term defect generally denotes departure from regularity in the arrangement of the constituent particles (atoms, ions and molecules) in a crystal. Defects in crystals greatly influence their physical properties (e.g., electrical conductivity and diffusion) and sometimes even their chemical properties.

Point Defects .A crystal lattice is a regular repeating arrangement of points representing relative locations of its constituent particles. If any of these particles is either missing from the crystal lattice or is dislocated to an interstitial position, a point defect is created. In such defects, electrical neutrality of the crystal is maintained and stoichiometry is not affected.

Schottky Defect Missing of equivalent numbers of cations and anions from their normal positions in the crystal lattice, creates voids or vacancies in the crystal. Existence of such paired vacancies at lattice points is called *Schottky defect.* It decreases the density of the crystal.

Frenkel Defect When an ion, generally cation which is smaller in size shifts to interstitial position, leaving a vacancy in its own position is called *Frenkel defect.*

Interstitials When the atoms of some other substance occupy the vacant sites (tetrahedral or octahedral voids) in the crystals of a particular substance, then this defect is called interstitials.

Semi-conductors Ionic compounds are ordinarily non-conductors in the solid state at room temperature. However, when the temperature is raised, there is usually a significant increase in conductivity due to defects in the crystal structure. These defects, make it easier for ions to move through the solid under the influence of an electrical field. A few ionic solids, e.g., oxides and sulphides of heavy metals (like PbS) have high conductivity and are of great industrial importance. Such solids are classified as *semiconductors.*

Silicon and germanium also behave as semiconductors but due to defect of quite different type. Small amounts of impurities incorporated into the crystal lattice of these elements covert them from insulators to semi-conductors widely used in transistors, and solar batteries. Extremely pure samples of Ge or Si contain virtually no free electrons as four valencies of each atom are used in forming covalent bonds with the neighbouring atoms. These elements, are therefore, non-conductors. When a small number of foreign atoms (even one in million) get introduced into the crystal lattice, the situation changes drastically. For example, an atom of As or Sb with five valency electrons can fit into the tetrahedral lattice of Ge or Si only by setting one of the five electrons relatively free. Such electrons moving under the influence of an electrical field give rise to ***n-type semi-conductors*** (named so because negative charge flows). However, when a trivalent boron atom is introduced, electron deficiency is created at the site of foreign atom. Such a defect is known as formation of a *positive hole* in the lattice. Under the influence of an electric field, electron from a neighbouring atom fills this hole, leaving a positive hole around the atom which it leaves. Such semiconductors are known as ***p-type semiconductors.*** A semiconductor, in which there is a junction between electron rich and electron-poor region, acts as a rectifier i.e., it can covert AC power supply to DC. These are also called ***n-p junctions.***

Solved Problems

1. In the structure determination of a solid, *X*-rays of wave length 5.84×10^{-11}m were used. A differaction angle equal to 16.8° was obtained for $n = 1$. What is the interplaner distance d in this solid?

(a) 2×10^{-8}m (b) 4×10^{-8}m
(c) 2×10^{-10}m (d) 4×10^{-10}m

Solution (c) Bragg's Law is

$$n\lambda = 2d \sin\theta$$

Here $n = 1$

$\lambda = 5.84 \times 10^{-11}$m

$2\theta = 16.8°$ so $\theta = 8.4$

Substituting these values

$$d = \frac{n\lambda}{2\sin\theta} = \frac{1 \times 5.84 \times 10^{-11}}{2 \times \sin 8.4}\text{ m}$$

$$= \frac{5.84 \times 10^{-11}}{2 \times 0.146} = 2 \times 10^{-10}\text{m}$$

2. A solid *AB* has the NaCl structure. If the radius of the cation is 100 pm, then the radius of the anion B^- is

(a) 146 pm (b) $\sqrt{3} \times 100$ pm
(c) 100 pm (d) 241.5 pm

Solution (d) In NaCl type solids

$$\frac{r_{A^+}}{r_{B^-}} \text{ should be } 0.414$$

$$r_{B^-} = \frac{r_{A^+}}{0.414} = \frac{100}{0.414} = 241.5 \text{ pm}$$

3. A crystal of Lead (II) sulphide has NaCl structure. In this crystal the shortest distance between a Pb^{2+} ion and S^{2-} ion is 297 pm. What is the length of the edge of unit cell in lead sulphide?

(a) 2.97×10^{-10}cm (b) 5.94×10^{-10}m

(c) 5.94×10^{-10}cm (d) 2.97×10^{-10}m

Solution (b) Since PbS has NaCl structure

edge length (a) = 2 × distance between Pb^{2+} and S^{2-} ions

= 2 × 297 pm

= 594 pm

= 594×10^{-12}m = 5.94×10^{-10}m

4. An element (atomic mass = 100 g mol^{-1}) having bcc structure has unit cell edge 400 pm. The density of the element is

(a) 10.4 g cm^{-3} (b) 5.2 g cm^{-3}

(c) 7.3 g cm^{-3} (d) 2.1 g cm^{-3}

Solution (b) Density $(\rho) = \dfrac{Z \times M}{a^3 \times N_A}$

where $Z = 2$ (because it has bcc structure)

$M = 100$ g mol^{-1}

$a = 400 \text{ pm} = 4 \times 10^{-8}$cm

$N_A = 6.02 \times 10^{23}$ mol^{-1}

Substituting the values

$$\rho = \frac{2 \times 100}{(4 \times 10^{-8})^3 \times 6.02 \times 10^{23}} = 5.2 \text{ g cm}^{-3}$$

5. The unit cell of an element of atomic mass 100 and density 12 g cm^{-3} is a cube with edge length of 300 pm. What is the structure of crystal lattice?

(a) B C C (b) F C C

(c) simple cubic (d) None of these

Solution (a) No of particles per unit cell,

$$Z = \frac{\rho \times a^3 \times N_A}{M}$$

Given that $\rho = 12$ g cm^{-3}

$a = 300 \text{ pm} = 3 \times 10^{-8}$cm

$M = 100$ g mol^{-1}

$N_A = 6.02 \times 10^{23}mol^{-1}$

Substituting these values

$$Z = \frac{12 \times (3 \times 10^{-8})^3 \times 6.02 \times 10^{23}}{100} = 1.96 \simeq 2$$

Since the number of particles per unit cell is 2 so it is a body centred cubic (B C C) lattice.

Problems for Practice

1. When one substance crystallizes in more than one form, the phenomenon is called

(a) Isomorphism (b) Polymorphism

(c) Efflorescence (d) Deliquescence

2. Which type of crystal should be the softest and have the lowest melting point?

(a) Molecular crystals (b) Metallic crystals

(c) Ionic crystals (d) Covalent crystals

3. Tetragonal crystal system has

(a) $a = b = c, \alpha = \beta = \gamma = 90°$

(b) $a = b \neq c$ and $\alpha = \beta = \gamma = 90°$

(c) $a \neq b \neq c, \alpha = \beta = \gamma = 90°$

(d) $a = b \neq c$ and $\alpha = \beta = 90°, \gamma = 120°$

4. Amorphous solids

(a) possess sharp melting points

(b) undergo clean cleavage

(d) do not undergo clean cleavage

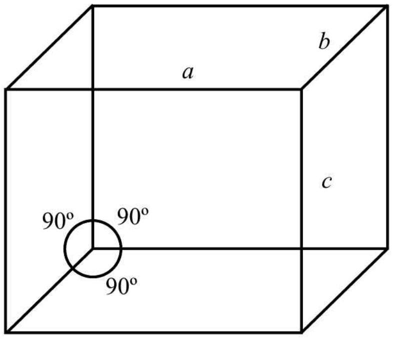
(d) possess orderly arrangement over long distance

5. In a crystal, all the lattice sites are found to be occupied by covalent molecules. This solid is

(a) ionic solid (b) molecular solid

(c) covalent solid (d) Any of these

6. The unit cell with the structure below refers to ... crystal system



(a) Cubic (b) Orthorhombic

(c) Tetragonal (d) Trigonal

7. The number of octahedral sites in a cubical close pack array of N spheres is

(a) N/2 (b) 2 N

(c) 4 N (d) N

8. The phenomenon in which different chemical substances exhibit the same crystalline form is known as

(a) Isomorphism (b) Polymorphism

(c) Enantiotropy (d) Monotropy

9. Fraction of the total volume occupied by atoms in a simple cube is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\sqrt{3}\pi}{8}$

(c) $\frac{\sqrt{2}\pi}{6}$ (d) $\frac{\pi}{6}$

10. Silicon dioxide is an example of

(a) Metallic crystal (b) Ionic crystal

(c) Covalent crystal (d) None of these

11. If a is the length of unit cell, then which one is correct relationship?

(a) For simple cubic lattice,

Radius of metal atom = $\frac{a}{2}$

(b) For bcc lattice,

Radius of metal atom = $\frac{\sqrt{3}a}{4}$

(c) For fcc lattice,

Radius of metal atom = $\frac{a}{2\sqrt{2}}$

(d) All of these

12. A compound formed by the elements A and B crystallises in cubic structure in which A atoms are at the corners with B atoms at the centre of the faces. The formula of the compound is

(a) AB (b) AB_2

(c) A_2B_3 (d) AB_3

13. Out of hexagonal closed packed structure and cubic closed packed structure, the density of

(a) hcp structure is higher
(b) ccp structure is higher
(c) Both (a) and (b) are equal
(d) Any one may be higher or lower

14. The name given to ABC. ABC. ABC type of arrangement is

(a) Cubic close packing
(b) Hexagonal close packing
(c) Body centred close packing
(d) Square close packing

15. In a face centred cubic arrangement the number of atoms per unit cell is

(a) 8 (b) 2

(c) 1 (d) 4

16. In *hcp* arrangement, the coordination number is

(a) 6 (b) 12

(c) 8 (d) 10

17. CsCl crystal consists of close packing of

(a) CsCl molecules (b) Cs^+ and Cl^- ions

(c) Cs and Cl atoms (d) Mixture of all these

18. Which of the following defect lowers the density of solids?

(a) Frenkel (b) Schottky

(c) Interstitial (d) Edge dislocation

19. An alloy of copper, silver, and gold is found to have copper constituting the *ccp* lattice. If silver atoms occupy the edge centres and gold is present at body centre, the alloy has the formula

(a) Cu_4Ag_2Au (b) Cu_4Ag_4Au

(c) Cu_4Ag_3Au (d) CuAgAu

20. The structure of CsCl crystal is

(a) Body centred cubic lattice

(b) Face centred cubic lattice

(c) Octahedral

(d) None of these

21. The co-ordination number of a body centred atom is

(a) 4 (b) 6

(c) 8 (d) 12

22. The mass of a unit cell of CsCl corresponds to

(a) 8 Cs^+ and 8 Cl^- (b) 6 Cs^+ and 6 Cl^-

(c) 1 Cs^+ and 1 Cl^- (d) 4 Cs^+ and 4 Cl^-

23. In hcp arrangement, the co-ordination number is

(a) 6 (b) 12

(c) 8 (d) 10

24. Due to Schottky defect in NaCl

(a) The density increases

(b) Lattice energy increases

(c) Both (a) and (b)

(d) None of these

25. For tetrahedral co-ordination, the radius ratio (r^+/r^-) should be

(a) 0.414 – 0.732 (b) 0.732 – 1.0

(c) 0.156 – 0.225 (d) 0.225 – 0.414

26. How many atoms are there in a cubic unit cell, having one atom on each corner and two atoms on each body diagonal of the cube?

(a) 9 (b) 8

(c) 6 (d) 4

27. Certain crystals produce electric signals on application of pressure. This phenomenon is called

(a) Pyroelectricity (b) Ferroelectricity

(c) Piezoelectricity (d) Pressoelectricity

28. In the zinc blende structure (ZnS), S^{2-} adopt ccp arrangement and Zn^{2+} occupy

(a) Octahedral sites (b) Hexagonal sites

(c) Tetrahedral sites

(d) Both octahedral and tetrahedral sites

29. Which of the following is a pseudo solid?

(a) CaF_2 (b) Glass

(c) NaCl (d) All

30. Bragg's law is given by equation

(a) $n\lambda = 2d \sin\theta$ (b) $n\lambda = \frac{2}{d} \sin\theta$

(c) $2n\lambda = d \sin\theta$ (d) $n\lambda = \frac{1}{2} d \sin\theta$

31. If one cation and one anion are lost simultaneously from the crystal lattice, the defect is called as

(a) Schottky defect (b) Frenkel defect

(c) Crystal defect (d) Ionic defect

32. The displacement of a cation into the interstitial site of a crystal is found in

(a) Schottky defect (b) Frenkel defect

(c) Ionic defect (d) Non-ionic defect

33. A compound formed by elements *A* and *B* form cubic structure in which '*A*' atoms are at the corners of a cube and '*B*' atoms are at the face centre. The formula of the compound is

(a) AB_3 (b) AB_2
(c) A_2B (d) AB

34. When an element of group 14 is doped with an element of group 15

(a) *P*-type of semi-conductors are formed
(b) *N*-type of semi-conductors are formed
(c) Zeolites are formed
(d) Electrolytes are formed

35. In anti-ferromagnetism

(a) Alignment of magnetic moments is additive
(b) Alignment of magnetic moments in one direction is compensated by alignment in the opposite direction
(c) Alignment of magnetic moments does not take place
(d) Alignment of magnetic moments varies with the nature of the material

36. Which is true about Piezoelectric crystals?

(a) They produce an electric current on heating
(b) They produce an electric current when a mechanical stress is applied
(c) They are insulators
(d) They are magnetic in nature

37. Pyroelectric crystals produce feeble electric current

(a) On deformation
(b) On dissolving in a solvent
(c) On heating
(d) On sublimation

38. TlCl has structure similar to CsCl, the coordination number of Tl^+ is

(a) 4 (b) 6
(c) 10 (d) 8

39. When NaCl crystal is doped with $MgCl_2$, the nature of defect produced is

(a) Interstitial defect (b) Schottky defect
(c) Frenkel defect (d) None of these

40. Silicon doped with arsenic is an example of which type of semiconductor?

(a) *n*-type (b) *p*-type
(c) *n*/*p*-type (d) *p*/*n*-type

41. In close packed array of N spheres, the number of tetrahedral holes are

(a) N/2 (b) 4 N
(c) 2 N (d) N

42. In calcium fluoride, the coordination number of cations and anions are respectively

(a) 6, 6 (b) 4, 4
(c) 8, 4 (d) 4, 8

43. A solid is made up of two elements *A* and *B*. Atoms of *B* are in ccp arrangement, while atoms of *A* occupy all the tetrahedral sites. The formula of the compound is

(a) AB_2 (b) AB
(c) AB_3 (d) A_2B

44. Because of Schottky and Frenkel defects, the electrical conductivity:

(a) Increases
(b) Decreases
(c) Neither decreases nor increases
(d) None of these

45. Solid CO_2 is an example of

(a) ionic solid (b) molecular solid
(c) covalent solid (d) metallic solid

46. In NaCl crystal, each Na^+ is surrounded by

(a) 2 Cl^- (b) 4 Cl^-
(c) 6 Cl^- (d) 8 Cl^-

47. The edge length of face centred cubic unit cell is 508 pm. If the radius of the cation is 110 pm, the radius of anion is

(a) 110 pm (b) 144 pm
(c) 618 pm (d) 398 pm

48. Which of the following does not represent a type of crystal system?

(a) Triclinic (b) Monoclinic
(c) Rhombohedral (d) Isotropical

49. Which of the following metal oxides is anti ferromagnetic in nature?

(a) MnO_2 (b) TiO_2
(c) NO_2 (d) CrO_2

50. A semiconductor of Ge can be made *p*-type by adding

(a) trivalent impurity
(b) tetravalent impurity
(c) pentavalent impurity
(d) any of these

ANSWERS

1. (b)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (d)	**10.** (c)
11. (d)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (d)	**16.** (b)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (a)
21. (c)	**22.** (c)	**23.** (b)	**24.** (d)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (a)
31. (a)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (c)	**38.** (d)	**39.** (b)	**40.** (a)
41. (c)	**42.** (c)	**43.** (d)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (d)	**49.** (a)	**50.** (a)

4 Solutions

BRIEF REVIEW OF THE CONCEPTS

A solution is a homogeneous mixture of two or more components whose concentration can be varied within certain limits. One of the component of the solution is called **solute** and the other is called **solvent.**

Solute is the component of a solution which gets dispersed.

Solvent is the component of a solution which has the same physical state as that of the solution.

A solution consisting of two components is called a binary solution. Similarly, it is called ternary or quaternary, if it is composed of three or four components, respectively.

Table 4.1 Types of binary solutions

S. No	*Solute*	*Solvent*	*Types of solution*	*Examples*
1.	Solid	Solid	Solid in solid	Alloys, Amalgams
2.	Liquid	Solid	Liquid in solid	Hydrated salts
3.	Gas	Solid	Gas in solid	Hydrogen in palladium
4.	Solid	Liquid	Solid in liquid	Salt or sugar in water
5.	Liquid	Liquid	Liquid in liquid	Alcohol in water
6.	Gas	Liquid	Gas in liquid	Aerated water
7.	Solid	Gas	Solid in gas	Iodine vapour in air
8.	Liquid	Gas	Liquid in gas	Humidity in air
9.	Gas	Gas	Gas in gas	Air

Depending upon the amount of solute dissolved in a solvent the solutions are classified as

(*i*) **Saturated solution** A solution in which no more of the solute can be dissolved at a given temperature is called a saturated solution and this point is known as saturation point. The quantity of solute dissolved per 100g of solvent in its saturated solution at a given temperature is called its *solubility.*

(*ii*) **Unsaturated solution** A solution in which more of the solute can be dissolved at a given temperature is called an unsaturated solution.

(*iii*) **Supersaturated solution** A solution which contains more solute than would be necessary to saturate it at a given temperature is called a supersaturated solution. It is a metastable state.

A substance dissolves in water if its hydration energy is grater than that of its lattice energy.

Methods for expressing the concentration of a solution

Gram Per Litre (gL^{-1}) is the number of grams of solute dissolved in one litre of the solution.

$$gL^{-1} = \frac{\text{Mass (in g) of solute}}{\text{Volume (in L) of solution}}$$

Mass Per cent is the number of mass units of solute present per 100 mass units of solution.

$$\text{Mass \%} = \frac{\text{Mass of solute}}{\text{Total mass of solution}} \times 100$$

Molarity (M) of a solution is the number of moles of the solute dissolved in 1 litre of solution.

$$\text{Molarity} = \frac{\text{Mass of solute}}{\text{Molar mass of solute}} \times \frac{1000}{\text{Vol. of solution (mL)}}$$

Molarity of a solution changes with temperature.

Molality (m) of a solution is the number of moles of the solute in 1 kg of the solvent.

$$\text{Molality} = \frac{\text{Moles of solute}}{\text{Mass of solvent (kg)}}$$

$$= \frac{\text{Mass of solute}}{\text{Molar Mass of solute} \times \text{Mass of solvent (kg)}}$$

Normality (N) is the number of gram equivalents of solute present in 1 litre of solution.

$$\text{Normality} = \frac{\text{Mass of solute}}{\text{Equiv. mass of solute}} \times \frac{1000}{\text{Vol. of solution (mL)}}$$

Parts Per Million (ppm) is the parts of solute in one million parts of a solution. It is used for very very low concentrations.

$$\text{ppm} = \frac{\text{Mass of solute}}{\text{Mass of solution}} \times 10^6$$

Mole Fraction (χ) of a component in a mixture is the ratio between the number of moles of that component to the total number of moles of all the components of the mixture. Mole fraction of component A

$$\chi_A = \frac{\text{Moles of } A}{\text{Total moles of all the components of the mixture}}$$

Mole fraction and molality are independent of temperature.

In a mixture the sum of the mole fractions of all the components is always equal to 1, i.e.,

$$\chi_A + \chi_B + \ldots + \chi_n = 1.$$

Liquids evaporate to form vapours. When this process occurs in a closed vessel, an equilibrium is established between vapours and liquid. The pressure exerted by vapours while in equilibrium with the liquid is called its **vapour pressure.** Each liquid exerts a characteristic vapour pressure at a given temperature.

Vapour Pressure at a particular temperature is pressure exerted by the vapours over a liquid when evaporation and condensation are at equilibrium, also called equilibrium vapour pressure.

Raoult's Law

Solution of volatile liquids: The partial vapour pressure of any component at a given temperature is equal to the product of the vapour pressure of the pure component at this temperature and the mole fraction of this component in the solution, i.e.,

$$p_A = p_A^0 \cdot \chi_A$$

Solution of non-volatile solute in a volatile solvent: The vapour pressure of a solution containing a non-volatile solute is directly proportional to the mole fraction of solvent. The proportionality constant being equal to the vapour pressure of pure solvent, i.e.,

$$p_{\text{soln.}} = p_{\text{solvent}} \cdot \chi_{\text{solvent}}$$

The **relative lowering of vapour pressure** for a solution is equal to the mole fraction of the solute when solvent alone is volatile, i.e.,

$$\frac{p^{\circ}_{\text{solvent}} - p_{\text{soln.}}}{p^{\circ}_{\text{solvent}}} = \text{Mole fraction of solute, } \chi_{\text{solute}}$$

For a dilute solution,

$$\frac{p^{\circ}_{\text{solvent}} - p_{\text{soln.}}}{p^{\circ}_{\text{solvent}}} = \frac{\text{no. of moles of solute}}{\text{no. of moles of solvent} + \text{no. of moles of solute}}$$

For very dilute solutions, it can be taken as equal to

$$= \frac{\text{no. of moles of solute}}{\text{no. of moles of solvent}} = \frac{\text{W}_{\text{solute}} \times \text{M}_{\text{solvent}}}{\text{W}_{\text{solvent}} \times \text{M}_{\text{solute}}}$$

Ideal Solution obeys Raoult's law at all the concentrations and temperatures. For an ideal solution $\Delta V_{\text{mix}} = 0$ and $\Delta H_{\text{mix}} = 0$ i.e., on mixing the individual components there is neither volume change nor enthalpy change.

Non-ideal Solutions are those solutions (i) which do not obey Raoult's law, (ii) for which $\Delta H_{\text{mixing}} \neq 0$ and $\Delta V_{\text{mixing}} \neq 0$.

Positive Deviation from Raoult's law This type of deviation is shown by liquid-pairs for which *A–B* molecular forces are weaker than the *A–A* or *B–B* molecular forces, e.g., water-ethanol, benzene-acetone.

Negative Deviation This type of deviation is shown by liquid-pairs for which *A–B* attractions are stronger than *A–A* or *B–B* attractions, e.g., methanol-acetic acid, acetone-chloroform.

Properties of non-ideal solutions of two liquids, *A* and *B*

Positive deviation from Raoult's Law	*Negative deiation from Raoult's Law*
A...*B* forces less than *A*...*A* or *B*...*B* forces,	*A*...*B* forces greater than *A*...*A* or *B*...*B* forces,
ΔH is positive	ΔH is negative
ΔV is positive	ΔV is negative

Colligative Properties: Those properties of solution which depend only upon the number of solute and solvent particles in the solution and not on their nature are called colligative properties.

Normal Boiling point The boiling point of a liquid is the temperature at which the vapour pressure of the liquid becomes equal to the atmospheric pressure. If the atmospheric pressure is equal to 1 atmosphere the boiling point in called the normal boiling point.

Elevation of Boiling Point, ΔT_b: The presence of a non-volatile solute in a solution raises the boiling point of the pure solvent. It is known as elevation in the boiling point, ΔT_b,

$$\Delta T_b = K_b m = K_b \cdot \frac{\text{no. of moles of solute}}{\text{mass of solvent (kg)}}$$

K_b is molal boiling point elevation constant, *m* is the molality of non-volatile solute in the solution.

$$\Delta T_b = \frac{1000 \times K_b \times W_B}{M_B \times W_A}$$

or

$$\Delta T_b = \frac{1000 \times K_b \times W_{\text{solute}}\text{ (g)}}{\text{Molar mass of solute } (M_B) \times W_{\text{solvent}}\text{ (g)}}$$

Molal Boiling Point Elevation Constant K_b is the increase in the boiling point of the pure solvent when 1 mole of an unionised, non-volatile solute is dissolved in 1 kg of the solvent.

Freezing Point is the temperature at which the solid and the liquid states of a substance have the same vapour pressure.

Depression of Freezing Point, ΔT_f is the decrease in the freezing point of a pure solvent when a non-volatile solute is dissolved into it.

$$\Delta T_f = K_f m$$

ΔT_f is the decrease in the freezing point of the pure solvent, K_f is the molal freezing point depression constant and *m* is the molality of the solute in the solution.

$$\Delta T_f = \frac{1000\times K_f\times W_B}{M_B\times W_A}$$

or $$\Delta T_f = \frac{1000\times K_f.\times W_{\text{solute}}\,(\text{g})}{\text{Molar mass of solute }(M_{\text{solute}})\times W_{\text{solvent}}(\text{g})}$$

Molal Freezing Point Depression Constant, K_f is the decrease in the freezing point of the pure solvent when the molality of the solution is 1 i.e., 1 mole of unionised, non-volatile solute is dissolved in 1 kg of the solvent.

Azeotropic Mixture is a solution which boils at a constant temperature and distills over without undergoing any change in its composition.

Osmosis is the spontaneous flow of solvent particles from a dilute solution (or solvent) into a concentrated solution when the two are separated by a semipermeable membrane.

Exosmosis is the outward flow of water from a cell containing an aqueous solution through semipermeable membrane e.g., grape is sugar solution.

Endosmosis is the inward flow of water into a cell containing an aqueous solution through semipermeable membrane e.g., dried grapes in water.

Desalination The removal of ions, especially Na^+ and Cl^-, from water is called desalination.

Osmotic Pressure, π: If a solution is separated from the solvent by a semipermeable membrane, the external pressure that must be applied to the solution side to stop osmosis i.e., to prevent the flow of solvent particles into solution is called osmotic pressure.

$$\pi = \frac{n_B}{V}RT \text{ or } \pi = cRT$$

Here π is the osmotic pressure, n_B is the number of moles of solute dissolved in V litres of solution. R is the gas constant and T temperature in kelvin. Osmotic pressure is generally measured by Berkley and Hartley method.

Isotonic Solutions are solutions which have the same value of osmotic pressure at the same temperature. If two isotonic solutions are separated by a semipermeable membrane, no osmosis will occur. The isotonic solutions also have same molar concentrations.

Hypotonic Solution A solution is called hypotonic if its osmotic pressure is lower than that of the solutions separating it by a semipermeable membrane. In such a system, the flow of solvent due to osmosis, will be from the hypotonic solution into the other solution across the semipermeable membrane.

Hypertonic Solution. A solution is called hypertonic if its osmotic pressure is higher than that of the solution separating it by a semipermeable membrane. In such a system, the flow of solvent due to osmosis, will be from the solution across the semipermeable membrane into the hypertonic solution.

1. **Osmotic pressure** is the best colligative property to determine the molar mass of a non-volatile solute.
2. **K_b and K_f are also** called *ebullioscopic* and *cryoscopic constants,* respectively.

Abnormal molar masses Colligative property depends upon the number of solute particles present in the solution. When a solute undergoes dissociation, or association in the solution, the effective number of solute particles changes. This change in the number of particles will affect the determined value of colligative property. Thus, the molar mass determined by the measurement of colligative property under these conditions will have an abnormal value.

Van't Hoff Factor '*i*' Solutes which undergo dissociation or association give abnormal value of colligative properties. Van't Hoff factor '*i*' is a correction factor defined as

$$i = \frac{\text{observed value of colligative property}}{\text{normal value of colligative property}}$$

$$= \frac{\begin{array}{c}\text{normal molar mass}\\ \text{(calculated from the molecular formula)}\end{array}}{\text{observed molar mass}}$$

For association $i < 1$, for dissociation $i > 1$. Van't Hoff factor modifies the equations relating the various colligative properties.

Degree of dissociation, $\alpha = \dfrac{i-1}{n-1}$

Where i is the Van't Hoff factor and n is no. of ions obtained by the dissociation of 1 molecule of solute.

Degree of association, $\alpha = (1 - i)\dfrac{n}{n-1}$

Modified equations for colligative properties:

$$\frac{p^\circ - p_s}{p^\circ} = i.\chi_B$$

$$\Delta T_b = i \times K_b \times m;\ \Delta T_f = i \times K_f \times m;\ \pi = icRT$$

Solved Problems

1. 2.12g of Na_2CO_3 (molar mass 106) is dissolved in water and the solution is made to 100 cm^3 in a measuring flask. Calculate the molarity of the solution.

Solution Molarity = mass dissolved/molar mass × volume of solution in litres.

Given that mass dissolved = 2.12 g

molar mass = 106 g mol^{-1}

Volume of solution = 100 cm^3 = 0.1 L

Substituting the values

$$\text{Molarity} = \frac{2.12\text{ g}}{106\text{ g mol}^{-1}} \times \frac{1}{0.1\text{L}} = 0.2 \text{ mol L}^{-1}$$

2. Calculate the molality of a 1.0 M solution of sodium nitrate. The density of the solution is 1.25 g cm^{-3}.

Solution

$$\text{Molality} = \frac{\text{mass of solute}}{\text{molar mass of solute} \times \text{mass of solvent (kg)}}$$

Given,

Molarity = 1M

Molar mass of $NaNO_3$ = 85g mol^{-1}

Density = 1.25g cm^{-3}

1M solution of $NaNO_3$ contains 85g of $NaNO_3$ in 1000 cm^3 of solution.

Mass of 1000 cm³ solution

$$= 1000\ \text{cm}^3 \times 1.25\ \text{g cm}^{-3}$$
$$= 1250\ \text{g}$$

1250 g solution contains 85g $NaNO_3$.

$$\text{Mass of water} = 1250 - 85 = 1165\ \text{g}$$
$$= 1.165\ \text{kg.}$$

Substituting the values

$$\text{Molality} = \frac{85\text{g}}{85\text{g mol}^{-1} \times 1.165\text{kg}}$$

$$= 0.8582\ \text{mol kg}^{-1}$$

3. **Calculate the number of molecules of oxalic acid ($H_2C_2O_4.2H_2O$) in 100 mL of 0.2 N oxalic acid solution.**

Solution Given,

Normality of oxalic acid = 0.2 N;
Volume of oxalic acid = 100 mL
No. of milli equiv. of acid $= N \times V$
$= 0.2 \times 100 = 20$
Mass of oxalic acid $= 20 \times 10^{-3} \times 63$ g

(since its equivalent mass is 63)

$$\text{Moles of oxalic acid} = \frac{20 \times 10^{-3} \times 63\text{g}}{126\text{g mol}^{-1}} = 10^{-2}\ \text{mol}$$

No. of molecules = No. of moles × Avogadro's number
$= 10^{-2} \times 6.02 \times 10^{23}$
$= 6.02 \times 10^{21}$.

4. **Calculate the change in freezing point of water if 0.01 mol sodium sulphate is dissolved in 1 kg of water.**

$K_f(H_2O) = 1.86\ \text{K kg mol}^{-1}$

Solution For a non-ideal solution

$$\Delta T_f = i\ K_f.m$$

Since Na_2SO_4 ionises as

$$Na_2SO_4 \rightleftharpoons 2Na^+ + SO_4^{2-}$$

Here $i = 3$

Substituting the values

$$\Delta T_f = 3 \times 1.86\ \text{K kg mol}^{-1} \times 0.01\ \text{mol kg}^{-1}$$
$$= 0.056\ \text{K}$$

The change in freezing point of water will be 0.056 K

5. **Calculate the freezing point of a one molar aqueous solution (density 1.04 g mL⁻¹) of KCL.**

(K_f for water = 1.86 K kg mol⁻¹, Atomic masses : K = 39, Cl = 35.5).

Solution Given Solution is of CKl

Molarity = 1M
Density = 1.04g mL⁻¹
K_f for water = 1.86 K kg mol⁻¹
atomic mass of K = 39
atomic mass of Cl = 35.5
∴ molar mass of KCl = 74.5

1 molar solution mean that 1000 mL of this solution contains 1 mol or 74.5g of KCl.

Mass of 1000 ml of

$$\text{KCl solution} = 1000\ \text{mL} \times 1.04\text{g mL}^{-1}$$
$$= 1040\text{g.}$$
$$\text{Mass of water} = 1040 - 74.5 = 965.5\text{g}$$
$$= 0.9655\ \text{kg.}$$
$$\text{Molality of KCl solution} = \frac{1\ \text{mol}}{0.9655\text{kg}}$$

KCl ionizes as

$$KCl \rightleftharpoons K^+ + Cl^-$$

here $i = 2$

Using the relation

$$\Delta T_f = i\ K_f\ \text{m}$$

Or $$\Delta T_f = 2 \times 1.86\ \text{K kg mol}^{-1} \times \frac{1\ \text{mol}}{0.9655\text{kg}}$$
$$= 3.85\ \text{K}$$

Thus KCl solution will freeze at (273.2 – 3.85) K
= 269.35 K

6. **A solution containing 18g of a non-volatile solute in 200g water freezes at 272.07K. Calculate the molar mass of the solute (K_f = 1.86K/m)**

Solution Given

mass of solute = 18g
mass of solvent = 200g
$T_f = 272.07\text{K}$
$\Delta T_f = 273.2 - 272.07 = 1.13\text{K}$
$K_f = 1.86\ \text{K/m}$

Let molar mass of non-volatile solute be M_B

$$\Delta T_f = \frac{w_B \times 1000 \times K_f}{M_B \times w_A}$$

Substituting the values

$$M_B = \frac{18\text{g} \times 1000 \times 1.86\text{K/m}}{1.13\text{K} \times 200g} = 180\text{g mol}^{-1}$$

7. **The boiling point of water is 100°C and it becomes 100.52°C if 3 g of a non-volatile solute is dissolved in 20 ml of it. Calculate the molar mass of the solute. (K_b for water is 0.52K kg mol⁻¹).**

Solution Formula used is

$$\Delta T_b = K_b\ \text{m} \qquad ...(1)$$

Let the molar mass of solute be = M_B

Molality,

$$m = \frac{\text{Mass of solute}}{\text{Molar mass of solute} \times \text{Mass of solvent (kg)}}$$

$$= \frac{3\text{g}}{M_B \times 20\text{g}} \times \frac{1000\text{g}}{1\text{kg}} = \frac{150\,g}{M_B \times \text{kg}}$$

Substituting the values in equn. (1)

$$0.52\ \text{K} = \frac{0.52\text{K kg mol}^{-1} \times 150\text{g}}{M_B \times \text{kg}}$$

Or $$M_B = \frac{0.52\text{K kg mol}^{-1} \times 150\text{g}}{0.52\text{K} \times \text{kg}}$$

$$= 150\ \text{g mol}^{-1}$$

8. Calculate the osmotic pressure of 0.6% aqueous solution of a non-volatile, non-electrolyte solute, urea (NH_2CONH_2) at 25°C. Take densities to be1 g mL^{-1} and assume ideal behaviour of the solution.

Gas constant, $R = 0.082$ L atm K^{-1} mol^{-1}

Solution Formula giving osmotic pressure

$$\pi = \frac{n_B}{V} \cdot RT$$

Given

$R = 0.082$ L atm K^{-1} mol^{-1}
mass of urea = 0.6 g
mass of solution = 100 g
density of solution = 1.0 g mL^{-1}

Volume of solution

$$V = \frac{\text{Mass}}{\text{Density}} = \frac{100\text{ g}}{1.0\text{ g mL}^{-1}} = 100\text{ mL}$$

$$= 0.1\text{ L}$$

Temperature, $T = 25 + 273 = 298$K

Molar mass of urea = 60g mol^{-1}

(From formula NH_2 $CONH_2$)

$$\text{Moles of urea, } n_B = \frac{0.6\text{g}}{60\text{g mol}^{-1}} = 0.01\text{ mol}$$

Substituting the values in the above formula

$$\pi = \frac{0.01\text{ mol} \times 0.082\text{L atm K}^{-1}\text{mol}^{-1} \times 298\text{K}}{0.1\text{L}}$$

$$= 2.45\text{ atm}$$

9. Calculate the mass of KCl per kilogram of water to lower its freezing point to –8.0°. $K_f(H_2O) = 1.88$°C kg mol^{-1}. (At. mass; K = 39, Cl = 35.5).

Solution Given

Mass of water = 1kg
$\Delta T = 8$°C
$K_f = 1.88$°C kg mol^{-1}
molar mass of KCl = 74.5 g mol^{-1}

KCl ionises as

$$KCl \rightleftharpoons K^+ + Cl^-$$

for it Van't Hoff factor $i = 2$

Let w g of KCl be added in 1kg of water.

Then molality, = wg/74.5 g mol^{-1}

Using the formula

$$\Delta T_f = i\, K_f\, m$$

Substituting the values

$$8°\text{C} = 2 \times 1.88°\text{C kg mol}^{-1} \times \frac{w\text{ g}}{74.5\text{g mol}^{-1}}$$

Or $w = 158.1$g

10. Osmotic pressure of a solution containing 2 g of dissolved protein per 300 cm^3 of solution is 20 mm of Hg at 27°C. Calculate the molecular mass of protein. (R = 0.0821 L atm K^{-1} mol^{-1})

Solution Formula used

$$\pi = \frac{n_B}{V} \cdot RT = \frac{w_B}{M_B V(L)} \cdot RT$$

Here $w_B = 2$ g
$V = 300\text{ cm}^3 = 0.3$ L
$\pi = 20$ mm Hg = 20/760 atm
$T = 27°\text{C} = 27 + 273 = 300$ K
$R = 0.0821$ L atm K^{-1} mol^{-1}

Substituting the values

$$M_B = \frac{2.0\text{g} \times 0.0821\text{L atm K}^{-1}\text{ mol}^{-1} \times 300\text{K} \times 760}{0.3\text{L} \times 20\text{ atm}}$$

$$= 9360\text{ g mol}^{-1}$$

Problems for Practice

1. The mole fraction of oxygen in a mixture of 7 g of nitrogen and 8 g of oxygen is
(a) 8/15 (b) 0.5
(c) 0.25 (d) 1.0

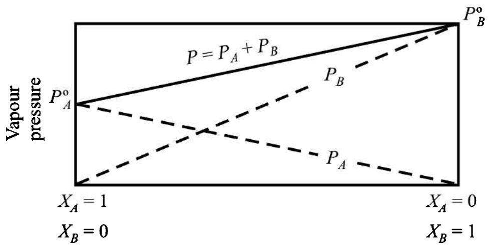
2. In which solvent should KCl be most soluble?
(a) Octane (b) Ethoxyethane
(c) Methanol (d) Benzene

3. If 200 ml of 1.50 M HCI is diluted with water to a volume of 500 ml. Then the molar concentration of the diluted solution is
(a) $\left(\frac{200}{500}\right)\left(\frac{1}{1.50}\right)$ (b) $\left(\frac{1.50}{1}\right)\left(\frac{200}{700}\right)$
(c) $\left(\frac{1.50}{1}\right)\left(\frac{200}{500}\right)$ (d) $\left(\frac{1.50}{1}\right)\left(\frac{500}{200}\right)$

4. In which mode of expression the concentration of a solution remains independent of temperature?
(a) molarity (b) normality
(c) molality (d) formality

5. The pair of compounds which cannot exist together in solution is
(a) $NaHCO_3$ and NaOH
(b) Na_2CO_3 and NaOH
(c) Na_2CO_3 and $NaHCO_3$
(d) NaCI and $NaHCO_3$

6. A vapour pressure graph obtained for a liquid-liquid solution is given below:



Which of the following pair of liquids would be most likely to show a vapour pressure composition curve as shown above?

(a) methanol and cyclohexane
(b) benzene and chloroform
(c) hexane and heptane
(d) water and nitric acid

7. For a dilute solution, Raoult's law states that
(a) the lowering of vapour pressure is equal to the mole fraction of solute.
(b) the relative lowering of vapour pressure is equal to the mole fraction of solute.
(c) both are true
(d) none of these

8. To change the strength of 25 ml of 0.15 M HCl to 0.1 M the amount of water that is to be added is
(a) 25 mL (b) 20.0 mL
(c) 12.5 mL (d) 37.5 mL

9. The molarity of pure water is
(a) 55.6 (b) 50
(c) 100 (d) 18

10. 2.5 litre of 1 M NaOH solution is mixed with another 3 litre solution of 0.5 M NaOH solution. Then the molarity of the resulting solution is
(a) 0.80 M (b) 1.0 M
(c) 0.73 M (d) 0.50 M

11. At room temperature, the mole fraction of a solution is 0.25 and the vapour pressure of the solvent is 0.80 atm. Then the lowering of vapour pressure is
(a) 0.75 (b) 0.60
(c) 0.20 (d) 0.80

12. The boiling point of a solution of 0.11 g of a substance in 15 g of ether was found to be 0.1°C higher than that of pure ether. The molecular weight of the substance will be (K_b = 2.16)
(a) 148 (b) 158
(c) 168 (d) 178

13. An aqueous solution freezes at – 0.186 °C, (K_f = 1.86, k_b = 0.512). What is the elevation in boiling point?
(a) 0.186 (b) 0.512
(c) 0.80 (d) 0.0512.

14. The elevation of boiling point method is used for the determination of molecular weight of
(a) Non-volatile and soluble solute
(b) Non-volatile and insoluble solute
(c) Volatile and soluble solute
(d) Volatile and insoluble solute

15. Water has a vapour pressure of 23.76 torr at 298 K. What is the vapour pressure of a solution of sucrose, if the mole fraction of sucrose is 0.250?
(a) 5.9 torr (b) 17.8 torr
(c) 23.5 torr (d) 24.01 torr

16. A pair of solutions having the same osmotic pressure under a given set of experimental conditions is called:
(a) Isobaric (b) Isotonic
(c) Isomeric (d) None of these

17. 1.0 mole of heptane (*V.P.* = 92 torr) was mixed with 4.0 mole of octane (*V.P.* = 31 torr). The vapour pressure of the resulting ideal solution is
(a) 46.2 torr (b) 43.2 torr
(c) 61 torr (d) 6.02 torr

18. The freezing point of equimolar aqueous solutions will be highest for
(a) $C_6H_5NH_3^+Cl^-$ (b) $Ca(NO_3)_2$
(c) $NaNO_3$ (d) $C_6H_{12}O_6$

19. What will be the freezing point of an aqueous solution which contains 50.0 g ethylene glycol in 700 g of water (k_f = 1.86)?
(a) 283.15 K (b) 272.08 K
(c) 271.01 K (d) 261.15 K

20. *Y* g of a non-volatile organic compound of molar mass *M* is dissolved in 250 g benzene. Molal boiling point elevation constant of benzene is k_b. Elevation in the boiling point of benzene is given by
(a) $\frac{4M}{K_bY}$ (b) $\frac{4K_bY}{M}$
(c) $\frac{K_bY}{4M}$ (d) $\frac{K_bY}{M}$

21. 6.0 g of urea is dissolved in 90.0 g water. The relative lowering of vapour pressure is equal to
(a) 0.196 (b) 0.392
(c) 0.0196 (d) 1.96

22. Which of the following statements is not true for solutions which show negative deviation from Raoult's law?
(a) *A*...*B* forces less than *A*...*A* or *B*...*B* forces
(b) *A*...*B* forces greater than *A*...*A* or *B*...*B* forces
(c) Dissolution process is exothermic.
(d) Solubility decreases on increasing the temperature.

23. The vapour pressure of a solution (*P*) and the vapour pressure of the solvent (P°) are related to each other as (χ_1 is the mole fraction of solvent)
(a) $P = P^\circ\chi_2$ (b) $P = P^\circ\chi_1$
(c) $P^\circ = P\chi_1$ (d) $P^\circ = P\chi_2$.

24. The solubility of a gas in water depends upon
(a) Nature of the gas
(b) Temperature
(c) Pressure of the gas
(d) All of the above.

25. A solution contains m_2g of a non volatile solute of molecular mass M_2. Which of the following can be used to calculate the molecular mass of solute in terms of osmotic pressure?
(a) $M_2 = \left[\frac{m_2}{\pi}\right]VRT$ (b) $M_2 = \left[\frac{m_2}{V}\right]\frac{RT}{\pi}$
(c) $M_2 = \left[\frac{m_2}{V}\right]nRT$ (d) $M_2 = \left[\frac{m_2}{V}\right]\frac{\pi}{RT}$

26. Which one of the following aqueous solution will have the lowest freezing point?
(a) 0.1 m sucrose (b) 0.1 m CH_3COOH
(c) 0.1 m NaCl (d) 0.1 m $CaCl_2$

27. The molal freezing point constant for water is 1.86. If 342 gm of cane sugar ($C_{12}H_{22}O_{11}$) are dissolved in 1000 g water, the solution will freeze at:
(a) –1.86°C (b) 1.86°C
(c) 3.72°C (d) –3.42°C

28. Assuming salts to be 90% dissociated, which of the following will have highest osmotic pressure?

(a) Decimolar $Al_2(SO_4)_3$

(b) Decimolar $BaCl_2$

(c) Decimolar Na_2SO_4

(d) A solution obtained by mixing equal volumes of (b) and (c) and filtering

29. Which of the following aqueous solutions has maximum freezing point?

(a) 0.01 M NaCl (b) 0.005 M C_2H_5OH

(c) 0.005 M MgI_2 (d) 0.01 M $MgSO_4$

30. Which one of the following pairs of solution can we expect to be isotonic at the same temperature?

(a) 0.1 M urea and 0.1 M NaCl

(b) 0.1 M urea and 0.2 M $MgCl_2$

(c) 0.1 M NaCl and 0.1 M Na_2SO_4

(d) 0.1 M $Ca(NO_3)_2$ and 0.1 M Na_2SO_4

31. Azeotropic mixtures are

(a) Constant boiling mixtures

(b) Those which boil at different temperatures

(c) Mixture of two solids

(d) None of the above.

32. Which of the following is not a colligative property?

(a) ΔT_f (b) π

(c) ΔT_b (d) K_b.

33. What is the molality of C_2H_5OH in water solution which will freeze at –10°C? [mol. *wt.* of C_2H_5OH = 46, K_f for water = 1.86]

(a) 6.315 m (b) 63.15 m

(c) 3.540 m (d) 5.3 m

34. At high altitudes, the boiling point of water decreases because

(a) Atmospheric pressure is low

(b) Temperature is low

(c) Atmospheric pressure is high

(d) Water is cold

35. For determining the molar mass of colloids, polymers and protein which colligative property is used

(a) Diffusion pressure (b) Atmospheric pressure

(c) Osmotic pressure (d) Turgor pressure.

36. Two solutions have different osmotic pressures. The solution of higher osmotic pressure is called

(a) Isotonic solution (b) Hypertonic solution

(c) hypotonic solution (d) None of these

37. *A* 0.6% solution of urea (mol. *wt.* 60) would be isotonic with

(a) 0.1 M glucose

(b) 0.1 M NaCl

(c) 0.6% NaCl

(d) 0.6% glucose

38. The boiling points of C_6H_6, CH_3OH, $C_6H_5NH_2$ and $C_6H_5NO_2$ are 80°C, 65°C, 184°C and 212°C respectively. Which of the following will have highest vapour pressure at room temperature?

(a) C_6H_6 (b) CH_3OH

(c) $C_6H_5NH_2$ (d) $C_6H_5NO_2$

39. The law which gives relationship between solubility of a gas and pressure is

(a) Boyle's law (b) Charle's law

(c) Henry's law (d) Raoult's law

40. At a certain temperature, the vapour pressure of water is 90 mm. At the same temperature the vapour pressure of solution containing non volatile solute is 81 mm. The relative lowering of vapour pressure of the solution is

(a) 0.1 (b) 0.05

(c) 1.5 (d) 0.15

41. The degree of dissociation (α) and vant Hoff factor, *i* of a substance are related as

(a) $\alpha = \frac{i-1}{n-1}$ (b) $\alpha = \frac{i}{n-1}$

(c) $\alpha = \frac{n-1}{i-1}$ (d) $\alpha = \frac{n}{i-1}$

42. If 0.1 M solution of glucose and 0.1 M solution of urea are placed on two sides of the semipermeable membrane to equal heights, then it will be correct to say that

(a) There will be no net movement across the membrane

(b) Glucose will flow towards urea solution

(c) Urea will flow towards glucose solution

(d) Water will flow from urea solution to glucose solution.

43. The molal freezing point constant for water is 1.86° C/m. Therefore, freezing point of 0.1 M NaCl solution in water is expected to be

(a) –1.86°C (b) –0.372°C

(c) –0.186°C (d) 0.372°C.

44. Which one of the following salts will have the same value of vant Hoff's factor (*i*) as that of $K_4[Fe(CN)_6]$?

(a) $Al_2(SO_4)_3$ (b) NaCl

(c) $Al(NO_3)_3$ (d) Na_2SO_4.

45. An aqueous solution containing 1 g of urea boils at 100.25°C. The aqueous solution containing 3 g of glucose in the same volume will boil at

(a) 100.75°C (b) 100.5°C

(c) 100°C (d) 100.25°C

46. When the vapour pressure of solution of two liquids is less than that expected from ideal solution, it is said to show

(a) Positive deviation from ideal behaviour

(b) Negative deviation from ideal behaviour

(c) Positive deviation for lower concentrations and negative deviation for higher concentrations

(d) None of these

47. The Vant' Hoff factor '*i*' for a dilute aqueous solution of glucose is

(a) zero (b) 1

(c) less than 1 (d) more than 1.

48. Two solutions which have the same value of osmotic pressure at 298 K must have same

(a) concentration in *g*/L

(b) molality

(c) molarity

(d) normality

49. The concentration units independent of temperature would be

(a) Normality (b) Mass-volume percent

(c) Molality (d) Molarity.

50. According to Raoult's law, the relative lowering of vapour pressure of solution is equal to the

(a) Mole fraction of solute
(b) Moles of solute
(c) Mole fraction of solvent
(d) Moles of solvent.

51. Which of the following solutions will have the highest boiling point?

(a) 0.5 molal $BaCl_2$
(b) 1.0 molal KBr
(c) 1.8×10^{24} glucose molecules per litre of solution
(d) 60 g urea (NH_2CONH_2) in 1 litre solution

52. The Arabian sea contains 4600 ppm bromide ion. What is the molality of the ion?

(a) 0.058 m
(b) 0.58 m
(c) 1.16 m
(d) 1.32 m

53. Osmotic pressure can be increased by

(a) Increasing temperature
(b) decreasing temperature
(c) increasing volume
(d) none of these

54. Lowering in vapour pressure is the highest for

(a) 0.2 M urea
(b) 0.1 M glucose
(c) 0.1 M $MgSO_4$
(d) 0.1 M $BaCl_2$

55. The relationship between osmotic pressure at 273 K when 10 g glucose (P_1), 10 g urea (P_2) and 10 g sucrose (P_3) are dissolved in 250 mL of water is

(a) $P_1 > P_2 > P_3$
(b) $P_3 > P_1 > P_2$
(c) $P_2 > P_1 > P_3$
(d) $P_2 > P_3 > P_1$

56. Vant Hoff factor of $Ca(NO_3)_2$ is

(a) 1
(b) 2
(c) 3
(d) 4.

57. Which of the following 0.10 m aqueous solution will have the lowest freezing point?

(a) $Al_2(SO_4)_3$
(b) $C_6H_{12}O_6$
(c) $C_{12}H_{22}O_{11}$
(d) KI

58. Which of the following will have the highest boiling point at 1 atm pressure?

(a) 0.1 M NaCl
(b) 0.1 M Sucrose
(c) 0.1 M $BaCl_2$
(d) 0.1 M Glucose

59. What will be the molality of a solution having 18 g of glucose (Mol. wt. 180) dissolved in 500 g of water?

(a) 1 m
(b) 0.5 m
(c) 0.2 m
(d) 2 m

60. Vapour pressure of CCl_4 at 25°C is 143 mm Hg. 0.5 g of a non volatile solute (mol weight 65) is dissolved in 100 mL CCl_4. Find the vapour pressure of the solution. (Density of CCl_4 = 1.58 g/cm³)

(a) 141.93 mm
(b) 94.39 mm
(c) 199.34 mm
(d) 143.99 mm

61. The relative lowering of vapour pressure is equal to the mole fraction of the solute. This law is called

(a) Henry's law
(b) Raoult's law
(c) Ostwald's law
(d) Arrhenius law

62. *A* 5% solution of cane sugar (*MW* = 342) is isotonic with 1% solution of substance *X*. The molecular weight of *X* is

(a) 171.2
(b) 68.4
(c) 34.2
(d) 136.2

63. The molar solution of sulphuric acid is equal to

(a) *N* solution
(b) 2 *N* solution
(c) *N*/2 solution
(d) 3 *N* solution

64. Which of the following is not a colligative property?

(a) Osmotic pressure
(b) Elevation of boiling point
(c) Depression in freezing point
(d) increase in freezing point

65. 1.8 g of fructose ($C_6H_{12}O_6$) is added to 2 kg of water. The freezing point of the solution

(a) 0.186°C
(b) – 0.0186°C
(c) –0.0093°C
(d) 0.0093°C

ANSWERS

1. (b)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (c)
11. (c)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (b)
21. (c)	**22.** (a)	**23.** (b)	**24.** (d)	**25.** (b)	**26.** (d)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (d)
31. (a)	**32.** (d)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (a)
41. (a)	**42.** (a)	**43.** (b)	**44.** (a)	**45.** (d)	**46.** (b)	**47.** (b)	**48.** (c)	**49.** (c)	**50.** (a)
51. (c)	**52.** (a)	**53.** (a)	**54.** (d)	**55.** (c)	**56.** (c)	**57.** (a)	**58.** (c)	**59.** (c)	**60.** (a)
61. (b)	**62.** (b)	**63.** (b)	**64.** (d)	**65.** (c)					

5 Chemical Thermodynamics

BRIEF REVIEW OF THE CONCEPTS

Thermodynamics is the study of energy changes accompanying physical and chemical processes.

System Any part of the universe selected for a particular investigation separated from its surroundings by real or imaginary boundary is called a system.

Surroundings is every thing in the universe that is not the part of the system.

State functions or State variables are variables such as temperature, pressure and volume whose values depend only on the state of the system and not on the path followed to change the state i.e., from initial to final. q and w are not state functions.

Isolated System is one which can exchange neither energy nor matter with the surroundings e.g., hot or cold water in a thermos flask.

Closed System can exchange only energy but not matter with the surroundings e.g., hot or cold water in a closed bottle.

Open System can exchange both energy and matter with the surroundings.

Isothermal process is a process which takes place at a constant temperature, i.e., $\Delta T = 0$.

Adiabatic process takes place when there is no heat exchange ($\Delta q = 0$) between the system and the surroundings, though temperature is allowed to change.

Isobaric process takes place at constant pressure i.e., $\Delta P = 0$.

Isochoric process takes place at constant volume i.e., $\Delta V = 0$.

Intensive property is one whose value does not depend on size, volume or amount (simple test: it does not change on division) e.g., temperature, density, refractive index, surface tension etc.

Extensive property is one whose value depends upon the size, volume of the system i.e., value changes on division e.g., mass, energy, entropy, enthalpy etc.

Reversible process. In a reversible process the change is carried out very slowly so that the system and the surroundings are always at equilibrium. The driving force is only infinitesimally greater than the opposing force.

Irreversible process. If the change is carried out rapidly and the system does not have a chance to attain equilibrium then the process is called irreversible. All natural processes are thermodynamically irreversible and hence spontaneous.

Work is defined as force (F) multiplied by distance (d) through which the force acts. The work w, is given by

$$w = F.d$$

where F is the force and d is the distance through which body moves.

Pressureor **volume work, w** is equal to pressure × change in volume

$$w = P\Delta V$$

w is positive if work is done on the system and negative when the system does work on the surroundings.

Internal energy, U is the total energy of the system. It is made up of

(i) Transitional energy of molecules,
(ii) Rotational energy of molecules,
(iii) Vibrational energy of molecules,
(iv) Interaction energy of the constituent particles and
(v) Coulombic energy between the electrons and the nuclei in the atoms.

The absolute value of the internal energy of a system cannot be determined but the change in internal energy, i.e. ΔU of a process can be measured,

$$\Delta U = U_{\text{final}} - U_{\text{initial}}$$

Internal energy is a state function and depends on temperature.

First law of thermodynamics (also known as law of conservation of energy). "Energy can neither be created nor destroyed".

Or

"The total energy of the universe is constant".

Mathematical expression

$$\Delta U = q + w$$

q is positive if the system absorbs heat, w is positive, when the work is done on the system.

Both the processes increase internal energy.

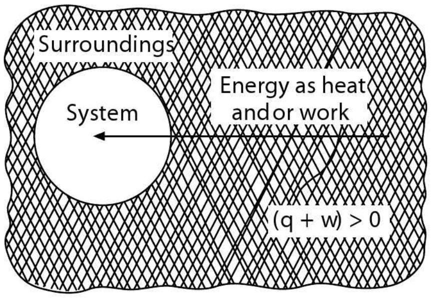


Fig. 5.1

Since $w = P\Delta V$ and at constant volume, $\Delta V = 0$,

Thus $\Delta U = q_v$ i.e., heat absorbed at constant volume.

Enthalpy or Heat content (*H*). It is the sum of internal energy and the pressure volume work. Thus

$$H = U + PV$$

Enthalpy is a state function. Change in enthalpy at constant pressure is given by

$$\Delta H = \Delta U + P\Delta V$$

Heat absorbed at constant pressure, q_p is given by

$$q_p = \Delta H$$

Exothermic process is one that releases heat from the system to the surroundings. ΔH is negative for an exothermic process.

Endothermic process is one in which the system absorbs heat from the surroundings. ΔH is positive for an endothermic process.

Note: Earlier the symbol used for internal energy was '*E*' *but* now '*U*' is used for it. You may find *E* in many questions given at the end of this chapter because, these questions have been taken from previous years examination papers.

Sign Conventions

"Whenever anything comes out of the system, it is negative and whenever anything goes into the system it is positive".

q is positive when a system absorbs heat.

q is negative when a system loses heat.

ΔH is positive when heat is absorbed by a system i.e., for an endothermic process.

ΔH is negative when heat is evolved i.e., process is exothermic.

w is positive when work is done on the system.

w is negative when system does work.

Hess's law of constant heat summation "The enthalpy change, ΔH, for a chemical or physical process is always same whether the process is carried out in one step or in multiple steps.

Relation between ΔU and ΔH

$$\Delta H = \Delta U + P\Delta V$$

$$PV = n_g RT$$

$$P\Delta V = \Delta n_g RT$$

or $$\Delta H = \Delta U + \Delta n_g RT$$

Δn_g = number of moles of gaseous products–number of moles of gaseous reactants.

When $\Delta V = 0$

or $\Delta n_g = 0$

$$\Delta H = \Delta U$$

Standard state of any substance is the physical state of its pure form at any specified temperature.

Standard enthalpy of formation, ΔH_f° is the enthalpy change or the amount of heat absorbed or evolved when one mole of the substance, in its standard state, is formed from its constituent elements present in their standard states.

The standard enthalpy of formation of an element, ΔH_f° is taken as zero (by convention).

Relation between enthalpy change, ΔH and standard enthalpy of formation of substances, ΔH_f°

$$\Delta H = \Sigma\Delta H_f^\circ \text{ (products) } -\Sigma\Delta H_f^\circ \text{ (reactants)}$$

Enthalpy of combustion is the enthalpy change when one mole of the given substance is completely burnt in excess of air or O_2 (*g*) in standard state i.e., under a pressure of 1 bar and temperature 298 K.

All combustion reactions are exothermic.

Enthalpy of vaporization ΔH°_{vap}, is the enthalpy change when 1 mole of the substance in the liquid state changes to the vapour state at 298 K and 1 atm (1 bar) pressure.

Enthalpy of neutralization, ΔH is the enthalpy change when 1 gram equivalent of an acid is completely neutralised by 1 gram equivalent of a base in dilute aqueous solution.

It is also taken as the enthalpy change for the reaction

$$H^+ + OH^- \rightarrow H_2O$$

i.e., when 1 mole of H^+ ions combine with 1 mole of OH^- ions to from 1 mole of undissociated water (H_2O) molecules.

Bond energy. The bond energy is the average amount of energy, per mole, required to break a particular type of bond in a gaseous molecule and separate the resulting atoms or radicals from one another. For example, water has two bonds i.e., H–O–H which dissociate in two steps.

(i) H–O–H (g) → H(g) + OH (g) $\Delta H = 498$ kJ mol^{-1}

(ii) OH (g) → O(g) + H (g) $\Delta H = 430$ kJ mol^{-1}

Bond energy of H—O bond in water is average of these two energies i.e., $(498 + 430)/2 = 464$ kJ mol^{-1}, In case of diatomic molecules it is equal to dissociation energy.

Bond dissociation energy is the amount of energy, per mole required to break a given bond in a gaseous molecule and separate the resulting atoms or molecules from one another.

Distinction between bond dissociation energy and bond energy Energy is always required to break a bond, hence bond breaking is always endothermic. Consider breaking of various bonds in methane which takes place in four successive steps,

(i) CH_3 –H(g) → CH_3 (g) + H(g) ΔH° = akJmol^{-1}

(ii) CH_3 (g) → CH_2 (g) + H(g) ΔH° = bkJmol–1

(iii) CH_2 (g) → CH(g) + H(g) ΔH° = ckJmol^{-1}

(iv) CH(g) → C(g) + H (g) ΔH° = dkJmol^{-1}

Each of the above 4 steps gives dissociation energy of a particular type of bond e.g., in CH_4 gaseous molecule the **bond dissociation energy** for C-H bond to from CH_3 free radical and H radical both in the gaseous state is a kJ mol^{-1} whereas the bond energy of C–H bond in CH_4 molecule is the average of all the four dissociation energies. In this case bond energy = $\left(\frac{a+b+c+d}{4}\right)$kJ mol^{-1}.

For a diatomic molecule like H_2, N_2 both bond energy and dissociation energy are equal.

Heat capacity is the heat required to raise the temperature of the system by one degree. For one mole of a pure substance it is called molar heat capacity.

Heat capacity at constant volume, $Cv = \left[\frac{\partial U}{\partial T}\right]_v$

Heat capacity at constant pressure, $Cp = \left[\frac{\partial H}{\partial T}\right]_p$

For an ideal gas, $Cp - Cv = R$

Entropy (S) is a measure of disorder within a system. Higher the value of entropy more is the disorder in the system.

The change in entropy, ΔS for an isothermal change is defined as

$$\Delta S = \frac{q_{rev}}{T}$$

here q_{rev} is the heat absorbed reversibly at constant temperature, T.

Second law of thermodynamics states "the entropy of the universe is constantly increasing".

Or

The entropy of the universe increases in the course of every spontaneous (natural) change.

Entropy change in phase transformations.

$$\Delta S_{\text{vap}} = \frac{\Delta H_{vap}}{T}$$

$$\Delta S_{\text{fusion}} = \frac{\Delta H_{\text{fusion}}}{T}$$

Gibb's free Energy, G is the amount of energy available to a system to do useful work, It is defined as

$$G = H - TS$$

and at constant temperature, and pressure the change in free energy, ΔG, is given by

$$\Delta G = \Delta H - T\Delta S$$

(This equation is valid at constant temperature and pressure). The reaction is spontaneous if $\Delta G < 0$; at equilibrium $\Delta G = 0$.

Table 5.1

ΔH	ΔS	ΔG	*Remarks*
–	+	–	Reaction will be spontaneous at all temperatures.
+	–	+	Reaction will be non-spontaneous at all temperatures.
–	–	–if $\lvert\Delta H\rvert > \lvert T\Delta S\rvert$	Reaction will be spontaneous
		+ if $\lvert\Delta H\rvert < \lvert T\Delta S\rvert$	Reaction will be non-spontaneous
+	+	+if $\lvert\Delta H\rvert > \lvert T\Delta S\rvert$	Reaction will be non-spontaneous
		– if $\lvert\Delta H\rvert < \lvert T\Delta S\rvert$	Reaction will be spontaneous

Standard free energy change, ΔG^o for a reaction is given by $\Delta G^o = \Sigma\Delta G^o_f$ (products) – $\Sigma\Delta G^o_f$ (reactants)

ΔG^o_f is the standard free energy change of formation of a compound.

The standard free energy change, ΔG^o and equilibrium constant, K for a reaction are related as

$$\Delta G^o = -R\ T \text{ in } K$$

or $$\Delta G^o = -2.303\ R\ T \log K$$

Free energy change ΔG and emf of cell E_{cell} are related as

$$\Delta G = -nFE_{\text{cell}}$$

or $$\Delta G^o = -nFE^o_{\text{cell}}$$

Third law of thermodynamics. At absolute zero the entropy of a perfectly crystalline substance is taken as zero.

Standard entropy changes ΔS^o and standard entropy S^o of substances are related as,

$$\Delta S^o \text{ (for a reaction)} = \Sigma S^o \text{ (product)} - \Sigma S^o \text{ (Reactants)}$$

Solved Problems

1. What is the change in internal energy for the process in which 670 J of heat is absorbed by a system while 285 J of work is done by the system?

Solution Here $q = +670$ J because system absorbs heat, work is done by the system, therefore $w = -285$ J.

$$\Delta U = q + w$$

Substituting the value of q and w, we have

$$\Delta U = +670\text{ J} - 285\text{ J}$$
$$= 385\text{ J}$$

2. Calculate standard enthalpy change for the reaction:

$$SO_2(g) + 2H_2S(g) \rightarrow 3S(s) + 2H_2O(l)$$

Given that $\Delta H^o_f\ SO_2(g) = -295.7$ kJ mol^{-1}

$\Delta H^o_f\ H_2S(g) = -20.2$ kJ mol–1

$\Delta H^o_f\ H_2O\ (l) = -286.7$ kJ mol^{-1}

Solution ΔH^o (reaction) = $\Sigma\Delta H^o_f$ (products) – $\Sigma\Delta H^o_f$ (reactants)

or $\Delta H^o = \{3\Delta H^o_f S(s) + 2\Delta H^o_f H_2O\ (l)\}$

$- \{\Delta H^o_f\ SO_2(g) + 2\Delta H^o_f H_2S(g)\}$

$= (3 \times 0 + 2 \times -286.7) - (-295.7 + 2 \times -20.2)$

$= -237.3$ kJ mol^{-1}

3. Ethanol boils at 78.4°C and the standard enthalpy of vaporization of ethanol is 42.4 kJ mol^{-1}. Calculate the entropy of vaporization of ethanol.

Solution Given $T = 273 + 78.4 = 351.4$ K

$\Delta H^o_{\text{vap}} = 42.4$ kJ mol^{-1}

$\Delta S^o_{\text{vap}} = ?$

$$\Delta S^o_{\text{vap}} = \frac{\Delta H^o_{\text{vap}}}{T} = \frac{42.4 \times 10^3 \text{J mol}^{-1}}{351.4\text{ K}}$$

$= 120.66$ JK^{-1} mol^{-1}.

4. Calculate ΔH°, the enthalpy change for the following reaction at 298 K:

$$2\,H_2\,O_2(l) \rightarrow 2H_2O(l) + O_2(g)$$

Given ΔH°_f for $H_2\,O_2(l) = -188$ kJ mol^{-1}

ΔH°_f for $H_2O\,(l) = -\,288$ kJ mol^{-1}

ΔH°_f for $O_2\,(g) = 0$ kJ mol^{-1}

Solution ΔH° for a reaction is given by

$$\Delta H^\circ = \Sigma\Delta H^\circ_f\,(\text{products}) - \Sigma\Delta H^\circ_f\,(\text{Reactants})$$

$$\Delta H^\circ = \{2 \times \Delta H^\circ_f\,H_2O(l) + \Delta H^\circ_f\,O_2(g)\} - 2 \times \Delta H^\circ_f\,H_2\,O_2(l)$$

Substituting the values

$$\Delta H^\circ = (2 \times -288 \text{ kJ mol}^{-1} + 0) - (2 \times -188 \text{ kJ mol}^{-1})$$

$$= -200 \text{ k J mol}^{-1}$$

5. Calculate the value of ΔH° for the reaction:

$$2H_2S(g) + 3O_2(g) \rightarrow 2H_2O(l) + 2SO_2(g)$$

Given $\Delta H^\circ_f H_2S(g) = -20.2$ kJ mol^{-1}

$\Delta H^\circ_f\,H_2O(l) = -286.0$ kJ mol^{-1}

$\Delta H^\circ_f\,SO_2(g) = -296.9$ kJ mol^{-1}

Solution ΔH° (reaction) = $\Sigma\Delta H^\circ_f$ (products) – $\Sigma\Delta H^\circ_f$ (Reactants)

$$= \{2 \times \Delta H^\circ_f\,H_2O(l) + 2 \times \Delta H^\circ_f\,SO_2(g)\} - \{2 \times \Delta H^\circ_f\,H_2S(g) + 3 \times \Delta H^\circ_f\,O_2(g)\}$$

Substituting the values of ΔH°_f

$$\Delta H^\circ = [(2 \times -286.0) + 2 \times -296.9)] - (2 \times -20.2 + 3 \times 0)$$

$$= -1165.8 + 40.4 = -1125.4 \text{ kJ}$$

6. Using the following thermochemical data, calculate the enthalpy of formation of ethanoic acid:

(*i*) $C(s) + O_2(g) \rightarrow CO_2\,(g);\ \Delta H = -394.8$ kJ ...(i)

(*ii*) $H_2(g) + \frac{1}{2}O_2 \rightarrow H_2O(l);\ \Delta H = -285.6$ kJ ...(ii)

(*iii*) $CH_3COOH\,(l) + 2O_2(g) \rightarrow 2CO_2\,(g) + 2H_2O(l);\ \Delta H = -882$ kJ ...(iii)

Solution The expression for ΔH based on eq. (iii) can be written as

$$\Delta H = \{2\Delta H^\circ_f\,CO_2(g) + 2\Delta H^\circ_f H_2O(l)\} - \{\Delta H^\circ_f\,CH_3\,COOH(l) + 2\Delta H^\circ_f\,O_2(g)\} \quad ...(iv)$$

Substituting the values of ΔH and ΔH°_f in eq. (*iv*) we have

$$-882 \text{ kJ} = (2 \times (-394.8 \text{ kJ}) + 2 \times (-285.6 \text{ kJ})) - \{\Delta H^\circ_f\,CH_3\,COOH\,(l) + 0\}$$

or $\Delta H^\circ_f\,CH_3\,COOH(l)$

$$= -1360.8 \text{ kJ} + 882.0 \text{ kJ}$$

$$= -478.8 \text{ kJ mol}^{-1}.$$

7. Calculate the standard heat of formation ΔH°_f, of glucose, $C_6\,H_{12}\,O_6$ (s) at 298 K from the following data:

Heat of combustion of glucose,

$\Delta H_{comb}\ C_6\,H_{12}\,O_6(s) = -2810$ kJ mol^{-1}

$\Delta H^\circ_f\,CO_2(g) = -390.5$ kJ mol^{-1}

$\Delta H^\circ_f\,H_2O(l) = -286.0$ kJ mol^{-1}

Solution The equation for the combustion of glucose can be written as

$$C_6\,H_{12}\,O_6(s) + 6O_2(g) \rightarrow 6CO_2\,(g) + 6H_2O(l)$$

Heat of reaction i.e., combustion in this case,

$$\Delta H_{comb} = \Sigma\Delta H^\circ_f(\text{Products}) - \Sigma\Delta H^\circ_f\,(\text{Reactants})$$

$$= \{6 \times \Delta H^\circ_f\,CO_2(g) + 6 \times \Delta H^\circ_f\,H_2O(l)\} - \{\Delta H^\circ_f C_6\,H_{12}O_6\,(s) + 6\Delta H^\circ_f\,O_2(g)\}$$

Let $\Delta H^\circ_f\,C_6\,H_{12}O_6(s)$ be Q kJ mol^{-1}

Substituting the values of ΔH_{comb} and various heats of formation in the above equation, we get

$$-\,2810 \text{ kJ mol}^{-1} = \{6 \times -390.5 \text{ kJ mol}^{-1} + 6 \times -286.0 \text{ kJ mol}^{-1}\} - \{Q + 6 \times 0\}$$

$$= (-2343.0 \text{ kJ mol}^{-1} - 1716.0 \text{ kJ mol}^{-1}) - Q \text{ kJ mol}^{-1}$$

$$Q = -1249 \text{ kJ mol}^{-1}$$

therefore $\Delta H^\circ_f\,C_6H_{12}O_6(s) = -1249$ KJ mol^{-1}

8. Calculate the enthalpy change, ΔH of decomposition of $CaCO_3$ (s) into CaO (s) and CO_2 (g) from the following data:

$\Delta H^\circ_f\,CaCO_3(s) = -288.5$ kJ mol^{-1}

$\Delta H^\circ_f\,CaO(s) = -152.0$ kJ mol^{-1}

$\Delta H^\circ_f\,CO_2(g) = -94.0$ kJ mol^{-1}

Solution ΔH° (decomposition of $CaCO_3$)

$$= \Sigma\Delta H^\circ_f\,(\text{products}) - \Sigma\Delta H^\circ_f\,(\text{reactants})$$

Substituting the values of ΔH°_f of various substances, we have

$$\Delta H^\circ = \{\Delta H^\circ_f\,CaO(s) + \Delta H^\circ_f\,CO_2\,(g)\} - \Delta H^\circ_f\,CaCO_3\,(s)$$

$$= 152.0 \text{ kJ mol}^{-1} + (-94.0) \text{ kJ mol}^{-1} - (-288.5) \text{ kJ mol}^{-1}$$

$$= +\,42.5 \text{ kJ mol}^{-1}$$

9. For a reaction $M_2O(s) \rightarrow 2M(s) + 1/2\ O_2(g)$; ΔH = 30 kJ mol^{-1} and $\Delta S = 0.07$ kJ K^{-1} mol^{-1} (at 1 atm)

Calculate upto which temperature the reaction would not be spontaneous.

Solution For a reaction to be spontaneous ΔG, should be negative i.e., $\Delta G < 0$.

For the given reaction

$\Delta H = 30$ kJ mol^{-1} and

$\Delta S = 0.07$ kJ K^{-1} mol^{-1}

$\Delta G = \Delta H - T\Delta S$

Reaction will be at equilibrium when $\Delta G = 0$.

or $T\Delta S = \Delta H$

or $T = \dfrac{\Delta H}{\Delta S}$

Substituting the values

$$T = \frac{30 \text{ kJ mol}^{-1}}{0.07 \text{ kJ } K^{-1} mol^{-1}} = 428.5 \text{ K}$$

The reaction would not be spontaneous below 428.5K.

10. For the reaction

$$2\,NO(g) + O_2(g) \rightarrow 2NO_2(g)$$

calculate ΔG at 700 K when enthalpy and entropy changes (ΔH and ΔS) are –113.0 kJ mol^{-1} and –145 JK^{-1} mol^{-1} respectively.

Solution

Given $T = 700$ K

$\Delta H = -113.0$ kJ mol^{-1}

$$= -113.0 \times 10^3 \text{ J mol}^{-1}$$

$$\Delta S = -145 \text{ JK}^{-1} \text{ mol}^{-1}$$

ΔG is given by

$$\Delta G = \Delta H - T\Delta S$$

Substituting the values

$$\Delta G = -113.0 \times 10^3 \text{ J mol}^{-1} - 700 \text{ K} \times (-145 \text{ JK}^{-1} \text{ mol}^{-1})$$

$$= -113.0 \times 10^3 \text{ J mol}^{-1} + 101.5 \times 10^3 \text{ J mol}^{-1}$$

$$= -11.5 \text{ kJ mol}^{-1}$$

11. ΔH and ΔS for vaporization of water at 1 atm pressure are 40.63 kJ mol^{-1} and 108.8 JK^{-1} mol^{-1} respectively. Calculate the temperature at which the free energy change for this transformation will be zero.

Solution $\Delta H = 40.63 \text{ kJ mol}^{-1} = 40630 \text{ J mol}^{-1}$

$$\Delta S = 108.8 \text{ JK}^{-1} \text{ mol}^{-1}$$

When $\Delta G = 0$ then T is given by

$$T = \frac{\Delta H}{\Delta S} = \frac{40630 \, J \text{ mol}^{-1}}{108.8 \, JK^{\ 1} \text{ mol}^{-1}}$$

$$= 373.4 \text{ K}$$

12. Enthalpy and entropy changes of a reaction are 40.63 kJ mol^{-1} and 108.8 JK^{-1} mol^{-1} respectively. Predict the feasibility of the reaction at 27°C.

Solution Given $\Delta H = 40630 \text{ J mol}^{-1}$

$$\Delta S = 108.8 \text{ JK}^{-1} \text{ mol}^{-1}$$

$$T = 27°\text{C} = 273 + 27 = 300 \text{ K}$$

At 300 K the value of ΔG is calculated,

$$\Delta G = \Delta H - T\Delta S$$

$$= 40630 \text{ J mol}^{-1} - 300 \text{ K} \times 108.8 \text{ JK}^{-1} \text{ mol}^{-1}$$

$$= (40630 - 32640) \text{ J mol}^{-1}$$

$$= 7990 \text{ J mol}^{-1}$$

Since ΔG is positive, the reaction is not feasible.

13. Will the reaction

$$I_2 \text{ (s)} + H_2S(g) \rightleftharpoons 2HI \text{ (g)} + S(s)$$

proceed spontaneously in the forward direction at 298 K.

$$\Delta G^o_f \, HI = 1.8 \text{ KJ mol}^{-1}$$

$$\Delta G^o_f H_2 \text{ S} = -33.8 \text{ kJ mol}^{-1}$$

Solution $\Delta G^o = \Sigma\Delta G_f^o \text{ (products)} - \Sigma\Delta G^o_f \text{(reactants)}$

$$= \{2 \times \Delta G^o_f \text{ HI(g)} + \Delta G^o_f \text{ S(s)}\} - \{\Delta G^o_f I_2\text{(s)} + \Delta G^o_f H_2S\text{(g)}\}$$

Substituting the values

$$\Delta G^o = \{2 \times 1.8 \text{ kJ mol}^{-1} + 0) - (0 - 33.8 \text{ kJ mol}^{-1})$$

(The standard free energy of formation ΔG^o_f of an element in its standard state is zero).

$$= (3.6 + 33.8) \text{ kJ}$$

$$= 37.4 \text{ kJ}$$

Since ΔG^o for the above reaction is positive. The reaction will not proceed spontaneously in the forward direction.

14. The value of K for the following reaction is 4 at 25°C. $CH_3COOH(l) + C_2H_5OH(l) \rightarrow CH_3COOC_2H_5(l) + H_2O(l)$ is this reaction spontaneous at 25°C.

$$R = 8.31 \text{ JK}^{-1}$$

Solution $\Delta G^o = -2.303 \, RT \log \text{K}$

$$T = 25°\text{C} = 298 \text{ K}$$

Substituting the values in the above expression gives

$$\Delta G^o = -2.303 \times 8.31 \text{ JK}^{-1} \text{ mol}^{-1} \times 298 \text{ K} \times \log 4$$

$$= -(2.303 \times 8.31 \times 298 \times 0.621) \text{ J mol}^{-1}$$

$$= -3433 \text{ J mol}^{-1}$$

Since the value of ΔG^o is negative, the above reaction is spontaneous at 25°C.

15. For the reaction $NH_4 \text{ Cl(s)} \rightarrow NH_3(g) + HCI(g)$, at 25°C, enthalpy change, $\Delta H^o = +177$ kJ mol^{-1} and entropy change, $\Delta S^o = +285$ J $\text{mol}^{-1}\text{K}^{-1}$. Predict whether the reaction is spontaneous or not.

Solution Reaction is $NH_4Cl(s) \rightarrow NH_3(g) + HCl(g)$

Given $T = 25°\text{C} = 298 \text{ K}$

$$\Delta H^o = +177 \text{ kJ mol}^{-1}$$

$$\Delta S^o = +285 \text{ J mol}^{-1} \text{ K}^{-1}$$

$$= 285 \times 10^{-3} \text{ kJ mol}^{-1} \text{ K}^{-1}$$

$$\Delta G^o = \Delta H^o - T\Delta S^o$$

$$= 177 \text{ kJ mol}^{-1} - 298 \text{ K} \times 285 \times 10^{-3} \text{ kJK}^{-1} \text{ mol}^{-1}$$

$$= 177 \text{ kJ mol}^{-1} - 84.93 \text{ kJ mol}^{-1}$$

$$= 92.07 \text{ kJ mol}^{-1}$$

Since $\Delta G^o > 0$ the reaction is not spontaneous.

Problems for Practice

1. Which of the following is an intensive property?

(a) Enthalpy (b) Heat capacity

(c) Surface tension (d) Volume

2. Which of the following is a state function?

(a) q (b) w

(c) T (d) q/w

3. At constant pressure q equals

(a) ΔE (b) ΔH

(c) ΔG (d) $P\Delta V$

4. For which of the following substances, the enthalpy of formation in the standard state is zero?

(a) O_2 (g) (b) H_2O (l)

(c) CO_2 (g) (d) NaCI (s)

5. For an endothermic reaction where ΔH represents the enthalpy change of the reaction in kJ/mol, the minimum value for the energy of activation will be
(a) less than ΔH (b) Zero
(c) more than ΔH (d) equal to ΔH

6. Vaporization is an example of a process for which
(a) ΔH and ΔS are positive at all temperatures.
(b) ΔH and ΔS are negative at high temperatures.
(c) ΔH and ΔG are positive at all temperature.
(d) ΔS and ΔG are positive at all temperature.

7. Bomb calorimeter experiment gives heat of combustion of a substance at constant
(a) Pressure
(b) temperature
(c) volume
(d) temperature and pressure

8. The difference between heats of reaction at constant pressure and constant volume for the reaction at 298 K
$2C_6H_6\,(l) + 15\,O_2(g) \rightarrow 12\,CO_2\,(g) + 6H_2O(l)$ in kJ is
(a) – 7.43
(b) + 3.72
(c) – 3.72
(d) + 7.43

9. Identify the intensive quantity from the following:
(a) Enthalpy (b) Temperature
(c) Volume (d) Number of moles

10. Evaporation of water is
(a) An exothermic change
(b) An endothermic change
(c) A process where no heat changes occur
(d) A process accompanied by chemical reaction.

11. In which of the following neutralization reaction, the heat of neutralization will be highest?
(a) NH_4OH and H_2SO_4
(b) HCI and NaOH
(c) CH_3COOH and KOH
(d) CH_3COOH and NH_4OH

12. Hess's law deals with
(a) Changes in heat of reaction
(b) Rate of reaction
(c) Equilibrium constant
(d) Influence of pressure on volume of a gas.

13. $\Delta H_{\text{neutralization}}$ is always
(a) positive
(b) negative
(c) zero
(d) can be positive or negative

14. If $C(s) + O_2(g) \longrightarrow CO_2(g);\ \Delta H = r$
and $CO(g) + \frac{1}{2}O_2\ (g) \longrightarrow CO_2(g);\ \Delta H = s$
then the heat of formation of CO is
(a) $s - r$ (b) rs
(c) $r + s$ (d) $r - s$

15. All the naturally occurring processes proceed spontaneously in a direction which leads to
(a) decrease of entropy
(b) increase of enthalpy
(c) increase of free energy
(d) decrease of free energy

16. Consider the following reaction occurring in a automobile engine
$2C_8H_{18}(g) + 25\ O_2(g) \longrightarrow 16CO_2(g) + 18H_2O(g)$
the signs of ΔH, ΔS and ΔG for above reaction would be
(a) +, –, + (b) –, +, –
(c) –, +, + (d) +, +, –.

17. The standard free energy change (ΔG^o) is related to equilibrium constant (K) as
(a) $\Delta G^o = -2.303\ RT\ \text{In}\ K$
(b) $\Delta G^o = 2.303\ RT \log K$
(c) $\Delta G^o = RT \log K$
(d) $\Delta G^o = -2.303\ RT \log K$.

18. An exothermic reaction is one in which the reacting substances
(a) have more energy than the products
(b) have less energy than the products
(c) have the same energy as the products
(d) are at a higher temperature than the products.

19. Heat of neutralization of strong acid against strong base is constant and is equal to
(a) –13.7 kcal (b) –57 kJ
(c) -5.7×10^4 J (d) All of the above

20. The Heats evolved in combustion of rhombic and monoclinic sulphur are – 70960 and –71030 cal mol^{-1} respectively. What will be heat of conversion of rhombic sulphur to monoclinic?
(a) 70960 calories (b) 71030 calories
(c) –70 calories (d) +70 calories.

21. In an electrochemical cell, if E^o is the e.m.f. of the cell involving 'n' mole of electrons, then ΔG^o is
(a) $\Delta G^o = nFE^o$ (b) $\Delta G^o = -nFE^o$
(c) $E^o = nF\Delta G^o$ (d) $\Delta G = nF/E^o$

22. If the enthalpy of vaporisation of water is 186.5 J mol^{-1}, the entropy of its vaporisation will be
(a) 0.5 $JK^{-1}\ mol^{-1}$ (b) 1.0 $JK^{-1}\ mol^{-1}$
(c) 1.5 $JK^{-1}\ mol^{-1}$ (d) 2.0 $JK^{-1}\ mol^{-1}$

23. The heat of combustion of yellow and red phosphorus are –9.91 kJ mol^{-1} and –8.78 kJ mol^{-1} respectively. The heat of transition of yellow $P \longrightarrow$ red P is
(a) –18.69 kJ (b) + 1.13 kJ
(c) + 18.69 kJ (d) –1.13 kJ.

24. For an isothermal physical process. The entropy change can be calculated using
(a) $\Delta S = \dfrac{Q_{\text{rev}}}{\Delta T}$ (b) $\Delta S = \dfrac{\Delta H}{RT}$
(c) $\Delta S = \dfrac{Q_{\text{rev}}}{T}$ (d) $\Delta S = \dfrac{\Delta G}{\Delta T}$

25. For a spontaneous endothermic reaction
(a) $\Delta G > 0$ (b) $\Delta H < 0$
(c) $\Delta S > \dfrac{\Delta H}{T}$ (d) $\Delta G = 0$

26. The standard enthalpy of combustion at 25°C for hydrogen, cyclohexene (C_6H_{10}) and cyclohexane (C_6H_{12}) are –241, –3800 and –3920 kJ/mol respectively. What is the heat of hydrogenation of cyclohexene?

(a) – 121 kJ (b) – 241 kJ

(c) + 121 kJ (d) + 241 kJ

27. What is Δn_{gas} for the combustion of 1 mole of acetic acid (l). When both reactants and products are at 298 K?

(a) – 1 (b) zero

(c) + 1 (d) + 2

28. In which of the following reaction would the heat absorbed at 298 K and 1 atm pressure be ΔH°_f for ethanol (C_2H_5OH)?

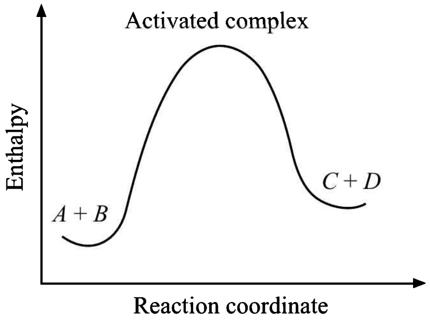
(a) $2C(s) + 3H_2(g) + 1/2\ O_2(g) \rightarrow C_2H_5OH(l)$

(b) $2C(s) + 3H_2(g) + 1/2O_2(g) \rightarrow C_2H_5OH(g)$

(c) $2CO_2(g) + 3H_2O(g) \rightarrow C_2H_5OH(l) + 3O_2(g)$

(d) $2C(s) + 3H_2(g) + O_2(g) \rightarrow C_2H_5OH(l)$

29. The enthalpies associated with the reaction $A + B \rightarrow C + D$ are shown in figure. The fact that the enthalpy of $(C + D)$ is higher than that of $(A + B)$ tells that



(a) the reaction is endothermic

(b) the reaction is exothermic

(c) a catalyst for the reaction is necessary

(d) the activation energy required for the reverse reaction is higher than for the forward reaction

30. For the reaction

$H_2 + I_2 \rightleftharpoons 2HI$, $\Delta H = 12.40$ kcal. The heat of formation (ΔH_f) of HI is

(a) 12.4 kcal (b) –12.4 kcal

(c) –6.20 kcal (d) 6.20 kcal.

31. The free energy change $\Delta G = 0$, when

(a) the system is at equilibrium

(b) catalyst is added

(c) reactants are initially mixed thoroughly

(d) the reactants are completely consumed

32. The heat of formation of the compound in the following reaction is

$$H_2(g) + Cl_2(g) \longrightarrow 2HCl\,(g) + 44 \text{ kcal}$$

(a) –44 kcal mol^{-1}

(b) –22 kcal mol^{-1}

(c) 11 kcal mol^{-1}

(d) –88 kcal mol^{-1}

33. When water is added to quick lime, the reaction is

(a) explosive (b) endothermic

(c) exothermic (d) photochemical

34. $S + \frac{3}{2}O_2 \longrightarrow SO_3 + 2x$ kcal

$SO_2 + \frac{1}{2}O_2 \longrightarrow SO_3 + y$ kcal

Find out the heat of formation of SO_2

(a) $(2x - y)$ (b) $(2x + y)$

(c) $(x + y)$ (d) $2x/y$.

35. The heat of neutralization of strong base and strong acid is –57.0 kJ. The heat released when 0.5 moles of HNO_3 are added to 0.20 moles of NaOH

(a) 57.0 kJ (b) 11.40 kJ

(c) 28.5 kJ (d) 34.9 kJ

36. For a reaction to occur spontaneously

(a) $(\Delta H - T\Delta S)$ must be negative

(b) $(\Delta H + T\Delta S)$ must be negative

(c) ΔH must be negative

(d) ΔS must be negative

37. Identify the correct statement regarding entropy

(a) At absolute zero of temperature, the entropy of perfectly crystalline substance is positive

(b) At absolute zero of temperature entropy of perfectly crystalline substance is taken to be zero

(c) At 0°C the entropy of a perfectly crystalline substance is taken to be zero

(d) At absolute zero of temperature, the entropy of all crystalline substances is taken to be zero

38. The reaction is spontaneous if the cell potential is

(a) Positive (b) Negative

(c) Zero (d) Infinite

39. In an endothermic reaction, the value of ΔH is

(a) Zero (b) Positive

(c) Negative (d) Constant

40. The entropy change for a phase transformation is equal to

(a) Zero (b) $\frac{\Delta H}{T}$

(c) $\Delta G - T\Delta H$ (d) $\Delta H - T\Delta S$

41. The combustion of 0.10 g of methane causes a temperature rise of 2.0°C in a bomb calorimeter that has a heat capacity of 500 joules/°C. What is the value of ΔE per mole of methane for the combustion process?

(a) + 160 kJ (b) + 1kJ

(c) –1 kJ (d) – 160 kJ

42. $CO(g) + \frac{1}{2}O_2(g) \rightarrow CO_2(g)$

The change in internal energy for the above reaction, ΔE is –282 kJ mol^{-1} at 300 K. What is ΔH for the same reaction assuming the gases are ideal?

(a) –1527 kJ mol^{-1} (b) –1245 kJ mol^{-1}

(c) –283 kJ mol^{-1} (d) –281 kJ mol^{-1}

43. What is the standard free energy change, ΔG°, for the reaction at 298 K?

$Zn(s) + Cu^{2+}(aq) \rightarrow Cu(s) + Zn^{2+}(aq)$

$E^\circ = 1.10$ V

(a) −106.15 kJ (b) + 106.15 kJ
(c) −212.3 kJ (d) + 212.3 kJ

44. Heat of combustion ΔH for C(s), H_2 (g) and CH_4 (g) are −94, −68 and −213 k cal/mol.

Then ΔH for $C(s) + 2\ H_2\ (g) \longrightarrow CH_4\ (g)$ is

(a) −17 kcal (b) −111 kcal
(c) −170 kcal (d) −85 kcal

45. What is the sign of ΔG for the process of ice melting at 263 K?

(a) $\Delta G > 0$
(b) $\Delta G = 0$
(c) $\Delta G < 0$
(d) ΔG can have value equal, less or more than zero,

46. In which of the following cases is a reaction possible at all the temperatures?

(a) $\Delta H < 0$; $\Delta S > 0$ (b) $\Delta H > 0$; $\Delta S > 0$
(c) $\Delta H < 0$; $\Delta S < 0$ (d) $\Delta S > 0$; $\Delta H = 0$

47. In which of the following reactions entropy increases?

(a) Crystallization of sucrose from its solution.
(b) Dissolving I_2 in CCl_4
(c) $N_2(g) + 3H_2\ (g) \rightarrow 2NH_3\ (g)$
(d) $2SO_2\ (g) + O_2\ (g) \rightarrow 2SO_3\ (g)$

48. For the reaction

$2Al\ (s) + 3Cl_2\ (g) \rightarrow 2AlCl_3\ (s)$

the entropies of Al (s), Cl_2 (g) and $AlCl_3$ (s) are 28.3, 222.96 and 110.7 $JK^{-1}\ mol^{-1}$ respectively. What is the standard entropy change for the above reaction?

(a) −668.88 J K^{-1} mol^{-1} (b) −504.08 J K^{-1} mol^{-1}
(c) −221.4 J K^{-1} mol^{-1} (d) + 725.48 J K^{-1} mol^{-1}

49. Which of the following statements is false?

(a) $P\Delta V$ can have units of joules.
(b) For an irreversible process, entropy is conserved.
(c) The enthalpy of a perfect crystal at 0K is zero.
(d) For reactions involving only solids or liquids $\Delta H = \Delta E$

50. Iodine sublimes according to the reaction

$$I_2\ (s) \rightleftharpoons I_2\ (g)$$

Which of the following expressions represents the temperature at which solid iodine is in equilibrium with gaseous iodine?

ΔH = change in enthalpy
ΔG = change in free energy
ΔS = change in entropy

(a) $T = \dfrac{\Delta G + \Delta H}{\Delta S}$ (b) $T = \dfrac{\Delta G - \Delta H}{\Delta S}$

(c) $T = \dfrac{\Delta S}{\Delta H}$ (d) $T = \dfrac{\Delta H}{\Delta S}$

51. The maximum possible work from a spontaneous process at isobaric and isothermal conditions is equal to

(a) $\dfrac{\Delta G - \Delta H}{\Delta T}$
(b) change in ethalpy
(c) change in free energy
(d) ΔG° + 2.303 RT

52. What is the standard entropy change in the reaction

$2\ SO_2\ (g) + O_2\ (g) \rightarrow 2\ SO_3\ (g)$

at 298 K; S° for SO_3 (g), SO_2 (g) and O_2 (g) are 61.34, 59.30 and 49.0 cal/mole.

(a) 14.37 (b) 4.10
(c) − 44.92 (d) 57.26

53. A reaction mixture consists of H_2, N_2 and NH_3. At 298 K, the value of ΔG° for the reaction below is

$$N_2\ (g) + 3H_2\ (g) \rightleftharpoons 2NH_3\ (g)$$

(a) $2.303 \times 298\ R \log \dfrac{P^2_{NH_3}}{P_{N_2} \times P^3_{H_2}}$

(b) $2.303\ RT\ \text{In}\ \dfrac{P_{N_2} \times P^3_{H_2}}{P^2_{NH_3}}$

(c) $-2.303 \times 298\ R \log \dfrac{P_{NH_3}}{P_{N_2} \times P_{H_2}}$

(d) $-2.303 \times 298\ R \log \dfrac{P^2_{NH_3}}{P_{N_2} \times P^3_{H_2}}$

54. At 298 K the entropy of diamond is 2.44 J $K^{-1} mol^{-1}$ and the entropy of diamond structure tin is 40.50 JK^{-1} mol^{-1}. As the temperature is reduced to 0K, the difference in entropy between these substances will

(a) decrease to zero
(b) decrease to −2.44
(c) decrease to (40.50 −2.44) J K^{-1} mol^{-1}
(d) remains unchanged

55. $C_2H_6\ (g) + 3.5\ O_2\ (g) \rightarrow 2\ CO_2\ (g) + 3H_2O\ (l)$

Given ΔH°_f (k J mol^{-1}) values for C_2H_6 (g), CO_2 (g) and H_2O (l) are − 85, −394, −286 respectively.

ΔH° for the above reaction is

(a) $\Delta H^\circ = -\ 394 - 286 + 85$
(b) $\Delta H^\circ = -\ 286 - \{2\ (-\ 85) + 3\ (-\ 394)\}$
(c) $\Delta H^\circ = -\ 85 - \{2 \times (-\ 394) + 3 \times (-\ 286)\}$
(d) $\Delta H^\circ = 2\ (-394) + 3\ (-286) - (-85)$

56. Which of the following statements is true?

(a) ΔG may be lesser or greater or equal to ΔH
(b) ΔG is always proportional to ΔH
(c) ΔG is always greater than ΔH
(d) ΔG is always less than ΔH.

57. A heat engine absorbs heat Q_1 at temperature T_1 and heat Q_2 at temperature T_2. Work done by the engine is $(Q_1 + Q_2)$. This data

(a) violates Ist law of thermodynamics
(b) violates Ist law of thermodynamics if Q_1 is negative
(c) violates Ist law of thermodynamics if Q_2 is negative
(d) does not violate first law of thermodynamics.

58. If an endothermic reaction is non-spontaneous at freezing point of water and becomes feasible at its boiling point, then

(a) ΔH is −*ve*, ΔS is + *ve*
(b) ΔH and ΔS both are + *ve*
(c) ΔH and ΔS both are − *ve*
(d) ΔH is + *ve* and ΔS is − *ve*

59. Enthalpy of

$CH_4 + \frac{1}{2}O_2 \longrightarrow CH_3OH$ is negative.

If enthalpy of combustion of methane and CH_3OH are x and y respectively, then which relation is correct?

(a) $x > y$ (b) $x < y$

(c) $x = y$ (d) $x \geq y$

60. In a closed insulated container a liquid is stirred with a paddle to increase the temperature. Which of the following is true

(a) $\Delta E = w \neq 0, q = 0$

(b) $\Delta E = w = q \neq 0$

(c) $\Delta E = 0, w = q \neq 0$

(d) $w = 0, \Delta E = q \neq 0$

61. C (diamond) $\longrightarrow$ C (Graphite), $\Delta H = -ve$. This shows that

(a) Graphite is more stable than diamond

(b) Graphite has more energy than diamond

(c) Both are equally stable

(d) Stability cannot be predicted

62. If a reaction involves only solids and liquids, which of the following is true

(a) $\Delta H < \Delta E$ (b) $\Delta H = \Delta E$

(c) $\Delta H > \Delta E$ (d) $\Delta H = \Delta E + RT\,\Delta n$

63. When 1 mol of a gas is heated at constant volume, temperature is raised from, 298 to 308 K. Heat supplied to gas is 500 J. Then which statement is correct?

(a) $q = w = 500$ J; $\Delta U = 0$

(b) $q = \Delta U = 500$ J; $w = 0$

(c) $q = w = 500$ J; $\Delta U = 500$

(d) $\Delta U = 0$; $q = w = -500$ J

ANSWERS

1. (c)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (b)
11. (b)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (d)	**20.** (d)
21. (b)	**22.** (a)	**23.** (d)	**24.** (c)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (d)
31. (a)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (a)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (b)
41. (d)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (b)	**49.** (c)	**50.** (d)
51. (c)	**52.** (c)	**53.** (d)	**54.** (a)	**55.** (d)	**56.** (a)	**57.** (d)	**58.** (b)	**59.** (a)	**60.** (a)
61. (a)	**62.** (b)	**63.** (b)							

Equilibrium

6

BRIEF REVIEW OF THE CONCEPTS

Equilibrium is the state of constant properties i.e. when there is no observable change in the composition, temperature and pressure of a system with the passage of time. Equilibrium involving changes in the physical states of the same substance is called **physical equilibrium.** Various types of physical equilibria are

(i) solid – liquid equilibrium.
(ii) liquid – vapour equilibrium.
(iii) solid – vapour equilibrium
(iv) equilibrium in solutions

Temperature at which solid and liquid states of a substance are at equilibrium at one atmospheric pressure is called its **melting point.** The temperature at which liquid-gas equilibrium for a substance is attained under the atmospheric pressure is called **normal boiling point.**

Dissolution of gases in liquids is governed by Henry's Law. It states that, the mass of gas dissolved in a given mass of a solvent at any temperature is directly proportional to pressure of the gas above the solvent.

$$m \propto p$$

or $$m = kp$$

Where m is the mass of the gas dissolved under equilibrium pressure p and k is the constant.

Chemical equilibrium involves reversible chemical reactions and may be homogeneous (one phase) or heterogeneous (two or more phases)

Law of chemical equilibrium At a given temperature, the product of the concentration of products each raised to the corresponding stoichiometric coefficients in the balanced chemical equation divided by the product of the concentrations of the reactants raised to the corresponding stoichiometric coefficients has a constant value.

For example, for a reversible reaction

$$aA + bB \rightleftharpoons cC + dD$$

the equilibrium constant K_c at constant temperature can be expressed as

$$K_c = \frac{[C]^c \times [D]^d}{[A]^a \times [B]^b}$$

In the above expression, formula of a substance enclosed between square brackets represents its molar concentration. This law is also known as *Law of mass Action*.

If the chemicals are all gaseous, then equilibrium constant K_p, in terms of equilibrium partial pressures is represented as

$$K_p = \frac{p_C^c \times p_D^d}{P_A^a \times P_B^b}$$

Units of K_p and K_c The expression for K_c is written in terms of concentration and that for K_p in terms of pressure so K_c and K_p will have some units, unless $a + b = c + d$. In fact, strictly speaking, expression for equilibrium constants are written in terms of activity which is a dimensionless quantity. Therefore, equilibrium constants are also dimensionless quantities and have no units.

Relation between K_p and K_c

$$K_p = K_c (RT)\Delta ng$$

where Δn_g = total no. of moles of gaseous products – total no. of moles of gaseous reactants

= change in number of moles of gaseous species

If total number of moles of gaseous products and reactants are same i.e., $\Delta n_g = 0$, then $K_p = K_c$

Characteristics of equilibrium constant (K)

1. It is independent of original concentration of the reactants
2. It has definite value at particular temperature and changes with temperature.
3. Its value is independent of pressure and is unaffected by the presence of a catalyst
4. Its value is not influenced by the nature and number of steps in the reaction as long as stoichiometry is not changed.
5. Its value is independent of the inert materials added to the system.
6. The larger the value of K, the greater is the reaction towards the product side.

Factors affecting the value of K_c or K_p.

(i) **The mode of representing the equilibrium state.** Calculate the value of K_c for the reversible reactions.

$$H_2 + I_2 \rightleftharpoons 2HI, \text{ and}$$

$$2HI \rightleftharpoons H_2 + I_2, \text{ respectively}$$

Let the equilibrium constants for the first and the second reactions be K_c and K'_c respectively.

Then

$$K_c = \frac{[HI]^2}{[H_2][I_2]}$$

and

$$K_c' = \frac{[H_2][I_2]}{[HI]^2}$$

$$K_c = \frac{1}{K'_c}$$

Therefore, the mode of expressing a chemical equilibrium state influences the value of K. If an equation is reversed, the value of K_c is inverted.

(ii) **Method of writing a balanced chemical equation.** The reaction between nitrogen and ammonia can be represented as follows,

$$N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$$

Equilibrium constant K_c, For this system is given by

$$K_c = \frac{[NH_3]^2}{[N_2][H_2]^3}$$

The above reaction can also be written as:

$$\frac{1}{2}N_2(g) + \frac{3}{2}H_2(g) \rightleftharpoons NH_3(g)$$

Then the expression for the equilibrium constant, K_c would be

$$K_c = \frac{[NH_3]}{[N_2]^{\frac{1}{2}}[H_2]^{\frac{3}{2}}}$$

(iii) **Change of temperature.** Equilibrium constant changes with the change in temperature. Increase in temperature usually increases the rate of reaction. But the extent to which forward and reverse reactions are affected is different, hence, K is also different.

(iv) **Change in the units of expressing concentration.** For K_c concentration is expressed in moles per litre and for K_p partial pressure is expressed in atmosphere or bar. If Δn is not equal to zero, any change in the units expressing concentrations or partial pressure of the constituents of a reaction mixture will alter the numerical value of the equilibrium constant.

Factors which do not influence the values of k_c and k_p. Following factors do not alter the value of equilibrium constant:

(i) Initial concentrations of reactants or products.

(ii) Addition of either of the constituents to the reaction mixture.

(iii) Addition of an inert gas does not change the value of equilibrium constant.

(iv) **Presence of a catalyst.** A catalyst does not take part in the chemical reaction, as such it does not figure in the chemical equation and for that reason in the expression for chemical equilibrium constant, K_c. Thus, we can say a catalyst does not affect the value of equilibrium constant. However, a catalyst increases the rate of forward as well as reverse reactions and thereby it will help the equilibrium to establish quickly.

Predicting the Direction of a reaction. The numerical value of the equilibrium constant is determined experimentally. At any stage of the reaction the concentrations of the species involved in the reaction can be put in the expression. For a generalized reversible reaction $aA + bB \rightleftharpoons xX + yY$, the reaction quotient is

$$Q = \frac{[X]^x[y]^y}{[A]^a[B]^b}$$

On substituting the concentrations of various species in the expression the value of reaction quotient 'Q' is obtained. The numerical value of Q helps us in predicting the direction in which the reaction could take place.

(i) $Q = K_c$. When the concentrations of varies species involved in the reaction are equilibrium concentrations in that case the value of Q is equal to K_c.

(ii) $Q < K_c$. When the calculated value of reaction quotient is less than the numerical value of K_c, then the reaction proceeds from left to right, i.e., forward reaction takes place until the equilibrium is established.

(iii) $Q > K_c$. When the value of reaction quotient is greater than that of the equilibrium constant K_c, then in such a situation reaction proceeds to the left from the right until equilibrium is attained.

Le Chatelier's Principle. If any kind of stress (Such as change in concentration, temperature or pressure) is applied on equilibrium, it shifts in a direction that tends to undo the effect of the stress.

Thus if a system is in equilibrium and a change is made in any of the factors, e.g., temperature, concentration or pressure, on which the equilibrium depends, the position of equilibrium will be adjusted in such a manner that the effect of induced change is nullified as far as possible. Accordingly:

(i) An increase of temperature will move the equilibrium in the direction in which the effect of added heat is minimised. Thus an endothermic reversible reaction is favoured at high temperature whereas an exothermic reaction will be favoured at low temperature.

(ii) When a reversible reaction takes place with a change in volume, increase of pressure will try to shift the equilibrium in the direction which occurs with the decrease of volume.

(iii) For the effect of concentration on equilibrium, consider the reaction.

$$A + B \rightleftharpoons C + D$$

At equilibrium, definite concentrations of A, B, C and D are present. Now, if we add more of A and B at this stage, more of C and D will be produced because in that way the added A or B is used up. Thus according to Le Chatelier's principle the addition of A and B will produce larger concentrations of C and D at equilibrium.

The principle as enunciated above is general one and is equally applicable to both physical as well as chemical equilibria.

Application of Le Chatelier's Principle in Physical Equilibria

(i) *Melting of ice*

$$\text{Ice} \rightleftharpoons \text{Water}; \Delta H = +Q \text{ kJ}$$

The melting of ice occurs with the absorption of heat and by a decrease of volume. Therefore, both the increase of pressure and temperature will cause more and more ice to melt.

(ii) *Vaporization of water*

$$\text{Water} \rightleftharpoons \text{Vapour}; \Delta H = +Q \text{ kJ}$$

When water is heated, it takes up heat and part of it is converted into vapour. Vaporization of water is accompanied by a large increase in volume. According to the above principle, increase of pressure will cause the liquefaction (to decrease the volume) and increase of temperature will produce more vapour (to cause more and more absorption of heat).

Application of Le Chatelier's Principle in Chemical Equilibria

Formation of ammonia by Haber's Process

$$N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g);\ \Delta H = -93.6 \text{ kJ}$$

1 vol. 3 vol. 2 vol.

Since this reaction is exothermic, i.e., heat is evolved when the forward reaction takes places, a low temperature will favour the forward reaction. Thus if the temperature is raised, equilibrium concentration of ammonia will decrease and those of N_2 and H_2 increase.

1 Volume of N_2 and 3 volumes of H_2 combine to from 2 volumes of ammonia, the reaction proceeds in the forward direction with the decrease of volume. Therefore, high pressure will favour the formation of ammonia because at a constant temperature, the increase of pressure will shift the equilibrium in the direction which is accompanied by decrease of volume (i.e., forward reaction in this case).

Since, according to Le Chatelier's principle, the increase of any component of the reaction mixture will shift the equilibrium in the direction in which the added components are used up. addition of N_2 or H_2 will result in a higher yield of ammonia.

Solved Problems

1. What is the K_p for the equilibrium $I_{2(g)} \rightleftharpoons 2I_{(g)}$ at a certain temperature and a total pressure of 10^5 *Pa*, when the iodine vapours contain 40% by volume of I atoms?

Solution Equilibrium is

$$I_2(g) \rightleftharpoons 2I(g)$$

For this equilibrium, K_p is given by,

$$K_p = \frac{P_I^2(g)}{P_{I_2}(g)}$$

Since I atoms in vapour are 40%, then 60% by volume will be $I_2(g)$.

Further, the number of moles are proportional to volume. The volume per cent can also be taken as mole per cent.

Since, partial pressure, p = mole fraction × total pressure

$$\text{mole fraction of } I(g) = \frac{40}{100} = 0.4$$

Then mole fraction of $I_2(g) = 1 - 0.4 = 0.6$

$$p_I(g) = 0.4 \times 10^5\ Pa$$

$$P_{I2}(g) = 0.6 \times 10^5\ Pa$$

Substituting the values of partial pressures of $I(g)$ and $I_2(g)$ in eq. (*i*), gives K_p,

$$K_p = \frac{(0.4\times10^5 Pa)^2}{0.6\times10^5\ Pa}$$

$$= 2.67 \times 10^4\ Pa$$

2. Which of the following reactions involve heterogeneous equilibria?

(a) $2N_2O(g) \rightleftharpoons 2N_2(g) + O_2(g)$

(b) $2NH_3(g) \rightleftharpoons N_2(g) + 3H_2(g)$

(c) $2Cu(NO_3)_2(s) \rightleftharpoons 2CuO(s) + 4NO_2(g) + O_2(g)$

(d) $CH_3COOC_2H_5(aq) + H_2O(l) \rightleftharpoons CH_3COOH(aq) + C_2H_5OH(aq)$

Solution When the reactants and products are in the same phase, the equilibrium is homogeneous and when there is more than one phase, the equilibrium is said to be heterogeneous. So the answer is '*c*'.

3. 2 moles of PCl_5 are heated is a closed two litre vessel. When equilibrium is attained, the pentachloride is 40% dissociated into PCl_3 and Cl_2. Calculate the equilibrium constant.

Solution The equilibrium is represented as

$$PCl_5 \rightleftharpoons PCl_3 + Cl_2$$

and
$$K_c = \frac{[PCl_3][Cl_2]}{[PCl_5]}$$

Initially, the number of moles of PCl_5 = 2

Degree of dissociation (α) is $\frac{40}{100} = 0.4$

So the number of moles of PCl_5 at equilibrium

$$= 2\ (1 - 0.4)$$

$$= 1.2 \text{ moles}$$

and number of moles of each of PCl_3 and $Cl_2 = 2 \times 0.4$

$$= 0.8 \text{ moles}$$

Volume of the vessel = 2L

So the concentrations

$$[PCl_5] = \frac{1.2}{2} = 0.6 \text{ mol L}^{-1}$$

$$[PCl_3] = [Cl_2] = \frac{0.8}{2} = 0.4 \text{ mol L}^{-1}$$

$$K_c = \frac{0.4\times0.4}{0.6} = 0.267$$

4. Find out the value of K_c for the following equilibrium from the value of K_p

$2NOCl(g) \rightleftharpoons 2NO(g) + Cl_2(g)$, $K_p = 1.8 \times 10^{-2}$ at 500 K

Solution The relationship between K_p and K_c is $K_p = K_c\ (RT)^{\Delta n}$

Here Δn =

Number of moles of the gaseous product – Number of moles of the gaseous reactants.

$$\Delta n = 3 \text{ moles} - 2 \text{ moles} = 1 \text{ mole}$$

$$T = 500 \text{ K};$$

$$R = 0.0831 \text{ bar } L/\text{mol } K$$

$$K_p = 1.8 \times 10^{-2}$$

Substituting these values we obtain

$$K_c = \frac{K_p}{(RT)^{\Delta n}} = \frac{1.8\times10^{-2}}{(0.0831\times500)} = 4.3 \times 10^{-4}$$

5. One mole of H_2O and one mole of CO are taken in a 10 litre vessel and heated to 725 K. At equilibrium 40 per cent of water (by mass) reacts with carbon monoxide according to the equation.

$$H_2O(g) + CO(g) \rightleftharpoons H_2(g) + CO_2(g)$$

Calculate the equilibrium constant for the reaction.

Solution The equilibrium is

	$H_2O(g)$ +	$CO(g)$ $\rightleftharpoons$	$H_2(g)$ +	$CO_2(g)$
Initial amount	1 mol	1 mol	0 mol	0 mol
Amount reacted	0.4 mol	0.4 mol	–	–
Amount at equilibrium	1 – 0.4 = 0.6 mol	1 – 0.4 = 0.6 mol	0.4 mol	0.4 mol
Concentration	$\frac{0.6}{10}$ mol L^{-1}	$\frac{0.6}{10}$ mol L^{-1}	$\frac{0.4}{10}$ mol L^{-1}	$\frac{0.4}{10}$ mol L^{-1}

From the reaction it is clear that 0.4 mol of water will react with 0.4 mol of CO to from 0.4 mol each of H_2 and CO_2.

The expression for K_c is

$$K_c = \frac{[H_2][CO_2]}{[H_2O][CO]}$$

Substituting the equilibrium concentrations in the above expression gives,

$$K_c = \frac{\frac{0.4}{10} \times \frac{0.4}{10}}{\frac{0.6}{10} \times \frac{0.6}{10}} = 0.44$$

Problems for Practice

1. Which one of the following is a reversible process?
   (a) Mixing of gases by diffusion
   (b) Dissociation of a solute in pure solvent
   (c) Expansion of gas in vacuum
   (d) Melting of ice without rise in temperature

2. For the system $2A(g) + B(g) \rightleftharpoons 3C(g)$, the expression for the equilibrium constant K is
   (a) $\frac{[2A][B]}{[3C]}$ (b) $\frac{[A]^2[B]}{[C]^3}$
   (c) $\frac{[3C]}{[2A][B]}$ (d) $\frac{[C]^3}{[A]^2[B]}$

3. In which of the following does the reaction go almost to completion?
   (a) $K_c = 10^3$ (b) $K_c = 10^2$
   (c) $K_c = 10^{-2}$ (d) $K_c = 10^{-3}$

4. For the reaction

   $ZnCO_3(s) \rightleftharpoons ZnO(s) + CO_2(g)$

   expression for the partial pressure constant for the above reaction would be
   (a) $K_p = \frac{[ZnO][CO_2]}{[ZnCO_3]}$
   (b) $K_p = \frac{p_{ZnO} \times p_{CO_2}}{p_{ZnCO_3}}$
   (c) $K_p = p^2_{ZnO} \times p_{CO2}$
   (d) $K_p = p_{CO2}$

5. Two moles of N_2 and two moles of H_2 are taken in a closed vessel of 5 litres capacity and suitable conditions are provided for the reaction. When the equilibrium is reached, it is found that a half mole of N_2 is used up. The equilibrium concentration of NH_3 is
   (a) 0.3 (b) 0.4
   (c) 0.2 (d) 0.1

6. The unit of equilibrium constant for the reversible reaction $H_2 + I_2 \rightleftharpoons 2HI$ is
   (a) $Mole^{-1}$ litre (b) $Mole^{-2}$ litre
   (c) Mole $litre^{-1}$ (d) None of these

7. In melting of ice, which one of the conditions will be more favourable?
   (a) High temperature and high pressure
   (b) Low temperature and low pressure
   (c) Low temperature and high pressure
   (d) High temperature and low pressure

8. When 3.00 mole of A and 1.00 mole of B are mixed in a 1.00 litre vessel the following reaction takes place

   $A(g) + B(g) \rightleftharpoons 2C(g)$

   The equilibrium mixture contains 0.5 mole of C. What is the value of equilibrium constant for the reaction?
   (a) 0.12 (b) 6
   (c) 1.5 (d) None of these

9. Which one of the following equilibrium moves backward when pressure is applied?
   (a) $N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$
   (b) $N_2(g) + O_2(g) \rightleftharpoons 2NO(g)$
   (c) Water $\rightleftharpoons$ Ice
   (d) $I_2(g) \rightleftharpoons I_2(s)$

10. 3 mole of reactant A and one mole of reactant B are mixed in a vessel of volume 1 litre. The reaction taking place is $A + B \rightleftharpoons 2C$. If 1.5 mol of C is formed at equilibrium, the value of K_c is
    (a) 0.12 (b) 0.50
    (c) 4.00 (d) 0.25

11. The equilibrium that is not affected by the increase in pressure is
    (a) $2SO_2(g) + O_2(g) \rightleftharpoons 2SO_3(g)$
    (b) $PCl_3(g) + Cl_2(g) \rightleftharpoons PCl_5(g)$

(c) $N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$

(d) $N_2(g) + O_2(g) \rightleftharpoons 2NO(g)$

12. In which of the following equilibrium, change in the volume of the system does not alter the number of moles?

(a) $N_2(g) + O_2(g) \rightleftharpoons 2NO(g)$

(b) $PCl_5(g) \rightleftharpoons PCl_3(g) + Cl_2(g)$

(c) $N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$

(d) $SO_2Cl_2(g) \rightleftharpoons SO_2(g) + Cl_2(g)$.

13. By the addition of an inert gas to the system $2HI \rightleftharpoons H_2 + I_2$, the degree of dissociation of HI will be

(a) Suppressed (b) Not altered

(c) Increased (d) Doubled

14. A large value for equilibrium constant K shows that

(a) The reaction has gone to near completion towards right

(b) The reaction has not yet started

(c) The reaction has gone to near completion towards left.

(d) None of these

15. When ethyl alcohol and acetic acid are mixed together in equimolecular proportions, equilibrium is attained when two-third of the acid and alcohol are consumed. The equilibrium constant of the reaction will be

(a) 0.4 (b) 4

(c) 40 (d) 0.04

16. A gas bulb is filled with NO_2 gas and immersed in an ice bath at 0°C which becomes colourless after sometime. This colourless gas will be

(a) NO_2 (b) N_2O

(c) N_2O_4 (d) N_2O_5

17. What are the most favourable conditions for the reaction

$$SO_2 + 1/2\ O_2 \rightleftharpoons SO_3;\ \Delta H = -ve$$

to occur

(a) low T and high P (b) low T and low P

(c) high T and low P (d) high T and high P.

18. Which of the following will not change the concentration of ammonia in the equilibrium?

$N_2(g) + 3H_2(g) \rightleftharpoons 2\ NH_3(g);\ \Delta H = -x$ kJ

(a) Increase of pressure

(b) Increase of temperature

(c) Decrease of volume

(d) Addition of catalyst

19. For the equilibrium $2NO_2(g) \rightleftharpoons N_2O_4(g) + 14.55$ Kcal; Increase in temperature would

(a) Favour the formation of N_2O_4

(b) Favour the decomposition of N_2O_4

(c) Not alter the equilibrium

(d) Stop the reaction

20. In the equilibrium

$$N_2 + 3H_2 \rightleftharpoons 2NH_3 + 22 \text{ kcal}$$

the formation of ammonia is favoured by

(a) Increasing the pressure

(b) Increasing the temperature

(c) Decreasing the pressure

(d) Adding ammonia

21. The effect of increasing the pressure on the following equilibrium.

$$2A + 3B \rightleftharpoons 3C + 2D \text{ is}$$

(a) Forward reaction is favoured

(b) Backward reaction is favoured

(c) No effect

(d) Depends upon the volume of the vessel

22. The equilibrium constant for the reversible reaction $N_2 + 3H_2 \rightleftharpoons 2NH_3$ is K and for the reaction $\frac{1}{2}N_2 + \frac{3}{2}H_2 \rightleftharpoons NH_3$, the equilibrium constant is K'. K and K' will be related as

(a) $K = K'$ (b) $K' = \sqrt{K}$

(c) $K = \sqrt{K'}$ (d) $K \times K' = 1$

23. The factor which changes equilibrium constant of the reaction

$$A_2(g) + B_2(g) \rightleftharpoons 2AB(g) \text{ is}$$

(a) Total pressure (b) Amounts of A_2 and B_2

(c) Temperature (d) Catalyst

24. Given the reaction:

$$2X\ (g) + Y(g) \rightleftharpoons 2Z(g) + 80 \text{ kcal}$$

which combination of pressure and temperature gives the highest yield of Z at equilibrium?

(a) 1000 atm and 500°C

(b) 500 atm and 500°C

(c) 1000 atm and 100°C

(d) 500 atm and 100°C

25. For a reaction

$H_2 + I_2 \rightleftharpoons 2HI$ at 721 K, the value of equilibrium constant is 50. If 0.5 moles each of H_2 and I_2 is added to the system the value of equilibrium constant will be

(a) 0.02 (b) 0.2

(c) 50 (d) 25.

26. In which of the following cases, the reaction yields minimum amount of products

(a) $K = 10^3$ (b) $K = 10^{-2}$

(c) $K = 10$ (d) $K = 10^2$

27. In an exothermic reaction, a 10° rise in temperature will

(a) decrease the value of equilibrium constant

(b) double the value of K_c

(c) not produce any change in K_c

(d) produce some increase in K_c

28. The equilibrium constant for a reaction

$$N_2(g) + O_2(g) \rightleftharpoons 2NO(g)$$

is 4×10^{-4} at 2000 K. In the presence of catalyst, the equilibrium constant is attained 10 times faster. The equilibrium constant in the presence of catalyst, at 2000 K is

(a) 40×10^{-4} (b) 4×10^{-5}

(c) 4×10^{-4} (d) 0.25×10^{-4}

29. $CaCO_3 \rightleftharpoons CaO + CO_2$. When this reaction is carried out in a lime furnace, then it reaches to completion due to the following.

(a) CaO and CO_2 do not combine

(b) CaO is very stable as compared to $CaCO_3$

(c) Temperature of the furnace is very high
(d) CO_2 is expelled as it is formed

30. The yield of product in the reaction

$$A_2(g) + 2B(g) \rightleftharpoons C(g) + Q\text{kJ}$$

would be higher at
(a) high temperature and high pressure
(b) high temperature and low pressure
(c) low temperature and high pressure
(d) low temperature and low pressure

31. The state of equilibrium refers to
(a) State of rest (b) Dynamic state
(c) Stationary state (d) State of inertness

32. 4 moles of A are mixed with 4 moles of B, when 2 moles of C are formed at equilibrium according to the reaction

$$A + B \rightleftharpoons C + D$$

The value of equilibrium constant is
(a) 4 (b) 1
(c) 1/2 (d) 1/4.

33. For the reaction $H_2(g) + I_2(g) \rightleftharpoons 2HI(g)$ equilibrium constant K_p changes with
(a) total pressure
(b) catalyst
(c) amount of H_2 and I_2 present
(d) temperature

34. For the equilibrium

$$2NO_2(g) \rightleftharpoons N_2O_4(g) + 14.6 \text{ kcal}$$

increase of temperature would.
(a) favour the formation of N_2O_4
(b) favour the decomposition of N_2O_4
(c) not affect the equilibrium
(d) stop the reaction

35. $CH_3COOH + C_2H_5OH \rightleftharpoons CH_3COOC_2H_5 + H_2O$. In the above reaction one mole of each of acetic acid and alcohol are heated in the presence of little conc. H_2SO_4. $\frac{1}{3}$ mole of CH_3COOH was present unreacted when equilibrium was attained.
(a) 1 mole of ethyl acetate is formed
(b) 2 moles of ethyl acetate are formed
(c) 1/3 mole of ethyl acetate is formed
(d) 2/3 mole of ethyl acetate is formed.

36. For which one of the following reactions is a combination of high pressure and high temperature helpful in obtaining a high equilibrium yield?
(a) $2NF_3(g) \rightleftharpoons N_2(g) + 3F_2(g) -54.40$ kcal
(b) $N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g) + 22.08$ kcal
(c) $Cl_2(g) + 2O_2(g) \rightleftharpoons 2ClO_2(g) - 49.40$ kcal
(d) $2Cl_2O_7(g) \rightleftharpoons 2Cl_2(g) + 7O_2(g) + 126.8$ kcal

37. 4.0 moles of PCl_5 dissociate at 760 K in a 2 litre flask as $PCl_5(g) \rightleftharpoons PCl_3(g) + Cl_2(g)$. At equilibrium 0.8 mole of Cl_2 was present in the flask. The equilibrium constant would be
(a) 1.0×10^{-1} (b) 1.0×10^{-4}
(c) 1.0×10^{-2} (d) 1.0×10^{-3}

38. Two moles of NH_3 gas are introduced into a previously evacuated one litre vessel in which it partially dissociates at high temperature as $2NH_3(g) \rightleftharpoons N_2(g) + 3H_2(g)$ At equilibrium, one mole of $NH_3(g)$ remains. The value of K_c is
(a) 3 (b) 27/16
(c) 3/2 (d) 27/64

39. 1 mole of NO_2 and 2 moles of CO are enclosed in a one line vessel to attain the following equilibrium $NO_2 + CO \rightleftharpoons NO + CO_2$. It was estimated that at the equilibrium, 25% of initial amount of CO is consumed. The equilibrium constant K_p is
(a) 1 (b) 1/2
(c) 1/4 (d) 1/3

40. The equilibrium constant K_p for a homogeneous gaseous reaction is 10^{-8}. The standard Gibb's free energy change $\Delta G°$ for the reaction (using $R = 2$ cal K^{-1} mol^{-1}) is
(a) 10.98 kcal (b) –1.8 kcal
(c) –4.1454 kcal (d) + 4.1454 kcal.

41. In a reversible chemical reaction having two reactants in equilibrium, if the concentration of the reactants are doubled then the equilibrium constant will
(a) also be doubled
(b) be halved
(c) becomes one fourth
(d) remains the same

42. Which of the following gaseous reactions will be favoured by low pressure?
(a) $H_2 + I_2 \rightleftharpoons 2HI$
(b) $PCl_5 \rightleftharpoons PCl_3 + Cl_2$
(c) $N_2 + 3H_2 \rightleftharpoons 2NH_3$
(d) $N_2 + O_2 \rightleftharpoons 2NO$

43. For the system $3A + 2B \rightleftharpoons C$, the expression for equilibrium constant K is
(a) $\frac{[3A] \times [2B]}{[C]}$ (b) $\frac{[A]^3 \times [B]}{[C]}$
(c) $\frac{[C]}{[A]^3 \times [B]^2}$ (d) $\frac{[C]}{[3A] \times [2B]}$

44. Equilibrium constant for the reaction:

$$2NO(g) + Cl_2(g) \rightleftharpoons 2NOCl(g)$$

is correctly given by the expression
(a) $K = \frac{[2NOCl]}{[2NO][Cl_2]}$ (b) $K = \frac{[NOCl]^2}{[NO]^2\,[Cl_2]}$
(c) $K = \frac{[NO]^2\,[Cl_2]^2}{[NO]^2 + [Cl_2]}$ (d) $K = \frac{[NO]^2 + [Cl_2]^2}{[NOCl]}$

45. A chemical reaction $A \rightleftharpoons B$ is said to be in equilibrium when
(a) complete conversion of A to B has taken place
(b) conversion of A to B is only 50% complete
(c) only 10% conversion of A to B has taken place.
(d) the rate of transformation of A to B is just equal to rate of transformation of B to A in the system,.

46. For the reaction

$$2A(g) + B(g) \rightleftharpoons 3C(g) + D(g)$$

two moles each of A and B were taken into a flask. The following must always be true when the system attained equilibrium

(a) $[A] = [B]$ (b) $[A] < [B]$
(c) $[B] = [C]$ (d) $[A] > [B]$

47. Which of the following factors will favour the reverse reaction in a chemical equilibrium?

(a) Increase in concentration of one of the reactants
(b) Increase in concentration of one of the products
(c) Removal of one of the products regularly
(d) None of these

48. For the manufacture of ammonia by the reaction

$$N_2 + 3H_2 \rightleftharpoons 2NH_3 + 21.9 \text{ kcal}$$

the favourable conditions are

(a) Low temperature, low pressure and catalyst
(b) Low temperature, high pressure and catalyst
(c) High temperature, low pressure and catalyst
(d) High temperature, High pressure and catalyst

49. At a constant temperature, when the concentration of a reactant is increased, the equilibrium constant of a reversible reaction

(a) increases
(b) decreases
(c) remains unaffected
(d) first increases and then decreases

50. Does Le Chatelier's principle predict a change of equilibrium concentration for the following reaction if the gas mixture is compressed?

$$N_2O_4(g) \rightleftharpoons 2NO_2(g)$$

(a) Yes, reactant favoured
(b) Yes, product favoured
(c) No change
(d) No information

51. In the gaseous equilibrium

$$H_2X_2 + \text{heat} \rightleftharpoons 2HX$$

the formation of HX will be favoured by

(a) high pressure and low temperature
(b) high temperature and low pressure
(c) low temperature and low pressure
(d) high temperature and high pressure

52. For a system of gases, A, B, C and D at equilibrium

$$A + 2B \rightleftharpoons C + 3D$$

the partial pressures are found to be A, 0.20; B, 0.10; C, 0.30 and D, 0.50 atm. What is the numerical value of the equilibrium constant?

(a) 11.25 (b) 18.75
(c) 5.0 (d) 3.75

53. According to Le Chatelier's principle, adding heat to a solid and liquid in equilibrium will cause the

(a) Amount of solid to decrease
(b) Amount of liquid to decrease
(c) Temperature to rise
(d) Temperature to fall

54. For the system

$$A(g) + 2B(g) \rightleftharpoons C(g)$$

the equilibrium concentration is

$A = 0.06$ mol L^{-1}, $B = 0.12$ mol L^{-1}

$C = 0.26$ mol L^{-1}. The K_c for the reaction is

(a) 250 (b) 416
(c) 4×10^{-3} (d) 125

55. The Haber's process for the manufacture of ammonia is usually carried out at about 500°C. If a temperature of about 250°C was used instead

(a) no ammonia would be formed at all
(b) the percentage of ammonia in the equilibrium mixture would be too low
(c) a catalyst would be of no use at all at this temperature
(d) the rate of formation of ammonia would be too slow.

56. Sodium sulphate dissolves in water with evolution of heat. Consider a saturated solution. (i.e. a solution in equilibrium with solid solute) of sodium sulphate. If the temperature is raised, then according to Le Chatelier's principle

(a) more solid will dissolve
(b) some solid will precipitate out from the solution
(c) the solution will become supersaturated
(d) solution concentration will remain unchanged

57. Raising the temperature of an equilibrium system

(a) favours the exothermic reaction only
(b) favours the endothermic reaction only
(c) favours both the exothermic and endothermic reactions
(d) favours neither the exothermic nor endothermic reactions

58. The decomposition of N_2O_4 to NO_2 is carried out at 280 K in chloroform. When equilibrium has been established, 0.2 mol of N_2O_4 and 2×10^{-3} mol of NO_2 are present in a 2L solution. The equilibrium constant for the reaction $N_2O_4 \rightleftharpoons 2NO_2$ is

(a) 1×10^{-2} (b) 2×10^{-3}
(c) 1×10^{-5} (d) 2×10^{-5}

59. In which of the following equilibrium the value of K_p is less than K_c?

(a) $H_2 + I_2 \rightleftharpoons 2HI$
(b) $N_2 + 3H_2 \rightleftharpoons 2NH_3$
(c) $N_2 + O_2 \rightleftharpoons 2NO$
(d) $CO + H_2O \rightleftharpoons CO_2 + H_2$

60. A cylinder fitted with a movable piston contains liquid water in equilibrium with water vapour at 25°C. Which one of the following operations result in a decrease in the equilibrium vapour pressure?

(a) Moving the piston downward a short distance
(b) Removing a small amount of vapour
(c) Removing a small amount of the liquid water
(d) Dissolving salt in the water

ANSWERS

1. (d)	**2.** (d)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (c)
11. (d)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (a)
21. (c)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (a)	**28.** (c)	**29.** (d)	**30.** (c)
31. (b)	**32.** (b)	**33.** (d)	**34.** (b)	**35.** (d)	**36.** (c)	**37.** (a)	**38.** (b)	**39.** (d)	**40.** (a)
41. (d)	**42.** (b)	**43.** (c)	**44.** (b)	**45.** (d)	**46.** (b)	**47.** (b)	**48.** (b)	**49.** (c)	**50.** (a)
51. (b)	**52.** (b)	**53.** (a)	**54.** (a)	**55.** (d)	**56.** (b)	**57.** (b)	**58.** (c)	**59.** (b)	**60.** (a)

Ionic Equilibrium

7

BRIEF REVIEW OF THE CONCEPTS

Electrolyte Substances whose solutions conduct electric current by migration of ions are called electrolytes.

Strong and weak Electrolytes The electrolytes that are almost completely dissociated into ions are called strong electrolytes while those that are dissociated only to a small extent are called weak electrolytes.

Examples of strong electrolytes HCl, HBr, HI, $HClO_4$ (perchloric acid), NaOH, KOH, $Ca(OH)_2$, $Ba(OH)_2$ etc.

Examples of Weak electrolytes HNO_2, H_3PO_4, HF, H_2CO_3, HCN, CH_3COOH, NH_4OH, etc.

Concepts of Acids and Bases

Arrhenius concept. Arrhenius (1887) defined an acid as a substance that will dissociate to yield hydrogen ions while a base is one that will dissociate to yield hydroxyl ions in aqueous solution.

$$\underset{\text{(Acid)}}{HCl(l)} \rightleftharpoons H^+(aq) + Cl^-(aq)$$

$$\underset{\text{(Base)}}{NaOH} \rightleftharpoons Na^+(aq) + OH^-(aq)$$

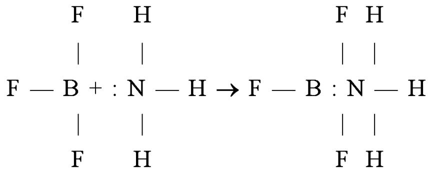
Lowry-Bronsted Concept (also known as Proton Concept) The Arrhenius concept was extended by Lowry and Bronsted (1923). They observed that an acid reacted with water in a manner opposite to that of a base. For example, HCl (an acid) donates a proton to a water molecule while NH_3 (a base) accepts a proton from a water molecule.

$$\underset{\text{(Acid)}}{HCl} + H_2O \rightleftharpoons H_3O^+ + Cl^-;$$

$$\underset{\text{(Base)}}{NH_3} + H_2O \rightleftharpoons NH_4^+ + OH^-$$

On this basis they postulated that compounds having a tendency to donate a proton are acids and those which accept a proton are bases. Thus, in the above first reaction, HCl is an acid and water is a base. In the other reaction H_2O is an acid and NH_3 is a base. Considering the above reactions in the reverse direction H_3O^+ is an acid, Cl^- is a base; NH_4^+ is an acid and OH^- is a base. Hence, we are led to the conclusion that for any acid-base equilibrium, there are two conjugate acid-base pairs. The base formed from an acid is referred to as the conjugate base of the acid. Correspondingly, the acid formed from a base is called the conjugate acid of the base **An acid-base pair differing by a proton constitutes a conjugate pair.** For the above reaction, Cl^- is the conjugate base of HCl and H_2O is the conjugate base of H_3O^+; HCl is the conjugate acid of base Cl^- and H_3O^+ is the conjugate acid of base H_2O.

Lewis Acids and Bases Lewis acid is a molecule or ion capable of accepting a pair of electrons in the formation of a covalent bond, while Lewis base is a molecule or ion capable of donating a pair of electrons in the formation of a covalent bond. This distinction extends the concept of acids to the substance that do not contain hydrogen and cannot donate a proton. For example, boron trifluoride and aluminium chloride are Lewis acids.



$$F-\underset{F}{\overset{F}{B}} + :\underset{H}{\overset{H}{N}}-H \rightarrow F-\underset{F}{\overset{F}{B}}:\underset{H}{\overset{H}{N}}-H$$

Here NH_3 is a Lewis base.

Strengths of Acids and Bases Since acids are substances capable of losing a proton, the tendency to lose a proton would be a measure of its acid strength. Similarly, the strength of a base is determined by its tendency to accept a proton. In this respect a strong acid would have least affinity for a proton and get completely dissociated to give proton. This means that its conjugate base cannot hold the proton and is thus a weak base. For example, in an aqueous solution, HCl is a strong acid and its conjugate base, Cl^- is a weak base. In general, the stronger the acid, the weaker its conjugate base. Conversely, the conjugate base of a weak acid is a strong base, e.g., CN^- is a strong conjugate base of the weak acid, HCN.

A strong base on the other hand, has a large affinity for a proton. Therefore, the conjugate acid of a strong base is a weak acid and the conjugate acid of a weak base is a strong acid. The stronger the base, the weaker is its conjugate acid. In water, CO_3^{2-} a strong base and HCO_3^- is its weak conjugate acid. Similarly, NH_4^+ is a strong conjugate acid of the weak base, NH_3.

Degree of Dissociation (α) The fraction of an electrolyte which dissociates into ions is known as degree of dissociation i.e., start with 1 mol of an electrolyte, the fraction of it which ionizes/ dissociates is called its degree of ionization/dissociation.

Ionization of Weak Electrolytes Ionic equilibrium of a weak electrolyte, *AB* is represented as

$$AB + H_2O(l) \rightleftharpoons A^+_{(aq)} + B^-_{(aq)}$$

We can write expression for the equilibrium constant K as

$$K = \frac{[A^+(aq)][B^-(aq)]}{[AB]}$$

where K includes the concentration of water (solvent) which remains practically unchanged. K is called **ionization constant or dissociation constant** of the electrolyte.

Since AB is a weak electrolyte, its ionization is not complete. Let α be its degree of ionization. If initial concentration of the electrolyte is c mol L^{-1}, The amount of AB ionized at equilibrium would be $c\alpha$ resulting in the formation of ions. Hence, concentration of $A^+(aq)$ and $B^-(aq)$ ions will be $c\alpha$ mol L^{-1} each. The concentration of unionised AB will be $c\,(1 - \alpha)$. These concentrations, when put into the formula for K, we get

$$K = \frac{(c\alpha)(c\alpha)}{c\,(1-\alpha)} = \frac{c\alpha^2}{1-\alpha}$$

For weak electrolytes $\propto$ is very small and can be neglected in comparison to one. With this approximation, the expression for K simplifies to

$$K = c\alpha^2$$

and $$\alpha = \sqrt{\frac{K}{c}}$$

If the electrolyte is a weak acid like acetic acid, phosphoric etc, the ionization constant is called **acid ionization constant** and is denoted as K_a. Similarly, in case of weak base like ammonium hydroxide, aniline etc, The ionization constant is called **base ionization constant** and denoted by K_b.

Ionic product of water Water is a weak electrolyte and dissociates slightly into H^+ ions and OH^- ions

$$H_2O \rightleftharpoons H^+ + OH^- \qquad \therefore\ K = \frac{[H^+][OH\ ]}{[H_2O]}$$

i.e., $K \times [\text{constant}] = [H^+]\,[OH^-]$

($\because$ $[H_2O]$ remains nearly constant)

$K_w = [H^+]\,[OH^-]$ where K_w is called ionic product of water and its **value at 25°C is 10^{-14}**. Thus for water $[H^+]\,[OH^-] = 1 \times 10^{-14}$ and $[H^+] = [OH^-] = 1 \times 10^{-7}$

pH Concept. The acidic or basic character of a solution can be given in terms of the H_3O^+ ion concentration of the solution. To express it more conveniently, Sorensen suggested the use of the pH scale. The pH of a solution is defined as the negative logarithm (to the base 10) of the H_3O^+ ions concentration in mol L^{-1}.

Mathematically: $pH = -\log_{10} [H_3O^+]$

For pure water: $[H_3O^+] = 10^{-7}$ M so, pH = 7 at 298 K.

Larger the value of $[H_3O^+]$, lower is the pH. When $[H_3O^+] = 1.0$ M the pH is zero and for $[H_3O^+]$ larger than this the pH becomes negative. For pure water pH = 7. Thus, when the pH is below 7, the solution is acidic and when it is above 7, the solution is basic at 298 K. Since in water the $[H^+] = [OH^-]$, pOH (i.e., $-\log [OH^-]$ is also 7. Thus for water pH + pOH = 14.

Solubility Product Insoluble substances like AgCl, $BaSO_4$, $PbCl_2$, etc., are in fact not completely insoluble when present in aqueous medium. Very small amount of these substances dissolves which is present as ions. Further, there exists an equilibrium between the undissolved and the dissolved salt. For AgCl, the equilibrium may be written as:

$$AgCl(s) \rightleftharpoons AgCl\,(aq) \rightarrow Ag^+(aq) + Cl^-(aq)$$

or, $$AgCl(s) \rightleftharpoons Ag^+\,(aq) + Cl^-(aq)$$

Applying the law of mass action,

$$K = \frac{[Ag^+][Cl^-]}{[AgCl]}$$

[AgCl] is assumed to be constant because of the fact that very little of this solid dissolves in aqueous solution.

Thus, $$K \times [AgCl] = K_{sp} = [Ag^+]\,[Cl^-]$$

The new constant, K_{sp} is known as *solubility product* and the expression $[Ag^+]\,[Cl^-]$, the *ion product*. The ion product is equal to the solubility product (K_{sp}), in a saturated solution of the sparingly soluble electrolytes. For a salt M_mX_n, dissolved in water, $K_{sp} = [M^{n+}]^m [X^{m-}]^n$ where $[M^{n+}]$ and $[X^{m-}]$ are molar concentrations of ionic species M^{n+} and X^{m-} in the saturated solution

The solubility product for CaF_2 may be written as:

$$CaF_2(s) \rightleftharpoons Ca^{2+}\,(aq) + 2F^-(aq)$$

$$K_{sp} = [Ca^{2+}]\,[F^-]^2$$

Common-ion Effect *The solubility of an ionic substance is decreased by the presence of another solute that furnishes an ion in common.* This is called common-ion effect. Thus, if through a saturated solution of NaCl, hydrogen chloride, having Cl^- ions in common with NaCl, is passed then some NaCl gets separated out due to decrease in solubility of NaCl.

To the solution of a weak electrolyte, another electrolyte having a common ion is added, then , dissociation of the weak electrolyte decreases. e.g., NH_4OH dissociates very little

$$NH_4OH \rightleftharpoons NH_4^+ + OH^-$$

$$K_b = \frac{[NH_4^+][OH\ ]}{[NH_4OH]}$$

When NH_4Cl is added, it being a strong electrolyte, ionizes completely $NH_4Cl \rightarrow NH_4^+ + Cl^-$

Therefore, concentration of NH_4^+ ions increases and to maintain constant value of K_b, $[OH^-]$ ions should decrease and $[NH_4OH]$ should increase, i.e., NH_4^+ ions should combine with OH^- ions forming NH_4OH.

Similarly, if HCl is added to H_2S which is a weak electrolyte then the dissociation of H_2S is suppressed.

Applications of Common-ion Effect

(i) **Use of HCl in 2nd Group of qualitative analysis:** Group reagent of 2nd group is H_2S

$$H_2S \rightleftharpoons 2H^+ + S^{2-}$$

Concentration of S^{2-} ion is enough to precipitate sulphides of 2nd and 3rd group cations. To prevent precipitation of 3rd group cations, concentration of S^{2-} should be decreased. This is done by adding HCl

$$HCl \rightarrow H^+ + Cl^-$$

Due to common H^+ ions, dissociation of H_2S is suppressed and S^{2-} ions are just enough to precipitate 2nd group cations.

(ii) **Use of NH_4Cl in 3rd Group** Precipitating reagent for group 3rd is NH_4OH

$$NH_4OH \rightleftharpoons NH_4^+ + OH^-$$

The $[OH^-]$ is enough to precipitate hydroxides of *III*, *IV* and *V* group cations. In the presence of NH_4Cl, due to common ion effect; $[OH^-]$ decreases and precipitation *IV* and *V* group cations is prevented.

(iii) **Purification of common salt** On passing HCl gas through a saturated NaCl solution, concentration of Cl^- is increased and the ionic product exceeds the solubility product of NaCl and hence NaCl precipitates out.

(iv) **Salting out of Soap (using NaCl)** Here the concentration of Na^+ ions increases on addition of NaCl and hence the ionic product of soap (Na salt of fatty acids) exceeds the solubility product of soap so it precipitates out.

Hydrolysis of Salts and the pH of their solutions Hydrolysis is a process in which ions of a salt react with water to form an acid and a base:

$$\text{Salt + Water} \rightleftharpoons \text{Acid + Base}$$

Hydrolysis constant (K_h). Constant, K_h measures the extent to which hydrolysis takes place.

(i) *Salts formed from strong acids and strong bases.* For example, NaCl is formed from strong acid, HCl, and strong base NaOH. Other examples include KCl, $NaNO_3$, $CaCl_2$, $BaCl_2$, etc., Their aqueous solutions are neutral and they do not undergo any pH change. Thus they are not hydrolyzed.

(ii) *Salts formed from strong acids and weak bases.* Salts like NH_4Cl, $(NH_4)_2SO_4$, etc., may be cited as examples. When they are dissolved in water, the resulting solution is acidic owing to the formation of a strong conjugate acid, H_3O^+ on hydrolysis.

$$NH_4^+ + 2H_2O \rightleftharpoons NH_4OH + H_3O^+$$

The hydrolysis constant, K_h for the equilibrium is given by

$$K_h = \frac{[NH_4OH][H_3O^+]}{[NH_4^+]} \qquad ...(i)$$

Multiplying the numerator and denominator by $[OH^-]$ and rearranging, we have

$$K_h = [H_3O^+][OH^-] \times \frac{[NH_4OH]}{[NH_4^+][OH^-]} \qquad ...(ii)$$

$$= K_w/K_b \left(\because K_b = \frac{[NH_4^+][OH^-]}{[NH_4OH]} \right) \qquad ...(iii)$$

Calculation of Degree of Hydrolysis, *h*, and pH of such Solutions

Let h be the degree of hydrolysis of a salt of strong acid and weak base (Say NH_4Cl) when the concentration of the solution is c moles per litre. Then

$$[NH_4^+] = c\,(1 - h)$$

$$[NH_4OH] = [H_3O^+] = ch$$

Putting these values in eq. (*iii*) gives

$$K_h = \frac{K_w}{K_b} = \frac{(ch)^2}{c\,(1-h)} = \frac{ch^2}{1-h}$$

if h is small then $1 - h \simeq 1$

So $K_h = ch^2$

or $h = \sqrt{\frac{K_h}{c}} = \sqrt{\frac{K_w}{K_b \times c}}$

At 298 K, pH is given by

$$pH = 7 - \frac{1}{2}(\log c + pK_b)$$

(iii) *Salts formed from weak acids and strong bases.* The familiar examples are CH_3COONa. Na_2CO_3, etc. Their aqueous solutions are alkaline because of the formation of a strong conjugate base, OH^-.

$$CH_3COO^- + H_2O^- \rightleftharpoons CH_3COOH + OH^-$$

For the hydrolysis of sodium acetate K_h, is given by

$$K_h = \frac{[CH_3COOH][OH^-]}{[CH_3COO^-]}$$

Multiplying the above expression by $[H_3O^+]/[H_3O^+]$ and rearranging, we have

$$K_h = [H_3O^+][OH^-] \frac{[CH_3COOH]}{[CH_3COO^-][H_3O^+]} = K_w/K_a$$

If h is the degree of hydrolysis when the concentration of the salt is c moles L^{-1}. Then,

$$K_h = \frac{ch^2}{1-h} = ch^2 \qquad (\therefore\ 1 - h \simeq 1)$$

$$h = \sqrt{\frac{K_h}{c}} = \sqrt{\frac{K_w}{K_a \times c}}$$

At 298 K, pH is given by

$$pH = 7 + \frac{1}{2}(pK_a + \log c)$$

(iv) *Salts formed from weak acids and weak bases.* Ammonium acetate, CH_3COONH_4 for example, is formed from weak acetic acid and weak ammonium hydroxide. In aqueous solution, it liberates H_3O^+ ions as well as OH^- ions, which balance each other. Thus the solution of such a salt is almost neutral (pH 7) and for such a salt.

$$K_h = \frac{K_w}{K_a \times K_b}$$

pH of such solutions at 298 K is calculated from:

$$pH = 7 + \frac{1}{2}(pk_a - pK_b)$$

Buffer Solution is a solution which resists change in pH on the addition of small amount of an acid, base or water. Good quality buffer solutions are prepared by mixing a weak acid and its salt with strong base or a weak base and its salt with strong acid. Examples of some buffers are as follows.

Acidic buffer: acetic acid and sodium acetate

Basic buffer: NH_4OH and NH_4Cl

Buffer Action The process by which added H_3O^+ ions or OH^- ions are removed, so that pH remains constant is known as Buffer Action.

For example, an acidic buffer made up of a weak acid HA and its salt NaA; due to the common A^- ions, dissociation of HA is negligible. Therefore, solution contains HA, Na^+ ions and A^- ions. A^- acts as reserved base and HA as reserred acid. When acid is added to this buffer, it neutralises the acid as follows:

$$H^+ + A^- \rightarrow HA.$$

When base is added to it, it reacts with reserved acid HA to form A^-.

$$OH^- + HA \rightarrow H_2O + A^-$$

Henderson's equation or a relation for pH of a Buffer Solution: Consider an acidic buffer made of a weak acid HA and its salt NaA. HA dissociates slightly.

$$HA \rightleftharpoons H^+ + A^-, \text{ Dissociation constant,}$$

$$K_a = \frac{[H^+][A^-]}{[HA]} \text{ or } [H^+] = K_a \cdot \frac{[HA]}{[A^-]}$$

In presence of common A^- ions, dissociation of HA is negligible and $[A^-]$ can be considered to be concentration of salt and concentration of HA can be taken as initial concentration.

$$\therefore \quad [H^+] = K_a \cdot \frac{[Acid]}{[Salt]}$$

Taking log,

$$\log [H^+] = \log K_a + \log \frac{[\text{Acid}]}{[\text{Salt}]}$$

Changing the sign, $-\log [H^+] = -\log K_a + \log \frac{[\text{Salt}]}{[\text{Acid}]}$

or $$\text{pH} = pK_a + \log \frac{[\text{Salt}]}{[\text{Acid}]}$$

Similarly, for basic buffer, $\text{pOH} = pK_b + \log \frac{[\text{Salt}]}{[\text{base}]}$

Any buffer can be used as a buffer upto two pH units only i.e., $pK_a \pm 1$. In other words, the molar ratio of the salt and acid or base can be 10 or 1/10. If the ratio exceeds these limits, then it cannot be used as a buffer. Best buffer is one in which the molar ratio is 1, and pH = pK_a or pOH = pK_b because it can remain as a buffer for longer time.

Buffer Capacity It refers to the amount of acid or base that may be added to a buffer solution before its pH is appreciably changed. It is determined by finding out the volume of a molar alkali or molar acid or number of millimoles of mono acidic alkali or mono basic acid required to change the pH of a litre of buffer solution by 1 unit.

Solved Problems

1. The ioniziation constant of phenol is 1.0×10^{-10}. What is the concentration of phenolate ion in 0.05 M solution of phenol?

Solution The ionization of phenol can be written as

$$C_6H_5OH + H_2O \rightleftharpoons C_6H_5O^- + H_3O^+$$

α, the degree of ionization = $\sqrt{\frac{Ka}{c}} = \sqrt{\frac{1.0\times10^{-10}}{0.05}}$

$= 4.47 \times 10^{-5}$

The concentration of phenolate ion in 0.05 M solution is given by

$$c\alpha = 0.05 \times 4.47 \times 10^{-5}$$
$$= 2.2 \times 10^{-6} \text{ mol L}^{-1}$$

2. Assuming complete dissociation, calculate the pH of 0.003 M HCl.

Solution $HCl + H_2O \rightleftharpoons H_3O^+ + Cl^-$

Since it is completely dissociated so the concentration of $[H_3O^+] = 3 \times 10^{-3}$ M

$$p\text{H} = -\log [3 \times 10^{-3}] = 3.0 - 0.4771$$
$$= 2.52$$

3. Calculate the pH of 0.05 M propionic acid solution in water when its ionization constant is 1.32×10^{-5}.

Solution Acid ionizes as

$$CH_3 - CH_2 - COOH + H_2O \rightleftharpoons CH_3 - CH_2 - COO^- + H_3O^+$$

Degree of ionization

$$\alpha = \sqrt{\frac{K_a}{c}} = \sqrt{\frac{1.32\times10^{-5}}{0.05}} = 1.62 \times 10^{-2}$$

$[H_3O^+] = c\alpha$

$= 0.05 \times 1.62 \times 10^{-2} = 8.1 \times 10^{-4}\text{mol L}^{-1}$

$\text{pH} = -\log [8.1 \times 10^{-4}]$

$= 3.09$

4. Calculate the pH of a solution obtained by dissolving 0.56 g of KOH in water to give 200 mL.

Solution Molarity = $\frac{0.56\times1000}{200\times56}$ = 0.05 M

Since KOH ionises completely

$$[K^+] = [OH^-] = 0.05 \text{ M} = 5 \times 10^{-2}\text{mol L}^{-1}$$

Hydrogen ion concentration in the solution is

$$[H^+] = \frac{10^{-14}}{[OH^-]} = \frac{10^{-14}}{5\times10^{-2}} = 2 \times 10^{-13}\text{mol L}^{-1}$$

So $\text{pH} = -\log [H^+] = -\log [2 \times 10^{-13}]$

$= 12.70.$

5. Determine the solubility of lead chloride at 298 K When its $K_{sp} = 1.6 \times 10^{-5}$

Solution Let the solubility of $PbCl_2$ be S mol L^{-1}

$$PbCl_2 \rightleftharpoons Pb^{2+} + 2\,Cl^-$$

Concentration of $[Pb^+] = S$ and that of $[Cl^-] = 2S$

$K_{sp} = [Pb^{2+}]\,[Cl^-]^2$

$= S \times (2S)^2 = 4S^3$

$4S^3 = 1.6 \times 10^{-5}$

$S^3 = \frac{1.6\times10^{-5}}{4} = 4 \times 10^{-6}$

$S = (4 \times 10^{-6})^{1/3} = 1.59 \times 10^{-2} \text{ mol L}^{-1}$

6. Calculate the solubility of calcium fluoride in 0.01 M solution of potassium fluoride. K_{sp} of CaF_2 is 3.2×10^{-11} M.

Solution $KF \longrightarrow K^+ + F^-$

$\therefore$ $[F^-] = [KF] = 0.01$ M

K_{sp} of $CaF_2 = [Ca^{2+}]\,[F^-]^2$

or $[Ca^{2+}] = \frac{K_{sp}}{[F^-]^2} = \frac{3.2\times10^{-11}}{0.01\times0.01}$

$= 3.2 \times 10^{-7}$ M

Solubility of $CaF_2 = 3.2 \times 10^{-7}$ **mol L^{-1}**

7. Calculate the pH of 0.73% of mass volume solution of HCl.

Solution Molarity of 0.73% mass volume solution of HCl

$= \frac{0.73\times1000}{36.5\times100} = 2 \times 10^{-1}$M

$\therefore$ Conc of $[H_3O^+] = [HCl] = 2 \times 10^{-1}\text{mol L}^{-1}$

$p\text{H} = -\log(2 \times 10^{-1})$

$= 1 - \log 2 = 0.69$

8. In what ratio should sodium acetate and acetic acid be mixed so as to get the buffer solution with pH = 4.70 K_a for acetic acid = 1.75×10^{-5}.

Solution For buffer solution,

$$pH = pK_a + \log\frac{[\text{Salt}]}{[\text{Acid}]}$$

$$\therefore \quad 4.70 = -\log(1.75 \times 10^{-5}) + \log\frac{[\text{Salt}]}{[\text{Acid}]}$$

$$= 5 - \log 1.75 + \log\frac{[\text{Salt}]}{[\text{Acid}]}$$

$$= 4.75 + \log\frac{[\text{Salt}]}{[\text{Acid}]}$$

$$\log\frac{[\text{Salt}]}{[\text{Acid}]} = 4.70 - 4.75 = -0.05$$

$$\frac{[\text{Salt}]}{[\text{Acid}]} = 0.0891$$

Thus sodium acetate and acetic acid be mixed is the ratio of 0.0891 : 1.

Problems for Practice

1. Select the species which can act both as an acid and as a base:
(a) SO_2^- (b) HSO_4^-
(c) H_3O^+ (d) CO_3

2. 100 ml of 0.1 N NaOH is mixed with 100 ml of 0.1N H_2SO_4. The pH of the resultant solution is:
(a) < 7 (b) > 7
(c) = 7 (d) Can not be predicted

3. Solubility product principle can be applied when:
(a) A solid is insoluble in a liquid
(b) A liquid is insoluble in another liquid
(c) Any ionic compound is sparingly soluble in a liquid
(d) Substance is ionic

4. The solubility product of AgCl is K_{sp}. Then the solubility of AgCl in xM KCl is
(a) $K_{sp} \times x^2$ (b) $\frac{x}{K_{sp}}$
(c) $\frac{K_{sp}}{x^2}$ (d) $\frac{K_{sp}}{x}$

5. The correct representation for the K_{sp} of SnS_2 is
(a) $[Sn^{2+}][S^{2-}]^2$ (b) $[Sn^{4+}][S^{2-}]^2$
(c) $[Sn^{2+}][2S^{2-}]$ (d) $[Sn^{4+}][2S^{2-}]^2$

6. A monoprotic acid in a 0.1 M solution ionises to 0.001%. Its ionization constant is
(a) 1.0×10^{-3} (b) 1.0×10^{-6}
(c) 1.0×10^{-8} (d) 1.0×10^{-11}

7. The product of ionic concentrations in a saturated solution of an electrolyte at a given temperature is constant and is known as the
(a) ionic product of the electrolyte
(b) solubility product
(c) ionization constant
(d) dissociation constant

8. Which salt can be classified as an acid salt?
(a) Na_2SO_4 (b) BiOCl
(c) Pb(OH)Cl (d) Na_2HPO_4

9. Aqueous solution of a salt is alkaline. This shows that the salt is made from
(a) a strong acid and a strong base
(b) a strong acid and a weak base
(c) a weak acid and a weak base
(d) a weak acid and a strong base

10. K_{sp} for sodium chloride is 36 mol²/litre². The solubility of sodium chloride is
(a) $\frac{1}{36}$ (b) $\frac{1}{6}$
(c) 6 (d) 3600

11. If the concentration of CrO_4^{2-} ion in a saturated solution of silver chromate is 2×10^{-4} M, solubility product of silver chromate will be
(a) 4×10^{-8} (b) 8×10^{-12}
(c) 32×10^{-12} (d) 6×10^{-12}

12. The gastric juice in our stomach contains enough HCl to make the hydrogen ion concentration about 0.01 mole/litre. The *p*H of gastric juice is
(a) 0.01 (b) 1
(c) 2 (d) 14

13. The ionic product of water will increase if
(a) pressure is decreased
(b) H^+ are added
(c) OH^- are added
(d) temperature is increased

14. Buffer solution can be obtained by mixing aqueous solution of
(a) CH_3COONa and excess HCl
(b) CH_3COONa and CH_3COOH
(c) NaOH + NaCl
(d) CH_3COOH and excess NaOH

15. When the pH of a solution is 2, the hydrogen ion concentration in moles per litre is
(a) 1×10^{-14} (b) 1×10^{-2}
(c) 1×10^{-7} (d) 1×10^{-12}

16. An example of a salt dissolved in water to give acidic solution is
(a) Ammonium chloride
(b) Sodium acetate
(c) Potassium nitrate
(d) Barium bromide

17. At 100°C, $K_w = 10^{-12}$. pH of pure water at 100°C will be
(a) 7.0 (b) 6.0
(c) 8.0 (d) 12.0

18. The Ostwald dilution law can be written as:
(a) $K_a = C \times V$ (b) $K_a = (1 + \alpha)V$
(c) $K_a = \alpha^2 V$ (d) $K_a = \alpha^2/(1 - \alpha)V$

19. A 0.2 molar solution of formic acid is ionized 3.2%, its ionization constant is:
(a) 9.6×10^{-3}
(b) 2.1×10^{-4}
(c) 1.25×10^{-6}
(d) 2.1×10^{-8}

20. A solution of pH = 2 is more acidic than that of pH = 6 by a factor of
(a) 4
(b) 12
(c) 400
(d) 10,000

21. Which equilibrium can be described as an acid-base reaction using the Lewis acid-base definition but not using the Bronsted-Lowry definition?
(a) $NH_3 + CH_3COOH \rightleftharpoons CH_3COO^- + NH_4^+$
(b) $H_2O + CH_3COOH \rightleftharpoons H_3O^+ + CH_3COO^-$
(c) $4NH_3 + [Cu(H_2O)_4]^{2+} \rightleftharpoons [Cu(NH_3)_4]^{2+} + 4H_2O$
(d) $2NH_3 + H_2SO_4 \rightleftharpoons 2NH_4^+ + SO_4^{2-}$

22. H_2O can act as either an acid or a base. Which of the following reactions best illustrates the behaviour of water as a base?
(a) $HCl + H_2O \rightleftharpoons H_3O^+ + Cl^-$
(b) $HCl + NaOH \rightleftharpoons NaCl + H_2O$
(c) $H_2O + NH_2^- \rightleftharpoons NH_3 + OH^-$
(d) $H_2O + NH_3 \rightleftharpoons NH_4^+ + OH^-$

23. In the reaction $2H_2O \rightleftharpoons H_3O^+ + OH^-$ water is
(a) a weak base
(b) a weak acid
(c) both a weak acid and a weak base
(d) neither acid nor a base

24. What will be the solubility of AgCl in a 0.10 M NaCl solution?
(K_{sp} for AgCl = 1.2×10^{-10})
(a) 0.1 M
(b) 1.2×10^{-6} M
(c) 1.2×10^{-9}
(d) 1.2×10^{-10} M

25. The conjugate base of HCO_3^- is
(a) CO_3^{2-}
(b) H_2CO_3
(c) H_2O
(d) CO_2

26. pH of 10^{-10} molar HCl solution at 25°C is approximately
(a) 10
(b) 7
(c) 1
(d) 14.

27. The pH of a 10^{-8} molar solution of HCl in water is
(a) 8
(b) –8
(c) Between 7 to 8
(d) Between 6 to 7.

28. When 1.0 ml of dilute HCl is added to 100 ml of a buffer solution of pH 4, the pH of the solution
(a) Becomes 7.0
(b) Almost remains same
(c) Becomes 2.0
(d) Becomes 10.0.

29. Which of the following would be an acidic solution?
(a) Solution having hydrogen ion concentration of 10^{-7} M
(b) Solution having hydrogen ion concentration of 10^{-13} M
(c) Solution having hydrogen ion concentration of 10^{-2} M
(d) Solution having hydrogen ion concentration of 10^{-12} M.

30. The value of K_w of water at 298 K
(a) 1×10^{-14}
(b) 0.1×10^{-12}
(c) 0.1×10^{-14}
(d) 0.2×10^{-12}

31. The ions whose compounds are generally soluble in water is
(a) Acetate
(b) Nitrate
(c) Both (1) and (2)
(d) None of these

32. pH of 0.1 M NH_4Cl solution is
(a) More than 7
(b) Less than 7
(c) Nearly 7
(d) Highly acidic

33. Degree of hydrolysis (h) of a salt of weak acid and a strong base is given by
(a) $h = \sqrt{\dfrac{K_h}{c}}$
(b) $h = \sqrt{K_h}$
(c) $h = \sqrt{\dfrac{c}{K_h}}$
(d) $h = \sqrt{\dfrac{K_w}{K_b}}$

34. The K_{sp} for a sparingly soluble Ag_2CrO_4 is 4×10^{-12}, The molar solubility of the salt is
(a) 2.0×10^{-6} mol L^{-1}
(b) 1.0×10^{-4} mol L^{-1}
(c) 2.0×10^{-12} mol L^{-1}
(d) 1.0×10^{-15} mol^{-1}

35. The suppression of ionization of a weak acid or weak base by adding one of its own ion is known as
(a) Buffer capacity
(b) Buffer action
(c) Common ion effect
(d) Ionization effect

36. The pH of H_2O at 373 K is equal to
(a) 6
(b) 7
(c) 9
(d) 7.5

37. BF_3 is
(a) Lewis acid
(b) Lewis base
(c) Bronsted acid
(d) Arrhenius acid

38. pH of a salt of a strong base with weak acid
(a) $pH = \frac{1}{2}pK_w + \frac{1}{2}pK_a + \frac{1}{2}\log C$
(b) $pH = \frac{1}{2}pK_w - \frac{1}{2}pK_a - \frac{1}{2}\log C$
(c) $pH = \frac{1}{2}pK_w + \frac{1}{2}pK_a - \frac{1}{2}\log C$
(d) None of these

39. The pH of blood is
(a) < 7
(b) > 7 but < 8
(c) > 8 but < 9
(d) > 10

40. The use of NH_4Cl in detection of the third group radicals is to:
(a) To precipitate third group readicals as their chlorides
(b) Supress the conc. of OH^- ions
(c) Prevent the interference of acid radicals
(d) Decrease the conc. of NH_4^+ ions

41. Precipitation occurs only if Ionic-Product
(a) Equals K_{sp}
(b) Exceeds K_{sp}
(c) Less than K_{sp}
(d) is very small

42. Which the following has the highest pH?
(a) NaCl
(b) Na_2CO_3
(c) NH_4Cl
(d) $NaHCO_3$

43. An aqueous solution of ammonium acetate is
(a) Faintly acidic
(d) Fairly acidic
(c) Faintly alkaline
(d) Almost neutral.

44. Which of the following salts undergoes hydrolysis?
(a) CH_3COONa
(b) KNO_3
(c) NaCl
(d) K_2SO_4

45. Which of the following dissolves in water to give the most alkaline solution?
(a) CH_3COONa
(b) NaCN
(c) NaOH
(d) $NaNO_3$

46. 100 c.c of *N*/10 NaOH solution is mixed with 100 c.c of *N*/5 HCl solution and the whole volume is made to 1 litre. The pH of the resulting solution will be

(a) 1	(b) 2
(c) 3	(d) 4

47. The following reactions are known to occur in the body

$$CO_2 + H_2O \rightleftharpoons H_2CO_3 \rightleftharpoons H^+ + HCO_3^-$$

If CO_2 escapes from the system

(a) pH will decrease
(b) Hydrogen ion concentration will diminish
(c) H_2CO_3 concentration will be promoted
(d) The forward reaction will be promoted

48. The precipitate of CaF_2 ($K_{sp} = 1.7 \times 10^{-10}$) is obtained when equal volumes of the following are mixed:

(a) 10^{-4} M Ca^{2+} + 10^{-4} M F^-
(b) 10^{-2} M Ca^{2+} + 10^{-3} M F^-
(c) 10^{-4} M Ca^{2+} + 10^{-3} M F^-
(d) 10^{-3} M Ca^{2+} + 10^{-5} M F^-

49. The addition of solid sodium carbonate to pure water causes:

(a) An increases in the hydronium ion concentration
(b) An increase in pH
(c) No change in pH
(d) A decreases in the hydroxide ion concentration

50. Solubility of BaF_2 in a solution of $Ba(NO_3)_2$ will be represented by the concentration term.

(a) $[Ba^{++}]$	(b) $[F^-]$
(c) 1/2 $[F^-]$	(d) $2[NO_3^-]$

51. The solubility product of a sparingly soluble salt. *AB* at room temperature is 1.21×10^{-6}. Its molar solubility is

(a) 1.21×10^{-6}	(b) 1.21×10^{-3}
(c) 1.1×10^{-4}	(d) 1.1×10^{-3}

52. Which of the following salts undergoes hydrolysis?

(a) CH_3COONa	(b) KNO_3
(c) NaCl	(d) K_2SO_4

53. The conjugate base of ammonium ion is

(a) OH^-	(b) NH_3
(c) NH_4OH	(d) NH_4Cl

54. A solution of $FeCl_3$ in water reacts acidic due to

(a) acidic impurities	(b) ionization
(c) hydrolysis	(d) dissociation

55. A saturated solution of Ag_2SO_4 is 2.5×10^{-2} M. The value of its solubility product is

(a) 62.5×10^{-6}	(b) 6.25×10^{-4}
(c) 15.625×10^{-6}	(d) 31.25×10^{-6}

56. If the solubility of $PbCl_2$ at 25°C is 6.3×10^{-3} mol/litre, its solubility product at that temperature is

(a) $(6.3 \times 10^{-3}) \times (6.3 \times 10^{-3})$
(b) $(6.3 \times 10^{-3}) \times (12.6 \times 10^{-3})$
(c) $(6.3 \times 10^{-3}) \times (12.6 \times 10^{-3})^2$
(d) $(12.6 \times 10^{-3}) \times (12.6 \times 10^{-3})$

57. pH of the solution at 25°C is 2. If the pH is to be doubled then the hydronium ion concentration of the solution should be

(a) halved
(b) doubled
(c) increased to 100 times
(d) decreased to 100 times

58. A mixture of a weak acid (say acetic acid) and its salt with a strong base (say sodium acetate) is a buffer solution. Which other pair of substance from the following may have a similar property?

(a) HCl and NaCl	(b) NaOH and $NaNO_3$
(c) KOH and KCl	(d) NH_4OH and NH_4Cl

59. Aniline is a weak base in water. The solvent in which the aniline will become a strong base is

(a) Water	(b) NH_4OH
(c) NaOH	(d) CH_3COOH

60. When 0.1 mole of ammonia is dissolved in sufficient water to make 1 litre of solution. The solution is found to have a hydroxide ion concentration of 1.34×10^{-3} M. The dissociation constant of ammonia is

(a) 1.8×10^{-5}
(b) 1.6×10^{-6}
(c) 1.34×10^{-3}
(d) 1.8×10^{-4}

61. To a mixture of acetic acid and sodium acetate a further amount of sodium acetate is added. Then pH of the mixture

(a) Increases
(b) Decreases
(c) Is unaltered
(d) Unpredictable

62. A salt of strong acid and weak base is dissolved in water, its hydrolysis in solution is

(a) Unaffected on heating
(b) Increased by adding strong acid
(c) Suppressed by diluting
(d) Suppressed by adding strong acid

63. A 0.1 N solution of sodium bicarbonate has a pH value of

(a) 5.6	(b) 7.0
(c) 8.4	(d) 4.0

64. A solution of NaOH contain 0.04 gm of NaOH per litre its pH is

(a) 10	(b) 9
(c) 11	(d) 12

65. The conjugate acid of a strong base is a

(a) strong acid	(b) weak acid
(c) strong base	(d) weak base

66. According to Bronsted concept, base is a substance which is

(a) a proton donor
(b) an electron-pair acceptor
(c) a proton acceptor
(d) an electron-pair donor

67. A solution of sodium bicarbonate in water turns

(a) phenolphthalein pink	(b) methyl orange yellow
(c) methyl orange red	(d) blue litmus red

68. Neutralization of an acid with a base invariably results in the production of

(a) H_3O^+
(b) H_2O
(c) H^+ and OH^-
(d) OH^-

69. Which of the following is a Lewis acid?

(a) PCl_3	(b) $AlCl_3$
(c) NCl_3	(d) $AsCl_3$

70. A soda water bottle has pH

(a) < 7
(b) > 7
(c) $= 7$
(d) unpredictable

71. The term pH comes from
(a) pure hydrogen content
(b) "*pure voir hydrogene*" which implies potential of hydrogen
(c) purity of hydrogen ions in solution
(d) name of the scientist associated.

72. Which one is not an acid salt?
(a) NaH_2PO_4 (b) NaH_2PO_2
(c) NaH_2PO_3 (d) All of the above are acid salts.

73. Which of the following salts when dissolved in water hydrolyse?
(a) NaCl (b) NH_4Cl
(c) KCl (d) Na_2SO_4

74. The number of moles of acid or base required by 1 litre of buffer to alter its pH by one unit is called
(a) Buffer action
(b) *p*OH
(c) Buffer capacity
(d) Efficiency

75. Which is the correct representation for the solubility product constant of Ag_2CrO_4?
(a) $[Ag^+]^2\,[CrO_4^{2-}]$
(b) $[2Ag^+]\,[CrO_4^{2-}]$
(c) $[2Ag^+]\,[CrO_4^{2-}]$
(d) $[2Ag^+]^2\,[CrO_4^{2-}]$

76. The pH of an aqueous solution is 6. The $[OH^-]$ of the solution will be
(a) 10^{-8} M (b) 10^{-6} M
(c) 10^{-7} M (d) 10^{+8} M

77. An acidic buffer solution can be prepared by mixing equimolar amounts of
(a) formic acid and sodium formate
(b) NH_4Cl and NH_4OH
(c) H_2SO_4 and Na_2SO_4
(d) CH_3CONH_2 and KOH

78. Which one of the following has highest pH?
(a) Distilled water
(b) 1 M $NH_3(aq)$
(c) 1 M NaOH
(d) Water saturated with chlorine

79. When 100 ml of 0.1 M NaOH are added to 50 mL of 0.2 M HCl, at 25°C, the pH of the resulting solution is
(a) 7
(b) greater than 7
(c) less than 7
(d) zero.

80. A saturated solution of Ag_2SO_4 has solubility 2.5×10^{-2} M. The value of its solubility product is
(a) 62.5×10^{-6}
(b) 6.25×10^{-4}
(c) 15.625×10^{-10}
(d) 3.125×10^{-6}

ANSWERS

1. (b)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (c)
11. (c)	**12.** (c)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (d)
21. (c)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (d)	**28.** (d)	**29.** (c)	**30.** (a)
31. (c)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (a)	**37.** (a)	**38.** (a)	**39.** (b)	**40.** (b)
41. (b)	**42.** (b)	**43.** (d)	**44.** (a)	**45.** (c)	**46.** (b)	**47.** (b)	**48.** (b)	**49.** (b)	**50.** (c)
51. (d)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (a)	**56.** (c)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (a)
61. (a)	**62.** (d)	**63.** (c)	**64.** (c)	**65.** (b)	**66.** (c)	**67.** (b)	**68.** (b)	**69.** (b)	**70.** (a)
71. (b)	**72.** (d)	**73.** (b)	**74.** (c)	**75.** (a)	**76.** (a)	**77.** (a)	**78.** (c)	**79.** (a)	**80.** (a)

Electro Chemistry

8

BRIEF REVIEW OF THE CONCEPTS

Oxidation involves gain of oxygen or loss of hydrogen or loss of electrons.

Reduction involves gain of hydrogen or loss of oxygen or gain of electrons.

Oxidizing agent (or an oxidant) is an atom, molecule or ion which accepts electrons from the other reactant. An oxidizing agent oxidizes the other species and gets itself reduced.

Reducing agent (or an reductant) is an atom, molecule or ion which donates electrons to the other reactant. A reducing agent reduces the other species and gets itself oxidized.

The process of oxidation and reduction are complementary to each other and hence always occur together. Reactions in which oxidation and reduction occurs are called oxidation-reduction reactions or simply **redox reactions**. The process of oxidation and reduction can also be looked into using the concept of **oxidation number**. Oxidation can also be defined as any process in which oxidation number of an atom increases whereas reduction involves decrease in oxidation number.

Oxidation number is defined as the charge that an atom appears to have in a given species when the bonding electrons are counted using certain rules. The oxidation number may be positive or negative, whole number or a fraction. Chemical equations can also be balanced using oxidation numbers.

An **electric current** is flow of electric charge, its S.I. unit is **ampere, A.**

Coulomb (C) is defined as the quantity of electricity passed when a current of one ampere flows for one second.

Coulombs = amperes × seconds

Watt is unit of power.

Watt = current (amperes) × potential difference (volts)

Faraday One faraday is the quantity of charge of one mole of electrons.

$$\begin{aligned} 1F &= N \times e^- \\ &= 6.022 \times 10^{23} \times 1.602 \times 10^{-19}\text{C} \\ &\sim 96500 \text{ coulombs mole}^{-1} \end{aligned}$$

Electrical energy (Joule) = current (*A*) × time (*s*) × potential difference (*V*)

Metallic conductance In the metallic conductance, the charge is carried by free electrons, and no chemical reaction takes place.

Electrolytic conductance is the flow of electric current through a solution. In electrolytic conduction the charge is carried by ions and a redox reaction takes place. Electrolytic conductance increases with the rise in temperature.

Electrolytes are those substances which either in the molten state or in solution allow electric current to pass through them.

Strong electrolytes, when dissolved in water are completely ionized.

Weak electrolytes are those substances whose aqueous solutions are only partially ionized.

Ohm's law The current, I flowing through a conductor of resistance, R, is related to the potential difference, V, between the two electrodes as

$$V = IR$$

Here V is in volts, I is in amperes and R is in *ohms*.

Conductance (*C*) is the reciprocal of resistance, R.

$$C = \frac{1}{R}$$

The S.I. unit of conductance is Siemens (S), with units ohm^{-1} or mhos being commonly used.

Specific conductance or Conductivity A solution offers resistance to the flow of current through it. This resistance, R , is proportional to the length, l, and inversely proportional to the cross-sectional area, A, of the electrodes

$$R \propto \frac{l}{A}$$

or $$R = \rho \times \frac{l}{A}$$

ρ is called specific resistance or resistivity.

$$\frac{1}{\rho} = \frac{1}{R} \times \frac{l}{A}$$

The reciprocal of specific resistance is called specific conductance, κ (kappa) or conductivity.

or $$\kappa = C \times \frac{l}{A}$$

Here C is conductance.

If $l = 1$ and $A = 1$, then, $\kappa = C$

Thus specific conductance of a solution can also be defined as the *conductance of the solution in a cube of 1 cm length when current flows from one face to the other perpendicularly.*

Cell constant For a cell, the quantity $\frac{l}{A}$ is constant and is known as *cell constant.*

Specific conductance = Observed conductance × Cell constant

The units of cell constant are cm^{-1} or m^{-1}.

Equivalent conductivity, Λ_{eq} of an electrolytic solution may be defined as the conductance of a solution containing one gram equivalent of the electrolyte such that, the entire solution is placed between the two large enough electrodes placed one centimeter apart. Its units are $ohm^{-1}\ cm^2\ equiv^{-1}$ or $S\ cm^2\ equiv^{-1}$.

Equivalent conductivity, Λ_{eq} is related to specific conductivity, κ, as

$$\Lambda_{eq} = \kappa \times V$$

Here V is the volume in cm^3 which contains 1 gram equivalent of the electrolyte and κ is the specific conductance of the solution.

or $$\Lambda_{aq} = \frac{\kappa \times 1000\ cm^3}{C_{eq}} = \frac{\kappa \times 1000\ cm^3}{\text{Normality}}$$

C_{eq} is the number of gram equivalents present in one litre of the solution i.e., normality of the solution.

Molar conductivity, Λ_m is defined as the conductance of the solution which contains one mole of the electrolyte such that the entire solution is placed between the two electrodes kept one centimeter apart.

$$\Lambda_m = \kappa \times V$$

where V is the volume in cm^3 containing one mole of the electrolyte, or

$$\Lambda_m = \frac{\kappa \times 1000\ cm^3}{M}$$

Here M is number of moles of the electrolyte (*i.e.*, molarity) present in one litre of the solution. The units of molar conductivity are $ohm^{-1}\ cm^2\ mol^{-1}$ or $S\ cm^2\ mol^{-1}$.

Relation between equivalent conductivity, Λ_{eq}, and molar conductivity, Λ_m is

$$\frac{\Lambda_m}{\Lambda_{eq}} = \frac{\text{Molar mass of the electrolyte}}{\text{Equivalent mass of the electrolyte}}$$

Variation of specific conductance on dilution, Since on dilution the number of ions per unit volume decreases, the specific conductance, κ, decreases on dilution.

Variation of equivalent or molar conductivity with concentration:

(i) **For strong electrolytes.** The initial value of Λ_m or Λ_{eq} for strong electrolytes is quite high because even in concentrated solutions they are almost completely ionized. *There is increase in the value of* Λ_m or Λ_{eq} on dilution. This increase is mainly due to decrease in the inter ionic attractions whereby the speed of ions increases.

At low concentrations the relation between molar conductivity and concentration is

$$\Lambda_m = \Lambda_m^\infty - b\sqrt{c}$$

Here b is a constant c is the molar concentration and Λ_m^∞ is the molar conductivity at infinite dilution or zero concentration.

(Λ_m^∞ for strong electrolytes is determined by extrapolating the curve between Λ_m and $\sqrt{c}$).

(ii) **For weak electrolytes** At high concentration the value of Λ_m or Λ_{eq} for weak electrolytes is quit low because of partial ionization of these electrolytes at high concentration, but on dilution there is a significant increase in their values rather it increases exponentially.

Kohlrausch's law of independent migration of ions It states that, at infinite dilution, when dissociation is complete, each ion makes a definite contribution towards the molar conductivity of the electrolyte irrespective of the nature of the other ions present in the solution and that the value of molar conductivity at infinite dilution for any electrolyte is given by the sum of contributions from its individual ions. If the molar conductance of the cation is denoted $\lambda c\infty$ and that of the anion λ_a^∞, then

$$\Lambda_m^\infty = \nu_c\lambda_c^\infty + \nu_a\lambda_a^\infty$$

where ν_c and ν_a are the number of cations and anions per formula unit of electrolyte (e.g., in case of $CaCl_2$, $\nu_c = 1$ and $\nu_a = 2$).

Relation between the degree of dissociation, α of weak electrolyte and the molar conductivity

$$\alpha = \frac{\Lambda_m^c}{\Lambda_m^\infty}$$

Here Λ_m^c and Λ_m^∞ are molar conductivities at the concentration, c and at infinite dilution respectively.

Electrolysis *is the decomposition of a compound (molten or in solution) by the passage of electric current through it.* The products of electrolysis appear only at the two conducting electrodes from where the current actually enters or leaves the electrolyte.

Quantitative aspect of electrolysis. Consider the reactions

$$M^{n+}(aq) + ne^- \longrightarrow M(s) \qquad ...(i)$$

$$2\,Cl^-(aq) \longrightarrow Cl_2(g) + 2e^- \qquad ...(ii)$$

in Eq. (*i*) n moles of electrons liberate 1 mole of M whereas in Eq. (*ii*) two moles of Cl^- ions produce one mole of $Cl_2(g)$ and 2 moles of electrons.

The charge, Q of n moles of electrons is given by $Q = n \times F$

Half cell. Each half cell consists of a metallic electrode which is dipped into an electrolyte. The electrochemical reaction, either oxidation or reduction which takes places at this electrode is called the **half-cell reaction**.

Cell is a device for producing an electric current by chemical action or vice versa. It consists of two half cells joined together.

Salt bridge is a glass tube bent at right angles at two places, which contains a solution of an electrolyte, usually KCl, in agar agar. It connects the solutions in the two half-cells.

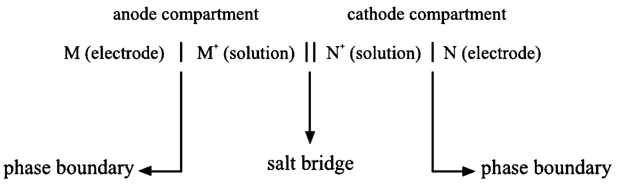
Galvanic or voltaic cell is a device in which chemical energy is converted into electrical energy. In this case work is done by the system, e.g., Daniel cell, dry cell, lead storage battery etc.

Representation of Galvanic cell

(i) *Oxidation always occurs* at the anode which is written on the left hand side.

(ii) *Reduction always takes place at the cathode* which is written on the right hand side.

A cell is depicted as follows:



where *M* and *N* are metals and a single line (I) indicates a phase difference between a solid electrode and solution. The double line (II) is a notation for the *salt bridge* connecting the anodic and cathodic compartments. The notations tell us that the reaction is

$$M + N^+ \longrightarrow M^+ + N$$

$$M(s) + N^+(aq) \longrightarrow M^+(aq) + N(s)$$

The concentrations of electrolytes may also be mentioned in brackets.

Electrode potential When a metal strip (*M*) is placed into a solution of its own ions (M^{n+}), a potential difference is set up between the metal and the solution. This is known as single electrode potential. It can also be defined as the tendency of the metal in equilibrium with its ions to gain or lose electrons.

$$M(s) \rightleftharpoons M^{n+}(aq) + ne^-$$

Standard electrode potential, E^o The electrical potential developed between the metal electrode in contact with a solution of its own ions having unit concentration (i.e., one mole per litre). All standard electrode potential E°_{el} are measured when coupled with a standard hydrogen electrode.

Standard/Normal hydrogen electrode It consists of a platinized foil sealed in tube which is dipped into one molar pure HCl free from all impurities. H_2 gas at 1atm pressure is maintained above the HCl. The temperature is 298 K. The half-cell is represented as

$$Pt, H_2(g, 1\ atm) \mid H^+\ (1\ M)$$

The standard electrode potential of *NHE* (*SHE*) is always taken to be zero.

Sign of electrode potential When at the electrode which is coupled with the standard hydrogen electrode (*SHE*), reduction takes place then at the *SHE* oxidation will take place, the electrode potential is taken as positive. The half reactions taking place at the electrode and *SHE* can be represent as

At Electrode $Mn^+(aq) + ne^- \longrightarrow M(s)$

At *SHE* $H_2(g) \longrightarrow 2H^+\ (aq) + 2e^-$

Standard electrode potentials, E°_{el} are always associated with the reduction process occurring at the electrode. Electrodes which have higher positive value of electrode potentials are easily reduced and hence act as strong oxidizing agents e.g., $F_2/2F^-$ with E°_{el} value + 2.85V is a very strong oxidizing agent.

EMF of a cell, E^o_{cell} can be calculated from E^o_{el} values using the formula

$$E^o_{cell} = E^o_{cathode} - E^o_{anode}$$

Electrochemical series or Activity series The different values of standard electrode potentials for various elements are arranged in the increasing order of to give the electrochemical series. *Li*⁺/*Li* electrode has the lowest value of E°_{el} (– 3.04 V) is at the top of the ***ECS***.

Applications of *ECS*

(i) Calculation of standard emf of a cell using E°_{el} values.

(ii) Any metal will displace another metal lower in the series, from the solutions of its ions e.g., iron displaces copper from the solution of Cu^{2+} ions.

(iii) Metals above hydrogen in the *ECS*, will liberate H_2 gas on reaction with an acid. For example,

$$Zn(s) + H_2SO_4 \longrightarrow ZnSO_4 + H_2(g)\uparrow$$

But a metal with a positive value of E°_{el} i.e., which is below hydrogen in *ECS* will not produce H_2 when it reacts with an acid.

$$Cu(s) + 4HNO_3 \longrightarrow Cu(NO_3)_2 + 2NO_2 + 2H_2O$$

(iv) *Predicting a reaction.* The direction in which a given redox reaction proceeds is determined by E°_{el} values.

For example: Given, $E^{\circ}_{Fe^{2+}/Fe}$ = – 0.44 V and $E^{\circ}_{Cu^{2+}/Cu}$ = + 0.34 V.

Here standard electrode potential of Cu^{2+}/Cu is higher than that of Fe^{2+}/Fe, thus Fe has greater reducing tendency compared to Cu^{2+}. So Cu^{2+} ions will be reduced in preference to Fe^{2+} ions. The reaction will be

$$Fe(s) + Cu^{2+}(aq) \longrightarrow Fe^{2+}(aq) + Cu(s)$$

Nernst equation for electrode and cell potentials For a hypothetical reaction

$$M^{n+} + ne^- \longrightarrow M$$

the electrode potential at different concentrations is calculated with the help of Nernst equation.

For the above reaction

$$E^o_{M^{n+}/M} - \frac{RT}{nF} In \frac{1}{[M^{n+}]}$$

or

$$E^o_{M^{n+}/M} + \frac{RT}{nF} In [M^{n+}]$$

Substituting the values of constants and converting natural logarithms to common logarithms, we get

$$E^o_{M^{n+}/M} - \frac{0.059}{n} \log [M^{n+}]$$

Nernst equation for the cell potential at 298 K

$$E_{cell} = E^o_{cell} - \frac{0.059}{n} \log \frac{[Products]}{[Reactants]}$$

or

$$E_{cell} = E^o_{cell} + \frac{0.059}{n} \log \frac{[Reactants]}{[Products]}$$

Equilibrium constant, K_c, and the standard cell potential Consider the redox reaction taking place in a Daniel cell

$$Zn(s) + Cu^{2+}(aq) \rightleftharpoons Zn^{2+}(aq) + Cu(s)$$

Equilibrium constant, K_c, is given by

$$K_c = \frac{[Zn^{2+}(aq)]_{equi^m}}{[Cu^{2+}(aq)]_{equi^m}}$$

Using Nernst equation, the cell potential is given by

$$E_{cell} = E^o_{cell} - \frac{2.303\,RT}{nF} \log \frac{[Zn^{2+}(aq)]_{equi^m}}{[Cu^{2+}(aq)]_{equi^m}}$$

At equilibrium $E_{cell} = 0$. Then

$$E^o_{cell} = \frac{2.303\,RT}{nF} \log \frac{[Zn^{2+}(aq)]_{equi^m}}{[Cu^{2+}(aq)]_{equi^m}}$$

or

$$E^o_{cell} = \frac{2.303\,RT}{nF} \log K_c$$

or at 298 K

$$E^o_{cell} = \frac{0.059}{n} \log K_c$$

or

$$\log K_c = \frac{nE^o_{cell}}{0.059}$$

When a voltaic cell operates, it does work against the surroundings. If a charge equivalent to nF is passed under a cell potential, E_{cell} the decrease in free energy, ΔG, or work done is given by

$$-\Delta G = w_{max} = nFE_{cell} \text{ or } -\Delta G^o = nFE^o_{cell}$$

All spontaneous cell reactions will have a positive value of cell potential.

If an electrolytic cell contains more than one substance, the substance which has the highest electrode potential is reduced first at the cathode.

In aqueous solution ions which have higher reduction potential than that of water,

$$\frac{1}{2}O_2(g) + 2H^+(aq) + 2e^- \longrightarrow H_2O(l) \quad E^\circ = 1.23 \text{ V}$$

will be reduced first at the cathode. Substance which have lower electrode potential than that of water (+ 1.23V) will be oxidized at the anode.

If aqueous solution of Na^+, Ca^{2+} or Al^{3+} ions is electrolysed, then on electrolysis it is H_2O and not either of Na^+, Ca^{2+} or Al^{3+} ions, which is reduced since the electrode potential of H_2O is much higher than that of Na^+, Ca^{2+} or Al^{3+} ions.

Aqueous solution of Cu^{2+} or Ag^+ ions is when electrolysed, it is Cu^{2+} or Ag^+ ions which are reduced since their electrode potential is higher than that of H_2O.

Battery is two or more voltaic cells connected in series.

Dry cell

Anode is zinc container. **Cathode** is graphite rod placed in the zinc container and surrounded by powdered MnO_2 and carbon. **Electrolyte** is paste of NH_4Cl and $ZnCl_2$ in water which is filled in the space between zinc container and powdered MnO_2 and C.

Electrode reactions

$$Zn(s) \longrightarrow Zn^{2+} + 2e^- \qquad (\textit{anode})$$

$$MnO_2 + NH_4^+ + e^- \longrightarrow MnO(OH) + NH_3 \qquad (\textit{cathode})$$

Here Mn^{4+} is reduced to Mn^{3+}.

Ammonia so evolved combines with Zn^{2+} to form $Zn(NH_3)_4^{2+}$ ion.

Cell voltage is 1.25 – 1.50 V.

Mercury cell

Anode: Zinc mercury amalgam

Cathode: HgO and C

Electrolyte: KOH + ZnO as a paste

Cell reaction

Anode reaction:

$$Zn(amalgam) + 2OH^- \longrightarrow ZnO(s) + H_2O(l) + 2e^-$$

Cathode reaction:

$$HgO(s) + H_2O(l) + 2e^- \longrightarrow Hg(l) + 2OH^-$$

Overall reaction:

$$Zn(amalgam) + HgO(s) \longrightarrow ZnO(s) + Hg(l)$$

No gas is evolved and the cell potential is approximately 1.35 V.

Storage batteries/Accumulators or Secondary cells are cells which can be recharged by passing electric current through it which reverses the redox reaction.

Lead accumulators It consists of a number of voltaic cells connected in series. Usually three to six such cells are connected to give 6 to 12 V battery. Each cell consists of *anode* made of porous lead and **cathode** is made of lead dioxide (PbO_2) and lead.

Electrolyte is H_2SO_4 having a density of 1.30 g ml^{-1}.

Anode reaction:

$$Pb(s) + SO_4^{2-}(aq) \longrightarrow PbSO_4(s) + 2e^-$$

Cathode reaction:

$$PbO_2(s) + 4H^+ + 2SO_4^{2-} + 2e^- \longrightarrow PbSO_4(s) + 2H_2O(l)$$

Cell reaction:

$$Pb(s) + PbO_2(s) + 4H^+ + 2SO_4^{2-} \longrightarrow 2PbSO_4(s) + 2H_2O(l)$$

Fuel cells In fuel cells the energy is produced from the combustion of fuel gases such as hydrogen, carbon monoxide or methane directly, in the form of electrical energy.

Hydrogen-oxygen fuel cell Porous carbon electrodes are used as anode and cathode placed into concentrated sodium hydroxide solution in which hydrogen and oxygen gases are fed in a controlled manner.

Anode reaction:

$$2H_2(g) + 4OH^-(aq) \longrightarrow 4H_2O(l) + 4e^-$$

Cathode reaction:

$$O_2(g) + 2H_2O(l) + 4e^- \longrightarrow 4OH^-(aq)$$

Cell reaction:

$$2H_2(g) + O_2(g) \longrightarrow 2H_2O(l)$$

Solved Problems

1. Calculate the number of coulombs required to deposit 5.4 g of Al when the electrode reaction is

$$Al^{3+} + 3e^- \rightarrow Al$$

(Atomic mass of Al = 27.0 g mol^{-1}: F = 96500C)

Solution From the electrode reaction. 3F of charge is required to produce one mole or 27.0 g of Al.

Charge required for 27 g of Al = 3 × 96500 C

Charge required for 5.4 g of Al

$$= \frac{3 \times 96500\,C}{27\,g} \times 5.4 \text{ g}$$

$$= 57900 \text{ C}$$

2. A solution of $Ni(NO_3)_2$ is electrolysed between platinum electrodes using a current of 5.0 amperes for 30 minutes. What mass of Ni will be produced at the cathode? (Atomic mass of Ni is 58.7 g mol^{-1})

Solution Cathode reaction is

$$Ni^{2+} + 2e^- \longrightarrow Ni\ (s)$$

1 mol i.e., 58.7 g of Ni is deposited by 2 × 96500 C of charge.

Charge, Q = current (amperes) × time (second)

$= 5 \times 30 \times 60$ C

2 × 96500 C produce 58.7 g Ni

$$5 \times 30 \times 60 \text{ C will produce } \frac{58.7g \times 5 \times 30 \times 60\,C}{2 \times 96500\,C}$$

$= 2.736$ g Ni

3. Calculate the equivalent conductivity of 1M H_2SO_4 solution. If its conductivity is 2.6×10^{-2} ohm^{-1} cm^{-1}.

Solution $\Lambda_{eq} = \frac{\kappa \times 1000}{N}$

here κ is conductivity

N is normality

1 M H_2SO_4 contains 1 mole or 2 gram equivalents of H_2SO_4 per litre of acid solution. Thus normality of acid solution is 2.

Substituting the values

$$\Lambda_{eq} = \frac{2.6 \times 10^{-2} \text{ohm}^{-1}\text{ cm}^{-1} \times 1000}{2 \text{ g equiv. cm}^{-3}}$$

$= 13.0$ ohm^{-1} cm^2 equiv^{-1}

4. Calculate molar conductance at infinite dilution for acetic acid, given

Λ_m^∞ HCl = 425 ohm^{-1} cm^2 mol^{-1}

Λ_m^∞ NaCl = 188 ohm^{-1} cm^2 mol^{-1}

Λ_m^∞ CH_3COONa = 96 ohm^{-1} cm^2 mol^{-1}

Solution Λ_m^∞ CH_3COOH = Λ_m^∞ CH_3COONa + Λ_m^∞ HCl – Λ_m^∞ NaCl

Substituting the values

Λ_m^∞ CH_3COOH = 96 ohm^{-1} cm^2 mol^{-1} + 425 ohm^{-1} cm^2 mol^{-1} – 188 ohm^{-1} cm^2 mol^{-1}

= 333 ohm^{-1} cm^2 mol^{-1}

5. The resistance of 0.01M NaCl solution in a conductivity cell was found to be 210 Ω. The specific conductance of this solution is 4.5×10^{-3} S cm^{-1}. What is the cell constant of this cell?

Solution Given: Resistance, $R = 210\Omega$

Specific conductance = 4.5×10^{-3} S cm^{-1}

Molarity of NaCl solution = 0.01 M

Cell constant = ?

You know

Specific conductance = Conductance × Cell constant

or Specific conductance = $\frac{\text{Cell constant}}{\text{Resistance}}$

Therefore,

Cell constant = Specific conductance × Resistance

Substituting the values

Cell constant = 4.5×10^{-3} S cm^{-1} × 210 ohm

= 945×10^{-3} cm^{-1}

6. Predict the reaction, if any, that occurs between Ag^+(aq) and Cu(s).

(Given $E^o_{Ag^+/Ag} = 0.80$ V, $E^o_{cu^{2+}/cu} = +\,0.34$V)

Solution Given $E^o_{Ag^{2+}/Ag} = 0.80$ V

$E^o_{cu^{2+}/cu} = 0.34$ V

Since the electrode potential of Ag^+/Ag electrode is higher than that of Cu^{2+}/Cu electrode, it will function as a cathode, and Cu^{2+}/Cu electrode will be anode,

$$E^o_{\text{cell}} = E^o_{\text{cathode}} - E^o_{\text{anode}}$$
$$= 0.80 \text{ V} - 0.34 \text{ V} = 0.46 \text{ V}$$

Since the *e.m.f.* of the cell is positive the reaction will be, where Ag^+ is reduced and Cu is oxidized. Cell reaction

$$2Ag^+\ (aq) + Cu(s) \rightarrow Cu^{2+}(aq) + 2Ag(s)$$

7. Calculate the cell potential for the cell

Mg(s)/Mg^{2+}(1M) || Cu^{2+} (1M) | Cu(s).

($E^o_{Mg^{2+}/Mg} = -\,2.37$ V and $E^o_{Cu^{2+}/Cu} = 0.34$ V)

Solution Cell potential $E^o_{\text{cell}} = E^o_{\text{cathode}} - E^o_{\text{anode}}$

$= 0.34 \text{ V} - (-2.37 \text{ V})$

$= 2.71$ V

8. Calculate the E.M.F. of cell

Mg(s) /Mg^{2+}(0.1 M)| | Ag^+ (1×10^{-4} M) | Ag (s).

$E^o_{Ag^+/Ag} = 0.80$ V, $E^o_{Mg^{2+}/Mg} = -\,2.37$ V

Solution $E^o_{\text{cell}} = E_{\text{cathade}} - E_{\text{anode}}$

$= 0.80 \text{ V} - (-2.37 \text{ V})$

$= +\,3.17$ V

Anode reaction $Mg(s) \rightarrow Mg^{2+} + 2e^-$

Cathode reaction $2Ag^+ + 2e^- \rightarrow 2Ag$

Overall cell reaction:

$$Mg(s) + 2Ag^+ \rightarrow Mg^{2+} + 2Ag$$

Using Nernst equation

$$E_{\text{cell}} = E^o_{\text{cell}} - \frac{0.059}{n}\log\frac{[Mg^{2+}]}{[Ag^+]^2}$$

Substituting the values

$$E_{\text{cell}} = 3.17 - \frac{0.059}{2}\log\frac{[0.1]}{[1\times10^{-4}]^2}$$

$$= 3.17 - \frac{0.059}{2}\log 10^7$$
$$= 2.96 \text{ V}$$

9. Calculate the e.m.f. at 25°C for a cell

Ag(s)|$AgNO_3$(0.01 M) || $AgNO_3$(1.0 M)| Ag(s).

Solution This is a concentration cell.

The e.m.f. of a concentration cell of given by:

$$E_{\text{cell}} = \frac{0.059}{n}\log\frac{C_2}{C_1}$$

Here C_2 and C_1 are concentrations in the cathodic and anodic compartments respectively.

Here $n = 1$, $C_1 = 0.01$ M, $C_2 = 1.0$ M

Substituting the values

$$E_{\text{cell}} = \frac{0.059}{1}\log\frac{1.0}{0.01}$$

$= 0.059 \times 2 = 0.118$ V

10. Determine the value of standard free energy change, ΔG° for the cell

$Mg|Mg^{2+}$ (0.001 M) | | Cu^{2+} (0.0001 M) | Cu

Given $E^\circ(Cu^{2+}/Cu) = 0.34$ V

$E^\circ(Mg^{2+}/Mg) = -2.37$ V

$F = 96500$ C mol^{-1}

Solution Anode reaction: $Mg(s) \rightarrow Mg^{2+} + 2e^-$

Cathode reaction: $Cu^{2+} + 2e^- \rightarrow Cu(s)$

Cell reaction: $Mg(s) + Cu^{2+} \rightarrow Mg^{2+} + Cu$.

$$E^\circ_{cell} = E_R - E_L$$

$$= 0.34\text{ V} - (-2.37\text{ V})$$

$$= 2.71\text{ V}$$

$$E_{cell} = E^\circ_{cell} - \frac{0.0592}{2}\log\frac{[Mg^{2+}]}{[Cu^{2+}]}$$

$$= 2.71\text{ V} - 0.0296\log\frac{(0.001)}{(0.0001)}$$

$$= (2.71 - 0.0296)\text{ V}$$

$$= 2.68\text{ V}$$

$$\Delta G^\circ = -nFE^\circ$$

$$= -2 \times 96500\text{ C} \times 2.71\text{ V}$$

$$= -523\text{ kJ}$$

Problems for Practice

1. The combination joule/volt of electrical units when expressed as a single unit is

(a) coulomb (b) ohm

(c) ampere (d) watt

2. A solution of copper (II) sulphate is electrolysed using copper electrodes by a current of 0.5 A for exactly 1 hour. Which of the following statements is *true* for the above electrolysis?

(a) 2 moles of electrons are liberated at the anode

(b) 2 moles of electrons are liberated at the cathode

(c) concentration of Cu^{2+} (*aq*) increases in the solution

(d) the original Cu^{2+} ions concentration of the solution remains unchanged

3. For a redox reaction to proceed spontaneously in a cell, the e.m.f. of the cell must be

(a) positive (b) zero

(c) 1.0 V (d) negative

4. How many faradays are required to reduce a mole of Fe^{3+} to Fe?

(a) 6 (b) 3

(c) 2 (d) 1

5. What is the anode reaction during electrolysis of copper (II) sulphate using Pt electrodes?

(a) $2H_2O \rightarrow O_2 + 4H^+ + 4e^-$

(b) $2SO_4^{2-} \rightarrow S_2O_8^{2-} + 2e^-$

(c) $2H_2O + 2e^- \rightarrow H_2 + 2OH^-$

(d) $2H^+ + 2e^- \rightarrow H_2$

6. If a steady current of one ampere is passed into molten NaCl for 5.0 minutes. How much Cl_2 (g) will be liberated?

(a) 0.110 g (b) 0.177 g

(c) 1.775 g (d) 3.550 g

7. Calculate the *e.m.f.* of the following cell

Pt; $H_2(g)$ | H^+ (10^{-8} M) | | H^+ (0.001 M) $H_2(g)$; Pt.

(a) – 0.059 V (b) – 0.295 V

(c) 0.059 V (d) 0.295 V

8. Which of the following metals is the best reducing agent?

$E^\circ_{Ag^+/Ag} = 0.80$ V $E^\circ_{Hg^{2+}/Hg} = 0.79$ V

$E^\circ_{cr^{3+}/cr} = -0.74$ V $E^\circ_{Mg^{2+}/Mg} = -2.37$ V

(a) Mg (b) Cr

(c) Ag (d) Hg

9. What is the oxidizing agent in a lead storage battery?

(a) Pb (b) PbO_2

(c) H_2SO_4 (d) $PbSO_4$

10. In which of the following, oxidation number of chlorine is + 5

(a) Cl_2O_7 (b) ClO_3^-

(c) ClO^- (d) ClO_4^-

11. In which of the following compounds, iron has lowest oxidation state?

(a) $FeSO_4 \cdot (NH_4)_2SO_4 \cdot 6H_2O$

(b) $K_4[Fe(CN)_6]$

(c) Fe_2O

(d) K_2FeO_4

12. Electrochemical equivalent of a substance is equal to its quantity liberated on passing electricity equal to

(a) One coulomb (b) One ampere

(c) One volt (d) 96,500 coulombs

13. How many atoms of hydrogen are liberated at cathode when 965 C of charge is passed through water?

(a) 6.02×10^{21} (b) 6.02×10^{23}

(c) 6.02×10^{19} (d) $1/6.02 \times 10^{23}$

14. In a reaction between zinc and iodine, zinc iodide is formed. What is being oxidized?

(a) Zinc ions (b) Iodide ions

(c) Zinc atom (d) Iodine

15. Silver nitrate solution turns blue when a piece of copper is added to it because of

(a) Oxidation of Ag (b) Reduction of Cu

(c) Oxidation of Cu (d) Reduction of Ag

16. One mole of Al is deposited by *X* coulombs of electricity passing through aluminium nitrate solution. The number of moles of silver deposited by *X* coulombs of electricity from silver nitrate solution is

(a) 3 (b) 4

(c) 2 (d) 1

17. The Kohlrauch's law is connected with

(a) Conductance of ions at infinite dilution

(b) Independent migration of ions

(c) Both of these

(d) None of these

18. Electrolysis of an aqueous solution of NaOH between platinum electrodes yields

(a) H_2 at cathode

(b) O_2 at anode

(c) H_2 at cathode, O_2 at anode

(d) H_2 at anode, O_2 at cathode

19. Electrolytes when dissolved in water dissociate into their constituent ions. The degree of dissociation of an electrolyte increases with

(a) increasing concentration of electrolyte

(b) decreasing concentration of electrolytes

(c) decreasing temperature

(d) the presence of substance yielding common ions

20. On carrying out electrolysis of acidified water, the volume of hydrogen liberated at *N.T.P.* is 22.4 L. The volume of oxygen liberated is

(a) 22.4 L (b) 44.8 L

(c) 11.2 L (d) 2.24 L

21. Which of the following cells has maximum E_{cell}

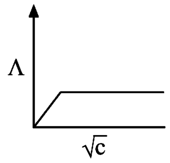
(a) $Zn_{(s)} \mid Zn^{2+}_{(0.2\ M)} \parallel Cu^{2+}_{(0.2\ M)} \mid Cu_{(s)}$

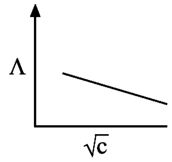
(b) $Zn_{(s)} \mid Zn^{2+}_{(0.002\ M)} \parallel Cu^{2+}_{(0.2\ M)} \mid Cu_{(s)}$

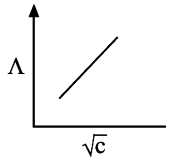
(c) $Zn_{(s)} \mid Zn^{2+}_{(0.2\ M)} \parallel Cu^{2+}_{(0.002\ M)} \mid Cu_{(s)}$

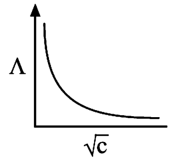
(d) All of these have same value

22. The variation of equivalent conductance of strong electrolyte with concentration is correctly shown in which figure?

(a)


(b)


(c)


(d)


23. The standard electrode potentials of two half-cells are given below

$Ni^{++} + 2e = Ni;$ $E° = -0.25$ Volt

$Zn^{++} + 2e = Zn;$ $E° = -0.77$ Volt

The voltage of the cell formed by combining the two half-cells would be

(a) – 1.02 Volt (b) + 0.52 Volt

(c) + 1.02 Volt (d) – 0.52 Volt

24. When an electric current is passed through acidulated water, 112 ml of hydrogen gas at STP collect at the cathode in 965 seconds. The current passed, in ampere, is

(a) 1.0 (b) 0.5

(c) 0.1 (d) 2.0

25. Given standard electrode potentials

$E°$

$Fe^{++} + 2e^- \rightarrow Fe\ -0.440$ V

$Fe^{+++} + 3e^- \rightarrow Fe\ -0.036$ V

The standard electrode potential (E°) for

$Fe^{+++} + e^- \rightarrow Fe^{++}$ is

(a) – 0.476 V (b) + 0.404 V

(c) + 0.404 V (d) + 0.771 V

26. One faraday of current was passed through the electrode cells placed in series containing solutions of Ag^+, Ni^{++} and Cr^{+++} respectively. The amount of Ag (at. wt. 108), Ni (at. wt. 59) and Cr (at. wt. 52.0) deposited will be

Silver	Nickel	Chromium
(a) 108 g	29.5 g	17.3 g
(b) 108 g	59.0 g	52.0 g
(c) 108 g	108.0 g	108.0 g
(d) 108 g	117.5 g	166.0 g

27. At 298 K standard reduction potential of half reduction reactions are given:

$Zn^{+2} + 2e^- \rightarrow Zn\ (s) - 0.762$

$Cr^{+3} + 3e^- \rightarrow Cr(s) - 0.740$

$2H^+ + 2e^- \rightarrow H_2(g) + 0.000$

$Fe^{+3} + e^- \rightarrow Fe^{+2} + 0.770$

Which one is the strongest reducing agent?

(a) Zn(*s*) (b) Cr(*s*)

(c) $H_2(g)$ (d) $Fe^{+2}(aq)$

28. Which of the following occur at cathode during electrolysis of fused lead bromide?

(a) $Pb \rightarrow Pb^{2+}\ 2e^-$ (b) $Br + e^- \rightarrow Br^-$

(c) $Br^- \rightarrow Br + e^-$ (d) $Pb^{2+} + 2e^- \rightarrow Pb$

29. In the electrolysis of water, one faraday of electrical energy would evolve

(a) One mole of oxygen

(b) One g of oxygen

(c) 8 g of oxygen

(d) 22.4 litres of oxygen

30. Maximum number of moles of oxygen gas that can be obtained by the electrolytic decomposition of 90 g of water will be

(a) 1 (b) 2.5

(c) 5 (d) 9

31. A solution containing one mole per litre of each $Cu(NO_3)_2$; $AgNO_3$; $Hg_2(NO_3)_2$; $Mg(NO_3)_2$ is being electrolysed by using inert electrodes. The values of standard electrode potentials in volts (reduction potentials) are $Ag^+/Ag = +0.80$, $Hg_2^{2+}/2Hg = +0.79$, $Cu^{2+}/Cu = +0.34$, $Mg^{2+}/Mg = -2.37$. With increasing voltage, the sequence of deposition of metals on the cathode will be

(a) Ag, Hg, Cu, Mg

(b) Mg, Cu, Hg, Ag

(c) Ag, Hg, Cu

(d) Cu, Hg, Ag

32. Two platinum electrodes were immersed in a solution of cupric sulphate and electric current passed through the solution. After some time it was found that the colour of copper sulphate disappeared with evolution of gas at the electrode. The colourless solution contains

(a) platinum sulphate (b) copper hydroxide

(c) copper sulphate (d) sulphuric acid

33. The number of faradays required to deposit one mole atoms of a metal M, when a solution of its ions M^{2+} is electrolyzed, is

(a) 1/2 (b) 1

(c) 2 (d) 96500

34. Which of following has highest value of standard reduction potential?

(a) F_2 (b) Cl_2

(c) Br_2 (d) I_2

35. The same quantity of electricity that liberated 2.158 g of silver was passed through a solution of a gold salt and 1.314 g gold was deposited. The equivalent weight of silver is 107.9. The equivalent weight of gold is

(a) 65.7 (b) 131.4

(c) 197.1 (d) 49.275

36. Which of following is not correct?

(a) Molar conductance of a solution increases with dilution

(b) Equivalent conductance increases with dilution

(c) Specific conductance increases with dilution

(d) At infinite dilution each ion (cation or anion) plays a definite role towards electrical conductance

37. The standard oxidation potential, E°, for the half reactions are as

$Zn \rightarrow Zn^{2+} + 2e^-$; E° = + 0.76 V

$Fe \rightarrow Fe^{2+} + 2e^-$; E° = + 0.41 V

The EMF for the cell reaction

$Fe^{2+} + Zn \rightarrow Zn^{2+} + Fe$ is

(a) – 0.35 V (b) + 0.35 V

(c) + 1.17 V (d) – 1.17 V

38. $E_0 = \frac{RT}{nF} \ln K_{eq}$

The above equation is called

(a) Gibb's equation

(b) Gibb's-Helmholtz equation

(c) Nernst equation

(d) van der Waals equation

39. The standard oxidation potentials of zinc and silver in water at 298 K are

$Zn(s) \rightarrow Zn^{++} + 2e^-$ E° = 0.76 V

$Ag(s) \rightarrow Ag^+ + e^-$ E° = – 0.80 V

Which of the reactions actually take place?

(a) $Zn(s) + 2Ag+(aq) \rightarrow Zn^{++}(aq) + 2Ag(s)$

(b) $Zn^{++} + 2Ag(s) \rightarrow 2Ag^+(aq) + Zn(s)$

(c) $Zn(s) + Ag(s) \rightarrow Zn^{++}(aq) + Ag^+(aq)$

(d) $Zn^{++}(aq) + Ag^+(aq) \rightarrow Zn(s) + Ag(s)$

40. Hydrazine, N_2H_4, can be used as the reducing agent in a fuel cell

$$N_2H_4(aq) + O_2(g) \rightarrow N_2(g) + 2H_2O\ (l)$$

If $\Delta G°$ for this reaction is – 60 kJ, what will be the theoretical voltage, E° for the cell

(a) 1.25 V (b) 1.50 V

(c) 1.57 V (d) 1.75 V

41. Electrolysis of dilute H_2SO_4 produces $H_2(g)$ at the cathode. If a minimum voltage required is 1.24 V. How many joules of energy will be used up to produce 1.0 kg of $H_2(g)$?

(a) 1.18 J (b) 11.8×10^5 J

(c) 118×10^3 J (d) 1.18×10^8 J

42. Which of the following statement is not true for corrosion?

(a) Corrosion involves transfer of electrons.

(b) Presence of CO_2 in water retards the rate of corrosion.

(c) Corrosion is faster at high temperature.

(d) Cathode protection of iron from corrosion involves connecting iron with aluminium wire.

43. For which of the following solutions, water is oxidized at anode on electrolysis?

(a) KBr (aq) (b) KI (aq)

(c) NaF (molten) (d) NaF (aq)

44. Which of the following halogens is the best oxidizing agent?

(a) F_2 (b) Cl_2

(c) Br_2 (d) I_2

45. In the electrochemical corrosion of metals, the metal undergoing corrosion

(a) becomes anode (b) becomes cathode

(c) becomes inert (d) None is correct

46. How many faradays are required to generate one gram atom of magnesium from $MgCl_2$?

(a) 1 (b) 2

(c) 3 (d) 4

47. In a salt bridge, KCl is used because:

(a) It is an electrolyte

(b) It is good conductor of electricity

(c) The transport number of K^+ and Cl^- ions are nearly same

(d) It is ionic compound

48. In the following equation

$3Br_2 + 6CO_3^{2-} + 3H_2O \rightarrow 5Br^- + BrO_3^- + 6HCO_3^-$

(a) Bromine is oxidised and carbonate is reduced

(b) Bromine is reduced and carbonate is oxidised

(c) Bromine is neither reduced nor oxidised

(d) Bromine is reduced as well oxidised

49. A current of 5.00 ampere flowing for 30 minutes deposits 3.048 g of zinc at the cathode. The equivalent weight of Zn is

(a) 32.7 (b) 65.4

(c) 98.1 (d) 35.7

50. The EMF of a chemical cell is positive when free energy change of reaction

(a) > 0

(b) < 0

(c) = 0

(d) No relationship of free energy change and e.m.f.

51. A solution of $CuSO_4$ is electrolysed for 7 minutes with a current of 0.6 ampere. The amount of electricity passed is equal to

(a) 4.2 F (b) 2.6×10^{-3} F
(c) 126 F (d) 36 F

52. Which of the following will be able to react with dilute HCl to give hydrogen gas?

(a) Cu (b) Mg
(c) Hg (d) Ag

53. Which one is correct about conductivity water?

(a) The conductance of water is very very small
(b) The water obtained after 7-8 times disillation
(c) Kohlrausch prepared the conductivity water for the first time
(d) All of these

54. The increase in molar conductance of HCl with dilution is due to

(a) Increase in the self ionization of water
(b) Hydrolysis of HCl
(c) Decrease in self ionization of water
(d) Decrease in the interionic forces

55. In a cell containing zinc electrode and normal hydrogen electrode (*NHE*), the zinc electrode acts as:

(a) Anode
(b) Cathode
(c) Neither cathode nor anode
(d) Both anode and cathode

56. The oxidation number of C in CH_4, CH_3Cl, CH_2Cl_2, $CHCl_3$ and CCl_4 is respectively

(a) –4, –2, 0, 2, 4
(b) 2, 4, 0, –2, –4
(c) 4, 2, 0, –2, 4
(d) 0, 2, –2, 4, 4

57. Chlorine is in +3 oxidation state in

(a) HCl (b) $HClO_4$
(c) ICl (d) ClF_3

58. Oxidation number of oxygen atom in O_3 molecule is

(a) 0 (b) – 2
(c) + 2 (d) – 1/2

59. The brown ring complex compound is formulated as $[Fe(H_2O)_5 NO]SO_4$. The oxidation state of iron is

(a) + 1 (b) + 2
(c) + 3 (d) + 6

60. If 0.1 faraday of electricity deposits 10.8g of silver from a silver nitrate solution, the amount of copper deposited by the same amount of electricity from copper sulphate solution is

(a) 3.18 g (b) 6.36 g
(c) 31.8 g (d) 0.318 g

61. When iron is added to $CuSO_4$ solution, copper is precipatated. It is due to

(a) Oxidation of Cu^{+2}
(b) Reduction of Cu^{+2}
(c) Hydrolysis of $CuSO_4$
(d) Ionization of $CuSO_4$

62. Which of the following reactions involves oxidation and reduction?

(a) $NaBr + HCl \longrightarrow NaCl + HBr$
(b) $HBr + AgNO_3 \longrightarrow AgBr + HNO_3$
(c) $H_2 + Br_2 \longrightarrow 2HBr$
(d) $Na_2O + H_2SO_4 \longrightarrow Na_2SO_4 + H_2O$

63. The number of Faradays required to deposit 1 g equivalent of aluminium (At. wt. 27) from a solution of ACl_3

(a) 1 (b) 2
(c) 3 (d) 4

64. The standard reduction potential of Pb and Zn electrodes are – 0.126 and – 0.763 volts respectively. The e.m.f. of the cell

$Zn|Zn^{2+}$ (0.1 M) || Pb^{2+} (1 M)|Pb is

(a) 0.637 V (b) < 0.637 V
(c) > 0.637 V (d) 0.889

65. Three faraday of electricity is passed through three electrolytic cells connected in series containing Ag^+, Ca^{2+} and Al^{+3} ions respectively. The molar ratio in which the three metal ions are liberated at the electrodes is;

(a) 1 : 2: 3 (b) 3: 2: 1
(c) 6: 3: 2 (d) 3: 4: 2

66. A current of 9.65 ampere flowing for 10 minutes, deposits 3.0 g of a metal. The equivalent weight of the metal is

(a) 10 (b) 30
(c) 50 (d) 96.5

67. The mass deposited at an electrode is directly proportional to

(a) at. wt. (b) eq. wt.
(c) mol. wt. (d) atomic number

ANSWERS

1. (a)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (b)
11. (c)	**12.** (a)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (c)	**20.** (c)
21. (b)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (d)	**26.** (a)	**27.** (a)	**28.** (d)	**29.** (c)	**30.** (b)
31. (c)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (a)	**36.** (c)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (c)
41. (d)	**42.** (b)	**43.** (d)	**44.** (a)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (d)	**49.** (a)	**50.** (b)
51. (b)	**52.** (b)	**53.** (d)	**54.** (d)	**55.** (a)	**56.** (a)	**57.** (d)	**58.** (a)	**59.** (a)	**60.** (a)
61. (b)	**62.** (c)	**63.** (a)	**64.** (c)	**65.** (c)	**66.** (c)	**67.** (b)			

9 Chemical Kinetics

BRIEF REVIEW OF THE CONCEPTS

Chemical Kinetics is the branch of chemistry which deals with the

(i) study of rates of chemical reactions

(ii) factors which influence the rate of reactions and

(iii) mechanism of the chemical reactions.

The rate of a reaction is defined as the decrease in the concentration of the reactant or the increase in the concentration of a product per unit time

$$\text{Rate} = \frac{\text{[Final concentration of product – Initial concentration of product]}}{\text{Time}} \quad ...(i)$$

$$= -\frac{\text{[Final concentration of reactant – Intial concentration of reactant]}}{\text{Time}} \quad ...(ii)$$

Rate of a reaction is always positive. Its units are mol L^{-1} $time^{-1}$.

Time could be in seconds, minutes, hours or in years.

Since the rate of a reaction decreases with time it can be expressed either as (i) or (ii)

$$\textbf{Average rate} = \frac{\text{[Change in concentration of a reactant or a product]}}{\text{Time interval}}$$

or

$$\text{Rate} = -\frac{\Delta\,[\text{Reactant}]}{\Delta t} = \frac{\Delta\,[\text{Product}]}{\Delta t}$$

Consider a general reaction

$$A + 2B \rightarrow 2C + D$$

The reaction rate may be expressed in the form of the change of concentration of either of the reactant or product as

$$-\frac{\Delta\,[A]}{\Delta t} = -\frac{1}{2}\frac{\Delta\,[B]}{\Delta t} = \frac{1}{2}\frac{\Delta\,[C]}{\Delta t} = \frac{\Delta\,[D]}{\Delta t}$$

Since two moles of B react with one mole of A, thus, concentration of B which disappears in a certain interval of time would be just double of the concentration of reactant, A, which disappears in the same interval of time. Same is true for the product, in a certain period of time two moles of C are formed for each mole of D formed.

Instantaneous Rate is the rate of change of concentration of any one of the reactants or products at that particular instant of time. The instantaneous rate of a reaction can be obtained graphically.

For expressing instantaneous rate, the time interval Δt is made so small that is dt ($\Delta t \rightarrow 0$), then the corresponding change in concentration is dx. Mathematically the instantaneous rate may be expressed as dx/dt.

Rate Law: An equation which relates the rate of a reaction to the concentration of reactants is called the *rate law* or *rate equation*.

Rate Constant or Specific Reaction Rate: Suppose in a reaction A and B are two reactants and the rate depends on concentration of A raised to some power x and the rate also depend on the concentration of B raised to power y. Then the rate equation is written as

$$\text{Rate}\ \alpha\ [A]^x\,[B]^y$$

or

$$\text{Rate} = k\,[A]^x\,[B]^y$$

In this equation, the constant of proportionality, k is called the *rate constant* or *specific reaction rate*.

Let $[A] = [B] = 1$

Then,

$$\text{Rate} = k$$

Thus the rate constant, k, is equal to the rate of the reaction when the concentration of each reactant is unity *i.e.*, one mole per litre. Rate constant is also called the *Specific Reaction Rate*.

The value of the rate constant, k, is independent of the concentration of reactants but depends upon

(i) the temperature of the reaction and

(ii) the particular reaction considered.

The units of rate constant, k depends upon the order of the reaction.

Order of Reaction is the sum of the powers of the concentration terms in the experimentally determined rate equation. As in the above equation order of the reaction is $x + y$. Order w.r.t. A is x and that w.r.t. B is y. Order of a reaction is generally a whole number, but it can also be zero or fraction in some cases.

The order of a reaction is always experimentally determined. It cannot be obtained from balanced chemical equation.

Molecularity of a Reaction is defined as the number of atoms, ions or molecules taking part in each step leading to the chemical reaction. For a multiple step reaction, overall molecularity is taken corresponding to the rate determining step.

Zero order reactions are those in which the rate of reaction is independent of the concentration of the reactants.

$$\text{Rate} = k\,[A]^\circ = k$$

units for the rate constant, k, for the zero order reactions are mol L^{-1} time^{-1}.

First order reactions are those in which the rate of a reaction is directly to the concentration of a reactant, or

$$\text{rate} = k\,[A]$$

where $[A]$ is the concentration of the reactant.

Units for, k, for first order reactions are

$$k = \frac{\text{rate}}{[A]} = \frac{[\text{concentration}]}{\text{time}} \times \frac{1}{[\text{concentration}]}$$

$$= \frac{1}{\text{time}} = (\text{time})^{-1}$$

i.e., s^{-1}, min^{-1}, hr^{-1} or year^{-1}

The Integrated Rate Equation for a First Order Reaction Integrated rate equation of a first order reaction can be written as

$$k = \frac{2.303}{t}\log\frac{C_0}{C_t}$$

Here k is the rate constant. C_o and C_t are the initial and final concentration of the reactant respectively after the time, t,. The same equation can also be written as

$$k = \frac{2.303}{t}\log\frac{a}{a-x}$$

where, k, is the rate constant, a is the initial concentration and x is the amount of the reactant that has reacted during the time t, hence $(a - x)$ represents the concentration of the reactant after time t.

Half-life period, $t_{1/2}$ is the time in which half of the reaction is completed, or the initial concentration (s) of reactant (s) is/are reduced to half. For a **first order reaction, half-life period, $t_{1/2}$** is

$$t_{1/2} = \frac{0.693}{k}$$

where k is the reaction rate constant.

The half-life period, $t_{1/2}$, for a first order reaction is independent of the initial concentration of the reactant.

For a zero order reaction

$$t_{1/2} = \frac{a}{2k}$$

For a first order reaction a plot of log $[C_t]$ against t should be a straight line with a slope equal to $\frac{-k}{2.303}$.

Thus the rate constant, k, can be calculated from the slope of the straight line.

It has been found that the rate of every reaction increases with rise in temperature and in most of the cases it doubles for every 10° rise in temperature.

Pseudo-unimolecular reactions are those reactions which are not truely first order but show first order kinetics under special condtions. For example acidic hydrolysis of esters involves two reactants but overall kinetics shows dependence of rate only on the concentration of ester because water is taken in excess and its concentration remains practically constant.

$$CH_3COOC_2H_5 + \underset{\text{(large excess)}}{H_2O} \xrightarrow{H^+} CH_3COOH + C_2H_5OH$$

The rate equation for 2nd order reaction is

$$k = \frac{1}{t}\cdot\frac{x}{a\,(a-x)}$$

and half life for a second order reaction

$$t_{1/2} = \frac{1}{ka}$$

Activation Energy, E_a The minimum energy which the molecules should possess so that their collisions result in a chemical reaction is called **threshold energy**. The collisions which result in the formation of products are called **effective collisions**. Collisions which result in overall energy less than the threshold energy are not the effective collisions and do not result in the formation of products. The additional energy required by the molecules to attain threshold energy is called **activation energy.**

Therefore, the *activation energy is the energy that must be absorbed from the surroundings by a system in order to react.* **It is independent of temperature.**

Effect of temperature on Reaction Rate: Arrhenius equation Arrhenius derived a mathematical expression to give a quantitative relationship between rate constant, k, and temperature.

$$k = A.e^{-Ea/RT}$$

or $$\text{In } k = \text{In } A - \frac{E_a}{RT}$$

or $$\log k = \log A - \frac{E_a}{2.303\,RT}$$

(here A is Arrhenius constant or frequency factor, k is rate constant, R = gas constant, T = temperature and E_a = activation energy). Another relation derived from it which is used for the calculation of activation energy for a reaction is

$$\log\frac{k_2}{k_1} = \frac{E_a}{2.303\,R}\left(\frac{T_2 - T_1}{T_1\,T_2}\right)$$

here k_1 and k_2 are the rate constants at two different temperatures T_1 and T_2, respectively.

Catalysts Substances which increase the rate of a reaction, and are recovered unchanged after the reaction is over, are called catalysts. A catalyst provides an alternative path of lower activation energy so more reactant molecules are able to cross over the energy barrier and hence the rate of the reaction is increased.

Photochemical Reactions There are many chemical reactions whose rates are influenced by radiations, particularly ultraviolet and visible light. Such reactions are called photochemical reactions. Few examples of this type of reactions are photosynthesis, photography, blue printing and vision. Photochemical reactions are very fast because they proceed through the formation of free radicals.

Photosensitization There are cases where molecules which absorb light do not undergo a reaction, but they transfer their extra energy to other molecules which undergo a reaction. This process is called photosensitization.

Autocatalysis is a phenomenon in which one of the products formed, itself acts as a catalyst and therefore enhances the rate. For example, in the reaction

$$2MnO_4^- + 6H^+ + 5\,(COOH)_2 \longrightarrow 2Mn^{2+} + 10CO_2 + 8H_2O$$

Mn^{2+} formed in the reaction acts as a catalyst.

Solved Problems

1. For the reaction $2A + B + C \rightarrow A_2B + C$, rate law has been determined to be:

$$\text{rate} = k\,[A]\,[B]^2$$

If the rate constant, $k = 2.0 \times 10^{-6}$ mol^{-2} $L^2 s^{-1}$, what will be initial rate of the reaction with $[A] = 0.1$ mol L^{-1}, $[B] = 0.2$ mol L^{-1} and $[C] = 0.8$ mol L^{-1}?

Solution Rate $= k[A]\,[B]^2$

Given $k = 2.0 \times 10^{-6}$ mol^{-2} L^2 s^{-1}

$[A] = 0.1$ mol L^{-1}

$[B] = 0.2$ mol L^{-1}

and $[C] = 0.8$ mol^{-1}

Substituting the values in rate law, we have

$$\text{Rate} = 2.0 \times 10^{-6}\ \text{mol}^{-2}\ \text{L}^2\ \text{s}^{-1} \times (0.1\ \text{mol L}^{-1})\,(0.2\ \text{mol L}^{-1})^2$$

$$= 8.0 \times 10^{-9}\ \text{mol L}^{-1}\ \text{s}^{-1}$$

2. A first order reaction is 75% complete in 60 minutes. Find the half life of this reaction.

Solution Given order of reaction is one

Time = 60 min

Let C_o be 100 then

$C_t = 100 - 75 = 25$

Substituting the values in the rate law

$$k = \frac{2.303}{t}\log\frac{C_o}{C_t} = \frac{2.303}{60\ \text{min}}\log\frac{100}{25}$$

$$= \frac{2.303}{60\ \text{min}} \times 0.6021$$

We know

$$t_{1/2} = \frac{0.693}{k}$$

Thus $$t_{1/2} = \frac{0.693 \times 60\ \text{min}}{2.303 \times 0.6021} = 30\ \text{min}$$

3. For a first order reaction, it takes 5 minutes for the initial concentration of 0.6 mol L^{-1} to become 0.4 mol L^{-1}. How long in all will it take for the initial concentration to become 0.3 mol L^{-1}?

Solution Given: order of reaction = 1

time = 5 min.

$C_o = 0.6$ mol L^{-1}

$C_t = 0.4$ mol L^{-1}

$$k = \frac{2.303}{t}\log\frac{C_O}{C_t} = \frac{2.303}{5\ \text{min}}\log\frac{0.6}{0.4}$$

$$= \frac{2.303}{5} \times 0.1761\ \text{min}^{-1}$$

Concentration from 0.6 mol L^{-1} is to change to 0.3 mol^{-1} L^{-1} i.e., just half, thus time required is $t_{1/2}$ which is given by

$$t_{1/2} = \frac{0.693}{k} = \frac{0.693 \times 5}{2.303 \times 0.1761}\ \text{min}$$

$$= 8.55\ \text{min}$$

4. A reaction that is of first order with respect to reactant A has a rate constant of 6 min^{-1}. If we start with $[A] = 5.0$ mol L^{-1} when would $[A]$ reach the value of 0.05 mol L^{-1}?

Solution Given $k = 6$ min^{-1}

$[A]_o = 5.0$ mol L^{-1}

$[A]_t = 0.05$ mol L^{-1}

Time(t) = ?

$$t = \frac{2.303}{k}\log\frac{[A]_0}{[A]_t}$$

$$= \frac{2.303}{6\ \text{min}^{-1}}\log\frac{5\ \text{mol L}^{-1}}{0.05\ \text{mol L}^{-1}} = \frac{2.303}{6\ \text{min}^{-1}} \times 2$$

= 46 seconds.

5. Time required to decompose SO_2Cl_2 to half its initial amount is 60 minutes. If the decomposition is a first order reaction, find the rate constant of this reaction.

Solution $t_{1/2} = 60$ min

$$k = \frac{0.693}{t_{1/2}} = \frac{0.693}{60\ \text{min}} = 1.156 \times 10^{-2}\ \text{min}^{-1}$$

6. The rate of formation of a dimer in a second order dimerisation reaction is 5.8×10^{-6} mol L^{-1} s^{-1} at 0.01 mol L^{-1} monomer concentration. Calculate the rate constant.

Solution Since the reaction is second order, the rate is given by

$$\text{Rate} = k\,[\text{Monomer}]^2$$

or $$k = \frac{\text{Rate}}{[\text{Monomer}]^2}$$

Given Rate $= 5.8 \times 10^{-6}$ mol L^{-1} s^{-1}

[Monomer] = 0.01 mol L^{-1}

Substituting the values of rates of reaction and initial concentration of monomer gives rate constant, k,

$$k = \frac{5.8 \times 10^{-6}\ \text{mol L}^{-1}\text{s}^{-1}}{[(0.01)\ \text{mol L}^{-1}]^2}$$

$$= 5.8 \times 10^{-2}\ \text{mol}^{-1}\ \text{L s}^{-1}$$

7. Show that in case of a first order reaction, the time required for 99.9% of the reaction to take place is ten times that required for half of the reaction.

Solution Let time required for 99.9% completion be t_1 and for 50% completion t_2 respectively.

When 99.9% reaction is completed a will be 100 and $(a - x)$ will be $100 - 99.9 = 0.1$

For 50% competition, $a = 100$ and $(a - x) = 50$

Substituting the values in the first order rate law

$$t = \frac{2.303}{k}\log\frac{a}{a - x}$$

gives

$$t_1 = \frac{2.303}{k}\log\frac{100}{100 - 99.9} = \frac{2.303}{k}\log 10^3$$

$$= \frac{2.303}{k} \times 3 \qquad \text{...(i)}$$

Similarly

$$t_2 = \frac{2.303}{k}\log\frac{100}{100-50}$$

$$= \frac{2.303}{k}\log 2 = \frac{2.303}{k}\times 0.301 \qquad \text{...(ii)}$$

Dividing eq. 1 by eq. 2

$$\frac{t_1}{t_2} = \frac{2.303}{k}\times\frac{3\times k}{2.303\times 0.301} = 10$$

8. The following rate data were obtained at 300 K for the reaction $2A + B \rightarrow C + D$:

Experiment No.	$[A]$ mol L^{-1}	$[B]$ mol L^{-1}	Rate of formation of D mol L^{-1} min^{-1}
1.	0.1	0.1	6.0×10^{-3}
2.	0.3	0.2	7.2×10^{-2}
3.	0.3	0.4	2.88×10^{-1}
4.	0.4	0.1	2.4×10^{-2}

Calculate order of reaction with respect to A, B and overall order of reaction.

Solution Let the rate law for the above reaction be rate = $k\,[A]^a\,[B]^b$

Considering experiment 4/experiment 1, we obtain

$$\frac{2.4\times10^{-2}}{6.0\times10^{-3}} = \frac{k\,[0.4]^a\,[0.1]^b}{k\,[0.1]^a\,[0.1]^b}$$

$$4 = [4]^a$$

Thus $a = 1$ i.e., order of reaction w.r.t. A is 1

Similarly experiment 3/ experiment 2 gives

$$\frac{2.88\times10^{-1}}{7.2\times10^{-2}} = \frac{k\,[0.3]^a\,[0.4]^b}{k\,[0.3]^a\,[0.2]^b}$$

$$4 = [2]^b$$

Thus $b = 2$ i.e., order of reaction w.r.t. B is 2

Overall order of reaction is 1 + 2 = 3

9. The rate of a particlular reaction doubles when temperature changes from 27°C to 37°C. Calculate the energy of activation for such a reaction.

(R = 8.314 JK^{-1} mol^{-1})

Solution We know

$$\log\frac{k_2}{k_1} = \frac{Ea}{2.303\,R}\left[\frac{T_2 - T_1}{T_1T_2}\right]$$

or

$$E_a = \frac{2.303\,RT_1T_2}{(T_2 - T_1)}\log\frac{k_2}{k_1}$$

The rate of reaction doubles since rate of a reaction is proportional to rate constant, for $\frac{k_2}{k_1}$, the rate of reactions ratio can be taken

Thus

$$\frac{k_2}{k_1} = 2$$

$$T_1 = 273 + 27 = 300 \text{ K}$$

$$T_2 = 273 + 37 = 310 \text{ K}$$

Substituting the values

$$E_a = \frac{2.303\times 8.314\ \text{JK}^{-1}\ \text{mol}^{-1}\times 300\text{K}\times 310\text{K}}{(310-300)\text{K}}\times\log 2$$

$$\text{or } E_a = \frac{2.303\times 8.314\ \text{J mol}^{-1}\times 310\times 300\times 0.3010}{10}$$

$$= 53.6\ \text{kJ mol}^{-1}$$

10. A first order reaction takes 100 min. for completion of 60% of reaction. Find the time when 90% of the reaction will be completed.

Solution The rate law for first order reaction is

$$t = \frac{2.303}{k}\log\frac{C_0}{C_t}$$

Case 1: Let initial conentration, C_0 be 100 M

Concentration C_t after 100 min

= 100 – 60 = 40 M

$$k = \frac{2.303}{k}\log\frac{C_0}{C_t} = \frac{2.303}{100\text{ min}}\log\frac{100\,M}{40\,M}$$

$$= \frac{2.303\times 0.3979}{100\text{ min}}$$

In the case 2, C_o = 100 M

C_t = 100 – 90 = 10 M

We have to find t for 90% completion

$$t = \frac{2.303}{k}\log\frac{100\,M}{10\,M} = \frac{2.303}{k}$$

Substituting the value of k, from the case 1.

$$t = 2.303\times\frac{100\text{ min}}{2.303\times 0.3979} = 245.6\text{ min}$$

Problems for Practice

1. Consider the decomposition of N_2O_5 as

$$2N_2O_5(g) \rightarrow 4NO_2(g) + O_2(g)$$

which of the following rate expression does not represent the rate of the above reaction correctly?

(a) $-\frac{1}{2}\frac{\Delta[N_2O_5]}{\Delta t}$ (b). $\frac{1}{2}\frac{\Delta[N_2O_5]}{\Delta t}$

(c) $\frac{\Delta[O_2]}{\Delta t}$ (d) $\frac{1}{4}\frac{\Delta[NO_2]}{\Delta t}$

2. Which of the following statements is not true for rate constant, k?

(a) Large value of k indicates fast reactions.

(b) The value of k is independent of the concentration of reactants.

(c) The value of k is independent of the concentration of products.

(d) The value of the rate constant, k, as the name suggests is always constant for a particular reaction at all temperatures.

3. The reaction:

$$2N_2O_5(g) \rightarrow 4NO_2(g) + O_2(g)$$

takes place in a 1 litre closed flask. In 5 seconds the concentration of NO_2 increased by 2.0×10^{-2} mol L^{-1}. What is the rate of this reaction?

(a) 4.0×10^{-3} mol L^{-1} s^{-1}

(b) 2.0×10^{-3} mol L^{-1} s^{-1}

(c) 1.0×10^{-3} mol L^{-1} s^{-1}

(d) 0.4×10^{-3} mol L^{-1} s^{-1}

4. The unit of rate constant, k, for a zero order reaction is

(a) mol L^{-1} s^{-1} (b) L mol^{-1} s^{-1}

(c) s^{-1} (d) L^2 mol^{-2} s^{-1}

5. The reaction

$$2SO_2 + O_2 \rightarrow 2SO_3$$

is carried out in the excess of O_2. The order of reaction with respect to O_2 is

(a) zero (b) one

(c) two (d) three

6. What is the half life of a radioactive substance if 75% of its given amount disintegrates in 60 min?

(a) 30 min (b) 45 min

(c) 75 min (d) 2 hrs.

7. A zero order reaction is one whose rate is independent of

(a) the temperature of the reaction

(b) the concentration of the reactants

(c) the concentration of the products

(d) the material of the vessel in which the reaction is carried out

8. The rate of chemical reaction

(a) increases as the reaction proceeds

(b) decreases as the reaction proceeds

(c) may increase or decrease during the reaction

(d) remains constant as the reaction proceeds

9. A first order reaction is 50% complete in 10 minutes. The rate constant of the reaction is

(a) 0.347 min^{-1} (b) 0.0693 min^{-1}

(c) 0.013 min^{-1} (d) 0.50 min^{-1}

10. If initial concentration is doubled, the time for half reaction is also doubled. The order of reaction is

(a) zero (b) first

(c) second (d) third

11. The number of molecules of the reactants taking part in a single step of the reaction tells about

(a) molecularity of the reaction

(b) mechanism of the reaction

(c) order of the reaction

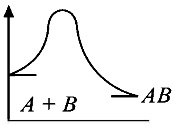
(d) none of the above

12. For an exothermic chemical process occurring in two steps as:

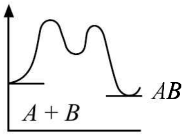
(i) $A + B \rightarrow X$(slow); (ii) $X \rightarrow AB$ (fast)

The progress of the reaction can be best described by:

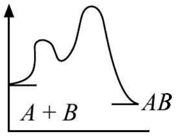
(a)



(b)



(c)



(d) None of these

13. In a reaction $2A \rightarrow$ Products; the concentration of A decreases from 0.5 mol $litre^{-1}$ to 0.4 mol $litre^{-1}$ in 10 minutes. The rate of reaction during this interval is:

(a) 0.05 M min^{-1} (b) 0.005 M min^{-1}

(c) 0.5 M min^{-1} (d) 5 M min^{-1}

14. 50% completion of a first order reaction takes place in 16 hours. What is the fraction that would react in 32 hours?

(a) 25% (b) 60%

(c) 75% (d) 80%

15. The first order rate constant for the decomposition of H_2O_2 is 6.2×10^{-4} s^{-1}. What is the half-life, $t_{1/2}$, for this decomposition.

(a) 6.2×10^4s (b) 3175 s

(c) 11.18×10^2s (d) 6.2×10^2s

16. Factors that determine reaction rates include all the following, except the:

(a) specific rate constant

(b) reaction temperature

(c) concentration of reactants

(d) magnitude of the equilibrium constant

17. For the net reaction $A + 2B \rightarrow C$, the rate law for the formation of C is:

(a) rate = $k[A]$ $[B]$

(b) rate = $k[A][B]^2$

(c) rate = $\dfrac{[C]}{[A][B]^2}$

(d) cannot tell from the data

18. A catalyst can

(a) diminish the activation energy of a reaction

(b) diminish the enthalpy of a reaction

(c) make a reaction thermodynamically more feasible

(d) shift the equilibrium of a reaction

19. The units of rate constant for a zero order reaction is

(a) L s^{-1} (b) L mol^{-1} s^{-1}

(c) mol L^{-1} s^{-1} (d) mol s^{-1}

20. The rate constant of a reaction depends on

(a) temperature

(b) initial concentration of the reactants

(c) time of reaction

(d) extent of reaction

21. A large increase in the rate of reaction for a rise in temperature is due to

(a) increase in the number of collisions.

(b) increase in the number of activated molecules

(c) lowering of activation energy

(d) shortening of the mean free path

22. For reaction

$$4A + B \rightleftharpoons 2C + 2D$$

which of the following statements is not correct?

(a) The rate of disappearance of B is one fourth the rate of disappearance of A

(b) The rate of appearance of C is one half the rate of disappearance of B

(c) The rate of formation of D is one half the rate of consumption of A

(d) The rates of formation of C and D are equal

23. The rate law for the reaction

$$\mathrm{RCl + NaOH}(aq) \rightarrow \mathrm{ROH + NaCl}$$

is given by, Rate = k_1[RCl]. The rate of the reaction will be

(a) doubled on doubling the concentration of sodium hydroxide

(b) halved on reducing the concentration of alkyl halide to one half

(c) increased on decreasing the temperature of the reaction

(d) unaffected by increasing the temperature of the reaction.

24. In a multistep reaction, the overall rate of the reaction is

(a) Equal to the rate of slowest step

(b) Equal to the rate of fastest step

(c) Equal to the average rate of the various steps

(d) Equal to the rate of the last step

25. For the hypothetical reaction $2A \rightarrow 3C$, the reaction rate 'r' in terms of the rate of change of the concentration is given by

(a) $r = -\frac{d[A]}{dt}$ (b) $r = -\frac{1}{2}\frac{d[A]}{dt}$

(c) $r = \frac{1}{3}\frac{d[A]}{dt}$ (d) $r = \frac{d[A]}{dt}$

26. In a hypothetical reaction $A \longrightarrow B$, the activation energies for the forward and backward reactions are 15 and 9 kJ mol^{-1} respectively. The potential energy of A is 10 kJ mol^{-1}

(a) The threshold energy of the reaction is 2 kJ

(b) The potential energy of B is 19 kJ

(c) Heat of the reaction is 25 kJ

(d) The reaction is endothermic

27. For the chemical reaction $A \rightarrow E$ it is found that the rate of the reaction doubles when the concentration of A is increased four times. The order w.r.t. A for this reaction is

(a) two (b) one

(c) zero (d) half

28. The rate equation for the decomposition of N_2O_5 in CCl_4 is

$$\text{Rate} = k[N_2O_5]$$

and $k = 6.3 \times 10^{-4}$ s^{-1} at 320 K. What is the initial rate of decomposition of N_2O_5 when $[N_2O_5] = 0.10$ M?

(a) 6.3×10^{-6} mol L^{-1} s^{-1}

(b) 6.3×10^{-5} mol L^{-1} s^{-1}

(c) $\frac{1}{6.3}\times10^{-4}$ mol^{-1} L s^{-1}

(d) 6.3×10^{-4} mol L^{-1} s^{-1}

29. A catalyst increases the rate of reaction by

(a) decreasing the concentrations of reactants

(b) lowering activation energy only of the reactants

(c) increasing activation energy

(d) lowering the activation energy of both reactants as well as products.

30. For a first order reaction a straight line is obtained when a graph is plotted between

(a) concentration *vs* time

(b) log $(a - x)$ *vs* time

(c) $\frac{1}{(a-x)}$ *vs* time

(d) log k *vs* time

31. In the reaction, $N_2 + 3H_2 \rightarrow 2NH_3$, the rate of reaction was measured as

$$\text{Rate} = -\frac{\Delta[N_2]}{\Delta t} = 1.0 \times 10^{-4}\ \text{mol L}^{-1}\ \text{s}^{-1}$$

the rate of reaction expressed in terms of NH_3 would be

(a) 1.0×10^{-4} mol L^{-1} s^{-1}

(b) 2.0×10^{-4} mol L^{-1} s^{-1}

(c) 3.0×10^{-4} mol L^{-1} s^{-1}

(d) -1.0×10^{-4} mol L^{-1} s^{-1}

32. What are the units of the rate constant, k, for a second order reaction?

(a) s^{-1}

(b) mol L^{-1} s^{-1}

(c) L mol^{-1} s^{-1}

(d) L^2 mol^{-2} s^{-1}

33. Rate of a reaction

(a) increases with increase in temperature

(b) decreases with increase in temperature

(c) does not depend on temperature

(d) does not depend on concentration

34. As the concentration of solution increases, the number of effective molecular collisions may be expected to

(a) Increase (b) Decrease

(c) Remain the same (d) Vary irregularly

35. Consider the data below for a reaction $A \rightarrow B$

Time (sec)	0	10	20	30
Rate	1.60×10^{-2}	1.60×10^{-2}	1.60×10^{-2}	1.59×10^{-2}

From the above data the order of reaction is

(a) Zero (b) 1

(c) 2 (d) 3

36. Check, which of the following statements is false?

(a) A catalyst does not differentiate between forward and backward reaction

(b) Large activation energy is association with low reaction rate

(c) Maxwell's distribution of velocities remains unaltered under all conditions of temperature and pressure

(d) A catalyst does not affect the equilibrium state of reaction

37. For the reaction $A + B \rightarrow$ products, it is found that order of A is 2 and the order of B is 3 in the rate expression. When the concentrations of both A and B are doubled the rate will increase by a factor

(a) 10
(b) 16
(c) 32
(d) 28

38. The order of a Radioactive decay is

(a) one
(b) two
(c) zero
(d) three

39. A large increase in the rate of a reaction for a rise in temperature is due to

(a) The increase in the number of collisions
(b) The increase in the number of activated molecules
(c) The shortening of the mean free path
(d) The lowering of the activation energy

40. For a reaction $A \longrightarrow B$, the rate of reaction is quadrupled when the concentration is doubled. The rate expression of the reaction is $r = k[A]^n$ where the value of n is

(a) 3
(b) 2
(c) 1
(d) Zero

41. According to Arrhenius equation the rate constant of a reaction is equal to

(a) $A \cdot e^{-Ea/R}$
(b) $A \cdot e^{Ea/RT}$
(c) $A \cdot e^{-Ea/RT}$
(d) $A^{-Ea/RT}$

42. The ionic reactions are usually very fast because

(a) The energy of interaction between charged ions is greater than that between neutral molecules
(b) It does not involve bond breaking
(c) The number of collisions between ions per unit volume per second are very large
(d) These reactions are highly exothermic

43. The rate of chemcial reaction usually varies with

(a) Concentration
(b) Temperature
(c) Time
(d) All of these

44. The effectiveness of a catalyst depends upon its ability to

(a) Decrease the activation energy
(b) Increase K_c
(c) Increase reactant concentration
(d) Increase temperature

45. For the reaction $N_2 + 3H_2 \rightarrow 2NH_3$, the rate of change of concentration for hydrogen is $- 0.3 \times 10^{-4}$ Ms^{-1}. The rate of change of concentration of amonia is

(a) -0.2×10^{4}
(b) 0.2×10^{-4}
(c) 0.1×10^{-4}
(d) 0.3×10^{-4}

46. A reaction is 50% complete in 2 h and 75% complete is 4 h. The order of reaction is

(a) 0
(b) 1
(c) 2
(d) 3

47. Given $t_{1/2} = 3$ h. Then how many gram of a substance will remain after 18th hour from 320 g of a substance

(a) 40 g
(b) 20 g
(c) 10 g
(d) 5 g

48. The order of reaction can be determined from:

(a) Chemical equation
(b) Experiment
(c) Rate constant
(d) Thermochemical equation

49. The activation energy of reaction is the energy sufficient to

(a) Change reactants into an actiivated complex
(b) Separate a compound into its constituent subatomic particles
(c) Cause an exothermic reaction
(d) None of these

50. For a Ist order reaction, a straight line is obtained if you plot

(a) Long conc. vs time
(b) Conc. vs time
(c) 1/conc. vs time
(d) Long conc. vs 1/time

51. The half life period of radium is 1620 years. In how much time 1 g of radium will reduce to 0.25 g?

(a) 3240 years
(b) 4860 years
(c) 2430 years
(d) 6480 years

52. The difference between the potential energy of reactants and that of products is called

(a) Activation energy
(b) Activated complex
(c) Heat of reaction
(d) Reaction pathway

53. Effect of temperature on reaction rate is given by

(a) Clausius-Clapeyron equation
(b) Gibbs-Helmholtz equation
(c) Arrhenius equation
(d) Kirchoff's equation

54. What type of reaction is

$$\text{R COOR'} + H_2O \xrightarrow{H^+} \text{R COOH} + \text{R'} - \text{OH}$$

(a) second order
(b) First order
(c) pseudo first order
(d) zero order

55. For a given reaction $t_{1/2} = \frac{1}{ka}$. The order of reaction is

(a) 3
(b) 2
(c) 1
(d) 0

56. The half-life period for a first order reaction is 693 seconds. The rate constant of this reaction would be

(a) 0.1 sec^{-1}
(b) 0.01 sec^{-1}
(c) 0.001 sec^{-1}
(d) 0.0001 sec^{-1}

57. A chemical reaction was carried out at 300 K and 280 K. The rate constants were found to the k_1 and k_2 respectively. Then

(a) $k_2 = 4k_1$
(b) $k_2 = 2k_1$
(c) $k_2 = 0.25\ k_1$
(d) $k_2 = 0.5\ k_1$

58. Units of rate constant for the first and zero order reactions in terms of molarity (M) are, respectively

(a) sec^{-1}, M sec^{-1}
(b) sec^{-1}, M
(c) M sec^{-1}, sec^{-1}
(d) M, sec^{-1}

59. The rate of reaction is doubled for 10° rise in temperature. The increase in the reaction rate as a result of temperature rise from 10° to 100° is

(a) 90 (b) 180

(c) 512 (d) 1024

60. $3A \rightarrow 2B$, the rate of this reaction $+\frac{d[B]}{dt}$ is

(a) $-\frac{3}{2}\frac{d[A]}{dt}$ (b) $-\frac{2}{3}\frac{d[A]}{dt}$

(c) $-\frac{1}{3}\frac{d[A]}{dt}$ (d) $+2\frac{d[A]}{dt}$

ANSWERS

1. (b)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (a)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (d)	**17.** (d)	**18.** (a)	**19.** (c)	**20.** (a)
21. (b)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (d)	**27.** (d)	**28.** (b)	**29.** (d)	**30.** (b)
31. (b)	**32.** (c)	**33.** (a)	**34.** (a)	**35.** (a)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (b)
41. (c)	**42.** (b)	**43.** (d)	**44.** (a)	**45.** (b)	**46.** (b)	**47.** (d)	**48.** (b)	**49.** (a)	**50.** (a)
51. (a)	**52.** (c)	**53.** (c)	**54.** (c)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (c)	**60.** (b)

10

Surface Chemistry

BRIEF REVIEW OF THE CONCEPTS

Adsorption and Absorption

The unbalanced forces acting on the surface of a solid result in a tendency to attract and retain molecules of a gas or a liquid which come in contact with it. The molecules remain only at the surface and do not go deeper into the bulk. *This phenomenon of attracting and retaining the molecules of a gas or a liquid on the surface of a solid resulting into a higher concentration of the molecules on the surface is called* **adsorption**. The substance thus adsorbed on the surface is called the *adsorbate* and the solid substance, which adsorbs is called the *adsorbent.*

The phenomenon of adsorption is different from absorption. The latter term implies that a substance is uniformly distributed throughout the body of a solid.

Adsorption occurs on surfaces of almost all solids. However the extent of adsorption depends upon the following factors.

(i) *Nature* and *surface area* of the *adsorbent.*

(ii) *Nature of the* gas *adsorbed.* It has been observed that the gases which are more easily liquefiable (e.g., SO_2, NH_3 etc.) are more easily adsorbed than others.

(iii) *Temperature*

It is generally observed that the extent of adsorption decreases with rise in temperature and vice versa. For example, one gram of charcoal adsorbs about 10 cm^3 of N_2 at 0°C, 20 cm^3 at – 25°C and 45 cm^3 at – 78° C. Adsorption is an exothermic process. The change in *enthalpy when one mole of a substance is adsorbed is called the* **enthalpy of adsorption.** The process of adsorption is just similar to the condensation. The reverse process is called *desorption.* It is endothermic in nature and just similar to evaporation. When a gas is kept in contact with a solid adsorbent in a closed container, in due course, a dynamic equilibrium is established between them.

$$\underset{\text{(adsorbate)}}{\text{gas}} + \underset{\text{(adsorbent)}}{\text{solid}} \underset{\text{desorption}}{\overset{\text{adsorption}}{\rightleftharpoons}} \underset{\text{(adsorbed on solid)}}{\text{gas}} + \text{heat}$$

Applying the Le Chatelier's principle it can be seen that increase of temperature would decrease the adsorption.

(iv) *Pressure of the gas* – At contant temperature the extent of adsorption increases with increase in the pressure of the gas.

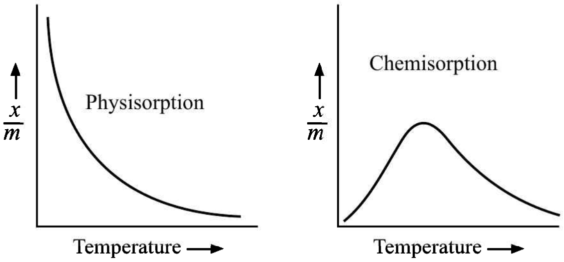
It has been found experimentally that adsorption is of two types—physical and chemical

Physical Adsorption or Physisorption

This is the type of adsorption that we usually come across. The basic feature of **physical adsorption** is that the adsorbate molecules are held at the surface of the adsorbent by van der Waals forces. These forces are the general intermolecular forces which exist between particles of all matter. Because of the general nature of the forces, the adsorption on surface of a solid is quit general. It is characterized by low enthalpy of adsorption, that is, about 10-40 kJ mol^{-1}. Physical adsorption of a gas by a solid is reversible and an equilibrium is established between adsorbent and adsorbate. Increase of pressure increases the adsorption and the decrease of pressure desorbs the gas. Similarly, lowering of temperature increases adsorption and heating releases the adsorbed gas [See fig. 10.1(a)]. In physical adsorption, generally *several layers* of adsorbate are formed one over the other.

Chemical Adsorption or Chemisorption

If a chemical combination is possible between the adsorbent and the adsorbate, it would result in a very strong adsorption. This type of adsorption, caused by forces similar to chemical bonds between an adsorbent and adsorbate is called **chemisorption** or chemical adsorption. The enthalpy of chemisorption is high and is 40-400 kJ mol^{-1} which is comparable to enthalpies of chemical bonds. Chemisorption is highly specific and is possible only between specific adsorbent and adsorbate. Like most of the chemical changes it is usually irreversible. Attempts to release the adsorbed gas gives the gas and some amount of a definite compound and not the free gas alone. For example, oxygen is chemisorbed on tungsten. It is released from the surface as a mixture of oxygen and tungsten oxide. Unlike physical adsorption, chemisorption first increases with temperature; only on further raising the temperature it decreases [fig. 10.1(b)]. This shows that chemisorption has an energy of activation. During chemisorption only *one layer* of adsorbate molecules is formed.



(*a*) (*b*)

Fig. 10.1 Effect of temperature on (*a*) physisorption and (*b*) chemisorption

Adsorption Isotherms

Adsorption is measured in terms of the quantity x/m, where x is the mass of the gas (adsorbate) adsorbed at equilibrium on mass m of the adsorbent. x/m is the mass of adsorbate adsorbed per unit mass of the adsorbent. It is called *extent of adsorption* or *specific adsorption.*

(i) *Adsorption of Gases*

The extent of adsorption of gas on a solid depends upon its pressure. Figure given below shows that x/m, the extent of adsorption of a gas on the surface of a solid adsorbent increases with increase in pressure of the gas, p, at a constant temperature. At a certain pressure, x/m attains a maximum value.

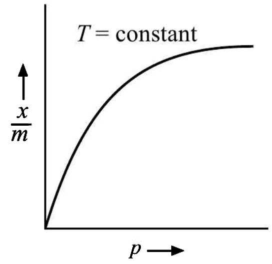


Fig. 10.2

Such curves which depict the variation in the extent of adsorption at constant temperature are called the *adsorption isotherms.* The mathematical relation between x/m and the pressure of the gas p is given by the relation which is also called *Freundlich adsorption isotherm*

$$\frac{x}{m} = kp^{\frac{1}{n}}$$

In this relation k is a constant at a given temperature and depends upon the nature of the adsorbate and adsorbent. The value of n is 1 at low pressures and it increases at higher pressure. On taking logarithm of both sides, we get

$$\log\frac{x}{m} = \log k + \frac{1}{n}\log p$$

This is an equation of a straight line and a plot of log x/m against log p should be a straight line with a slope $1/n$. In actual practice a straight line is often obtained provided the data obtained at very low and high pressures are neglected.

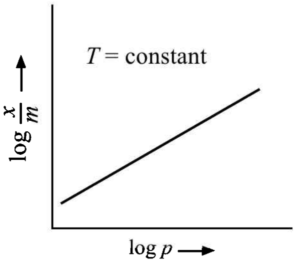


Fig. 10.3

Plot of log x/m against log p at a constant temprature

(ii) *Adsorption from Solutions*

The extent of adsorption is dependent upon the *concentration* of the solution which replaces the pressure in the isotherm given earlier for adsorption of gases. The Freundlich isotherm using the concentration of the solution is applicable to adsorption from solutions, that is

$$\frac{x}{m} = kc^{\frac{1}{n}}$$

Here c is the concentration of the solute in the solution. Taking logarithms

$$\log\frac{x}{m} = \log k + \frac{1}{n}\log c$$

A plot of log (x/m) against log c is expected to be a straight line. It is actually found to be so if very low and high concentrations are avoided. In fact, the Freundlich isotherm applies to adsorption from solutions more effectively than to adsorption of gases.

Applications of Adsorption

The phenomenon of adsorption finds extensive applications. A few examples are:

1. Activated charcoal is used in gas masks in which toxic gases are adsorbed by the charcoal while the pure air passes through it.
2. Very high vacuum can be created by connecting a bulb of charcoal cooled in liquid air to a vessel which has already been evacuated as far as possible by a vacuum pump. The remaining traces of air are adsorbed by the charcoal almost completely, thus creating a high vacuum.
3. Silica gel packed in small cloth bags is used for adsorbing moisture from bottles containing medicines and in small electronic instruments.
4. Animal charcoal is used for decolourizing the sugar and many other compounds during their manufacture.
5. Adsorption plays an important role in heterogeneous catalysis, for example in manufacture of ammonia in presence of finely divided iron.
6. Dyeing with the help of mordants which fix dyes by adsorption.
7. In chromatographic analysis, the selective adsorption of certain substances from a solution by a particular solid adsorbent helps in the separation of the components of a mixture.
8. Eosin and dichlorofluorescein dyes are used as adsorption indicators in precipitation titrations, for instance between $AgNO_3$ and KBr.
9. Surface area of a solid adsorbent can be estimated from the quantity of adsorbate adsorbed to form a monomolecular layer on the surface of the adsorbent if the area of cross-section of one molecule of the adsorbate is known.

COLLOIDS

Between true solutions and ordinary suspensions there exists an intermediate situation where we have a mixture in which the solute-like particles are of intermediate size. They are so finely divided that they do not settle down on keeping yet they are large enough to make the mixture cloudy (such as glue, or gum in water). Such suspensions are called colloids or colloidal suspensions. The main distinction between true solutions, colloidal suspensions and ordinary suspensions is in the size of the suspended particles. In true solutions the solute particles are the molecules or ions with diameters between 0.1 to 1 nm (assuming that they are all spherical in shape). In suspensions the particle size is more than 100 nm. The colloidal particles have the intermediate size in the range of 1 to 100 nm.

Classification of Colloids

The terms solute and solvent as used in solutions are replaced by *dispersed phase* and *dispersion medium* while discussing colloids. The **dispersed phase** is the substance which is distributed as small particles throughout the **dispersion medium**. Just as in case of true solutions, solutes and solvents can be solids, liquids or gases, so can the dispersed phase and the dispersion medium in a colloidal suspension. In fact, colloids are classified according to the physical states of dispersed phase and dispersion medium. Table given below lists classes of colloids. It may be noted, however, that two gases cannot form a colloidal system, as they are always completely miscible.

Table 10.1 Types of Colloids with Examples

Dispersed phase	*Dispersion medium*	*Common name*	*Example*
Gas	liquid	foam	shaving cream, whipped cream.
Gas	solid	solid foam	sponge, foam rubber, pumice stone, styro-foam, cork.
Liquid	gas	aerosols of liquids	fog, mist, clouds, aerosol spray of scents and insecticides.
Liquid	liquid	emulsions	milk, face cream, many syrups.
Liquid	solid	gel	cheese, butter, jelly, shoepolish.
Solid	gas	aerosols of solids	smoke, auto exhaust.
Solid	liquid	sol	paints, mud, gum, pastes.
Solid	solid	solid sol	porcelain, many alloys such as steel, coloured gems.

Colloidal suspensions are generally called sols. Colloidal suspensions in water are called *hydrosols* or *aquosols.* Those in alcohol are called *alcosols,* those in benzene are called *benzosols* and those in air are called *aerosols.*

Some substance have a tendency to form stable sols. Such sols are called **lyophilic** (solvent-loving) sols. Other substances form only unstable sols. They are called **lyophobic** (solvent fearing) sols. When water is the dispersing medium in sol, the colloids are classified as *hydrophilic* or *hydrophobic* sols.

Table 10.2 Properties of Lyophilic and Lyophobic sols

Lyophilic sols	*Lyophobic sols*
1. On evaporation of solvent they give solids, which easily form sols again by the action of solvent.	1. On similar treatment they give solids which do not readily form sols again by the action of solvent.
2. They are not precipitated by relatively small amounts of electrolyte. (May be salted out by large amount of electrolyte.)	2. They are precipitated by relatively small amounts of electrolyte.
3. They generally have higher viscosity and lower surface tension than the dispersion medium.	3. They have surface tension and viscosity similar to those of the dispersion medium.
4. Under the influence of an electric field, the particles may not migrate at all, or may move in either direction.	4. The colloidal particles all migrate in one direction under the influence of an electrical field.
5. The particles are not easily detected in the ultramicroscope.	5. The colloidal particles are easily detected in the ultra-microscope.
Examples: gums, starch, proteins, gelatin	Examples: Colloidal metals, metallic sulphides and hydroxides; other inorganic colloids.

Colloids are also classified on the basis of nature of particles constituting dispersed phase. These are of three types:

(i) **Multimolecular colloids** When the particles of substances which constitute dispersed phase are of molecular dimensions *i.e.*, less than 1nm, but a large number of such atoms or molecules group together into larger aggregates of colloidal size (1-100 nm). These are called multimolecular colloids. For example sulphur (S_8), AgCl colloids etc.

(ii) **Macromolecular colloids** In some substances such as starch, proteins, asbestos fibres, the size of their particles is of colloidal dimensions *i.e.*, between 1-100 nm. Such substance are known as macromole-cular colloids.

(iii) **Micelles** These are charged species which behave as true solutions at low concentration but behave as colloidal systems at higher concentration. These associated particles are called micelles or **associated colloids,** e.g., , aqueous solutions of soaps.

Properties of Colloids. Colloids exhibit following properties which differentiate them from true solutions

(i) **Brownian movement** is the zig-zag motion of colloidal particles. It is due to the continuous bombardment of colloidal particles by the particles of dispersion medium.

(ii) **Tyndall effect** When a beam of light is passed through a colloidal solution and viewed perpendicular to the path of the incident light, *the path* of light becomes visible. This phenomenon is called Tyndall effect. It is due to the scattering of light by colloidal particles.

(iii) **Electrophoresis** Colloidal particles are charged particles. They carry either positive or negative charge. When an electric field is applied to this system, colloidal particles move towards one of the electrodes which is oppositely charged. This movement, of colloidal particles towards one of the electrodes under the influence of an applied electric field is called electrophoresis.

Peptization is a process of converting a freshly prepared precipitate into colloidal state. This is done by adding a suitable electrolyte.

Dialysis is a process to separate a crystalloid from a colloid by diffusion through a semipermeable membrane (parchment paper). When dialysis is carried out under the influence of an electric field, this process is called **electro-dialysis.**

Coagulation or Flocculation is the phenomenon of the precipitation of a colloid by the addition of an electrolyte. The minimum concentration of an electrolyte in millimoles per litre of the electrolyte solution which is required to cause the coagulation or flocculation of colloid is called *Flocculation value.*

$$\text{Flocculation value} \propto \frac{1}{\text{coagulation power}}$$

The coagulating behaviour of electrolytes is governed by Hardy Schulze rule according to which

(i) The ion which carries the charge opposite to the charge on the colloid is responsible for the coagulation of the colloid. These ions are called flocculating ions.

(ii) Higher the valency of the precipitating ion the greater will be its coagulating power. It can be said, "Coagulating power of a precipitating ion is proportional to its valency".

Iso-electric point A colloidal particle carries positive or negative charge. By adding ions carrying charge opposite to that of sol this charge is decreased. A stage is thus reached when colloidal particles carry no charge. This stage is called the iso-electric point.

Protective Colloids A colloid system may be stabilized by adding a second colloid called a protective colloid. The protective colloids are of lyophilic type and their protective power is represented by a number called *Gold number*. It is defined as *the number* of *milligrams of dry protective colloid required to just prevent the coagulation of 10mL of red gold sol when 1 mL of 10% solution of sodium chloride is added to it.* Lower is the value of gold number, greater is the protective power.

Emulsions

An emulsion is a colloidal system in which the dispersed phase and the dispersion medium both, are liquids which are otherwise immiscible with each other. When kerosene and water are shaken together, the kerosene is dispersed as small droplets throughout the water and an emulsion is formed. On keeping, the kerosene particles aggregate and soon it forms a separate layer. Such a temporary emulsion may be made more permanent by addition of an *emulsifying agent* such as soap. *An* **emulsifying agent** *is a protective colloid which forms a film about each tiny oil droplet and prevents their coalescence.* Other emulsifying agents are long chain sulphonic acids (detergents) and many lyophilic colloids. Milk is an emulsion of drops of fat in water. Here casein acts as an emulsifying agent. Emulsions can be divided into two types:

(i) **Oil-in-water** *type in which oil is the dispersed phase and water is the dispersion-medium.*

(ii) **Water-in-oil** *type in which water is the dispersed phase and oil is the dispersion medium.*

The type of emulsion formed depends upon the relative amounts of the two liquids. When water is present in excess it becomes the dispersion medium and oil-in-water type emulsion is formed; otherwise water-in-oil-type emulsion is formed.

Some Important Applications of Colloids

1. *Purification of water:* The colloidal impurities can be removed from water by adding certain electrolytes like alum.
2. *Sewage disposal:* The sewage contains colloidal particles of dirt, mud etc and they carry electric charge. The suspended matter is removed by passing it between plates kept at high electric potential. Due to electrophoresis the colloidal particles move to the charged plates and get coagulated.
3. *Precipitation of smoke:* The suspended particles in smoke are precipitated by passing it through a Cottrell precipitator. This reduces air pollution.
4. *Tanning of leather:* The particles in animal hides are of colloidal size which carry positive charge. They are coagulated by soaking them in tannin which is a negatively charged sol. This imparts hardness to leather.
5. *Artificial rain:* Clouds are aerosols of water in air and carry electric charges. Artificial rain can be induced by spraying oppositely charged colloidal dust of sand or precipitates like AgI on to a cloud. The charge on colloidal water gets neutralized and it coagulates to form bigger drops which results in artificial rain.
6. *Paints and inks* are colloidal suspensions.
7. *Blood* is a colloid of haemoglobin in water. A solution of alcoholic ferric chloride causes its coagulation and results in formation of a clot.
8. *Blue colour of the sky* is due to scattering of light by the suspended colloidal dust particles in air.

Shape selective catalysis depends on the pore structure. Zeolites are generally used as catalysts. Depending on the size of the reactant and product molecules compared to the size of the cages or pores of the zeolites, reactions proceed in a particular manner. *Zeolites* are microporous aluminosilicates of the general formula $M_{x/n}[(AlO_2)_x (SiO_2)_y]\ mH_2O$.

The crystals contain a series of cages which are linked on six sides with similar cages. These links form channels which are from 0.5 to 0.13 nm in diameter. Sodium ions or other cations reside in these channels and are not attached rigidly to the aluminosilicate structure. The diameters of openings in zeolites vary with the size of the cations. If structures with openings of correct size are chosen, zeolites can be used to absorb small molecules *selectively.* It is due to this property that zeolites are referred to as *molecular sieves.* Many small molecules such as ammonia, carbon dioxide, ethanol and methanol are reversibly absorbed by them. Such molecular sieves are used industrially in drying up gases and liquids, in separating nitrogen from air and in separating straight and branched chain hydrocarbons. Zeolites are extremely important in the chemical industry because the sodium ions in their channels can be replaced by ions of various transition metals which are catalytically active. This makes them catalysts which have active sites of definite size and shape. The geometry of the site will permit formation of only those products whose molecules can be accommodated in the channels of the zeolite. For example, in making gasoline from hydrogen-methane mixture. Such synthetic zeolites are used as catalyst which permit formation of hydrocarbons with selective geometry and size.

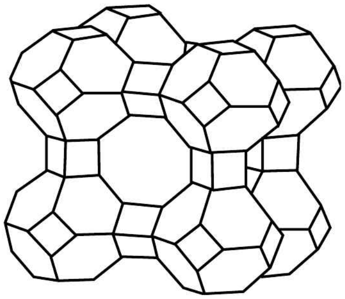


Fig. 10.4 A synthetic zeolite

Problems for Practice

1. Which is not a colloidal solution?
(a) Smoke
(b) Ink
(c) Air
(d) Blood

2. Lyophobic colloids are
(a) Reversible colloids
(b) Irreversible colloids
(c) Protective colloids
(d) Gum, proteins

3. Pumice stone is an example of a colloid in which
(a) solid is dispersed in a liquid
(b) liquid is dispersed in a solid
(c) gas is dispersed in a solid
(d) solid is dispersed in a solid

4. Regarding criteria of catalysis which one of the following statements is not true?
(a) The catalyst is unchanged chemically at the end of the reaction
(b) A small quantity of catalyst is often sufficient to bring about a considerable amount of reaction
(c) In a reversible reaction the catalyst alters the equilibrium position
(d) The catalyst accelerates the reaction

5. Chemisorption is
(a) multimolecular
(b) reversible
(c) often highly specific
(d) not very specific

6. A colloidal solution of arsenious sulphide is most readily coagulated by the addition of a normal solution of
(a) NaCl (b) $CaCl_2$
(c) Na_3PO_4 (d) $Al_2(SO_4)_3$

7. The substances which readily form sol when brought in contact with water is called
(a) Hydrophobic (b) Hydrophilic
(c) Crystalloid (d) None of these

8. Which one of the following is a property of physisorption?
(a) Non-specific nature
(b) High specificity
(c) Irreversibility
(d) None of these

9. Which of the following is not applicable to chemisorption?
(a) Effect of pressure is given by Freundlich adsorption isotherm
(b) There is formation of monomolecular layer
(c) It occurs at low temperature
(d) It involves the formation of chemical bonds between adsorbent and adsorbate.

10. Which of the following method is not employed for the purification of colloids?
(a) Electrodialysis
(b) Dialysis
(c) Ultracentrifugation
(d) Peptisation

11. Which of the following indicates the charge on colloidal particles?
(a) Brownian movement
(b) Electrophoresis
(c) Electrolysis
(d) Tyndall effect

12. Gold number is a measure of the
(a) protective action by a lyophilic colloid on a lyophobic colloid
(b) protective action by a lyophobic colloid on a lyophilic colloid
(c) number of mg of gold in a standard red gold sol
(d) stability of a gold sol

13. Which one of the following methods is used for sol destruction?
(a) Condensation
(b) Dialysis
(c) Diffusion through animal membrane
(d) addition of an electrolyte

14. Fe $(OH)_3$ when treated with $FeCl_3$ soln. a reddish-brown solution is formed. The process involved is

(a) dispersion (b) exchange of solvent
(c) peptization (d) none

15. Which of the following will have highest coagulating power for As_2S_3 colloid?
(a) PO_4^{3-} (b) SO_4^{2-}
(c) Na^+ (d) Al^{3+}

16. Colloidal solution of silver is prepared by
(a) colloidal milk
(b) double decomposition method
(c) Bredig's method
(d) peptization

17. Tyndall effect in colloidal solution is due to
(a) scattering of light
(b) reflection of light
(c) absorption of light
(d) presence of electrically charged particles

18. Which of the following electrolytes is least effective in causing flocculation of ferric hydroxide sol?
(a) $K_3[Fe(CN)_6]$ (b) K_2CrO_4
(c) KBr (d) K_2SO_4

19. Metals like silver and copper can be obtained in the colloidal state by
(a) Peptization (b) Bredig's are method
(c) Dialysis (d) Coagulation

20. Emulsifier is an agent which
(a) Accelerates the dispersion
(b) Stabilises the emulsion
(c) Homogenizes the emulsion
(d) None of these

21. Colloidal solution of arsenious sulphide can be prepared by
(a) electrodispersion method
(b) peptization
(c) double decomposition
(d) hydrolysis

22. Gold number gives the indication of
(a) Gram-molecules of gold per 1000 ml of colloidal solution
(b) Percentage of gold in the suspension
(c) Charge on the colloids
(d) Protective power of a colloid

23. The oxidation of oxalic acid by acidified $KMnO_4$ increases in velocity as the reaction progresses. This is due to the following reason
(a) presence of Mn^{++}
(b) presence of SO_4^{--}
(c) presence of K^+
(d) presence of MnO_4^-

24. Which is not a colloidal solution of gas in liquid?
(a) Froths (b) Foams with tiny bubbles
(c) Mist (d) Whipped cream

25. Which of the following has minimum protecting power?
(a) Gelatin (Gold no = 0.01)
(b) Dextrin (Gold no = 15)
(c) Potato starch (Gold no = 25)
(d) Albumin (Gold no = 0.25)

26. The protective power of lyophilic sol is
(a) Dependent on the size of colloidal particles
(b) Expressed in terms of gold number
(c) Expressed by x/m
(d) Directly proportional to the magnitude of charge on it

27. Freundilch adsorption isotherm gives a straight line on plotting
(a) x/m vs P (b) Log x/m vs P
(c) Log x/m vs log P (d) x/m vs $1/P$

28. "The coagulating effect of an ion depends upon its valency." This is known as
(a) Faraday law (b) Schulze-Hardy law
(c) electrophoresis (d) Tyndall effect

29. Cloud bursts occur due to one of the following reasons.
(a) The clouds are attracted towards the electrical charges on the earth
(b) Large amount of water is present in the cloud
(c) Dense clouds are present in the upper atmosphere
(d) Mutual discharge of oppositely charged clouds resulting in the coagulation of water droplets

30. A catalyst increases the rate of a chemical reaction by
(a) increasing the activation energy
(b) decreasing the activation energy
(c) reacting with reactant
(d) reacting with products

31. On addition of one ml solution of 10% NaCl to 10 ml gold sol in the presence of 0.0250 g of starch, the coagulation is just prevented. Starch has the following gold number.
(a) 0.025 (b) 0.25
(c) 2.5 (d) 25

32. The movement of colloidal particles under an applied electric field is called
(a) electrodeposition
(b) electrodialysis
(c) electro-osmosis
(d) electrophoresis

33. Which of the following ions has maximum flocculation value?
(a) $[Fe(CN)_6]^{4-}$ (b) Cl^-
(c) SO_4^{2-} (d) PO_4^{3-}

34. The separation of colloidal particles from the particles of molecular dimensions is known as
(a) photolysis (b) dialysis
(c) pyrolysis (d) peptization

35. Which is the correct statement in case of milk?
(a) Milk is an emulsion of fat in water
(b) Milk is an emulsion of protein in water
(c) Milk is destabilized by protein
(d) Milk is stabilized by fat

36. Flocculation value is expressed in terms of

(a) millimole per litre
(b) mole per litre
(c) gram per litre
(d) mole per millilitre

37. When dispersion medium is water, the colloidal system is called
(a) sol (b) aerosol
(c) organosol (d) aquasol

38. Movement of colloidal particles under the influence of electric field is called
(a) Electrophoresis (b) Dialysis
(c) Ionization (d) Electrodialysis

39. Colloidal solution are not purified by
(a) dialysis (b) electrodialysis
(c) electrophoresis (d) ultrafiltration

40. The Brownian motion is due to
(a) temperature fluctuations within the liquid phase
(b) attraction and repulsion between charges on the colloidal particles
(c) impact of the molecules of the dispersion medium on the colloidal particles
(d) convection current

41. Whipped cream is an example of .... colloid.
(a) gas in liquid
(b) liquid in gas
(c) liquid in liquid
(d) liquid in solid

42. Which of the following forms a colloid in water?
(a) NaCl (b) Glucose
(c) Starch (d) Barium nitrate

43. The colour of the sky seems to be blue due to
(a) dispersion effect
(b) reflection
(c) transmission
(d) scattering

44. Butter is a colloid of
(a) fat dispersed in solid casein
(b) fat globules dispersed in water
(c) water dispersed in fat
(d) suspension of casein in water

45. The catalyst iron, employed in the Haber process, contains some molybdenum, the function of which is
(a) to increase the rate of combination of gases
(b) to counterbalance for the presence of impurities in the gases
(c) to act as a catalyst promoter and increase its activities
(d) to make up for the adverse temperature and pressure conditions

46. Which one of the following statements is incorrect in the case of heterogeneous catalysis?
(a) The catalyst lowers the energy of activation
(b) The catalyst actually forms a compound with the reactant
(c) The surface of the catalyst plays a very important role
(d) There is no change in the energy of activation

47. Which reaction gives colloidal solution?
(a) $Cu + HgCl_2 \rightarrow CuCl_2 + Hg$
(b) $2HNO_3 + 3H_2S \rightarrow 3S + 4H_2O + 2NO$
(c) $2Mg + CO_2 \rightarrow 2MgO + C$
(d) $Cu + CuCl_2 \rightarrow Cu_2Cl_2$

48. Gelatin is often used as an ingredient in the manufacture of ice cream. The reason for this is:
(a) To prevent the formation of a colloid
(b) To stabilize the colloid and prevent crystal growth
(c) To cause the mixture to solidify
(d) To improve the flavour

49. In the adsorption of oxalic acid by activated charcoal, the activated charcoal is known as
(a) Adsorbent (b) Adsorbate
(c) Absorber (d) None of these

50. Medicines are more effective if they are used in
(a) Colloidal state (b) Solid state
(c) Solution state (d) Both a and c

51. Blood contains
(a) Positively charged particles
(b) Negatively charged particles
(c) Neutral particles
(d) Negatively as well as positively charged particle

52. Which statement is incorrect?
(a) Higher the gold number of lyophilic substance better is its protective action
(b) Lower the gold number of a lyophilic substance better is its protective action
(c) The Bredig arc method is usually suitable for preparing sols of metals
(d) The osmotic pressure method gives the average molar mass of a polymer

53. Point out the incorrect statement
(a) Adsorption may or may not involve the formation of bond between adsorbent and adsorbate
(b) At high pressure the Freundlich isotherm acquires a form $x/m \propto p$
(c) Physisorption involves formation of multimolecular layers
(d) Adsorption involves the concentration of one substance over the surface of other.

54. In aerosol, the dispersion medium is
(a) Solid (b) Liquid
(c) Gas (d) Any of these

55. A sol has positively charged colloidal particles. Which of the following solutions is required in lowest concentration for coagulation?
(a) NaCl (b) $K_4[Fe(CN)_6]$
(c) $ZnCl_2$ (d) Na_2SO_4

56. Milk is a colloid in which
(a) a liquid is dispersed in liquid

(b) a solid is dispersed in liquid
(c) a gas is dispersed in liquid
(d) some sugar is dispersed in water

57. Isoelectric point refers to the H^+ ion concentration at which of the colloidal particles
(a) coagulate
(b) become electrically neutral
(c) move towards an electrode in an electrical field
(d) reverse the electrical charges

58. Tanning of leather is
(a) Colouring of leather by chemicals
(b) Drying process to make the leather hard
(c) Polishing of leather to make it look attractive
(d) Coagulative hardening of the leather by chemicals

59. Coagulation of colloidal solution takes place
(a) By the action of atmospheric oxygen
(b) By the use of electrolytes
(c) By allowing it to stand for some time
(d) None of these

60. An increase in the concentration of a substance at the surface relative to its concentration in bulk phase is called
(a) Adsorption (b) Enthalpy
(c) Absorption (d) None of these

61. Which of the following is not a colloid?
(a) Chlorophyll (b) Smoke
(c) Ruby glass (d) Milk

62. The stability of lyophilic colloids is due to
(a) Charge on their particles
(b) A layer of medium of dispersion on their particles
(c) The smaller size of their particles
(d) The large size of their particles

63. Fog is an example of colloidal system of
(a) Liquid dispersed in gas
(b) Gas dispersed in gas
(c) Solid dispersed in gas
(d) Solid dispersed in liquid

64. The ability of an ion to bring about the coagulation of a given colloidal solution depends upon
(a) its size
(b) magnitude of its charge
(c) sign of its charge alone
(d) both magnitude and sign of its charge

65. Colloids can be purified by
(a) peptization (b) coagulation
(c) dialysis (d) Bredig's arc method

66. Milk can be preserved by adding a few drops of
(a) Formic acid solution
(b) Formaldehyde solution
(c) Acetic acid solution
(d) Acetaldehyde solution

67. When dilute aqueous solution of $AgNO_3$ (excess) is added to KI solution, positively charged sol particles of AgI are formed due to adsoption of ion
(a) K^+ (b) Ag^+
(c) I^- (d) NO_3^-

68. Gold number of a lyophilic sol is such a property that
(a) larger its value, the greater is the peptizing power
(b) lower its value, the greater is the peptizing power
(c) lower its value, the greater is the protecting power
(d) larger its value, the greater is the protecting power

69. The charge on As_2S_3 sol is due to the adsorbed
(a) H (b) OH^-
(c) O_2^- (d) S^{2-}

70. Dialysis is process of
(a) precipitating colloids by addition of electrolytes
(b) separating suspensions from solution
(c) removing electrolytes from colloidal solution
(d) dispersion of freshly precipitated substances into colloids

71. The size of a colloidal particle is
(a) 0.1 to 0.2 nm
(b) 0.2 to 0.5 nm
(c) 1 to 100 nm
(d) 200 to 500 nm

72. Light scattering takes place in
(a) electrolytic solutions
(b) colloidal solutions
(c) electrodialysis
(d) electroplating

ANSWERS

1. (c)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (d)
11. (b)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (b)
21. (c)	**22.** (d)	**23.** (a)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (d)	**30.** (b)
31. (d)	**32.** (d)	**33.** (b)	**34.** (b)	**35.** (a)	**36.** (a)	**37.** (d)	**38.** (a)	**39.** (c)	**40.** (c)
41. (b)	**42.** (c)	**43.** (d)	**44.** (c)	**45.** (c)	**46.** (d)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (d)
51. (b)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (b)	**56.** (a)	**57.** (b)	**58.** (d)	**59.** (b)	**60.** (a)
61. (a)	**62.** (b)	**63.** (a)	**64.** (d)	**65.** (c)	**66.** (b)	**67.** (b)	**68.** (c)	**69.** (d)	**70.** (c)
71. (c)	**72.** (b)								

11

Atomic Structure

BRIEF REVIEW OF THE CONCEPTS

Atom is the smallest particle of an element that retains all its properties and enters into a chemical reaction. The term atom was introduced by Dalton. Atom is composed of three fundamental particles called *electrons, protons* and *neutrons.*

Table 11.1 Charge and mass of fundamental subatomic particles

Name	*charge*	*mass*
Electron (e)	− 1 unit	9.1×10^{-31} kg
	or -1.602×10^{-19}C	or 5.5×10^{-4} u
	or -4.8×10^{-10} esu	
Proton (p)	+ 1 unit	1.677×10^{-27} kg
	or $+1.602 \times 10^{-19}$C	or 1.007 u
	or $+4.8 \times 10^{-10}$esu	
Neutron (n)	zero (neutral species)	1.678×10^{-27} kg

Thomson's Model of Atom According to Thomson, the atom is a sphere of positive charge in which the small negatively charged electron are embedded.

The credit for giving first successful model of an atom goes to ***Rutherford***. His model was based upon the results of alpha-rays scattering experiment. *He postulated that atom consists of two parts—nucleus and extra-nuclear part.*

Nucleus is rigid dense central core which carries a positive charge and the entire mass of the atom. It contains all protons and neutrons, hence these particles are also known as nucleons.

Extra nuclear part is the space around the nucleus where electrons keep revolving.

Dual nature of light and electromagnetic waves Some properties of light and electromagnetic waves can be explained only by considering the wave nature (diffraction and interference) while some require the particle nature (photoelectric effect and Compton effect). However, most of the properties can be explained by both these theories.

For waves, the wavelength λ, velocity c, frequency ν and wave number $\bar{\nu}$ are given by the relations, $\nu = \frac{c}{\lambda}$ and $\bar{\nu} = \frac{1}{\lambda}$.

Photoelectric effect When a beam of light of a certain minimum frequency known as the **threshold frequency** (below which the photoelectric effect does not occur) falls on a metallic surface, electrons are emitted. These are called *photoelectrons* and this effect is called *photoelectric effect.*

Niel Bohr modified the Rutherford's model by adding the concept of *quantisation of energy.* According to him the *electrons revolve around the nucleus on fixed circular paths called* ***orbits*** *without losing or gaining energy.* The energy of various orbits in an atom can be calculated by the expression

$$E_n = -\frac{2\pi^2 z^2 m e^4}{n^2 h^2}$$

and for hydrogen like species (*i.e.* having one electron only) $z = 1$ gives

$$E_n = -\frac{2.178 \times 10^{-18}}{n^2} \text{ J atom}^{-1} \text{ or}$$

$$E_n = -\frac{13.595}{n^2} \text{ eV atom}^{-1} \text{ or}$$

$$E_n = -\frac{1312}{n^2} \text{ kJ mol}^{-1}$$

Radius of a Bohr orbit is given by the relation

$$r_n = \frac{n^2 h^2}{4\pi^2 Z m e^2}$$

$$r_n = \frac{5.29 \times 10^{-11} n^2}{Z} m$$

The energy can be gained or lost by an electron in discrete amount when it jumps from orbit of lower energy to the orbit of higher energy or vice versa. Amount of energy absorbed or released can be calculated as

$$\Delta E = E_{\text{final}} - E_{\text{initial}}$$

The energy gain or loss takes place by the absorption or emission of radiations which are related as

$$\Delta E = h\nu = \frac{hc}{\lambda} = hc\bar{\nu}$$

If this transition is from the orbit of high energy (n_2) to orbit of low energy n_1. Then this transition results in the emission of a photon of frequency ν with energy $h\nu$

$$E = h\nu = (2.18 \times 10^{-18}\text{J})\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

or $$\nu = 3.29 \times 10^{15}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)\text{s}^{-1}$$

or $$\bar{\nu} = 109677\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)\text{cm}^{-1}.$$

Atomic Number (Z) The number of protons present in an atom is known as its atomic number. It is equal to the number of electrons also.

Mass Number (A) The sum of number of protons and neutrons is called the mass number of the atom.

Isotopes Species with same atomic number (Z) but different mass number (A). They contain different number of neutrons.

Isobars Species with same mass number (A) but different atomic number.

Isotones Species having same number of neutrons but different atomic number and also mass number.

> Mass number, being the sum of number of protons and neutrons so it is always a whole number whereas atomic mass is not a whole number because it is average mass of all existing isotopes of that element.

Dual nature of matter and de Broglie equation In 1924, de Broglie postulated dual nature of moving particles and showed that wavelength, λ, of the matter wave is related to the momentum (p) by the equation

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

m = mass, v = velocity and h is Planck's constant.

According to **Heisenberg uncertainty principle,** it is not possible to determine accurately at any given moment both the position and the momentum (or velocity) of a microscopic particle. Mathematically uncertainty in momentum (Δp) and that in position (Δx) are related by the expression

$$\Delta x \,.\, \Delta p \geq \frac{h}{4\pi}$$

or $$\Delta x \,.\, m \,.\, \Delta v \geq \frac{h}{4\pi}$$

(i) **Orbit** represents a definite circular path at a definite distance from the nucleus in which the electrons revolve around the nucleus.

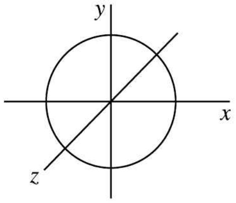
(ii) **Orbital** (ψ^2) is the region in space surrounding the nucleus of an atom where there is maximum probability of finding an electron. ψ is a solution of Schrödinger wave equation.

Node is the region around the nucleus where the probability of finding an electron is zero.

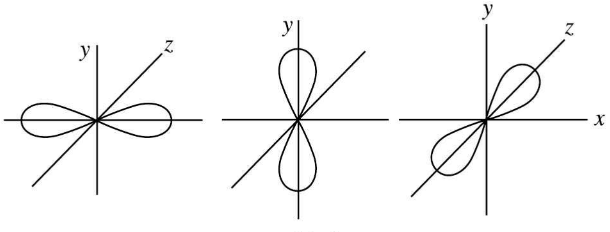
Quantum Numbers The set of four numbers which are required to define the state of an electron in an atom are called quantum numbers.

Principal Quantum Number (n) identifies the main energy level to which the electron belongs. It also specifies the average distance of the electron from the nucleus. It can have any integral value greater than zero *i.e.* n = 1, 2, 3, ... The various values of n are sometimes also designated by the letters K, L, M, N, ...

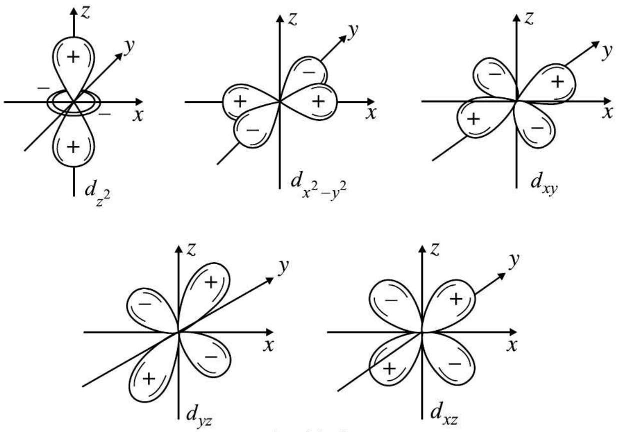
Azimuthal Quantum Number or Angular Momentum Quantum Number (l) determines the shape of an orbital and the angular momentum of an electron occupying that orbital. l can have values from 0 to (n – 1), each of which represents a different sub-energy level or sub-shell. These sub-shells are designated as s, p, d, f according to the value of l = 0, 1, 2, 3 respectively.



s-orbital



p-orbitals



d-orbitals

Fig. 11.1

Magnetic Quantum Number (m) determines the preferred orientations of orbitals in space. The permitted values for m varies from – l to + l including zero.

Spin Quantum Number (s or m_s) arises due to the spinning of the electron about its own axis. The spin can be clockwise represented by $+\frac{1}{2}$ or anti-clockwise represented by $-\frac{1}{2}$.

Pauli's Exclusion Principle. It states that no two electrons in an atom can have the same set of values of all the four quantum numbers. It can also be said that in an orbital, there can be maximum of two electrons and they should be of opposite spin.

Hund's Rule of Maximum Multiplicity According to this rule electron pairing in any of the p, d or f orbital does not occur until all the orbitals of that sub-shell are singly occupied.

Aufbau Principle A sub-shell (group of orbitals) with lower energy is filled up first before filling of the sub-shell with higher energy begins. In other words, the electron must occupy the sub-shell of the lowest energy. The order of energy can be remembered as follows—

(i) Lower the value of $(n + l)$, lower is the energy of the sub-shell.

(ii) If the two sub-shells have the same $(n + 1)$ value, the one with lower n value will have the lower energy.

Half-filled and completely filled sub-shells are more stable due to symmetry or we can say due to exchange of energy. Hence the configurations $(n - 1)d^5\ ns^1$ and $(n - 1)d^{10}ns^1$ are more stable than $(n - 1)d^4\ ns^2$ and $(n - 1)d^9ns^2$ respectively. This is reflected in the configuration of elements like Cr(z = 24) and Cu(z = 29).

Solved Problems

1. The energy of one photon of a beam of light with wavelength 3×10^{-5}m is

(a) 1.986×10^{-38}J (b) 6.62×10^{-34}J

(c) 2.21×10^{-29}J (d) 6.62×10^{-21}J

Solution (d) Energy of a photon(E) = $\dfrac{hc}{\lambda}$

Substituting the values of h, c and λ

$$E = \frac{(6.62\times10^{-34}\text{Js})\,(3\times10^{8}\text{ms}^{-1})}{3\times10^{-5}\text{m}}$$

$= 6.62 \times 10^{-21}$J

So answer is 'd'.

2. What is the ratio of energies of two radiations with wavelengths of 800 nm and 400 nm?

(a) 1 : 2 (b) 2 : 1

(c) 1 : 4 (d) 4 : 1

Solution (a) Ratio of energy of two radiations say E_1 & E_2 can be written as $\dfrac{E_1}{E_2} = \dfrac{hc}{\lambda_1} \times \dfrac{\lambda_2}{hc} = \dfrac{\lambda_2}{\lambda_1} = \dfrac{400}{800} = \dfrac{1}{2}$ so answer is 'a'

3. What is the energy of an electron (m = 9.1×10^{-31} kg) moving with a speed 5.0×10^{7} kilometre per second?

(a) 11.4×10^{-16}J (b) 11.4×10^{-10}J

(c) 22.75×10^{-24}J (d) 22.75×10^{-21}J

Solution (b) Energy = $\dfrac{1}{2}$ mv^2

Given $v = 5.0 \times 10^{7}\text{kms}^{-1} = 5.0 \times 10^{10}\text{ms}^{-1}$

Substituting the values $E = \dfrac{1}{2} \times 9.1 \times 10^{-31}$ kg $\times$

$(5.0 \times 10^{10}\text{ms}^{-1})^2 = 11.4 \times 10^{-10}$J

So the answer is 'b'

4. What is the momentum of a particle which has a wavelength of 2Å?

(a) 3.3×10^{-24} kg ms^{-1}

(b) 3.3×10^{-34} kg ms^{-1}

(c) 3.3×10^{-26} kg ms^{-1}

(d) 13.2×10^{-44} kg ms^{-1}

Solution (a) Momentum $p = \dfrac{h}{\lambda}$

Given λ = 2Å = 2×10^{-10}m and

$h = 6.6 \times 10^{-34}$ kg m^2 s^{-1}

Substituting the values $p = \dfrac{6.6\times10^{-34}\text{kg m}^2\text{s}^{-1}}{2\times10^{-10}\text{m}}$

$= 3.3 \times 10^{-24}$ kg ms^{-1}

so the answer is 'a'.

5. If the energy difference between the ground state of an atom and its excited state is 3.0×10^{-19}J, what is the wavelength of the photon required to produce this transition?

(a) 6.62×10^{-9}cm (b) 6.62×10^{-5}m

(c) 6.62×10^{-7}m (d) 6.62×10^{-7}cm

Solution (c) $\lambda = \dfrac{hc}{\Delta E}$

Substituting the values $h = 6.62 \times 10^{-34}$Js

$c = 3 \times 10^{8}$ ms^{-1}

and $\Delta E = 3 \times 10^{-19}$J

$$\lambda = \frac{(6.62\times10^{-34}\text{Js})\,(3.0\times10^{8}\text{ms}^{-1})}{3\times10^{-19}\text{J}}$$

$= 6.62 \times 10^{-7}$m

Problems for Practice

1. Rutherford's experiment showed for the first time that the atom has

(a) nucleus (b) neutrons

(c) protons (d) electrons

2. Which Bohr orbit is expected to have energy = –145.78 kJ mol^{-1}, where $E_n = -\dfrac{1312}{n^2}$ kJ mol^{-1}?

(a) $n = 1$ (b) $n = 2$

(c) $n = 3$ (d) $n = 4$

3. An increasing order for the values of e/m for electron (e), proton (p), neutron (n) and α particles is

(a) e, p, n, α (b) n, p, e, α

(c) n, p, α, e (d) n, α, p, e

4. A node is a surface, where the probability of finding an electron is

(a) 0
(b) 1%
(c) > 10%
(d) > 90%

5. A deutron contains

(a) 1 neutron and 2 protons
(b) 1 neutron and 1 proton
(c) 1 neutron and 1 positron
(d) 2 neutrons and 1 proton

6. The spectrum of He is expected to be similar to

(a) H
(b) Li^+
(c) Na
(d) He^+

7. Heaviest particle is:

(a) Meson
(b) Neutron
(c) Proton
(d) Electron

8. Those species are called isotones which have same

(a) atomic number
(b) mass number
(c) number of electrons
(d) number of neutrons

9. Which of the following pairs are isobars?

(a) $_{92}U^{235}$ and $_{94}Pu^{239}$
(b) $_{36}Kr^{83}$ and $_{36}Kr^{84}$
(c) $_{10}Ne^{19}$ and $_{9}F^{19}$
(d) $_{58}Ce^{138}$ and $_{58}Ce^{140}$

10. Isotopes of the same element have

(a) same number of neutrons
(b) same atomic mass
(c) same number of electrons
(d) different atomic number

11. Nucleons are

(a) protons and electrons
(b) protons and neutrons
(c) neutrons and electrons
(d) electrons, protons and neutrons

12. If the mass number of an atom is 96 and the number of neutrons is equal to 58, its atomic number is

(a) 58
(b) 38
(c) 96
(d) 154

13. Rutherford's experiment which established the nuclear model of the atom, used a beam of

(a) β-particles
(b) γ-rays
(c) helium atoms
(d) helium nuclei

14. Atom of an element has Z electrons and its atomic mass is $2Z + 3$. The number of neutrons in its nucleus will be

(a) $2Z$
(b) $Z + 3$
(c) $Z + 2$
(d) Z

15. If an isotope of hydrogen has two neutrons in its atom, its atomic number and mass number will be

(a) 2 and 1
(b) 3 and 1
(c) 1 and 1
(d) 1 and 3

16. The nucleus and an atom can be assumed to be spherical. The radius of the nucleus of mass number A is given by $1.25 \times 10^{-13} \times A^{1/3}$ cm. The atomic radius of atom is one Å. If the mass number is 64, the fraction of the atomic volume that is occupied by the nucleus is

(a) 1.0×10^{-3}
(b) 5.0×10^{-5}
(c) 2.5×10^{-2}
(d) $1/25 \times 10^{-13}$

17. The triad of nuclei that is isotonic is

(a) $^{14}_{6}C$, $^{15}_{7}N$, $^{17}_{9}F$
(b) $^{12}_{6}C$, $^{14}_{7}N$, $^{19}_{9}F$
(c) $^{14}_{6}C$, $^{14}_{7}N$, $^{17}_{9}F$
(d) $^{14}_{6}C$, $^{14}_{7}N$, $^{19}_{9}F$

18. Which of the following oxides of nitrogen is isoelectronic with CO_2?

(a) N_2O_2
(b) NO
(c) N_2O
(d) NO_2

19. Addition of two neutrons in an atom A would

(a) change the chemical nature of A
(b) produce an isobar of A
(c) produce an isotope of A
(d) produce another element

20. Bohr model can explain

(a) the spectrum of hydrogen atom only
(b) spectrum of an atom or ion containing one electron only
(c) the spectrum of hydrogen molecule
(d) the solar spectrum

21. Energy of orbit

(a) increases as we move away from nucleus
(b) decreases as we move away from nucleus
(c) remains same as we move away from nucleus
(d) none of these

22. The value of Planck's constant is:

(a) 6.6×10^{-32} g m^{2}s
(b) 6.6×10^{-34} kg m^{2}s^{-1}
(c) 6.6×10^{-33} mg m s^{-1}
(d) 6.6×10^{-34} g m^{2}s^{-1}

23. An element of atomic mass Z consists of two isotopes of mass number Z – 1 and Z + 2. Percentage abundance of the heavier isotope is

(a) 25
(b) 33.33
(c) 66.66
(d) 75

24. The difference between the successive energy levels as your move away from the nucleus

(a) Decreases
(b) Increases
(c) Remains constant
(d) Has no definite order

25. The third line in the Balmer series corresponds to an electronic transition between which Bohr orbit in hydrogen

(a) $5 \to 2$
(b) $4 \to 2$
(c) $3 \to 2$
(d) $6 \to 2$

26. Bohr's orbits are called stationary states because

(a) Electrons in them are stationary
(b) The orbits have fixed radii
(c) The electrons in them have fixed energy
(d) The protons remain in the nuclei and are stationary

27. The no. of protons, neutrons and electrons in the atom ${}_{82}^{207}A$ are:

(a) 82,207,82 (b) 125,82,125

(c) 82,125,82 (d) 82,289,82

28. According to Bohr's model, the electrons revolve around the nucleus in

(a) Orbitals (b) Orbits

(c) Electron clouds (d) Subshells

29. Transition of an electron from $n = 3$ to $n = 2$ results in which of the following series

(a) Balmer (b) Brackett

(c) Pfund (d) Lyman

30. Which of electronic level would allow the hydrogen to absorb a photon but not emit a photon?

(a) 3*s* (b) 2*p*

(c) 2*s* (d) 1*s*

31. The ionisation energy of H-atom (in the ground state) is x kJ. The energy required for an electron to jump from 2nd to 3rd orbit will be:

(a) $x/6$ (b) $5x$

(c) $7.2x$ (d) $5x/36$

32. No two electrons in an atom can have:

(a) The same principal quantum number

(b) The same azimuthal quantum number

(c) The same magnetic quantum number

(d) The same set of quantum numbers

33. The magnetic quantum number is related to

(a) Size (b) Orientation

(c) Spin (d) Shape

34. The principal and azimuthal quantum number of electrons in 4f orbitals are:

(a) 4, 2 (b) 4, 4

(c) 4, 3 (d) 3, 4

35. When the azimuthal quantum number is 1, the shape of the orbital will be:

(a) Spherical (b) Dumb bell

(c) Double dumb-bell (d) More complicated

36. Correct set of quantum numbers for the unpaired electron of chlorine is

(a) 2, 1, 0 (b) 2, 1, 2

(c) 3, 1, 1 (d) 3, 0, 0

37. The momentum of a particle which has a de Broglie wavelength of 0.1 nm is ($h = 6.6 \times 10^{-34}$Js)

(a) 3.2×10^{-24}kg ms^{-1}

(b) 4.3×10^{-22}kg ms^{-1}

(c) 5.3×10^{-22}kg ms^{-1}

(d) 6.62×10^{-24}kg ms^{-1}

38. For sodium atom, the number of electrons with $m = 0$ will be

(a) 2 (b) 7

(c) 9 (d) 8

39. de Broglie equation is a relationship between

(a) position of an electron and its momentum

(b) wavelength of an electron and its momentum

(c) mass of an electron and its energy

(d) wavelength of an electron and its frequency

40. What is the de Broglie wavelength of a 66 kg man skiing down Kufri hill in Simla at 1×10^{3}m sec^{-1}?

(Planck's constant = 6.6×10^{-34} Js)

(a) 1×10^{-36}m (b) 1×10^{-37}m

(c) 1×10^{-38}m (d) 1×10^{-39}m

41. At 200°C the velocity of hydrogen molecule is 2.4×10^{5} cm/sec. In this case the de-Broglie wave length is about

(a) 1 Å (b) 1000 Å

(c) 100 Å (d) 10Å

42. The de Broglie equation suggests that an electron has

(a) particle nature

(b) wave nature

(c) particle-wave nature

(d) radiation behaviour

43. Which one of the following is not the characteristic of Planck's quantum theory of radiation?

(a) The energy is not absorbed or emitted in whole number multiples of quantum

(b) Radiation is associated with energy

(c) Radiation energy is not emitted or absorbed continuously but in the form of small packets called quanta

(d) The magnitude of energy associated with a quantum is proportional to the frequency

44. The maximum number of electrons which can be accommodated in all the orbitals with principal quantum number 3 and azimuthal quantum number 2, is

(a) 2 (b) 8

(c) 10 (d) 18

45. The statement, "All the four quantum numbers for two electrons in an atom can not be the same", is known by which of the following ?

(a) Heisenberg uncertainty principle

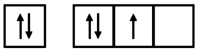
(b) Pauli's exclusion principle

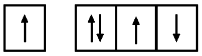
(c) Hund's rule

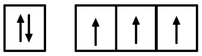
(d) Aufbau's principle

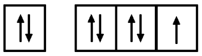
46. The orbital diagram in which the Aufbau principle is violated is

2*s* 2*p*

(a) 

(b) 

(c) 

(d) 

47. There is no difference between a 2*p* and a 3*p* orbital regarding

(a) shape (b) size

(c) energy (d) value of *n*

48. An atom has 2 electrons in *K* shell, 8 electrons in *L* shell and 6 electrons in *M* shell. The number of *s*-electrons present in that element is

(a) 6 (b) 5
(c) 7 (d) 10

49. Electronic energy is –ve because:

(a) Electron has –ve charge
(b) Energy is zero near the nucleus and decreases as the distance from nucleus increases
(c) Energy is zero at infinite distance from the nucleus and decreases as the electron comes towards nucleus
(d) There are interelectronic repulsions

50. The energy of an electron in the first Bohr orbit of hydrogen atom in –13.6 eV. The possible energy value of the excited state for electron in Bohr orbits of hydrogen is

(a) –3.4 eV (b) –4.2 eV
(c) –6.8 eV (d) +1.51 eV

51. Principal, azimuthal and magnetic quantum numbers are respectively related to

(a) Size, orientation and shape
(b) Size, shape and orientation
(c) Shape, size and orientation
(d) Shape, size and spin of electron

52. In a hydrogen atom, the orbital having maximum energy will be

(a) 7*s* (b) 6*p*
(c) 6*d* (d) 6*f*

53. The number of maximas in probability distribution curve of 2*s* orbital are

(a) 1 (b) 2
(c) 3 (d) 4

54. ψ^2 is always

(a) Positive
(b) Negative
(c) Either positive or negative
(d) None of these

55. Which of the following is most stable configuration?

(a) $1s^2, 2s^1$ (b) $1s^2, 2s^2\,2p^4$
(c) $1s^2, 2s^2, 2p^5$ (d) $1s^2, 2s^2\,2p^6$

56. After filling of 3*d* orbitals, the new electron will enter in

(a) 4*s* (b) 4*p*
(c) 4*d* (d) None of these

57. Which of the following set of quantum numbers is impossible.

(a) $n = 3, l = 2, m = -2, s = +\frac{1}{2}$

(b) $n = 4, l = 3, m = 3, s = +\frac{1}{2}$

(c) $n = 5, l = 4, m = 2, s = -\frac{1}{2}$

(d) $n = 3, l = 3, m = -3, s = -\frac{1}{2}$

58. An electron in an atom is excited to the third energy level from the first energy level. It will

(a) remain there permanently
(b) come back to original state either in one or more jumps.
(c) come back to ground state in one jump
(d) come back to second level only.

59. With increasing principle quantum number, the energy difference between adjacent energy levels in H atom

(a) decreases
(b) increases
(c) remains constant
(d) Decreases for low value of *Z* and increases for higher value of *Z*

60. How many electrons can be accommodated in a sub-shell for which $n = 3, l = 1$?

(a) 8 (b) 32
(c) 6 (d) 18

61. Magnetic quantum number specifies

(a) size of orbitals
(b) shape of orbitals
(c) nuclear stability
(d) orientation of orbitals in space

62. The total spin resulting from a d^7 configuration is

(a) $+\frac{1}{2}$ (b) $+\frac{3}{2}$
(c) 2 (d) 7

63. In third principal quantum number, the total number of orbitals will be

(a) 16 (b) 9
(c) 4 (d) 1

64. How many unpaired electrons are there in Ni^{2+} ($Z = 28$)?

(a) 0 (b) 2
(c) 4 (d) 8

65. Which of the following ions does not have the configuration of Ne?

(a) Cl^- (b) F^-
(c) Na^+ (d) Mg^{2+}

66. The nitrogen atom has 7 protons and 7 electrons. The nitride ion will have

(a) 7 proton and 10 electrons
(b) 4 protons and 7 electrons
(c) 4 protons and 10 electrons
(d) 10 protons and 7 electrons

67. Any '*p*' orbital can accommodate upto

(a) 4 electrons
(b) 6 electrons
(c) 2 electrons with parallel spins
(d) 2 electrons with opposite spins

68. The correct set of quantum numbers for the unpaired electron of chlorine atom is

(a) $n = 2, l = 1, m = 0$
(b) $n = 2, l = 1, m = 1$
(c) $n = 3, l = 1, m = 1$
(d) $n = 3, l = 0, m = 0$

69. The quantum number not obtained from the Schrodinger's wave equation is

(a) n (b) l
(c) m (d) s

70. Splitting of spectral lines under the influence of magnetic field is called

(a) Stark effect (b) Photoelectric effect
(c) Zeeman effect (d) None of these

71. If uncertainty in the position of an electron is zero, the uncertainty in its momentum would be

(a) zero (b) $> \frac{h}{4\pi}$
(c) $< \frac{h}{4\pi}$ (d) infinite

72. Which of the following electronic configuration is not possible?

(a) $2p^3$ (b) $2d^5$
(c) $4s^1$ (d) $5f^8$

73. For which one of the following set of quantum number an electron will have the highest energy?

(a) 3, 2, 1, 1/2 (b) 4, 2, –1, 1/2
(c) 4, 1, 0, –1/2 (d) 5, 0, 0, 1/2

74. Nitrogen has atomic number 7 and oxygen 8. The total number of electrons in a NO_3^- ion is

(a) 15 (b) 31
(c) 32 (d) 63

75. The electronic configuration of Cu(29) is

(a) $[Ar]4s^1 3d^5$ (b) $[Ar]4s^1 3d^{10}$
(c) $[Ar]4s^0 3d^4$ (d) $[Ar]4s^0 3d^9$

76. The electronic configuration of an element is $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^5\ 4s^1$. This represents its

(a) excited state (b) ground state
(c) cationic form (d) anionic form

77. Identify the least stable among the following:

(a) Li^- (b) Be^-
(c) B^- (d) C^-

78. Electronic configuration of Cu is

(a) $1s^2, 2s^2\ 2p^6, 3s^2\ 3p^6\ 3d^{10}, 4s^1$
(b) $1s^2, 2s^2\ 2p^6, 3s^2\ 3p^5\ 3d^{11}, 4s^1$
(c) $1s^2, 2s^2\ 2p^6, 3s^2\ 3p^5\ 3d^{10}, 4s^2$
(d) $1s^2, 2s^2\ 2p^6, 3s^1\ 3p^6\ 3d^{10}, 4s^2$

79. The order of filling of electrons in the orbitals of an atom will be

(a) $4s, 3d, 4p, 5s, 4d$ (b) $5s, 4p, 3d, 4d, 5s$
(c) $3d, 4p, 4s, 4d, 5s$ (d) $3d, 4s, 4p, 4d, 5s$

80. The two electrons in the K-shell will differ in

(a) principal quantum number
(b) azimuthal quantum number
(c) magnetic quantum number
(d) spin quantum number

81. The species X^{2+} has the electronic configuration $1s^2, 2s^2\ 2p^6, 3s^2\ 3p^6\ 3d^9$. Then atomic number of X is

(a) 27 (b) 25
(c) 29 (d) 31

82. The ground state electronic configuration of Fe^{3+} is

(a) $(Ar)^{18}3d^3 4s^2$
(b) $(Ar)^{18}3d^6 4s^2$
(c) $(Ar)^{18}3d^5$
(d) $(Ar)^{18}3d^6$

83. The electronic configuration $1s^2, 2s^2\ 2p^6, 3s^1\ 3p^1$ is correctly described by

(a) the ground state of Na
(b) the ground state of Si^{2+}
(c) the excited state of Mg^+
(d) the excited state of Al^+

84. The atom of the element having atomic number 14 should have

(a) one unpaired electron
(b) two unpaired electrons
(c) three unpaired electrons
(d) four unpaired electrons

ANSWERS

1. (a)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (c)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (d)
31. (d)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (d)	**38.** (b)	**39.** (b)	**40.** (c)
41. (a)	**42.** (c)	**43.** (a)	**44.** (c)	**45.** (b)	**46.** (b)	**47.** (a)	**48.** (a)	**49.** (c)	**50.** (a)
51. (b)	**52.** (a)	**53.** (b)	**54.** (a)	**55.** (d)	**56.** (b)	**57.** (d)	**58.** (b)	**59.** (a)	**60.** (c)
61. (d)	**62.** (b)	**63.** (b)	**64.** (b)	**65.** (a)	**66.** (a)	**67.** (d)	**68.** (c)	**69.** (d)	**70.** (c)
71. (d)	**72.** (b)	**73.** (b)	**74.** (c)	**75.** (b)	**76.** (b)	**77.** (b)	**78.** (a)	**79.** (a)	**80.** (d)
81. (c)	**82.** (c)	**83.** (d)	**84.** (b)						

Periodic Classification of Elements

12

BRIEF REVIEW OF THE CONCEPTS

At present, 115 elements are known and every element is different from every other element. But they do possess certain similarities also due to which some of them could be grouped together. Dobereiner in 1829 observed that some chemically related elements showed a constant difference in atomic mass. He could find out some sets of three element, called **triads,** where the atomic mass of the central element was arithmetic mean of the atomic masses of the other two. For example,

Element	Li	Na	K
Atomic mass	7	23	39

Mean of the atomic masses of first (Li) and third (K) element is

$\frac{7+39}{2}$ = 23 which is the atomic mass of middle element (Na).

Newland suggested a better criteria in form of **Law of Octaves.** He arranged the elements in increasing order of atomic masses and found that the eighth element starting from a given one was very similar to it in its properties.

The first thorough attempt to relate chemical properties to atomic masses was made by **Mendeleev** in the form of a **periodic law** which was stated as, "*the properties of the elements are periodic functions of their atomic masses.*"

The table showing the arrangement of elements according to the periodic law is known as **Periodic Table.** Mendeleev arranged then known elements in order of their increasing atomic masses and found that every ninth, nineteenth or thirty third element showed similarities in properties with other. This led him to arrange elements in several horizontal rows such that elements having similar properties got accommodated in the same vertical column (called group) of the periodic table.

Salient Features of the Mendeleev's Periodic Table

(a) The comparative studies of the elements were made easier.

(b) It shows the relationship in properties of elements of a group.

(c) Atomic masses of some elements (e.g., atomic mass of Au) were corrected. At the time of Mendeleev, the atomic mass of gold and platinum were known as 196.2 and 196.7 respectively. However, Mendeleev placed Au after Pt saying that atomic mass of Au is incorrect which was later on found to be 197.

(d) At the time of Mendeleev, about 63 elements were known. When he kept the elements with similar characteristics in similar group, he left some blank spaces where the element of the expected properties were not available. He predicted that some elements would be discovered with those expected properties. He named these unknown elements by placing a prefix **eka** before the names of elements above it. Such as **eka silicon** was the names given to an element which was expected below silicon. Later on, an element with similar properties was discovered and it was named as germanium.

Anomalies in Mendeleev's Periodic Table

(a) position of hydrogen could not be made clear.

(b) Some elements with similar properties were placed in different groups, whereas some with dissimilar properties were placed in the same group.

(c) Some elements with higher atomic masses were placed before an element of lower atomic mass in order to maintain the similar chemical nature of elements. Such as Argon (At. mass = 40) precedes potassium (At. mass = 39); Cobalt (At. mass = 59.9) precedes Nickel (At. mass 58.6) and Tellurium (At. mass = 127.6) precedes iodine (At. mass = 126.9).

(d) Isotopes did not find any place in this periodic table.

MODERN PERIODIC LAW

The problem of isotopes and the misfits in the Mendeleev's periodic table, disappeared when the basis of classification of elements was changed to atomic number instead of atomic mass. In 1912, it was Moseley who showed that the position of an element in the Periodic Table depended on its atomic number and not on its atomic mass as stated by Mendeleev. It was observed that the properties of the elements are better co-related with their atomic numbers and on this basis a new periodic law was given, called as **Modern Periodic Law.** It states that "the properties of the elements are periodic functions of their atomic numbers."

Long Form of Periodic Table or Modern Periodic Table

When the elements are arranged according to Modern Periodic Law, a new table is obtained which is named as **Modern Periodic Table** or **Long Form of Periodic Table** It differs in several respects from the Mendeleev's periodic table. It can be described as follows:

(i) Horizontal rows of elements are called **periods** and this periodic table contains **seven periods.**

(ii) There are **eighteen vertical** Columns called as **groups.** They are numbered as IA to VIIA, and IB to VIIB making 14 columns, number VIII consists of three columns and eighteenth column is numbered as 0. In the latest IUPAC system, these groups are simply numbered as 1 to 18.

This table could solve the problem of the placement of isotopes of elements. As we know that the isotopes of an element differ only in their atomic mass but have the same atomic number. So isotopes of an element occupy the same position as the element.

The sequence of filling up of electrons in orbitals lead to another type of grouping of elements in the long form of periodic table. They are called as ***s*-block, *p*-block, *d*-block and *f*-block** elements.

(i) **s-Block Elements** Elements of the groups IA and IIA comprise this block. They have the electronic configuration in outermost shell as ns^1 or ns^2, respectively.

(ii) **p-Block Elements** Elements of groups IIIA to VIIA and zero group (noble gases) constitute *p*-block elements. The electronic configuration of the outermost shell of these elements varies from $ns^2\ np^1$ to $ns^2\ np^6$.

(iii) **d-Block Elements** The elements of the groups IIIB to VII B, VIII, I B and II B belonging to the fourth, fifth, sixth and seventh periods are known as *d*-block elements. Their general electronic configuration for two outermost shells varies from ns^2, $(n-1)\ d^1$ to ns^2, $(n-1)d^{10}$. Since their properties are midway between those of *s*-block and *p*-block elements, they are also called **transition elements.**

(iv) **f-Block Elements** These elements are also called **inner transition elements.** These elements have three incomplete electronic shells in them. There are two series of *f*-block elements (*a*) Lanthanides (from atomic number 58 to 71) and (*b*) Actinides (from atomic number 90 to 103). In lanthanides, the last electron is being progressively filled in the 4*f* orbital, while in actinides, the last electron is progressively filled in the 5 *f* orbital.

ATOMIC PROPERTIES AND THEIR TREND IN THE PERIODIC TABLE

The use of long form of periodic table, which is based on atomic structures, is understandable for the study of atomic properties of the elements. When the elements are arranged in the order of their atomic numbers, there appears a periodicity, not only in their chemical properties, but also in their physical properties and in their electronic configurations.

The atomic properties of elements are determined by the following factors:

(a) the mass of nucleus of the atom.

(b) The magnitude of nuclear charge, *i.e.*, the total positive charge on the nucleus.

(c) The distribution of electrons around the nucleus.

(d) The electronic configuration of valence electrons, i.e., electrons in the outer electronic shell.

It is important to note that since it is obviously impossible to isolate an individual atom and determine its properties, we have to measure these properties of elements in bulk phase. Hence most of the atomic properties are therefore, average properties exhibited by atoms in their environments. Some of the properties and their variation along a period or group are discussed below:

1. Valency The valency of an element is directly related to the electronic configuration of its atom and is usually determined by the number of electrons in the outermost shell. This is because the outer electrons are the ones largely responsible for the chemical behavior, as the electrons in these orbitals usually participate in chemical bonding. The *outer shell electrons are often referred to as valence shell electrons* or simply *valence electrons.*

On moving along the period, the number of valence electrons increases from 1 to 8. Consequently, the valency of the elements with respect to hydrogen increases from 1 to 4 upto group IV A and then decreases to 1 as shown in the Table given below. However, valency with respect to oxygen increases from one to seven along the period.

On moving down the group, the number of valence electrons remains the same. Therefore, all the elements in a group have the same valency. For example, the elements of group IA have valency 1 and elements of group IIA have valency 2 whereas, elements of group VIA have valency 2 (which is 8 – 6).

Table 12.1 Valency of elements of 2nd period

Group	IA	IIA	IIIA	IVA	VA	VIA	VIIA
Element	Li	Be	B	C	N	O	F
Valency with respect to oxygen	1	2	3	4	5	6	7
Valency with respect to hydrogen	1	2	3	4	3	2	1

2. Atomic radii The term radius of an atom is generally considered as the distance from the centre of the nucleus to the outermost orbit of the atom. In other words, it means the radius of the sphere enclosing the space occupied by the electrons. But it is difficult to determine the size of an atom because of the following reasons.

(i) An atom is too small to be isolated.

(ii) The electronic probability distribution never becomes exactly zero, even at great distances from the nucleus.

(iii) The electronic probability distribution is affected by neighbouring atoms and hence the size of the atom may change in going from one condition to another, e.g., in going from one compound to another.

(iv) Multiplicity of the bond,i.e., whether the bond is a single, double or triple.

(v) The extent of ionic bonding or covalent bonding.

(vi) The oxidation number of the atoms, which indicates the number of electrons involved in the bonding.

(vii) The positive charge on the nucleus.

(viii) The repulsive forces between atoms, not directly bonded to each other.

It is obvious that any combination of the above factors may produce significant size variations. The atomic size (hence atomic radius) of a particular element may differ from one substance to another. Therefore, the distance designated as the boundary of the atom is an arbitrary choice and thus we have to define atomic radius depending upon the usage to which the atom is put in its environment.

Commonly three types of atomic radii are used. There are (i) *Van der Waals' Radius* (ii) *Crystal Radius* and (iii) *Covalent Radius.*

(i) **Van der Waals' Radius** An isolated atom can be compared to a sphere of definite radius. The sphere consists of a diffuse electron cloud surrounding a nucleus; '*the radius corresponds to a distance of closest approach within which atoms will interfere with one another*'. This radius is the so-called van der Waals' radius, and can be determined from the nearness of approach of atoms in a solid.

(ii) **Crystal Radius** In a molecule or crystal of a substance in which two similar atoms are joined together by a single bond (A-A), the crystal radius can be taken as one-half of the internuclear distance of two adjoining atoms in a crystal or a molecule. Thus in the I_2 molecule:

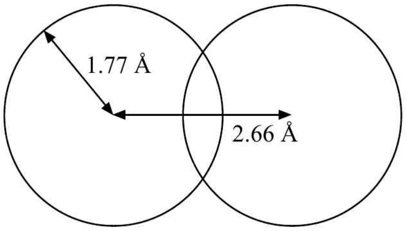


Fig. 12.1

Bond length (I_2) = 2.66Å

$$\text{Crystal radius (I)} = \frac{2.66}{2} = 1.33\text{Å}$$

Whereas van der Waals' radius of (I) = 1.77Å

The crystal radius is always less than van der Waals' radius, because the formation of a single bond involves overlapping of electron clouds of the two separate atoms. Crystal radii of various atoms have been worked out from X-ray studies. They are used to calculate the covalent or atomic radii of other elements.

(iii) **Covalent radius** For similar atoms joined by a single bond in a molecule, the crystal radius and the covalent radius are identical. This is half of the bond length. In the case of molecules where the two different atoms are joined by a single bond, the half of the bond length does not give the atomic radii of each atom. In such cases, the atomic radius of any atom can be calculated by subtracting the known atomic radius of other atom with which the first combines. Thus *covalent radius of an atom is either the same as its crystal radius or equal to the difference of the bond length and atomic radius* (or crystal radius) *of the other atom*. For example in the bond, C–F, single bond distance is 1.41Å and crystal radius of carbon is 0.77Å, therefore, covalent radius of

$$F = 1.41 - 0.77 = 0.64 \text{ Å}$$

As we move from **left to right across a period** in the periodic table, the **atomic radii** gradually **decrease** as electrons are added to a particular energy level. For example, for the second period (Li to Ne) electrons are added in the L-energy level only which is also the size determining shell. As the nuclear charge increases and electrons are added to the same principal energy level, the increased nuclear charge draws the electrons closer and closer to the nucleus, thereby decreasing the radii of atoms.

As we move **down a group,** atomic radii **increase** as electrons are added to larger orbitals in higher energy level. For example, for lithium (Group IA) the size determining energy level is L and for sodium, in the same group, it is M. In going down a group, though the nuclear charge increases but its effect is not so pronounced since the size determining shell is new and is situated farther away from the nucleus as shown in Table below.

Table 12.2 Atomic radii in a group

Element of Group IA	*Atomic radii (Å)*	*Nuclear Charge*	*Electronic Configuration*
Li	1.34	+3	2, 1
Na	1.54	+11	2, 8, 1
K	1.96	+19	2, 8, 8, 1
Rb	2.11	+37	2, 8, 18, 8, 1
Cs	2.25	+55	2, 8, 18, 18, 8, 1

(iv) **Ionic radii** Addition or removal of electrons from the valency shell (outermost) results in the change of *atom* into an ion, A positive ion is formed by removing one or more electrons from an atom. The positive ion (cation) is appreciably smaller as compared to the atom. This happens because the ratio of the positive charge on the nucleus to the number of orbital electrons is increased. Thus the *effective nuclear charge* is increased and the electrons are pulled in. *The more are the electrons removed, the smaller the positive ion becomes.* Thus:

Table 12.3

Atomic radius	*Ionic radii*
Li = 1.34 Å	Li^+ = 0.60 Å
Na = 1.54 Å	Na^+ = 0.98 Å
Fe = 1.17 Å	Fe^{+2} = 0.76 Å
	Fe^{+3} = 0.64 Å

When a negative ion (anion) is formed, one or more electrons are added to an atom. This results in the increase of the negative charge as compared to the nuclear positive charge. The effective nuclear charge is reduced and hence the electrons are pulled less, i.e., they move a little away and result in the increase in the size of the ion. *Negative ions are hence bigger than the correspo-nding atoms.* Thus:

Atomic radius of Cl (atom) = 1.40 Å

Atomic radius of Cl^- (ion) = 1.81 Å

3. Ionization energy (IE) Atom is electrically neutral and energy is required to dislodge an electron from its orbit. The mutual attraction between the nuclear charge and orbital electrons will oppose any effort to separate the electrons. This opposition is overcome by the performance of work, called ionisation energy.

The amount of energy required to remove the most loosely bound electron (that is the electron in the outermost orbit) from a neutral gaseous atom is termed as ionization energy or strictly speaking the first ionization energy (IE_1). The process may be represented as follows:

$$\text{Atom } (g) + E_1 \rightarrow \text{Positive ion } (g) + e^-$$

The energy required to remove the second electron is known as the second ionization energy (IE_2); the energy required to remove the third electron is known as third ionization energy (IE_3) and so on.

For any given element, the second ionization energy is greater than the first, similarly third ionisation energy is greater than the second and so on. This is because of the fact that after removal of the first electron, the **effective nuclear charge** increases so the force of attraction becomes more and hence more energy is required to remove the second electron. This also explains why trivalent and tetravalent ions are less common.

The ionization energy increases along a period as we move from left to right. This is because

(i) As we move along the period, the atomic number increases and hence the magnitude of the nuclear charge increases. The additional electrons are, however, being added up in the orbits already existing. Due to increased nuclear charge, the force of attraction on the valence electrons increases which results in an increase of I.E.

(ii) As we move along the period, because of increased force of attraction, the size of atoms decreases. This makes the force of attraction still higher and so the I.E. increases.

Table 12.4 Ionization Energies of Elements of 2nd Period

Element	Li	Be	B	C	N	O	F	Ne
Atomic Number	3	4	5	6	7	8	9	10
I.E. (KJ/Mole)	520	900	801	1085	1400	1314	1680	2080

The ionization energy decreases along a group as we move from top to the bottom. This is because

(i) As we move from top to the bottom in a group, the charge on the nucleus increases which tends to increase the ionisation potential but this is more than compensated by the addition of new orbits. The net result is that the size of the atom increases and so the force of attraction on the valence electrons decreases.

(ii) As we move from top to the bottom in a group, the electrons are being added up in the next higher and higher orbits. The intervening orbits act as screen or shield between the nucleus and the outermost electrons. This results in decrease of the force of attraction on the valence electrons and hence the I.E. decreases.

Table 12.5 Ionization Energies of the First Group Elements

Elements	*Atomic number*	*I.E. (joule/mole)*
H	1	1311
Li	3	520
Na	11	496
K	19	419
Rb	37	403
Cs	55	376

Ionization energy in also referred as **ionization potential.** Now it has been renamed as **ionization enthalpy (Δ_{ion} H°)** and is defined as the enthalpy change during the process of removing the most lossely held electron from a neutral gaseous atom in its ground state.

4. Electron affinity The *amount of energy released when an extra electron is added to an isolated neutral atom in the vapour state to form a negative ion, is termed as electron affinity.* The higher the energy given out, the greater is the electron affinity which is a measure of tightness with which the added electron is bound to the atom.

$$\underset{\text{(neutral atom)}}{A(g)} + e^- \longrightarrow \underset{\text{(Anion)}}{A^-} + Energy$$

Usually, only one electron is added, forming a uninegative ion. The negative ion so formed repels further electrons and energy is needed to add on a second electron. Hence, the *second electron affinity of elements is negative.*

Let us take the case of oxygen atom as an example. When an electron is added. We get O^- ion and energy is released. But when the second electron is added to form O^{2-} (oxide ion), energy is absorbed to overcome the strong repulsion from the already negatively charged O^- ion.

Electron affinity generally decreases in moving down the group. This is to be expected on account of increase in the size of the atom on moving down the group. Due to increase in the size of atoms, the effective nuclear attraction for electrons decreases and hence there is less tendency to attract additional electrons with increase in atomic number.

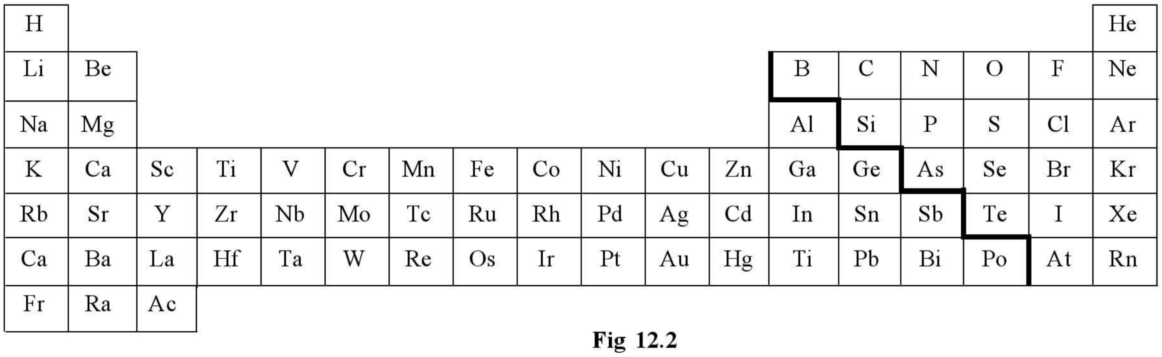
Electron affinity generally increases as we move from left to right in a period. This is due to the increase in the nuclear charge which results in greater attraction for electrons. In the second period, fluorine has the maximum value of electron affinity. As the electron affinity increases, the element becomes less metallic and more non-metallic.

Electron affinity has also been renamed as **electron gain enthalpy (Δ_{eg} H°).** The difference comes in the representation of values with sign only. Suppose the electron affinity of an element is 20 kJ mol^{-1} then its Δ_{eg} H° value will be – 20 kJ mol^{-1} because it is an exothermic process.

5. Position of metals and non-metals in periodic table In periodic table, a pattern is observed in physical properties of the elements. All the elements on the left side and in the middle of the periodic table (except hydrogen) are metallic elements or metals. Also, majority of the elements in the periodic table are metals. The metals are separated from non-metals by a diagonal step like line that runs from boron to astatine. Many of the elements that lie along the line that separates metals from non-metals such as antimony have properties that fall between those of metals and non-metals. These elements are often regarded as **semi metals** or **metalloids.** The metalloids are B, Si, Ge, As, Sb, Te and At.

The metallic character increases down the group and decreases along the period. The non-metal character increases along the period and decreases down the group.

The heavy black line marks the division between metals (on the left) and non-metals (on the right).



H																	He
Li	Be											B	C	N	O	F	Ne
Na	Mg											Al	Si	P	S	Cl	Ar
K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn	Sb	Te	I	Xe
Ca	Ba	La	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg	Ti	Pb	Bi	Po	At	Rn
Fr	Ra	Ac															

Fig 12.2

The change in the reactivity of metals and non-metals is shown in the Fig. given below.

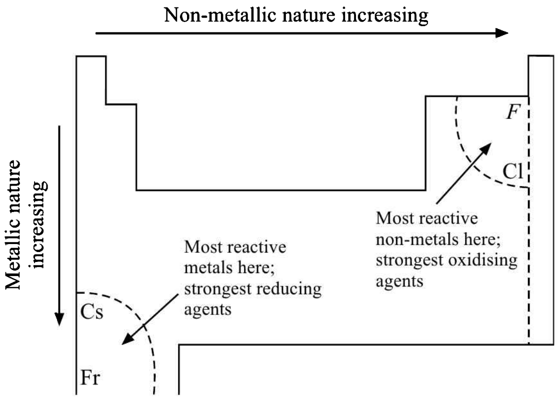


Fig. 12.3 Metallic and non-metallic reactivity change in the Periodic Table.

Problems for Practice

1. The removal of electrons from atom is best described quantitatively in terms of
   (a) Ionisation potential
   (b) Bond energy
   (c) Electron affinity
   (d) Electronegativity
2. Element with lowest ionisation potential is
   (a) Fluorine (b) Chlorine
   (c) Bromine (d) Iodine
3. The least electronegative element has the following electronic configuration
   (a) $ns^2\ np^5$ (b) $ns^2\ np^4$
   (c) $ns^2\ np^3$ (d) $ns^2\ np^6$
4. Consider the iso-electronic series : K^+, S^{2-}, CI^- and Ca^{2+}. The radii of the ions decrease as
   (a) $Ca^{2+} > K^+ > CI^- > S^{2-}$
   (b) $CI^- > S^{2-} > K^+ > Ca^{2+}$
   (c) $S^{2-} > CI^- > K^+ > Ca^{2+}$
   (d) $K^+ > Ca^{2+} > S^{2-} > CI^-$
5. Modern periodic law is
   (a) The physical and chemical properties of the elements are periodic function of their atomic numbers
   (b) The physical and chemical properties of the elements depend upon the energy of the electrons
   (c) The physical and chemical properties of the elements depend upon atomic mass
   (d) None of these
6. The elements on the right side of the periodic table are
   (a) Metals (b) Metalloids
   (c) Non-metals (d) Transition elements
7. Elements of the same vertical group of the periodic table have
   (a) Same atomic size
   (b) Same electronic configuration
   (c) Same number of electrons in outermost shell of their atom
   (d) Same number of protons
8. In the long from of periodic table, the elements having lowest ionisation potential are present in
   (a) I group (b) IV group
   (c) VII group (d) Zero group
9. Which of the following elements are analogous to the lanthanides?
   (a) Actinides (b) carbides
   (c) borides (d) Hydrides
10. Which of the following is most metallic?
    (a) P (b) As
    (c) Sb (d) Bi
11. Which of these is chemically most reactive?
    (a) Mg (b) Ca
    (c) Sr (d) Ba
12. In the periodic table, on moving from left to right across a period, the metallic charater of an element
    (a) Increases
    (b) Decreases
    (c) Increases and then decreases
    (d) Remains constant.
13. Iron triad of elements also contain
    (a) Cobalt and nickel
    (b) Manganese and chromium
    (c) Palladium and platinum
    (4) platinum and iridium
14. The ionic radii of S^{2-} and Te^{2-} are 1.84 and 2.21Å respectively. The ionic radii of Se^{2-} is expected to be
    (a) 1.83 Å (b) 2.22 Å
    (c) 4.5 Å (d) 2.02 Å

15. The element having lowest first ionization potential among Sr, As, S and F is
(a) Sr (b) As
(c) S (d) F

16. Which of the following is correct order of size?
(a) $I > I^- > I^+$ (b) $I > I^+ > I^-$
(c) $I^+ > I^- > I$ (d) $I^- > I > I^+$

17. The elements with atomic number 2, 10, 18, 36, 54, 86 are called
(a) Noble gases (b) Rare metals
(c) Noble metals (d) Halogens

18. Electronegativity of the following elements increases in the order
(a) C,N,Si,P (b) N,Si,C,P
(c) Si,P,C,N (d) P,Si,N,C

19. Which of the following elements should have the maximum electronegative character?
(a) Oxygen (b) Nitrogen
(c) Fluorine (d) Astatine

20. Which of the following element has the lowest first ionization potential?
(a) Na (b) Cs
(c) Cr (d) I

21. Which one of the following periodic groups consists entirely of metals?
(a) IIA (b) IIIA
(c) IVA (d) VIIA

22. Which is the most non-metallic of the following elements?
(a) Beryllium (b) Boron
(c) Magnesium (d) Zinc

23. Which one of the following elements resembles most in chemical properties to sulphur?
(a) Carbon (b) Fluorine
(c) Selenium (d) Silicon

24. Which of the following is NOT a correct formula for a substance at normal laboratory conditions?
(a) H_2S (g) (b) He (g)
(c) NaNe (s) (d) $CaCI_2$ (s)

25. Which one of the following statements is NOT TRUE for ionization energy (I.E.) of an element?
(a) I.E. is defined as the minimum amount of energy required to remove the most loosely bound electron from an isolated gaseous atom in its ground state.
(b) Ionization energies measure how tightly electrons are bound in atoms.
(c) I.E. is represented by

$$M\ (g) + e^- \rightarrow M^- + \text{Energy}$$

(d) It decreases within a group from top to bottom.

26. Atoms with high first ionization energies always have
(a) Large atomic radii
(b) Low electron affinities
(c) Metallic properties
(d) Tightly bound valence electrons

27. Which of the following has the largest radius?
(a) N (b) O
(c) F (d) S

28. Which of the following has the largest radius?
(a) K^+ (b) CI^-
(c) O^{2-} (d) Cr^{3+}

29. The energy 349 kJ mol^{-1} evolved in the following chemical reaction

$$CI\ (g) + e^- \rightarrow CI^-\ (g) + 349\ \text{kJ}$$

is called
(a) Ionization energy (b) Electron affinity
(c) Electronegativity (d) Bond energy

30. The size of the bromine species follow the order
(a) $Br^+ < Br^- < Br$ (b) $Br^+ < Br^- = Br$
(c) $Br^+ < Br < Br^-$ (d) $Br < Br^- < Br^+$

31. Each transition series contains
(a) 12 elements (b) 10 elements
(c) 8 elements (d) 14 elements

32. In a period, the alkali metals have?
(a) Highest ionizatgion energy
(b) Largest atomic radii
(c) Highest density
(d) Highest electronegativity

33. An atom with high electronegativity generally has
(a) Tendency to form +ve ions
(b) High ionization potential
(c) Large atomic size
(d) Low electron affinity

34. Lowest ionization potential in a period is shown by
(a) Alkali metals
(b) Halogen
(c) Transition elements
(d) Alkaline earth metals

35. The ionization potential of nitrogen is more than that of oxygen because of
(a) Greater attraction of electrons by the nucleus
(b) Extra stability the half filled *p*-orbitals
(c) Smaller size of nitrogen
(d) More penetrating effect

36. Ease of formation of the anion is favoured by
(a) Lower value of ionization potential
(b) Lower value of electron affinity
(c) Higher value of electron gain enthalpy
(d) Lower value of electronegativity

37. When a neutral atom is converted into a cation its
(a) Atomic mass increases
(b) Atomic mass decreases
(c) Size increases
(d) Size decreases

38. Which of the following has zero electron affinity?
(a) Rn (b) Na
(c) F (d) Li

39. Which of the following N^{3-}, O^{2-}, F^- is largest?
(a) N^{3-} (b) O^{2-}
(c) F^- (d) All are equal

40. Diagonal relationship in periodic table is shown by the elements of
(a) First with second period
(b) Second with third period
(c) Third with fourth period
(d) None of theses

41. Atomic number of next inert gas if discovered will be
(a) 87 (b) 104
(c) 118 (d) 132

42. The largest size out of Na^+, Ne, F^- is
(a) Na^+ (b) Ne
(c) F^- (d) All are equal

43. Which of the following is best oxidizing agent?
(a) F_2 (b) CI_2
(c) O_2 (d) Br_2

44. Which of the following elements has the highest atomic volume?
(a) Rn (b) Ra
(c) Fr (d) He

45. Characteristic of transition elements is
(a) Incomplete *d*-orbitals
(b) Incomplete *f*-orbitals
(c) Incomplete *s*-orbitals
(d) Incomplete *p*-orbitals

46. The correct order of atomic size is
(a) Be > C > F > Ne
(b) Be < C < F < Ne
(c) Be > C > F < Ne
(d) F < Ne < Be < C

47. Which indicates the correct variation in electronegativities?
(a) F < N > O > C (b) F > N > O > C
(c) F < N < O < C (d) F > N > O < C

48. Which of the following has the largest ionic radius?
(a) Be^{2+} (b) Mg^{2+}
(c) Ca^{2+} (d) Sr^{2+}

49. The first ionization energy of Na, Mg, AI and Si is in the order?
(a) Na < Mg > AI < Si
(b) Na > Mg > AI > Si
(c) Na < Mg < AI > Si
(d) Na > Mg > AI > Si

50. The maximum tendency to form unipositive ion is for the element with the following electronic configuration?
(a) $Is^2 2s^2 2p^6 3s^2$ (b) $Is^2 2s^2 2p^6 3s^2 3p^1$
(c) $Is^2 2s^2 2p^6 3s^2 3p^2$ (d) $Is^2 2s^2 2p^6 3s^2 3p^3$

51. Which element has the maximum electron affinity?
(a) Mg (b) Na
(c) AI (d) S

52. Beryllium shows diagonal relationship with:
(a) Mg (b) AI
(c) Li (d) Si

53. On going down any group in the periodic table (example Li to Cs in IA or Be to Ra in IIA). The expected trend of change in atomic radius is a
(a) Continuous increase
(b) Continuous decrease
(c) An increase followed by a decrease
(d) A decrease followed by increase

54. Elements in the same vertical group of the periodic table have generally the same
(a) Atomic number
(b) Number of isotopes
(c) Same no. of electrons
(d) Same no. of electrons in the valence shell

55. Elements which generally exhibit multiple oxidation states and whose ions are usually coloured are
(a) Metalloids (b) Transition elements
(c) Non-metals (d) Gases

56. Which of the following elements have strong tendency to form anions?
(a) P, S, CI, (b) As, Sb, Bi,
(c) Fe, Co, Ni, (d) Li, Be, B,

57 The first element of each period in the periodic table has
(a) Two electrons in the first shell
(b) One electron in the first shell
(c) One *s*-electron in the outermost shell
(d) 8 electrons in the outermost shell

58. The 3d transition series contains elements having atomic numbers from
(a) 22 to 30 (b) 21 to 30
(c) 21 to 31 (d) 21 to 29

59. Out of the hydrogen halides, which one has the highest boiling point
(a) HI (b) HBr
(c) HCI (d) HF

60. The lightest metal is
(a) Li (b) Mg
(c) Ca (d) Na

61. Which element has the greatest tendency to lose electrons?
(a) F (b) S
(c) Fr (d) C

62. As one moves along a given row in the periodic table, ionization energy
(a) Remains unchanged
(b) Increases from left to right
(c) Increases and then decreases
(d) Decreases from left to right

63. Which is the lightest metal in the periodic table?
(a) H (b) Mg
(c) Ca (d) Li

64. In the periodic table, with the increase in atomic number, the metallic character of an element
(a) Decreases in a period and increases in a group
(b) Increases in a period and decreases in a group

(c) Increases both in a period and the group
(d) Decreases both in a period and the group

65. A newly discovered elements X is placed in group IA of the periodic table because it forms
(a) An oxide which is acidic
(b) A volatile chloride having formula XCl
(c) An ionic chloride having formula XCl
(d) An insoluble XCO_3

66. Among the following group of elements, the one whose elements can have positive as well as negative oxidation states are
(a) H, F, O (b) Na, Mg, AI
(c) He, Li, Be (d) H, CI, Br

67. The correct order of relative stability of half filled and completely filled shells is
(a) $p^3 < d^5 < d^{10} < p^6$
(b) $d^5 > p^3 > d^{10} > p^6$
(c) $d^5 < p^3 < d^{10} < p^6$
(d) $p^3 < d^{10} < d^5 < p^6$

68. Ease of formation of cation is favoured by
(a) Lower value of ionization potential
(b) Higher value of ionization potential
(c) Higher value of electron affinity
(d) Higher value of electronegativity

69. Ionization potential of Na would be numerically the same as:
(a) Electron affinity of Na^+
(b) Electronegativity of Na^+
(c) Electron affinity of Ne
(d) Ionization potential of Mg

70. Which of the following is most soluble in water?
(a) $Be(OH)_2$ (b) $Mg(OH)_2$
(c) $Ca(OH)_2$ (d) $Ba(OH)_2$

71. Which has the largest first ionization potential?
(a) Li (b) K
(c) Na (d) Rb

72. Element with electronic configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s^2 5p^3$ belongs to which of the following group of the periodic table?
(a) 3rd (b) 5th
(c) 7th (d) 2nd

73. O_2^{2-} is isoelectronic with
(a) H_2 (b) N_2
(c) F_2 (d) S

74. The decreasing order of second ionization potential of K, Ca and Ba (Z : K = 19, Ca = 20, Ba = 56).
(a) K > Ca > Ba (b) Ca > Ba > K
(c) Ba > K > Ca (d) K > Ba > Ca

75. In the periodic table the trend of electropositive character from Na to Ar
(a) Increases
(b) Decreases
(c) Remains constant
(d) First increases and then decreases

76. The atom with least covalent radius is
(a) I (b) Cl
(c) Br (d) F

77. The cation which is least stable is
(a) Li^+ (b) K^+
(c) AI^{2+} (d) Si^{2+}

78. Which of the following has the smallest radius?
(a) O^{2-} (b) F^-
(c) Li^+ (d) Be^{2+}

79. Lanthanide contraction can explain
(a) Atomic number of the series
(b) Density of lanthanides
(c) Valancy of lanthanides
(d) Ionic radius of lanthanides

80. In which of the following pairs, the first atom or ion is not larger than the second?
(a) N, O (b) CI^-, CI
(c) O, S (d) Fe^{2+}, Fe^{3+}

81. Elements of the same group in the period table are characterised by the same
(a) Ionization potential
(b) Electronegativity
(c) Electorn affinity
(d) No. of valency electrons

82. The element which has most stable +2 state
(a) C (b) Si
(c) Sn (d) None

83. The first four ionization energy values of an element are 191, 578, 872 and 5962 kcal. The no. of valence electrons in the element is
(a) 1 (b) 2
(c) 3 (d) 4

84. The elements whose atoms have three outermost shells incomplete are called
(a) *s*-block elements (b) *p*-block elements
(c) *f*-block elements (d) *d*-block elements

85. Which of the following species has highest electron affinity?
(a) F^- (b) O
(c) O^- (d) Na^+

86. The biggest ion is
(a) AI^{3+} (b) Ba^{2+}
(c) Mg^{2+} (d) Na^+

ANSWERS

1. (a)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (d)
11. (d)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (a)	**16.** (d)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (c)
31. (b)	**32.** (b)	**33.** (b)	**34.** (a)	**35.** (b)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (b)
41. (c)	**42.** (c)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (c)	**47.** (a)	**48.** (d)	**49.** (a)	**50.** (b)
51. (d)	**52.** (b)	**53.** (a)	**54.** (d)	**55.** (b)	**56.** (a)	**57.** (c)	**58.** (b)	**59.** (d)	**60.** (a)
61. (c)	**62.** (b)	**63.** (d)	**64.** (a)	**65.** (c)	**66.** (d)	**67.** (c)	**68.** (a)	**69.** (a)	**70.** (d)
71. (a)	**72.** (b)	**73.** (c)	**74.** (a)	**75.** (b)	**76.** (d)	**77.** (c)	**78.** (d)	**79.** (d)	**80.** (c)
81. (d)	**82.** (c)	**83.** (c)	**84.** (c)	**85.** (d)	**86.** (b)				

13 Chemical Bonding

BRIEF REVIEW OF THE CONCEPTS

When two atoms strongly tend to remain together rather than separate from each other, they are said to be in chemical bonding with each other. **A chemical bond** may be defined as "*attraction between two atoms in a molecule*". A **molecule** may be defined as '*a small electrically neutral cluster* or *group or aggregate of mutually boned atoms*'.

Cause of Chemical Bonding

Inert gases are extraordinarily stable and hardly enter into any chemical reactions. On studying the electronic structures of these gases, it was found that in every case (except helium which has two electrons), the outermost shell contains eight electrons (an octet.)

It was concluded that their chemical stability is due to the presence of eight electrons (two in case of helium) in the outermost shell. This electronic arrangement was, therefore, regarded as a **stable configuration.**

In case of all other elements which are chemically reactive, the outermost shells are found to contain less than eight electrons (incomplete octet). These atoms tend to complete their octets and attain a stable configuration by entering into chemical combination with other atoms, which involves a rearrangement of electrons. Thus '*the cause of chemical combination is the urge on the part of atoms, of various elements to attain a stable configuration of eight electrons in their outermost shell.* However, for hydrogen and lithium, the stable configuration is of two electrons and is known as **duplet.** The atoms complete their octets by losing, gaining or sharing electrons. Hence, according to the electronic theory of valency, the **valency** of an element is '*'the number of electrons lost or gained or contributed for sharing by the atom of an element in order to attain a stable configuration of eitht electrons in the outermost shell or complete its octet"*.

Types of Chemical Bonds Broadly speaking, the chemical bonds may be of following different types:

(i) Ionic bond
(ii) Covalent Bond
(iii) Dative bond or Co-ordinate bond
(iv) Hydrogen bond
(v) Metallic bond

IONIC BOND

When a complete transfer of electron (or electrons) takes place from one atom to the other so as to complete their octet, the two atoms become oppositely charged ions. These ions are then held together by electrostatic forces of attraction. The bond thus formed between the two atoms is called an ionic bond or electrovalent linkage. The number of electrons lost or gained gives the *Electro-valency.*

Properties of Ionic Compounds The ionic compounds are found to possess the following characteristic properties:-

(a) *Polar nature.* These compounds do not exist as individual molecules. Instead they are often hard solids, always made up of ions held together by strong electrostatic forces of attraction.

(b) *Electrical conductivity.* When put into water, the ions fall apart. The solution can thus conduct electricity. Similarly, these compounds conduct electricity in the fused state also.

(c) *Melting and boiling points.* These compounds generally have high melting points. This is because a lot of thermal energy is required to break down the inter-ionic forces and form a liquid. Similarly, they possess high boiling points.

(d) *Solubility.* These compounds are usually soluble in water but insoluble in benzene or other organic solvents.

Variable Valency The electrons present in the outermost shell of an atom give the valency of that atom and hence are called *valency electrons.* If the valency electrons are removed, the remainder is called *Core or Kernel.* In case of normal atoms, the core is stable and hence the valency of normal elements is fixed. However in case of transition or ***d*-block** elements, the core is not stable. Thus after the removal of valency electrons, one or more electrons can be lost from the outermost shell of the core also. Hence, these elements show two or more different valencies, i.e., they have **variable valency.**

Examples:

(a) *Iron having atomic no. 26, shows a valency of 2 or 3 as follows:*

$_{26}Fe$ = 2, 8, 14, 2 or $1s^2, 2s^2, 2p^6; 3s^2, 3p^6, 3d^6; 4s^2$

Fe^{2+} = 2, 8, 14 or $1s^2; 2s^2, 2p^6; 3s^2, 3p^6, 3d^6$

Fe^{3+} = 2, 8, 13 or $1s^2; 2s^2, 2p^6; 3s^2; 3p^6, 3d^5$

(b) *Cobalt having atomic no. 27, shows a valency of 2 or 3, as follows:*

$_{27}Co$ = 2, 8, 15, 2 or $1s^2, 2s^2, 2p^6 ; 3s^2, 3p^6, 3d^7 ; 4s^2$

Co^{2+} = 2, 8, 15 or $1s^2 ; 2s^2, 2p^2 ; 3s^2, 3p^6, 3d^7$

Co^{3+} = 2, 8, 14 or $1s^2 ; 2s^2, 2p^6 ; 3s^2, 3p^6, 3d^6$

In case of Sn and Pb, the variable valencies of 2 and 4 are due to a different reason. Each of these has four electrons in their outermost shell. Firstly they lose two electrons only forming bivalent positive ions, the remaining two electrons do not play their role in ion formation. Such electrons are called an *inert pair* and this effect is called **inert pair effect**. When an inert pair is also lost, tetravalent ions are formed.

COVALENT BOND

When the electrons are shared equally by the two atoms so as to complete their octets, the bond formed between them is called *covalent bond.* The number of electrons contributed by each atom for sharing gives the *co-valency.*

This may the explained as under:

The two atoms are short of the stable number of electrons (two or eight) in the outermost shell. Both the atoms contribute some of their electrons for sharing in order to complete their octets (or duplets). These mutually shared electrons become the common property of both atoms and also count towards their stable configuration. Each pair of shared electrons is indicated by a line (–).

The covalent bond formed by the mutual sharing of one pair of electrons between two atoms is called a single bond. If the electron pairs shared between two atoms are two or three then the bond is said to be double or triple, respectively.

Properties of Covalent Compounds. The covalent compounds are found to possess the following general properties, which are opposite to those of ionic compounds:–

1. These substances are made up of individual covalent molecules with weak inter-molecular forces.
2. Because of the weak inter-molecular forces, the substances are gases, liquids or soft solids at room temperature.
3. These substances when put into water do not undergo ionisation. Hence these substances in the fused state or in solution do not conduct electricity.
4. These substances have low melting and boiling points because little energy is needed to break down the weak inter-molecular forces.
5. These substances are commonly soluble in benzenc and other organic solvents but insoluble in water.

DATIVE BOND OR COORDINATE BOND

When in the formation of a bond between two atoms, only one atom contributes the pair of electrons and the other atom simply participate in sharing, the bond formed is called **dative bond** or **Co-ordinate bond.** This may be explained as under:–

In this case, one atom has its octet complete and has at least one lone pair (unshared) of electrons. The lone pair of electrons belonging to the atom with the complete octet is shared with the other atom and thus a link is established. The atom contributing the shared pair of electrons is called the **donor** while the atom which accepts these shared electrons is called the **acceptor.** This linkage set up between two atoms is represented by an arrow starting from the donor to acceptor.

HYDROGEN BOND

Whenever a substance contains a hydrogen atom linked to a highly electronegative atom N, O or F, the latter attracts the pair of electrons more and becomes slightly negative whereas the hydrogen atom becomes a slightly positive end. The negative end of the molecule is attracted by the positive end (*i.e.*Hydrogen atom) of the other molecule and in this way a bond is formed. Such a bond is called a **hydrogen bond** and is represented by a dotted line. Thus the hydrogen bond between the hydrogen atom bonded initially to the electronegative atom *X* and the other electronegative atom *Y* may be represented as follows:—

$$X—H...Y$$

Thus the hydrogen atom is attached simultaneously to two electronegative atoms. Hence it acts as a bridge between the two and is, therefore, called the *hydrogen bridge.*

Inter-molecular hydrogen bonding When hydrogen bonding occurs between two or more molecules, it is called inter molecular hydrogen bonding. Some examples are–

(a) *Hydrides of Fluorine, Oxygen and Nitrogen.* In case of *HF*, fluorine is highly electronegative atom. Thus the fluorine atom attracts the pair of electrons so that *F* end becomes negative and the *H* end becomes positive. The hydrogen atom of one molecule will attract the fluorine atom of the second molecule resulting in the formation of the associated molecule $(HF)_n$ as shown below:–

$$H—F...H—F...H—F...H—F$$

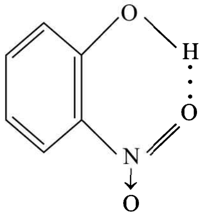
Similar association is found to occur in water (H_2O) and ammonia (NH_3). The existence of hydrogen bonds in these compounds is proved by the fact that the compounds show abnormally high melting and boiling points as compared with other hydrides in the same group of the periodic table.

It may be noted that since chlorine, bromine and iodine are not as highly electronegative as fluorine, therefore HCI, HBr and HI molecules do not show hydrogen bonding.

(b) The water molecules in ice form hydrogen bonds causing association due to which it has very open structure. This explains the low density of ice than that of water. Similarly, intermolecular hydrogen bonding exists in alcohols and carboxylic acids also.

Intra-molecular hydrogen bonding When hydrogen bonding occurs within a molecule, it is called intramolecular hydrogen bonding. It usually involves linking of two groups leading to the formation of a ring structure. An important example of intra-molecular hydrogen bonding is that of *o-nitrophenol.* Boiling point of *o*-nitrophenol is 214°C as compared to 290°C for *m* and 279°C for *p*-nitrophenol. Further *o*-nitrophenol is more volatile in steam and less soluble in water than the other two forms.

All the above facts can be explained on the assumption that *o*-nitrophenol contains hydrogen bond represented as



Due to the intra-molecular hydrogen bonding, the solubility of *o*-nitrophenol in water is also reduced. In *m*- and *p*-isomers, intra-molecular hydrogen bonding is not possible so inter-molecular hydrogen bonding takes place. This explains higher boiling points of *m*- and *p*-isomers.

Conditions for Hydrogen Bonding In case of inter molecular hydrogen bonding, the most important condition is that the molecules must contain one hydrogen atom linked to one highly electronegative atom.

In case of intra-molecular hydrogen bonding, the following conditions are favourable for hydrogen bonding:

(i) The molecule should contain two groups such that one group contains H-atom linked to a highly electro-negative atom and the other group should also contain a highly electronegative atom linked to a lesser electronegative atom.

(ii) The molecule should be planar.

Strength of Hydrogen Bond. The hydrogen bond is a weak bond. The strength of the strongest hydrogen bond is 20 – 40 kJmol^{-1} as compared with strength of 200 – 400 kJ mol^{-1} for normal covalent bonds.

Importance of Hydrogen Bond

(i) Hydrogen bonds are important in fixing properties such as solubilities, melting points and boiling points and in determining the form and stability of crystalline structures.

(ii) Hydrogen bonds are important in biological systems. Proteins, for example, contains both $>$CO and $>$NH groups and hydrogen bonds can be formed to bridge the space between H and O. The structure and hence the properties of proteins depend upon the existence of hydrogen bonds.

(iii) Hydrogen bonding within a single molecule is one of the chief factors in determining the structure of important biological substances.

(v) Metallic Bonding, Electron-sea model In case of metals, the electrons in the outermost shell (valence electrons) are lossely bound, hence they are also called as *free electrons.* The remainder portion of the atom is known as **Kernel** (which is positively charged sphere). These positively charged spheres are packed in a regular fashion. The free electrons are mobile in nature and move from one kernel to another throughout the metal lattice. Thus the metal crystal may be pictured as *an arrangement of positive ions immersed in* a '*Sea of mobile electrons'*. The electrons move in such a way that they are simultaneously near to two or more kernels and bind them together. The bond thus formed between the metal atoms is known as **Metallic bond.** *Hence a metallic bond may be defined as the* bond formed as a result of simultaneous attraction of an electron by two or more than two positive ions of the metal.

Problems for Practice

1. A trigonal bipyramidal molecule has bond angle of

(a) 90° (b) 109°28'

(c) 120° (d) 90°, 120°

2. Carbon tetrachloride is insoluble in water because

(a) Water is polar

(b) CCl_4 is non polar

(c) it can not form intermolecular hydrogen bonding with water

(d) They can not form intramolecular hydrogen bonding

3. Which of the following contain both ionic and covalent bonds?

(a) H_2O (b) NaOH

(c) C_6H_5Cl (d) CO_2

4. Compound having sp^3d hybridization is

(a) BF_3 (b) PF_5

(c) SF_6 (d) IF_7

5. Molecules obtained by sp^3d^2 hybridization has bond angle (s) of

(a) 90° (b) 109°28'

(c) 120° (d) 90° and 120°

6. The ion that is isoelectronic with CO is

(a) O_2^- (b) O_2^+

(c) CN^- (d) N_2^+

7. The number of electrons shared by each N atom in N_2 is

(a) 2 (b) 1

(c) 3 (d) 4

8. According to Fajan's rule, covalent bond is favoured by

(a) Large cation and small anion

(b) Large cation and large anion

(c) Small cation and small anion

(d) Small cation and large anion.

9. Among the following the electron deficient compound is

(a) CCl_4 (b) PCl_5

(c) $BeCl_2$ (d) BCl_3.

10. The most suitable method for the separation of a mixture of ortho and para-nitrophenol mixed in the ratio of 1 1 is

(a) distillation (b) crystallisation

(d) vapourization (d) All of these

11. O_2 molecule is

(a) paramagnetic (b) diamagnetic

(c) ferromagnetic (d) none of these

12. The bond order of O_2^- is

(a) 0.5 (b) 1.5

(c) 3.5 (d) 2.5

13. Which of the following will be octahedral?

(a) SF_6 (b) BF_4^-

(c) PCI_5 (d) BO_3^{3-}

14. The shape of gaseous $SnCI_2$ is

(a) Tetrahedral (b) Linear

(c) Angular (d) T-shaped

15. IF_5 has the following hybridization

(a) sp^3d^2 (b) sp^3d^3

(c) sp^3d (d) none of these

16. Antibonding MO is formed by

(a) Addition of atomic orbitals

(b) Subtraction of atomic orbitals

(c) Multiplication of atomic orbitals

(d) Division of atomic orbitals

17. On hybridization of one *s* and one *p* orbitals we get.

(a) Two mutually perpendicular orbitals

(b) Two orbitals at 180°

(c) Four orbitals directed tetrahedrally

(d) Three orbitals in a plane

18. Carbon tetrachloride has no net dipole moment because of
(a) Its planer structure
(b) Its regular tetrahedral structure
(c) Similar size of carbon and chlorine atoms
(d) Similar electron affinties of carbon and chlorine

19. Which is the weakest among the following types of bonds?
(a) Ionic bond (b) Metallic bond
(c) Covalent bond (d) Hydrogen bond

20. Polarisation is the distortion of the shape of an anion by an adjacently placed cation. Which of the following statements is correct?
(a) Maximum polarization is brought about by a cation of high charge
(b) Minimum polarization is brought about by a cation of low radius
(c) A large cation is likely to bring about a large degree of polarization
(d) A small anion is likely to undergo a large degree of polarization

21. Shape of H_3O^+ is
(a) Tetrahedral (b) Angular
(c) Pyramidal (d) Trigonal planar

22. The planar molecule among the following is
(a) BCl_3 (b) $SOCl_2$
(c) NH_3 (d) NF_3

23. Carbon in carbondioxide molecule is
(a) sp hybridized (b) sp^2 hybridized
(c) sp^3 hybridized (d) dsp^2 hybridized

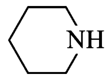
24. The compound 1, 2-butadiene has:
(a) Only sp hybridized carbon atoms
(b) Only sp^2 hybridized carbon atoms
(c) Both sp^2 and sp^3 hybridized carbon atoms
(d) sp, sp^2 and sp^3 hybridized carbon atoms

25. Which one of the statements is not true for salicylaldehyde?
(a) It is intra molecularly hydrogen bonded
(b) It is steam volatile
(c) Have higher solubitities in organic solvents
(d) Have high melting point and boiling point

26. The number of anti-bonding electron pairs in O_2^{2-} ion on the basis of MO theory is
(a) 4 (b) 3
(c) 2 (d) 5

27. Polarization of electrons in acrolein may be written as
(a) $\overset{\delta-}{CH_2}=\overset{\delta+}{CH}-CH=O$
(b) $\overset{+\delta}{CH_2}=CH-CH=\overset{-\delta}{O}$
(c) $\overset{\delta-}{CH_2}=CH-\overset{\delta+}{CH}=O$
(d) $\overset{\delta-}{CH_2}=CH-CH=\overset{\delta+}{O}$

28. The structure of ICl_2^- is
(a) Trigonal (b) Trigonal bipyramidal
(c) Octahedral (d) Square planar.

29. In piperidine  the hybrid state assumed by N is
(a) sp (b) sp^2
(c) sp^3 (d) dsp^2

30. The calculated bond order in H_2^- ion is
(a) 0 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) 1

31. The electronic configuration of metal M is $1s^2 2s^2 2p^6 3s^1$. The formula of its oxide will be
(a) MO (b) M_2O
(c) M_2O_3 (d) MO_2

32. The electronegativities of F, Cl, Br and I are 4·0, 3·0, 2· 8, 2·5 respectively. Hydrogen halide with a high percentage of ionic character is
(a) HF (b) HCl
(c) HBr (d) HI

33. Which pair among the following is isostructural?
(a) XeF_2, IF_2^- (b) NH_3, BF_3
(c) CO^{2-}_3, SO_3^{2-} (d) PCl_5, ICl_5

34. Which of the following has sp^3 hybridization of central atom?
(a) XeO_3 (b) BCl_3
(c) XeF_4 (d) BBr_3

35. The numerical value of the dipole moment of the molecule Cl_2 is
(a) Zero (b) 1.03
(c) 1.85 (d) 1.67

36. Of the following, the substance which has the highest boiling point is
(a) H_2S (b) H_2Se
(c) H_2Te (d) H_2O

37. The molecule which has pyramidal shape is
(a) PCl_3 (b) SO_3
(c) CO_3^{2-} (d) NO_3^-

38. Which of the following bonds is the strongest?
(a) F – F (b) I – I
(c) Cl – Cl (d) Br – Br

39. Bond angle in water molecules is 104.5° instead of 109.5° because of
(a) Lone pair-bond pair repulsion
(b) Bond pair-lone pair repulsion
(c) Lone pair-lone pair replusion
(d) Bond pair-bond pair repulsion

40. The shortest carbon-carbon bond distance is found in
(a) Diamond (b) Ethane
(c) Benzene (d) Acetylene

41. A number of ionic compounds, e.g. AgCl, CaF_2 $BaSO_4$ are insoluble in water. This is because
(a) Ionic compounds do not dissolve in water
(b) Water has a high dielectric constant
(c) Water is not a good ionizing solvent
(d) These molecules have exceptionally high attractive forces in lattice

42. The shape of CO_2 molecule is similar to
(a) H_2O (b) BeF_2
(c) SO_2 (d) None of these

43. Which is most viscous?
(a) CH_3OH (b) $C_2H_5 - OH$
(c) CH_2OH (d) None of these
$|$
CH_2OH

44. Which of the following statement is true?
(a) HF is less polar than HBr
(b) Absolutely pure water does not contain any ions
(c) Chemical bond formation takes place when forces of attraction overcome the forces of repulsion
(d) In covalency, the transferance of electrons takes place

45. Number of sigma bonds in P_4O_{10} is
(a) 6 (b) 7
(c) 17 (d) 16

46. In which of the following species is the underlined carbon having sp^3 hybridization?
(a) $CH_3\underline{C}OOH$ (b) $CH_3\underline{C}H_2OH$
(c) $CH_3\underline{C}OCH_3$ (d) $CH_2 = \underline{C}H - CH_3$

47. Sidewise overlap of *p-p* orbitals forms
(a) Sigma bond (b) Pi bond
(c) Coordinate bond (d) H-bond

48. H-bonding is not present in
(a) Glycerine
(b) Water
(c) Hydrogen sulphide
(d) Hydrogen fluoride

49. The boiling point of para nitrophenol is greater than that of ortho nitrophenol because.
(a) there is intermolecular hydrogen bonding in para nitrophenol and intramolecular hydrogen bonding in ortho nitrophenol.
(b) there is intramolecular hydrogen bonding in para nitrophenol and intermolecular hydrogen bonding in ortho nitrophenol.
(c) both have the same kind of hydrogen bonding.
(d) para introphenol is polar, while ortho nitrophenol is non-polar.

50. Number of π-bonds in naphthalenae is
(a) 6 (b) 3
(c) 4 (d) 5

51. An example of non-polar molecule is
(a) BF_3 (b) ClF_3
(c) PCl_3 (d) SO_2

52. Which of the following species possess giant network covalent structure?
(a) SiO_2 (b) Solid CO_2
(c) Ice (d) NaCI

53. How many σ and π bonds are there in the molecule $(CN)_2C = C(CN)_2$?
(a) Nine σ and seven π
(b) Five σ and nine π
(c) Five σ and eight π
(d) Nine σ and nine π

54. Ionic compounds in general possess both
(a) High melting point and non-directional bonds
(b) High melting points and low boiling points
(c) Directional bonds and low-boiling points
(d) High solubilities in polar and non-polar solvents

55. The shape PCl_5 molecule is
(a) Trigonal bipyramidal
(b) Pyramidal
(c) Octahedral
(d) None of these

56. NH_3 and BF_3 form an adduct which involves
(a) Ionic bond (b) Covalent bond
(c) Coordinate bond (d) H-bond

57. Which is most ionic?
(a) NaF (b) NaCl
(c) NaBr (d) NaI

58. The molecules BF_3 and NF_3 are both covalent compounds, but BF_3 is non-polar where as NF_3 is polar. The reason for this is
(a) B is a metal while nitrogen is a gas in uncombined state
(b) B-F bonds are non-polar white N-F bonds are polar
(c) BF_3 is planar but NF_3 is pyramidal
(d) Atomic size of boron is larger than that of nitrogen

59. Type of hybridization of central carbon in propadiene is
(a) sp^3 (b) sp^2
(c) sp (d) None of these

60. Which of the following contains covalent, ionic and coordinate bonds?
(a) NH_4Cl (b) NaCl
(c) $CaCl_2$ (d) CCl_4

61. In XeF_2, XeF_4 and XeF_6, the number of lone pairs of electrons is respectively
(a) 2, 3, 1 (b) 1, 2, 3
(c) 4, 1, 2 (d) 3, 2, 1

62. The bond order of O_2^+ is
(a) 1 (b) 1.5
(c) 2.5 (d) 3

63. Which of the following has zero dipole-moment?
(a) ClF (b) PCl_3
(c) SiF_4 (d) $CFCl_3$

64. The correct sequence of decrease in the bond angles of the following hydrides is
(a) $NH_3 > PH_3 > AsH_3 > SbH_3$
(b) $NH_3 > AsH_3 > PH_3 > SbH_3$
(c) $SbH_3 > AsH_3 > PH_3 > NH_3$
(d) $PH_3 > NH_3 > AsH_3 > SbH_3$

65. Fluorine molecule is formed by
(a) The axial *p-p* overlap
(b) The sidewise *p-p* overlap
(c) The axial *s-p* overlap
(d) The overlap of two sp^2 hybrid orbitals

66. The correct order of bond angles in the molecules, H_2O, NH_3, CH_4 and CO_2 is
(a) $H_2O > NH_3 > CH_4 > CO_2$
(b) $H_2O < NH_3 < CO_2 < CH_4$
(c) $H_2O < NH_3 > CO_2 > CH_4$
(d) $CO_2 > CH_4 > NH_3 > H_2O$

67. Which of the following arrangement of molecules is correct on the basis of their dipole moments?

(a) $BF_3 > NF_3 > NH_3$ (b) $NF_3 > BF_3 > NH_3$
(c) $NH_3 > BF_3 > NF_3$ (d) $NH_3 > NF_3 > BF_3$

68. Which of the following has highest dipole moment?

(a) NF_3 (b) NH_3
(c) NCl_3 (d) CCl_4

69. Solid CH_4 is a

(a) Molecular solid (b) Covalent solid
(c) Ionic solid (d) None of these

70. Which of the following has resonating structure?

(a) NO_3^- (b) NH_3
(c) BF_3 (d) CH_4

71. In graphite 'C' is

(a) sp^2 hybridized (b) sp^3 hybridized
(c) sp hybridized (d) None of these

72. Which of the following has intramolecular-H-bonding?

(a) o-nitro phenol (b) p-nitro phenol
(c) Phenol (d) Benzene

73. In solid argon, the atoms are held together by

(a) Ionic bonds
(b) Hydrogen bonds
(c) van der Waals forces
(d) Hydrophobic bonds

74. Of the following, the compound that obeys the octet rule is

(a) CO (b) BCl_3
(c) PCl_3 (d) SiF_6^{2-}

75. In which one of the following molecules central atom does not employ sp^3 hybrid orbital in bonding?

(a) H_2O (b) NH_3
(c) CO_2 (d) H_2S

76. The hybridization of sulphur in SO_2 is

(a) sp^2 (b) sp^3
(c) sp (d) spd^2

77. Two ice cubes are pressed over each other until they unite to form one block. Which one of the following forces dominates for holding them together?

(a) Dipole-dipole
(b) van der Waal forces
(c) Hydrogen bond formation
(d) Covalent attraction

78. The number of electrons that are paired according to MO theory in oxygen molecule is

(a) 16 (b) 12
(c) 7 (d) 14

79. In which of the following molecules, is the covalent bond most polar?

(a) HI (b) HBr
(c) HCl (d) H_2

80. Among the following compounds, which has a dipole moment?

(a) CCl_4 (b) C_6H_6
(c) BF_3 (d) HF

81. BCl_3 molecule is planar while NCl_3 is pyramidal because

(a) BCl_3 does not have lone pair on B but NCl_3 has
(b) B – Cl bond is more polar than N – Cl bond
(c) N atom is smaller than B
(d) N – Cl bond is more covalent than B – Cl bond.

ANSWERS

1. (d)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (c)	**8.** (d)	**9.** (d)	**10.** (a)
11. (a)	**12.** (c)	**13.** (a)	**14.** (c)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (b)	**19.** (d)	**20.** (a)
21. (c)	**22.** (a)	**23.** (a)	**24.** (d)	**25.** (d)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (b)
31. (b)	**32.** (a)	**33.** (a)	**34.** (a)	**35.** (a)	**36.** (d)	**37.** (a)	**38.** (c)	**39.** (c)	**40.** (d)
41. (d)	**42.** (b)	**43.** (a)	**44.** (c)	**45.** (d)	**46.** (b)	**47.** (b)	**48.** (c)	**49.** (a)	**50.** (d)
51. (a)	**52.** (a)	**53.** (d)	**54.** (a)	**55.** (a)	**56.** (c)	**57.** (a)	**58.** (c)	**59.** (c)	**60.** (a)
61. (d)	**62.** (c)	**63.** (c)	**64.** (a)	**65.** (a)	**66.** (d)	**67.** (d)	**68.** (b)	**69.** (a)	**70.** (a)
71. (a)	**72.** (a)	**73.** (c)	**74.** (c)	**75.** (c)	**76.** (a)	**77.** (c)	**78.** (d)	**79.** (c)	**80.** (d)
81. (a)									

14 Principles and Processes of Extraction of Elements

BRIEF REVIEW OF THE CONCEPTS

Most of the metals occur in nature in combination with the electronegative anions such as oxide, sulphide, chloride etc. Some metals occur as their carbonates, nitrates or silicates also. Only a few metals like gold, platinum etc.are found in free state and are called *native metals.* To some extent silver and copper are also found in native state.

The various compounds of metals which occur in nature and are obtained by mining are called minerals. These minerals are often associated with sand or other earthy impurities. All the minerals may not be useful for the extraction of metals from them. *Those minerals from which the metals can be profitably extracted are called ores.* All minerals can not be ores but all ores are minerals.

METALLURGY

Metallurgy is a process of obtaining pure metal from its ore. There is no common process for the extraction of all the metals because different ores have different nature. The procedure for obtaining a particular metal also depends on its chemical reactivity. However, the processes may be classified into following three steps:

1. Preliminary treatment involving concentration of ore or ore dressing.
2. Reduction of the ore to free metal.
3. Refining or purification of metal.

1. Preliminary Treatment

The ores, when mined from earth, usually contain large quantities of impurities such as stones, rocks, earthy matter, limestone, mica etc. These impurities are known as *gangue* or *matrix.* The removal of such impurities is known as ore dressing or concentrating the ore.

First of all such impurities are removed by hand picking (if possible) and then the ore is grinded into powder or smaller particles. After this it is concentrated by any of the following processes depending upon the nature of the ore.

(i) **Levigation or Hydraulic Washing**

This method is used for those ores, which are much heavier than the gangue particles present with it. The lighter gangue particles are washed away with a stream of running water. The heavier ore particles settle down. This is done on a specially designed slanting table with a series of obstacles on it. This table is known as *Wilfley table.* This method is generally used for the oxide ores of iron and tin.

(ii) **Froth Floatation Method**

This method is mainly used for the concentration of sulphide ores. This method is based on the difference in the wetting properties of ores and gangue particles. Sulphide ores are not wetted by water but are wetted by oils. In this method powdered ore is taken in a tank to which water and some quantity of pine or eucalyptus oil is added. Then air is blown through this mixture and froth is formed. The gangue particles sink to the bottom because they are wetted by water. The ore particles rise to the surface with the froth which is collected in another tank. After some time the froth subsides and the concentrated ore is obtained.

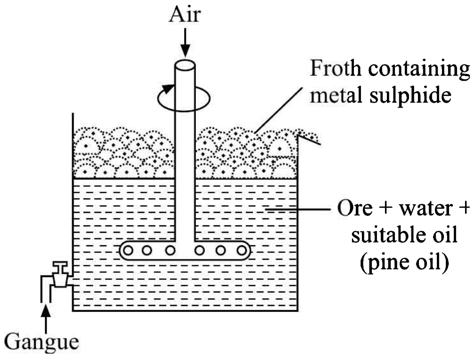


Fig. 14.1 Froth Floatation Method

(iii) **Magnetic Separation**

This method is useful for those ores which are magnetic in nature i.e., they are attracted by the magnet. In this method, the powdered ore is dropped over a belt which moves over two rollers. One of the rollers has a magnet in the centre. As the ore particles roll over the belt the magnetic component in the ore gets attracted by the magnet and falls nearer the roller. It forms a heap whereas gangue particles form a separate heap because they fall farther away.

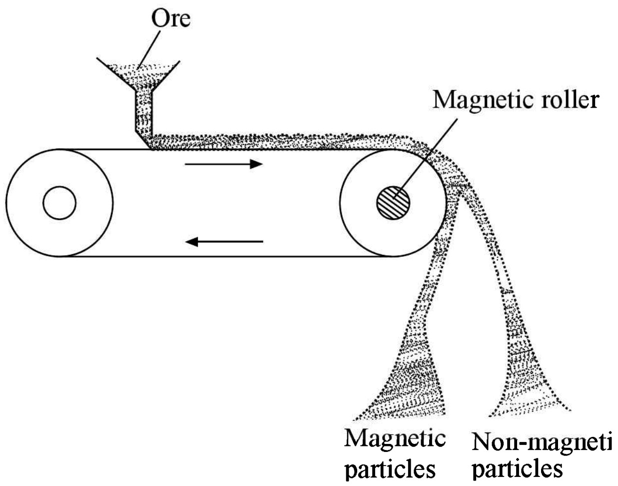


Fig. 14.2 Magnetic Separation

(iv) **Liquation**

This method is used to concentrate ores having lower melting points than the impurities. The powdered ore is heated on a sloping floor of a furnace. The temperature is maintained just above the melting point of the ore. The ore melts and flows away while the impurities are left behind. Stibnite (Sb_2S_3), an ore of antimony is concentrated by liquation.

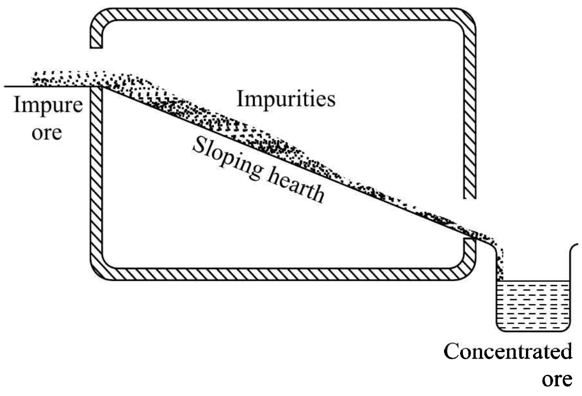


Fig. 14.3 Liquation

(v) **Leaching**

This method is based on the principle that ore may be dissolved in some chemical. For example, bauxite ($Al_2O_3.2H_2O$) contains impurities like Fe_2O_3, SiO_2 and can be purified by this method.

The powdered bauxite is leached with hot solution of sodium hydroxide. Al_2O_3, dissolves forming sodium aluminate ($NaAlO_2$) while impurities remain undissolved which are filtered off. Sodium aluminate solution is treated with little freshly precipitated aluminium hydroxide and diluted, when whole of aluminum is precipitated as $Al(OH)_3$. This is filtered, dried and ignited to obtain pure alumina (Al_2O_3).

$$\underset{\text{(Impure)}}{Al_2O_3} + 2NaOH \longrightarrow 2NaAlO_2 + H_2O$$

$$NaAlO_2 + 2H_2O \longrightarrow NaOH + Al(OH)_3\downarrow$$

$$2Al(OH)_3 \xrightarrow{\text{heat}} Al_2O_3 + 3H_2O$$

2. Reduction of the Ore into Free Metal

The extraction of a metal from one of its ores is basically a process of reduction. There are two common procedures for reduction. One is called *chemical reduction* and the other is *electrolytic reduction.* For chemical reduction, the concentrated ore is converted into the oxide by *roasting* or *calcination.* The oxide of metal thus obtained is reduced with the help of carbon or some other reactive metal. Some of the metals can not be obtained by chemical reduction because they themselves are very reactive. Such metals are obtained by carrying out the electrolytic reduction of their salts, e.g., aluminium, sodium etc.

Usually, ores contain many impurities in the form of sand, clay or limestone which are very difficult to melt.Various other substances are added which convert these impurities into some fusible compounds known as *slag*. The substances which are added for this purpose are known as *flux*. For example, in order to remove sand (SiO_2, also called silica) from an ore, limestone ($CaCO_3$) is added as flux. It combines with sand to form calcium silicate ($CaSiO_3$) which has low melting point and is separated as slag in molten state. Similarly, if an ore has the contamination of limestone then sand is added as flux. Reaction can be written as follows:

$$\underset{\text{Limestone}}{CaCO_3} \xrightarrow{\text{heat}} \underset{\text{Lime}}{CaO} + \underset{\text{Carbon dioxide}}{CO_2}$$

$$\underset{\text{Lime}}{CaO} + \underset{\text{Sand}}{SiO_2} \xrightarrow{\text{heat}} \underset{\substack{\text{Calcium silicate}\\ \text{(slag)}}}{CaSiO_3}$$

Below are mentioned some of the processes used for reduction.

(i) Roasting

Roasting is the process in which an ore is heated in excess of air. In this process, any impurities of sulphur or phosphorus are removed because they get converted into their oxides which are lost as gases.

$$\underset{\text{Sulphur}}{S} + \underset{\text{Oxygen}}{O_2} \longrightarrow \underset{\text{Sulphur dioxide}}{SO_2}$$

$$\underset{\text{Phophorus}}{P_4} + \underset{\text{Oxygen}}{5O_2} \longrightarrow \underset{\text{Phosphorus Pentoxide}}{P_4O_{10}}$$

Along with it the sulphide ores get converted into their oxides. For example, zinc sulphide on roasting gets converted into zinc oxide.

$$\underset{\text{Zinc sulphide}}{2ZnS} + \underset{\text{Oxygen}}{3O_2} \longrightarrow \underset{\text{Zinc oxide}}{2ZnO} + \underset{\text{Sulphur oxide}}{2SO_2}$$

(ii) Calcination

Calcination is the process in which ore is heated in absence of air. During this process, the moisture present in the ore is driven out and it becomes dry. Some metal hydroxides lose water and get converted into metal oxides. For example, aluminium hydroxide is converted into aluminium oxide.

Carbonate ores also lose carbon dioxide and get converted into metal oxides. For example, zinc carbonate is converted into oxide during calcination.

Reduction of the Metal Oxide into Metal

The oxide of the metal obtained after roasting or calcination is converted into free metal by reduction. This process of reduction can be done in number of ways depending upon the reactivity and the nature of the metal.

(i) *Reduction by carbon*

In this process, the metal oxide is mixed with carbon (coke) and heated to a temperature above its melting point. Carbon combines with oxygen (from ore) to form carbon monoxide and free metal is obtained. This process is generally used for the extraction of metals like zinc, copper, iron, lead and tin etc.

$$\underset{\text{Zinc oxide}}{ZnO} + \underset{\text{Coke}}{C} \longrightarrow \underset{\text{Zinc metal}}{Zn} + \underset{\text{Carbon monoxide}}{CO}$$

This process of reducing by carbon is also known as smelting.

(ii) *Reduction by a more reactive metal*

In some cases, highly reactive metals such as aluminium and magnesium are used to reduce the oxide of other metal. Such as chromium is obtained by the reaction of chromium oxide with aluminum powder.

$$\underset{\text{Chromium Oxide}}{Cr_2O_3} + \underset{\text{Aluminium metal}}{2Al} \longrightarrow \underset{\text{Chromium metal}}{2Cr} + \underset{\text{Aluminium oxide}}{Al_2O_3}$$

This method of reduction by aluminium is known as *alumino-thermite process*. It is also used in welding of railway track and heavy machines.

(iii) *Electrolytic reduction*

The oxides of highly electropositive and reactive metals like sodium, magnesium or aluminium are quite stable. They can not be reduced by heating with carbon. Then such metals are obtained by the electrolytic reduction of oxides, or chlorides etc. For electrolysis, the salts of such metals are taken in molten state. Sometimes, some other salt is also added to lower the melting point of the compound. During electrolysis, the positively charged metal ions move towards cathode where they gain electrons and are converted into metal atoms. Such as in case of aluminium oxide, aluminium ions (Al^{3+}) are reduced at cathode and aluminium metal is formed.

$$\underset{\text{Aluminium ion}}{Al^{3+}} + \underset{\text{Electrons}}{3e^-} \longrightarrow \underset{\text{Aluminium metal}}{Al}$$

Similarly, sodium is obtained by the electrolysis of molten sodium chloride.

$$\underset{\text{Sodium ion}}{Na^+} + \underset{\text{Electron}}{e^-} \longrightarrow \underset{\text{Sodium metal}}{Na}$$

3. Purification or Refining of the Metal

The metals obtained by any of the metallurgical operations are not pure. Sometimes they are contaminated with small quantities of other metals. It is also possible that materials used for reduction (e.g., carbon) also come with the metal. As iron obtained after reduction is always contaminated with carbon. The different methods which are used for refining are discussed below. The choice of a method for a particular metal depends on its nature and also on the type of the impurities present.

(i) Liquation

Metals like tin, lead, zinc, bismuth etc. have low melting points and can be purified by this method. Impure metal is placed in the upper part of a sloping hearth of a furnace. The temperature is maintained slightly above the melting point of the metal. So the metal melts and flows down along the slope and is collected whereas the impurities remain behind. (Remember that this method is also used for concentration of ore).

(ii) Distillation

Metals like mercury and zinc etc. having low boiling points can be purified by this method. In this method, metal is heated so that it goes into the vapour state and vapours are collected to give pure metal. Impurities whose boiling points are high will remain behind because they are not able to go into vapour state at that temperature.

(iii) Zone refining

This method is used to obtain the metals with very high purity. Impure metal is taken in the form of a rod. It is heated at one end with a circular heater. When this end of the rod melts then the heater is moved a little away towards the other end. At the previous end when cooling occurs, the metal solidifies and the impurities move with the molten zone. So ultimately, the impurities move with the moving heater towards the other ends of the rod where they are discarded.

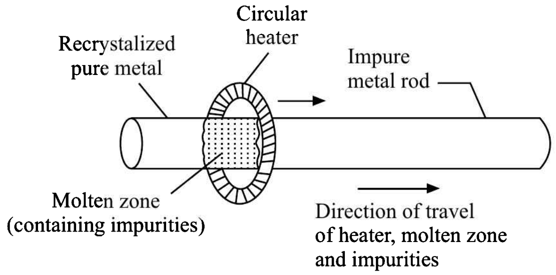


Fig. 14.4 Zone refining

(iv) Electrolytic refining

This method is mainly used for the purification of metals like copper, silver, zinc, tin, lead etc. The impure metal is made anode i.e., connected with the positive terminal of the battery. A thin sheet of pure metal is made cathode i.e., connected with the negative terminal. The vessel is filled with the aqueous solution of a salt of that metal. When electricity is passed through the solution, then at cathode, metal ions are reduced and are deposited on the sheet of the pure metal. At anode the metal atoms get oxidised and are converted into ions which move into the solution. So metal slowly dissolves from anode while the pure metal deposits at cathode. The impurities settle down and are known as *anode mud*.

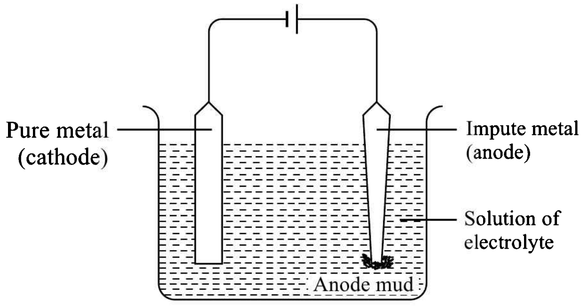


Fig. 14.5 Electrolytic refining of metals

For example, in the purification of copper a solution of $CuSO_4$ is taken in the tank. Impure copper is made anode and a sheet of pure copper is made the cathode. At anode oxidation occurs and copper ions are formed by the loss of electrons.

$$Cu \longrightarrow Cu^{2+} + 2e^-$$

At cathode, copper ions gain electrons and are converted into copper atoms which are deposited at cathode.

$$Cu^{2+} + 2e^- \longrightarrow Cu$$

(v) *Vapour Phase refining*

In this method, the impure metal is converted into some compound which is volatile and then it is decomposed by heating to yield the pure metal. For example, in **Mond Process** for the refining of nickel, it is heated with carbon monoxide to form volatile nickel carbonyl, Ni $(CO)_4$. This compound is subjected to still higher temperature when it gets decomposed to give pure nickel.

$$Ni + 4CO \xrightarrow{330-350K} Ni\,(CO)_4 \xrightarrow{450-470K} Ni + 4CO$$

Another example is **Van Arkel** method which is used for the purification of zirconium or titanium. Zirconium on heating with iodine forms the volatile zirconium tetraiodide which on heating at 2075 K decomposes to give zirconium.

$$Zr + 2I_2 \xrightarrow{870K} ZrI_4 \xrightarrow{2075K} Zr + 2I_2$$

FURNACES USED IN METALLURGICAL PROCESSES

A furnace is a heating device designed for the production, control, and continuous application of heat. It may be heated using electric current (electric furnace) or fired by fuels (combustion furnace). Fuels may be solid (coal), liquid (petroleum oil) or gaseous (petroleum gas, coke oven gas, etc.). Various types of furnaces are used in metallurgical operations. Some common types are described here.

1. ***Kiln***

   It is that arrangement in which fuel and material are mixed and air is freely admitted but no fusion takes place, e.g., lime kiln where quicklime is produced from limestone:

   $$CaCO_3 \rightarrow CaO + CO_2$$

2. ***Blast Furnace***

   In this type of furnace fuel and material (Ore + flux + fuel) are mixed and hot air blast is sent near the bottom and fusion of the charge occurs, e.g., metallurgy of iron, copper, and lead.

3. ***Reverberatory Furnace***

   In this furnace the fuel is burnt in a separate part of the furnace, called the fire-place. Flames and hot gases come in contact with the material to be heated. The furnace is used when it is required to heat the charge without mixing with the fuel. The main feature of the furnace is that the flames from the fire place strike against the arched roof to the furnace and are deflected on the charge on the hearth. Reverberatory furnace is used both for roasting and smelting.

4. ***Electric Furnace***

   Such a furnace is economical where electric power is cheap, and employed where very high temperatures are required. An example is the arc furnace used in steel manufacture. A temperature of about 3300 K can be obtained.

Table 14.1 List of some important ores for various elements

Elements	*Nature of Occurrence*	*Mineral*	*Composition*
	Sulphides		
Iron (Fe)		Iron pyrites	FeS_2
Copper (Cu)		Copper pyrites	$CuFeS_2$
Silver (Ag)		Argentite	Ag_2S
Zinc (Zn)		Zinc blende	ZnS
Mercury (Hg)		Cinnabar	HgS
Lead (Pb)		Galena	PbS
	Oxides		
Iron (Fe)		Haematite	Fe_2O_3
		Magnetite	$Fe_3\,O_4$
Aluminium (Al)		Bauxite	$Al_2O_3 . 2H_2O$
Titanium (Ti)		Rutile	TiO_2
Manganese (Mn)		Pyrolusite	MnO_2
Copper (Cu)		Cuprite	Cu_2O
Tin (Sn)		Cassiterite	SnO_2
	Carbonates		
Magnesium (Mg)		Magnesite	$MgCO_3$
		Dolomite	$CaCO_3 . MgCO_3$
Calcium (Ca)		Lime stone	$CaCO_3$
Zinc (Zn)		Calamine	$ZnCO_3$
	Halides		
Sodium (Na)		Rock salt	$NaCl$
Magnesium (Mg)		Carnallite	$KCl. MgCl_2 . 6H_2O$
Silver (Ag)		Horn Silver	$AgCl$
	Sulphates		
Calcium (Ca)		Gypsum	$CaSO_4 . 2H_2O$
Magnesium (Mg)		Epsom salt	$MgSO_4 . 7H_2O$

Problems for Practice

1. The process of extracting a metal in pure form from its ore is known as

(a) Refining (b) Purification
(c) Metallurgy (d) None of these

2. Froth floatation process for the concentration of ores is an illustration of the practical application of

(a) Adsorption (b) Absorption
(c) Coagulation (d) Sedimentation

3. Which of the following is not needed for the metallurgy of aluminium?

(a) Carbon electrodes
(b) A flux such as cryolite
(c) Sodium carbonate for refining bauxite
(d) Fe powder for reduction of Al_2O_3

4. The froth-floatation process is based upon

(a) Magnetic properties of gangue
(b) Specific gravity of ore particles
(c) Preferential wetting of ore particles by oil
(d) Preferential adsorption of gangue particles by oil

5. Chief ore of aluminium is

(a) Cryolite (b) Kaolin
(c) Bauxite (d) Felspar

6. The process of extraction of sodium on a commercial scale by the electrolysis of fused sodium chloride is called

(a) Down's process (b) Solvay process
(c) Nelson process (d) Castner process

7. Permanent magnets are made of alloys of

(a) Co (b) Zn
(c) Al (d) Pb

8. Which of the following alloys of steel is used for making utensils and automobile parts?

(a) Invar (b) Chrome steel
(c) Stainless steel (d) Tungsten steel

9. Which metal shows inertness due to its protective layer?

(a) Ag (b) Al
(c) Fe (d) Au

10. Pyrolusite is an ore of

(a) Mn (b) Al
(c) Cr (d) Zn

11. Gun metal is an alloy of

(a) Cu and Sn (b) Cu, Zn and Ni
(c) Cu, Zn and Sn (d) Cu and Al

12. Zinc is obtained on large scale by

(a) Electrolysis of $ZnCl_2$
(b) Reduction of ZnO
(c) Precipitation with Ag
(d) All are correct

13. Monal metal is an alloy of

(a) Cu-Ni (b) Mg-Al
(c) Cu-Sn (d) Zn-Sn

14. Malachite is a mineral of

(a) Mg (b) Mn
(c) Hg (d) Cu

15. In the formation of Al_2O_3 large amount of heat is evolved. This property is used in

(a) Deoxidation
(b) Confectionary
(c) Indoor photography
(d) Thermite welding

16. Blister copper is

(a) Pure copper
(b) Ore of copper
(c) Copper containing some impurity
(d) Alloy of copper

17. The process of zone refining is based upon

(a) Fractional crystallization
(b) Fractional distillation
(c) Magnetic properties of impurities
(d) Impurities are less fusible than metals

18. Reducing metal required in Alumino Thermic process is

(a) Chromium powder (b) Zinc powder
(c) Aluminium powder (d) Silver powder

19. Calamine, an ore of zinc is

(a) $ZnSO_4$ (b) ZnS
(c) $ZnCO_3$ (d) $CaCO_3$

20. An ore which is concentrated by magnetic separation is

(a) Galena (b) Cassiterite
(c) Alumina (d) Cinnabar

21. Which of the following metals is obtained by Mond's process?

(a) Ni (b) Co
(c) Fe (d) All of these

22. Carbon reduction method is used in the extraction of

(a) Sn (b) Zn
(c) Pb (d) All of these

23. The process of concentrating the silver ore is based on its solubility in

(a) HCl (b) HNO_3
(c) KCN (d) NaOH

24. The main function of roasting is

(a) Reduction (b) Oxidation
(c) Concentration (d) Separation

25. Anode mud obtained in the electrorefining of copper contains

(a) Ag and Au only
(b) Ag, Au and other insoluble impurities
(c) Ag, Au, Fe and insoluble impurities
(d) Ag, Au and Fe only

26. The process in which metal is converted into vapour form is

(a) Mond's process (b) Van Arkel process
(c) Amalgamation (d) Both (a) and (b)

27. Stainless steel usually contains

(a) Al (b) Cr
(c) Sn (d) Zn

28. Froth floatation process is used for the concentration of

(a) Oxide ore (b) Sulphide ore
(c) Carbonate ore (d) Sulphate ore

29. Which one of the furnaces among the following can produce highest temperature?

(a) Muffle furnace
(b) Blast furnace
(c) Reverberatory furnace
(d) Electric furnace

30. The oil used in the floatation method for the purification of ores is

(a) Coconut oil (b) Olive oil
(c) Pine oil (d) None

31. Of the following, which can not be obtained by the electrolysis of the aqueous solution of their salts?

(a) Ag (b) Mg and Al
(c) U (d) Cr

32. In metallurgy, the product of following reaction is known as

$$CaO + SiO_2 \rightarrow CaSiO_3$$

(a) anode mud (b) gangue
(c) flux (d) slag

33. The ore containing magnesium carbonate and calcium carbonate is known as

(a) Dolomite (b) Carnalite
(c) Magnesite (d) Bauxite

34. The main ore of iron is

(a) Chloride (b) Sulphate
(c) Nitrate (d) Oxide

35. Van Arkel method of purification of metals involves converting the metal to a

(a) Volatile stable compound
(b) Volatile unstable compound
(c) Non volatile stable compound
(d) None of these

36. The process of heating pyrite ores in air to remove sulphur is known as

(a) smelting (b) roasting
(c) calcination (d) auto oxidation

37. The purpose of smelting an ore is

(a) to oxidize it
(b) to reduce it
(c) to separate volatile impurities
(d) to obtain an alloy

38. In the Hall's process for extraction of Al, the ore is fused with

(a) $NaHCO_3$ (b) Na_2CO_3
(c) NaF (d) Na_3AlF_6

39. Process of removing layers of basic oxides from metals before electroplating is called

(a) Galvanising (b) Anodising
(c) Pickling (d) Poling

40. Which of the following metals is obtained by leaching process using a solution of sodium cyanide and then precipitating the metal by addition of zinc dust?

(a) Copper (b) Silver
(c) Nickel (d) Iron

41. The term 'anode mud' refers to

(a) Impure metal acting as anode
(b) Anode coated with mud
(c) Insoluble matter collected under the anode
(d) Calcium silicate

42. In the manufacture of iron from an iron oxide ore, limestone is added because it acts as

(a) An oxidizing agent
(b) A reducing agent
(c) A precipitating agent
(d) A flux

43. The electrolytic method of reduction is employed for the preparation of metals that are

(a) weakly electropositive
(b) weakly electronegative
(c) strongly electropositive
(d) easily oxidizable

44. The sulphide ores are converted into their oxides by roasting which is heating in the presence of air. The carbonate ores are converted into their oxides by

(a) Roasting (b) Calcination
(c) Smelting (d) None of these

45. The substance which is added to ores to remove the impurities is known as

(a) Gangue (b) Catalyst
(c) Slag (d) Flux

46. The process that is carried out in the blast furnace with respect to ore in the metallurgy of iron is

(a) Oxidation (b) Reduction
(c) Both (d) None of these

47. The lighter impurities can be removed from the ore by

(a) Magnetic separation
(b) Froth floatation process
(c) Gravity separation
(d) None of these

48. Cupellation process is used in the metallurgy of

(a) silver (b) copper
(c) aluminium (d) iron

49. An alloy is

(a) an intermetallic compound
(b) a solution of two or more metals
(c) a solution of two or more non-metals
(d) a mineral from which a metal can be extracted easily.

50. Van Arkel method involves converting the metal to

(a) volatile stable compound
(b) volatile unstable compound
(c) non-volatile stable compound
(d) any of these

ANSWERS

1. (c) **2.** (a) **3.** (d) **4.** (c) **5.** (c) **6.** (a) **7.** (a) **8.** (c) **9.** (b) **10.** (a)
11. (c) **12.** (b) **13.** (a) **14.** (d) **15.** (d) **16.** (c) **17.** (a) **18.** (c) **19.** (c) **20.** (b)
21. (a) **22.** (d) **23.** (c) **24.** (b) **25.** (b) **26.** (d) **27.** (b) **28.** (b) **29.** (d) **30.** (c)
31. (b) **32.** (d) **33.** (a) **34.** (d) **35.** (b) **36.** (b) **37.** (b) **38.** (d) **39.** (c) **40.** (b)
41. (c) **42.** (d) **43.** (c) **44.** (b) **45.** (d) **46.** (b) **47.** (c) **48.** (a) **49.** (b) **50.** (b)

Hydrogen

15

BRIEF REVIEW OF THE CONCEPTS

Hydrogen was discovered in 1766 by Henry Cavendish. He found that when metals such as iron, zinc or tin were dissolved in dilute acid, a combustible gas was evolved. He was also the first person to recognize in 1781, that water is the product of the combination of hydrogen and oxygen. Hydrogen plays crucial role in most of the biochemical reactions. Its oxide, water, is absolutely essential for the very existence of life on earth. The smaller atomic size, and the ways hydrogen forms bonds with other elements makes it unique. In combination with other elements, hydrogen is widely distributed. Apart from water, it is an important constituent of petroleum products, fats, proteins and carbohydrates.

Position of Hydrogen in the Periodic Table

Hydrogen ($Z = 1$) is the first member of the periodic table with only one electron in the valence shell defined by the principal quantum number, $n = 1$. This signifies that the electronic configuration of hydrogen atom in the ground state is $1s^1$. This single electron makes it practicable to classify hydrogen along with the alkali metals (Group IA), but unlike alkali metals hydrogen does not lose this single electron easily to permit such a classification. Similarly hydrogen may be classified along with halogens (Group VIIA) because a halogen atom needs only one electron to acquire a noble gas electronic configuration.

Therefore, the position of hydrogen in the periodic table finds an interesting situation.

(a) **Resemblance with alkali Metals**

(i) *Electronic configuration*: Similar to alkali atoms, hydrogen atom has one valence electron only.

(ii) *Formation of monopositive ion*. Like alkali metals, hydrogen has the tendency to lose its valence electron to form H^+. This way both the hydrogen as well as the alkali metals show + 1 valency.

(iii) *Electropositive nature*: Hydrogen resembles alkali metals in its electropositive nature and combines with electronegative elements such as halogens to form halides. The evidence provided for the electropositive nature of hydrogen is that it is liberated at the cathode when an aqueous solution of HCl is electrolyzed. Alkali metal halides also behave in the same way.

(iv) *Combination with* O_2: Hydrogen combines with oxygen to form H_2O similar to alkali metals forming e.g., Na_2O and K_2O.

(v) *Reducing behaviour*: Like alkali metals hydrogen is also a good reducing agent.

(b) **Differences with Alkali Metals**

(i) *Ionization energy*: Ionization energy of hydrogen as compared with those of alkali metals is perhaps the main objection in not placing hydrogen with alkali metals. Through it is not surprising to see the abnormal behaviour of the first element of the various groups, yet hydrogen is much off the track.

	H	Li	Na	K	Rb	Cs
Ionization Energy (kJ mol^{-1})	1311	520	495.7	418.6	402.9	375.6

It is this high value of ionization energy of hydrogen that makes the stability of a bare proton, H^+, impossible in water. In water it is the H_3O^+ ion that really exists and not H^+. This phenomenon is not shown by the alkali metals.

(ii) *Nature of oxides*: This is another objection which disfavours the inclusion of hydrogen in Group IA elements. Oxides of hydrogen, H_2O, is amphoteric whereas the oxides of alkali metals e.g., Na_2O, are basic.

(c) **Resemblance with Halogens**

(i) *Electronic configuration*: By acquiring one electron, hydrogen atom can resemble with electr-onic configuration of helium, $1s^2$. Halogens also behave similarly. The valence shell electronic configuration of halogens is $ns^2\ np^5$ and by acquiring just one electron this changes to $ns^2\ np^6$, the noble gas electronic configuration.

(ii) *Formation of negative ions*: Halogens form halide ions by gaining an electron which is energetically a favourable process. Similarly hydrogen also forms hydride ion, H^- by gaining an electron.

$$\text{Cl} \xrightarrow{+e^{-1}} \text{Cl}^- \text{ (chloride ion)}$$

$$\text{H} \xrightarrow{+e^{-1}} \text{H}^- \text{ (hydride ion)}$$

Thus, H_2 is liberated at the *anode* on electrolyzing molten anhydrous ionic hydrides such as NaH the way Cl_2 is liberated when fused NaCl is electrolyzed.

(iii) *Valency*: Because of the formation of hydride ion, H^-, hydrogen possesses a valency of – 1 (or – I oxidation state) like the halogens, F^-, Cl^- Br^- and I^-.

(iv) *Atomic state*: Like halogens hydrogen exists in the diatomic state, H_2.

(v) *Combination with non-metals:* Halogens (*e.g.*, Cl_2) combine with non-metals like C, Si to form covalent compounds CCl_4, $SiCl_4$, respectively. Similarly hydrogen combines with non-metals to form analogous compounds like CH_4 and SiH_4.

(vi) *Exchange of hydrogen and halogen atoms*: Hydrogen atoms can be exchanged with halogen atoms in organic compounds and vice-versa.

(d) **Differences with Halogens**

(i) *Nature of oxides*: Hydrogen oxide, H_2O, is neutral while those of halogens like Cl_2O_7 are acidic.

(ii) *Valency*: Hydrogen shows a valency of one whereas variable valency is seen among the heavier halogens e.g., iodine forms IF_3, IF_7 (*interhalogen* compounds).

Preparation of Hydrogen

Hydrogen may be obtained from its sources such as water, acids, alkalies, metallic hydrides and hydrocarbons in a number of ways. Hydrogen is also obtained as a by-product in the manufacture of caustic soda and during the preparation of heavy water.

(i) *By the reaction of metals with water*: Highly electropositive metals such as alkali metals liberate hydrogen from cold water, other require hot water or steam. In general, the vigour of action is greater, the higher a metal is placed in the E.M.F. series.

$$2Na + 2H_2O \longrightarrow 2NaOH + H_2$$

$$Zn + \underset{\text{(Steam)}}{H_2O} \longrightarrow ZnO + H_2$$

(ii) *By the action of acids on metals*: Hydrogen is often prepared in the laboratory by the action of dilute solution of strong acids with moderately active metals which lie above the E.M.F. series, e.g.,

$$Zn + 2HCl \longrightarrow ZnCl_2 + H_2$$

$$Fe + H_2SO_4 \longrightarrow FeSO_4 + H_2$$

$$2Al + 3H_2SO_4 \longrightarrow Al_2(SO_4)_3 + 3H_2$$

An oxidizing acid, such as HNO_3, will liberate hydrogen but rarely, as, for example very dilute (1 – 2%) HNO_3 with magnesium

$$Mg + 2HNO_3 \longrightarrow Mg(NO_3)_2 + H_2$$

(with concentrated nitric acid NO is obtained

$$3Mg + 8HNO_3 \longrightarrow 3Mg(NO_3)_2 + 4H_2O + 2NO$$

(iii) *By the action of certain metals on alkalies*: Certain amphoteric metals like aluminium, zinc and silicon react with caustic alkali solution to liberate hydrogen.

$$2Al + 2NaOH + 2H_2O \longrightarrow 2NaAlO_2 + 3H_2$$

(iv) *By the action of water on metallic hydrides*: Hydrolysis of ionic hydrides yields hydrogen and the alkali

$$LiH + H_2O \longrightarrow LiOH + H_2$$

$$CaH_2 + 2H_2O \longrightarrow Ca(OH)_2 + 2H_2$$

In fact CaH_2 is produced commercially as portable source of hydrogen under the name *hydrolith*.

Commercial Production of Hydrogen

(i) *From water by electrolysis*: A satisfactory way of decomposing water is by passing a direct current through water to which a small amount of H_2SO_4 or NaOH, has been added. In practice, solutions of NaOH are electrolyzed in cells with iron cathodes and nickel anodes; the cells are designed to keep the anode and cathode products separate.

(ii) *From water gas—Bosch process*: When steam is passed over coke, carbon monoxide and hydrogen are produced:

$$C + H_2O \xrightarrow{1300\text{ K}} CO + H_2$$

The mixture of CO *and* H_2 *is called water gas*. It is an important industrial fuel because both the substances are combustible. This mixture is treated with steam in the presence of a catalyst iron oxide or cobalt oxide to oxidize carbon monoxide to carbon dioxide:

$$\underset{\text{water gas}}{[CO + H_2]} + H_2O \xrightarrow[\text{Catalyst}]{673\text{ K}} CO_2 + 2H_2$$

The carbon dioxide is readily removed by passing the mixture of the two gases through water under pressure. The carbon dioxide dissolves; the hydrogen does not.

(iii) *Steam-hydrocarbon process*: Large amounts of commercial hydrogen are made by passing mixtures of hydrocarbons and steam over a nickel catalyst at high temperature:

$$CH_4 + 2H_2O \xrightarrow[1200\text{ K}]{Ni} CO_2 + 4H_2$$

Properties of hydrogen:

Molecular hydrogen is the lightest of all gases and is colourless, odourless and tasteless at room temperature. Hydrogen gas is adsorbed on certain metals such as platinum or palladium especially if the metal is finely divided or in colloidal state.

Chemical properties of hydrogen are due to its combination with other elements showing oxidation states of +1, –1 and 0. It acts as reducing agent in many processes and adds on to alkenes or alkynes to corresponding saturated compounds.

Uses or Hydrogen

(i) Hydrogen is used in the manufacture of ammonia by the Haber's process:

$$N_2 + 3H_2 \underset{200\text{ atm}}{\overset{\sim 823\text{ K}}{\rightleftharpoons}} 2NH_3$$

(ii) It is used in the refining of petroleum.

(iii) In the welding processes to produce high temperature oxy-hydrogen flame.

(iv) In the hydrogenation of vegetable oils to manufacture vanaspati ghee.

(v) For the production of synthetic methanol from carbon monoxide:

$$CO + 2H_2 \xrightarrow{\text{Catalyst}} CH_3OH$$

(vi) As a coolant for steam driven turbogenerators because it has the highest thermal conductance properties of any gas.

(vii) It is used as a rocket fuel in the liquid state.

(viii) Its future uses may include as an automobile fuel and in the fuel cells for the generation of electricity.

Ortho and Para Hydrogen

Heisenberg (1927) predicted from the quantum mechanical considerations that molecular hydrogen, H_2, exists in two isomeric forms on account of the spinning of the nuclei of the two atoms around their own axis. *When the two nuclei spin in the same direction the resulting molecule is called the ortho-hydrogen. On the other hand when the two nuclei spin in the opposite or antiparallel direction it is called para-hydrogen.*

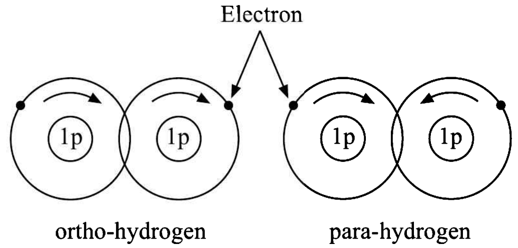


Fig 15.1

These two varieties are also called *nuclear isomers* of hydrogen. The *para*-form is symmetrical while the *ortho*-form is not. It is found that the total energy content of *para-hydrogen* is less than the *ortho-hydrogen*. Because of low energy *para-hydrogen* is favoured at low temperatures. At room temperature molecular hydrogen consists of 25% *para*-and 75% *ortho*-forms. Both *para*- and *ortho*-hydrogen are chemically similar. Like hydrogen, deuterium (an isotope of hydrogen) also exists in *ortho*- and *para*-forms.

Isotopes of Hydrogen

Natural hydrogen consists of three isotopes: ***protium*** ($_1^1H$): ***deuterium*** or heavy hydrogen ($_1^2H$ or D); and ***tritium*** ($_1^3H$ or T). Protium nucleus contains just one proton; the deuterium nucleus has one proton and one neutron while the tritium nucleus has one proton and two neutrons. The protium isotope is most abundant of the three isotopes. In nature protium is 99.985 per cent, deuterium 0.015 per cent while tritium occurs to the extent of about 1 atom per 10^{18} hydrogen atoms. It is formed as a result of nuclear reactions induced by cosmic rays in the upper atmosphere:

$$^{14}_{7}N + ^{1}_{0}n \longrightarrow ^{3}_{1}H + ^{12}_{6}C$$

The scarcity of tritium in nature is due to the instability and consequent radioactivity of its nucleus. Half-life of tritium is 12.35 years.

In chemical reactions, protium and deuterium show different equilibrium constants. Property differences arising from differences in atomic masses are called *isotopic effects*. For example, the dissociation constant of ordinary water in the equilibrium

$$H_2O + H_2O \rightleftharpoons H_3O^+ + OH^-$$

is 1.0×10^{-14} at room temperature. For the corresponding dissociation of heavy water.

$$D_2O + D_2O \rightleftharpoons D_3O^+ + OD^-$$

the equilibrium constant is 0.2×10^{-14}, which is significantly smaller.

Hard and Soft Water

Water obtained from natural sources contains many salts dissolved into it and is classified as soft and hard water depending upon its behaviour towards soap solution. Sample of water that produces foam with soap is called soft water and which does not produce foam is called hard water. Hardness of water is due to the presence of chlorides, sulphates and bicarbonates of magnesium and calcium. Hardness due to bricabonates is called as temporary hardness whereas chlorides and sulphates cause permanent hardness. Temporary hardness can be removed by (i) simply boiling the water or (ii) by treating with calculated amount of quick lime called *Clark's process*. In both these processes, bicarbonates get converted into insoluble carbonates which, are removed by filtration. Permanent hardness is removed by the following processes.

(i) *By treatment with sodium carbonate* which converts soluble chlorides or sulphates into insoluble carbonates which are removed by filtration.

(ii) *By ion exchange method* in which Ca^{2+} and mg^{2+} ions present in hard water are exchanged by Na^+ ions present in compounds called ion exchangers.

(iii) *By Calgon process* in which Ca^{2+} and Mg^{2+} ions present in hard water are made ineffective by treatment with sodium hexametaphosphate, $Na_2\,[Na_4\,(PO_3)_6]$, also called calgon.

Heavy Water (D_2O)

Heavy water is prepared by the electrolysis of 0.5 M NaOH solution. It has been found that 83 mL of 99% pure D_2O is obtained by the electrolysis of 2400 liters of ordinary water using nickel electrodes. Heavy water is also obtained by fractional distillation of ordinary water although the difference in the boiling points of ordinary water (373 K) and heavy water (374.42 K) is very small.

Heavy water is mainly used as a moderator and also as coolant in nuclear reactors because it slows down the fast moving neutrons. It is also used as a tracer compound in mechanistic studies of various reactions.

Hydrogen Peroxide (H_2O_2)

It can be obtained by the following processes:

(i) **From sodium peroxide:** Hydrogen peroxide is obtained by treating sodium peroxide with 20% sulphuric acid solution. This method is also called Merck's process.

$$Na_2O_2 + H_2SO_4 \longrightarrow Na_2SO_4 + H_2O_2$$

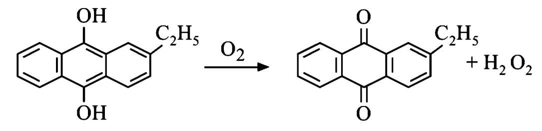
(ii) **From barium peroxide:** Barium peroxide reacts with sulphuric acid or phosphoric acid to give hydrogen peroxide.

$$BaO_2 + H_2SO_4 \longrightarrow BaSO_4 + H_2O_2$$
$$3BaO_2 + 2H_3PO_4 \longrightarrow Ba_3(PO_4)_2 + 3H_2O_2$$

(iii) **By the electrolysis of dilute sulphuric acid:** Electrolysis of 50% H_2SO_4, produces peroxodisulphuric acid which on hydrolysis yields hydrogen peroxide.

$$H_2SO_4(aq) \rightleftharpoons H^+_{(aq)} + HSO^-_{4\,(aq)}$$
$$\text{At anode } 2HSO_4^- \longrightarrow H_2S_2O_{8(aq)} + 2e^-$$
$$H_2S_2O_{8(aq)} + 2H_2O \longrightarrow 2H_2SO_4 + H_2O_2$$

(iv) **By oxidation- reduction of 2-ethylanthraquinol:** In this process air is passed through the solution of 2-ethylanthraquinol which gets oxidised to 2-ethylanthraquinone and hydrogen peroxide which is extracted with water.



2-Ethylanthraquinol 2-Ethylanthraquinone

Fig 15.2

2-Ethylanthraquinone formed is reduced back to 2-ethylanthraquinol with hydrogen in presence of Pd and it is used again in the first step.

Strength of Hydrogen Peroxide: The strength of an aqueous solution of hydrogen peroxide is generally expressed as volume strength. It is defined as the volume of oxygen liberated at NTP by the decomposition of 1 ml of that sample of hydrogen peroxide. For example 20 *volume* means 1 mL of that sample of hydrogen peroxide will produce 20 mL of oxygen at NTP.

Problems for Practice

1. *Para* and *ortho* hydrogen differ in
(a) atomic number (b) atomic mass
(c) spins of proton (d) number of neutrons

2. Heavy water is obtained by
(a) boiling water
(b) fractional distillation of H_2O
(c) prolonged electrolysis of H_2O
(d) Both b and c

3. What is heavy water?
(a) $H_2{}^{18}O$ (b) $H_2{}^{16}O$
(c) H_2O_3 (d) D_2O

4. The high density of water as compared to ice is due to
(a) Hydrogen bonding interactions
(b) Dipole-dipole interactions
(c) Dipole-induced dipole interactions
(d) Induced dipole-induced dipole interactions

5. Heavy water is used in atomic reactor as
(a) Coolant
(b) Moderator
(c) Both moderator and coolant
(d) Neither coolant nor moderator

6. When zeolite which is hydrated sodium aluminium silicate, is treated with hard water, the sodium ions are exchanged with
(a) H^+ ions (b) Ca^{2+} ions
(c) Mg^{2+} ions (d) Both Ca^{2+} and Mg^{2+}

7. Hydrogen does not combine with
(a) Antimony (b) Sodium
(c) Bismuth (d) Helium

8. Hydrogen peroxide is used as:
(a) Oxidizing agent
(b) Reducing agent
(c) Both as oxidizing and reducing agent
(d) Drying agent

9. In the laboratory, H_2O_2 is prepared by the action of
(a) Cold H_2SO_4 on BaO_2
(b) Aqueous alkali on Na_2O_2
(c) MnO_2 and cold H_2SO_4
(d) Dilute HCl and MnO_2

10. The velocity of neutrons in nuclear reactor is slowed down by
(a) Heavy water (b) Ordinary water
(c) Zinc rod (d) Fused caustic soda

11. The metal which displaces hydrogen from a boiling caustic soda solution is
(a) As (b) Zn
(c) Mg (d) Fe

12. When same amount of zinc is treated separately with excess of sulphuric acid and excess of sodium hydroxide solution, the ratio of volumes of hydrogen evolved is
(a) 1 : 1 (b) 1 : 2
(c) 2 : 1 (d) 9 : 4

13. Water is said to be permanently hard when it contains
(a) Sulphates of Mg and Ca
(b) Bicarbonates of Mg and Ca
(c) Carbonates of Na and K
(d) Sulphates of Na and K

14. Which of the following pairs of substances on reaction will not evolve H_2 gas?
(a) Fe and H_2SO_4 (aqueous)
(b) Copper and HCl (aqueous)
(c) Sodium and ethyl alcohol
(d) Iron and steam.

15. The structure of H_2O_2 is
(a) planar (b) non-planar
(c) spherical (d) linear

16. The reaction $H_2S + H_2O_2 \rightarrow S + 2H_2O$ show
(a) Acidic nature of H_2O_2
(b) Alkaline nature of H_2O_2
(c) Oxidising action of H_2O_2
(d) Reducing action of H_2O_2

17. H_2O_2 is manufactured these days:
(a) By the action of H_2O and BaO_2
(b) By the action of H_2SO_4 on Na_2O_2
(c) By electrolysis of 50% H_2SO_4
(d) By burning hydrogen in excess of oxygen

18. When a substance *A* reacts with water, it produces a combustible gas *B* and a solution of substance *C* in water. When another substance *D* reacts with this solution *C*, it produces the same gas *B* on warming. *D* can also produce gas *B* on reaction with dilute sulphuric acid at room temperature. *A* imparts a deep golden yellow colour to a smokeless flame of Bunsen burner. *A*, *B*, *C*, and *D* respectively are:
(a) Na, H_2, NaOH, Zn
(b) K, H_2, KOH, Al
(c) Ca, H_2, $Ca(OH)_2$, Sn
(d) CaC_2, C_2H_2, $Ca(OH)_2$, Fe.

19. Which of the following will determine whether the given colourless liquid is water or not?
(a) Melting
(b) Tasting
(c) Phenolphthalein
(d) Adding a pinch of anhydrous $CuSO_4$

20. Heavy water is
(a) H_2O
(b) D_2O

(c) Water at 4ºC

(d) Water obtained by repeated distillation

21. The hydride ion H^- is a stronger base than hydroxide ion, which of the following reactions will occur if sodium hydride (NaH) is dissolved in water?

(a) $H^- (aq) + H_2O(l) \rightarrow H_3O^+(aq)$

(b) $H^- (aq) + H_2O(l) \rightarrow OH^-(aq) + H_2(g)$

(c) $H^- + H_2O \rightarrow$ No reaction

(d) Any of these depending upon temperature.

22. Which of the following can adsorb large volume of hydrogen gas?

(a) Finely divided platinum

(b) Finely divided nickel

(c) Colloidal solution of palladium

(d) Colloidal hydroxide

23. The nuclei of tritium ($_1{}^3H$) atom would contain how many neutrons:

(a) 1 (b) 2

(c) 3 (d) 4

24. An oxide which gives H_2O_2 on treatment with dilute acid is

(a) PbO_2 (b) Na_2O_2

(c) MnO_2 (d) TiO_2

25. 1 mL of H_2O_2 solution gives 10 ml of O_2 at N.T.P. It is

(a) 10 vol H_2O_2 (b) 20 vol H_2O_2

(c) 30 vol H_2O_2 (d) 40 vol H_2O_2

26. Temporary hardness of water is removed by

(a) Mg (b) $CaCO_3$

(c) $Ca(OH)_2$ (d) $CaCl_2$

27. Oxygen in H_2O_2 is

(a) *sp* hybridized (b) sp^2 hybridized

(c) sp^3 hybridized (d) None of these

28. Which of the following is interstitial hydride?

(a) CaH_2 (b) CuH

(c) PH_3 (d) NaH

29. In ortho hydrogen

(a) Both nuclei have parallel spin

(b) Both nuclei have opposite spin

(c) Both (a) and (b)

(d) None of these

30. The oxidation number of O in H_2O_2 is:

(a) – 2 (b) – 1

(c) + 1 (d) + 2

31. Hydrogen cannot reduce:

(a) Hot CuO (b) Hot Fe_2O_3

(c) Hot SnO_2 (d) Hot Al_2O_3.

32. Temporary hardness of water is due to the presence of

(a) Magnesium bicarbonate

(b) Calcium chloride

(c) Magnesium sulphate

(d) Calcium carbonate

33. Calgon used as water softner is

(a) $Na_2[Na_4 (PO_3)_6]$ (b) $Na_4[Na_2(PO_3)_6]$

(c) $Na_2[Na_4(PO_4)_5]$ (d) $Na_4[Na_2(PO_4)_6]$

34. The O—O—H bond angle in H_2O_2 is

(a) 106° (b) 109° 28′

(c) 120° (d) 94.8°

35. 30 volume H_2O_2 means

(a) 30 mL of the solution contains 1g H_2O_2

(b) 1 mL of the solution liberates 30 mL of O_2 at NTP

(c) 30% H_2O_2

(d) 30 mL of the solution contains 1 mole of H_2O_2.

36. Hydrogen combines with other elements by

(a) losing an electron

(b) gaining an electron

(c) sharing an electron

(d) All of these

37. The oxidation state of hydrogen in BaH_2 is

(a) + 2 (b) – 2

(c) – 1 (d) 0

38. What is used as moderator in nuclear reactors?

(a) Hard water (b) Distilled water

(c) Heavy water (d) Cold water

39. The maximum number of hydrogen bonds, a water molecule can form is

(a) 1 (b) 2

(c) 3 (d) 4

40. The acid which is not suitable for the preparation of hydrogen on treatment with metals

(a) HNO_3 (b) HCl

(c) H_2SO_4 (d) CH_3COOH

ANSWERS

1. (c)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (a)	**10.** (a)
11. (b)	**12.** (a)	**13.** (a)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (d)	**20.** (b)
21. (b)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (b)
31. (d)	**32.** (a)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (a)

16

s-Block Elements

BRIEF REVIEW OF THE CONCEPTS

Left hand side of the periodic table contains *s*-block elements. The elements are further placed in two distinct groups—group 1 or IA (the alkali metals) and group 2 or II A (the alkaline earth metals).

Group 1 Elements (Alkali Metals)

This Group comprises of lithium (Li), sodium (Na), potassium (K), rubidium (Rb), cesium (Cs) and francium (Fr). Out of these, francium is a radioactive element. Na^+ and K^+ ions concentration in body cells play a vital role. These elements are highly reactive and are never found in the free state. They are called alkali metals because their oxides and hydroxides, when treated with water form alkaline solutions. All alkali metals are soft, silvery white and have bright lusture when freshly cut.

The general electronic configuration of alkali metals is ns^1. They have low value of ionization energy which further decreases down the group. They form M^+ cations by losing one electron from the valence shell. The atomic size increases down the group. Their densities, melting and boiling points are very low.

Alkali metals behave as strong reducing agents because of the low value of ionization energies. Alkali metals and their salts impart characteristic colours to the flame. These elements combine with oxygen and moisture when kept in air and their surface gets tarnished; hence they are stored in kerosene or paraffin oil. Lithium can not he stored in kerosene because it has very low density and floats to the surface so it is generally kept wrapped in paraffin wax.

Alkali metals react readily and vigorously with water forming hydroxides with the liberation of hydrogen.

$$2M + H_2O \rightarrow 2MOH + H_2$$

These hydroxides are strong bases and their basic character increases down the group.

These elements react with oxygen to different extents. Lithium forms only mono oxide.

$$4Li + O_2 \rightarrow 2Li_2O$$

Sodium forms peroxide and potassium, rubidium and cesium form super oxides

$$2Na + O_2 \rightarrow Na_2O_2$$

$$M + O_2 \rightarrow MO_2$$

$$(M = K, Rb, Cs)$$

Alkali metals react directly with hydrogen on heating to give hydrides

$$2M + H_2 \rightarrow 2MH$$

Only lithium combines directly with nitrogen to give lithium nitride

$$6Li + N_2 \rightarrow 2Li_3N$$

Alkali metals react with halogens to form the corresponding metal halides which are ionic in nature. Reactivity of halogens with a given alkali metal follows the order

$$F_2 > Cl_2 > Br_2 > I_2$$

Alkali metals dissolve in liquid ammonia to give deep blue solutions. The colour in these solutions is due to the absorption of light by solvated electrons. These free ammoniated electrons make the solution a very powerful reducing agent.

$$M + NH_3 \rightarrow M^+\,(NH_3)_x + e^-(NH_3)_y$$

On standing for some time, the blue colour starts disappearing, because, solvated electrons react with ammonia to liberate hydrogen

$$NH_3 + e^- \rightarrow NH_2^- + \frac{1}{2}H_2$$

With the increase in the concentration of the alkali metal, the metal ion cluster formation takes place and at very high concentration the solution attains the colour like that of metallic copper.

Except for lithium, the alkali metal carbonates are stable to heat

$$Li_2CO_3 \xrightarrow{\Delta} Li_2O + CO_2$$

In general elements of a group show strong similarities in their physical and chemical properties due to the same outer electronic configuration. But there are also some significant differences between the properties of the element of second period and lower elements. For example Li differs from other alkali metals, similarly Be, B, C, N, O and F also show significant difference in their properties in comparison to other elements of their respective group. This anomalous behavior of the first element of each group can be attributed to

(i) Small size
(ii) High ionization enthalpy
(iii) High charge density of ions and
(iv) non-availability of *d*-orbitals.

Anomalous Nature of Lithium in Group 1

(i) Lithium is harder while other alkali metals are soft.
(ii) The melting and boiling points of lithium are comparatively high. It is due to the very small size of lithium atoms which leads to strong metallic bonds and strong cohesive forces.

(iii) Lithium combines directly with nitrogen to form nitride (Li_3N) whereas other alkali metals do not combine with nitrogen.

(iv) Reaction of lithium with water is slow.

(v) Lithium hydroxide is a weak base while the hydroxides of other alkali metals are strong bases.

(vi) Lithium salts with anions of high charge density are less soluble in water than those of other alkali metals. The halides of lithium are more covalent than the other halides and are more soluble in organic solvents.

(vii) Lithium hydroxide (LiOH) and lithium carbonates (Li_2CO_3) decompose on heating to form lithium oxide (Li_2O).

(viii) Lithium forms monoxide with oxygen while other alkali metals form peroxides (M_2O_2) and superoxides (MO_2).

(ix) Lithium nitrate on heating decomposes to Li_2O, NO_2 and O_2 while other alkali metal nitrates decompose to give the corresponding nitrite and oxygen.

(x) Lithium forms more stable covalent bonds than other alkali metals and, therefore, forms more stable complex compounds. Also due to the very small size of Li^+, it is strongly hydrated in aqueous solutions.

Group 2 Elements (Alkaline Earth Metals)

The elements of this group are beryllium (Be), magnesium (Mg), calcium (Ca), strontium (Sr), barium (Ba) and radioactive elements radium (Ra).

The alkaline earth metals are greyish white, light, malleable and ductile with bright metallic lusture. Magnesium is present in chlorophyll which is an important plant pigment.

These are called alkaline earth metals because their oxides are alkaline and existed in earth's crust.

The general electronic configuration of the valence shell is ns^2. They have a strong tendency to lose two outer *s*-electrons in order to achieve noble gas configuration; hence they form M^{2+} ions. The atoms of alkaline earth metals are larger in size but smaller than the corresponding alkali metal.

The first ionization energy of each alkaline earth element is higher than the first ionization energy of the preceding alkali metal. All alkaline earth metals, except beryllium, react with water but these reactions are less violent than the corresponding reaction of an alkali metal. The reactivity of alkaline earth metals with water increases as we move down the group.

Beryllium does not react with steam even at red heat. It is probably due to the formation of a very strong protective oxide film on the surface. Magnesium reacts with hot water and forms oxide

$$Mg + H_2O \rightarrow MgO + H_2$$

Other elements react like alkali metals and form hydroxides

$$Ba + 2H_2O \rightarrow Ba(OH)_2 + H_2$$

Oxides of beryllium and magnesium dissolve in water to give corresponding hydroxides. These hydroxides show basic character, however beryllium hydroxide is amphoteric in nature. It reacts both with caustic alkali and mineral acids.

$$Be(OH)_2 + 2NaOH \rightarrow \underset{\text{(Sodium Berylate)}}{Na_2BeO_2} + 2H_2O$$

$$Be(OH)_2 + 2HCl \rightarrow BeCl_2 + 2H_2O$$

The solubility of alkaline earth metal hydroxides in water increases on moving down the group. Be $(OH)_2$ and Mg $(OH)_2$ are almost insoluble, while $Ca(OH)_2$ is fairly soluble and Sr $(OH)_2$ and Ba $(OH)_2$ are increasingly more soluble. Alkaline earth metal hydroxides $M(OH)_2$ combine with CO_2 and form corresponding carbonates. That is why CO_2 turns lime water milky because of the formation of insoluble calcium carbonate

$$Ca(OH)_2 + CO_2 \rightarrow CaCO_3 \downarrow + H_2O$$

These carbonates decompose on heating to give carbon dioxide and metal oxide.

$$MCO_3 \xrightarrow{\Delta} MO + CO_2$$

Their thermal stability increases on moving down the group.

These elements burn in nitrogen to form nitrides of the type M_3N_2. The metals from Mg to Ba or their oxides react with carbon on heating to form carbides of the type MC_2 which are ionic in nature. All the alkaline earth metals react with acids and replace hydrogen. Reaction of Be is very slow.

Complex formation is favoured in case of alkaline earth metals because of their small size and this tendency is maximum in case of beryllium. Except Be, all other elements when heated, combine with hydrogen to form hydrides of the type MH_2. MgH_2 is a covalent compound while other hydrides are ionic in nature.

Alkaline earth metals react slowly with oxygen and form the corresponding oxides. However, Ba and Sr form peroxides on heating with excess of oxygen.

$$2M + O_2 \rightarrow 2MO \ (M = Be, Mg, Ca)$$
$$M + O_2 \rightarrow MO_2 \ (M = Ba, Sr)$$

Alkaline earth metals (except Be) combine directly with halogens on heating to form metal halides of the general formula MX_2. These are also obtained by the action of halogen acids on metals, metal oxides, hydroxides and carbonates. Beryllium chloride can be prepared by passing CI_2 gas over a mixture of BeO and carbon.

$$BeO + C + CI_2 \xrightarrow{\Delta} BeCI_2 + CO$$

Beryllium halides are covalent in nature due to small size and relatively high charge on Be^{2+} ion. Halides of other elements are ionic solids. Beryllium halides are hygroscopic and fume in air due to hydrolysis. Calcium chloride has a strong affinity for water and is used as a popular dehydrating agent.

Anomalous Nature of Beryllium in Group 2

(i) The cohesive forces in beryllium are much stronger due to its small size than those of magnesium and other elements of the group. As a result, it is much harder and has higher melting and boiling points.

(ii) Beryllium does not react with water even at high temperatures while other elements of the group react with water.

(iii) Its higher polarizing power leads to all its compounds being largely covalent with lower melting and boiling points.

(iv) Beryllium oxide is amphoteric whereas oxides of other alkaline earth metals are basic.

(v) Beryllium salts do not impart any colour to the flame

(vi) Beryllium sulphate is soluble in water while the sulphates of other metals are either sparingly soluble or insoluble in water.

Diagonal Relation Between Lithium and Magnesium

Lithium resembles magnesium in many properties

(i) Their atomic and ionic radii are quite close.

(ii) Both form nitrides (Li_3N and Mg_3N_2)

(iii) Hydroxides of both of these decompose on heating to give oxides.

$$2LiOH \rightarrow Li_2O + H_2O$$

$$Mg(OH)_2 \rightarrow MgO + M_2O$$

(iv) Both react with oxygen to give mono oxides (Li_2O and MgO)

(v) Nitrates of both of these decompose to produce NO_2 and O_2.

$$4LiNO_3 \rightarrow 2Li_2O + 4NO_2 + O_2$$

$$2Mg(NO_3)_2 \rightarrow 2MgO + 4NO_2 + O_2$$

Problems for Practice

1. Which of the following configurations corresponds to an alkaline earth metal?
(a) Ne $2s^22p^2$ (b) Ar $3d^24s^2$
(c) Ar $3d^{10}4s^2$ (d) Ar $4s^2$

2. An aqueous solution of sodium carbonate is alkaline because sodium carbonate is a salt of
(a) Weak acid and weak base
(b) Strong acid and weak base
(c) Weak acid and strong base
(d) Strong acid and strong base.

3. Lithophone used as white pigment is a mixture of
(a) $CaSO_4 + ZnS$ (b) $BaSO_4 + ZnS$
(c) $BaSO_4 + CaSO_4$ (d) $CaSO_4 + ZnS$

4. Excess of Na^+ ions in human system cause
(a) Diabetes
(b) Anaemia
(c) Low blood pressure
(d) High blood pressure.

5. Reason for alkali metals to be soft is
(a) They are less metallic in nature
(b) There is only one valency electron per atom
(c) They do not have close packed structures
(d) They have high I.E.

6. An atom with atomic number 20 is most likely to combine chemically with the atom whose atomic number is
(a) 11 (b) 14
(c) 16 (d) 10.

7. Which is the most basic of the following?
(a) Na_2O (b) BaO
(c) As_2O_3 (d) Al_2O_3.

8. Among the following NaOH, $Ca(OH)_2$, KOH and $Zn(OH)_2$. The weakest base is
(a) NaOH (b) $Ca(OH)_2$
(c) KOH (d) $Zn(OH)_2$.

9. Amongst the metals Be, Mg, Ca and Sr of group 2 of the periodic table, the least ionic chloride would be formed by
(a) Mg (b) Be
(c) Ca (d) Sr.

10. The compounds of alkaline earth metals have the following magnetic nature
(a) Diamagnetic (b) Paramagnetic
(c) Ferromagnetic (d) Antiferromagnetic.

11. Which one of the following substances is used in the laboratory for a fast drying of neutral gases?
(a) Phosphorus Pentoxide
(b) Active charcoal
(c) Anhydrous calcium chloride
(d) Na_3PO_4.

12. Which of the following is correct for thermal stability?
(a) $K_2CO_3 < MgCO_3 < CaCO_3 < BeCO_3$
(b) $BeCO_3 < MgCO_3 < CaCO_3 < K_2CO_3$
(c) $BeCO_3 < MgCO_3 < K_2CO_3 < CaCO_3$
(d) $CaCO_3 < MgCO_3 < BeCO_3 < K_2CO_3$.

13. Which of the following is not an acidic salt?
(a) NaH_2PO_2 (b) NaH_2PO_3
(c) NaH_2PO_4 (d) Na_2HPO_4.

14. Which of the following compounds absorbs CO_2 and reacts with water violently?
(a) ZnO (b) H_2SO_4
(c) CaO (d) $CaCO_3$

15. What is X in the following reaction?
$MgCl_2 + H_2O \rightarrow X + 2HCI$
(a) MgO (b) Mg
(c) $Mg(OH)_2$ (d) Mg(OH)Cl.

16. Which metals are radioactive in the *s*-block?
(a) Cs and Fr (b) Cs and Ra
(c) Fr and Ra (d) Ba and Ra

17. On mixing with an adequate quantity of water, plaster of Paris sets into a hard solid having the composition:
(a) $CaSO_4 . 2H_2O$ (b) $CaSO_4 . H_2O$
(c) $CaSO_4 . Ca(OH)_2$ (d) $(CaSO_4)_2 . H_2O$

18. Alkaline earth metals form dipositive ions instead of unipositive ions because:
(a) The second ionization energy is almost double than the first ionization energy
(b) Unipositive ions do not have stable configuration
(c) The hydration energy of dispositive ions more than compensates the higher value of the second ionization energy
(d) Unipositive ions are highly hydrated.

19. The alkali metals
(a) Form salt like hydrides
(b) Form salts which are predominantly covalent
(c) Show decreased chemical reactivity with dry oxygen in going form Li to Cs
(d) Show increasing electronegativity form Li to Cs.

20. Main process for the manufacturer of sodium carbonate is
(a) Carbon process (b) Solvay process
(c) Down's process (d) Nelson Process.

21. Washing soda has formula
(a) $Na_2CO_3.\ 7H_2O$
(b) $Na_2CO_3.\ 10H_2O$
(c) $Na_2CO_3.\ 3H_2O$
(d) Na_2CO_3.

22. The alkali metal that reacts with nitrogen directly to form nitride is
(a) Li (b) Na
(c) K (d) Rb.

23. The reaction of slaked lime with Cl_2 gas gives
(a) Only $Ca(OCl)_2$
(b) Only $CaCl_2$
(c) A mixture of $Ca(OCl)_2$, $CaCl_2$ and H_2O
(d) Baryta water.

24. Which among the following shows the tendency to form peroxide?
(a) Li (b) Mg
(c) Be (d) Ba

25. The solubility of sulphates in water decreases from $MgSO_4$ to $BaSO_4$. It is due to the fact that
(a) Ionic nature increases
(b) Size of M^{2+} ion increases
(c) Lattice energy decreases
(d) Hydration enthalpy of M^{2+} ions decreases.

26. Which of the following statements is incorrect for Be?
(a) Most of its compounds are largely covalent
(b) It forms a covalent chloride $BeCl_2$ which has bridge covalent structure and is a linear molecule
(c) It has anomalous properties due to smaller size and high electronegativity
(d) It forms Be^{2+} ions because of the lower value the sum of 1st and 2nd I.E.

27. A fire works gives out crimson coloured light, it contains a salt of
(a) Na (b) Ca
(c) Ba (d) Sr.

28. Crystalline sodium chloride is a bad conductor of electricity because
(a) It contains only molecules
(b) It does not possess ions
(c) The ions present in it are not free to move
(d) It contains free molecules.

29. When K_2O is added to water, the solution is basic because it contains a significant concentration of
(a) O_2^{2-} (b) O^{3-}
(c) OH^- (d) K^+

30. Which of the following alkali metal ions has lowest ionic mobility in aqueous solution?
(a) Rb^+ (b) Cs^+
(c) Li^+ (d) Na^+

31. The pair of compounds which cannot exist together in solution is
(a) $NaHCO_3$ and NaOH
(b) $NaHCO_3$ and H_2O
(c) $NaHCO_3$ and Na_2CO_3
(d) Na_2CO_3 and NaOH.

32. Sodium sulphate is soluble in water but barium sulphate is sparingly soluble because
(a) The hydration energy of Na_2SO_4 is more than its lattice energy
(b) The lattice energy of $BaSO_4$ is more than its hydration energy
(c) The lattice energy has no role to play in solubility
(d) Both (a) and (b) are correct.

33. Alkali metals readily dissolve in liquid ammonia to give blue coloured solutions. The blue colour is believed to be due to
(a) Ammoniated cations
(b) Ammoniated anions
(c) Ammoniated electrons
(d) Ammoniated cations and ammoniated electrons

34. Alkali metals are powerful reducing agents because
(a) These are metals
(b) These are non-metals
(c) Their ionic radii are large
(d) Their ionization potentials are low.

35. The difference of water molecules in gypsum and Plaster of Paris is
(a) $\frac{5}{2}$ (b) 2
(c) $\frac{1}{2}$ (d) $1\frac{1}{2}$.

36. Strongest bond is between
(a) CsF (b) NaCl
(c) CsI (d) NaBr

37. The correct order of melting points of Be, Mg and Ca is
(a) Be > Mg > Ca (b) Be > Ca > Mg
(c) Ca > Mg > Be (d) Ca > Be > Mg.

38. KNO_3 on heating gives:
(a) O_2 (b) NO_2
(c) $O_2 + NO_2$ (d) None of these.

39. Which of the following does not illustrate the anomalous properties of Li?
(a) The *m.p.* and *b.p.* of Li are comparatively high
(b) Li forms a nitride Li_3N unlike group 1 metals
(c) Li is much softer than the other 1 group metals.
(d) Li^+ ion and its compounds are more heavily hydrated than those of the rest of the group.

40. The thermal stability of alkaline earth carbonates increases from Be to Ba. This is because:
(a) Covalent nature increases
(b) Lattice energy increases
(c) Electropositive nature increases
(d) None of these

41. Which of the following has largest size?
(a) Na (b) Na^+
(c) Na^- (d) All of these have same

42. Which compound will show the highest lattice energy?
(a) RbF (b) CsF
(c) NaF (d) KF

43. The standard reduction potentials at 25°C of Li^+/Li, Ba^{2+}/Ba, Na^+/Na and Mg^{2+}/Mg are – 3.05, –2.73, –2.71 and –2.37 volt respectively. Which one of the following is the strongest oxidising agent?
(a) Na^+ (b) Li^+
(c) Ba^+ (d) Mg^{2+}.

44. A combustible gas is liberated when caustic soda solution is heated with

(a) S (b) NH_4Cl
(c) I_2 (d) Zn.

45. Identify the correct statement.
(a) Gypsum contains a lower percentage of calcium than Plaster of Paris
(b) Gypsum is obtained by heating Plaster of Paris
(c) Plaster of Paris can be obtained by hydration of gypsum
(d) Plaster of Paris is obtained by partial oxidation of gypsum.

46. Fusion mixture is
(a) $K_2CO_3 + Na_2CO_3$ (b) $KHSO_4 + NaHSO_4$
(c) $K_2CO_3 + NaHSO_4$ (d) $KHSO_4 + Na_2SO_3$.

47. The metal extracted by electrolysis of its fused salt is
(a) Iron (b) Sodium
(c) Copper (d) Lead.

48. Oxone is
(a) CaO (b) N_2O
(c) Na_2O_2 (d) $NaBO_3$.

49. Which of the following has the highest melting point ?
(a) NaCl (b) NaF
(c) NaBr (d) NaI.

50. Alkali metals are usually prepared by
(a) Reduction of their oxides with aluminum
(b) Reduction of their oxides with carbon
(c) Electrolysis of their molten chlorides
(d) Thermal decomposition of their carbonates

51. Which compound possesses the highest lattice energy?
(a) CsI (b) KF
(c) LiF (d) NaCl

52. Which one of the following has lowest first ionization energy?
(a) Li (b) Na
(c) Rb (d) Cs

53. Which is manufactured by the electrolysis of aqueous sodium chloride?
(a) NaOH (b) $NaHCO_3$
(c) Na_2SO_4 (d) Na_2CO_3

54. Which of the following properties is not true for an alkali metal?
(a) Low atomic volume
(b) Low ionization energy
(c) Low density
(d) Low electronegativity.

55. Which of the following is a man made element?
(a) Ra (b) Fr
(c) Rn (d) Cr

56. The formula of Plaster of Paris is
(a) $CaSO_4.\ 1\frac{1}{2}H_2O$ (b) $CaSO_4.\ 2H_2O$
(c) $2CaSO_4.\ H_2O$ (d) $CaSO_4.\ H_2O$.

57. Lithium shows diagonal relationship with
(a) Magnesium (b) Beryllium
(c) Aluminium (d) Boron.

58. A metal *X* on heating in nitrogen gas gives *Y*. *Y* on treatment with H_2O gives a colourless gas which when passed through $CuSO_4$ solution gives a blue colour. *Y* is
(a) $Mg(NO_3)_2$ (b) Mg_3N_2
(c) NH_3 (d) MgO.

59. When NaOH is manufactured, the gas released at the cathode is
(a) Cl_2 (b) H_2
(c) O_2 (d) H_2O.

60. Calcium is obtained by
(a) Roasting of lime stone
(b) Electrolysis of solution of calcium chloride in water
(c) Reduction of calcium chloride with carbon
(d) Electrolysis of molten anhydrous calcium chloride.

61. A neutral white solid (sodium salt) on heating liberates a gas and leaves an alkaline residue. The gas turns lime water milky. The solid is:
(a) Na_2CO_3 (b) $NaHSO_4$
(c) $NaHCO_3$ (d) $NaNH_4HPO_4$

62. The alkali metals used in photoelectric cell is
(a) K (b) Cs
(c) Rb (d) Fr.

63. Which one of the following has highest electropositive character?
(a) Cu (b) Cs
(c) Ba (d) Cr

64. The electronic configuration of metal M is $1s^2, 2s^2, 2p^6, 3s^1$. The formula of its oxide would be
(a) MO (b) M_2O
(c) M_2O_3 (d) MO_2

65. The reactivity of sodium with water, is made use of
(a) In drying of alcohol
(b) In drying of benzene
(c) In drying of ammonia solution
(d) As a general drying agent

66. Which of the following electronic configurations corresponds to an element with the lowest ionization energy?
(a) $1s^2\ 2s^2\ 2p^3$ (b) $1s^2\ 2s^2\ 2p^5$
(c) $1s^2\ 2s^2\ 2p^6$ (d) $1s^2\ 2s^2\ 2p^6\ 3s^1$.

67. A solution of sodium metal in liquid ammonia is strongly reducing due to the presence of
(a) Sodium atoms (b) Sodium hydride
(c) Sodium amide (d) Solvated electrons

68. Cesium oxide will be
(a) Very strongly basic (b) Acidic
(c) Weakly basic (d) Amphoteric

69. Which alkali metal has the lowest *m.p.*?
(a) Li (b) K
(c) Na (d) Cs

70. To a clear solution of compound (*X*) a solution of $BaCl_2$ is added and a heavy white precipitate is formed which does not dissolve in dilute HCl. The compound *X* is
(a) A nitrate (b) A bromide
(c) A sulphate (d) A carbonate

71. Which of the following compounds has the lowest anion to cation size ratio?
(a) LiF (b) NaF
(c) CsI (d) CsF

72. Which of the following alkali metal hydroxides is the strongest base?

(a) LiOH (b) NaOH
(c) KOH (d) CsOH

73. Compared with the alkaline earth metals, the alkali metals exhibit
(a) Smaller ionic radii
(b) Highest boiling points
(c) Greater hardness
(d) Lower ionization energies

74. Anhydrous mixture of KF and HF contains which type of ions?
(a) K^+, H^+, F^- (b) $(KF)^+$ $(HF)^-$
(c) KH^+, F^- (d) K^+, HF_2^-

75. On heating sodium metal in a current of dry ammonia, the compound formed is
(a) Sodium amide (b) Sodium azide
(c) Sodium nitride (d) Sodium hydride

76. A sudden large jump between the values of second and third ionization energies of an element would be associated with the electronic configuration
(a) $1s^2\ 2s^2\ 2p^6\ 3s^1$ (b) $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^1$
(c) $1s^2\ 2s^2\ sp^6\ 3s^2\ 3p^2$ (d) $1s^2\ 2s^2\ 2p^6\ 3s^2$.

77. Which of the following has correct increasing basic strength?
(a) MgO < BeO < CaO < BaO
(b) BeO < MgO < CaO < BaO
(c) BaO < CaO < MgO < BeO
(d) CaO < BaO < BeO < MgO

78. In view of their low ionization energies, the alkali metals are
(a) Weak oxidising agents
(b) Strong reducing agents
(c) Strong oxidising agents
(d) Weak reducing agents

79. Baking powder has one of the following constituents
(a) Na_2CO_3 (b) Na_2SO_4
(c) $NaHCO_3$ (d) K_2CO_3.

80. The first ionisation energies of alkaline earth metals are higher than those of the alkali metals. This is because
(a) There is increase in the nuclear charge of the alkaline earth metals.
(b) There is decrease in the nuclear charge of the alkaline earth metals.
(c) There is no change in the nuclear charge.
(d) None of the above.

81. Which of the following carbonates decomposes on heating?
(a) $MgCO_3$ (b) Na_2CO_3
(c) K_2CO_3 (d) All

82. Mg is an important component of which biomolecule occurring extensively in living world?
(a) Haemoglobin (b) Chlorophyll
(c) Glycogen (d) ATP

83. Epsom salt's chemical formula is
(a) $MgSO_4.\ 7H_2O$ (b) $Mg(OH)_2$
(c) $2CaSO_4.\ H_2O$ (d) $BaSO_4$.

84. For two ionic solids CaO and KI, identify the wrong statement among the following.
(a) Lattice energy of CaO is much higher than that of KI.
(b) KI is soluble in benzene.
(c) CaO has high *m.p*.
(d) KI is soluble in water.

ANSWERS

1. (d)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (d)	**9.** (b)	**10.** (a)
11. (c)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (b)
21. (b)	**22.** (a)	**23.** (c)	**24.** (d)	**25.** (d)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (c)	**30.** (c)
31. (a)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (d)	**36.** (a)	**37.** (b)	**38.** (a)	**39.** (c)	**40.** (c)
41. (c)	**42.** (c)	**43.** (d)	**44.** (d)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (c)	**49.** (b)	**50.** (c)
51. (c)	**52.** (d)	**53.** (a)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (a)	**58.** (b)	**59.** (a)	**60.** (d)
61. (c)	**62.** (b)	**63.** (b)	**64.** (b)	**65.** (b)	**66.** (d)	**67.** (d)	**68.** (a)	**69.** (d)	**70.** (c)
71. (d)	**72.** (d)	**73.** (d)	**74.** (d)	**75.** (a)	**76.** (d)	**77.** (b)	**78.** (b)	**79.** (c)	**80.** (a)
81. (a)	**82.** (b)	**83.** (a)	**84.** (b)						

17

p-Block Elements

BRIEF REVIEW OF THE CONCEPTS

Groups 13 to 18 (III A to VII A and 0) comprise *p*-block of the periodic table. The characteristics of the elements of these groups are briefly discussed below.

Group 13 Elements (The Boron Family)

The elements of this group are boron (B), aluminium (Al), gallium (Ga), indium (In) and thalium (Tl).

Their general electronic configuration in the valence shell is ns^2np^1. Although the atomic radius increases down the group, but the atomic radius of Ga is less than that of Al.

The first ionisation energy of the group 13 elements are less as compared to the corresponding energies of group 2 elements. But second ionisation energy of group 13 elements is considerably higher, because, after the loss of *p*-electron, the other electrons have to be removed from the *s*-orbital which has a pair of electrons.

Boron is a non-metal, whereas other elements show metallic properties. As these elements have three electrons in their valence shell so they can show +3 oxidation state. Boron does not lose all the valence electrons to form B^{3+} ion. Ga, In and Tl show +1 and +3 oxidation states and +1 state becomes more prominent as we move down the group. It is due to inert pair effect.

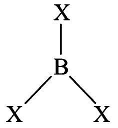
Boron forms only covalent compounds, aluminium forms covalent as well as ionic compounds and compounds of other elements are mainly ionic.

All elements of group 13 form hydrides. Boron forms two series with general formula B_nH_{n+4} and B_nH_{n+6} which are known as boranes. Aluminium and gallium form complex anionic hydrides such as $LiAlH_4$ which are used as powerful reducing agents in organic chemistry. B_2H_6 is the simplest hydride of boron. It is a dimer of BH_3 and is prepared by the reduction of BF_3 with lithium hydride.

$$2BF_3(g) + 6LiH \xrightarrow{450K} B_2H_6\,(g) + 6LiF(s)$$

The structure of diborane is discussed separately. All the elements of group 13 react with oxygen and form oxides of general formula M_2O_3 except thalium.

All the elements of this group form halides of the type MX_3. Boron in such compounds has six electrons around it and is capable of accepting a pair of electrons; hence such halides behave as Lewis acids. This tendency to accept a pair of electrons decreases with the increase in the size of the cation.



The power to accept lone pair of electrons in case of boron halides follows the order

$$BI_3 > BBr_3 > BCl_3 > BF_3$$

This order is reverse to what is expected. This can be explained due to the relative tendency of the halogen atom to back donate a pair of electrons to the boron atom.

The term *alum* was originally used to name potassium aluminium sulphate ($K_2SO_4.Al_2SO_4.24H_2O$) but now it is a general term given to double sulphates of the type $M_2SO_4.M'_2(SO_4)_3.24H_2O$ where M is a monovalent metal and M′ is a trivalent metal.

Anomalous Nature of Boron in Group 13

(i) Boron is a non-metal and is a bad conductor of electricity whereas other elements of the group are metals.

(ii) Boron exists in two allotropic forms— crystalline and amorphous.

(iii) The melting and boiling points of boron are much higher than those of other elements of the group

(iv) Boron forms only covalent compounds, aluminum forms some ionic compounds whereas other elements form mainly ionic compounds.

(v) The hydroxides and oxides of boron are acidic in nature, those of aluminum are amphoteric whereas those of other elements are basic in nature.

(vi) Boron forms a number of hydrides (e.g. B_2H_6, B_4H_{10}) which are covalent in nature and are quite stable.

(vii) The halides of boron are purely covalent in nature, exist as monomers and are hydrolysed by water forming boric acid. The halides of aluminum are covalent only in the gaseous and solid states but in aqueous solution give Al^{3+} ions. Halides of other elements of the group are largely ionic in nature.

DIMERS–BANANA BOND

Many compounds of elements exist as dimers - that is two molecules remain associated with each other. It generally occurs when the central atom has not been able to achieve the completion of its octet by formation of bonds using its valence electrons. Such compounds are termed as *electron deficient compounds*. This phenomenon is observed in halides and hydride of group 2 and group 13 elements.

One type of dimer formation occurs in which one of the atoms donates a lone pair of electrons (to form a coordinate bond) to the electron deficient atom of another molecule. Such coordinate bonds hold the two molecules together. Such as beryllium chloride ($BeCl_2$)

and aluminum chloride ($AlCl_3$) both exist as dimers in the vapour state. Their dimeric structures can be written as

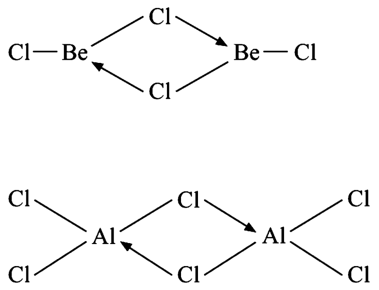


Fig. 17.1 Dimers of beryllium chloride and aluminium chloride

Another type of dimers are formed where the atoms do not have lone pairs of electrons to form the coordinate bond but still some bonding forces exist. An example is boron hydride (BH_3) which exists as a dimeric structure called diborane (B_2H_6). The structure of diborane is of great interest from the point of view of its electronic arrangement. In it not enough valence electrons are present to form the expected number of covalent bonds. It has total of 12 valence electrons (three from each boron and one from each hydrogen atom) whereas corresponding carbon compound, ethane (C_2H_6) possesses 14 valence electrons. X-rays differaction studies show the presence of two different types of B-H bonds. It has been observed that there are four B-H_t bonds with 119 pm bond length and two B-H_b bonds with 133 pm bond length. It has also been observed that all the six hydrogen atoms do not lie on the same plane. The structure of diborane is shown below. In this, four hydrogen atoms and two boron atoms lie in the same plane. Two other hydrogen atoms lie above and below this plane and form bridge between two boron atoms and hence are called *bridge hydrogens* (H_b). Two hydrogen atoms attached to boron on the left and two on the right are called *terminal hydrogens* (H_t).

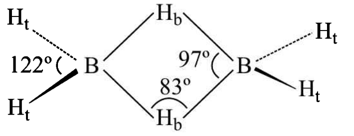


Fig. 17.2 Structure of diborane molecule

The four terminal B-H bonds are normal covalent bonds and are formed by sharing of one electron each from boron and hydrogen. Such bonds are called *two centre-two electron (2c-2e) bonds.*The two bridge bonds (B......H.....B) are not like normal covalent bonds. Each hydrogen is linked to two boron atoms but total number of electrons involved is only two (one from hydrogen and one from boron). Such bonds are called *three centre-two electron (3c-2e) bonds*. These are weak bonds and due to their shape, are also called *banana bonds*.

Nature of this banana bond can be explained on the basis of orbital-overlap also. Boron is sp^3 hybridised, in which three orbitals contain one electron each and one orbital is empty. Two of these half-filled orbitals overlap with half filled 1*s* orbital of hydrogen atoms to form two normal covalent bonds. One half filled orbital from one boron atom and one vacant orbital from another boron atom overlap with half-filled 1s orbital of hydrogen to from the B-H-B bridge bond. This is involving three centres but has two electrons only, hence called as 3*c*-2*e* bond. This overlap of orbitals is shown below.

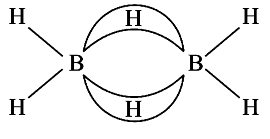


Fig. 17.3 Orbital overlap diagram of diborane

Group 14 Elements (Carbon Family)

Group 14 consists of the elements carbon (C), silicon (Si), germanium (Ge,) tin (Sn) and lead (Pb).

Carbon is found to occur in free state in the form of coal, graphite, diamond and fullerine. In combined state, it occurs as petroleum, natural gas and along with coal deposits. Silicon is found abundantly in earth's crust in the form of silica (SiO_2) and silicates.

Carbon and silicon are non-metals. Germanium is a semimetal whereas tin and lead behave as metals.

Ionization energy decreases gradually from carbon to tin.

The general electronic configuration of the elements of this group is ns^2np^2; so they have four electrons in their valency shell. They exhibit the oxidation states of +4 and +2. In +4 oxidation state, the compounds are covalent, while in +2 state, compounds are ionic.

Carbon exhibits a remarkable property of catenation which forms the basis of a large number of organic compounds. The tendency of catenation decreases in the order

$$C >> Si > Ge \approx Sn >> Pb$$

Carbon in the form of graphite is a good conductor of electricity.

Hydrides of carbon are called alkanes. Similarly, hydrides of silicon are called silanes.Thermal stability of hydrides decreases from carbon to lead.

All elements of this group form two types of oxides. They are monoxides (MO) and dioxides (MO_2). In addition, carbon forms a suboxide C_3O_2 and lead forms a mixed oxide Pb_3O_4 ($2PbO.PbO_2$).

CO_2 is a gas while dioxides of other elements are crystalline solids of high melting point. Compounds of silicon with oxygen are called silicates

All these elements form tetrahalides when reacted with halogens. These halides are covalent in nature and tetrahedral in shape.

Polymeric compounds of the general formula $(R_2SiO)_n$ are known as silicones. Silicones are obtained in the form of oils, rubber or resins depending upon the extent of polymerization.

Carbon is not affected by alkalis, while other members react on heating.

CO reacts with many metals to form complexes named as metal carbonyls. In carbonyls the oxidation state of metals is always zero. The poisonous nature of CO is also due to its ability to form a bond with iron present in haemoglobin.

Silicon and germanium are used in preparation of semiconductors.

Tetraethyl lead is used as anti-knock compounds in petrol.

Anomalous Nature of Carbon in Group 14

(i) Carbon is very hard in the form of diamond while other members of the group are soft. It has much higher melting and boiling points.

(ii) Carbon dioxide is a gas while the dioxides of other elements of the group are solids.

(iii) Carbon forms multiple bonds with carbon itself or other elements such as sulphur, oxygen, nitrogen etc. but the other members have very less tendency of form multiple bonds.

(iv) Carbon tetrachloride cannot be hydrolysed with water but $SiCl_4$ is very easily hydrolysed.

(v) Alkalies have no action on carbon while silicon forms silicates

$$Si + 2NaOH + H_2O \longrightarrow Na_2SiO_3 + 2H_2$$

(vi) The hydrides of carbon (alkanes) are very stable compounds and it does not form complexes.

Silicates

These are the compounds with a base unit of $(SiO_4)^{4-}$ anion in which the Si atom is linked directly to four oxygen atoms tetrahedrally. These tetrahedra link themselves by corners to give various types of structures.

Orthosilicates are ionic silicates in which SiO_4^{4-} anions are there and many metal ions act as cations e.g. sodium silicate (Na_4SiO_4), magnesium silicate (Mg_2SiO_4) etc.

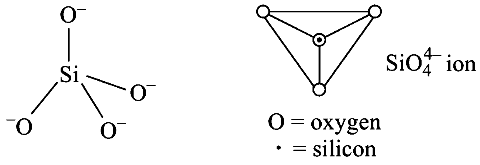


Fig. 17.4

Pyrosilicates contain two tetrahedral units sharing one oxygen atom between them. They have a base unit of $(Si_2O_7)^{6-}$ and are also called *Island structures.*

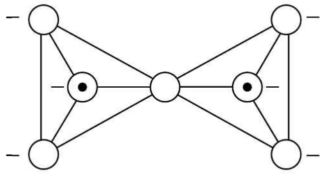
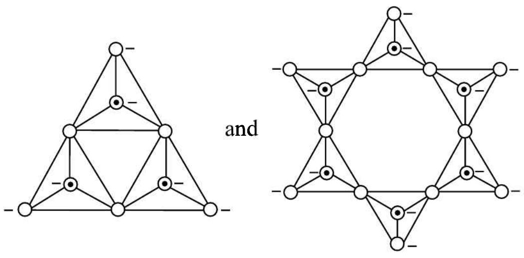


Fig. 17.5

Cyclic or ring silicates result when two oxygen atoms per tetrahedron are shared to form a closed ring. They have two types of ions $(Si_3O_9)^{6-}(Si_6O_{18})^{12-}$. Some common examples are emerald and wollastonite.



$(Si_3O_9)^{6-}$ $(Si_6O_{18})^{12-}$

Fig. 17.6

Chain silicates are obtained by the sharing of two oxygen atoms per tetrahedron with other tetrahedra. This gives the formula $(SiO_3)_n^{2n-}$. Some common examples are asbestos and hornblende.

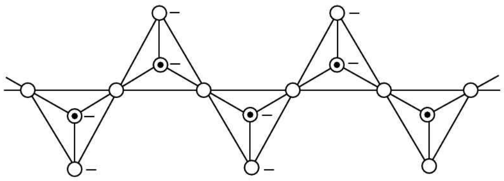


Fig. 17.7

Sheet silicates: In these each tetrahedral unit shares three oxygen atoms amongst them which results in the formation of a two-dimensional sheet such as in mica, talc and clays etc.

Three-dimensional silicates are formed by sharing of all the four oxygen atoms of the SiO_4 tetrahedra. All crystalline forms of SiO_2 e.g., quartz, cristohalite etc. are examples of this type of silicates.

Group 15 Elements (The Nitrogen Family)

The elements of this group are nitrogen (N), phosphorus (P), arsenic (As), antimony (Sb) and bismuth (Bi).

Nitrogen is a gas at room temperature. Phosphorus is a soft waxy solid. Arsenic is hard but lustreless whereas antimoy and bismuth are solids with metallic lustre. The ionization energy of nitrogen is very high and it decreases from nitrogen to bismuth. The electronegativity of nitrogen is very high due to which it forms N^{3-} ion. The electronegativity decreases from N to Bi which also indicates change from non-metallic to metallic character.

All the elements except bismuth exhibit allotropy. Nitrogen exists in two allotropic forms, α-and β-nitrogen. Phosphorus exists as white, red, scarlet, violet and black allotrops.

General electronic configuration is $ns^2\ np^3$. When only *p*-electrons are involved in bonding, +3 oxidation state results, while +5 oxidation state is caused when both *s*-and *p*-electrons are involved. The stability of +3 oxidation state increases down the group due to inert-pair effect.

All the elements of this groups combine with hydrogen giving volatile hydrides of the type MH_3. They and NH_3 (ammonia), PH_3 (Phosphine), AsH_3 (arsine), SbH_3 (stibine) and BiH_3 (bismuthine). They are all covalent compounds and their stability decreases gradually from NH_3 to BiH_3. These hydrides behave as Lewis bases because of the presence of a pair of electron on the central atom which can take part in co-ordinate bonding. Basicity decreases from NH_3 to BiH_3.

Except nitrogen, all the elements of this groups form trihalides and pentahalides. Nitrogen is not able to form pentahalides due to the absence of *d*-orbitals.

All the elements form oxides of the general formula X_2O_3, X_2O_4 and X_2O_5. Nitrogen forms a number of oxides with oxidation state from +1 to +5. These are nitrous oxide (N_2O), nitrice oxide (NO), dinitrogen trioxide (N_2O_3), nitrogen dioxide (NO_2) and dinitrogen pentaoxide (N_2O_5).

Anomalous Nature of Nitrogen in Group 15

(i) Nitrogen is a gas at room temperature whereas the other elements are solids and exist in different allotropic forms.

(ii) It forms $p\pi - p\pi$ bond and exists as diatomic molecule (N_2) whereas other elements form tetraatomic molecules such as P_4, As_4, Sb_4 and even more complicated arrangements.

(iii) It is capable of forming hydrogen bonding while the other elements do not form hydrogen bonding.

(iv) Nitrogen forms a number of oxides such as N_2O, NO, and NO_2 whereas other elements do not form corresponding oxides. They form only trioxides and pentoxides.

(v) Nitrogen cannot form complexes by accepting electron pairs. The other elements form complexes due to the presence of *d*-orbitals.

Group 16 Elements (The Oxygen Family)

The group consists of oxygen (O), sulphur (S), selenium (Se), tellurium (Te) and polonium (Po). The outer electronic configuration of these elements is ns^2np^4. All the elements, except polonium are called chalcogens (means ore forming). Oxygen is a gas at room temperature whereas other elements are solids.

Oxygen is most electronegative in this group and electronegativity decreases down the group. Ionization energy also decreases down the group. Due to which metallic character increases from oxygen to polonium. Oxygen to a small extent and sulphur to a large extent show the property of catenation.

All the elements of this group, except oxygen, exhibit common oxidation stated of –2, +2, +4 and +6. The normal oxidation state of oxygen is –2 and it shows positive oxidation state only in fluorides.

All the elements of this group exhibit allotropy and they are polymorphic. Oxygen exists in two forms—oxygen (O_2) and ozone (O_3).

All the elements of this group form covalent hydrides of general formula H_2X. They are named as water (H_2O), hydrogen sulphide (H_2S), hydrogen selenide (H_2Se), hydrogen telluride (H_2 Te) and hydrogen polonide (H_2Po). Their thermal stability decreases down the group.

Water is a liquid whereas H_2S is a gas. It is due to the association of water molecules due to hydrogen bonding.

All the elements of this group, except oxygen, form halides of the type M_2X_2, MX_2, MX_4 and MX_6. Oxygen reacts with fluorine giving OF_2 and O_2F_2. With other halogens, oxygen gives many compounds which are better called oxides.

Anomalous Nature of Oxygen in Group 16

(i) Oxygen is diatomic and a gas whereas other elements of the group are polyatomic and solids.

(ii) Molecular oxygen (O_2) is paramagnetic in nature while other elements have diamagnetic behaviour.

(iii) Oxygen is more electronegative so its compounds are more ionic in nature.

(iv) Hydrogen bonding is also important in some oxygen compounds.

(v) Oxygen generally shows the oxidation state of –2 in its compounds (exceptions OF_2 and peroxides) while other elements exhibit oxidation states of –2, +2, +4 and +6.

Group 17 Elements (The Halogen Family)

Fluorine (F), chlorine (Cl), bromine (Br), iodine (I) and astatine (At) constitue this group and they are collectively known as halogens.

All these elements have general outer electronic configuration of ns^2np^5.

Sinece these elements have seven electrons in their outermost shell, they have a strong tendency to take up one electron so as to attain the noble gas configuration.

Fluorine is the most reactive and iodine is the least reactive amongest the halogens. The first element, fluroine differs in properties from other elements because of its small size. Fluorine and chlorine are gases, bromine is brown liquid and iodine is a solid.

All these elements are non-metals in nature due to high electronegativity. They have high value for ionization energy which decreases down the group from fluorine to iodine.

Halogens react with different elements to form ionic as well as covalent compounds. Fluorine shows only –1 oxidation state. All other halogens exhibit different oxidation states which are –1, +1, +3, +4, +5, +6 and +7.

Halogens are good oxidising agents since they have strong tendency to accept electrons. The oxidising power decreases from fluorine to iodine.

The more reactive halogen also displaces the less reactive halogen from the salt solution. So fluorine can displace chlorine, bromine, and iodine whereas chlorine can displace bromine and iodine only.

Halogens combine with hydrogen to form hydrogen halides of the type HX. The stability of the hydrides decreases in the sequence HF > HCl > HBr > HI. They all behave as acids and the acid strength follows the sequence HF > HCl > HBr >HI.

All the halogens form oxides on reaction with oxygen but the oxides of fluorine are called fluorides because fluorine is more electronegative than oxygen.

On account of large electronegativity and size difference, halogens combine with one another to form a number of colvalent compounds, known as interhalogen compounds.

Anomalous Nature of Fluorine in Group 17

(i) Fluorine has high electronegativity, due to which it is highly non-metallic element. The compounds of fluorine have maximum ionic character.

(ii) It has low F-F bond energy in comparison to the other elements of the group. It is due to the repulsive interactions of lone pairs of electrons around the small fluorine atoms.

(iii) It is most reactive among the halogens due to its low value of bond dissociation energy.

(iv) It shows oxidation state of –1 only while other elements of the group show different oxidation states.

(v) The hydride of fluorine (HF) is liquid at room temperature whereas hydrides of other elements of the group are gases. It is due to formation of intermolecular hydrogen bonding.

(vi) H-F is also a weak acid in comparison to other hydrogen halides due to strong H-F bond.

Group 18 Elements (Rare, Noble or Inert Gases)

This group consists of helium (He), neon (Ne), argon (Ar), krypton (Kr), xenon (Xe) and radon (Rn).

All these gases, except radon, are present in atmosphere, in very small amount hence are called rare gases.

Their outer shell configuration is $ns^2\ np^6$ (except for He, which contains only two electrons) which has completely filled orbitals. Due to this they are chemically inert hence named as inert or noble gases.

Since their outermost shells have stable configuration, they possess very high ionisation energies. Due to complete octet, noble gas atoms have no tendency to take extra electron hence they have almost zero electron affinities.

All noble gases are adsorbed by activated wood charcoal, or by metals like Pd and Pt.

The first compound of xenon, $XePtF_6$, was discovered in 1962. Xenon reacts with fluorine to give three compounds viz. XeF_2, XeF_4 and XeF_6. Other known compounds of xenon are XeO_3, XeO_4, $XeOF_2$, XeO_2F_2 etc. No compound of He, Ne and Ar has been prepared.

Problems for Practice

1. Boron compounds behave as Lewis acid because of their
(a) acidic nature (b) covalent nature
(c) electron deficiency (d) all of these

2. Which of the following is not a Lewis acid?
(a) SiF_4 (b) $FeCl_3$
(c) BF_3 (d) C_2H_4

3. Quartz is a crystalline variety of
(a) Si (b) SiO_2
(c) Na_2SiO_3 (d) SiC

4. The laughing gas is:
(a) Nitrous oxide (b) Nitric oxide
(c) Nitrogen trioxide (d) Nitrogen pentoxide

5. Cassiterite is an ore of
(a) Mn (b) Ni
(c) Sb (d) Sn

6. An element M has an atomic mass 19 and atomic number 9. Its ion is represented by
(a) M^+ (b) M^{2+}
(c) M^- (d) M^{2-}

7. Which of the following is anhydride of nitric acid?
(a) NO (b) NO_2
(c) N_2O_5 (d) N_2O_3

8. Which of the following is most stable?
(a) Sn^{2+} (b) Ge^{2+}
(c) Si^{2+} (d) Pb^{2+}

9. The variety of glass used in making lenses and prisms is
(a) Soda glass (b) Borosilicate glass
(c) Flint glass (d) Crooke's glass

10. Alum helps in purifying water by
(a) forming sigma complex with clay particles
(b) sulphate part which combines with dirt and removes it
(c) aluminium which coagulates the mud particles
(d) making mud water soluble

11. Nitrogen is used to fill electric bulbs because it
(a) is lighter than air
(b) makes the bulb to give more light
(c) does not support combustion
(d) is non toxic

12. Conc. nitric acid oxidizes phosphorus to
(a) H_3PO_4 (b) P_2O_5
(c) H_3PO_3 (d) $H_2P_2O_7$

13. Which of the following halogen does not exhibit positive oxidation state in its compounds?
(a) Cl (b) Br
(c) I (d) F

14. Ionic radii of As^{3+}, Sb^{3+} and Bi^{3+} follow the order
(a) $As^{3+} > Sb^{3+} > Bi^{3+}$
(b) $Sb^{3+} > Bi^{3+} > As^{3+}$
(c) $Bi^{3+} > As^{3+} > Sb^{3+}$
(d) $Bi^{3+} > Sb^{3+} > As^{3+}$

15. In which of the following the bond angle is maximum?
(a) NH_3 (b) NH_4^+
(c) PCl_3 (d) SCl_2

16. A considerable part of harmful ultraviolet radiation of sun does not reach earth's surface. This is because earth's atmosphere has a layer of
(a) CO_2 (b) H_2
(c) O_3 (d) NH_3

17. The type of hybridisation of boron in diborane is
(a) *sp*-hybridisation
(b) sp^2-hybridisation
(c) sp^3-hybridisation
(d) sp^3d^2-hybridisation

18. The equivalent weight of phosphoric acid (H_3PO_4) in the reaction
$NaOH + H_3PO_4 \longrightarrow NaH_2PO_4 + H_2O$ is
(a) 25 (b) 49
(c) 59 (d) 98

19. Aluminium (III) chloride forms a dimer because
(a) Higher co-ordination can be achieved by Al
(b) Al has high ionisation energy
(c) Al belongs to third group
(d) It can not form a trimer

20. When chlorine reacts with cold and dilute solution of sodium hydroxide, the products obtained are
(a) $Cl^- + ClO^-$ (b) $Cl^- + ClO_2^-$
(c) $Cl^- + ClO_3^-$ (d) $Cl^- + ClO_4^-$

21. In nitrogen family, the H– M– H bond angle in the hydrides gradually becomes closer to 90° on going from N to Sb. This shows that gradually
(a) The basic strength of the hydrides increases
(b) Almost pure *p*-orbitals are used for M – H bonding
(c) The bond energies of M – H bonds increase
(d) The bond pairs of electrons become nearer to the central atom

22. In PO_4^{3-} ion, the formal charge on each oxygen atom and P – O bond order, respectively are
(a) −0.75. 1.25 (b) −3, 1.25
(c) −0.75, 1.0 (d) −0.75, 0.6

23. The basic character of hydrides of the 15 group elements decreases in the order
(a) $SbH_3 > PH_3 > AsH_3 > NH_3$
(b) $NH_3 > SbH_3 > PH_3 > AsH_3$
(c) $NH_3 > PH_3 > AsH_3 > SbH_3$
(d) $SbH_3 > AsH_3 > PH_3 > NH_3$

24. Which one of the following allotropic forms of carbon is isomorphous with crystalline silicon?
(a) Graphite (b) Coal
(c) Coke (d) Diamond

25. A clathrate may be defined as a
(a) Cage compound (b) Liquid crystal
(c) Mixture (d) Solid solution

26. Which of the following oxides of nitrogen is anhydride of HNO_2 (Nitrous acid)?

(a) NO (b) N_2O_3

(c) N_2O_4 (d) N_2O_5

27. Three centred bond is present in

(a) NH_3 (b) B_2H_6

(c) BCl_3 (d) $AlCl_3$

28. Which of the following fluorides does not exist?

(a) NF_5 (b) PF_5

(c) AsF_5 (d) SbF_5

29. Ge (II) compounds are powerful reducing agents whereas Pb(IV) compounds are strong oxidants. It can be due to

(a) Pb is more electropositive than Ge

(b) Ionization potential of lead is less than that of Ge

(c) Ionic radii of Pb^{2+} and Pb^{4+} are larger than those of Ge^{2+} and Ge^{4+}

(d) More pronounced inert pair effect in lead than in Ge

30. Which of the following statements is not true about noble gases?

(a) Their ionization energies are very high

(b) Their electron affinities are nearly zero

(c) They do not form any chemical compound

(d) They are not easily liquefied

31. Helium-oxygen mixture is used by deep sea divers in preference to nitrogen-oxygen mixture because

(a) Helium is much less soluble in blood than nitrogen

(b) Nitrogen is much less soluble in blood than helium

(c) Due to high pressure deep under the sea nitrogen and oxygen react to give poisonous nitric oxide

(d) Nitrogen is highly soluble in water

32. CaC_2 reacts with N_2 to form fertilizer called

(a) Nitrolim (b) Urea

(c) NH_4NO_3 (d) None of these

33. Fluorine is more electronegative than either boron or phosphorus. What conclusion can be drawn from the fact that BF_3 has no dipole moment but PH_3 does?

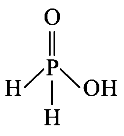
(a) BF_3 is not spherically symmetrical but PF_3 is

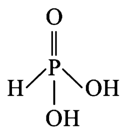
(b) BF_3 molecule must be linear

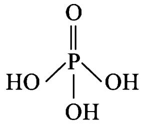
(c) The atomic radius of P is larger than that of B

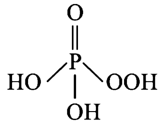
(d) The BF_3 molecule must be planar triangular

34. The structural formula of hypophosphorous acid is

(a)


(b)


(c)


(d)


35. Oxygen is prepared in laboratory from

(a) Sodium and water

(b) Electrolysis of water

(c) $KClO_3$ and MnO_2

(d) HNO_3

36. Which of the hydrogen halides forms salts like KHX_2 (where X is a halogen atom)?

(a) HF (b) HCl

(c) HI (d) HBr

37. What causes nitrogen to be chemically inert?

(a) Multiple bond formation in the molecule

(b) Absence of bond polarity

(c) Short internuclear distance

(d) High bond energy

38. The chemical composition of white lead is

(a) Na_2CO_3 (b) Pb_3O_4

(c) $Pb(OH)_2$. $2PbCO_3$ (d) PbO

39. Which is the strongest oxidizing agent?

(a) Fluorine (c) Chlorine

(c) Bromine (d) Iodine

40. Which of the following reactions is possible?

(a) $I_2 + 2NaBr \rightarrow Br_2 + 2NaI$

(b) $I_2 + 2NaCl \rightarrow Cl_2 + 2NaCl$

(c) $Br_2 + 2NaCl \rightarrow Cl_2 + 2NaBr$

(d) $Cl_2 + 2NaBr \rightarrow Br_2 + 2NaCl$

41. The electron affinity of the halogens follows the order

(a) $F > Cl > Br > I$ (b) $F < Cl < Br < I$

(c) $F < Cl > Br > I$ (d) $F < Cl < Br > I$

42. The electronegativity follows the order

(a) $F > O > Cl > Br$ (b) $F > Cl > Br > O$

(c) $O > F > Cl > Br$ (d) $Cl > F > O > Br$

43. What is hybridization of P in PCl_5?

(a) sp^3 (b) sp^3d^2

(c) sp^3d (d) sp^2

44. In OF_2 molecule, the total number of bond pairs and lone pairs of electrons present respectively are

(a) 2, 6 (b) 2, 8

(c) 2, 10 (d) 2, 9

45. Which of the following isoelectronic ions has the lowest ionization energy?

(a) K^+ (b) Ca^{2+}

(c) Cl^- (d) S^{2-}

46. The halide which is not hydrolysed is

(a) $SiCl_4$ (b) SiF_4

(c) CCl_4 (d) $PbCl_4$

47. Glass is a

(a) Liquid

(b) Solid

(c) Super cooled liquid

(d) Transparent organic polymer

48. Unlike other halogens, fluorine does not show higher oxidation states because

(a) It is highly electronegative

(b) It has no *d*-orbitals

(c) Its atomic radius is very small

(d) The F^- ion is stable and isoelectronic with neon

49. Which of the following is incorrect with respect to property indicated?

(a) Electronegativity : F > Cl > Br

(b) Electron affinity : Cl > Br < F

(c) Oxidizing power : $F_2 > Cl_2 > Br_2$

(d) Bond energy : $F_2 > Cl_2 > Br_2$

50. The shape of gaseous $SnCl_2$ is

(a) Tetrahedral (b) Linear

(c) Angular (d) T-shaped

51. The oxide which *cannot* act as reducing agent is

(a) SO_2 (b) NO_2

(c) CO_2 (d) ClO_2

52. Mark the oxide which is amphoteric in character

(a) CO_2 (b) SiO_2

(c) SnO_2 (d) CaO

53. Which of the following is the correct corder of their bond strengths?

(a) HF > HCl > HBr > HI

(b) HI > HBr > HCl > HF

(c) HF > HBr > HCl > HI

(d) HCl > HF > HBr > HI

54. NO_2 can not be prepared by heating

(a) KNO_3 (b) $Pb(NO_3)_2$

(c) $Cu(NO_3)_2$ (d) $AgNO_3$

55. Galena is an ore of

(a) Gallium (b) Lead

(c) Tin (d) Germanium

56. NH_4NO_3 on heating gives

(a) N_2O (b) NO

(c) N_2 (d) N_2O_3

57. Which of the following causes damage to the building containing calcium and responsible for cough and choking in human?

(a) Sulphur (b) Carbon

(c) Nitrogen dioxide (d) Sulphur dioxide

58. Which of the following statements is correct?

(a) BCl_3 and $AlCl_3$ are both Lewis acids and BCl_3 is stronger than $AlCl_3$

(b) BCl_3 and $AlCl_3$ are both Lewis acids and $AlCl_3$ is stronger than BCl_3

(c) BCl_3 and $AlCl_3$ are both equally strong Lewis acids

(d) Both BCl_3 and $AlCl_3$ are not Lewis acids

59. The number of σ bonds in P_4O_{10} is

(a) 6 (b) 16

(c) 20 (d) 7

60. The coloured discharge tubes for advertisement mainly contain

(a) Xenon (b) Helium

(c) Neon (d) Argon

61. Which of the following halides is most acidic?

(a) PCl_3 (b) $SbCl_3$

(c) $BiCl_3$ (d) CCl_4

62. The formula of ozone is O_3, it is

(a) An allotrope of oxygen

(b) Compound of oxygen

(c) Isotope of oxygen

(d) Isobar of oxygen

63. Which of the following noble gas is not present in atmosphere?

(a) He (b) Ne

(c) Ar (d) Rn

64. Thermite is a mixture of iron oxide and

(a) zinc powder (b) sodium shavings

(c) potassium metal (d) aluminium powder

65. Among noble gases (from He to Xe) only xenon reacts with fluorine to form stable xenon fluorides because xenon

(a) has the largest size

(b) has the lowest ionization enthalpy

(c) has the highest heat of vaporization

(d) is the most readily available noble gas

66. The oxidation number of Xe in $XeOF_2$ is

(a) 0 (b) 2

(c) 4 (d) 3

67. $Na_2S_2O_3$ is prepared by

(a) Reacting $H_2S_2O_3$ with NaOH

(b) Reducing Na_2SO_4 with S in alkaline medium

(c) Heating NaOH and S

(d) Reducing Na_2SO_4 with S in acidic medium

68. What is the hybridization of S in SF_4?

(a) sp^3d^2 (b) sp^3d

(c) sp^3d^3 (d) sp^3

69. In the reaction

$2Br^- + X_2 \longrightarrow Br_2 + 2X^-$, X_2 is

(a) Cl_2 (b) Br_2

(c) I_2 (d) N_2

70. The reaction of $KMnO_4$ and HCl results in

(a) Oxidation of Mn in $KMnO_4$ and production of Cl_2

(b) Reduction of Mn in $KMnO_4$ and production of H_2

(c) Oxidation of Mn in $KMnO_4$ and production of H_2

(d) Reduction of Mn in $KMnO_4$ and production of Cl_2

71. Which of the following has $p\pi - d\pi$ bonding?

(a) NO_3^- (b) SO_3^{2-}

(c) BO_3^{3-} (d) CO_3^{2-}

72. Bleaching action of SO_2 is due to its

(a) oxidizing property (b) acidic property

(c) reducing property (d) basic property

73. Among F, Cl, Br and I the lowest ionization potential will be of

(a) Fluorine (b) Chlorine

(c) Bromine (d) Iodine

74. The three important oxidation states of phosphorus are

(a) –3, + 3 and +5 (b) –3, +3 and –5

(c) –3, + 4 and –4 (d) –3, + 3 and +4

75. "Fluorosis" disease is caused due to the reaction of ....with excess of fluoride in the body

(a) Ca (b) Mg

(c) Fe (d) K

76. NH_3 reacts with excess of Cl_2 to give

(a) NCl_3 (b) NH_4Cl

(c) N_2 (d) NCl_5

77. In the manufacture of cement which of the following is used?

(a) Clay and silica
(b) Lime stone and silica
(c) Lime stone and clay
(d) Lime stone and gypsum

78. Which of the following statements is false?
(a) Radon is obtained from the decay of radium
(b) Helium is an inert gas
(c) Xenon is the most reactive among the rare gases
(d) The most abundant rare gas found in the atmosphere is helium

79. An oxide of lead which is used in lead storage batteries, in safety matches and is a powerful oxidising agent is
(a) PbO (b) PbO_2
(c) Pb_3O_4 (d) $2PbO.PbO_2$

80. Na^+, Mg^{2+}, Al^{3+} and Si^{4+} are isoelectronic, their ionic size will follow the order:
(a) $Na^+ > Mg^{2+} > Al^{3+} < Si^{4+}$
(b) $Na^+ < Mg^{2+} < Al^{3+} < Si^{4+}$
(c) $Na^+ > Mg^{2+} > Al^{3+} > Si^{4+}$
(d) $Na^+ < Mg^{2+} > Al^{3+} > Si^{4+}$

81. In the clathrates of xenon with water, the nature of bonding between xenon and water molecule is
(a) Covalent
(b) Hydrogen bonding
(c) Coordinate
(d) Dipole-induced dipole interaction

82. The compounds of boron and hydrogen are collectively called
(a) Diboranes (b) Borazoles
(c) Boracits (d) Boranes

83. Noble gases do not react with other elements because
(a) They are monoatomic
(b) They are found in abundance
(c) The size of their atoms is very small
(d) They have completely paired up electrons and stable electron shells

84. The composition of the common glass is
(a) $Na_2O.CaO.6SiO_2$
(b) $Na_2O.Al_2O_3.2SiO_3$
(c) $CaO.Al_2O_3.SiO_2$
(d) $Na_2O.CaO.Al_2O_3.6SiO_2$

85. Hybridization in SO_3 is
(a) sp^2 (b) sp^3
(c) sp^2d (d) sp^3d^2

86. Which of the following will be octahedral?
(a) SF_6 (b) BF_4^-
(c) PCl_5 (d) BO_3^{3-}

87. When oxalic acid is dehydrated by conc. H_2SO_4, then it forms
(a) $C + CO_2$ (b) CO
(c) CO_2 (d) $CO + CO_2$

88. Which of the following shows bonds in silicones?
(a) Si–Si–Si–Si (b) Si–C–Si–O–Si
(c) Si–C–Si–C–Si (d) –Si–O–Si–O–Si

89. Which of the following fluorides of xenon is impossible?
(a) XeF_2 (b) XeF_3
(c) XeF_4 (d) XeF_6

90. XeF_2 molecule is
(a) Linear (b) Trigonal planar
(c) Pyramidal (d) Square planar

91. A particular element belongs to group 13 and second period of the periodic table; it is
(a) Gas, slightly metallic
(b) Liquid, non-metallic
(c) Solid, non-metallic
(d) Solid, less metallic

92. The structure and hybridization of $Si(CH_3)_4$ is
(a) Bent, *sp* (b) Trigonal, sp^2
(c) Octahedral, d^2sp^3 (d) Tetrahedral, sp^3

93. The shape of water molecule is same as that of
(a) C_2H_2 (b) CO_2
(c) NH_3 (d) Cl_2O

94. The chemical composition of cryolite mineral is
(a) Al_2O_3 (b) $Al_2O_3.12H_2O$
(c) $KAlSi_3O_8$ (d) Na_3AlF_6

95. Contact process is better than chamber process for the manufacture of H_2SO_4 because
(a) In contact process pure acid is obtained
(b) In contact process control of plant is easier
(c) Contact plant is cheaper
(d) No waste gases are given out

96. The spectrum of helium is expected to be similar to that of
(a) H (b) Li^+
(c) Na (d) He^+

97. Which of the following hydrides shows the highest boiling point?
(a) H_2O (b) H_2S
(c) H_2Se (d) H_2Te

98. BCl_3 does not exist as dimer because:
(a) B has small size
(b) B has low electronegativity
(c) B has low electron affinity
(d) B has no vacant orbitals

99. In B_2H_6
(a) There is a B-B-B bond
(b) The structure is similar to C_2H_6
(c) The boron atoms are linked through hydrogen bridges
(d) All the atoms lie in one plane

100. The geometry of $(SiO_4)^{4-}$ion is:
(a) Tetrahedral (b) Square planer
(c) Octahedral (d) Planar triangular

101. Carborundum is the commercial name of:
(a) Al_2O_3 (b) SiC
(c) SiO_2 (d) HPO_3

102. NCl_5 is not formed because:
(a) It is unstable
(b) Nitrogen is inert
(c) Nitrogen has no vacant *d*-orbital
(d) Nitrogen has small atomic radius

103. Which one of the following is an oxyacid?
(a) $Ba(OH)_2$ (b) $Mg(OH)_2$
(c) $HPO(OH)_2$ (d) HCl

104. Pyrosilicate ion is
(a) SiO_2^{2-} (b) $Si_2O_7^{6-}$
(c) $Si_2O_6^{7-}$ (d) SiO_3^-

105. In the electrolysis of alumina, cryolite is added to
(a) Lower the melting point of alumina
(b) Minimise anodic effect
(c) Remove impurities from alumina
(d) None of these

106. Which bond is most polar?
(a) Cl–F (b) Br–F
(c) I–F (d) F–F

107. $2NH_3 + 3CuO \rightarrow N_2 + 3Cu + 3H_2O$, NH_3 is
(a) Reducing agent (b) Oxidizing agent
(c) Both (a) and (b) (d) None of these

108. Which of the following oxides reacts with HCl and NaOH?
(a) CaO (b) ZnO
(c) N_2O_5 (d) CO_2

109. When SO_2 is passed through cupric chloride solution
(a) A white precipitate is obtained
(b) The solution becomes colourless
(c) The solution becomes colourless and a white percipitate of Cu_2Cl_2 is obtained
(d) No visible change takes place

110. When conc. H_2SO_4 comes in contact with sugar it becomes black due to
(a) Hydrolysis
(b) Hydration
(c) Decolourization
(d) Dehydration

111. Which of the following oxides is the most acidic?
(a) N_2O_5
(b) P_2O_5
(c) As_2O_5
(d) Sb_2O_5

112. The shape of the ClO_3^- is
(a) Triangular pyramid (b) Tetrahedral
(c) Triangular planar (d) Triangular bi-pyramid

113. There is S–S bond in
(a) $H_2S_2O_7$ (b) $H_2S_2O_8$
(c) $H_2S_2O_6$ (d) $H_2S_2O_3$

114. The hybridization in ICl_7 is
(a) $sp^3 d^3$ (b) $d^2 sp^3$
(c) $sp^3 d$ (d) sp^3

115. The shape of the molecule SF_3Cl_3 is
(a) trigonal bipyramidal
(b) square pyramidal
(c) octahedral
(d) tetrahedral

116. Copper turning when heated with concentrated sulphuric acid gives
(a) SO_2 (b) SO_3
(c) H_2S (d) O_2

117. Which one is known as Caro's acid?
(a) H_2SO_3 (b) H_2SO_5
(c) $H_2S_2O_8$ (d) $H_2S_2O_7$

118. If phosphoric acid is allowed to react with sufficient quantity of NaOH, the product obtained is
(a) Na_3PO_4 (b) NaH_2PO_4
(c) Na_2HPO_4 (d) $NaHPO_3$

119. Which of the following shows oxidising behaviour of H_2SO_4?
(a) $PCl_5 + H_2SO_4 \rightarrow POCl_3 + H_2O + SO_2Cl_2$
(b) $2NaOH + H_2SO_4 \rightarrow Na_2SO_4 + 2H_2O$
(c) $NaCl + H_2SO_4 \rightarrow NaHSO_4 + HCl$
(d) $2HI + H_2SO_4 \rightarrow I_2 + SO_2 + 2H_2O$

120. When thiosulphate ion is oxidised by iodine, which one of the following ion is produced?
(a) SO_3^{2-} (b) SO_4^{2-}
(c) $S_4O_6^{2-}$ (d) $S_2O_6^{2-}$

121. Dry ice is composed of
(a) Solid He
(b) Solid CO_2
(c) Solid SO_2
(d) Solid C_6H_6

122. White phosphorus contains
(a) P_2 molecules
(b) P_6 molecules
(c) P_4 molecules
(d) P_5 molecules

123. Phosphine is generally prepared in the laboratory
(a) by heating phosphorus in a current of hydrogen
(b) by decomposition of P_2H_4 at 110°C
(c) by heating red phosphorus with an aqueous solution of caustic soda
(d) by heating white phosphorus with caustic potash

124. Strength of halogen acids is in the order:
(a) $HF > HCl > HBr > HI$
(b) $HI > HBr > HCl > HF$
(c) $HF < HCl > HBr > HI$
(d) $HF > HCl < HBr < HI$

125. When moist coloured flowers are added into SO_2 gas the flowers are decolourized because
(a) SO_2 absorbs colouring matter
(b) SO_2 oxidises vegetable colouring matter
(c) SO_2 reduces vegetable colouring matter
(d) SO_2 gives colourless product

126. Mark the incorrect statement.
(a) $SnCl_2$ is a solid. $SnCl_4$ is liquid
(b) CO_2 and SiO_2 are both linear and gases at room temperature
(c) Zeolites can act as sieves
(d) Carborundum is SiC

127. HF is not stored in glass bottles because
(a) It reacts with visible part of light
(b) It reacts with sodium oxide of the glass
(c) It reacts with the aluminium oxide of the glass
(d) It reacts with SiO_2 of the glass

ANSWERS

1. (c)	**2.** (d)	**3.** (b)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (c)	**8.** (d)	**9.** (c)	**10.** (c)
11. (c)	**12.** (a)	**13.** (d)	**14.** (d)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (c)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (c)
31. (a)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (d)	**38.** (c)	**39.** (a)	**40.** (d)
41. (c)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (d)	**46.** (c)	**47.** (c)	**48.** (b)	**49.** (d)	**50.** (c)
51. (c)	**52.** (c)	**53.** (a)	**54.** (a)	**55.** (b)	**56.** (a)	**57.** (d)	**58.** (b)	**59.** (b)	**60.** (c)
61. (a)	**62.** (a)	**63.** (d)	**64.** (d)	**65.** (b)	**66.** (c)	**67.** (c)	**68.** (b)	**69.** (a)	**70.** (d)
71. (b)	**72.** (c)	**73.** (d)	**74.** (a)	**75.** (a)	**76.** (a)	**77.** (b)	**78.** (d)	**79.** (b)	**80.** (c)
81. (d)	**82.** (d)	**83.** (d)	**84.** (a)	**85.** (a)	**86.** (a)	**87.** (d)	**88.** (d)	**89.** (b)	**90.** (a)
91. (c)	**92.** (d)	**93.** (d)	**94.** (d)	**95.** (a)	**96.** (b)	**97.** (a)	**98.** (a)	**99.** (c)	**100.** (a)
101. (b)	**102.** (c)	**103.** (c)	**104.** (b)	**105.** (a)	**106.** (c)	**107.** (a)	**108.** (b)	**109.** (c)	**110.** (d)
111. (a)	**112.** (a)	**113.** (c)	**114.** (a)	**115.** (c)	**116.** (a)	**117.** (b)	**118.** (a)	**119.** (d)	**120.** (d)
121. (b)	**122.** (c)	**123.** (d)	**124.** (b)	**125.** (c)	**126.** (b)	**127.** (d)			

18 Transition Metals Including Lanthanides

BRIEF REVIEW OF THE CONCEPTS

The elements with partly filled *d*-orbitals in their atoms or in common oxidation states are known as transition elements. The world transition shows their position in periodic table as they are placed in between electropositive metals and electronegative non-metals. The elements with ns^2 $(n-1)$ d^{10} electronic configuration are not regarded as transition elements (e.g., Zn, Cd, Hg) although they belong to *d*-block of periodic table.

There are three series of transition elements, each consisting of 10 elements where the electrons are successively filled in 3*d*, 4*d* and 5*d* subshell. There is a fourth transition series (also called super heavy elements) starts with element having atomic number 104 and goes up to 112. These elements have been synthesized by artificial nuclear reactions and are very unstable with respect to α-decay or spontaneous fission with very short half-lives. The *f*-block elements comprise two series of elements-the lanthanide series and the actinide series according to the filling of 4*f* and 5*f* orbitals, respectively. A group of fourteen elements following lanthanum in the periodic table are called lanthanides. Similarly, the fourteen elements following actinium in the periodic table are termed as actinide elements. The elements following uranium have been produced artificially and are collectively called transuranium elements. The lanthanides (also called lanthanoids or lanthanons) and actinides (also known as actinoids or actinons) are listed separately in two series at the bottom of the periodic table.

IUPAC NOMENCLATURE OF 6*d* TRANSITION SERIES ELEMENTS

All the elements in the periodic table have been given definite names and their symbols are derived by either taking the first alphabet, or by taking first alphabet and one more alphabet from the name of the elements. But the 6*d* transition series elements are given special names and symbols according to IUPAC rules, which are given below:

1. The name of the element is directly derived from the atomic number of the element as per the following numerical roots:

0	1	2	3	4	5	6	7	8	9
nil	un	bi	tri	quad	pent	hex	sept	oct	enn

2. These roots are written together in order of appearance of the digits in the atomic number and terminated by-ium. If enn comes before nil, the last n of enn is elided. Similarly, the final i of bi and tri is omitted when it occurs before-ium.
3. Taking the first letter of the initial roots that make up the number derives the symbols of the elements.

Thus, the name of the elements with atomic number 104 will be unnilquadium and the symbol will be Unq. The name and symbol of elements having atomic numbers 104- 112 are given below.

It is pertinent to note that these IUPAC names are assigned to elements only temporarily till the claim/s about their synthesis are confirmed after which they are assigned proper names and symbols. Elements with atomic numbers till 109 have already been assigned proper names. Recently, the element with atomic number 110 has been named as Darmstadtium (Ds).

Table 18.1 IUPAC names and symbols of fourth transition series elements

Atomic Number	*Name of Element*	*Symbol*	*Atomic Number*	*Name of Element*	*Symbol*
104	Unnilquadium	Unq	109	Unnilennium	Une
105	Unnilpentium	Unp	110	Ununnilium	Unn
106	Unnilhexium	Unh	111	Unununium	Uuu
107	Unnilseptium	Uns	112	Ununbium	Unb
108	Unniloctium	Uno			

General Properties of Transition metals

(i) Atomic and Ionic Radii

The atomic radii of the *d*-block elements of a given series generally decrease with increase in the atomic number up to group 10 and increase again towards the end of the series. This is explained in terms of the increased force of repulsion among the added electrons (in *d*-orbitals) which exceeds the attractive force due to increased nuclear charge.

The ionic radii of the *d*-block elements follow the same trend (i.e., decreases with increase in atomic number) as their atomic radii discussed above.

(ii) Density

Densities of *d*-block elements are very high as compared to those of group 1 and 2 elements. This is due to their small radii and close packed structure.

All the *d*-block elements are metals and except mercury (which is a liquid) they are hard, malleable and ductile.

(iii) Melting and boiling points

Melting and boiling points of transition metals are very high which suggest stronger interatomic attractions. Melting and boiling point increase on moving from left to right, reach to a maximum near group 5 or 6 and then start decreasing.

(iv) Ionization Energies

The values of their ionization energies lie between those of *s*-block and *p*-block elements. Along a series, the ionization energies increase as we move from left to right, but the increase is not much pronounced. For example, in the first series, the ionization energies of Sc, Ti, V and Cr are fairly close to one-another. Similarly, values for Fe, Co, Ni and Cu are almost similar. This is due to the fact that the increase in the nuclear charge is almost offset by the screening by *d*-electrons.

(v) Oxidation states

d-Block elements exhibit a variety of oxidation states in their compounds because both *ns* and $(n-1)d$ electrons are available for bonding. In each case, the minimum oxidation state is given by the number of outer *s*-electrons and maximum oxidation state is given by the sum of *s* and *d*-electrons.

(vi) Complex formation

A significant feature of the cations of *d*-block elements is their tendency to form complex ions with ligands. They have high tendency to form complexes because of their small size and high effective nuclear charge, therefore they accept lone pairs of electrons from other molecules or ions.

(vii) Coloured compounds formation

The majority of transition metal ions are coloured. The colour depends on the number of unpaired *d*-electrons which absorb a portion of white light falling on them.

(viii) Catalytic activity

Many transition metals and their compounds, particularly the oxides, show catalytic properties. The catalytic activity is due to their ability to exhibit multiple oxidation states and hence available free valencies. In some cases, transition elements provide unpaired *d*-electrons to form the unstable intermediate comp-ound with the reactant, while in other the transition metals provide a large surface area for the reactants to be adsorbed. Some important catalysts are Fe in Haber's process, Pt or V_2O_5 in contact process, Pd or Pt or Ni in catalytic hydrogenation, etc.

(ix) Alloy formation

Since *d*-block elements have almost similar atomic sizes, can mutually substitute one another in crystal lattices to form alloys.

(x) Interstitial compounds formation

Transition elements have a unique ability to form interstitial compounds by accommodating non-metal atoms such as B, C or N, etc. In their crystal lattice. These small atoms easily fit into the voids present in the lattice of transition elements e.g., in carbides, nitrides. Such compounds show difference in physical properties.

(xi) Magnetic properties

Transition elements and their compounds exhibit different magnetic properties i.e., paramagnetic, diamagnetic and ferromagnetic. These properties depend upon the number of unpaired electrons present in the atom or ion.

General properties of first Row Transition Metal Compounds

The transition elements combine with oxygen to form a number of oxides in different oxidation states having the formula MO, M_2O_3, M_3O_4, MO_2, M_2O_5, MO_3, etc. These oxides are basic in lower oxidation states and their acidic character increases in higher oxidation states. The oxides in higher oxidation states are also powerful oxidizing agents e.g., CrO_3, $K_2Cr_2O_7$, $KMnO_4$, etc.

Halogens react with transition elements on heating (fluorine being the most reactive while iodine is the least reactive) to give halides. Higher oxidation states of the elements are stabilised by highly oxidising fluorine. Fluorides are ionic in nature and ionic character decreases from fluorides to iodides. Halides in higher oxidation states can be hydrolysed easily resulting in the corresponding oxides.

Transition metals form sulphides in their low oxidation states. Higher oxidation states cannot be stabilized by the large sulphur atom. Sulphides can be obtained by H_2S gas from ions or by the direct union of metal and sulphur.

The Lanthanides and the Actinides

The elements which have partly filled $(n-2)$ *f*-orbitals in atomic state or in any ionic state are called *f*-block or inner-transition elements.

If $(n-2)$ *f*-orbital is 4*f* then all the elements of this series are called lanthanides. These are called lanthanides because they follow lanthanum in the periodic table. The elements in which 5*f* orbitals are being successively filled are called actinides.

Electronic configuration for the valence shell of lanthanides can be represented as $4f^{0-14}5\,d^{0-1}6s^2$.

The main oxidation state exhibited by all the lanthanides is +3. Some elements also show +2 and +4 states but +3 is more stable.

As we move form La to Lu along the series, there is a constant decrease in the ionic radii in +3 oxidation state and is called *Lanthanide contraction.* The reason for this is that as we move along the series, the nuclear charge increases by one unit and the new electron enters the '*f*' orbital, whose shielding effect is very little. This causes a constant increase in pull of electrons by the nucleus so the atomic and ionic size decrease.

Lanthanide contraction plays an important role in determining the chemistry of lanthanides. Basic strength of oxides and hydroxides decreases with increase in atomic number. This is due to the fact that decrease in size of M^{3+} cation increases the covalent character thereby reducing the basic character.

The covalent radii of the corresponding elements of 4*d* and 5*d* series become very close, because the lanthanide contraction cancels the expected increase due to the addition of new shell (covalent radii of Zr and Hf are 1.45 Å and 1.44 Å respectively).

Lanthanides are very reactive metals like alkaline form metals. However, they show every little difference in their chemical reactivity because the 4*f* electrons are very effectively shielded from interaction with other electrons by the overlying 5*s*, 5*p* and 6*s* electrons.

Lanthanides do not have much tendency to form complexes. The tendency to form complexes and their stability increases with increasing atomic number.

Certain alloys of the lanthanide elements, known as *mish metals* containing predominantly 30-35% of Ce together with small quantities of other light lanthanides are used as reducing agents in metallurgical operations. Mg-alloys containing about 30% mish metal and 1% Zr are used for making parts of jet engine.

Actinides are analogous to the lanthanides and result from the filling of the 5*f* orbitals. Before 1940, only three actinides, thorium, protactinium and uranium were known to occur in nature. All other actinides have been made artificially by nuclear reactions. These elements are also called *transuranic elements.* The valence shell electronic configuration for actinides can be represented as $5f^{0-14}6d^{0-1}7s^2$.

Main characteristics of actinides are

(i) They are silvery white metals

(ii) They are highly reactive and get tarnished when react with alkalies.

(iii) They show the common oxidation states of +2, +3, +4, +5 and +6. Most common is the +3 oxidation state.

Problems for Practice

1. In general, the transition elements show the highest oxidation state in compounds with elements like

(a) S and N (b) P and C
(c) F and O (d) All of these

2. Which of the following does not show the variable oxidation state?

(a) Zn (b) Mn
(c) Cu (d) Fe

3. Ammonia does not form complexes with

(a) Ag^+ (b) Cd^{2+}
(c) Cu^{2+} (d) Pb^{2+}

4. Deep blue colour obtained on addition of ammonia to Cu^{2+} salts is due to the formation of

(a) $[Cu(OH)_3]^{2-}$ (b) $[Cu(NH_3)_4]^{2+}$
(c) $[Cu(OH)_2(NH_3)_2]$ (c) $[Cu(H_2O)_4]^{2+}$.

5. General electronic configuration of transition elements is

(a) $(n-1)d^{1-5}$ (b) $(n-1)d^{1-10}ns^1$
(c) $(n-1)d^{1-10}ns^{1-2}$ (d) None of these.

6. Which of the following elements involve gradual filling of electrons in 5*f* orbitals?

(a) Lanthanides (b) Actinides
(c) Transition metals (d) Active metals

7. Manganese exhibits maximum oxidation state in

(a) K_2MnO_4 (b) $KMnO_4$
(c) MnO_2 (d) Mn_3O_4.

8. The highest magnetic moment is shown by the transition metal ion with the outer electronic configuration

(a) $3d^{10}$ (b) $3d^9$
(c) $3d^7$ (d) $3d^5$

9. Transition elements are frequently used as catalysts because of

(a) unpaired *d*-electrons
(b) high ionic charge
(c) variable valency
(d) large surface area.

10. Colour in transition metal compounds is attributed to

(a) small size metal ions
(b) absorption of light in UV region
(c) complete ns shell
(d) incomplete $(n-1)d$ subshell

11. A transition elements M has a configuration $[Ar]3d^4$ in its 3+ oxidation state. Its atomic number is

(a) 25 (b) 26
(c) 22. (d) 27

12. Formation of interstitial compound makes the transition metal

(a) more soft (b) more ductile
(c) more metallic (d) more brittle

13. The transition metal present in vitamin B_{12} is

(a) Fe (b) Co
(c) Ni (d) Cu

14. Acidified $KMnO_4$ solution is not decolourized by

(a) Mohr's salt (b) Oxalic acid
(c) Bleaching powder (d) Alkenes.

15. Which of the following has lowest percentage of carbon?

(a) Cast iron (b) Pig iron
(c) Wrought iron (d) Steel.

16. Standard reduction potential of transition elements is generally

(a) Negative
(b) Positive
(c) Zero
(d) Can be either negative or positive.

17. In the reduction of dichromate by Fe^{2+}, number of electrons involved per chromium atom is

(a) 1 (b) 4
(c) 5 (d) 3.

18. The lowest degree of paramagnetism is shown by which of the following

(a) $MnSO_4.\ 4H_2O$ (b) $FeSO_4.\ 6H_2O$
(c) $CuSO_4.\ 5H_2O$ (d) $NiSO_4.\ 6H_2O$.

19. Which of the following is white in cold and yellow in hot?
(a) ZnS (b) ZnO
(c) $ZnCl_2$ (d) $ZnSO_4$

20. The most abundant ore of iron is
(a) Haematite (b) Limonite
(c) Magnetite (d) Siderite

21. General electronic configuration of lanthanides is
(a) $(n-2)f^{1-14}(n-1)\ s^2p^6\ d^{0-1}ns^2$
(b) $(n-2)f^{0-14}(n-1)d^{0-1}ns^2$
(c) $(n-2)f^{0-14}(n-1)d^{10}\ ns^2$
(d) $(n-2)f^{0-14}(n-1)\ f^{0-14}ns^1$.

22. Mercury sticks to the surface of the glass when it comes in contact with
(a) Hg_2O (b) HNO_3
(c) O_3 (d) Grease

23. Oxidation state of osmium (Os) in OsO_4 is
(a) +4 (b) +6
(c) +7 (d) +8.

24. $CuSO_4$ reacts with KCN solution and forms
(a) Cu(CN) (b) $Cu(CN)_2$
(c) $K_2[Cu(CN)_4]$ (d) $K_4[Cu(CN)_6]$

25. The properties of Zr and Hf are similar because
(a) Both belong to *d*-block
(b) Both belong to same group of the periodic table
(c) Both have similar radii
(d) Both have same number of electrons

26. The highest oxidation state shown by any transition elements is:
(a) +2 (b) +6
(c) +7 (d) +8

27. Which of the following configuration is correct for iron?
(a) $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^5$
(b) $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 4s^2\ 3d^5$
(c) $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 4s^2\ 3d^7$
(d) $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 4s^2\ 3d^6$.

28. Galvanization of iron denotes coating with
(a) Al (b) Sn
(c) Pd (d) Zn.

29. Super conductors are derived from compounds of
(a) *p*-Block elements (b) Lanthanides
(c) Actinides (d) Transition elements

30. Arrange Ce^{3+}, La^{3+}, Pm^{3+} and Yb^{3+} in *increasing order* of their ionic radii
(a) $Yb^{3+} < Pm^{3+} < Ce^{3+} < La^{3+}$
(b) $Ce^{3+} < Yb^{3+} < Pm^{3+} < La^{3+}$
(c) $Yb^{3+} < Pm^{3+} < La^{3+} < Ce^{3+}$
(d) $Pm^{3+} < La^{3+} < Ce^{3+} < Yb^{3+}$.

31. Which metal has the lowest melting point?
(a) Cs (b) Na
(c) Hg (d) Sn

32. When intimate mixture of potassium dichromate and potassium chloride is heated with conc. H_2SO_4 which of the following is produced in the form of red vapours?
(a) CrO_3 (b) Cr_2O_3
(c) CrO_2Cl_2 (d) $CrCl_2$.

33. The protection of steel by chrome plating is due to
(a) Cathodic protection
(b) Anodic protection
(c) Covering of steel surface
(d) Formation of alloy with iron.

34. Which of the following compounds has colour but no unpaired electrons?
(a) $KMnO_4$ (b) K_2MnO_4
(c) $MnSO_4$ (d) $MnCl_2$

35. In which of the following lanthanides, oxidation state +2 is most stable?
(a) Ce (b) Eu
(c) Tb (d) Dy

36. Which of the following oxides of Cr is amphoteric?
(a) CrO_2 (b) Cr_2O_3
(c) CrO_5 (d) CrO_3

37. Which one of the following elements constitutes a major impurity in pig iron?
(a) Silicon (b) Oxygen
(c) Sulphur (d) Graphite

38. Which one of the following ion has least magnetic moment?
(a) Cu^{2+} (b) Ni^{2+}
(c) CO^{2+} (d) Fe^{2+}

39. German silver is an alloy of
(a) Cu, Sn, Al (b) Cu, Zn, Ni
(c) Cu, Zn, Ag (d) Fe, Cr, Ni.

40. The elements which exhibit both vertical and horizontal similarities are
(a) Inert gas elements
(b) Representative elements
(c) Rare elements
(d) Transition elements.

41. A White solid halide of mercury forms a black mixture with NH_4OH. The halide is
(a) $HgCl_2$ (b) HgI_2
(c) Hg_2I_2 (d) Hg_2Cl_2.

42. Permanent magnets are generally made of alloys of
(a) Co (b) Zn
(c) Cu (d) Pb

43. The image on an exposed and developed photographic film is due to:
(a) Ag Br (b) $[Ag\ (S_2O_3)_2]^{3-}$
(c) Ag (d) Ag_2O

44. Ammonium dichromate is used in fireworks. The green coloured powder blown in the air is
(a) CrO_3 (b) Cr_2O_3
(c) Cr (d) $CrO(Cl)_2$

45. If a non metal is added to the interstitial sites of a metal, then the metal becomes
(a) Softer
(b) Less tensile
(c) Less malleable
(d) More ductile

46. Which is colourless in water?
(a) Ti^{3+}
(b) V^{3+}
(c) Cu^{2+}
(d) Sc^{3+}

47. An explosion takes place when conc. H_2SO_4 is added to $KMnO_4$. Which of the following is formed?
(a) Mn_2O_7
(b) MnO_2
(c) $MnSO_4$
(d) M_2O_3

48. Stainless steel contains iron and
(a) Cr
(b) Cr + Zn
(c) Zn + Pb
(d) Cr + Ni

49. The purple colour of $[Ti(H_2O)_6]^{3+}$ ion is due to
(a) Unpaired *d*-electron
(b) Transfer of an electron
(c) Intermolecular vibrations
(d) Presence of an electron.

50. In Nessler's reagents, the ion present is
(a) HgI^{2-}
(b) $HgI^{2-}{}_4$
(c) Hg^+
(d) Hg^{2+}

51. Which of the following elements shows maximum number of different oxidation states in its compounds?
(a) Eu
(b) La
(c) Gd
(d) Am.

52. Which one of the following ionic species will impart colour to an aqueous solution?
(a) Ti^{4+}
(b) Cu^+
(c) Zn^{2+}
(d) Cr^{3+}

53. Which of the following gives blood red colour with KCNS?
(a) Cu^{2+}
(b) Fe^{3+}
(c) Al^{3+}
(d) Zn^{2+}

54. Zinc and mercury do not show variable valency like *d*-block elements because
(a) They are soft
(b) Their *d*-shells are complete
(c) They have only two electrons in the outer most shell
(d) Their *d*-shells are incomplete

55. Which of the following statement is *not correct* ?
(a) $La(OH)_2$, is less basic than $Lu(OH)_3$
(b) In lanthanide series ionic radius of M^{3+} ions decreases
(c) La is actually an element of transition series rather than lanthanide series
(d) Atomic radii of Zr and Hf are same because of lanthanide contraction.

56. In the extraction of iron, slag is produced which is
(a) CO
(b) $FeSiO_3$
(c) $MgSiO_3$
(d) $CaSiO_3$

57. Which of the following has more unpaired *d*-electrons?
(a) Zn^+
(b) Fe^{2+}
(c) Ni^+
(d) Cu^+.

58. The equilibrium $Cr_2O^{2-}{}_7 \rightleftharpoons 2CrO^{2-}{}_4$
(a) Exists in acidic medium
(b) Exists in basic medium
(c) Exists in neutral medium
(d) It does not exist

59. Which one of the following pairs of ions have the same electronic configuration?
(a) Cr^{3+}, Fe^{3+}
(b) Fe^{3+}, Mn^{2+}
(c) Fe^{3+}, Co^{3+}
(d) Sc^{3+}, Cr^{3+}.

60. Most common oxidation states shown by cerium are
(a) +2, +4
(b) +3, +4
(c) +3, +5
(d) +2, +3.

61. The reason for the stability of Gd^{3+} ion is
(a) 4*f* subshell – half filled
(b) 4*f* subshell – completely filled
(c) Possesses the general electronic configuration of noble gases
(d) 4*f* subshell empty

ANSWERS

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (d)
11. (a)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (d)	**18.** (c)	**19.** (b)	**20.** (a)
21. (a)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (d)	**28.** (d)	**29.** (d)	**30.** (a)
31. (c)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (a)	**37.** (d)	**38.** (a)	**39.** (b)	**40.** (d)
41. (d)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (c)	**46.** (d)	**47.** (a)	**48.** (d)	**49.** (a)	**50.** (b)
51. (d)	**52.** (d)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (d)	**57.** (b)	**58.** (c)	**59.** (b)	**60.** (b)
61. (a)									

Coordination Compounds and Organometallics

19

BRIEF REVIEW OF THE CONCEPTS

Compounds which contain some ionic or molecular species attached to metal atom (or ion) through coordinate bonds are called coordination compounds (or more often called complexes). The metal is called central metal atom (or ion) and groups attached by coordinate bonds are called **ligands**. The ligands donate a pair of electrons to form coordinate bond.

The number of coordinate bonds formed by ligands to the metal is known as its **coordination number.**

Ligands are classified as mono dentate, bidentate, tridentate, etc. depending upon the number of pairs of electrons which it can use for coordinate bond formation. (i.e., no. of coordinate bonds each ligand can form).

When a ligand is attached to metal by more than one coordinate bond, then a ring structure results which is known as **chelate.** The central metal atom and the ligands attached to it are collectively termed as **coordination sphere** and is written in square bracket.

Werner's Coordination Theory

Werner proposed a theory to explain the bonding in coordination compounds. According to this theory, a metal possesses two types of valencies. One is called primary (or ionisable) valency and the other is called secondary (or nonionisable) valency. Primary valencies are satisfied by negative ions whereas the secondary valencies are satisfied either by ions or neutral molecules.

Nomenclature of complex compounds

Coordination compounds are named according to IUPAC system as discussed in the following steps.

(i) As in case of simple salts, the positive part of the coordination compound is named first followed by the negative part.

(ii) While naming a coordination sphere (complex part), ligands are named first and then the metal. After this the oxidation state of the metal is written in roman numerals placed in parentheses.

(iii) If the complex part is an anion then the name of the metal ends in **ate** e.g., cobaltate, ferrate, etc.

(iv) The names of negatively charged ligands end in-'o' (e.g. chloride becomes chloro). Neutral ligands are named as the name of the molecule (except for water and ammonia which are named as aqua and ammine respectively). The names of positive ligands end in *ium.*

(v) While writing the name of the complex part, the names of the ligands are written in alphabetical order regardless of the number of each. The number of ligands of each type are given the prefix di, tri, tetra, etc for two, three or four respectively.

(vi) In order to indicate the number of polydentate ligands, prefixes used are bis, tris, tetrakis, etc.

Table 19.1 Formulae and IUPAC names of some coordination compounds

Formula	*IUPAC name*
(i) $[Cr(NH_3)_6]^{3+}$	(i) Hexaamminechromium (III) ion
(ii) $[Cr(H_2O)_6]^{3+}$	(ii) Hexaaquachromium (III) ion
(iii) $[Cr(CN)_6]^{3-}$	(iii) Hexacyanochromate (III) ion
(iv) $[Ni(NH_3)_6]CI_2$	(iv) Hexamminenickel (II) chloride
(v) $[Co(en)_3]^{3+}$	(v) Tris (ethylenediamine) cobalt (III) ion
(vi) $[Fe(CN)_6]^{4-}$	(vi) Hexacyanoferrate (II) ion
(vii) $[Co(en)_2CINO_2]^+$	(vii) Chlorobis (ethylenediamine) nitrocobalt (III) ion
(viii) $[Pt(C_5H_5N)_4][Pt\ Cl_4]$	(viii) Tetrapyridineplatinum (II) tetrachloroplatinate (II).
(ix) $[Co(NH_3)_5Cl]Cl_2$	(ix) Pentaamminechlorocobalt (III) chloride
(x) $[Cu(NH_3)_4]SO_4$	(x) Tetramminecopper (II) sulphate

Formula	*IUPAC name*
(xi) $[Ag(NH_3)_2]Cl$	(xi) Diamminesilver (I) chloride
(xii) $K_3[Co(CN)_6]$	(xii) Potassium hexacyanocobaltate (III)
(xiii) $[Pt(NH_3)_2Cl_2]$	(xiii) Diamminedichloroplatinum (II)
(xiv) $[Fe(CO)_5]$	(xiv) Pentacarbonyliron (O)
(xv) $[NiCl_4]^{2-}$	(xv) Tetrachloronickelate (II) ion
(xvi) $[Co(NH_3)_6]^{3+}$	(xvi) Hexaamminecobalt (III) ion.

Isomerism in coordination compounds

Two or more compounds that have the same molecular formula but differ in properties are called isomers. Coordination compounds show **structural isomerism** as well as **stereoisomerism.**

Structural isomerism is mainly of four types:

(i) *Ionization isomerism:* This is due to the exchange of groups between the complex and the ions outside. They are distinguished on the basis that they provide different ions when dissolved in water.

e.g., $[Co(NH_3)_5Br]SO_4$ and $[Co(NH_3)_5SO_4]Br$

(ii) *Hydrate isomerism:* In some crystalline complexes water can occur in more than one way–in the coordination sphere as ligand or outside the coordination sphere as water of crystallization e.g., $[Cr(H_2O)_6]Cl_3$ and $[Cr(H_2O)_5Cl]Cl_2.H_2O$.

(iii) *Linkage isomerism:* This type of isomerism arises due to the presence of two different donor atoms in the same ligand which may thus attach through either of the two atoms. e.g.,

$[Co(NH_3)_5NO_2]Cl_2$ and $[Co(NH_3)_5(ONO)]Cl_2$

(iv) *Coordination isomerism:* This type of isomerism is observed in those coordination compounds whose both the parts *i.e.* cation and anion are complex. The isomerism arises due to interchange of ligands between two complex ions. An example is $[Co(NH_3)_6][Cr(CN)_6]$ and $[Cr(NH_3)_6][Co(CN)_6]$.

Stereoisomers differ from each other only in spatial arrangement of ligands around the central metal atom. It is of the following two types:

(i) *Geometrical isomerism:* This type of isomerism is exhibited by complexes with 4 and 6 coordination numbers having square planar and octahedral arrangements respectively. When similar ligands occupy adjacent position, it is called cis-isomer and when similar ligands are opposite to each other then it is trans-isomer. Examples are:

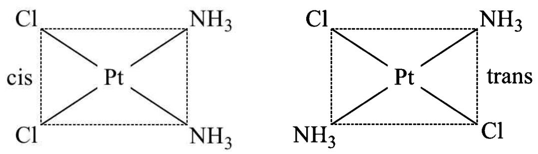


Fig. 19.1

(ii) *Optical isomerism:* This type-of isomerism exists when a molecule is not superimposable on its mirror image. These isomer are optically active and are said to be enantiomers. Octahadral compounds with three bidentate ligands, with two bidentate and two monodentate ligands or $[MA_2B_2C_2]$ type show optical isomerism. For example, $[CO(en)_3]^{3+}$ exists in two optically active isomeric forms.

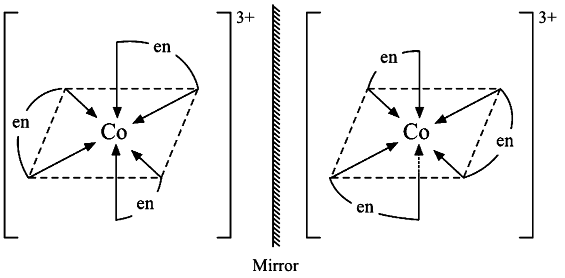


Fig. 19.2

Similarly $[Co(en)_2Cl_2]^+$ Can show geometrical isomerism as well as optical isomerism as shown below.

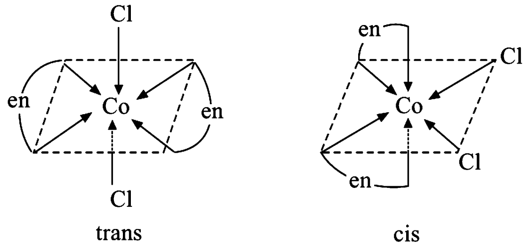


Fig. 19.3

The cis isomer can show optical isomerism also because its mirror image is non-superimposable.

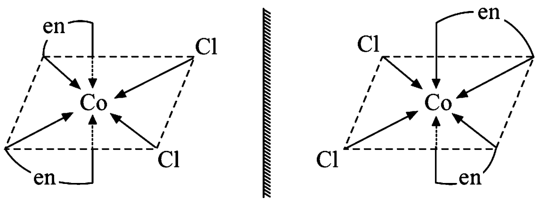


Fig. 19.4

Stability of a complex

The stability constant for a complex formation reaction can be represented as follows :

$$M^{x+} + nL^{y-} \rightleftharpoons [ML_n]^{z+}$$

$$K = \frac{[ML_n^{z+}]}{[M^{x+}][L^{y-}]^n}$$

Fig. 19.5

It shows that higher the value of *K*, more stable is the complex.

Bonding in Coordination Compounds

Valence Bond theory Since majority of complexes are formed by the transition metals and they have incomplete *d* orbitals of penultimate shell which are near in energy to *s* and *p* orbitals of the outermost shell.

These empty orbitals can accommodate electrons donated by the ligands. Different orbitals (*s*, *p* and *d*) of central metal ion hybridize to give a set of equivalent orbitals which take part in chemical bonding with the ligands. The shape of the complex will depend upon the type of hybridization of the central metal ion.

If the complex contains unpaired electrons, the complex is paramagnetic and if it does not contain unpaired electrons, the complex is diamagnetic in nature. Under the influence of a strong ligand, the electrons can be forced to pair up against the Hund's rule of maximum multiplicity. Shapes and magnetic behaviour of some of the complexes is discussed below.

(i) $[Co(NH_3)_6]^{3+}$

The oxidations state of cobalt in this complex is 3+. the valence shell electronic configuration for Co^{3+} can be represented as

$$1s^2, 2s^2, 2p^6, 3s^2, 3p^6, 3d^6, 4s^0$$

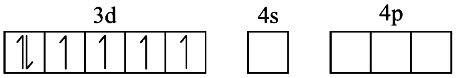


Fig. 19.6

During complex formation these '*d*' electrons pair up and two '*d*' orbitals become vacant.

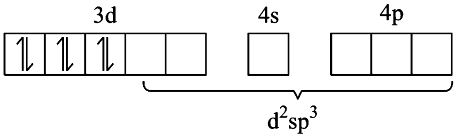


Fig. 19.7

These orbitals along with *s* and *p* orbitals undergo d^2sp^3 hybridisation which leads to octahedral geometry.

This complex is expected to be diamagnetic as it has no unpaired electron.

(ii) $[Cr(NH_3)_6]^{3+}$

In this complex the oxidation state of chromium is 3+ and the valence shell electronic configuration for Cr^{3+} can be shown as

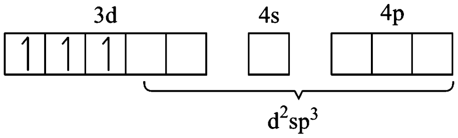


Fig. 19.8

The vacant two 3*d*, one 4*s* and three 4*p* orbitals hybridized to form $d^2\ sp^3$ hybridised orbitals which accept the pair of electrons from six NH_3 molecules. This hybridization leads to octahedral geometry. This complex is not diamagnetic because it has three unpaired electrons in 3*d* orbitals.

(iii) $[Ni(CO)_4]$

The oxidation state of Nickel in $[Ni(CO)_4]$ is zero. Its outer electronic configuration can be represented as

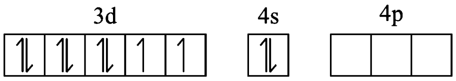


Fig. 19.9

Now in this case, four vacant orbitals are obtained by shifting of 4*s* electrons into 3*d* orbitals where these electrons get paired up.

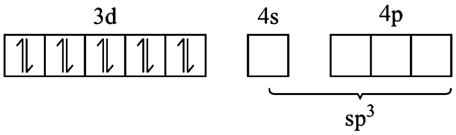


Fig. 19.10

The four vacant orbitals undergo sp^3 hybridisation and $[Ni(CO)_4]$ acquires a tetrahedral geometry.

This molecule shows diamagnetic behaviour because in it all the electrons are paired up and there is no unpaired electron.

(iv) $[Ni(CN)_4]^{2-}$

The oxidation state of nickel in this complex is 2+ and the valence shell electronic configuration for Ni^{2+} can be represented as

$$3s^23p^63d^8$$

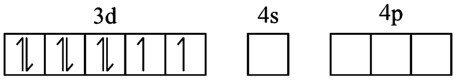


Fig. 19.11

During complex formation pairing of '*d*' electrons takes place because CN^- is a strong ligand. One '*d*' orbital so vacated, one '*s*' orbital and two '*p*' orbitals undergo dsp^2 hybridisation which results in the formation of square planar complex.

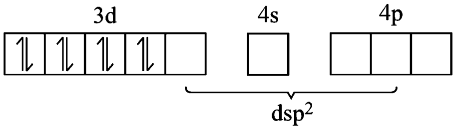


Fig. 19.12

This compound is also expected to show diamagnetic behaviour as it has no unpaired electron.

(v) $[Fe(CN)_6]^{3-}$

Electronic configuration of Fe^{3+} is

$$1s^2, 2s^2, 2p^6, 3s^2, 3p^6, 3d^5$$

Valence shell configuration can be shown as

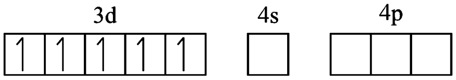


Fig. 19.13

During complex formation *d*-electrons get paired up because CN^- is a strong ligand. It undergoes d^2sp^3 type of hybridisation of vacant orbitals.

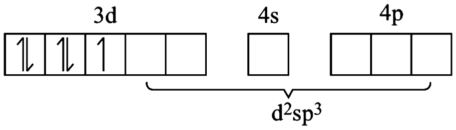


Fig. 19.14

Ferricyanide ion is paramagnetic because of the presence of one unpaired electron in 3*d* orbital.

(vi) $[Pt(NH_3)_2Cl_2]$

Oxidation state of Pt in this complex is 2+. The valence shell electronic configuration of Pt^{2+} can be shown as

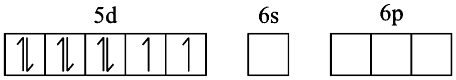


Fig. 19.15

During complex formation these *d* electrons pair up and orbitals used for bonding with ligands are one *d*, one *s* and two *p*, Hence the hybridization is dsp^2 and shape of this complex will be square planar. It shows diamagnetic behaviour as it has no unpaired electrons.

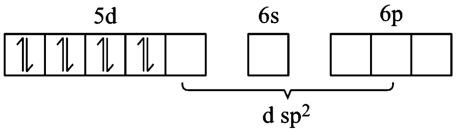


Fig. 19.16

Uses of Coordination Compounds

Coordination compounds are very often used

- (i) As catalyst.
- (ii) In electroplating industry
- (iii) Qualitative and quantitative analysis of many ions e.g., EDTA in estimation of hardness of water.
- (iv) Used in cancer therapy e.g., cisplatin.
- (v) In photography i.e., a complex of silver and thiosulphate ion $[Ag(S_2O_3)_2]^{3-}$ forms the basis of fixing process in photography.
- (vi) In extraction of metal e.g., silver, gold, etc.
- (vii) In water treatment e.g., complexation of Ca^{2+} or Mg^{2+} ions with polyphosphates in hard water.

Organometallic Compounds

Organometallic compounds are those in which carbon atom of an organic group is bound to metal atoms, for example, Grignard reagents where alkyl group is linked to magnesium metal atom. Metal carbonyls e.g., $Fe(CO)_5$, also considered as coordination compounds, are important organometallic compounds.

Organometallic compounds are classified into two types on the basis of bonding involved in them.

- (i) σ-bonded organometallics—which have a metalcarbon σ bond in them e.g., Grignard reagent, tetraethyllead, alkyllithium, etc.

  Various metals are known to form compounds involving metalcarbon σ-bonds e.g., Mg, Li, Ti, Pb, Cd, etc.
- (ii) π-bonded organometallics—which have a carbon group attached to metal by a π-bond e.g., metal carbonyls, sandwich compouns like ferrocene, etc.

π-Bonded coordination compounds are also called π-complexes. These are the compounds of transition metals with ligands that have vacant π-orbitals in addition to lone pairs.

The first organometallic compound was prepared in 1827 by Zeise and is know as Zeise's salt. It has the stoichiometry $K[Pt(Cl_3(C_2H_4)]$ and its structure was established in 1953.

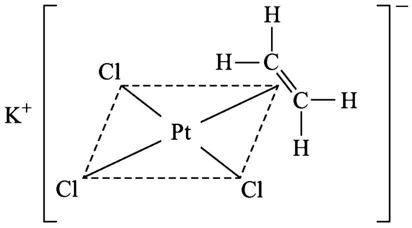


Fig. 19.17

The bonding of alkenes involve the donation of π-electrons form the double bond of the alkene to the vacant orbitals of the transition metal. The transition metal in turn pushes the electrons into the vacant anitbondting orbitals of the alkene. The bonding is referred to as back bonding .

The bonding in metal carbonyls involves donation of lone pair of electrons by the carbon of carbon monoxide into the vacant orbitals of the transition metal. This results in the formation of C → M sigma bond. By donating a pair of electrons CO makes the transition metal negatively charged which in turn pushes these electrons into the antibonding orbitals of carbon monoxide. This results in the formation of a M → C π bond, also called back bonding.

Several cyclic molecules like benzene form complexes involving transition metals. These are called sandwich compounds. One important example is ferrocene $Fe(\eta^5 - C_5H_5)_2$. It is a compound in which Fe^{2+} ion is sandwiched between two cyclopentadienyl anions. Its structure is shown as follows. (η^5 indicates all the five carbons are attached to metal).

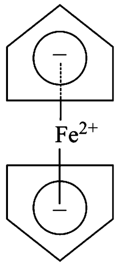


Fig. 19.18

Organometallic compounds which are soluble in organic solvents act as homogeneous catalysts. One such example is Wilkinson's catalyst $[(C_6H_5)_3P]_3$ RhCl which is used in hydrogenation of alkenes.

Problems for Practice

1. Coordination compounds are mainly formed by

(a) *s*-block elements (b) *p*-block elements

(c) *d*-block elements (d) *f*-block elements

2. Ligands are

(a) Negative ions only

(b) Neutral molecules only

(c) Positive ions only

(d) All of these

3. The coordination number of Cr in $[Cr(NH_3)_3(H_2O)_3]Cl_3$ is

(a) 3 (b) 4

(c) 6 (d) 9

4. Which one of the following has transition metal atom in the oxidation state +2?

(a) $K_4[Fe(CN)_6]$ (b) $K_3[Fe(CN)_6]$

(c) $K_2[Pt(CN)_6]$ (d) $[Co(NH_3)_5Br]SO_4$

5. Ethylenediamine is

(a) Monodentate (b) Bidentate
(c) Tridentate (d) Hexadentate

6. Which of the following is a chelate complex?
(a) $K_3[Co(CN)_6]$ (b) $[Co(H_2O)_6]Cl_3$
(c) $[Co(NH_3)_4Cl_2]Cl$ (d) $[Co(en)_3]Cl_3$

7. EDTA is used for the estimation of
(a) Na^+ ions in blood
(b) Ca^{2+} and Mg^{2+} ions in water
(c) K^+ ions in nerves
(d) All of these

8. According to Werner's theory, the secondary valencies of a central metal atom correspond to its
(a) Oxiation state
(b) Co-ordination number
(c) Sum of the two
(d) None of these

9. Which of the following will have least conductivity in aqueous solution?
(a) $[Co(NH)_3)_6]Cl_3$ (b) $[Co(NH_3)_5Cl]Cl_2$
(c) $[Co(NH_3)_3Cl_3]$ (d) $[Co(NH_3)_4Cl_2]Cl$

10. The number of ions produced from one molecule of $[Pt(NH_3)_5Cl]Cl_3$ in aqueous solution will be
(a) 3 (b) 4
(c) 5 (d) 6

11. Geometrical isomerism in coordination compounds is exhibited by
(a) Square planar and octahedral complexes
(b) Square phanar and tetrahedral complexes
(c) Square planar complexes only
(d) Tetrahedral and octahedral complexes

12. Which of the following compounds is not coloured?
(a) $Na_2[CuCl_4]$ (b) $Na_2[CdCl_4]$
(c) $K_4[Fe(CN)_6]$ (d) $K_3[Fe(CN)_6]$

13. $CuSO_4$ dissolves in NH_3 due to formation of
(a) $Cu(OH)_2$ (b) $[Cu(NH_3)_4]SO_4$
(c) $[Cu(NH_3)_4(OH)_2]$ (d) CuO

14. In hexacyanomanganate (II) ion, the Mn atom assume d^2sp^3 hybrid state. The number of unpaired electrons in the complex is
(a) 1 (b) 2
(c) 3 (d) Zero

15. Hexafluoroferrate (III) ion is an outer orbital complex. The number of unpaired electrons present in it is
(a) 1 (b) 5
(c) 4 (d) Unpredictable

16. Hybridization in $[Ni(CN)_4]^{2-}$ is
(a) dsp^2 (b) d^2sp
(c) sp^2d (d) sp^3

17. Which is not true for ligand metal complex?
(a) Larger the ligand, the more stable is the metal ligand complex.
(b) Highly charged ligands form stronger bonds
(c) Larger the permanent dipole moment of ligand the more stable is the bond
(d) Greater the ionization potential of central metal, stronger is the bond.

18. The oxidation number of Pt in $[Pt(C_2H_4)Cl_3]^-$ is
(a) +1 (b) +2
(c) +3 (d) +4

19. Ligands in a complex are
(a) Anions linked by coordinate bonds to a central metal atom or ion.
(b) Cations linked by coordinate bonds to a central metal atom or ion.
(c) Molecules linked by coordinate bonds to a central metal atom or ion.
(d) Ions or molecules linked by coordinate bonds to a central metal atom or ion.

20. A group of atoms can function as a ligand only when.
(a) It is a small molecule
(b) It has an unshared electron pair
(c) It is a negatively charged ion
(d) It is a positively charged ion

21. Ferrocene is
(a) $Fe(\eta^5 - C_5H_5)_2$ (b) $Fe(\eta^2 - C_5H_5)_2$
(c) $Cr(\eta^5 - C_5H_5)_5$ (d) $Os(\eta^5 - C_5H_5)_2$

22. The incorrect among the following is:
(a) Pyridine is a monodentate ligand
(b) $[Ni(CN_4]^{2-}$ is tetrahedral and diamagnetic
(c) Organometallic compounds contain at least one metal-carbon bond
(d) The oxidation state of nickel in $[Ni(CO)_4]$ is zero

23. A complex of platinum, ammonia and chlorine produces four ions per molecule in the solution. The structure consistent with the observation is
(a) $[Pt(NH_3)_6]Cl_4$ (b) $[Pt(NH_3)_2CI_4]$
(c) $[Pt(NH_3)_5Cl]Cl_3$ (d) $[Pt(NH_3)_4Cl_2]Cl_2$

24. The Ni $(CO)_4$ complex is diamagnetic in nature. The hybrid state of Ni is
(a) sp^3 (b) sp^3d
(c) sp^2d (d) spd^3

25. How many ions are produced from $[Co(NH_3)_6]Cl_3$ in solution?
(a) 6 (b) 4
(c) 3 (d) 2

26. The oxidation state of Ag in Tollen's reagent is
(a) 0 (b) 1
(c) 2 (d) +1.5

27. AgCI dissolves in NH_4OH due to the formation of
(a) $[Ag(NH_4)_2Cl]$ (b) $[Ag(NH_4)_3]Cl$
(c) $[Ag(NH_3)_2]Cl$ (d) $[Ag(NH_3)_2OH]$

28. In any ferric salt, on adding potassium ferrocyanide, a prussian blue colour is obtained, which is mainly due to the formation of
(a) $K_3[Fe(CN)_6]$ (b) $KFe[Fe(CN)_6]$
(c) $FeSO_4.Fe(CN)_6$ (d) $Fe_4[Fe(CN)_6]_3$

29. $K_3\,[Al\,(C_2O_4)_3]$ is called
(a) Potassium aluminooxalate
(b) Potassium alumino (III) oxalate
(c) Potassium trioxalatoaluminate
(d) Potassium trioxalatoaluminate (III)

30. Which of the following is not an example of organometallic compound?
(a) Trimethyl boron
(b) Trimethyl aluminium
(c) Trimethoxy titanium chloride
(d) Tetracarbonyl nickel

31. Which statement is incorrect?
(a) $[Ni(CO)_4]$-Tetrahedral, paramagnetic
(b) $[Ni(CN)_4]^{2-}$ Square planar, diamagnetic
(c) $[Ni(CO)_4]$-Tetrahedral, diamagnetic
(d) $[NiCl_4]^{2-}$-Tetrahedral, paramagnetic

32. Which of the following complex species involves sp^3d^2 hybridization?
(a) $[CoF_6]^{3-}$ (b) $[Co(NH_3)_6]^{3+}$
(c) $[Fe(CN)_6]^{3-}$ (d) $[Cr(NH_3)_6]^{3+}$

33. The type of isomerism shown by
$[Co(en)_2\,(NCS)_2]CI$ and $[Co\,(en)_2(NCS)Cl]\,NCS$ is:
(a) Co-ordination (b) Ionisation
(c) Linkage (d) All of these

34. The co-ordination number and oxidation number of X in the following compound
$[X\,(SO_4)\,(NH_3)_5]Cl$ will be:
(a) 10 and 3 (b) 2 and 6
(c) 6 and 3 (d) 6 and 4

35. The formula dichlorobis (urea) copper (II) is
(a) $[Cu\{O = C(NH_2)_2\}_2]Cl_2$
(b) $[CuCl_2\,\{O = C(NH_2)_2\}_2]$
(c) $[Cu\,\{O = C(NH_2)_2\}Cl]\,Cl$
(d) None of these

36. The complex ion $[Co(en)_2Cl_2]^+$ exhibits
(a) Cis-trans isomerism only
(b) Cis-trans and linkage isomerism
(c) Cis-trans and optical isomerism
(d) Optical and linkage isomerism

37. An ambident ligand is one which
(a) Is linked to the metal atom through two donor atoms
(b) Has two donor atoms, but only one of them has the capacity to form a coordinate bond
(c) Has two donor atoms, and either of two can form a coordinate bond
(d) Forms chelate rings

38. $K_4\,[Fe(CN)_6]$ is named as
(a) Potassium hexacyanoferrate (II)
(b) Potassium ferricyanide
(c) Potassium hexacyanoferrate (III)
(d) Prussian blue

39. Which of the following compounds is square planar and does not have any unpaired electron?
(a) $[Ni(CO)_4]$ (b) $[Ni(H_2O)_6]^{2+}$
(c) $[NiCl_4]^{2-}$ (d) $[Ni(CN)_4]^{2-}$.

40. The unpaired electrons in $Ni(CO)_4$ are
(a) Zero (b) One
(c) Three (d) Four

41. Which of the following complex species does not involve d^2sp^3 hybridization?
(a) $[CoF_6]^{3-}$ (b) $[Co(NH_3)_6]^{3+}$
(c) $[Fe(CN)_6]^{3+}$ (d) $[Cr(NH_3)_6]^{3+}$

42. Which of the following complexes can form d and l isomers?
(a) $[Co(NH_3)_4\,CI_2]^+$ (b) $[Co(NH_3)_3\,CI_3]$
(c) trans $[Co(en)_2\,Cl_2]$ (d) $[Co\,(en)_2\,Cl_2]^+$

43. Out of the following, which will not show geometrical isomerism?
(a) $[Pt(NH_3)_2\,Cl_2]$ (b) $[Co(NH_3)_5NO_2]Cl_2$
(c) $[Co\,(en)_2\,Cl_2]Cl$ (d) $[Cr\,(NH_3)_4\,Cl_2]Cl$

44. The currect IUPAC name of $Mn_3\,(CO)_{12}$ is:
(a) Dodecarbonyl manganate (0)
(b) Dodecarbonyl manganese (0)
(c) Dodecarbonyl trimanganese (0)
(d) Manganic dodecarbonyl (0)

45. Which of the following is most likely structure of $CrCl_3.\,6H_2O$ if 1/3 of total chlorine of the compound is precipitated by adding $AgNO_3$ to its aqueous solution?
(a) $CrCl_3\cdot 6H_2O$
(b) $[Cr(H_2O)_3Cl_3](H_2O)_3$
(c) $[CrCl_2\,(H_2O)_4]\,Cl\cdot 2H_2O$
(d) $[CrCl.\,(H_2O)_5]\,Cl_2\cdot H_2O$

46. The co-ordination number of Cr in $[Cr\,(NH_3)_3\,(H_2O)_3]Cl_3$ is
(a) 3 (b) 4
(c) 6 (d) 2

47. The complex ion $[Pt(NH_3)_4]^{2+}$ has ...structure
(a) Pyramidal (b) Tetrahedral
(c) Square planar (d) Pentagonal

48. Wilkinson's catalyst is:
(a) $TiCl_4$ and $Al(C_2H_5)_3$
(b) $[(C_6H_5)_3P]_3\,Rh\,Cl$
(c) $TiCl_4$ and $(C_6H_5)_3P$
(d) $Al(C_2H_5)_3.\ TiCl_4$ and $(C_6H_5)_3P.$

49. The complex ions $[Co(NH_3)_5\,(NO_2)]^{2+}$ and $[Co(NH_3)_5(ONO)]^{2+}$ are
(a) Geometrical isomers
(b) Ionisation isomers
(c) Linkage isomers
(d) Coordination isomers

50. Optical isomerism will be exhibited by
(a) $[Fe(NH_3)_6]Cl_3$
(b) $[Fe(H_2O)_6]Cl_3$
(c) $K_3[Fe(CN)_6]$
(d) $K_3[Fe(C_2O_4)_3]$

51. Nessler's reagent contains
(a) $[HgI_4]^-$ (b) $[HgI_2]^{2-}$
(c) $[HgI_4]^{3-}$ (d) $[HgI_4]^{2-}$

52. Which of the following is π-complex?
(a) Trimethylaluminium
(b) Ferrocene
(c) Diethylzinc
(d) Nickelcarbonyl

53. A coordination complex compound of cobalt has molecular formula containing five ammonia molecules, one nitro group and two chlorine atoms for one cobalt atom. One mole of this compound produces three mole ions in an aqueous solution. On reacting this solution with excess of silver nitrate solutions, two moles of AgCl get precipitated. The ionic formula of this complex would be

(a) $[Co(NH_3)_4NO_2Cl][(NH_3)Cl$

(b) $[Co(NH_3)_5Cl][(NO_2)$

(c) $[Co(NH_3)_5(NO_2)]Cl_2$

(d) $[Co(NH_3)_5]\ (NO_2)_2Cl_2$

54. According to Werner's theory, the secondary valencies of the central metal atom corresponds to its.

(a) Oxidation state

(b) Coordination number

(c) Either (a) or (b)

(d) None of these

55. The compounds $[Co(NO_2)\ (NH_3)_5]\ Cl_2$ and $[Co(ONO)\ (NH_3)_5]\ Cl_2$ are example of

(a) Geometrical isomers

(b) Linkage isomers

(c) Ligand isomers

(d) Ionisation isomers

56. The hypothetical complex chlorodiaquatriamminecobalt (III) chloride can be represented as

(a) $[CoCl(NH_3)_3(H_2O)_2]Cl_2$

(b) $[Co(NH_3)_3(H_2O)_2Cl]$

(c) $[Co(NH_3)_3(H_2O)_2Cl_3]$

(d) $[Co(NH_3)_3(H_2O)_2]Cl_3$

57. Ionization of $K[Ag(CN)_2]$ in water will give

(a) K^+ and $[Ag(CN)_2]^-$ ions

(b) KCN and AgCN

(c) K^+, Ag^+ and CN^- ions

(d) K^+ ions and AgCN

58. The oxidation state of nickel in $[Ni(CO)_4]$ is

(a) 2 (b) 4

(c) 0 (d) 1

59. Which isomerism is exhibited by the compound $[Co(NH_3)_3(H_2O)_3]Cl_3$?

(a) Linkage isomerism

(b) Cis-trans isomerism

(c) Co-ordination isomerism

(d) None of these

60. Hexaflurocobaltate (III) ion is found to be high spin complex. The probable hybrid state of cobalt in it is

(a) d^2sp^3 (b) sp^3

(c) sp^3d (d) sp^3d^2

61. The complex $Hg[Co(CNS)_4]$ is correctly named as

(a) Mercury tetrathiocyanato cobaltate (II)

(b) Mercury cobalt tetra sulphocyano (II)

(c) Mercury tetra thio cyanide cobaltate (II)

(d) Mercury sulphocyanato cobalt (II)

62. Which one of the following ligands can bring about highest oxidation state in a transition metal?

(a) F^- (b) Cl^-

(c) Br^- (d) S^-

63. Which of the following is expected to be a paramagnetic complex?

(a) $[Ni(H_2O)_6]^{2+}$ (b) $[Ni(CO)_4]$

(c) $[Zn(NH_3)_4]^{2+}$ (d) $[Co(NH_3)_6]^{3+}$

64. Coordination number of Ni in $[Ni(C_2O_4)_3]^{4-}$ is

(a) 3 (b) 6

(c) 4 (d) 5

ANSWERS

1. (c)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (d)	**20.** (b)
21. (a)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (d)	**30.** (c)
31. (a)	**32.** (a)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (d)	**40.** (a)
41. (a)	**42.** (d)	**43.** (b)	**44.** (c)	**45.** (c)	**46.** (c)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (d)
51. (d)	**52.** (b)	**53.** (c)	**54.** (b)	**55.** (b)	**56.** (a)	**57.** (a)	**58.** (c)	**59.** (b)	**60.** (d)
61. (a)	**62.** (a)	**63.** (a)	**64.** (b)						

20

Nuclear Chemistry

BRIEF REVIEW OF THE CONCEPTS

Nuclear chemistry deals with nuclei and reactions that cause changes in nuclei.

Radioactivity is the spontaneous disintegration of unstable atomic nuclei with emission of particles, or electromagnetic radiation, or both. The phenomenon of radioactivity was discovered by Henri Becquerel, in 1896. Naturally occurring radioactive elements when undergo decay, emit three types of radiations.

Alpha (α)-rays are doubly positive charged helium atoms i.e., He-atoms which have lost two electrons and are denoted as $^{4}_{2}He$.

Beta (β)-rays are negatively charged particles with negligible mass and one unit negative charge i.e., electrons denoted by $^{0}_{-1}e$ or β.

Gamma (γ)-rays are electromagnetic radiations. They have no mass and no charge. They are extremely penetrating and are not deflected in an electric or magnetic field.

Characteristics of these three types of rays are summarized below.

Table 20.1 Characteristics of α, β and γ-rays

		α-rays	*β-rays*	*γ-rays*
1.	Symbol	$^{4}_{2}He$	$^{0}_{-1}e$	γ
2	Charge	+2	–1	0
3.	Mass (in g)	6.65×10^{-24}	9.11×10^{-28}	Nil
4.	Relative ability to ionize air	10,000	100	1
5.	Speed (in kms^{-1})	$\sim 10^{4}$	$\sim 10^{5}$	$\sim 10^{5}$
6.	Limited penetrating power, sheets of paper or clothing can stop them.		More penetrating than α-particles 3 mm sheet of' Al can stop.	High penetrating power. Thick layers of Pb are needed to stop them.

Nuclear Reactions: Nuclear reactions result from changes taking place within atomic nuclei. Various types of nuclear reactions are discussed below.

(i) **Alpha Emission (or α - decay):** When an α-particle is ejected from the nucleus of an atom, the resulting element has the atomic number reduced by 2 and mass number by 4 units. Examples of α-emission are

$$^{238}_{92}U \longrightarrow {}^{234}_{90}Th + {}^{4}_{2}He$$

$$^{210}_{84}Po \longrightarrow {}^{206}_{82}Ra + {}^{4}_{2}He$$

(ii) **Beta Emission (or β-decay):** Beta decay refers to the emission of an electron from the nucleus. Beta particle may be considered to result from the transformation of a neutron into a proton. The emission of a beta particle causes the atomic number to increase by 1 but the mass number remains unchanged. Examples are

$$^{234}_{90}Th \longrightarrow {}^{234}_{91}Pa + {}^{0}_{-1}e$$

$$^{14}_{6}C \longrightarrow {}^{14}_{7}N + {}^{0}_{-1}e$$

(iii) **γ-emission:** Gamma radiations are electromagnetic radiations similar to X-rays.

Their emission is caused by energy changes within the nucleus in which the mass number or the atomic number of the nuclide does not change. After an α- or β-emission, the nuclide produced may contain more energy than in its normal ground state. This excess energy may be emitted as gamma radiation.

(iv) **Positron Emission (or $_{+1}^{0}$e-decay):** It is the emission of a positively charged electron from the nucleus symbolized by $_{+1}^{0}e$.Positron possesses the same mass as that of an electron but has opposite charge. Positron arises from the conversion of a nuclear proton into a neutron. Hence the daughter nucleide so formed has atomic number reduced by one, although mass number does not change. Examples are

$$_{30}^{65}Zn \longrightarrow {}_{29}^{65}Cu + {}_{+1}^{0}e$$

$$_{8}^{15}O \longrightarrow {}_{7}^{15}N + {}_{+1}^{0}e$$

(v) **Electron Capture (or K- capture)**

In the electron capture process the nucleus captures an electron form the K-shell, and the captured electron converts a nuclear proton into a neutron. This results in a nucleide with one proton less, hence, the atomic number of the daughter nucleide is one less than that of the parent nucleide. Examples are

$$_{29}^{64}Cu + {}_{-1}^{0}e \longrightarrow {}_{28}^{64}Ni$$

$$_{26}^{55}Fe + {}_{-1}^{0}e \longrightarrow {}_{25}^{55}Mn$$

Radioactive series. Disintegration of nuclear species ultimately leads to a more stable species. This takes place in a series of steps till a stable nuclei is formed. All these steps which are involved in forming a stable nuclei from an unstable nuclei form a series called disintegration series or radioactive series.

There are three disintegration series involving only alpha and beta emission that occur in nature. These are Thorium-232 (or 4n), Uranium - 238 (or 4n + 2) and Uranium-235 or Actinium (or 4n +3) series. These three series terminate with stable lead nucleides. There also exists a foruth series which is known as plutonium-241 or Neptunium-237 (or 4n + 1) series. This series was discoverd following the synthesis of cartain elements not found in nature. The main difference of (4n + 1) series with the other series is that bismuth is the end product instead of lead. These series are summarized in the table given below.

Artificial radioactivity is the disintegration of man made elements.

Table 20.2

Series	*Name*	*Starting element*	*Last stable element*	*No of α-particlers emitted*	*No of β-particlers emitted.*
4n	Thorium Series	$_{90}^{232}Th$	$_{82}^{208}Pb$	6	4
4n+1	Plutonium Series (or Neptunium Series)	$_{94}^{241}Pu$	$_{83}^{209}Bi$	8	5
4n+2	Uranium Series	$_{92}^{238}U$	$_{82}^{206}Pb$	8	6
4n+3	Actinium Series	$_{92}^{235}U$	$_{82}^{207}Pb$	7	4

Group Displacement Law

The radioactive decay or artificial nuclear reactions result in a new nucleide which can be predicted by the *Group Displacement law* also known as *Soddy-Fajan rule.* It states

The emission of an α- particle from a radioactive element decreases the mass nummber of the nucleide by four and atomic number by two resulting in the formation of a daughter element which lies two groups to the left in the periodic tabel. The emission of a β-particle does not change the mass number but increases the atomic number by one. The resulting nucleide moves one group to the right in the periodic table. For example

$$\underset{\text{(Groups 16)}}{_{84}^{215}Po} \xrightarrow{-\alpha} \underset{\text{(Group 14)}}{_{82}^{211}Pb} \xrightarrow{-\beta} \underset{\text{(Group 15)}}{_{83}^{211}Bi}$$

At the time when this law was proposed, radioactive decay process involving positron emission was not known. Now we can also include the positron emission in the very sense of group displacement law. Positron emission shifts the new nucleide one place to the left of the parent nucleide. An example is

$$\underset{\text{(Group 2)}}{_{12}^{23}Mg} \longrightarrow \underset{\text{(Group 1)}}{_{11}^{23}Na} + {}_{+1}^{o}e$$

Binding energy of a nucleus is a measure of its stability.

Binding energy, ΔE, is given by Einstein equation,

$$\Delta E = mc^2$$

here m is the mass loss and c is the velocity of light 3.0×10^8 ms^{-1}.

The energy released in nuclear reactions is very large and is usually expressed in terms of MeV (mega electron volt)

$$1 \text{ MeV} = 1.622 \times 10^{-13} J$$

Mass loss is the mass difference between the sum of the individual nucleons present and mass of the atom. Mass loss of 1 amu = 931.48MeV.

Rate of Radioactive Decay

Redioactive decay obeys the kinetics of the first order reaction.

The disintegration constant, λ, is given by

$$\lambda = \frac{2.303}{t} \log \frac{N_0}{N}$$

Here N_0 is the initial quantity of radionucleide (in any mass units or as the number of nuclei), N is the quantity of radionucleide left after the time t.

Half-life, $t_{1/2}$ is time gap when the intial quantity, of radionucleide is reduced to half i.e., $t_{1/2}$ is given by

$$t_{1/2} = \frac{0.693}{\lambda}$$

If n is the number of half-lives elapsed,

$$\text{Fraction of original quantity remaining} = \left(\frac{1}{2}\right)^n$$

Transmutation is the conversion of one element into antother. It can be carried out using α-particles, protons, deutrons, neutrons or some other nuclei.

Nuclear fission In a nuclear fission, one atom of a heavy fissionable isotope breaks up into two atoms of lighter nuclei and several neutrons.

Fission can be initiated by bombardment with many types of particles or by γ-radiation, and many heavy nucleides can undergo fission. Most common nuclear fission reactions are initiated by bombardment with slow neutrons.

$$^{235}_{92}U + \frac{1}{0}n \rightarrow {}^{236}_{92}U \rightarrow {}^{144}_{56}Ba + {}^{90}_{36}Kr + 2^{1}_{0}n$$

Fission produces more than one neutron per fission reaction. Thus enough neutrons are available to keep the fission going.

A large amount of energy is released during nuclear fission because of a loss in mass, the sum of the masses of two lighter nucleides formed and neutrons released is less than the sum of the masses of isotope undergoing fission and one neutron.

Critical mass of a fissionable material is the smallest mass that will support a self-sustaining chain reaction under a given set of conditions.

Nuclear reactor is a device for controlling the rate of fission in a nuclear power plant.

In a nuclear or atomic reactor, the rate of fission is controlled by inserting cadmium rods or other "neutron absorbers" into the reactor. The rods absorb neutrons and by withdrawing or inserting the rods, the rate of the reaction can be increased or decreased. The rods are so adjusted that only one neutron per fission is available for the next fission reaction.

Not all nuclei can undergo nuclear fission on bombarding with a neutron, $^{235}_{92}U$ and $^{239}_{94}Pu$ are two nucleides where fission is possible.

Nuclear Fusion In nuclear fusion two or more atoms of lighter nuclei combine to form a hevy nucleus, *e.g.*,

$${}^{2}_{1}H + {}^{3}_{1}H \longrightarrow {}^{4}_{2}He + {}^{1}_{0}n$$

Breeder Reactors A breeder reactor is the one that produces its own fuel. In the breeder reactor, $^{235}_{92}U$ or $^{239}_{94}Pu$ is the regular fuel and supplenental amounts of $^{238}_{92}U$ and $^{232}_{90}Th$ are added to the fuel. Breeding occurs because some of the neutrons produced by the fission are captured by $^{238}_{92}U$ nuclei, therely setting up the series of reactions that produce the fissionable $^{239}_{94}Pu$. Fissonable $^{235}_{92}U$ is produced in a similar manner from $^{232}_{90}Th$. In an ideal breeder reactor the fissionable fuel produces two neutrons, one for itself for further fission and one for $^{238}_{92}U$ or $^{232}_{90}Th$ to converted them to fissionable nucleides.

Problems for Practice

1. The atoms of different elements having the same number of neutrons are known as
   (a) isobars (b) isomers
   (c) isotones (d) isotopes
2. The elements of atomic number larger than 92 show
   (a) artificial radioactivity
   (b) natural radioactivity
   (c) artificial as well as natural radioactivity
   (d) no radioactivity
3. Electrons do not exist inside the nucleus and β-particles are nothing but electrons coming from the nucleus. This is because
   (a) neutron decays into a proton and an electron
   (b) proton breaks into a neutron and an electron
   (c) neutrons and protons interact to produce electrons
   (d) all the above
4. α-particles are double charged
   (a) helium nuclei
   (b) hydrogen nuclei
   (c) hydrogen ion
   (d) deutrons
5. In α-decay neutron-protion ratio N/P
   (a) N/P decreases (b) N/P increases
   (c) N/P remains constant
   (d) none of the above
6. The loss of a β-particle is equivalent to
   (a) increase of one proton
   (b) decrease of one neutron
   (c) combination of both (a) and (b)
   (d) none of the above
7. The emission of β-particle by an atom of an element results in the formation of its
   (a) isotope (b) isomer
   (c) isomorph (d) isobar
8. If $^{235}_{92}U$ decays only by emitting two α and one β particles, the resulting nucleide is
   (a) $^{234}_{89}Ac$ (b) $^{235}_{89}Ac$
   (c) $^{236}_{89}Ac$ (d) $^{227}_{89}Ac$
9. The number of α-and β-particles emitted during the transformation of $^{232}_{90}Th$ to $^{208}_{82}Th$
   (a) 2, 2 (b) 4, 2
   (c) 8, 6 (d) 6, 4
10. When $^{7}_{3}Li$ bombarded with protons, γ-rays are produced. The nucleide formed is
    (a) $^{6}_{4}Be$ (b) $^{8}_{4}Be$
    (c) $^{7}_{4}Be$ (d) $^{9}_{4}Be$
11. Which of the following is artificial radioactive series?
    (a) $4n$ (b) $4n + 1$
    (c) $4n + 2$ (d) $4n + 3$
12. Breeder reactor involves use of
    (a) $^{238}_{92}U$ (b) $^{239}_{92}U$

(c) ${}^{239}_{93}\text{Np}$ (d) ${}^{239}_{94}\text{Pu}$

13. ${}^{92}_{46}\text{Mo} + {}^{1}_{1}\text{H} \rightarrow {}^{92}_{47}\text{Tc} + {}^{1}_{0}\text{n}$ reaction is of the type

(a) α, *n* (b) *n*, *p*
(c) *n*, α (d) *p*, *n*

14. In the reaction ${}^{7}_{3}\text{Li} + \text{Z} \rightarrow {}^{7}_{4}\text{Be} + {}^{1}_{0}\text{n}$

The bombarding projectile Z is

(a) neutron (b) α-particle
(c) proton (d) deutron

15. Nuclear fission and fusion have which one of the following as common property?

(a) In both processes neucleus splits into lighter nuclei
(b) In both processes the lighter nuclei combine to give heavier nucleus
(c) Both processes release large amount of energy
(d) Both processes require large amount of energy

16. In carbon dating

(a) the decay rate of ${}^{12}_{6}\text{C}$ is studied
(b) the rate of accumulation of ${}^{12}_{6}\text{C}$ is studied
(c) the rate of formation of ${}^{13}_{6}\text{C}$ is studied
(d) the rate of formation of ${}^{14}_{6}\text{C}$ is studied

17. The half life period of a radioactive isotope is 15 hours. How long will it take for its activity to be reduced to 1/32 of its original value?

(a) 60 h (b) 75 h
(c) 120 h (d) 240 h

18. If 8 g of a radioactive isotope has a half life of 10 h., the half life of 2 g of the same substance is

(a) 205 h (b) 5.0 h
(c) 10 h (d) 1.25 h

19. A radioactive isotope decays at such a rate that after 96 min only 1/8th of the original amount remains. The value of $t_{1/2}$ of this nucleide is

(a) 12 min (b) 32 min
(c) 24 min (d) 96 min

20. Which of the following is used as neutron absober in the nuclear reactor?

(a) Water (b) Deuterium
(c) Uranium (d) Cadmium

21. In determining the age of organic material one measures

(a) The time required for half of the C-14 in the sample to decay
(b) The ratio of C-14 in the given material and living organism
(c) The percentage of carbon in the sample
(d) The time required for half of the organic material to decay

22. An isotone of ${}^{76}_{32}\text{Ge}$ is

(a) ${}^{77}_{32}\text{Ge}$ (b) ${}^{78}_{33}\text{As}$
(c) ${}^{77}_{34}\text{Se}$ (d) ${}^{78}_{34}\text{Se}$

23. The positron is nearly as heavy as:

(a) alpha particle (b) Proton
(c) Electron (d) Deutron

24. An alkaline earth metal is radioactive. It decays by emitting one α and two β particles in succession. In what group the resulting element be found

(a) Zero (b) 6
(c) 4 (d) 2

25. Nuclear reaction accompanied with emission of neutrons is

(a) ${}^{27}_{13}\text{Al} + {}^{4}_{2}\text{He} \longrightarrow {}^{30}_{15}\text{P}$
(b) ${}^{12}_{6}\text{C} + {}^{1}_{1}\text{H} \longrightarrow {}^{13}_{7}\text{N}$
(c) ${}^{30}_{15}\text{P} \longrightarrow {}^{30}_{14}\text{Si} + {}_{+1}\text{e}^{0}$
(d) ${}^{240}_{95}\text{Am} + {}^{4}_{2}\text{He} \longrightarrow {}^{244}_{97}\text{Bk}$

26. If ${}^{235}_{92}\text{U}$ is assumed to decay only by emitting two α-and one β-particles. The possible product of decay is

(a) ${}^{231}_{89}\text{Ac}$ (b) ${}^{235}_{89}\text{Ac}$
(c) ${}^{237}_{89}\text{Ac}$ (d) ${}_{89}{}^{227}\text{Ac}$

27. When ${}^{35}_{17}\text{Cl}$ undergoes (*n*, *p*) reaction, the radioisotope formed is

(a) ${}^{32}_{15}\text{P}$ (b) ${}^{35}_{16}\text{S}$
(c) ${}^{34}_{16}\text{S}$ (d) ${}^{34}_{15}\text{P}$

28. A method which uses radioactivity for determining the age of pre-historic materials is called

(a) Carbon dating
(b) Deuterium dating
(c) Radium dating
(d) Uranium dating

29. The counting rate observed from a radioactive source at *t* = 0 second was 1600 counts/sec. The counting rate observed in count per sec at *t* = 6 sec will be ($t_{1/2}$ = 2 sec)

(a) 400 (b) 300
(c) 200 (d) 150

30. If the half life of Sr is 20 years what part of the original quantity of a portion of strontium will remain after 60 years?

(a) 1/2 (b) 1/4
(c) 1/8 (d) 1/16

31. The end product of the natural radioactive series is:

(a) Bismuth
(b) Thorium
(c) Any isotope of lead
(d) Any isotope of calcium

32. In the nuclear reaction ${}_{92}{}^{238}\text{U} \longrightarrow {}_{82}{}^{206}\text{Pb}$ the number of α and β particles emitted are

(a) 7 α, 5 β (b) 6 α, 4 β
(c) 4 α, 3 β (d) 8 α, 6 β

33. Gamma rays are

(a) High energy electrons
(b) Low energy electrons
(c) High energy electro-magnetic waves
(d) High energy positrons

34. The source of energy of the sun is

(a) Natural radioactivity
(b) Artificial radioactivity
(c) Nuclear fission
(d) Nuclear fusion

35. In a certain radioactive decay, an electron is emitted. It comes out from

(a) Outermost orbit of the atom
(b) Inner shells of the atom
(c) Nucleus of the atom
(d) None of these

36. Which one of the following particles is used to bombard $^{27}_{13}Al$ to give $^{30}_{15}P$ and a neutron?
(a) $^{2}_{1}H$ (b) γ
(c) α (d) β

37. Which of the following nuclear process is accompanied by emission of X-rays?
(a) α-emission (b) β-emission
(c) γ-emission (d) Electron caputre

38. When $^{238}_{92}U$ decays it emits an α-particle, the new nucleide in turn emits a β-particle to give another nucleide X. The mass number and atomic number of X are respectively
(a) 234 and 91 (b) 234 and 96
(c) 232 and 88 (d) 234 and 88

39. When $^{238}_{92}U$ emits α-particle, the product has a mass number and atomic number
(a) 236, 92 (b) 234, 90
(c) 238, 90 (d) 136, 90

40. Living things contains both ^{12}C and ^{14}C isotopes out of which ^{14}C decays and declines in proprotrional quantity. The technique that uses this principle for determining tha age of fossil is called
(a) C-12 dating
(b) Radiocarbon dating
(c) Carbon age
(d) Fossil carbon

41. The nuclear process that takes place when a hydrogen bomb is exploded is of the same nature as the process taking place
(a) In the centre of the earth
(b) On the suns and stars
(c) During red dust storm
(d) During atom bomb fission

42. ^{14}C is
(a) an artificial radioactive isotope
(b) a natural radioactive isotope
(c) a natural non-radioactive isotope
(d) an artificial non-radioactive isotope

43. $^{238}_{92}U \xrightarrow{-\alpha} A \xrightarrow{-\beta} {}^{y}_{x}B$
what are the values of x and y?
(a) 90, 234 (b) 92, 234
(c) 93, 234 (d) 91, 234

44. Isobar of a nucleide is formed when it loses
(a) 1 β-particle
(b) 1 α-particle
(c) 1α and 1β particle
(d) 2α and 1β particle

45. In the following reaction 'X' is
$^{23}_{11}Na + {}^{1}_{1}H \longrightarrow {}^{23}_{12}Mg + X$
(a) an electron (b) a deutron
(c) a neutron (d) a positron

ANSWERS

1. (c)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (d)	**8.** (d)	**9.** (d)	**10.** (b)
11. (b)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (c)	**19.** (b)	**20.** (d)
21. (b)	**22.** (d)	**23.** (c)	**24.** (d)	**25.** (a)	**26.** (d)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (c)
31. (c)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (c)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (b)	**40.** (b)
41. (b)	**42.** (b)	**43.** (d)	**44.** (a)	**45.** (c)					

Classification and Nomenclature of Organic Compounds

21

BRIEF REVIEW OF THE CONCEPTS

In almost all the organic compounds, carbon has been found to be present in tetravalent state. Although its ground state electronic configuration is $1s^2\ 2s^2\ 2p_x^1\ 2p_y^1$ which indicates that it should be bivalent. It is explained on the basis that one electron from $2s$ orbital jumps to vacant $2p_z$ orbital which makes four half filled orbitals. These four orbitals hybridize to give four sp^3 hybridized orbitals. This type of hydridization is encountered in organic compounds which are saturated (e.g., alkanes). In compounds containing C═C bonds (e.g., alkenes) sp^2 hybridization and in compounds containing C≡C bonds (e.g. alkynes) sp hybridization is observed. Shapes around sp^3 hybridised carbon is tetrahedral, around sp^2 hybridised carbon is trigonal planar and around sp hybridised carbon is linear.

The C—C, C═C and C≡C bond lengths are 154 pm, 134 pm and 120 pm respectively. Bond lengths of C—H bond in alkanes (sp^3–s overlap) alkenes (sp^2–s overlap) and alkynes (sp–s overlap) are 111 pm, 110 pm and 108 pm, respectively.

> When a carbon atom forms four sigma bonds in a molecule, the hybridization of carbon atom is sp^3. When a carbon atom in a molecule forms three sigma and one pi bond, its hybridization state is sp^2. When a carbon atom forms two sigma and two *pi* bonds, the hybridization of carbon is sp.

In organic compounds carbon atoms are classified as primary (1°), secondary (2°), tertiary (3°) and quaternary (4°) depending upon whether the carbon atom is linked to one, two, three or four other carbon atoms, respectively. Similarly hydrogen atoms are classified as primary (1°), secondary (2°) or tertiary (3°) when they are attached to 1°, 2° or 3° carbon atoms, respectively. There can be no quaternary (4°) hydrogen.

Classification of Organic Compounds

The classification of organic compounds is based on basic skeletion of carbon atoms. The molecules made up of only carbon and hydrogen are known as hydrocarbons. All other compounds are considered to be the derivatives of hydrocarbons obtained by the replacement of one or more hydrogen atoms with other atoms or groups called **functional groups.** Such a group largely governs the properties of compounds.

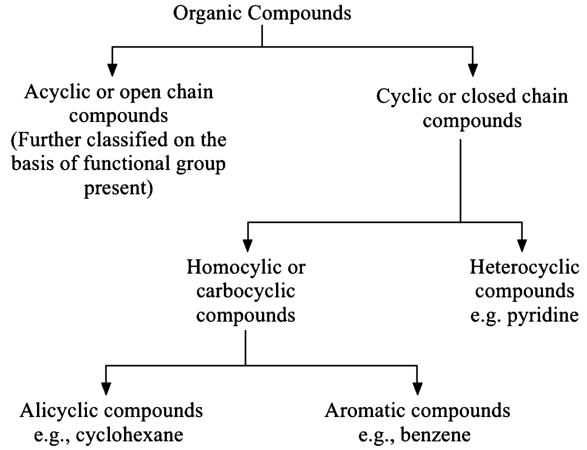


Fig. 21.1

Additional information on IUPAC System of Nomenclature of Polyfunctional Compounds

1. When an organic compound contains two or more different functional groups, then one of the functional group is selected as the principal functional group, while all other groups are treated as substituents. The choice of principal functional group is made on the basic of the following order of preference.

Sulphonic acid > carboxylic acid > anhydride > esters > acid halide > acid amide > nitrile > aldehyde > ketone > alcohol > amine.

The principal functional group so selected is indicated as a secondary suffix while all other groups are treated as substituents and indicated as prefixes. The prefixes for these groups are given below.

Table 21.1

Class	*Formula*	*Prefix*
Sulphonic acid	$—SO_3H$	sulpho-
Carboxylic acid	—COOH	carboxy-
Ester	—COOR	alkyl-oxycarbonyl-
Acid halide	—COX	haloformyl
Amide	$—CONH_2$	carbamoyl-
Nitrile (cyanide)	$—C{\equiv}N$	cyano-
Aldehyde	—CHO	formyl-
Ketone	$>C{=}O$	oxo-
Alcohol	—OH	hydroxy-
Amine	$—NH_2$	amino-
Thiol	—SH	sulphonyl-

The carbon of the substituent group is not counted in the principal chain while using the prefixes such as formyl, cyano, carbony, carbamoyl etc. When prefix -oxo is used for ketone, its carbon is counted in the principal chain. Aldehyde group can also be represented by -oxo group but then carbon of – CHO group is counted in the chain.

2. **Selecting the principal chain:** While selecting the principal chain present in a polyfunctional compound care should be taken that it contains maximum of substituents and the principal group i.e., the suffix. If however, two or more chains contains same number of groups, then that chain is selected which contains maximum number of carbon-carbon multiple bonds.
3. **Numbering the principal chain:** The principal chain present in a polyfunctional compound must be so numbered that the principal functional group gets the lowest possible number, followed by double bond, triple bond and substituents.
4. If the given organic compound contains two similar terminal groups, then principal chain is selected in such a way that these groups occur at the two ends of the principal chain. In such a case the carbon atoms of both such group are counted in the principal chain and both such groups are indicated by a suitable suffix along with the numerical prefix '*di*'. For example,

$$HOO\overset{5}{C}—\overset{4}{C}H_2—\overset{3}{C}H_2-+\overset{2}{C}H_2—\overset{1}{C}OOH$$

Pentanedioic acid

$$NH_2\overset{4}{C}O—\overset{3}{C}H_2—\overset{2}{C}H_2—\overset{1}{C}ONH_2$$

Butanediamide

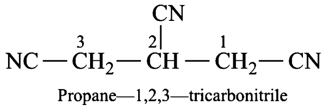
$$N{\equiv}\overset{5}{C}—\overset{4}{C}H_2—\overset{3}{C}H_2—\overset{2}{C}H_2—\overset{1}{C}{\equiv}N$$

Pentanedinitrile

$$OH\overset{5}{C}—\overset{4}{C}H_2—\overset{3}{C}H_2—\overset{2}{C}H_2—\overset{1}{C}HO$$

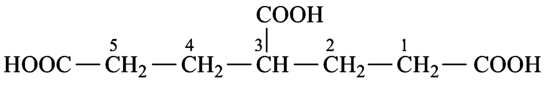
Pentanedial

5. If an unbranched carbon chain is directly linked to more than two like functional groups, the organic compound is named as derivative of the parent alkane which does not include carbon atoms of the functional groups, For example,



$$NC—\overset{3}{C}H_2—\overset{2}{\underset{|}{C}}H(CN)—\overset{1}{C}H_2—CN$$

Propane—1,2,3—tricarbonitrile
(3—Cyanopetane—1,5—dinitrile)

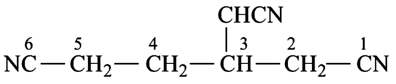


$$HOOC—\overset{5}{C}H_2—\overset{4}{C}H_2—\overset{3}{C}H(COOH)—\overset{2}{C}H_2—\overset{1}{C}H_2—COOH$$

Pentane—1, 3, 5—tricarboxylic acid
(4—carboxyheptane—1, 7—dioic acid)

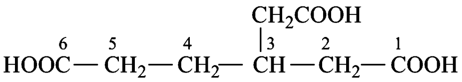
Earlier such compound were named as given in the brackets.

If however, all the three like groups are not directly linked to the unbranched carbon chain, the carbon atoms of the two like groups are included in the parent chain while the third which forms the side chain is considered as a substituent group, For example,



$$\overset{6}{N}C—\overset{5}{C}H_2—\overset{4}{C}H_2—\overset{3}{C}H(CHCN)—\overset{2}{C}H_2—\overset{1}{C}N$$

3—(cyanomethyl) hexane—1, 6—dinitrile

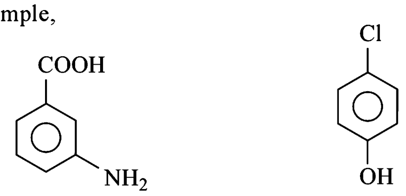


$$HOO\overset{6}{C}—\overset{5}{C}H_2—\overset{4}{C}H_2—\overset{3}{C}H(CH_2COOH)—\overset{2}{C}H_2—\overset{1}{C}OOH$$

3—(Carboxymethyl) hexane—1, 6—dioic acid

Additional Information on Nomenclature of Polyfunctional Aromatic Compounds

1. The principal functional group is selected and the compound is named as a derivative of the compound with this group at position number 1 on the ring. For example,

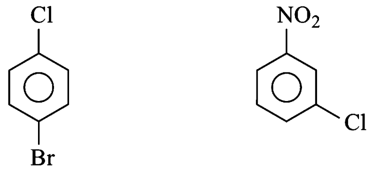


3-Aminobenzoic acid 4-Chlorophenol

3-Hydroxybenzoic acid 2-Bromoaniline

Fig. 21.2

2. If there are two groups with equal preference then lowest locant is given to the group which comes earlier in alphabetical order. For example,



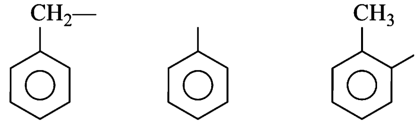
1-Bromo-4-chlorobenzene 1-chloro-3-nitrobenzene

Fig. 21.3

The aryl group obtained by removing one hydrogen atom from benzene is named as phenyl and not benzyl.

The aryl group obtained by removing a hydrogen atom from side chain in case of toluene is named as benzyl.

The aryl group obtained by the removal of one hydrogen atom from ring of toluene is named as tolyl.



Benzyl Phenyl *o*-Tolyl

Fig. 21.4

IUPAC rules for nomenclature of polycyclic compounds

In compounds containing two or more rings, the carbon atoms common to both the rings are called **bridge head atoms**. Each bond or chain of carbon atoms connecting both the bridge heads is called a bridge. The bridge may, contain 0, 1, 2... etc., carbon atoms as shown in the following examples.

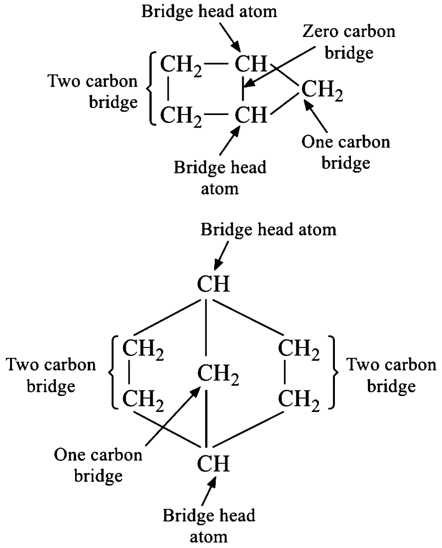
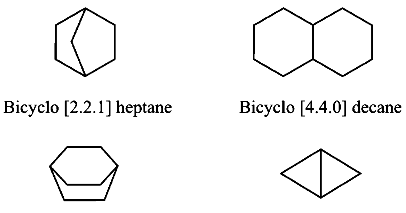


Fig 21.5

A bicyclic compound is named by writing the prefix bicyclo before the name of the hydrocarbon containing the same number of carbon atoms. The number of carbon atoms in each of the bridge is indicated by arabic numerals viz. 0, 1, 2, 3--- etc. These are written in descending order in a square bracket and separated from each other by full stops. After this the name of the corresponding alkane is written. For example


Bicyclo [2.2.1] heptane Bicyclo [4.4.0] decane

Bicyclo [2. 2.2] octane Bicyclo [1.1.0] butane

Fig 21.6

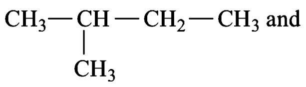
Isomerism It is the phenomenon by virtue of which compounds having same molecular formula possess some difference in their properties. Isomerism is classified mainly into two types.

(A) Structural Isomerism This type of isomerism exists in compounds due to different arrangement of atoms. It can be defined as the compounds having same molecular formula but different structure. Structural isomerism is further classified into six types.

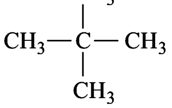
(i) Chain isomerism It is shown by compounds which differ in the structure of the carbon chain forming the skeleton of the molecule. For example,

$CH_3-CH_2-CH_2-CH_2-CH_3$,

(Pentane)


$CH_3-\underset{\substack{|\\ CH_3}}{CH}-CH_2-CH_3$ and

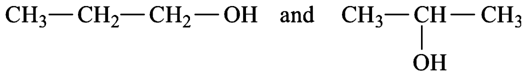
(2-Methylbutane)


$CH_3-\overset{\substack{CH_3\\ |}}{\underset{\substack{|\\ CH_3}}{C}}-CH_3$

(2, 2-Dimethypropane)

(ii) Position isomerism Compounds having same carbon skeleton but differing in the position of functional group are said to be position isomers. For example,

$CH_3-CH_2-CH=CH_2$ and $CH_3-CH=CH-CH_3$
(But-1-ene) (But-2-ene)

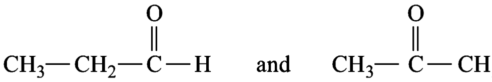

$CH_3-CH_2-CH_2-OH$ and $CH_3-\underset{\substack{|\\ OH}}{CH}-CH_3$

(Propan-1-ol) (Propan-2-ol)

(iii) Functional isomerism: Compounds having the same molecular formula but different functional groups are called functional isomers. It can also be said that the compounds with same molecular formula but belonging to different homologous series are called functional isomers. For example diethyl ether and butan-1-ol (both with molecular formula $C_4H_{10}O$) are isomers of this type.

$CH_3-CH_2-O-CH_2-CH_3$ $CH_3-CH_2-CH_2-CH_2-OH$
(Diethyl either) (Butan-1-ol)

Another example can be cited as that of propanal (an aldehyde) and propanone (a ketone), both with molecular formula (C_3H_6O).

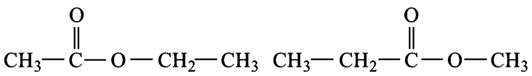

$CH_3-CH_2-\overset{\substack{O\\ \|}}{C}-H$ and $CH_3-\overset{\substack{O\\ \|}}{C}-CH_3$

(propanal) (propanone)

(iv) Metamerism: This type of isomerism is shown by compounds of the same homologous series in which the isomers differ by the nature of alkyl groups attached to the functional group e.g., diethly ether (ethoxy ethane and methly n-propyl ether (1-methoxypropane).

$CH_3-CH_2-O-CH_2-CH_3$ and $CH_3-O-CH_2-CH_2-CH_3$
(Ethoxyethane) (1-Methoxypropane)

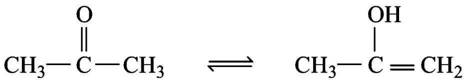
This type of isomerism can be seen in esters also e.g., Ethyl ethanoate and methyl propanoate.


$CH_3-\overset{\substack{O\\ \|}}{C}-O-CH_2-CH_3$ $CH_3-CH_2-\overset{\substack{O\\ \|}}{C}-O-CH_3$

(Ethyl ethanoate) (Methyl propanoate)

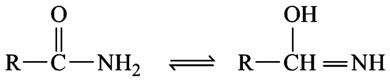
(v) Tautomerism: It is a special case of functional isomerism where the two compounds possessing different functional group are inter convertible by the movement one hydrogen atom. It is accompanied by the rearrangement of bonds also. Tautomerism is further classified by naming the functional groups e.g.,

Keto-enol tautomerism


$CH_3-\overset{\substack{O\\ \|}}{C}-CH_3 \rightleftharpoons CH_3-\overset{\substack{OH\\ |}}{C}=CH_2$

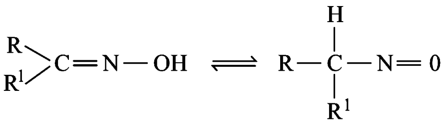
(Propanone) (Prop-1-en-2-ol)
(Keto-form) (enol-form)

Amide-iminol Tautomerism


$R-\overset{\substack{O\\ \|}}{C}-NH_2 \rightleftharpoons R-\overset{\substack{OH\\ |}}{CH}=NH$

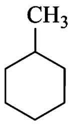
(Amide-form) (Iminol-form)

Oximino-nitroso Tautomerism

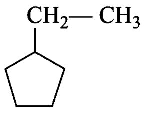


(Oxime form) (Nitroso form)

(vi) Ring-chain isomerism: This type of isomerism is shown by compounds where ring is attached to some alkyl chain. *e.g.*



and



(Methyl cyclohexane) (Ethyl cyclopentane)

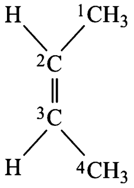
(B) Stereoisomerism: This type of isomerism is exhibited by compounds having the same structure but differing in the spatial arrangement of atoms.

In other words compounds which have the same attachment of atoms but have different orientation in space, are called **stereoisomers** and the phenomenon is known as **stereoisomerism.**

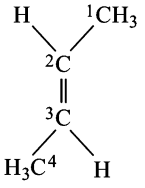
Configuration. The arrangement of atoms in space which characterizes a particular stereoisomer is called its configuration. Therefore the stereoisomers are also known as **configurational isomers.**

Stereoisomerism is of two types

(i) Geometrical isomerism: Restricted rotation across a carbon-carbon double bond is responsible for this type of isomerism. When two carbon atoms with two different substituents on them are linked by a double bond then they give rise to this type of isomerism. Such as 2-butene can have two such arrangements (I and II) which can not be interconverted unless a bond between the carbon number 2 and 3 is broken hence they are geometrical isomers



I
(cis-2-Butene)



II
(trans-2-Butene)

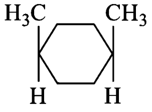
When the similar groups lie on the same side (as in I), it is called cis-isomer. When similar groups lie on the opposite sides (as in II), it is called a trans-isomer.

Thus a compound will show geometrical isomerism if it meets the two requirements:

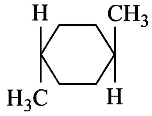
(a) It should possess a carbon-carbon double bond and

(b) The two group attached to each doubly bonded carbon should be different.

Geometrical isomerism can also exist if the restricted rotation is due to a ring, e.g., 1, 4-dimethyl cyclohexane exist in two isomeric forms shown below

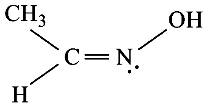


and

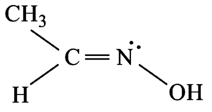


cis trans

Other functional group e.g. oximes can also possess this types of isomerism. Acetoxime can exist in the following two forms.



(anti-form)



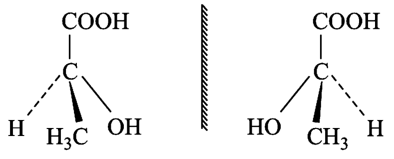
(syn-form)

(ii) Optical isomerism: In this case, the isomers differ only in their behaviour towards plane polarized light. This property of compounds is called optical activity. If a compound does not rotate the plane of plane -polarised light then it is said to be optically inactive. Any compound that has the ability to rotate the plane of plane-polarized light, is said to be *optically active*. The optical activity of any substance is measured by an instrument called *polarimeter*. Compounds which rotate the plane of plane polarised light in clockwise direction are said to be dextrorotatory (*d* or +) and the isomers which rotate the plane in anticlockwise direction are said to be laevorotatory (*l* or–)

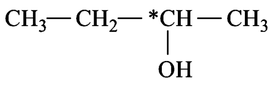
The optical activity of any compound is expressed in terms of specific rotation, [α]. It is defined as the rotation observed when the compound is taken in concentration of 1 g per mL and is taken in a tube of 1 dm length.

$$[\alpha] = \frac{\text{observed rotation}\,(\alpha_{obs})}{\text{length of the tube}\,(dm) \times \text{concentration}\,(g\,\text{mL}^{-1})}$$

This type of isomerism is show by those compounds whose molecules are chiral. A molecule is said to be chiral if its mirror image is non superimposable over it as shown below for lactic acid.

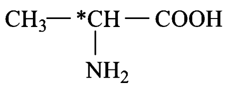


All organic compounds which contain one *asymmetric carbon* atom are chiral and exist in two isomeric forms which are nonsuperimposable images of each other. Such a pair of isomers are also called *enantiomers*. (Asymmetric carbon atom is that carbon which has four different groups attached to it). A mixture of enantiomers will be optically inactive because the effect of one type of isomer is exactly cancelled by other isomer. Such a mixture is called racemic mixture (dl or ±) and cause of its optical inacitivity is attributed to external compensation. Some examples of compounds which contain one asymmetric carbon atom and hence show enantiomersim are given below:



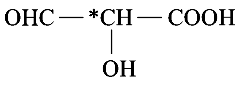
$$CH_3-CH_2-\underset{\substack{|\\OH}}{^*CH}-CH_3$$

(2-Butanol)



$$CH_3-\underset{\substack{|\\NH_2}}{^*CH}-COOH$$

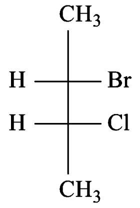
(Alanine)



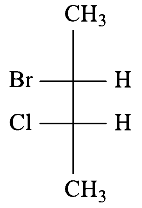
$$OHC-\underset{\substack{|\\OH}}{^*CH}-COOH$$

(Glyceraldehyde)

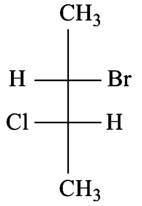
If a compoud contains more than one asymmetric carbon atom then the maximum number of possible optical isomers is given by 2^n where n = no. of asymmetric carbon atoms. For example the four possible isomers for 2-bromo-3-chlorobutane can be shown as follows



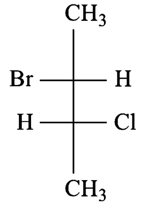
I



II



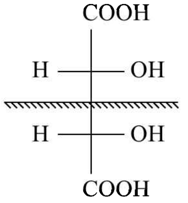
III



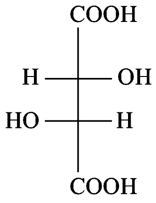
IV

Isomers I and II and also III and IV are non superimposable and hence represent the pair of enatiomers. To show the relation between I and III or II and III or I and IV or II and IV, a new term is introduced - such pairs are called *diastereoisomers*. Two diastereoisomers differ in physical as well as in chemical properties.

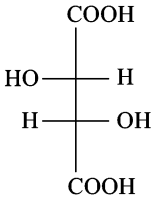
If in a compound two asymmetric carbon atoms have similar substituents. Then the number of isomers is reduced to three instead of expected four. As shown below are the three isomers for tartaric acid.



I
(meso-)



II



III

(Enantiomers)

Isomer I and its mirror image are superimposable and hence it is optically inactive also. Such compounds which contain asymmetric carbon atoms and still are optically inactive are called *meso compounds*. Cause of their optical inactivity is attributed molecular symmetry or internal compensation. Because in this molecule one half of the molecule is mirror image of other half and effect of each asymmetric carbon atom on plane polarized light is mutually cancelled. Such molecules are also said to possess a plane of symmetry.

Some other important terms

Plane of symmetry The plane of symmetry is a plane which bisects the molecule in two equal halves in such a way that one half of the molecules is mirror image of other half.

Centre of symmetry A centre of symmetry is a point such that, if a line is drawn from any element to this point and then extended an equal distance beyond the point, another identical element will be found at the end of the line. It is also known as centre of inversion.

Stereoselective reaction A reaction that yields predominantly one stereoisomer of several diastereomeric possibilities is called a stereoselective reaction.

Stereospecific reaction A reaction in which stereochemically different reactants give stereochemically different products is called a stereospecific reaction.

Resolution of a racemic mixture Process of separating a racemic mixture into the enantiomeric components is known as resolution.

Problems for Practice

1. Which alkane would have only primary and tertiary carbon?

(a) Pentane

(b) 2-Methylbutane

(c) 2, 2-Dimethylpropane

(d) 2,3-Dimethylbutane

2. In a homologous series

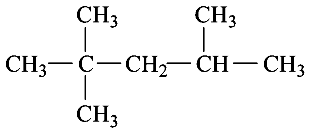
(a) All members show similar chemical properties

(b) There is a difference of CH_2 between two consecutive members of the series

(c) All members can be represented by the same general formula

(d) All statements are correct

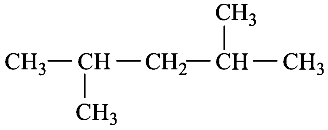
3. The number of tertiary carbon atoms in the following structure is



$$CH_3-\underset{CH_3}{\overset{CH_3}{\underset{|}{\overset{|}{C}}}}-CH_2-\overset{CH_3}{\overset{|}{CH}}-CH_3$$

(a) Four (b) Three
(c) Two (d) One

4. In the structure



$$CH_3-\underset{CH_3}{\underset{|}{CH}}-CH_2-\overset{CH_3}{\overset{|}{CH}}-CH_3$$

the number of carbons are:

(a) One primary, one secondary and one tertiary
(b) Four primary, one secondary and two tertiary
(c) One primary, one secondary, one tertiary and one quaternary
(d) Five primary, one secondary, one tertiary and one quaternary

5. Which of the following compounds has only secondary carbon atoms?

(a) Cyclohexane (b) Methylcyclopentane
(c) 2-Methylpentane (d) Pentane

6. The compound has simplest formula CH_2. To which series does it belong?

(a) Alkanes (b) Cycloalkanes
(c) Alkynes (d) None of these

7. Which of the following alkanes has all the four types of carbons, i.e., 1°, 2°, 3°, and 4°?

(a) $(CH_3)_3CH$
(b) $CH_3—CH(CH_3)—CH(CH_3)—CH_3$
(c) $CH_3—CH(CH_3)—CH_2—CH_3$
(d) $(CH_3)_3C—CH_2—CH(CH_3)_2$

8. Which of the following is an aromatic hydrocarbon?

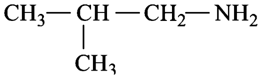
(a) Cyclopentene (b) Cyclohexane
(c) Naphthalene (d) 1,3-Butadiene

9. The IUPAC name of

$CH_3—CHBr—CH_2OH$ is:

(a) 3-Hydroxy 2-bromopropane
(b) 2-Bromopropanol-1
(c) 2-Bromopropanol
(d) 3-Hydroxy isopropyl bromide

10. The IUPAC name of the following compound is



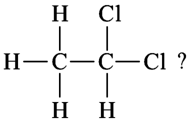
$$CH_3-\underset{CH_3}{\underset{|}{CH}}-CH_2-NH_2$$

(a) 1-Methyl-1-aminopropane
(b) 2-Methyl-1-propanamine
(c) 1-Methyl-1-aminopropane
(d) 2-Aminobutane

11. The IUPAC name of $Cl_3C.CHO$ is:

(a) Trichloroacetaldehyde
(b) 1, 1, 1-Trichloroethanal
(c) 2, 2, 2-Trichloroethanal
(d) Chloral

12. What is the IUPAC name of



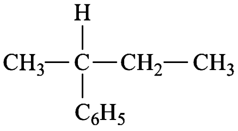
$$H-\underset{H}{\underset{|}{\overset{H}{\overset{|}{C}}}}-\underset{H}{\underset{|}{\overset{Cl}{\overset{|}{C}}}}-Cl\ ?$$

(a) 1, 2-Dichloroethane
(b) 2, 2-Dichloroethane
(c) 1, 1-Dichloroethane
(d) Dichloroethane

13. The IUPAC name of $CH_3OC_2H_5$ is

(a) Ethoxymethane
(b) Methoxyethane
(c) Methyl ethyl ether
(d) Ethyl methyl ether

14. IUPAC name of the following compound is



$$CH_3-\underset{C_6H_5}{\underset{|}{\overset{H}{\overset{|}{C}}}}-CH_2-CH_3$$

(a) 2-Cyclohexylbutane
(b) 2-Phenylbutane
(c) 3-Cyclohexylbutane
(d) 1-(2′-butyl) benzene

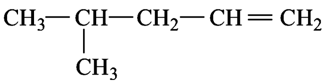
15. IUPAC name of CH_3 CH (CH_3) COOH is

(a) Dimethyl acetic acid
(b) 2-Methylpropanoic acid
(c) Isobutyric acid
(d) 2-Methylbutanoic acid

16. The IUPAC name of CH_3CHO is

(a) Acetaldehyde (b) Methyl aldehyde
(c) Formyl methane (d) Ethanal

17. The IUPAC name of the following compound is



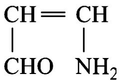
$$CH_3-\underset{CH_3}{\underset{|}{CH}}-CH_2-CH=CH_2$$

(a) 2-Methylpentene
(b) 4-Methylpentene
(c) 4-Methylpentene-1
(d) 2-Methylpentene-4

18. IUPAC name of $(C_2H_5)_2$ $CHCH_2OH$ is

(a) 2-Ethylbutanol-1 (b) 2-Methylpentanol-1
(c) 2-Ethylpentanol-1 (d) 3-Ethylbutanol-1

19. The name of



$$\underset{CHO}{\underset{|}{CH}}=\underset{NH_2}{\underset{|}{CH}}$$

is

(a) 1-Aminoprop-2 enal
(b) 3-Aminoprop-2-enal
(c) 1-Amino-2-formylethene
(d) 3-Amino-1-oxoprop-2-ene

20. The IUPAC name of

$(CH_3)_3C—CH{=}CH_2$ is

(a) 2, 2-Dimethylbut-2-ene
(b) 2, 2-Dimethylpent-3-ene
(c) 3, 3-Dimethylbut-1-ene
(d) Hex-1-ene.

21. IUPAC name of the compound

$ClCH_2—CH_2—CH_2—CH_2—CH_2Br$ is

(a) 1-Chloro-5-bromopentane
(b) 5-Bromo-1-chloropentane
(c) 1-Bromo-5-chloropentane
(d) 5-Chloro-1-bromopentane

22. IUPAC name of the compound

$BrCH_2—CHCl—CHCl_2$ is

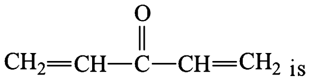
(a) 1-Bromo-2, 3, 3-trichloropropane
(b) 1, 1, 2-Trichloro-3-bromopropane
(c) 3-Bromo-1, 1, 2-trichloropropane
(d) None of these

23. IUPAC name of the compound

$CH_3—CH=CH—C\equiv C—C\equiv CH$ is

(a) Hepta-2-ene-4, 6-diyne
(b) Hepta-5-ene-1,3-diyne
(c) Hepta-1,3-diyne-5-ene
(d) None of these

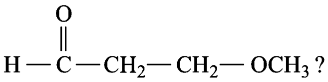
24. The IUPAC name of the compound



$CH_2{=}CH{-}\overset{\overset{O}{\|}}{C}{-}CH{=}CH_2$ is

(a) Divinyl ketone
(b) 3-Keto penta-1,4-diene
(c) 1, 4-Pentadiene-3-one
(d) None of these

25. What is the IUPAC name of

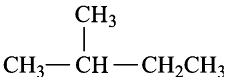


$H-\overset{\overset{O}{\|}}{C}-CH_2-CH_2-OCH_3$?

(a) 2-Formylmethoxyethane
(b) Methoxypropanal
(c) 2-Methoxypropanal
(d) 3-Methoxypropanal

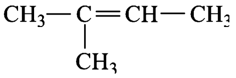
26. 2-Methylbut-2-ene will be represented as

(a)



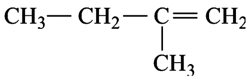
$CH_3-\overset{CH_3}{\overset{|}{CH}}-CH_2CH_3$

(b)



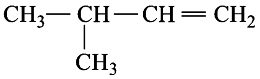
$CH_3-\underset{CH_3}{\underset{|}{C}}=CH-CH_3$

(c)



$CH_3-CH_2-\underset{CH_3}{\underset{|}{C}}=CH_2$

(d)



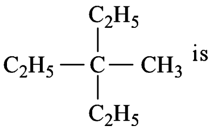
$CH_3-\underset{CH_3}{\underset{|}{CH}}-CH=CH_2$

27. The IUPAC name of

$CH_2=CH—CH_2—C\equiv CH$ is:

(a) Pent-2-en-5-yne
(b) Pent-1-en-4-yne
(c) Pent-4-en-1-yne
(d) Pent-4-en-2-yne

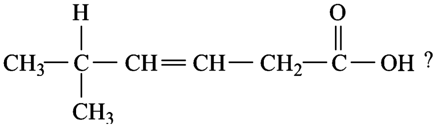
28. The IUPAC name of the compound



$C_2H_5-\underset{C_2H_5}{\underset{|}{\overset{C_2H_5}{\overset{|}{C}}}}-CH_3$ is

(a) 3-Ethyl-3-methylpentane
(b) Triethylmethylmethane
(c) 2, 2-Diethylbutane
(d) 3, 3-Diethylbutane

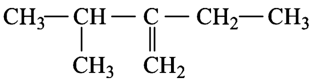
29. What is the correct IUPAC name for



$CH_3-\underset{CH_3}{\underset{|}{\overset{H}{\overset{|}{C}}}}-CH=CH-CH_2-\overset{\overset{O}{\|}}{C}-OH$?

(a) 4-Methyl-2-hexenoic acid
(b) 5-Carboxyl-2-methyl pentene
(c) 4-Isopropyl-3-butenoic acid
(d) 5-Methyl-3-hexenoic acid

30. The IUPAC name for the given compound is



$CH_3-\underset{CH_3}{\underset{|}{CH}}-\underset{CH_2}{\underset{\|}{C}}-CH_2-CH_3$

(a) 2-Ethyl-3-methylbut-1-ene
(b) 2-Isopropylbutene
(c) 2-Methyl-3-ethyl-3-butene
(d) 3-Methyl-3-hexene

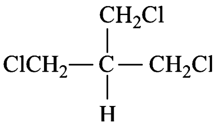
31. IUPAC name of tert. butyl bromide is

(a) 2-Bromo-2-methylpropane
(b) 2-Bromo-3-methylpropane
(c) 1-Bromobutane
(d) 2-Bromo-1-methylpropane

32. Which of the following IUPAC names is correct?

(a) 2-Methyl-3-ethylpentane
(b) 3-Ethyl-2-methylpentane
(c) 2-Ethyl-3-methylpentane
(d) 3-Methyl-2-ethylpentane

33. IUPAC name of



$ClCH_2-\underset{H}{\underset{|}{\overset{CH_2Cl}{\overset{|}{C}}}}-CH_2Cl$ is

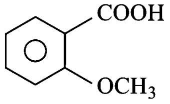
(a) 2-(chloromethyl) 1,3-dichloro propane
(b) Trichloromethyl methane
(c) 1, 2, 3-Trichloropropane
(d) None of these

34. IUPAC name of

$CH_2=CH—CH(CN)—CHO$ is

(a) 2-formyl-3-butene nitrile
(b) 3-oxo-1-butene nitrile
(c) 2-cyano-3-buten-1-al
(d) 3-cyano-1-buten-1-al

35. IUPAC name of



is

(a) 2-methoxybenzoic acid
(b) 2-carboxymethoxide
(c) 2-carboxyphenyl methoxide
(d) 2-methoxy carboxybenzene

36. Acetonitrile in IUPAC system is

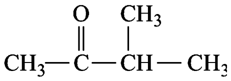
(a) Cyanomethane (b) Cyanoethane

(c) Formyl nitrile (d) Ethane nitrile

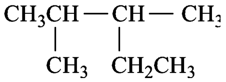
37. IUPAC name of $(CH_3)_2\ CHCH(CH_3)_2$ is

(a) 1, 1, 2, 3-Tetramethyl ethene
(b) 1, 2-Di-isopropylethane
(c) 2, 3-Dimethylbutane
(d) 2, 3, 3-Trimethylbutane

38. The incorrect IUPAC name is

(a)

$CH_3-\overset{\overset{O}{\|}}{C}-\overset{\overset{CH_3}{|}}{CH}-CH_3$

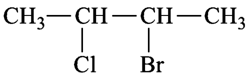
2-Methylbutan-3-one

(b)

$CH_3\underset{\underset{CH_3}{|}}{CH}-\underset{\underset{CH_2CH_3}{|}}{CH}-CH_3$

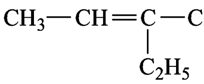
2, 3-Dimethylpentane

(c) $CH_3-C\equiv CCH(CH_3)_2$

4-Methylpent-2-yne

(d)

$CH_3-\underset{\underset{Cl}{|}}{CH}-\underset{\underset{Br}{|}}{CH}-CH_3$

2-Bromo-3-chlorobutane

39. The compoud

$CH_3-CH=\underset{\underset{C_2H_5}{|}}{C}-CH_3$ is called

(a) 1-Ethyl-2-butene
(b) 3-Methyl-3-pentene
(c) 3-Methyl-2-pentene
(d) 2-Ethyl-2-butene

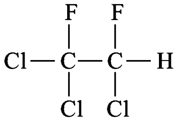
40. The IUPAC name of the compound having the formula $(CH_3)_3$ $C-CH=CH_2$ is:

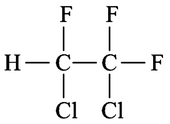
(a) 3, 3-Dimethyl-1-butene
(b) 3-Dimethyl-3-butene
(c) 2-Dimethyl-1-butene
(d) 3-Dimethyl-2-butene

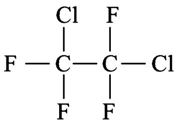
41. The structure of 4-methylpent-2-ene is:

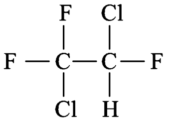
(a) $(CH_3)_2\ CH\ CH_2CH = CH_2$
(b) $(CH_3)_2\ CH-CH = CH-CH_3$
(c) $(CH_3)_2\ CH-CH_2-CH = CH-CH_2$
(d) $(CH_3)_2\ C = CH_2CH_2CH_3$

42. What is the correct chemical formula for 1, 2-dichloro tetrafluoroethane?

(a)

$Cl-\underset{\underset{Cl}{|}}{\overset{\overset{F}{|}}{C}}-\underset{\underset{Cl}{|}}{\overset{\overset{F}{|}}{C}}-H$

(b)

$H-\underset{\underset{Cl}{|}}{\overset{\overset{F}{|}}{C}}-\underset{\underset{Cl}{|}}{\overset{\overset{F}{|}}{C}}-F$

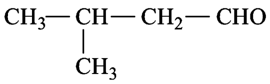
(c)

$F-\underset{\underset{F}{|}}{\overset{\overset{Cl}{|}}{C}}-\underset{\underset{F}{|}}{\overset{\overset{F}{|}}{C}}-Cl$

(d)

$F-\underset{\underset{Cl}{|}}{\overset{\overset{F}{|}}{C}}-\underset{\underset{H}{|}}{\overset{\overset{Cl}{|}}{C}}-F$

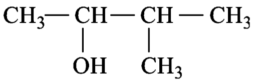
43. Which of the following compound has wrong IUPAC name?

(a) $CH_3CH_2-CH_2COO-CH_2CH_3$

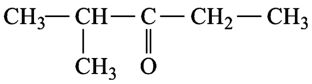
(Ethyl butanoate)

(b)

$CH_3-\underset{\underset{CH_3}{|}}{CH}-CH_2-CHO$

(3-Methylbutanal)

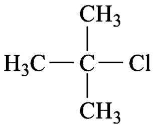
(c)

$CH_3-\underset{\underset{OH}{|}}{CH}-\underset{\underset{CH_3}{|}}{CH}-CH_3$

(2-Methyl-3-butanol)

(d)

$CH_3-\underset{\underset{CH_3}{|}}{CH}-\underset{\underset{O}{\|}}{C}-CH_2-CH_3$

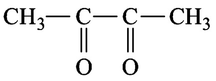
(2-Methyl-3-pentanone)

44. IUPAC name of the compound having the following formula is


$H_3C-\underset{\underset{CH_3}{|}}{\overset{\overset{CH_3}{|}}{C}}-Cl$

(a) tert. Butyl chloride
(b) 1-Chloro-1, 1-dimethylethane
(c) 2-Chloro-2-methylpropane
(d) 2-Methyl-2-propyl chloride

45. The IUPAC name of the following compound

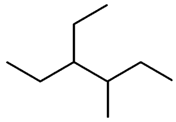

$CH_3-\underset{\underset{O}{\|}}{C}-\underset{\underset{O}{\|}}{C}-CH_3$

(a) Butandione (b) 2, 3-Butanediketone
(c) Butane-2, 3-dione (d) None of these

46. IUPAC name for the compound having formula $C(CH_3)_4$ is

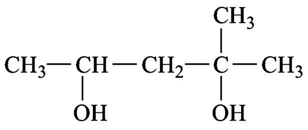
(a) Tetra methyl methane
(b) 1, 1, 1, 1-Tetramethylmethane
(c) 2, 2-Dimethylpropane
(d) None of these

47. IUPAC name for the compound is

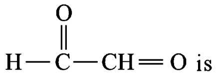


(a) 3-Methyl-4-ethylhexane
(b) 3-Ethyl-4-methylhexane
(c) 4-Ethyl-3-methylhexane
(d) None of these

48. Correct IUPAC name for


$CH_3-\underset{\underset{OH}{|}}{CH}-CH_2-\underset{\underset{OH}{|}}{\overset{\overset{CH_3}{|}}{C}}-CH_3$ is

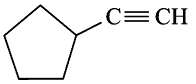
(a) 1, 1-Dimethylbutane-1, 3-diol
(b) 1, 3, 3-Trimethylpropane-1, 3-diol
(c) 2-Methylpentane-2, 4-diol
(d) 1, 3, 3-Trimethyl-1, 3-propanediol

49. The IUPAC name of

$H-\overset{\overset{O}{\|}}{C}-CH=O$ is

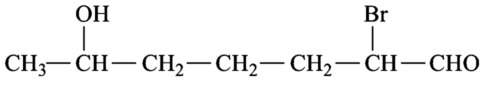
(a) Formylmethanal (b) 1, 2-Ethanedione
(c) Formyl methanoate (d) Ethane-1, 2-dial

50. IUPAC name of Succinic acid is
(a) Ethanedioic acid
(b) Propane-1, 3-dioic acid
(c) Propane-1, 2-dioic acid
(d) Butane-1, 4-dioic acid

51. IUPAC name of the compound



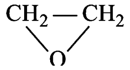
(a) Cyclopentylethyne
(b) Ethynylcyclopentane
(c) Cyclopentyl acetylene
(d) None of these

52. 

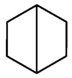
$CH_3-\overset{\overset{\displaystyle OH}{|}}{C}H-CH_2-CH_2-CH_2-\overset{\overset{\displaystyle Br}{|}}{C}H-CHO$ is

(a) 1-Bromo-5-hydroxy-1-hexanal
(b) 1-Bromo-4-hydroxypentanal
(c) 2-Hydroxy-5-bromo-6-hexanal
(d) 2-Bromo-6-hydroxyheptanal

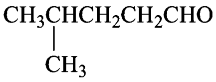
53. The IUPAC name of the compound



(a) Ethene oxide (b) Ethylene oxide
(c) Epoxy ethane (d) Ethylene epoxide

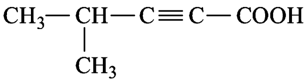
54. The IUPAC name of the compound  is
(a) Cyclo hexane
(b) Bicyclo (2 · 2 · 0) hexane
(c) Bicyclo (0 · 2 · 2) hexane
(d) Bicyclohexane

55. Indicate the wrongly named compound

(a) 

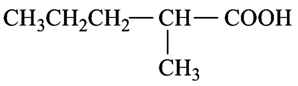
$CH_3\underset{\underset{\displaystyle CH_3}{|}}{C}HCH_2CH_2CHO$

(4-Methylpentan-1-al)

(b) 

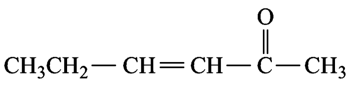
$CH_3-\underset{\underset{\displaystyle CH_3}{|}}{C}H-C\equiv C-COOH$

(4-Methylpent-2-yn-1-oic acid)

(c) 

$CH_3CH_2CH_2-\underset{\underset{\displaystyle CH_3}{|}}{C}H-COOH$

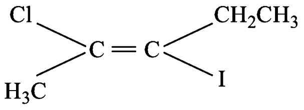
(2-Methylpentanoic acid)

(d) 

$CH_3CH_2-CH=CH-\overset{\overset{\displaystyle O}{\|}}{C}-CH_3$

(Hex-3-en-5-one)

56. IUPAC name for the compound given below is

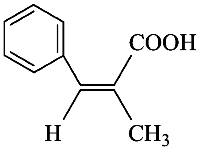


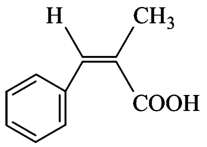
(a) *trans*-2-Chloro-3-iodopent-2-ene
(b) *cis*-2-Chloro-3-iodopent-2-ene
(c) *trans*-3-Iodo-4-chloro-3-pentene
(d) *cis*-3-Iodo-4-chloro-3-pentene

57. The number of isomeric dibromobutanes is
(a) 6 (b) 7
(c) 8 (d) 9

58. The number of isomeric hexanes is
(a) 4 (b) 5
(c) 6 (d) 7

59. The types of isomerism shown by the following two compounds is





(a) optical (b) positional
(c) geometrical (d) no isomerism

60. Which of the following compound is not chiral?
(a) $DCH_2CH_2CH_2Cl$
(b) CH_3CHDCH_2Cl
(c) $CH_3CHClCH_2D$
(d) CH_3CH_2CHDCl

61. Cyanides and isocyanides are isomers of the type
(a) Position isomers
(b) Tautomers
(c) Functional isomers
(d) None of these

62. The number of optical isomers of the compound $CH_3CHBrCHBrCOOH$ is
(a) 0 (b) 1
(c) 3 (d) 4.

63. *n*-Propyl alcohol and isopropyl alcohol are examples of
(a) Position isomerism
(b) Chain isomerism
(c) Tautomerism
(d) Geometrical isomerism

64. Which of the following exhibit geometrical isomerism?
(a) $CH_2{=}CClBr$ (b) $CH_3CH{=}CClBr$
(c) $CH_3CH{=}CCl_2$ (d) $(CH_3)_2C{=}CClBr$

65. Which of the following compounds will show metamerism?
(a) $CH_3-O-COC_3H_7$
(b) $CH_3SC_2H_5$
(c) CH_3-O-CH_3
(d) $CH_3-O-C_2H_5$

66. Which of the following compounds will exhibit geometrical isomerism?
(a) 2-Butene
(b) 2-Butyne
(c) 2-Butanol
(d) Butanal

67. 1-Chloropropane and 2-Chloropropane are
(a) Position isomers
(b) chain isomers

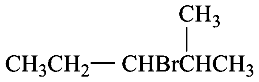
(c) functional isomers
(d) optical isomers

68. The total number of possible isomeric trimethylbenzene is

(a) 2 (b) 3
(c) 4 (d) 6

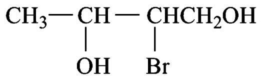
69. Which one of the following compounds will not exhibit optical isomerism?

(a) $CH_3CH(OH)Br$
(b) $CH_3CH(OH)CH_3$
(c)



$CH_3CH_2-CHBr\overset{CH_3}{\overset{|}{C}}HCH_3$

(d)



$CH_3-\underset{OH}{\underset{|}{CH}}-\underset{Br}{\underset{|}{C}}HCH_2OH$

70. Which of the following will exhibit geometrical isomerism?

(a) 1-Phenyl-2-butene
(b) 3-Phenyl-1-butene
(c) 2-Phenyl-1-butene
(d) 1, 1-Dipheny-1-propene

71. Which of the following is an isomer of propanal?

(a) acetone (b) propane
(c) propanol (d) propionic acid

72. Which is the chiral molecule?

(a) CH_3Cl (b) CH_2Cl_2
(c) $CHBr_3$ (d) $CHClBrI$

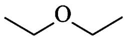
73. Which of the following molecule contains asymmetric carbon atom?

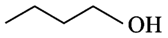
(a) $CH_3CHClCOOH$
(b) CH_3CH_2COOH
(c) $ClCH_2CH_2COOH$
(d) $Cl_2CHCOOH$

74. Optically active isomers but not mirror images are called

(a) Enantiomers (b) Mesomers
(c) Tautomers (d) Diastereoisomers

75. What type of isomerism is shown by the following compounds





(a) Metamerism (b) Positional
(c) Functional (d) Tautomerism

ANSWERS

1. (d)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (b)
11. (c)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (c)
21. (c)	**22.** (c)	**23.** (b)	**24.** (c)	**25.** (d)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (a)
31. (a)	**32.** (b)	**33.** (a)	**34.** (c)	**35.** (a)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (a)
41. (b)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (d)
51. (a)	**52.** (d)	**53.** (c)	**54.** (b)	**55.** (d)	**56.** (a)	**57.** (d)	**58.** (b)	**59.** (d)	**60.** (a)
61. (c)	**62.** (d)	**63.** (a)	**64.** (b)	**65.** (a)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (b)	**70.** (a)
71. (a)	**72.** (d)	**73.** (a)	**74.** (d)	**75.** (c)					

Basic Concepts of Organic Chemistry

22

BRIEF REVIEW OF THE CONCEPTS

ELECTRON DISPLACEMENT EFFECTS

There are four types of electron displacements, frequently observed in organic molecules which are briefly described below.

(a) Inductive effect This is a permanent displacement of electrons which arises due to the difference in the electronegativity of two atoms. Consider a chain of carbon atoms in which one terminal carbon atom is joined to a chlorine atom. Chlorine is more electronegative than carbon. Therefore the electron pair forming the covalent bond between the chlorine atom and C_1 will be displaced towards the chlorine atom. This causes the chlorine atom to acquire a small negative charge and C_1 a small positive charge.

$$- C_3 - C_2 - \overset{\delta+}{C_1} - \overset{\delta-}{Cl}$$

Since C_1 is slightly positively charged, it will attract towards itself the electron pair forming the covalent bond between C_1 and C_2. This will cause C_2 to acquire a small positive charge but the charge will be smaller than that on C_1 because the effect of the chlorine atom has been passed through C_1 to C_2. Similarly C_3 acquires a positive charge which will be less than that on C_2. This type of electron shift or displacement along a carbon chain is known as the inductive effect. Inductive effect is permanent and decreases rapidly as the distance from the source (C1) increases. From the practical point of view, it may be ignored after the third carbon atom. The inductive effect may be represented with the help of arrows as shown below

$$—C_4 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow Cl$$

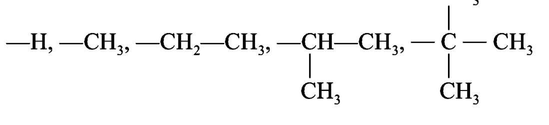
The symbol I is used to denote inductive effect. For comparison of inductive effect of various groups, hydrogen is chosen as the standard.

The groups which are electron withdrawing, they are said to cause – I effect and electron releasing groups are said to cause + I effect.

Groups with decreasing — I effect.

$$– NO_2, – CN, — COOH, – F, – Cl, – Br, – I, – OCH_3, – OH, – C_6H_5$$

Groups with increasing + I effect.



$$—H, —CH_3, —CH_2—CH_3, —\underset{\displaystyle CH_3}{\underset{|}{CH}}—CH_3, —\overset{\displaystyle CH_3}{\overset{|}{\underset{\underset{\displaystyle CH_3}{|}}{C}}}— CH_3$$

(b) Electromeric effect: This is a temporary effect in which there is complete transfer of shared pair of electrons to one or the other atom joined by a multiple bond, i.e., a double or a triple bond. The electromeric effect is brought into play only at the requirement of the attacking reagent and takes place almost instantaneously. Let us consider a case where two atoms A and B are joined by two shared pairs of electrons. If one of the two shared pairs of electrons is transferred to one of the atoms, say B, at the time of attack of the reagent, one bond is lost. This gives the atom A, a positive charge while B having taken complete control of the electron pair, gets a negative charge.

$$A \overset{\curvearrowright}{=} B \quad A^{+} – B^{-}$$

The curved arrow shows the displacement of the shared pair beginning at the position where the pair was present originally and ending where the pair has migrated.

Removal of the attacking reagent causes the charged molecule to revert to its original electronic condition.

$$A = B \underset{\text{reagent removed}}{\overset{\text{reagent added}}{\rightleftharpoons}} A^{+} – B^{-}$$

Inductive effect plays a vital role in knowing the direction of the transfer of shared electron pair to one of the atoms. Thus

(i) In a carbonyl group $>C = 0$ present in aldehydes or ketones, the displacement is towards the oxygen atom. This is due to inductive effect of oxygen which is more strongly electron-attracting than carbon.

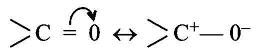
$$>C \overset{\curvearrowright}{=} 0 \rightarrow >C^{+} – 0^{-}$$

(ii) In propene, the displacement of shared electron pair is towards the carbon atom which is away from methyl group. This is again due to the inductive effect of methyl group which is electron repelling.

$$\underset{\text{Propene}}{CH_3 — CH \overset{\curvearrowright}{=} CH_2} \rightarrow CH_3 — CH^{+} — CH_2^{-}$$

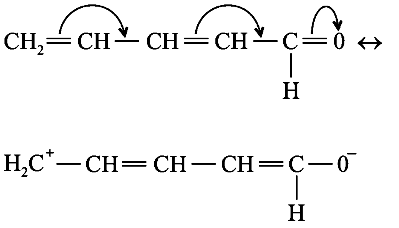
(c) Mesomeric (conjugative) effect: It is another case of electron displacement in the molecules which cause permanent polarization. This electron displacement is relayed through π-electrons of multiple bonds in the carbon chain of the molecule. Unlike the inductive effect which operates in molecules having σ-bonds, the mesomeric effect operates in only those systems which have an extended chain with conjugate π-bonds (alternate σ and π bonds). For this reason this effect is also referred as *conjugative effect.*

When an electron donating or electron withdrawing group is conjugated with a π bond or a set of alternately arranged σ and π bonds, the electron displacement is transmitted through π-electrons in the chain. Consider a carbonyl group ($>C = 0$) linked with a conjugated system as $CH_2 = CH - CH = CH - CH = 0$. The π-electrons of the carbon-oxygen bond get displaced towards oxygen, the more electronegative atom, giving the resonance structure as



$$>C = 0 \leftrightarrow >C^+—0^-$$

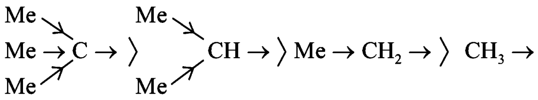
In this whole molecule, this polarisation can be transmitted further via the π-electrons as



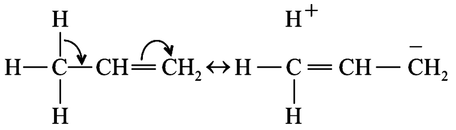
$$H_2C^+—CH=CH—CH=C(H)—0^-$$

The difference between the transmission through a conjugated system of π-bonds and the inductive effect along a σ-bond chain lies in the fact that this effect suffers very little decrease in intensity. The terminal carbon is almost as positive as the first carbon.

(d) Hyperconjugation: The relative magnitude of the inductive effect of alkyl groups is normally found to follow the order:



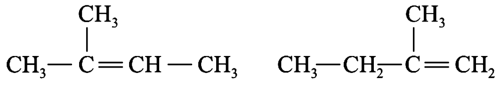
However, when the alkyl groups are attached to an unsaturated system, e.g., a double bond or a benzene nucleus, this order is found to be disturbed, and in the case of some conjugated systems actually reversed. It thus appears that alkyl groups are capable, under some circumstances, of giving rise to electron release by a mechanism different from the inductive effect. This has been explained as proceeding by an extension of the conjugative or mesomeric effect, delocalization taking place by the involvement of σ-electrons of C-H bond.



I II

This effect, called *hyperconjugation,* has been used successfully to explain a number of otherwise unconnected phenomena. It would be emphasised that it is not suggested that a proton actually becomes free in (I), for if it moved from its original position, one of the conditions necessary for delocalisation to occur would be controverted.

Hyperconjugation can explain the greater thermodynamic stability of alkenes in which the double bonds are not terminal , e.g., (III) compared with isomeric compound in which double bond is terminals e.g., (IV). In (III) there are nine 'hyperconjugable' α-hydrogen atoms, compared with only five in (IV)



III IV

Reaction Intermediates

Depending upon their nature, the attacking reagents my be classified into three categories:

(a) free radicals

(b) Electrophilic reagents or electrophiles

(c) Nucleophilic reagents or nucleophiles

These reagents are obtained by the cleavage of a covalent bond. A covalent bond between two atoms may be broken in following two different ways:

(i) *Homolysis or homolytic cleavage*

It involves the cleavage of a bond between two atoms in such a way that each atom carries one electron of the shared pair.

$$X - Y \longrightarrow X^\bullet + Y^\bullet$$

This leads to the formation of neutral species containing an unpaired electron. Such species are known as **free radicals** which are highly reactive. They have sp^2 hybridized central carbon and overall shape is planer. The unpaired electron is present in the *p*-orbital which is perpendicular to the plane of hybridized orbitals.

The decreasing order of stability of free radicals is

Benzyl > Allyl > 3° > 2° > 1° > methyl > vinyl

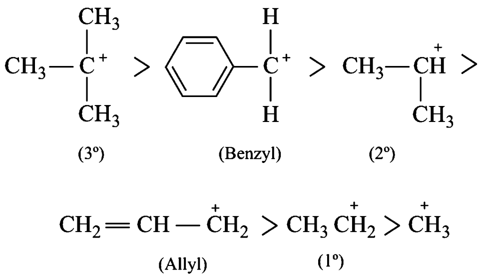
(ii) *Heterolysis or heterolytic cleavage:*

It involves the cleavage of a bond between two atoms in such a way that both the electrons of the shared pair are taken away by one of the atoms. This leads to the formation of charged species.

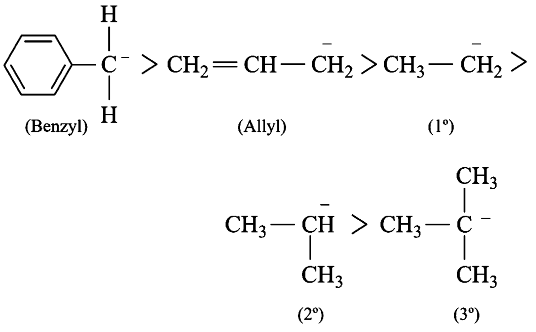
$$X - Y \longrightarrow X^+ + Y^- \text{ (or } X^- + Y^+)$$

Positively charged reagents are likely to attack the electron rich centre in the substrate molecule, hence they are called *electrophiles*. Whereas the negatively charged reagents prefer the electron deficient site in substrate and are called *nucleophiles*. Some other electron deficient, neutral species also behave as electrophiles, for example BF_3, $Al\ Cl_3$ etc. Similarly, many neutral species having a lone pair of electrons behave as nucleophiles, for example NH_3, H_2O, $R\text{-}NH_2$ etc.

A group containing a positively charged carbon atom is known as **carbocation**. In this the positively charged carbon atom is sp^2 hybridized and has planar geometry. The order of stability of various carbocations is



If in the species formed, the negative charge is on the carbon atom, then they are called as **carbanions**. In such species the carbon is sp^3 hybridized and has a pyramidal geometry. The order of stability of various carbanions is



One more type of reactive intermediate formed are known as **carbenes**. Such species are neutral, contain a di-valent carbon and two electrons. For example,: CH_2, : CCl_2 etc.

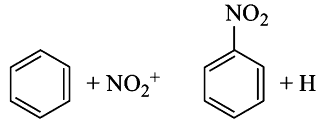
If the two unshared electrons are present in one orbital as a pair then the corbon is sp^2 hybridized and it is called a **singlet carbene**. If the two electrons are present in two different orbitals then the carbon is sp hybridized and it is called a **triplet carbene**. In general, a triplet carbene is more stable than a singlet carbene.

TYPES OF ORGANIC REACTIONS

Depending upon the change which the substrate molecules suffer, the reactios are classified into following four types:

(a) *Substitution reactions:* The replacement of an atom or group from a molecule by different atom or group is known as substitution or displacement reaction, e.g.,

$$CH_3OH + HBr \longrightarrow CH_3Br + H_2O$$



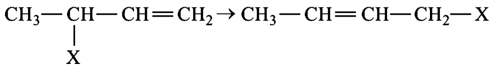
(b) *Addition reactions:* Reactions in which atoms or group of atoms are added to a molecule (i.e., there is simply net gain of the reagent atoms in the product molecule) are known as addition reactions. This type of reaction occurs only when there is a centre of unsaturation in the molecule which is generally due to the presence of a multiple bond between two atoms, e.g.,

$$CH_2 = CH_2 + HBr \longrightarrow CH_3 - CH_2Br$$

(c) *Elimination reactions:* These reactions are essentially the reversal of addition reactions and involve loss of atoms or group of atoms from a molecule to form a multiple linkage. Most commonly, loss of atoms or groups occurs from adjacent carbon atoms to yield an olefin, e.g.,

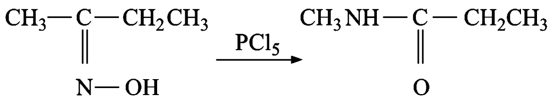
$$CH_3CH_2Cl \longrightarrow CH_2 = CH_2 + HCl$$

(d) *Rearrangements:* Rearrangement reactions involve the migration of a functional group to another position in the molecule e.g.,



$$CH_3-\underset{X}{\underset{|}{CH}}-CH=CH_2 \rightarrow CH_3-CH=CH-CH_2-X$$

It may also involve the reshuffling of the sequence of atoms forming the basic carbon skeleton of the molecule to form a product with new structure, e.g.



$$CH_3-\underset{N-OH}{\underset{\|}{C}}-CH_2CH_3 \xrightarrow{PCl_5} CH_3NH-\underset{O}{\underset{\|}{C}}-CH_2CH_3$$

Methyl ethyl ketoxime N-methyl propanamide

Sometimes an overall complex reaction may fall into more than one of the above types, but the individual steps which make up complex reaction can always be placed in one of the categories listed above.

Each of the above types of reaction can occur by the involvement of any of the three types of reagents mentioned earlier. For example a substitution reaction occurring by the involvement of free radicals will be called *free radical substitution reaction.* Similarly an addition reaction occurring by the involvement of nucleophiles will be called *nucleophilic addition* reaction.

PURIFICATION OF ORGANIC COMPOUNDS

One of the important job of an organic chemist is to determine the structural formulae of various compounds, whether isolated from natural sources or synthesised in the laboratory. Organic compounds obtained from both of these sources are generally impure. Hence it becomes necessary that they should be purified before the analysis could be carried out. The various methods employed for effecting purification depend upon the nature of the compound and the impurities present in it. The processes commonly used for the separation and purification of organic compounds are briefly described below.

(i) *Crystallization:* The process of crystallization is based upon the difference in the solubilities and is most commonly employed for the purification of solids. The impure solid is dissolved in a suitable solvent by heating and then filtered while hot to remove the insoluble impurities. The filtrate on cooling yields the crystals of the pure compound.

(ii) *Fractional crystallization:* This method is used to separate the mixture of two compounds which are soluble in the same solvent. The mixture is dissolved in the hot solvent and allowed to cool. Most of the less soluble compound and a little of the more soluble compound crystallize out first. A second crop of crystals will contain more of the more soluble compound. The whole process is repeated separately with the two fractions of the crystals till both the compounds are obtained in pure state.

(iii) *Sublimation:* Some organic compounds pass directly from solid to vapour state on heating . The vapours give back the solids on cooling. This process is known as sublimation. A volatile compound may be separated from non-volatile impurities by this method. Some of the substances which can be readily purified by this process are naphthalene, camphor, benzoic acid etc.

(iv) *Distillation:* In this method the impure liquid is heated to form its vapours and the impurities are left behind. The vapours are condensed to form the liquid again which is collected in a separate vessel. A mixture of two liquids can be separated by this method if their boiling points differ at least by 20°.

(v) *Fractional distillation:* Fractional distillation is used for separating a mixture of two or more liquids whose boiling points are quite near. In this method the vapours of the liquids are allowed to pass through a fractionating column which has a large cooling surface. Due to which, first the vapours of liquid with low boiling point are obtained and then that of high boiling point are obtained. This method is used for separating different components of petroleum.

(vi) *Distillation under reduced pressure:* Some liquids decompose on boiling at atmospheric pressure. Such liquids cannot be purified by simple distillation. They are generally purified by distillation under reduced pressure because lowering the pressure on the surface of a liquid causes lowering in its boiling point. For example, glycerol is purified by this method.

(vii) *Steam distillation:* In steam distillation, impure compounds are distilled in a current of steam. Mixture of steam and vapours of the compound is obtained when the sum of their vapour pressures becomes equal to the atmospheric pressure. This temperature will obviously be less than the normal boiling point of that compound. Mixture of the vapours is condensed and then the organic compound is separated from water. This method is very useful for purification of amines. For example, aniline can be distilled by this method at 96.5° C whereas its normal boiling point is 184°C.

(viii) *Extraction with a solvent:* Organic compounds are generally more soluble in organic solvents than in water. Hence, they are separated from other water soluble impurities by extraction with a suitable organic solvent. The solvent used should be immiscible with water. Ether is an ideal organic solvent for this purpose. Some other common solvents are chloroform, benzene, ethyl acetate etc.

(ix) *Chromatography.* Chromatography is based on the general principle of distributing the components of a mixture of organic compounds between two phases–a stationary phase and a moving phase. The stationary phase can be a solid or liquid supported on a solid, while the moving phase is a liquid or a gas. When the stationary phase is a solid, the basis of separation is adsorption and when it is a liquid, the basis is partition.

Hence chromatography can be defined as the technique for the separation of a mixture of compounds where the separation is brought about by the differential movement of the individual compounds through a porous medium under the influence of a moving solvent. The different chromatographic techniques used are column chromatography, thin layer chromatography, paper chromatography and ion-exchange chromatography etc. Paper chromatography is an important and useful class of partition chromatography. In this technique, the stationary phase is considered to be made up of water molecules bound to the cellulose network (inert support) of the paper. The mobile phase, known as the developing solvent consists of either one solvent or a mixture of different solvents. Separation of the mixture into pure compounds takes place by the partitioning of different compounds between these two liquid phases (one is water on paper and other is the organic solvent used as developer). The mobile phase travels by capillary action through the paper.

(x) *Chemical methods:* The chemical methods are employed in cases of mixtures of substances that are chemically different. The separation of basic and acidic compounds of coal tar is a well-known example of this type as in this concentrated sulphuric acid is employed. Of the numerous other cases of chemical purification and separation of organic substances, the following may be cited as examples

(a) Purification of acetylene by passing the gas through an ammoniacal solution of cuprous chloride and decomposing the product (brown precipitate of cuprous acetylide) with hot, dilute hydrochloric acid so as to regenerate acetylene.

(b) Separation of acetic acid from "pyroligneous acid" of the wood-distillation industry, which is based on the conversion of the acid into calcium acetate and decomposition of the latter with concentrated hydrochloric acid.

(c) The preparation of pure methyl alcohol, free from acetone, is based on the conversion of the alcohol into the crystalline derivative methyl oxalate, which is then decomposed by boiling with a solution of sodium hydroxide.

QUALITATIVE ANALYSIS OF THE ORGANIC COMPOUNDS

Once an organic compound is obtained in pure state, the next step is the detection of the elements present in it. This is called qualitative anaysis of an organic compound. In general the organic compound may contain two or more than two elements from carbon, hydrogen, oxygen, nitrogen, sulphur, halogen, phosphorus and metals. Carbon is present in all organic compounds and hydrogen is also there in most of them.

(i) ***Detection of Carbon & Hydrogen:***

If the compound under examination is known to be definitely organic, there is no need to test for the presecne of carbon and hydrogen. However, these can be tested together by a common test.

A small quanity of the given dry organic compound is intimately mixed with 2-3 times the quantity of dry fine copper (II) oxide, CuO. The mixture is then heated in a dry hard glass test tube fitted with a delivery tube having a bulb. In the bulb is placed anhydrous copper sulphate (white) and the other end of the delivery tube is passed through lime water contained in another test tube.

Carbon and hydrogen of the organic compounds are oxidized by copper oxide to carbon dioxide and water, respectively. Carbon dioxide thus produced will turn lime water milky and water vapours coming out of the test tube will turn anhydrous copper (II) sulphate into hydrated copper (II) sulphate ($CuSO_4.\ 5H_2O$), which is blue in colour.

Turning of lime water milky and of anhydrous copper sulphate into blue $CuSO_4.\ 5H_2O$, show the presence of carbon and hydrogen, respectively in the compound.

$$C + 2CuO \longrightarrow CO_2 + 2Cu$$

$$CO_2 + Ca(OH)_2 \longrightarrow \underset{\text{(milky)}}{CaCO_3\downarrow} + H_2O$$

$$2H + CuO \longrightarrow H_2O + Cu$$

$$\underset{\text{(white)}}{CuSO_4} + 5H_2O \longrightarrow \underset{\text{(blue)}}{CuSO_4}\ .\ 5H_2O$$

(ii) ***Detection of Nitrogen, Sulphur and Halogens:*** These elements are detected by Lassaigne's test. In this method the organic compound is treated with sodium metal. The elements present in the organic compounds are converted into ionic salts. Thus, the nitrogen in presence of carbon gets converted into cyanide ions (CN^{θ}), Sulphur into sulphide ions (S^{2-}) and halogens into halide ions (X^-).

$$Na + C + N \rightarrow NaCN$$

$$2Na + S \rightarrow Na_2S$$

$$Na + X \rightarrow NaX\ (X = Cl, Br, I)$$

The above formed salts are all water soluble and the solution obtained is called Lassaigne's Extract (LE).

To a portion of LE, small amount of solid ferrous sulphate is added and then acidified with dil . H_2SO_4. A prussian blue precipitate or colouration indicates the presence of nitrogen.

$$Fe^{2+} + 2NaCN \longrightarrow Fe(CN)_2 + 2Na^+$$

$$Fe\ (CN)_2 + 4NaCN \longrightarrow Na_4\ [Fe(CN)_6]$$

$$3Na_4\ [Fe(CN)_6] + 4\ Fe^{3+} \longrightarrow \underset{\text{(prussian blue)}}{Fe_4[Fe(CN)_6]_3} + 12Na^+$$

The presence of sulphur in organic compound can be tested by applying the following two tests to Lassaigne's extract.

In one test, sodium nitroprusside solution is added to LE-appearance of purpule colour indicates the presence of sulphur.

$$Na_2S + \underset{\text{sodium nitroprusside}}{[Na_2Fe(CN)_5NO]} \rightarrow \underset{\text{purple coloured complex}}{Na_4[Fe(CN)_5NOS]}$$

Another portion of LE is acidified with acetic acid and few drops of lead acetate solution are added. Formation of black precipitate of lead sulphide shows the presence of sulphur

$$Na_2S + (CH_3\ COO)_2\ Pb \rightarrow \underset{\text{Black ppt}}{PbS} + 2CH_3COONa$$

In order to test the halogens, a portion of LE is boiled with concentrated nitric acid till its volume is reduced to one-half. To this solution, silver nitrate solution is added

- Formation of a white precipitate soluble in ammonium hydroxide indicates chlorine

- Formation of a pale yellow precipitate soluble in ammonium hydroxide with difficulty indicates bromine
- Formation of a yellow precipitate insoluble in ammonium hydroxide indicate iodine

$$NaX + AgNO_3 \longrightarrow AgX + NaNO_3$$

[X = Cl(white), Br (pale yellow), I(yellow)]

Presence of bromine and iodine can also be tested by another test called layer test. In this test LE is acidified with dil HNO_3 and then 1mL of CCl_4 or CS_2 is added which will form a separate layer. Now chlorine water is added dropwise with constant shaking, A yellow colour in organic layer indicates the presence of bromine and a violet layer indicates the presence of iodine. This test is based upon the fact that chlorine oxidises bromide ions and iodide ions present in LE to Br_2 and I_2 respectively. They dissolve in organic layer to impart yellow or violet colour

$$2Br^- + CI_2 \longrightarrow Br_2 + 2CI^-$$
$$2I^- + CI_2 \longrightarrow I_2 + 2CI^-$$

The presence of halogens in any organic compound can also be detected by beilstein test. In this test a red hot copper wire is touched to the compound and then taken to the flame of bunsen burner. If any halogen is present then it imparts green colour to the flame.

(iii) ***Detection of phosphorus and Arsenic***

For testing phosphorus or arsenic, organic compound is fused with a mixture of sodium carbonate and potassium nitrate, when phosphorus and arsenic get oxidised to sodium phosphate and sodium arsenate respectively. The fused mass is extracted with distilled water, boiled and filtered.

To a little filtrate concentrated nitric acid and freshly prepared solution of ammonium molybdate are added and then heated. A canary yellow precipitate indicates the presence of phosphorus or arsenic in the organic compound

$$Na_2CO_3 + As + (O) \longrightarrow Na_3AsO_4 + CO_2$$
$$Na_2CO_3 + P + (O) \longrightarrow Na_3PO_4 + CO_2$$
$$PO_4^{3-}\ (aq) + 12(NH_4)_2\ MoO_4\ (aq) \longrightarrow$$
$$(NH_4)_3\ PO_4 . 12MoO_3\ (s)$$

(Ammonium phosphomolybdate)

To distinguish between phosphorus and arsenic, the above filtrate portion is heated with silver nitrate. A chocolate precipitate confirms arsenic

$$Na_3AsO_4(aq) + 3AgNO_3(aq) \longrightarrow 3NaNO_3(aq) + \underset{\text{Chocolate}}{Ag_3AsO_4(aq)}$$

(iv) ***Detection of Metals:*** The presence of a metal in organic compound is tested by igniting it in nickel or platinum crucible in free air. Most of the elements present in the organic compound burn leaving a reside of the metal or its oxide. The residue can be dissolved in hydrochloric acid and then various tests characteristics of the metal ions can be performed with the solution by the usual scheme employed for inorganic salts.

Quantitative Analysis of the Organic Compounds

The esitmation of percentage composition of the various elements present in a compound is the next step in determining the formula of the compound. The different methods employed for the estimation of various elements are described below.

(i) *Estimation of carbon and hydrogen:*

These two elements are always estimated together by **Liebig's combustion method**. A weighed amount of the compound is heated strongly with excess of copper oxide in an atmosphere of air or oxygen. The constituents hydrogen and corbon are thus, oxidised to water and carbon dioxide, which are collected separately and weighed. The percentage of carbon and hydrogen in the compound can be calculated as given below.

$$\%C = \frac{12}{44} \times \frac{\text{Mass of } CO_2 \text{ formed}}{\text{Mass of the substance}} \times 100$$

$$\%H = \frac{2}{18} \times \frac{\text{Mass of } H_2O \text{ formed}}{\text{Mass of the substance}} \times 100$$

(ii) *Estimation of nitrogen*

Two methods commonly employed for the estimation of nitrogen are

(a) *Dumas' Method:* This method can be applied to all organic compounds containing nitrogen. The method is based on the principle that if an organic compound containing nitrogen is heated with copper oxide, free nitrogen and oxides of nitrogen are formed along with other products (carbon dioxide, water vapour, etc.)

The oxides of nitrogen are reduced to free nitrogen on passing over heated copper and the whole of nitrogen is collected over KOH solution. The volume of nitrogen collected is measured and from this the percentage present in the compound is calculated.

$$\%N = \frac{28}{22400} \times \frac{\text{Volume of nitrogen at NTP}}{\text{Mass of the compound}} \times 100$$

(b) *Kjeldahl's Method:* This method is commonly employed for estimating nitrogen in food materials and fertilizers. Though this method is simpler than Dumas' method, it is not applicable to all nitrogenous organic compounds.

In this method, nitrogen in the compound is quantitatively converted into ammonium sulphate by heating with concentrated sulphuric acid. Ammonium sulphate thus obtained is estimated from which the percentage of nitrogen is calculated.

Accurately weighed organic compound (0.5 to 1 g) is taken in a Kjeldahl's flask. It is then mixed with about 20 mL of concentrated sulphuric acid, 10 g of potassium sulphate and a little copper sulphate.

Now the flask is fitted with a loose glass bulb and heated. Potassium sulphate raises the boiling point of sulphuric acid, whereas copper sulphate acts as a catalyst. In the beginning, the liquid in the flask turns brown but the heating is continued till the liquid becomes clear. This indicates that the whole of nitrogen has been converted into ammonium sulphate.

The clear liquid is transferred into another flask and heated with sodium hydroxide. Sodium hydroxide reacts with ammonium sulphate to liberate ammonia.

$$(NH_4)_2\ SO_4 + 2NaOH \longrightarrow Na_2SO_4 + 2NH_3 + 2H_2O$$

The liberated ammonia is bubbled through a known volume of standard acid. Some of the acid is thus neutralized.

When the formation of ammonia stops, the remaining acid is titrated against a standard alkali. Knowing the volume of acid neutralized by ammonia, the percentage of nitrogen is calculated

$$\%N = \frac{14}{1000} \times \frac{\text{Vol. of acid used} \times \text{Normality of the acid}}{\text{Mass of the compound}} \times 100$$

(iii) *Estimation of Halogens*

Carius Method: A known mass of the organic substance containing halogen is heated with fuming nitric acid along with few crystals of silver nitrate in a sealed tube. The silver halide is formed which is separated, washed dried and weighed. From the mass of silver halide obtained, the percentage of halogen is calculated.

%Halogen

$$= \frac{\text{Atomic mass of halogen} \times \text{Mass of silver halide obtained} \times 100}{\text{Molar mass of silver halide} \times \text{Mass of compound taken}}$$

Carius method does not give satisfactory results with iodine as silver iodide is slightly soluble in nitric acid and some iodine is also produced even in the presence of excess of silver nitrate. Moreover, for highly halogenated aromatic compounds, the results are not accurate.

(iv) *Estimation of Sulphur.* Sulphur is also estimated by *Carius method.* In this case, the organic compound is heated only with nitric acid. Sulphur present in the compound is thus oxidized to sulphuric acid which is treated with barium chloride to precipitate barium sulphate. The precipitate of barium sulphate is washed, dried and weighed. From the weighed mass of barium sulphate obtained, the percentage of sulphur is calculated.

$$\% \text{ S} = \frac{32}{133} \times \frac{\text{Mass of BaSO}_4}{\text{Mass of the compound}} \times 100$$

(v) *Estimation of Phosphorus.* Phosphorus is estimated like sulphur, i.e., organic compound is heated with fuming nitric acid. Phosphorus in the compound is thus oxidized to phosphoric acid which is precipitated by adding magnesia mixture. The precipitate of magnesium ammonium phosphate is ignited to obtain magnesium pyrophosphate.

$$\underset{\text{Magnesium ammonium phosphate}}{2MgNH_4PO_4} \longrightarrow \underset{\text{Magnesium pyrophosphate}}{Mg_2P_2O_7} + 2NH_3 + H_2O$$

Magnesium pyrophosphate is weighed and the percentage of phosphorus calculate.

$$\% \text{ P} = \frac{62}{222} \times \frac{\text{Mass of Mg}_2\text{P}_2\text{O}_7}{\text{Mass of the compound}} \times 100$$

Determination of Empirical Formula of the Compound

The empirical formula of a compound is the simplest formula which expresses the relative number of atoms of constituent elements present in the molecule. Empirical formula is calculated from the perecntage composition and the steps involved in the calculation are as follows:

(i) The percentage of each constituent element is divided by its atomic mass. The different number so obtained are the relative numbers of different atoms present in the molecule.

(ii) The relative numbers of different atoms obtained in step (i) are divided by the lowest one amongst them. This gives the simple atomic ratio of atoms present in the molecule.

(iii) The numbers expressing the atomic ratio may or (may not) be whole numbers. If, it is not so, then they are multiplied by suitable common factor to convert each of them to whole numbers. Minor fraction are neglected.

(iv) Lastly, the symbols of each element present are written in a line side by side and the respective numbers giving the atomic ratio put as subscripts at the lower right corner of each respective element. This gives the empirical formula. It can be made clear by the following example.

Example An organic compound on analysis gave the following percentage composition of the elements.

Carbon 39.13%, hydrogen 8.64% and oxygen 52.23%. Calculate the empirical formula of the organic compound.

Solution

Table 22.1

Element	*At. mass*	*Percentage*	*Relative number of atoms*	*Atomic ratio*	*Simple whole number ratio*
Carbon	12	39.13	39.13/12 = 3.26	$\frac{3.26}{3.26} = 1$	3
Hydrogen	1	8.64	8.64/1 = 8.64	$\frac{8.64}{3.26} = 2.65$	8
Oxygen	16	52.23	52.23/16 = 3.26	$\frac{3.26}{3.26} = 1$	3

Empirical formula of the compound is $C_3H_8O_3$.

Determination of Molecular Formula of the Compound

The molecular formula of a compound expresses the actual number of atoms of various elements present in a molecule of the compound.

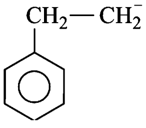
Molecular formula may be either the same as the empirical formula or a simple multiple of it.

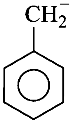
Molecular formula = (Empirical formula) × *n*, where *n* is a whole number. The value of *n* is obtained by dividing the molecular mass by empirical formula mass.

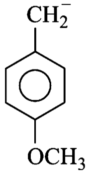
$$n = \frac{\text{Molecular mass}}{\text{Empirical formula mass}}$$

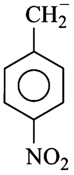
Problems for Practice

1. Glycerol decomposes at its boiling point. The purification of glycerol can be affected by
   (a) Crystallization
   (b) Simple distillation
   (c) Distillation under reduced pressure
   (d) Fractional crystallization
2. Which of the following method can be applied to purify aniline containing non volatile impurities?
   (a) Simple distillation
   (b) Fractional distillation
   (c) Crystallization
   (d) Steam distillation
3. A mixture of acetone and methanol can be separated by
   (a) Vacuum distillation
   (b) Steam distillation
   (c) Fractional distillation
   (d) None of these
4. In Duma's method for determining the nitrogen content of an organic compound, the nitrogen content is determined in the form of
   (a) Gaseous NH_3 (b) NaCN
   (c) Gaseous N_2 (d) $(NH_4)_2SO_4$
5. Two solids A and B have appreciably different solubilities in water but their melting points are very close. The mixture of A and B can be separated by
   (a) Sublimation
   (b) Fractional crystallization
   (c) Distillation
   (d) Specific method
6. Beilstein test is used for the detection of
   (a) Cl (b) Br
   (c) I (d) All of these
7. Consider the following carbocations:
   (I) $C_6H_5\overset{+}{C}H_2$ (II) $C_6H_5CH_2\overset{+}{C}H_2$
   (III) $C_6H_5\overset{+}{C}HCH_3$ (IV) $C_6H_5\overset{+}{C}(CH_3)_2$.
   The correct sequence for the stability of these is
   (a) II < I < III < IV (b) II < III < I < IV
   (c) III < I < II < IV (d) IV < III < I < II.
8. The blue colour developed during the Lassaigne's test for nitrogen is due to
   (a) $Fe_3[Fe(CN)_6]_4$ (b) $Fe_4[Fe(CN)_6]_3$
   (c) $K_4[Fe(CN)_6]$ (d) $Fe(CNS)_3$
9. The separation of mixture of two compounds by chromatographic technique is based upon
   (a) Differential solubilities
   (b) Different densities
   (c) Different absorption
   (d) Different adsorption
10. Which one of the following is strong nucleophile?
    (a) OH^- (b) CN^-
    (c) $C_2H_5O^-$ (d) $H—C{\equiv}C^-$
11. The formation of cyanohydrin from a ketone is an example of
    (a) Electrophilic addition
    (b) Nucleophilic addition
    (c) Nucleophilic substitution
    (d) Electrophilic substitution
12. Acetaldehyde is the rearrangement product of
    (a) Methyl alcohol (b) Allyl alcohol
    (c) Vinyl alcohol (d) All are correct
13. The reaction
    $(CH_3)_3CBr \xrightarrow{H_2O} (CH_3)_3C-OH$ is
    (a) elimination reaction
    (b) substitution reaction
    (c) free radical reaction
    (d) addition reaction
14. Which of the following behave both as a nucleophile and as an electrophile?
    (a) $H_3C—C{\equiv}N$ (b) $H_3C—OH$
    (c) $H_2C{=}CH—CH_3$ (d) $H_3C—NH_2$
15. Intermediate involved in Reimer-Tiemann reaction is
    (a) Carbocation (b) Carbanion
    (c) Carbene (d) Free radical
16. The most stable carbanion among the following is

(a) 

(b) 

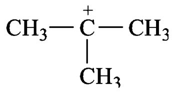
(c) 

(d) 

17. A gaseous hydrocabon has 85% C and its vapour density is 28. The formula of hydrocarbon is
    (a) C_4H_8 (b) C_2H_2
    (c) C_2H_4 (d) C_2H_6
18. 60 g of organic compound on analysis gave following results C = 24 g; H = 4 g and O = 32 g.
    The empirical formula of compound is
    (a) CH_2O_2 (b) C_2H_2O
    (c) $C_2H_2O_4$ (d) CH_2O
19. The substance which can be used as an adsorbent in column chromatography is
    (a) Na_2O (b) NaCl
    (c) Al_2O_3 (d) Alum
20. Which of the following is least stable?
    (a) $CH_3-CH_2-\overset{+}{C}H_2$

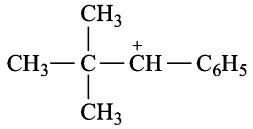
(b) $CH_3 - \overset{+}{C}H - CH_2 - CH_3$

(c)



$CH_3-\overset{+}{\underset{CH_3}{C}}-CH_3$

(d)



$CH_3-\overset{CH_3}{\underset{CH_3}{C}}-\overset{+}{C}H-C_6H_5$

21. The reaction,

$CH_2 = CH - CH_3 + HBr \longrightarrow CH_3CHBr - CH_3$ is

(a) Nucleophilic addition
(b) Electrophilic substitution
(c) Electrophilic addition
(d) Free radical addition

22. Electrophile in the case of chlorination of benzene in presence of $FeCl_3$ is

(a) Cl^+ (b) Cl^-
(c) Cl (d) $FeCl_3$

23. The violet colour in the Lassaigne's test of sulphur is due to

(a) $FeCl_3$ (b) $Na_2\,[Fe(CN)_6]$
(c) $Na_4[Fe(CN)_5NOS]$ (d) None of these

24. The function of boiling the sodium extract with concentrated nitric acid before testing halogens is

(a) To make solution clear
(b) To destroy CN^- and S^{2-} ions
(c) To make the solution acidic
(d) None of these

25. In which of the following, resonance will be possible?

(a) $CH_2 = CH - CH_2 - CHO$
(b) $CH_2 = CH- CH = O$
(c) CH_3COCH_3
(d) $CH_2 = CH - CH_2 - CH = CH_2$

26. Which of the following is correct regarding the – I effect of the substituents?

(a) $-NR_2 < -OR < -F$
(b) $-NR_2 > -OR < -F$
(c) $-NR_2 < -OR > -F$
(d) $-NR_2 > -OR > -F$

27. Catalyst used in Kjeldahl's method for estimation of nitrogen is

(a) Sodium (b) Cupper sulphate
(c) Mercury (d) Potassium sulphate

28. Which species represents the electrophile in aromatic nitration?

(a) NO_2^- (b) NO_2^+
(c) NO_2 (d) NO_3^-

29. The addition of HCN to a carbonyl compound is an example of

(a) nucleophilic substitution
(b) electrophilic addition
(c) nucleophilic addition
(d) electrophilic substitution

30. Which of the following is not a nucleophile?

(a) H_2O (b) CH_3OH
(c) H_2 (d) NH_3

31. Which of the following is an electrophile?

(a) H_2O (b) NH_3
(c) $AlCl_3$ (d) $C_2H_5NH_2$

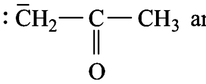
32. Distillation under reduced pressure is used to purify liquids which

(a) Are explosives
(b) Are highly volatile
(c) Decompose below their b.p.
(d) Have high boiling points

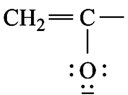
33. Which of the following will not give the Lassaigne's test for nitrogen?

(a) $NH_2 - NH_2$
(b) $C_6H_5 - N = N - C_6H_5$
(c) CH_3CONH_2
(d) CH_3CN

34.



$:\bar{C}H_2-\underset{O}{\underset{\|}{C}}-CH_3$ and



$CH_2=\underset{:\ddot{O}:^-}{C}-CH_3$ are

(a) Resonating structures
(b) Tautomers
(c) Geometrical isomers
(d) Optical isomers

35. Which one of the following acids would you expect to be the strongest?

(a) $I - CH_2COOH$ (b) $Cl - CH_2COOH$
(c) $Br - CH_2COOH$ (d) $F - CH_2COOH$

36. Amongst the following the most basic compound is

(a) Benzylamine (b) Aniline
(c) Acetanilide (d) *p*-nitroaniline

37. Analysis of an organic compound gave 74.0% C, 8.65% H and 17.3% N. What is the empirical formula of compound?

(a) C_5H_8N (b) $C_{10}H_{12}N$
(c) C_5H_7N (d) $C_{10}H_{14}N$

38. Heterolytic cleavage of a covalent bond gives only,

(a) Cationic species (b) Anionic species
(c) Both the above (d) Free radicals

39. The reaction

$$CH_3CH_2Br + OH^- \longrightarrow CH_3CH_2OH + Br^-$$

is an example of ;

(a) Electrophilic addition
(b) Electrophilic substitution
(c) Nucleophilic addition
(d) Nucleophilic substitution

40. Camphor and sand can be separated by

(a) Sublimation (b) Distillation
(c) Steam distillation (d) None of these

41. The most suitable method of separating mixture of o and p-nitrophenol is

(a) Chromatography (b) Steam distillation
(c) Distillation (d) None of these

42. Which of the following contains three pairs of electrons in the valence shell?

(a) Carbocations (b) Carbanions

(c) Free radicals (d) None

43. Heterolysis of carbon-chlorine bond produces

(a) Two free radicals

(b) Two carbocations

(c) Two carbanions

(d) One cation and one anion.

44. Electrophilic reagents are

(a) Electron pair donors

(b) Lewis acids

(c) Odd electron molecules

(d) None of the above

45. The most stable free radical among the following is

(a) $C_6H_5CH_2\dot{C}H_2$ (b) $C_6H_5\dot{C}HCH_3$

(d) $CH_3\dot{C}H_2$ (d) $CH_3\dot{C}HCH_3$

46. Which of the following is the most stable carbocation?

(a) $CH_3CH_2^+$ (b) $(CH_3)_2\overset{+}{C}H$

(d) $(CH_3)_3\overset{+}{C}$ (d) $C_6H_5\overset{+}{C}H_2$

47. The most stable carbocation is

(a) Methyl carbocation

(b) Primary carbocation

(c) Secondary carbocation

(d) Tertiary carbocation

48. Homolytic fission of C – C bond in ethane gives an intermediate in which carbon is

(a) sp^3-hybridized (b) sp^2-hybridized

(c) sp-hybridized (d) sp^2d hybridized

49. Which of the following is not a nucleophile?

(a) CN^- (b) OH^-

(c) NH_3 (d) BF_3

50. Steam distillation is applied for the separation of those compound which are

(a) Steam volatile and soluble in water

(b) Steam volatile and decompose in water

(c) Steam volatile and insoluble in water

(d) Capable of chemical reaction with steam

51. The process of differential extraction is based upon:

(a) Different solubilities

(b) Different molecular masses

(c) Different boiling points

(d) None of these

52. Which of the following is an electrophilic reagent?

(a) RO^- (b) BF_3

(c) NH_3 (d) ROH

53. The typical reactions of olefinic bond are

(a) Electrophilic substitution reactions

(b) Electrophilic addition reactions

(c) Nucleophilic substitution reactions

(d) Nucleophilic addition reactions

54. The homolytic fission of a hydrocarbon results in the formation of

(a) carbocations

(b) Free radicals

(c) Carbanions

(d) Carbenes

55. Which of the following is an example of elimination reaction?

(a) Chlorination of methane

(b) Dehydration of ethanol

(c) Nitration of benzene

(d) Hydroxylation of ethylene.

ANSWERS

1. (c)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (d)	**10.** (d)
11. (b)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (d)	**17.** (a)	**18.** (d)	**19.** (c)	**20.** (a)
21. (c)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (c)
31. (c)	**32.** (c)	**33.** (a)	**34.** (a)	**35.** (d)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (a)
41. (b)	**42.** (a)	**43.** (d)	**44.** (b)	**45.** (b)	**46.** (d)	**47.** (d)	**48.** (b)	**49.** (d)	**50.** (c)
51. (a)	**52.** (b)	**53.** (b)	**54.** (b)	**55.** (b)					

23 Hydrocarbons

BRIEF REVIEW OF THE CONCEPTS

Hydrocarbons These are organic compounds which contain carbon and hydrogen only. Natural gas, petroleum and coal are the main natural sources for hydrocarbons.

Natural Gas It contains about 90% methane and the remaining 10% being a mixture of higher gaseous hydrocarbons. Most of it is used as fuel without purification.It is used under the name CNG as fuel for vehicles also.

Petroleum It is obtained as viscous, dark coloured oil. It consists of a mixture of gaseous, liquid and solid hydrocarbons. The hydrocarbons present in crude oil consist of C_1 to C_{40} carbon atoms. It is separated into various fractions by a process known as refining of petroleum.

Table 23.1 Important fractions of petroleum refining

	Fraction	*Boiling range (K)*	*Approximate Carbon Content*
1.	Gaseous Hydrocarbons	Below 293	C_1–C_4
2.	Petroleum ether	293-343	C_5–C_8
3.	Gasoline	343-473	C_6–C_{11} and cycloalkanes
4.	Kerosene	448-598	C_{10}–C_{16} and aromatics
5.	Diesel Oil or Gas Oil	573-673	C_{15}–C_{18}
6.	Lubricating oil	above 673	C_{17}–C_{20}
7.	Paraffin wax	above 673	C_{20}–C_{30}
8.	Asphalt and coke	Non-volatile residue	—

Gasoline as Engine Fuel Gasoline finds its use as fuel for internal combustion engines in automobiles and aeroplanes. In the operation of such engines, the downstroke of the piston draws gasoline and air into the cylinder where it is compressed by the forward motion of the piston.The compressed mixture is then ignited by a spark which burns the fuel and the resulting expansion develops a pressure that forces the piston back and provides the driving power.

Knocking It is due to the precombustion of fuel. If the fuel gets self ignited due to increase in temperature before it is properly compressed, then the effect is like a hammer blow on the cylinder. This gives rise to a sharp metallic ping known as knocking. It has been found that the tendency of different hydrocarbons to produce knocking follows the sequence, straight chain > cyclic hydrocarbons > aromatics. The efficiency of a gasoline sample as a fuel is represented by a number called octane number.

Octane number It is defined as the volume percentage of iso-octane in a mixture of iso-octane and *n*-heptane that matches the gasoline sample in knocking properties. The fuel samples with higher octane number are better. These days fuels with octane number above 100 are also available.

Reforming Gasoline with high octane number can be prepared by enhancing the percentage of branched chain alkanes, alkenes and aromatics. This is accomplished by passing the straight chain alkanes over a suitable catalyst at a temperature above 673 K and high pressure, the process is called reforming.

Diesel oil as Engine Fuel The working of a diesel engine is different from that of a petrol engine. In diesel engine air is highly compressed and the temperature rises to 773 K or more. At this stage, oil is injected in the form of spray. The knocking characteristics of diesel oil are expressed in terms of cetane number.

Cetane number The cetane number is defined as the percentage by volume of cetane ($C_{16}H_{34}$) in a mixture of cetane and α-methyl naphthalene which is equivalent in ignition quality to diesel under test.

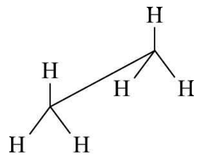
Coal Coal is a rock composed of different kinds of organic matter and carbon which vary in their proportions in samples obtained from different places. They are classified as *peat*, *lignite*, *bituminous* or *anthracite* coals. Coal, as mined from earth, has a sieve carbon structure in which complex aromatic molecules are entrapped. It is distilled in absence of air (at 1270-1570 K) and three useful fractions are obtained viz. *coal gas*, *coal tar* and *coke*. Coke is used as reducing agent in metallurgical processes and also as a fuel. Coal gas is also used as fuel in steel industry. Cool tar, a black viscous liquid, is a source of many aromatic compounds which are isolated by the process of distillation.

Table 23.2 Important fractions of coal tar

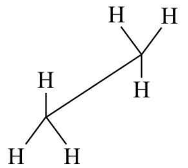
	Fraction	*Boiling range (K)*	*Chief components*	*Industrial uses*
1.	Crude light oil	up to 443	Benzene, toluene, xylenes	Solvents
2.	Middle oil or carbolic oil	443-503	Phenol and cresols	Drugs and dyes
3.	Heavy oil or Creosote oil	503-543	Naphthalene	
4.	Green oil or Anthracene oil	543-633	Anthracene and Phenanthrene	Dyes
5.	Pitch	Left as residue		Road surfacing and water proofiing

Conformations of Hydrocarbons Different arrangements of atoms in a molecules which are obtained by rotation across single bonds are called *conformations*. The number of conformations for any compound is always infinite, although names are given to those conformations only which are generally more stable or least stable. We will discuss here some aspects of this phenomenon with reference to ethane, *n*-butane and cyclohexane.

Let us consider the molecule of ethane, $CH_3–CH_3$ and rotate one methyl group across C—C bond axis. We get two extreme cases. These are called the eclipsed conformation (I) and staggered conformation (II). In the eclipsed conformation, the hydrogens of the one carbon are directly behind those of other. In the staggered conformation the hydrogens of the two carbon atoms are as far apart as possible.

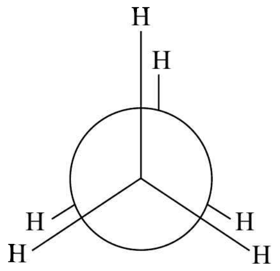


(Eclipsed)
(I)

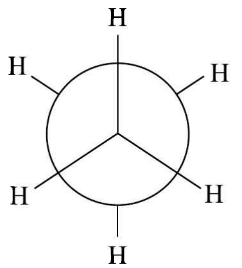


(Staggered)
(II)

These conformation may be represented by the newman's projection formulas also.



(Eclipsed)

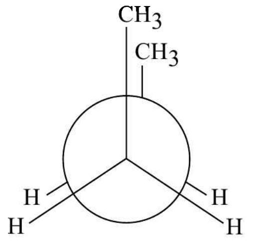


(Staggered)

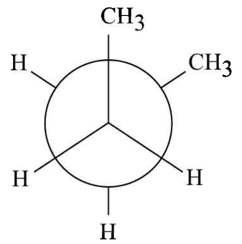
It has been observed that the staggered conformation is more stable than the eclipsed by about 12.6 kJ mole^{-1}. There could be many other arrangements possible in between the eclipsed form and staggered form, which are called *skew* forms. From the difference in energy of two extreme conformations we conclude that there is an energy barrier to rotation around the single bond. But this energy barrier is not large enough to prevent rotation. Even at ordinary temperature, the molecules possess sufficient thermal or kinetic energy to overcome this barrier thus conformations keep on changing from one form to the other.

The molecule of butane is some what more complex for conformational studies. If we consider the rotation around the single bond between C_2 and C_3, we have a molecule similar to ethane but with a methyl group in place of one hydrogen atom on each carbon,

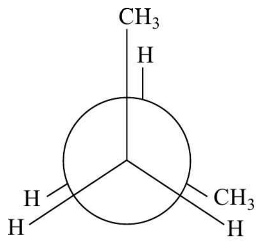
In constrast with ethane, *n*-butane can have several different staggered and eclipsed conformations. Four of the important conformations are shown below.



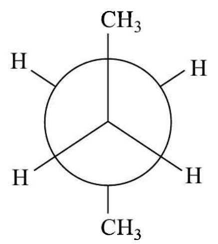
I
(Fully eclipsed)



II
(Gauche)



III
(Partially eclipsed)

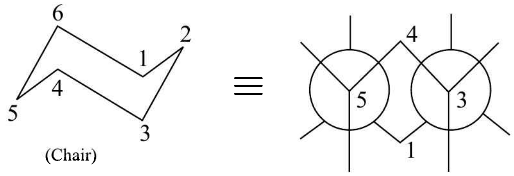


IV
(Anti)

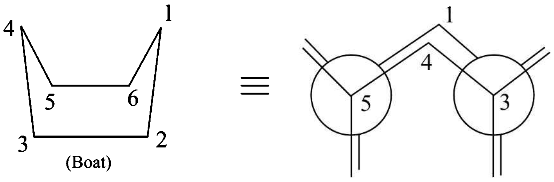
Of the two eclipsed conformation (I and III), the one having larger methyl groups eclipsing (I) will naturally experience more repulsive forces than the one where a methyl group is faced with a hydrogen atom (III). The III is more stable than I. In likewise manner, of the two staggered conformations (II and IV), anti conformation (IV) will be more stable than the gauche (II). So from this we can say that at any time, butane will have highest percentage of anti conformation and least percentage of fully eclipsed conformation.

Due to the possibility of rotation around carbon-carbon single bonds in them, cyclo alkanes can also exist in various conformation. Of the possible conformations of cyclohexane, the three important ones are shown below.

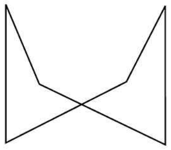
Out of these conformations, chair form is most stable, whereas boat is less stable and twist boat lies in between. From this we can also conclude that cycloalkanes are not planar ring structures rather they are puckered ring structures.



(Chair)



(Boat)



(Twist Boat)

Methods for the Preparation of Alkanes

(i) **By catalytic hydrogenation of alkenes and alkynes** Alkenes and alkynes react with hydrogen in presence of Pd, Pt or Ni to give alkanes.

$$H_2C = CH_2 + H_2 \xrightarrow{\text{Pd, Pt or Ni}} CH_3{-}CH_3$$

(ii) **By reduction of alkyl halides** Alkyl halides are reduced into alkanes on treatment with zinc and hydrochloric acid

$$R - X \xrightarrow{\text{Zn/HCl}} R - H + HX$$

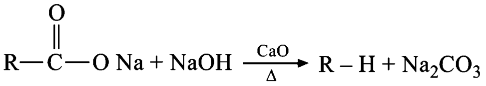
(iii) **By the hydrolysis of Grignard reagents** Alkly halides react with magnesium metal to form Grignard reagents which on reaction with water give alkanes

$$R - X + Mg \rightarrow R - Mg - X \xrightarrow{H_2O} RH + Mg(OH)X$$

(iv) **By Wurtz's reaction** This reaction involves the treatment of alkyl halides with sodium metal. It brings about the union of two alkyl groups thus producing the higher alkane.

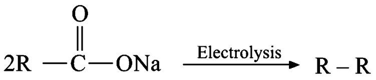
$$R - \boxed{X + 2Na + X} - R \rightarrow R - R + 2Na\,X$$

(v) **By decarboxylation of carboxylic acids** The sodium salts of carboxylic acids on heating strongly with soda lime (NaOH + CaO) give alkanes.



$$R{-}\overset{\overset{O}{\|}}{C}{-}O\,Na + NaOH \xrightarrow[\Delta]{CaO} R - H + Na_2CO_3$$

(vi) **Kolbe's synthesis** When a concentrated solution of sodium salt of carboxylic acid is electrolysed, an alkane is formed



$$2R{-}\overset{\overset{O}{\|}}{C}{-}ONa \xrightarrow{\text{Electrolysis}} R - R$$

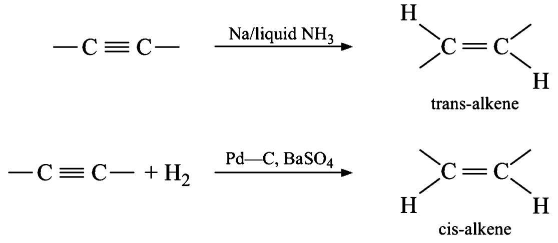
(vii) **By the hydrolysis of carbides** Methane can very conveniently be obtained by treating aluminium carbide or beryllium carbide with water. But the gas obtained by this method is generally contaminated with hydrogen and acetylene.

$$Al_4C_3 + 12H_2O \longrightarrow 4\,Al(OH)_3 + 3CH_4$$

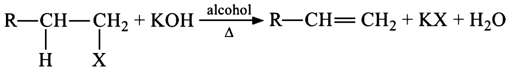
$$Be_2C + 4H_2O \longrightarrow 2\,Be(OH)_2 + CH_4$$

Methods for the Preparation of Alkenes

(i) **By partial reduction of alkynes** Reduction of an alkyne to the double bond stage can be carried out with sodium or lithium in liquid ammonia which yields trans-alkenes. The reduction to alkene stage with hydrogen can also be carried out using **Lindlar's catalyst** which consists of palladium spread over carbon and poisoned with barium sulphate. By this reduction cis-alkenes are specifically obtained.



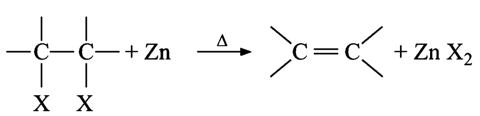
(ii) **By Dehydrohalogenation of Alkyl Halides** When an alkyl halide is heated with an alcoholic solution of potassium hydroxide, a molecule of hydrogen halide is eliminated and an alkane is formed.



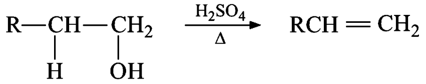
$$R{-}\underset{H}{\underset{|}{CH}}{-}\underset{X}{\underset{|}{CH_2}} + KOH \xrightarrow[\Delta]{\text{alcohol}} R{-}CH{=}CH_2 + KX + H_2O$$

In this reaction, whenever the possibility exists, more highly substituted alkene is the major product such as dehydrobromination of 2-bromobutane yields 2 -butene as the major product. This generalisation is known as **Saytzeff rule**.

(iii) **By Dehalogenation of Vicinal Dihalides** When heated with zinc in alcoholic solution, a vicinal dihalide (compound having two halogen atoms on adjacant carbon atoms) gets dehalogenated to form an alkene



(iv) **By Dehydration of Alcohols** When an alcohol is heated in the presence of concentrated sulphuric acid, a molecule of water is eliminated and an alkene is formed.



$$R{-}\underset{H}{\underset{|}{CH}}{-}\underset{OH}{\underset{|}{CH_2}} \xrightarrow[\Delta]{H_2SO_4} RCH{=}CH_2$$

The mechanism of dehydration of alcohols involves the formation of a carbocation ion as an intermediate. The reaction is believed to proceed in three steps. In the first step alcohol combines with a proton (from acid) to give protonated alcohol which loses a molecule of water in second step to give the carbocation.

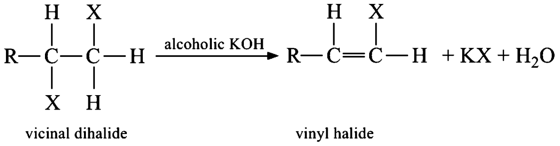
Finally, the carbocation loses a proton to yield the alkene. Here also, if possibility exists, the loss of proton takes place from that side so that to yield the highly substituted alkene.

Dehydration of an alcohol may also be brought by passing the vapours of alcohol over heated alumina (Al_2O_3) at 673 K. Other dehydrating agents which are sometimes used, are phosphorus pentoxide (P_2O_5) and ortho phosphoric acid (H_3PO_4).

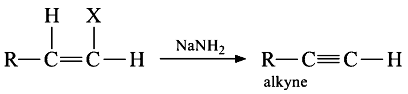
Methods for the Preparation of Alkynes

(i) **By dehydrohalogenation of vicinal dihalides** Alkynes are obtained by treating vicinal dihalides with alcoholic potassium hydroxide followed by reaction with sodium amide ($NaNH_2$).

This elimination reaction proceeds in two steps. The product of the first HX elimination is a substituted vinyl halide. The vinyl halides are unreactive and alcoholic KOH is not strong enough to remove the second HX molecule so a stronger base ($NaNH_2$) is employed.



$$\underset{\text{vicinal dihalide}}{R-\underset{X}{\overset{H}{C}}-\underset{H}{\overset{X}{C}}-H} \xrightarrow{\text{alcoholic KOH}} \underset{\text{vinyl halide}}{R-\overset{H}{C}=\overset{X}{C}-H} + KX + H_2O$$



$$R-\overset{H}{C}=\overset{X}{C}-H \xrightarrow{NaNH_2} \underset{\text{alkyne}}{R-C\equiv C-H}$$

(ii) **From smaller alkynes** Terminal alkynes on treatment with sodium metal in liquid ammonia give the sodium salt. Such as acetylene reacts to give sodium acetylide.

$$H-C\equiv C-H + Na \xrightarrow{\text{liquid } NH_3} H-C\equiv C^{-}Na^{+} + \frac{1}{2}H_2$$

$$R-C\equiv C-H + Na \xrightarrow{\text{liquid } NH_3} R-C\equiv C^{-}Na^{+} + \frac{1}{2}H_2$$

These sodium salts react with primary alkyl halides to form higher alkynes

$$R-C\equiv C^{-}Na^{+} + R_1-X \longrightarrow R-C\equiv C-R_1 + NaX$$

$$H-C\equiv C^{-}Na^{+} + C_2H_5-Br \longrightarrow H-C\equiv C-C_2H_5 + NaBr$$

(iii) **Special methods for the preparation of acetylene:** The smallest alkyne, ethyne (acetylene) is obtained by treating calcium carbide with water.

$$CaC_2 + 2H_2 \longrightarrow H-C\equiv C-H + Ca(OH)_2$$

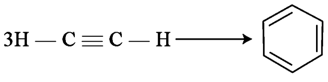
Acetylene is also produced when electric spark is produced between carbon electrodes in the atmosphere of hydrogen gas (Berthelot's Synthesis). It is also obtained by the reaction of haloforms with silver metal.

$$C + H_2 \xrightarrow{\text{electric arc}} H-C\equiv C-H$$

$$2CHX_3 + 6Ag \longrightarrow H-C\equiv C-H + 6AgX$$

Preparation of Aromatic Hydrocarbons

Two major sources of aromatic compounds are coal tar and petroleum. Open chain hydrocarbons are also converted into aromatic hydrocarbons by the process of reforming. Benzene itself can be obtained by passing acetylene through a heated iron tube (673 K).



Physical Properties of Hydrocarbons

The straight chain alkanes from C_1 to C_4 are gases; C_5 to C_{17} alkanes are liquids; C_{18} and larger alkanes are solid waxes. Amongst alkenes and alkynes, the C_2 to C_4 members are gases; the C_5 to C_{18} compounds are liquids and those above C_{18} are solids. It has been observed that the boiling point rises by 20-30° for each addition of a carbon atom in all the series of hydrocarbons. Branched-chain isomer has a lower boiling point than the corresponding straight chain isomer. The increase in melting point with increasing molecular weight is not regular. Hydrocarbons are miscible with each other or dissolve in non-polar solvents like CCl_4 or less polar solvents like ether and chloroform etc. They are insoluble in water. The density of hydrocarbons increase with increase in molecular mass and reaches to a maximum of 0.8 gcm^{-3}. Hence all the hydrocarbons are lighter than water and float on it.

Properties of alkanes

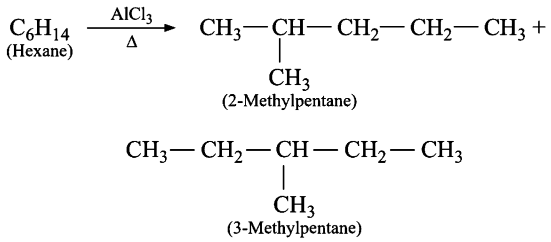
(i) **Halogenation** The halogens react with alkanes in presence of sunlight to give alkyl halides. Reaction with iodine is very slow, whereas fluorine reacts explosively hence fluorination is generally not carried out.

$$R-H + X_2 \xrightarrow{h\nu} R-X + HX$$

(ii) **Combustion** Alkanes combine with oxygen to form carbon dioxide, water and a large amount of heat is produced. The combustion of alkanes may be shown by taking an example

$$C_3H_8 + 5O_2 \longrightarrow 3CO_2 + 4H_2O + \text{heat.}$$

(iii) **Isomerization** Alkanes isomerise to form branched chain alkanes on heating with anhydrous aluminium chloride.



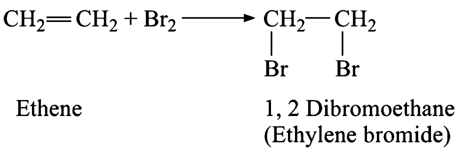
$$\underset{\text{(Hexane)}}{C_6H_{14}} \xrightarrow[\Delta]{AlCl_3} \underset{\text{(2-Methylpentane)}}{CH_3-\underset{CH_3}{CH}-CH_2-CH_2-CH_3} + \underset{\text{(3-Methylpentane)}}{CH_3-CH_2-\underset{CH_3}{CH}-CH_2-CH_3}$$

(iv) **Cracking** It is the process of breaking of bigger hydrocarbon molecules (usually kerosene and diesel oil) into simpler low boiling hydrocarbon of lower molecular mass. Two methods are commonly employed for this purpose. The first is known as **thermal cracking** or **pyrolysis** which involves heating under high pressure. The second method is called **catalytic cracking** in which higher alkanes are passed over silica or alumina kept at 600-650 K.

Properties of alkenes Alkenes undergo mainly electrophilic addition reactions.

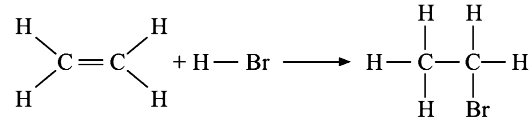
(i) **Addition of hydrogen** Refer to preparation of alkanes.

(ii) **Addition of Halogens:** Alkenes readily react with a molecule of chlorine or bromine to give vicinal dihalides, such as



$$\underset{\text{Ethene}}{CH_2=CH_2} + Br_2 \longrightarrow \underset{\text{1, 2 Dibromoethane (Ethylene bromide)}}{\underset{Br}{CH_2}-\underset{Br}{CH_2}}$$

(iii) **Addition of Hydrogen Halides** Alkenes readily undergo addition reactions with hydrogen halides (HCl, HBr or HI) to from alkyl halides.

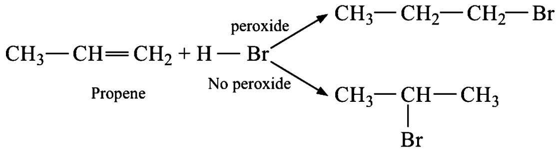


Ethene Bromoethene

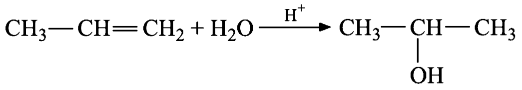
If the alkene is not symmetrical, then there is a possibility of formation of two products. It has been found that actually one product always predominates.

Based on the result of many observations, the Russian chemist Vladimir Markovnikov in 1869 summarized his findings in a statement known as **Markovnikov's rule.** It states that in addition of HX to unsymmetrical alkenes, the hydrogen goes to that double-bond carbon which bears the large number of hydrogen atoms.

It has also been observed that if the addition of hydrogen bromide to alkenes is carried out in the presence of organic peroxide (compounds of the type R—O—O—R), the product formed are the reverse of those expected from Markovnikov's addition. This type of product is referred to as an anti-Markovnikov's product, and the reaction is called an anti-Markovnikov addition. This phenomenon is known as peroxide effect. The Markovinkov and anti-markovnikov addition of HBr to propene is shown below:



(iv) **Addition of Water: Hydration of Alkenes** Water adds to the alkenes in the presence of acids to yield alcohols. This addition also follows Markovnikov's rule. Such as propane on addition of water in presence of few drops of sulphuric acid yields 2-propanol



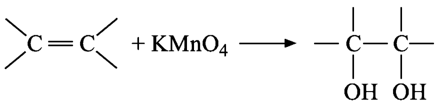
$$CH_3{-}CH{=}CH_2 + H_2O \xrightarrow{H^+} CH_3{-}\underset{\underset{\displaystyle OH}{|}}{CH}{-}CH_3$$

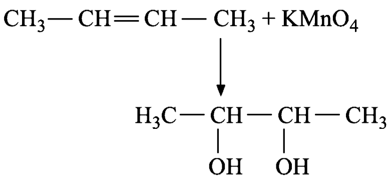
2-propanol

(v) **Polymerization** One important property of alkenes is their ability to form gaint molecules called *polymers* and the process is called *polymerization.* It involves the reaction of alkene units with themselves under the influence of various catalysts.

(vi) **Oxidation of Alkenes**

(a) **Hydroxylation** Alkenes are oxidized by a cold, dilute solution of potassium permanganate to yield dihydroxy compounds (glycols)

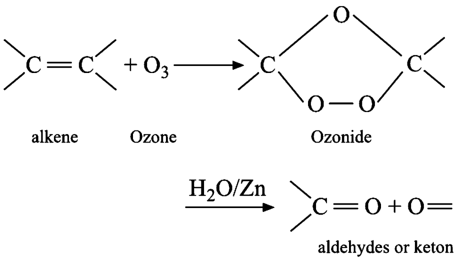




$$CH_3{-}CH{=}CH{-}CH_3 + KMnO_4 \longrightarrow H_3C{-}\underset{\underset{\displaystyle OH}{|}}{CH}{-}\underset{\underset{\displaystyle OH}{|}}{CH}{-}CH_3$$

The reaction is useful as a diagnostic test for the presence of carbon – carbon double bond because it involves an easily identifiable change of colour (purple colour of $KMnO_4$ disappears. It is also called *Baeyer' s test.*

(b) **Ozonolysis** Alkenes react with ozone to give ozonides which on further reaction with water in presence to zinc give aldehydes or ketones depending on the structure of the starting alkene.



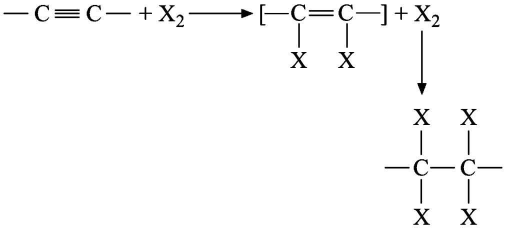
(c) **Combustion** Alkenes burn in presence of oxygen to form carbon dioxide and water along with the evolution of heat.

$$CH_2 = CH_2 + 3O_2 \longrightarrow 2CO_2 + 2H_2O + \text{heat}$$

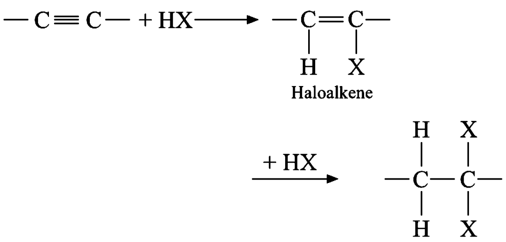
Properties of alkynes

(i) **Addition of hydrogen** Refer to preparation of alkanes and alkenes.

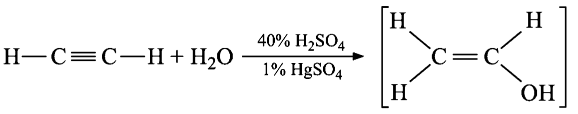
(ii) **Addition of Halogens** Alkynes react with two molecules of halogens to give tetrahalides.



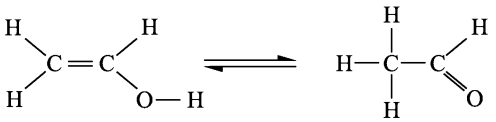
(iii) **Addition of Hydrogen Halides** The addition of hydrogen chloride, bromide or iodide to alkynes follows Markovnikov's rule. The reaction proceeds in two steps and may be stopped at the haloalkene stage or if allowed to proceed further then gem-dihalides are formed. The term gem-dihalide means both the halogens are on the same carbon atom.



(iv) **Addition of water** Water adds to alkynes in the presence of dilute sulphuric acid and mercuric sulphate. The addition of water follows Markovnikov's rule to give an adduct called enol. Enols undergo tautomerium and are converted immediately to more stable products which contain a carbonyl ($>C = O$) group. For example, the addition of water to acetylene gives first vinyl alcohol which rearranges to acetaldehyed, as shown below.



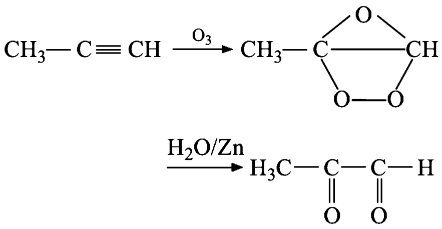
Vinyl alcohol (enol)



Acetaldehyde

(v) **Oxidation of Alkynes**

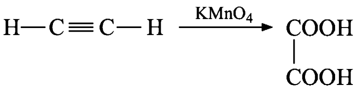
(a) *Ozonolysis:* Alkynes give diketo compounds on ozonolysis.



$$CH_3-C\equiv CH \xrightarrow{O_3} CH_3-C(O_3)CH \xrightarrow{H_2O/Zn} H_3C-\underset{\underset{O}{\|}}{C}-\underset{\underset{O}{\|}}{C}-H$$

(b) *Oxidation with Potassium Permanganate:* The oxidation of alkynes with alkaline potassium permanganate cleaves the molecule at the site of the triple bond to form carboxylic acids and carbon dioxide. Acylene under these conditions yields oxalic acid whereas propyne gives ethanoic acid (acetic acid) and carbon dioxide.

$$R-C\equiv C-R' \xrightarrow{KMnO_4} R-COOH + R'-COOH$$



$$H-C\equiv C-H \xrightarrow{KMnO_4} \begin{array}{c} COOH \\ | \\ COOH \end{array}$$

Oxalic acid

$$CH_3-C\equiv CH \xrightarrow{KMnO_4} \underset{\text{ethanoic acid}}{CH_3-COOH} + CO_2$$

(c) *Combustion:* Alkynes also burn in excess of oxygen to give carbon dioxide, water and large amount of heat. Acetylene is used in combination with oxygen in oxy-acetylene flame which is used for welding.

$$2H-C\equiv C-H + 5O_2 \longrightarrow 4CO_2 + 2H_2O + \text{heat}$$

(vi) **Aciditity of Alkynes** Hydrogen attached to a triply-bonded carbon, as in acetylene or any alkyne with terminal triple bond (R—C≡CH), show appreciable acidity. For example, sodium reacts with acetylene to give sodium acetylides and hydrogen

$$H-C\equiv C-H + Na \longrightarrow H-C\equiv C^-Na^+ + \frac{1}{2}H_2$$

The acidic acetylenes react with certain heavy metal ions, e.g. Ag^+ and Cu^+ to form insoluble acetylides. This reaction will be shown by terminal alkynes only and hence can be used to differentiate from non-terminal alkynes.

$$R-C\equiv C-H + \text{Ammoniacal } AgNO_3 \longrightarrow \underset{\text{(White ppt)}}{R-C\equiv C-Ag}$$

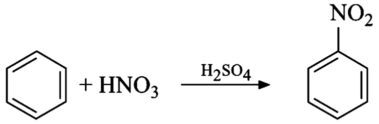
$$R-C\equiv C-H + \text{Ammoniacal cuprous chloride} \longrightarrow \underset{\text{(Red ppt)}}{R-C\equiv C-Cu}$$

Properties of aromatic hydrocarbons (arenes)

The typical reactions of benzene or arenes are the substitution reactions and not the addition reactions like alkenes. Due to the presence of π-electron clouds above and below the plane of benzene ring, it serves as a source of electrons and is easily attacked by electron seeking reagents. Hence the characteristic reactions of benzene or other arenes are *electrophilic substitution reactions.* Benzene undergoes some addition and oxidation reactions also which are also discussed below.

(i) *Nitration*

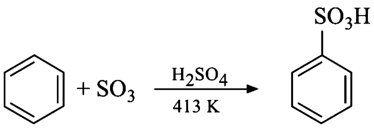
Benzene reacts with concentrated nitric acid in the presence of concentrated sulphuric acid to form nitrobenzene



(Nitrobenzene)

(ii) *Sulphonation*

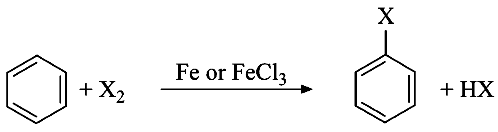
When benzene is heated with fuming sulphuric acid (oleum, $H_2SO_4 + SO_3$), SO_3H is substituted for H, the product is called benzene sulphonic acid.



(Benzene sulphonic acid)

(iii) *Halogenation*

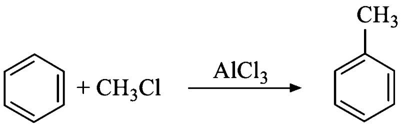
Bromine and chlorine react with benzene in the presence of a catalyst, generally iron filings or ferric chloride. The catalyst acts as a halogen carrier. Fluorination is not carried out directly because the reaction between benzene and fluorine is very vigorous. Iodo-benzene can be prepared but that requires the presence of an oxidising agent because HI, which is produced as side product is a strong reducing agent.



(X = Cl or Br) (Halobenzene)

(iv) *Friedel-Crafts Alkylation*

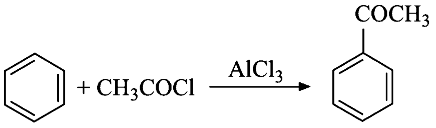
Benzene reacts with alkyl halides in the presence of a lewis acid (e.g. aluminium chloride) to give alkyl benzene. The reaction is called Friedel-Crafts alkylation.



(Toluene)

(v) *Friedel-Crafts acylation*

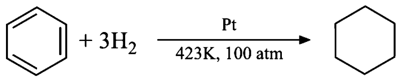
If acyl halide is taken instead of an alkyl halide, then the product is an alkyl phenylketone. Such as acetyl chloride reacts with benzene to give acetophenone. Such a reaction is known as Friedel-Crafts acylation.



(Acetophenone)

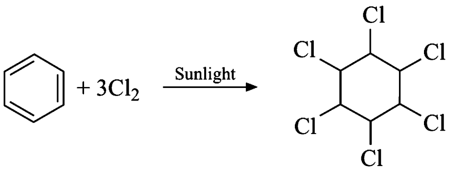
(vi) *Addition Reactions*

Benzene is so stable that it gives addition reactions much less readily. It can be reduced by catalytic hydrogenation, when three molecules of hydrogen add to it in presence of platinum as catalyst.



(Cyclohexane)

Addition of three molecules of chlorine also takes place to one molecule of benzene in presence of sunlight. The product formed is 1, 2, 3, 4, 5, 6-hexachlorocyclohexane (also called benzene hexachloride). This compound is used as an important insecticide and is sold commercially in the name of BHC or gammaxene.

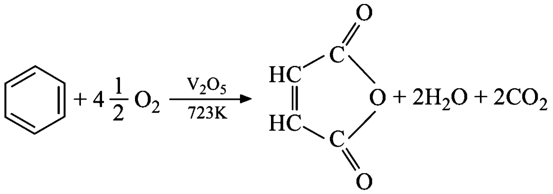


(1, 2, 3, 4, 5, 6-hexachlorocyclohexane)

(vii) *Oxidation*

Although oxidising agents such as $KMnO_4$ and $K_2Cr_2O_7$ have no action on benzene, but they oxidise the alkyl groups attached to benzene ring. The alkyl side chain, irrespective of its size or nature is completely oxidised to a carboxyl group. For example, toluene and butylbenzene both on oxidation give benzoic acid as the final product.

Oxidation of benzene with air in presence of vanadium pentoxide (V_2O_5) at 723K forms maleic anhydride.



(Maleic anhydride)

Problems for Practice

1. When alkyl magnesium halide is decomposed by water, the product obtained is

   (a) Alkyne (b) Alkane

   (c) Alkene (d) None of these

2. Mixture of CH_3I and C_2H_5 I was heated with Na and dry ether. The product obtained is

   (a) $CH_3 - CH_3$ (b) $C_2H_5 - C_2H_5$

   (c) $CH_3 - C_2H_5$ (d) All the three

3. A mixture of methane, ethane and acetylene is passed through a container having ammoniacal silver nitrate. The gas coming out is

   (a) Methane

   (b) Ethane

   (c) Mixture of ethane and methane

   (d) Acetylene

4. Starting from benzene, toluene is prepared by

   (a) Cannizaro's Reaction

   (b) Kolbe's Reaction

   (c) Reimer-Tiemann Reaction

   (d) Friedel-Craft's Reaction

5. A cylinder of compressed gas that bears no label is supposed to contain either ethene or propene. Combustion of the sample reveals that 12 mL of the gas require 54 mL of oxygen for complete combustion. This indicates that the gas is

   (a) Ethene (b) Propene

   (c) Mixture of both (d) None of these

6. Tetraethyl lead is added to petrol

   (a) as an anti-knock

   (b) to precipitate inorganic substances

   (c) to raise its boiling point

   (d) as an anti-freeze compound

7. Liquid hydrocarbons are converted to low molecular weight gaseous hydrocarbons by a process known as

   (a) Hydrogenation (b) Reforming

   (c) Cracking (d) Reduction

8. A metallic carbide on treatment with water gives a colourless gas which burns readily in air and gives a precipitate with tollen's reagent. The gas evolved is

   (a) Ethylene (b) Acetylene

   (c) Ethane (d) Methane

9. The empirical formula of a compound is CH_2. One mole of the compound has the mass 42 g. The molecular formula is

   (a) C_4H_8 (b) C_2H_4

   (c) C_3H_6 (d) C_3H_8

10. What volume of methane is formed at NTP by fusing 8.2 g of sodium acetate with soda lime?

    (a) 10 litres (b) 11.2 litres

    (c) 5.4 litres (d) 2.24 litres

11. Kerosene oil is a mixture of

(a) aromatic compounds
(b) alkanes
(c) different oils
(d) None of these

12. Domestic cooking gas is

(a) Mixture of different hydrocarbons
(b) Mixture of propane and butane
(c) Mixture of methane and ethane
(d) Methane mixed with little benzene

13. A hydrocabon has the molecular formula C_3H_4. To find out whether it contains two double bonds or a triple bond, which of the following tests should be performed?

(a) Treat with bromine water
(b) Pass through ammoniacal cuprous chloride
(c) Treat with Baeyer's reagent
(d) Treat with Fehling solution

14. Ethylene can be prepared by heating to 443K, a mixture of

(a) Concentrated sulphuric acid and ethyl alcohol
(b) Concentrated sulphuric acid and ethyl bromide
(c) Glacial acetic acid and ethyl alcohol
(d) Concentrated hydrochloric acid and ethyl alcohol

15. Coverting *n*-hexane into benzene in the presence of chromium oxide on alumina support is an example of reaction called

(a) Hydrogenation (b) Isomerization
(c) Aromatization (d) None of these

16. An olefinic compound upon ozonolysis gives one mlolecule each of formaldehyde, acetaldehyde and methyl glyoxal (CH_3COCHO). The parent compound is

(a) $CH_3 - CH = CH - CH_2 - CH = CH_2$
(b) $CH_3 - CH = C(CH_3)CH = CH_2$
(c) $CH_3 - CH = CH - CH = CH_2$
(d) $CH_2 = CH - CH = CH_2$

17. Oxidation of benzene with air at 723K in the presence of V_2O_5 gives

(a) Malic acid (b) Maleic acid
(c) Malonic acid (d) Maleic anhydride

18. 10 mL of a gaseous hydrocarbon requires 25mL of oxygen for complete combustion. The hydrocarbon is

(a) C_2H_2 (b) C_2H_4
(c) C_2H_6 (d) C_3H_8

19. When an alkene is formed by dehydrohalogenation of alkyl halide, the alkene which is most substituted predominates. This generalisation is called

(a) Markovnikov's Rule
(b) Saytzeff's Rule
(c) Huckel's Rule
(d) Peroxide effect

20. Addition of HBr to 2-methyl propene in the presence of peroxide mainly forms

(a) 1- Bromobutane
(b) 1- Bromo-2- methylpropane
(c) 2-Bromo-2-methylpropane
(d) None of these

21. Addition of HI to double bond of propene yields isopropyl iodide as major product. It is because the addition proceeds through

(a) More stable carbocation
(b) More stable carbanion
(c) More stable free radical
(d) Homolysis

22. The function of $AlCl_3$ in Friedel-Craft's reaction is

(a) to absorb water
(b) to absorb HCl
(c) to produce electrophile
(d) to produce nucleophile

23. Propene is reacted with HBr in presence of peroxide, the product is

(a) 2-bromopropane
(b) 1-bromopropane
(c) 3-bromopropane
(d) None of these

24. One mole of 1, 2-Dibromopropane on treatment with X moles of $NaNH_2$ followed by treatment with ethyl bromide gave 2-pentyne. The value of X is

(a) One (b) Two
(c) Three (d) Four.

25. The order of activity of the various *o*- and *p*- director is

(a) $-O^- > -OH > -OCOCH_3 > -COCH_3$
(b) $-OH > -O^- > -OCOCH_3 > -COCH_3$
(c) $-OH > -O^- > -COCH_3 > -OCOCH_3$
(d) $-O^- > -COCH_3 > -OCOCH_3 > -OH$

26. A gas decolourises bromine in CCl_4 and forms a precipitate with ammoniacal silver nitrate. The gas is

(a) C_2H_2 (b) C_2H_4
(c) C_2H_6 (d) CH_4

27. Formation of polythene from calcium carbide takes place as follows:

$CaC_2 + 2H_2O \longrightarrow Ca(OH)_2 + C_2H_2$
$C_2H_2 + H_2 \longrightarrow C_2H_4$
$nC_2H_4 \longrightarrow \left(CH_2 - CH_2 \right)_n$

The amount of polythene obtained from 64 kg of CaC_2 is

(a) 7 kg (b) 14 kg
(c) 21 kg (d) 22 kg

28. In the commercial gasoline, the type of hydrocarbons which are not desirable are

(a) branched hydrocarbons
(b) straight chain hydrocarbons
(c) linear unsaturated hydrocarbons
(d) toluene

29. Which one of these is not compatible with arenes?

(a) Greater stability
(b) Resonance
(c) Electrophilic addition
(d) Delocalisation of π-electrons

30. Iodoethane reacts with sodium in the presence of dry ether. The main product is

(a) ethane (b) Propyne
(c) Butane (d) Butene

31. The product formed on passing ethylene through a cold alkaline $KMnO_4$ solution is
(a) Ethyl alcohol (b) Ethylene glycol
(c) Ethylene oxide (d) Ethane

32. A gas which reacts with aqueous $KMnO_4$ solution but does not give precipitate with ammoniacal Cu_2Cl_2 solution is
(a) Ethylene (b) Methane
(c) Ethane (d) Acetylene

33. When 1- butene is treated with bromine, the expected product is
(a) 1, 1 - Dibromobutane
(b) 2, 2 - Dibromobutane
(c) 1, 2 - Dibromobutane
(d) All of the above

34. The typical reaction of olefinic bond is
(a) electrophilic substitution
(b) nucleophilic substitution
(c) nucleophilic addition
(d) electrophilic addition

35. An alkene which upon Ozonolysis gives CH_3CHOas one of the product is
(a) 1-Hexane (b) 1-Butene
(c) Ethylene (d) Propylene

36. Acetylene can be converted to acetaldehyde by
(a) passing through red hot tubes
(b) oxidation with $KMnO_4$
(c) hydration with $HgSO_4$ and H_2SO_4
(d) action of hypochlorous acid

37. Ethylene upon oxidation with Baeyer's reagent give
(a) Ethylene glycol (b) Ethene
(c) Formaldehyde (d) Acetic acid

38. Heating alcoholic solution of ethylene dibromide with zinc gives
(a) Ethane (b) Ethene
(c) Acetylene (d) Methane

39. Iodoform when heated with silver powder gives
(a) Ethane (b) Ethyne
(c) Ethene (d) None of these

40. The catalyst used for partial reduction of alkyne to alkene is
(a) Raney Nickel (b) Lindlar's catalyst
(c) Palladium (d) Platinum

41. Methane can be prepared by the action of water on
(a) zinc carbide (b) calcium carbide
(c) magnesium carbide (d) aluminium carbide

42. Both methane and ethane can be obtained in one step reaction from
(a) CH_3Br (b) C_3H_4
(c) CH_3OH (d) C_2H_5OH

43. A pentane that can form only one mono-substituted derivative is
(a) 2- Methylbutane
(b) 2, 2-Dimethylpropane
(c) *n*-Pentane
(d) Iso-Pentane

44. In Bergius process for the manufacture of petrol, the raw material used is
(a) Coal dust (b) Water gas
(c) Isobutane (d) Natural petroleum gas

45. The gas used for artificial ripening of fruits is:
(a) Ammonia (b) Acetylene
(c) Ethane (d) Ethene

46. The decomposition of compound by the action of heat is called
(a) Oxidation (b) Combustion
(c) Pyrolysis (d) Roasting

47. Benzene + Ozone $\longrightarrow$ Y. In this sequence Y is
(a) Benzene monoozonide
(b) Benzene diozonide
(c) Benzene triozonide
(d) Succinic acid

48. Benzene does not undergo addition reaction easily because
(a) It has a cyclic structure
(b) Double bonds in it are very strong
(c) Resonance stabilized system is to be preserved
(d) It has six hydrogen atoms

49. Acetylene reacts with acetic acid in presence of Hg^{++} ions at room temperature to give
(a) Ethyl acetate (b) Acetaldehyde
(c) Vinyl acetate (d) Methyl acetate

50. Heating a mixture of sodium benzoate and soda lime gives
(a) Benzene (b) Methane
(c) Biphenyl (d) Calcium benzoate

51. Octane number is zero for
(a) *n*-Heptane (b) Isooctane
(c) *n*-Hexane (d) Isoheptane

52. In Friedel-Craft's alkylation besides $AlCl_3$ the other reactants are
(a) $C_6H_6 + NH_3$ (b) $C_6H_6 + CH_4$
(c) $C_6H_6 + CH_3Cl$ (d) $C_6H_6 + CH_3COCl$

53. For preparing an alkane, a concentrated aqueous solution of sodium or potassium salt of saturated carboxylic acid is subjected to
(a) Hydrolysis (b) Oxidation
(c) Hydrogenation (d) Electrolysis

54. Which of the following species participate in sulphonation of benzene ring?
(a) H_2SO_4 (b) HSO_3^-
(c) SO_3 (d) SO^{-}_{2}

55. In its reaction with silver nitrate, acetylene shows
(a) Oxidising property (b) Reducing property
(c) Basic property (d) Acidic property

56. Which species represents the electrophile in aromatic nitration?

(a) NO_2^- (b) NO_2^+
(c) NO_2 (d) NO_3^-

57. The addition of HBr is easiest with
(a) $CH_2 = CHCl$ (b) $ClCH = CHCl$
(c) $CH_3 - CH = CH_2$ (d) $(CH_3)_2\ C = CH_2$

58. Propene can be converted into 1-propanol. Indicate which set of reagents amongst the following is ideal to effect the above conversion.
(a) $KMnO_4$ (alkaline)
(b) Osmium tetroxide (OsO_4/CH_2Cl_2)
(c) B_2H_6 and alk. H_2O_2
(d) O_3/Zn

59. Two organic compounds A and B both containing only carbon and hydrogen, on quantitative analysis gave the same percentage composition by weight:
$C = (12/13) \times 100\%,\ H = (1/13) \times 100\%$
A decolourizes bromine water but B does not. A and B respectively are
(a) C_2H_2 and C_6H_6 (b) C_6H_6 and C_2H_2
(c) C_2H_4 and C_2H_6 (d) C_2H_2 and C_2H_6

60. 1, 3-Butadiene when treated with Br_2 gives
(a) 1, 4-Dibromo-2-butene
(b) 1, 3-Dibromo-2-butene
(c) 3, 4-Dibromo-1-butene
(d) 2, 3-Dibromo-2-butene

61. The presence of unsaturation in organic compounds can be tested with
(a) Schiff's reagent (b) Tollen's reagent
(c) Fehling's reagent (d) Baeyer's reagent

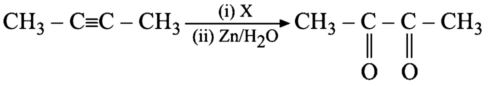
62. An unknown compound A has a molecular formula C_4H_6. When A is treated with excess of Br_2 a new substance B with formula $C_4H_6Br_4$ is formed. A forms a white ppt. with ammoniacal silver nitrate solution. A may be
(a) But-1-yne (b) But-2-yne
(c) But-1-ene (d) But-2-ene

63. *n*-Butylbenzene on oxidation will give
(a) benzoic acid
(b) butanoic acid
(c) 4-phenylbutanoic acid
(d) benzaldehyde

64. Which of the following has highest octane number?
(a) *n*-Hexane
(b) *n*-Heptane
(c) *n*-Pentane
(d) 2, 2, 4-Trimethylpentane

65. The product formed when acetylene is passed through red hot tube is
(a) Benzene (b) Cyclohexane
(c) Neoprene (d) Ethane

66.



$$CH_3 - C \equiv C - CH_3 \xrightarrow[\text{(ii) Zn/H}_2\text{O}]{\text{(i) X}} CH_3 - \underset{\underset{O}{\|}}{C} - \underset{\underset{O}{\|}}{C} - CH_3$$

In the above reaction, X is
(a) HNO_3 (b) O_2
(c) O_3 (d) $KMnO_4$

67. A fuel has the same knocking property as a mixture of 70% isooctane (2, 2, 4-trimethyl pentane) and 30% *n*-heptane by volume. The octane number of the fuel is
(a) 100 (b) 70
(c) 30 (d) 40

68. What is formed when calcium carbide reacts with heavy water?
(a) C_2D_2 (b) CaD_2
(c) Ca_2D_2O (d) CD_2

69. Which of the following possesses the highest melting point?
(a) Chlorobenzene (b) *o*-Dichlorobenzene
(c) *m*-Dichlorobenzene (d) *p*-Dichlorobenzene

70. Pure methane can be produced by
(a) Wurtz reaction
(b) Kolbe's electrolytic method
(c) Soda-lime decarboxylation
(d) Reduction with H_2

71. Which of the following is most reactive towards electrophilic substitution?
(a) $C_6H_5NO_2$ (b) C_6H_5Cl
(c) $C_6H_5CH_3$ (d) C_6H_6

72. Acetylene reacts with ammoniacal silver nitrate to from
(a) Silver mirror (b) Metal silver
(c) Silver acetate (d) Silver acetylide

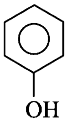
73. Bond length of C – C bond in Ethane (I), Ethene (II), Acetylene (III) and Benzene (IV) follows the order
(a) I > II > III > IV
(b) I > II > IV > III
(c) I > IV > II > III
(d) III > IV > II > I

74. When ethyl alcohol is heated with conc. H_2SO_4 at 443K,ethylene is formed by
(a) Intramolecular hydration
(b) Intermolecular hydration
(c) Intermolecular dehydration
(d) Intramolecular dehydration

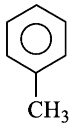
75. In the following reaction

$$C_2H_2 \xrightarrow[HgSO_4/H_2SO_4,\ 60°C]{H_2O} X \rightleftharpoons CH_3CHO$$

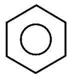
What is X ?
(a) CH_3CH_2OH (b) $CH_3 - O - CH_3$
(c) CH_3CH_2CHO (d) $CH_2 = CHOH$



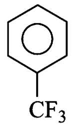
I



II



III



IV

76. The correct arrangement for decreasing order of electrophilic substitution reactions.
(a) I > II > III > IV
(b) IV > I > II > III
(c) III > IV > II > I
(d) II > IV > III > I

77. Reaction of benzene with alkyl halide in the presence of anhydrous $AlCl_3$ is called

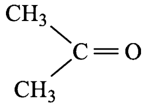
(a) Friedel-Carft reaction

(b) Wurtz reaction

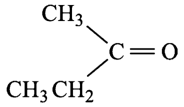
(c) Williamson's synthesis

(d) Baeyer's reaction

78. An alkene having molecular formula C_7H_{14} was subjected to ozonolysis in the presence of zinc dust. An equimolar amount of the following two compounds was obtained



and



The IUPAC name of the alkene is

(a) 3, 4-Dimethyl-3-pentene

(b) 2, 4-Dimethyl-2-pentene

(c) 2, 3-Dimethyl-3-pentene

(d) 2, 3-Dimethyl-2-pentene

79. Which reaction sequence would be best to prepare 3-chloroaniline from benzene?

(a) Chlorination, nitration, reduction

(b) Nitration, chlorination, reduction

(c) Nitration, reduction, chlorination

(d) Nitration, reduction, Acylation, chlorination, hydrolysis.

ANSWERS

1. (b)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (d)
11. (b)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (b)	**20.** (b)
21. (a)	**22.** (c)	**23.** (b)	**24.** (c)	**25.** (a)	**26.** (a)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (b)	**32.** (a)	**33.** (c)	**34.** (d)	**35.** (d)	**36.** (c)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (b)
41. (d)	**42.** (a)	**43.** (b)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (c)	**48.** (c)	**49.** (c)	**50.** (a)
51. (a)	**52.** (c)	**53.** (d)	**54.** (c)	**55.** (d)	**56.** (b)	**57.** (d)	**58.** (c)	**59.** (a)	**60.** (a)
61. (d)	**62.** (a)	**63.** (a)	**64.** (d)	**65.** (a)	**66.** (c)	**67.** (b)	**68.** (a)	**69.** (d)	**70.** (c)
71. (c)	**72.** (d)	**73.** (c)	**74.** (d)	**75.** (d)	**76.** (a)	**77.** (a)	**78.** (d)	**79.** (b)	

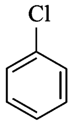
Haloalkanes and Haloarenes

24

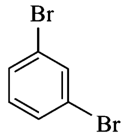
BRIEF REVIEW OF THE CONCEPTS

Compounds obtained by the replacement of hydrogen in alkanes by halogen atom (F, Cl, Br or I) are called haloalkanes or alkyl halides. They may be mono, di, tri, and poly halogen derivatives. The general formula of mono haloalkanes is C_nH_{2n+1} X where X is a halogen atom. Haloalkanes may be classified as 1°, 2° or 3° depending upon the carbon to which they are attached. They exhibit chain and position isomerism.

Haloarenes These are aromatic compounds in which the halogen atom is attached directly to the carbon atom of the aromatic ring. Examples,



Chlorobenzene

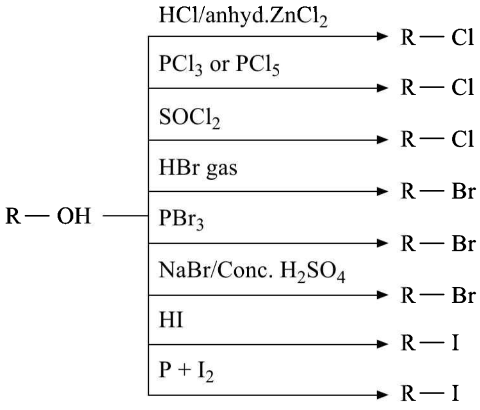


1, 3-Dibromobenzene
(m-Dibromobenzene)

Preparation of haloalkanes

(i) **From hydrocarbons** The direct halogenation of alkanes with halogens in presence of sunlight produces alkyl halides. Addition of halogen acids to alkenes also produce alkyl halides. Addition of halogens to alkenes and addition of halogen acids to alkynes give dihalogen derivativers.

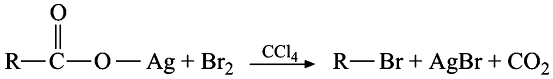
(ii) **From alcohols** Different methods of preparing alkyl halides from alcohols are summarised below.



(iii) **Finkelstein reaction** It is used for the preparation of iodoalkanes from alkyl chlorides or bromides.

$$R - Cl + NaI \xrightarrow[\Delta]{\text{acetone}} R - I + NaCl$$

(iv) **Hunsdiecker reaction** The silver salts of fatty acids react with Br_2 and produce alkyl bromides by losing a molecule of CO_2.



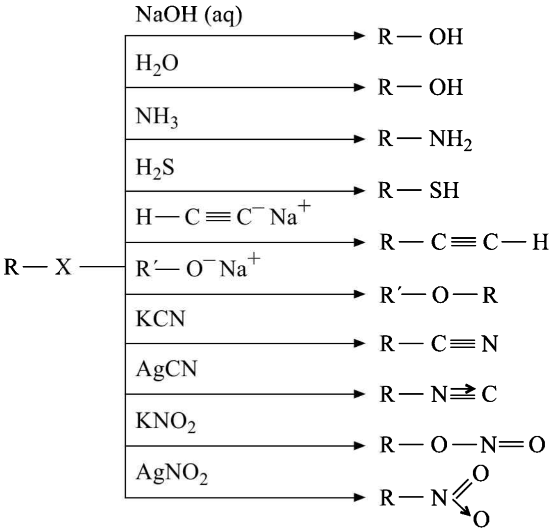
$$R—\overset{\overset{\large O}{\|}}{C}—O—Ag + Br_2 \xrightarrow{CCl_4} R—Br + AgBr + CO_2$$

Properties of haloalkanes

(i) **Nucleophilic substitution reactions** In alkyl halides the carbon carrying halogen acquires a slight (+) ve charge due to electronegativity of halogen atoms. Hence nucleophiles can attack this carbon and displace halogens. Different reactions of this category are summarised below:

$$R - X + Nu^-: \longrightarrow R - Nu + X^-:$$

where Nu^-: is a nucleophile

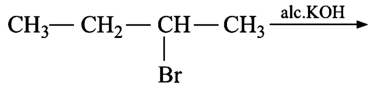


Two different mechanisms operate in this reaction viz., SN^1 and SN^2. The order of reactivity with respect to alkyl halides is 1° > 2° > 3° by SN^2 mechanism and 3° > 2° > 1° by SN^1 mechanism. The reactivity order with respect to halogen is R I > RBr > RCl > RF by both the mechanisms.

(ii) **Dehydrohalogenation** Alkyl halides are converted into alkenes on treatment with strong bases like alcoholic KOH. It involves a loss of a molecule of halogen acid.

If the dehydrohalogenation can yield more than one alkenes then“the more substituted alkene is obtained as the major product”.

It is known as **Saytzeff Rule.**



$$CH_3—CH_2—\underset{Br}{\underset{|}{CH}}—CH_3 \xrightarrow{\text{alc.KOH}}$$

$$\underset{\text{(major)}}{CH_3—CH{=}CH—CH_3} + \underset{\text{(minor)}}{CH_3—CH_2—CH{=}CH_2}$$

For a given alkyl halide, the ease of dehydrohalogenation is R – I > R – Br > R – Cl

(iii) **Wurtz reaction** Alkyl halides react with sodium metal in presence of dry ether to form alkanes.

$$2R - X + 2Na \xrightarrow{\text{dry ether}} R - R + 2\ NaX$$

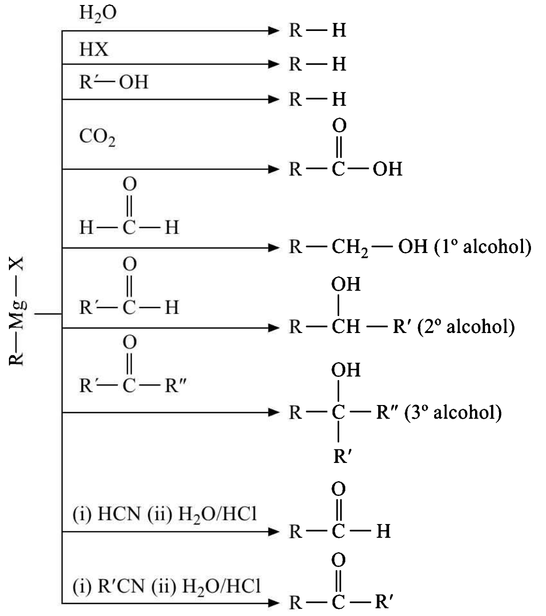
(iv) **Reduction** Alkyl halides are reduced to corresponding alkanes on reduction with Zinc and HCl or H_2/Ni.

$$R - X \xrightarrow{\text{Zn/HCl}} R - H$$

(v) **Grignard Reaction** Alkyl halides react with magnesium metal in presence of dry ether to give Grignard reagents which are named as alkyl magnesium halides

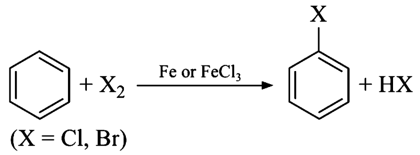
$$R - X + Mg \xrightarrow{\text{dry ether}} R - Mg - X$$

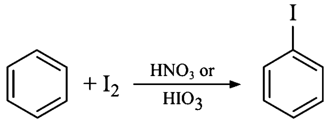
Grignard reagents are very reactive and some of their reactions are summarized below:



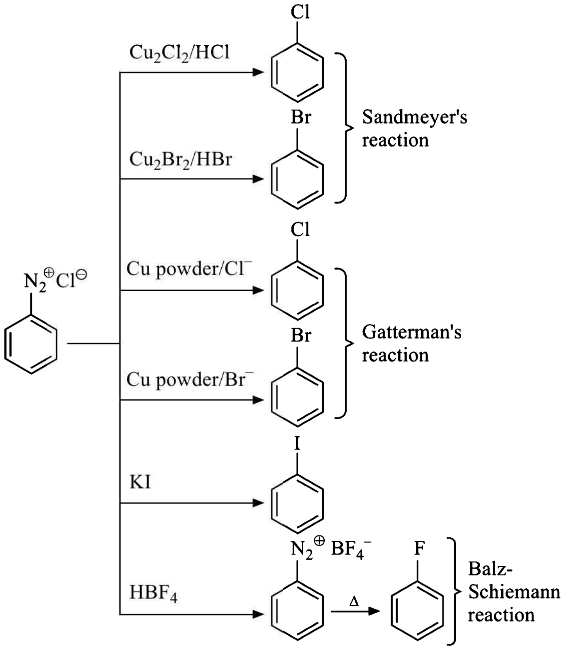
Preparation of haloarenes (Arylhalides)

(i) **By direct halogenation** Chloro and bromobenzene can be prepared by direct halogenation of benzene in presence of Fe or $FeCl_3$ whereas iodobenzene can be obtained by treating benzene with iodine in presence of nitric or iodic acid.



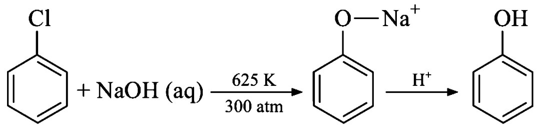


(ii) **From diazonium salts** Aromatic amines on treatment with HNO_2 at O–5°C are converted into diazonium salts which can be converted into haloarenes by various reactions as listed below:

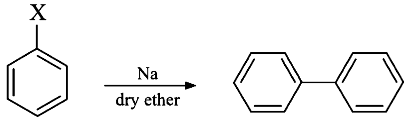


Properties of Arylhalides

(i) **Nucleophilic substitution Reactions** Arylhalides are relatively much less reactive towards such reactions and undergo substitution under drastic conditions. e.g., chlorobenzene undergoes this reaction to give phenol at pressure of about 300 atm. **(Dow's process).**

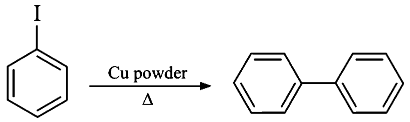


(ii) **Fittig Reaction** Arylhalides react with sodium metal to give diaryls.

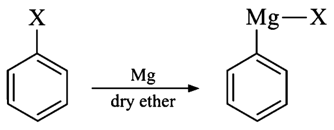


Biphenyl

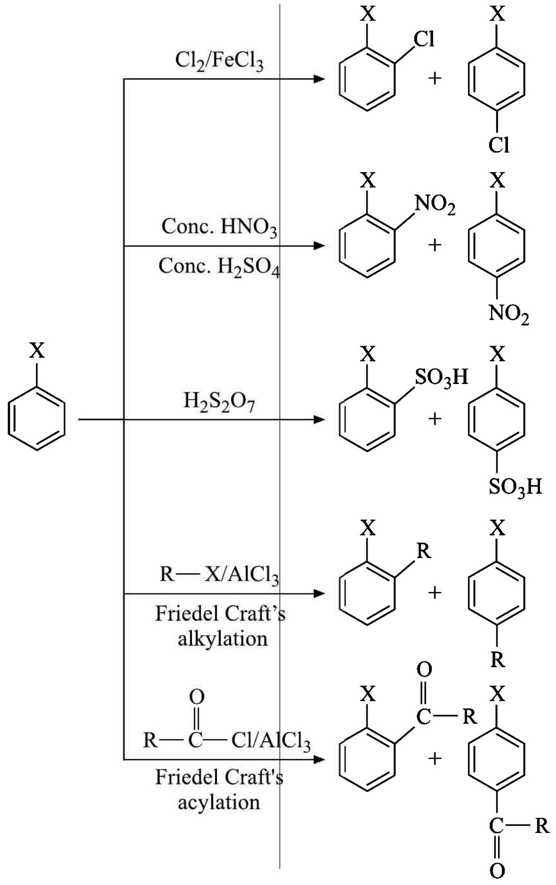
(iii) **Ullmann Reaction** When iodoarenes are heated with copper powder, diaryls are formed.



(iv) **Grignard Reaction** Arylhalides react with magnesium to form Grignard reagents which undergo all reactions discussed earlier for alkyl magnesium halides.



(v) **Ring substitution Reactions** Haloarenes undergo electrophilic substitution reaction and direct the incoming group to the ortho and para-positions.



Problems for Practice

1. Carbon tetrachloride has no net dipole moment because of
   (a) Its planar structure
   (b) Similar electron affinities
   (c) Different electron affinities of C and Cl atoms
   (d) Its regular tetrahedral structure
2. Out of the following compounds which one will have a zero dipole moment?
   (a) 1,1 Dichloroethylene
   (b) Cis-1,2 Dichloroethylene
   (c) Trans-1,2 Dichloroethylene
   (d) None of these compounds
3. Only two isomeric monochloro derivatives are possible for
   (a) n-Butane (b) 2, 4 Dimethylpentane
   (c) Benzene (d) 2 Methylpentane
4. Arrange CCl_4, $CHCl_3$, CH_2Cl_2 and CH_3Cl in the decreasing order of density
   (a) $CH_3Cl > CH_2Cl_2 > CHCl_3 > CCl_4$
   (b) $CCl_4 > CHCl_3 > CH_2Cl_2 > CH_3Cl$
   (c) $CCl_4 > CHCl_3 > CH_3Cl > CH_2Cl_2$
   (d) $CHCl_3 > CCl_4 > CH_2Cl_2 > CH_3Cl$
5. The alkyl halide that can be made by free radical halogenation of alkanes
   (a) RCl and RBr not RF or RI
   (b) RF, RCl and RBr not RI
   (c) RF, RCI, RBr and RI
   (d) RF, RCl and RI but not RBr
6. Product obtained by heating iodoform with Ag powder is
   (a) Ethane (b) Ethene
   (c) Ethyne (d) None of these
7. Allyl bromide and n-propyl bromide can be distinguished by
   (a) $AgNO_3$ (b) NaOH
   (c) Tollen's reagent (d) Baeyer's reagent
8. Which of the following compounds is formed when

$$C_6H_6 + \underset{\text{(excess)}}{Cl_2} \xrightarrow{\text{Sun light}} ?$$

   (a) Chlorobenze
   (b) Dichlorobenzene
   (c) Hexachlorobenzene
   (d) Benzene hexachloride
9. Haloforms are trihalogen derivatives of
   (a) ethane (b) methane
   (c) propane (d) benzene
10. Which of the following statements is correct?

(a) C_2H_5Br reacts with alcoholic KOH to form C_2H_5OH
(b) C_2H_5Br when treated with metallic sodium gives ethane
(c) C_2H_5Br when treated with sodium ethoxide forms diethyl ether
(d) C_2H_5Br with AgCN forms ethyl cyanide

11. The compound which will not give iodoform test is
(a) CH_3CHO (b) CH_3CH_2OH
(c) $C_6H_5COCH_3$ (d) $C_6H_5CH_2CH_2OH$

12. Which of the following is hydrolysed most easily?
(a) C_6H_5Cl (b) $(C_6H_5)_2CHCl$
(c) $C_6H_5CH_2Cl$ (d) $(C_6H_5)_3CCl$

13. An alkyl halide may be converted into an alcohol by
(a) addition
(b) substitution
(c) dehydrohalogenation
(d) elimination

14. DDT is obtained by the action of conc. H_2SO_4 on:
(a) Chloral and bromobenzene
(b) Chloral and chlorobenzene
(c) Chlorine and benzene
(d) Chlorine and toluene

15. When 2-chlorobutane is treated with alcoholic KOH, the major product is:
(a) 1-Butene (b) 2-Butene
(c) 1-Butyne (d) 2-Butyne

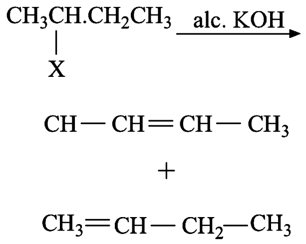
16. Chloroform undergoes condensation with acetone to form:
(a) Chloral (b) Chloretone
(c) Salicyladehyde (d) Acetyl chloride

17. Vapour density of an organic compound is 23.0. It contains 52.17% of carbon and 13% of hydrogen. The compound gives iodoform test. The compound is
(a) ethanol (b) dimethyl ether
(c) acetone (d) methanol

18. Which of the following compounds will give a yellow precipitate with iodine and alkali?
(a) Pentan-3-ol
(b) Acetophenone
(c) Methyl acetate
(d) Acetamide

19. Chloroform when treated with concentrated nitric acid yields
(a) Chloroprene
(b) Chloropicrin
(c) Nitropicrin
(d) Nitrochloropicrin

20. For the reaction



$$\underset{\substack{|\\ X}}{CH_3CH}.CH_2CH_3 \xrightarrow{\text{alc. KOH}}$$

$$CH—CH{=}CH—CH_3$$

$$+$$

$$CH_3{=}CH—CH_2—CH_3$$

(a) $CH_3–CH{=}CH–CH_3$ predominates
(b) $CH_2{=}CH–CH_2–CH_3$ predominates
(c) both are formed in equal amounts
(d) No such product is formed

21. Ethene is obtained from ethyl bromide by
(a) Nucleophlic substitution
(b) Hydrolysis
(c) Simple heating
(d) Dehydrohalogenation

22. When 1-chloropropane is treated with alcoholic KOH. It forms an alkene. This reaction is
(a) Substitution reaction
(b) Elimination reaction
(c) Addition reaction
(d) Dehydration reaction

23. For the reaction:

$$C_2H_5OH + HX \xrightarrow{ZnCl_2} \xrightarrow{ZnCl_2} C_2H_5\ X$$

the order of reactivity is:
(a) HBr > HI > HCl (b) HI > HCl > HBr
(c) HI > HBr > HCl (d) HCl > HBr > HI

24. If chloroform is left open in air in the presence of sun rays:
(a) Explosion takes place
(b) Poisonous gas phosegene is formed
(c) Polymerization takes place
(d) No reaction takes place

25. When chloroform is heated with aniline and caustic potash, we get a compound having
(a) rose odour
(b) jasmine odour
(c) bitter almond odour
(d) obnoxious odour

26. Which of the following process does not occur during formation of $CHCl_3$ from C_2H_5OH and bleaching powder?
(a) Hydrolysis (b) Oxidation
(c) Reduction (d) Chlorination

27. Which reagent cannot be used to prepare an alkyl halide from an alcohol?
(a) HCl + $ZnCl_2$ (b) NaCl
(c) PCl_5 (d) $SOCl_2$

28. The Cl-C-Cl angle in 1, 1, 2, 2 tetrachloroethene and tetrachloromethane will be about
(a) 120° and 109°.5° (b) 90° and 109.5°
(c) 109.5° and 90° (d) 109.5° and 120°

29. The chemical formula of the most commonly used Freon in refrigeration and air conditioning is:
(a) $CHCl_2F$ (b) CCl_2F_2
(c) CCl_3F (d) CF_3Cl

30. Dehydrohalogenation of alkyl halides occurs on heating with
(a) sodium ethoxide (b) sodium chloride
(c) sodium acetate (d) hydrochloric acid

31. Which compound undergoes the iodoform reaction?

(a) Formaldehyde (b) Acetaldehyde

(c) CH_3OH (d) CH_3COOH

32. Chloroform is kept in dark coloured bottles because

(a) it is inflammable

(b) it gives a peroxide

(c) it undergoes rapid chlorination

(d) it is oxidized to poisonous phosgene

33. Which of the following sequence would yield m-nitro chlorobenzene (Z) from benzene?

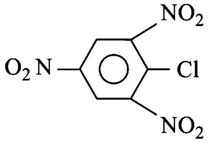
(a) Benzene $\xrightarrow{Cl_2/FeCl_3} X \xrightarrow{HNO_3/H_2SO_4} Z$

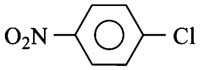
(b) Benzene $\xrightarrow{H_2SO_4/HNO_3} X \xrightarrow{HCl} Z$

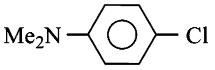
(c) Benzene $\xrightarrow{H_2SO_4/HNO_3} X \xrightarrow{FeCl_3/Cl_2} Z$

(d) None of these

34. Which chloroderivative of benzene among the following would undergo hydrolysis most readily with aqueous sodium hydroxide to furnish the corresponding hydroxy derivative?

(a) 

(b) 

(c) 

(d) C_6H_5Cl

35. The reaction products of the reaction between $C_6H_5NH_2$, $CHCl_3$ and KOH are

(a) $C_6H_5NC + KCl$

(b) $C_6H_5OH + NH_4Cl$

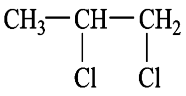
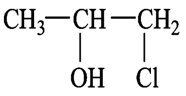
(c) $C_6H_5Cl + NH_4Cl + KCl$

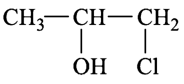
(d) $C_6H_5CN + KCl$

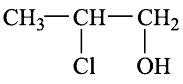
36. Identify 'Z' in

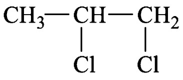
$CH_3.CH_2CH_2Br \xrightarrow{Aq.NaOH} X \xrightarrow[heat]{Al_2O_3} Y \xrightarrow{Cl_2/H_2O} Z$

(a) Mixture of


$CH_3-\underset{Cl}{CH}-\underset{Cl}{CH_2}$ and 
$CH_3-\underset{OH}{CH}-\underset{Cl}{CH_2}$

(b) 
$CH_3-\underset{OH}{CH}-\underset{Cl}{CH_2}$

(c) 
$CH_3-\underset{Cl}{CH}-\underset{OH}{CH_2}$

(d) 
$CH_3-\underset{Cl}{CH}-\underset{Cl}{CH_2}$

37. Iodoform is formed on warming iodine and sodium hydroxide with

(a) C_2H_5OH

(b) CH_3OH

(c) CH_3CH_2CHO

(d) $CH_3CH_2CHOHCH_2CH_3$

38. When ethylidene dichloride is warmed with aqueous alkali we get

(a) CH_3OH (b) CH_2CH_2OH

(c) CH_3CHO (d) CH_3COCH_3

39. Methyl chloride reacts with sodium acetylide to form:

(a) Acetylene (b) Ethylene

(b) Propyne (d) Propylene

40. Wurtz-Fittig reaction involves the reaction of sodium metal and

(a) Two molecules of an alkyl halide

(b) One molecule of an aryl halide and one molecule of an alkyl halide

(c) Two molecule of an aryl halide

(d) Two molecules of chloroform

41. Dipole moment is shown by

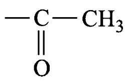
(a) 1, 4-Dichlorobenzene

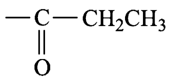
(b) Cis 1, 2-Dichloroethene

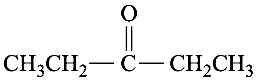
(c) Trans 1, 2-Dichloroethene

(d) Trans-2,3-Dichloro-2 butene

42. An organic substance gives haloform reaction. It indicates the presence of which group?

(a) 
$-\underset{\underset{O}{\|}}{C}-CH_3$

(b) 
$-\underset{\underset{O}{\|}}{C}-CH_2CH_3$

(c) 
$CH_3CH_2-\overset{\overset{O}{\|}}{C}-CH_2CH_3$

(d) CH_3Cl

43. Fluorobenzene can be conveniently prepared

(a) On heating benzenediazonium chloride with tetra-fluoroborate

(b) On treatment of benzene with F_2

(c) By the action of phenol with SiF_4

(d) By treatment of benzene with Freon in presence of anhydrous $AlCl_3$

44. Which one of the following compounds is least reactive towards alkali?

(a) Vinyl halide (b) Secondary halide

(c) Allyl halide (d) Same in all

45. $CH_2 = CHCl$ reacts with HCl to form

(a) $CH_2Cl - CH_2Cl$ (b) $CH_3 - CHCl_2$

(c) $CH_2 = CHCl.HCl$ (d) none of these

46. Chloropicrin is

(a) CCl_2HNO_2 (b) $C.ClH_2NO_2$

(c) CCl_3NO_2 (d) $CCl_2(NO_2)_2$

47. Treatment of ammonia with excess of ethyl chloride will yield

(a) diethyl amine

(b) ethane

(c) tetraethyl ammonium chloride

(d) methyl amine

48. Chlorination of toluene in presence of light followed by treatment with aqueous NaOH gives

(a) o-Cresol
(b) p-Cresol
(c) 2,4-Dihydroxytoluene
(d) Benzoic acid

49. Aryl halides are less reactive towards nucleophilic substitution reactions as compared to alkyl halides due to

(a) The formation of less stable carbocation
(b) Resonance stablization
(c) Longer carbon-halogen bond
(d) The inductive effect

50. The reaction of toluene with chlorine in presence of ferric chloride gives predominantly

(a) Benzoyl chloride (b) m-Chlorotoluene
(c) Benzyl chloride (d) o-and-p Chlorotoluene

51. The reaction conditions leading to the best yield of C_2H_5Cl are

(a) C_2H_6 (excess) + $Cl_2 \xrightarrow{\text{UV light}}$
(b) $C_2H_6 + Cl_2 \xrightarrow{\text{Dark}}$
(c) $C_2H_6 + Cl_2$ (excess) $\xrightarrow{\text{UV light}}$
(d) $C_2H_6 + Cl_2 \xrightarrow{\text{UV light}}$

52. Ethylene dichloride and ethylidene chloride are isomeric compounds. Identify the statement which is not applicable. Both of them

(a) react with alcoholic potash and give the same product
(b) react with aqueous potash and give the same product
(c) are dihalides
(d) answer Beilstein's test

53. Butanenitrile may be prepared by heating

(a) propyl alcohol with KCN
(b) butyl chloride with KCN
(c) butyl alcohol with KCN
(d) propyl chloride with KCN

54. The compound which is used as fire-extinguisher under the name pyrene is

(a) CH_3Cl (b) $CHCl_3$
(c) CCl_4 (d) CCl_3. CHO

55. Which of the following methods is used to prepare chlorobenzene?

(a) Action of PCl_5 on phenol
(b) Action of HCl on phenol in presence of $ZnCl_2$
(c) Action of diazonium chloride with CuCl/HCl
(d) Direct chlorination of toluene

56. The halogen atom in aryl halides is

(a) o-, p- Directing
(b) m-Directing
(c) Both o-, p- and m-directing
(d) None of these

57. When ethyl bromide is distilled with potassium hydrogen sulphide, the product formed is

(a) Ethyl alcohol
(b) Ethyl hydrogen sulphate
(c) Ethyl thioalcohol
(d) Thioether

58. Freon which is dichlorodifluoromethane is an important chemical is used

(a) as local anaesthetic
(b) in metallurgical operations for separating impurities
(c) in refrigerators
(d) in printing industry

59. Reaction of alkyl halide with sodium alkoxide yields

(a) alcohol (b) ketone
(c) ether (d) carboxylic acid

60. Ethylidene chloride is obtained when phosphorous pentachloride is treated with

(a) Acetyl chloride (b) Acetic anhydride
(c) Acetaldehyde (d) None the these

61. An alkyl halide by formation of Grignard's reagent followed by hydrolysis yields butane. What is the original alkyl halide?

(a) Ethyl halide (b) Propyl halide
(c) Butyl halide (d) Methylhalide

62. When ethyl iodide is treated with alcoholic caustic potash, we get

(a) ethyl alcohol (b) ethane
(c) ethylene (d) acetylene

63. Iodoform can be used in medicine as an

(a) Anaesthetic (b) Antiseptic
(c) Analgesic (d) Antipyretic

64. $CH_3—CH = CH_2$ + HBr gives

(a) 1- Bromopropane (b) 2-Bromopropane
(c) 3-Bromopropane (d) None of these

65. Which of the following compounds gives formic acid on hydrolysis?

(a) CH_3CH_2Cl (b) $CHCl_3$
(c) 3-Bromopropane (d) None of these

66. Phenyl isocyanide is formed when chloroform is treated with

(a) Aniline (b) Benzene
(c) Nitrobenzene (d) Nitroaniline

67. The carbylamine reaction is given by

(a) CCl_4 (b) CH_3Cl
(c) $CH_2 = CHCl$ (d) $CHCl_3$

68. A sample of chloroform being used as an anaesthetic is tested by

(a) $AgNO_3$ solution
(b) Fehling solution
(c) $AgNO_3$ solution after boiling with alc. KOH
(d) Tollen's reagent

69. The bad smelling substance formed by the action of alcoholic potash on chloroform and aniline is

(a) phenyl isocyanide (b) nitrobenzene
(c) acetylene (d) chloropicrin

70. The order of reactivity of alkyl halides towards nucleophiles is

(a) R I > R Br > R CI
(b) R Br > R I >R CI

(c) R CI > R Br > R I

(d) R I > R CI > R Br

71. The number of structural and configurational isomers of a bromo compound C_5H_9 Br formed by the addition of HBr to 2-pentyne, respectively are

(a) 1 and 2 (b) 2 and 4

(c) 4 and 2 (d) 2 and 1

72. Bleaching powder when heated with ethyl alcohol gives:

(a) Chloral (b) Ethyl chloride

(c) Chloroform (d) Methylene chloride

73. An organic halide with formula $C_6H_{13}Br$ on heating with alc. KOH gives two isomeric alkenes A and B with formula C_6H_{12}. On reductive ozonolysis of mixture of A and B the following compounds were obtained:

CH_3COCH_3, CH_3CHO, CH_3CH_2CHO and

$(CH_3)_2$ CHCHO. The organic halide is

(a) 2-Bromohexane

(b) 3-Bromo-2-methylpentane

(c) 2,2-Dimethyl-1-bromohexane

(d) None of these

74. Which one of the following alkyl halides is not hydrolyzed by *SN*¹ mechanism?

(a) $C_6H_5CH_2Br$ (b) CH_3Br

(c) $CH_2{=}CHCH_2Br$ (d) $(CH_3)_3$ CBr

75. The intermediate involved during the chlorination of propylene to allyl chloride is

(a) $\overset{\oplus}{C}H_2{-}CH{=}CH_2$ (b) $\overset{\ominus}{C}H{-}CH{=}CH$

(c) $\overset{\bullet}{C}H - CH = CH_2$ (d) None of these

76. n-Propyl iodide reacts with aq. KOH to form alcohol. It is:

(a) SN^1 reaction

(b) SN^2 reaction

(c) Both SN^1 and SN^2 reactions

(d) None of these reactions

77. Identify Z in the following series:

$C_2H_5I \xrightarrow{\text{alc KOH}} X \xrightarrow{Br_2} Y \xrightarrow{KCN} Z$

(a) $CH_3CH_2 - CN$

(b) $CN - CH_2 - CH_2 - CN$

(c) $Br - CH_2 - CH_2 - CN$

(d) $Br - CH = CHCN$

ANSWERS

1. (d)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (d)	**8.** (d)	**9.** (b)	**10.** (c)
11. (d)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (d)	**26.** (c)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (a)
31. (b)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (c)	**39.** (c)	**40.** (b)
41. (b)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (c)	**48.** (d)	**49.** (b)	**50.** (d)
51. (a)	**52.** (b)	**53.** (d)	**54.** (c)	**55.** (c)	**56.** (a)	**57.** (c)	**58.** (c)	**59.** (c)	**60.** (c)
61. (c)	**62.** (c)	**63.** (b)	**64.** (b)	**65.** (b)	**66.** (a)	**67.** (d)	**68.** (a)	**69.** (a)	**70.** (a)
71. (b)	**72.** (c)	**73.** (b)	**74.** (b)	**75.** (c)	**76.** (b)	**77.** (b)			

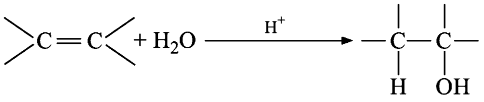
25 Alcohols, Phenols and Ethers

BRIEF REVIEW OF THE CONCEPTS

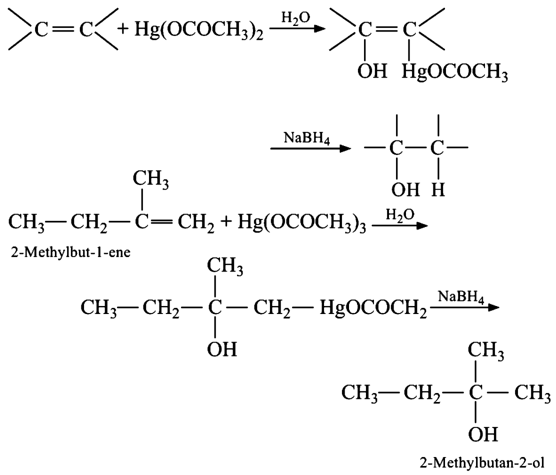
Alcohols are compounds which contain one or more hydroxyl (–OH) groups directly attached to a carbon chain whereas phenols contain –OH group directly attached to an aromatic ring. Ethers form a class of compounds in which oxygen is bonded to two carbon atoms with the general formula R – O – R′ where R and R′ may be alkyl or aryl groups.

Preparation of Alcohols

(i) **From alkenes:** Alcohols are obtained by the hydration of alkenes in acidic medium. The product obtained is according to Markovnikov's rule in case of unsymmetrical alkenes.



Addition of water to alkenes takes places easily by oxymercuration – demercuration also and the product formed is according to Markovnikov's rule.



$$CH_3-CH_2-\overset{CH_3}{\overset{|}{C}}=CH_2 + Hg(OCOCH_3)_3 \xrightarrow{H_2O}$$

2-Methylbut-1-ene

$$CH_3-CH_2-\underset{OH}{\underset{|}{\overset{CH_3}{\overset{|}{C}}}}-CH_2-HgOCOCH_2 \xrightarrow{NaBH_4}$$

$$CH_3-CH_2-\underset{OH}{\underset{|}{\overset{CH_3}{\overset{|}{C}}}}-CH_3$$

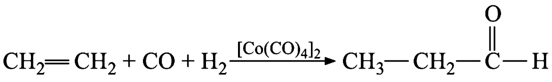
2-Methylbutan-2-ol

Hydroboration-oxidation of alkenes also gives alcohols but the product formed is different from that obtained by direct addition of water.

$$CH_3-CH=CH_2 \xrightarrow{B_2H_6} (CH_3-CH_2-CH_2)_3B$$

$$\xrightarrow{H_2O_2} CH_3-CH_2-CH_2-OH$$

Alkenes can also be converted into alcohols with one more carbon by *oxo process* as shown in the following example



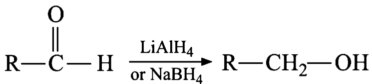
$$CH_2=CH_2 + CO + H_2 \xrightarrow{[Co(CO)_4]_2} CH_3-CH_2-\overset{O}{\overset{\|}{C}}-H$$

$$\xrightarrow{LiAlH_4} CH_3-CH_2-CH_2-OH$$

(ii) **From alkyl halides** Alkyl halides are converted into alcohols by their hydrolysis

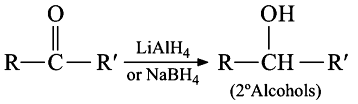
$$R-X \xrightarrow[\text{or moist }Ag_2O]{H_2O/OH^-} R-OH$$

(iii) **From aldehydes and ketones** Alcohols can be obtained by the reduction of aldehydes and ketones.



$$R-\overset{O}{\overset{\|}{C}}-H \xrightarrow[\text{or }NaBH_4]{LiAlH_4} R-CH_2-OH$$

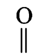
(1°Alcohols)



$$R-\overset{O}{\overset{\|}{C}}-R' \xrightarrow[\text{or }NaBH_4]{LiAlH_4} R-\overset{OH}{\overset{|}{C}H}-R'$$

(2°Alcohols)

Alcohols are also obtained by the addition of Grignard reagents to aldehydes and ketones

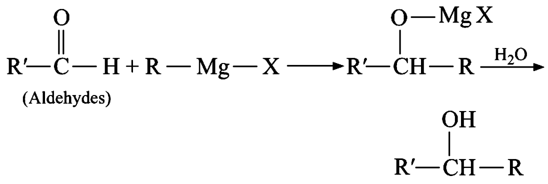


$$H-\overset{O}{\overset{\|}{C}}-H + RMgX \longrightarrow R-CH_2-OMg-X$$

(Formaldehyde)

$$\xrightarrow{H_2O} R-CH_2-OH$$

(1° Alcohols)

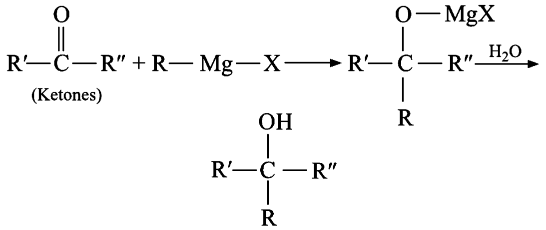


$$R'-\overset{O}{\overset{\|}{C}}-H + R-Mg-X \longrightarrow R'-\overset{O-MgX}{\overset{|}{C}H}-R \xrightarrow{H_2O}$$

(Aldehydes)

$$R'-\overset{OH}{\overset{|}{C}H}-R$$

(2° Alcohols)



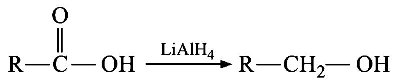
$$R'-\overset{O}{\overset{\|}{C}}-R'' + R-Mg-X \longrightarrow R'-\underset{R}{\underset{|}{\overset{O-MgX}{\overset{|}{C}}}}-R'' \xrightarrow{H_2O}$$

(Ketones)

$$R'-\underset{R}{\underset{|}{\overset{OH}{\overset{|}{C}}}}-R''$$

(3° Alcohols)

(iv) **From carboxylic acids** Carboxylic acids are reduced to corresponding primary alcohols on treatment with $LiAlH_4$



$$R-\overset{O}{\overset{\|}{C}}-OH \xrightarrow{LiAlH_4} R-CH_2-OH$$

(v) **From amines** Aliphatic primary amines on diazotization give diazonium salts which are unstable and decompose to give alcohols.

$$-NH_2 \xrightarrow[0-5°C]{NaNO_2/HCl} [R-N_2^+Cl^-] \xrightarrow{H_2O} [R-N_2^+OH^-]$$

(vi) **By fermentation** Ethyl alcohol is commercially obtained by the fermentation of starch or sucrose.

$$\text{Starch} \xrightarrow{\text{Diastase}} \text{Maltose } (C_{12}H_{22}O_{11})$$

$$\text{Maltose} \xrightarrow{\text{Maltase}} \text{Glucose } (C_6H_{12}O_6)$$

$$\text{Glucose } (C_6H_{12}O_6) \xrightarrow{\text{Zymase}} 2C_2H_5OH + 2CO_2$$

$$\text{Sucrose} \xrightarrow{\text{Invertase}} \text{Glucose + Fructose}$$

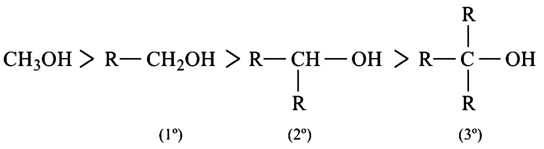
After the fermentation is complete, the mixture is distilled to obtain ethanol. The distillate contains about 95% ethanol and 5% water and is called as *rectified spirit.* When some methyl alcohol is added to make it unfit for drinking then it called *methylated spirit.* Sometimes other substances like pyridine, acetone and some colouring matter are added, then it is called *denatured spirit.* Rectified spirit can be converted into 100% ethyl alcohol called *absolute alcohol* by its azeotropic distillation with benzene. Absolute alcohol can also be obtained by treating rectified spirit with quick lime (CaO) followed by treatment with magnesium metal.

Properties of Alcohols

(i) **Acidic character** Alcohols are very weak acids and their hydrogen can be displaced only by treating with reactive metals like sodium.

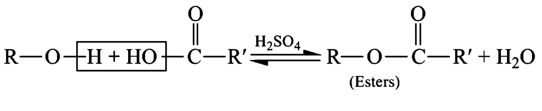
$$R-OH \xrightarrow{Na} R-O^-Na^+ + \frac{1}{2}H_2$$

The order of acidity of different alcohols is



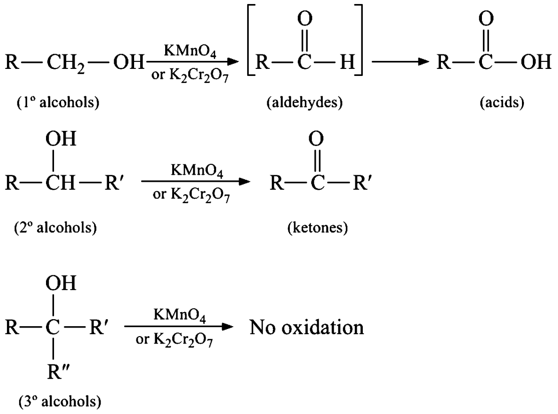
$$\underset{(1°)}{CH_3OH > R-CH_2OH} > \underset{(2°)}{R-\underset{R}{\underset{|}{CH}}-OH} > \underset{(3°)}{R-\overset{R}{\overset{|}{\underset{R}{\underset{|}{C}}}}-OH}$$

(ii) **Ester formation** They react with carboxylic acids to give esters



$$R-O-\boxed{H + HO}-\overset{O}{\overset{\|}{C}}-R' \underset{}{\overset{H_2SO_4}{\rightleftharpoons}} \underset{\text{(Esters)}}{R-O-\overset{O}{\overset{\|}{C}}-R'} + H_2O$$

(iii) **Oxidation**

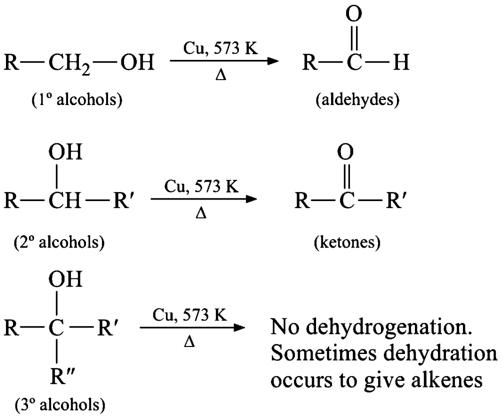


$$\underset{(1°\text{ alcohols})}{R-CH_2-OH} \xrightarrow[\text{or } K_2Cr_2O_7]{KMnO_4} \underset{\text{(aldehydes)}}{\left[R-\overset{O}{\overset{\|}{C}}-H\right]} \longrightarrow \underset{\text{(acids)}}{R-\overset{O}{\overset{\|}{C}}-OH}$$

$$\underset{(2°\text{ alcohols})}{R-\overset{OH}{\overset{|}{CH}}-R'} \xrightarrow[\text{or } K_2Cr_2O_7]{KMnO_4} \underset{\text{(ketones)}}{R-\overset{O}{\overset{\|}{C}}-R'}$$

$$\underset{(3°\text{ alcohols})}{R-\overset{OH}{\overset{|}{\underset{R''}{\underset{|}{C}}}}-R'} \xrightarrow[\text{or } K_2Cr_2O_7]{KMnO_4} \text{No oxidation}$$

(iv) **Dehydrogenation**



$$\underset{(1°\text{ alcohols})}{R-CH_2-OH} \xrightarrow[\Delta]{Cu,\ 573\ K} \underset{\text{(aldehydes)}}{R-\overset{O}{\overset{\|}{C}}-H}$$

$$\underset{(2°\text{ alcohols})}{R-\overset{OH}{\overset{|}{CH}}-R'} \xrightarrow[\Delta]{Cu,\ 573\ K} \underset{\text{(ketones)}}{R-\overset{O}{\overset{\|}{C}}-R'}$$

$$\underset{(3°\text{ alcohols})}{R-\overset{OH}{\overset{|}{\underset{R''}{\underset{|}{C}}}}-R'} \xrightarrow[\Delta]{Cu,\ 573\ K} \text{No dehydrogenation. Sometimes dehydration occurs to give alkenes}$$

(v) **Replacement by halogens** —OH of alcohols can be displaced by different halogens to give alkyl halides.

$$R-OH \xrightarrow{PCl_5 \text{ or } PCl_3 \text{ or } SOCl_2} R-Cl$$

$$R-OH \xrightarrow{HBr \text{ gas or } PBr_3 \text{ or } NaBr/H_2SO_4} R-Br$$

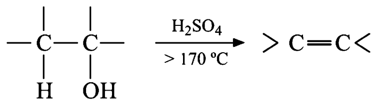
$$R-OH \xrightarrow{HI \text{ or } I_2/\text{red P}} R-I$$

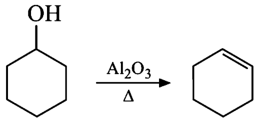
Lucas Test This test is used to distinguish 1°, 2° and 3° alcohols. It is based on the fact that the order of formation of chlorides from alcohols is 3°>2°>1°. In this test alcohols are treated with a mixture of concentrated hydrochloric acid and anhydrous zinc chloride, also called *Lucas Reagent.* The alkyl chlorides so formed produce a cloudiness because they are insoluble.

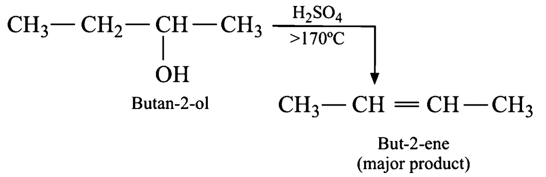
$$R-OH + HCl \xrightarrow{ZnCl_2} R-Cl + H_2O$$

In case of 3° alcohols, cloudiness appears within a minute. It takes about 5 minutes in case of 2° alcohols whereas the turbidity does not appear at room temperature in case of 1° alcohols.

(vi) **Dehydration** Alcohols on heating to high temperature in presence of conc. H_2SO_4 or Al_2O_3 give alkenes and the major product is obtained according to Saytzeff rule







The order of ease of dehydration in alcohols follows the sequence

$$3° > 2° > 1°$$

The product obtained from alcohols by the treatment with H_2SO_4 depends on temperature of the reaction as well as the quantity of H_2SO_4 used as exemplified below, taking C_2H_5OH as an example.

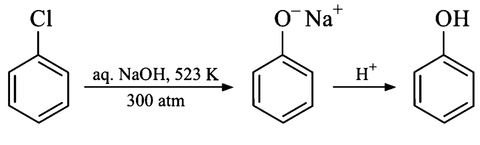
$$CH_3—CH_2—OH + \text{Conc. } H_2SO_4 \xrightarrow{-\ 90° C}$$

$$\underset{\text{(excess)}}{CH_3-CH_2-OH} + \underset{\text{(less amount)}}{\text{Conc.}H_2SO_4} \xrightarrow{-140° C} \underset{\text{(Diethyl ether)}}{CH_3—CH_2—O-CH_2-CH_3}$$

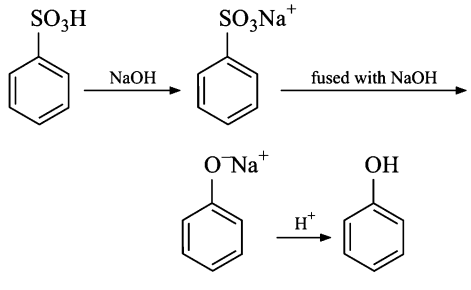
$$CH_3—CH_2—OH + \text{Conc. } H_2SO_4 \xrightarrow{>170° C} \underset{\text{(Ethene)}}{CH_2=CH_2}$$

Preparation of Phenols

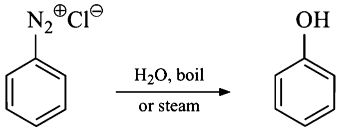
(i) **From chlorobenzene (Dow's process)**



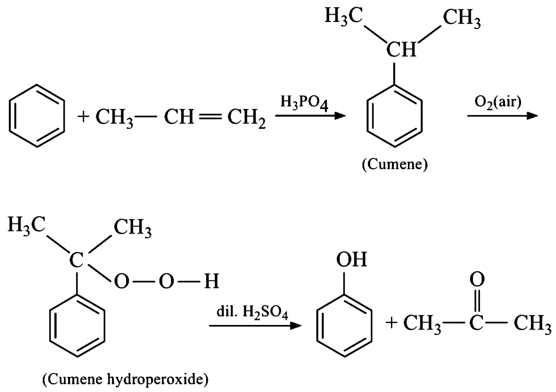
(ii) **From sulphonic acids**



(iii) **From diazonium salts**



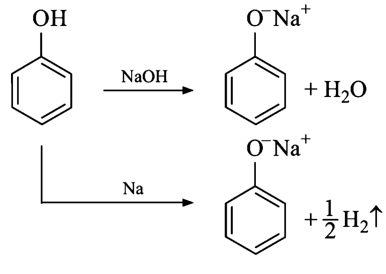
(iv) **Cumene process**



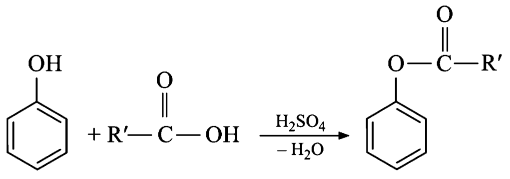
Properties of Phenols

(i) **Acidic character** Phenols are slightly more acidic than alcohols. They can react (i) with alkalies to form salts

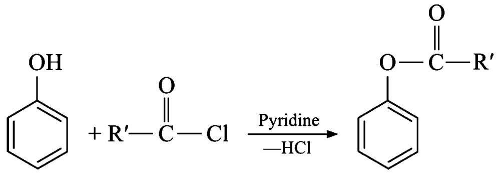
(ii) with more reactive metals to liberate hydrogen.



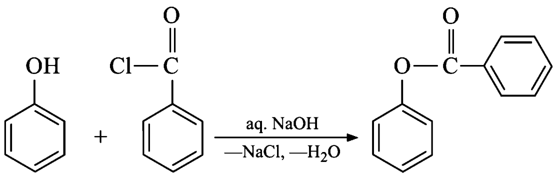
(ii) **Ester formation.** Phenols react with acids to form esters.



Reaction is more convenient with acid chlorides.

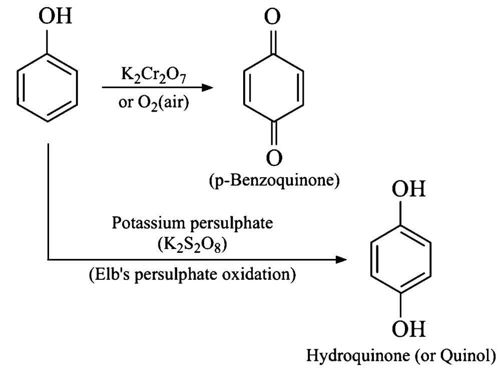


If we use an aromatic acid chloride and aq. NaOH the reaction is known as **Schotten-Baumann reaction.**

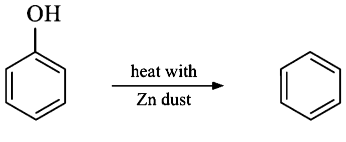


(Benzoyl chloride) (Phenyl benzoate)

(iii) **Oxidation**

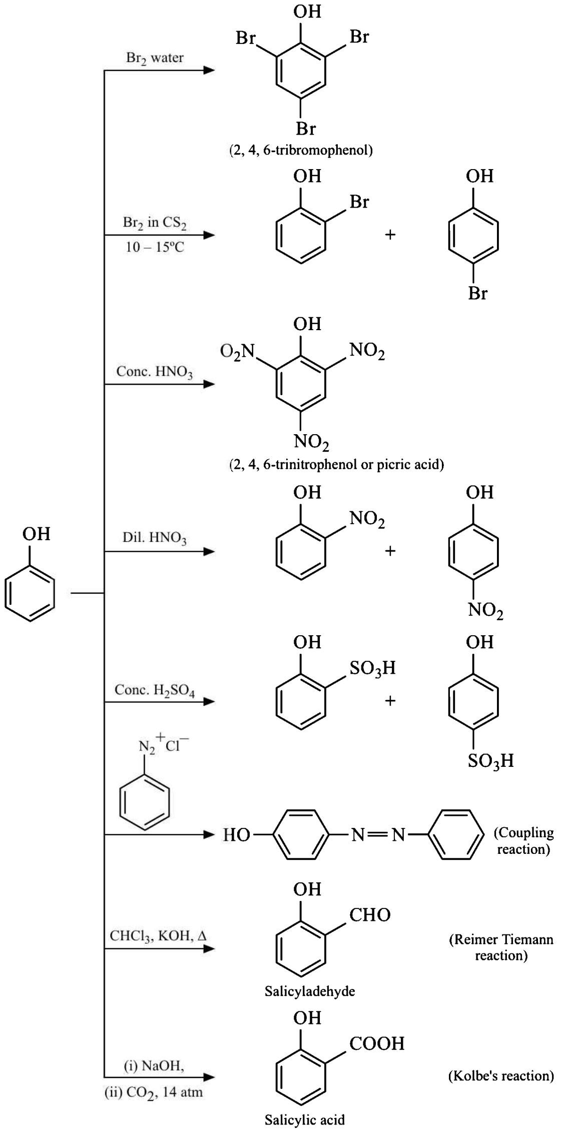


(iv) **Reduction**



(v) **Reaction with Fe^{3+} ions** Phenols give blue-purple colour with neutral ferric chloride. The colour is due to formation of complex with Fe^{3+} ions where phenol molecules act as ligands. This reaction is used as test for phenols.

(vi) **Ring substitution reaction** —OH groups in phenols is activating and ortho, para-directing group for electrophilic substitution reactions which are summarized below.



Methods of Preparation of Ethers

(i) **By Williamson's synthesis** In this method an alkyl halide is treated with sodium alkoxide (or phenoxide).

$$R—X + Na^+O^-—R' \longrightarrow R—O—R' + NaX$$

This method can be used for the preparation of symmetrical as well as unsymmetrical ethers.

(ii) **By dehydration of alcohols** In the presence of concentrated sulphuric acid, alcohols are dehydrated to ethers when excess of alcohol is heated with conc.sulphuric acid at 413K. The alcohol is continuously added to keep its concentration in excess.

$$R—OH + H_2SO_4 \longrightarrow ROSO_3H + H_2O$$

$$R—OH + ROSO_3H \xrightarrow{413\,K} R—O—R + H_2SO_4$$

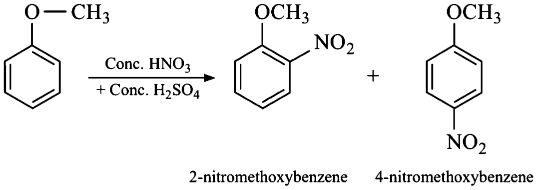
Properties of Ethers

(i) **Cleavage by acids** Ethers are cleaved at C—O bond, by heating with HI or HBr to 370K.

$$R—O—R + HX \xrightarrow{370\,K} R— OH + R—X$$

However if excess of acid is used the alcohol first formed may react further with halogen acid to give the corresponding alkyl halide. The cleavage of aryl alkyl ethers with hydrogen halide always yields phenol and a molecule of alkyl halide and not the aryl halide and alcohol.

(ii) **Ring substitution in aromatic ethers** Aromatic ethers undergo electrophilic substitution in the benzene ring at the ortho and para-positions. For example nitration of anisole gives the mixture of two products.



(iii) **Peroxide formation** When ethers are exposed to air and light for a long time, they are oxidised to form peroxide. These peroxides are highly unstable compounds and may explode violently on heating.

Industrial preparation of methanol

(i) **By destructive distillation of wood:** The aqueous distillate obtained is called pyroligneous acid. It contains about 3% methyl alcohol, 0.5% acetone and about 10% acetic acid. Methyl alcohol is obtained from it by a series of steps.

(ii) **From water gas:** Water gas ($CO + H_2$) is mixed with half of its volume of hydrogen and passed over heated catalyst ($ZnO + CrO_3$) at a pressure of about 300 atmosphere

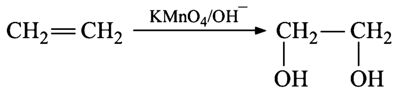
$$CO + 2H_2 \xrightarrow[\substack{573\,K \\ \text{and } 300\,\text{atm}.}]{ZnO + CrO_3} CH_3OH$$

(iii) **From natural gas:** Methane from natural gas is mixed with oxygen and passed over a catalyst under pressure of 100 atmosphere at a temperature of 533 K to give methanol.

$$CH_4 + \frac{1}{2}O_2 \longrightarrow CH_3OH$$

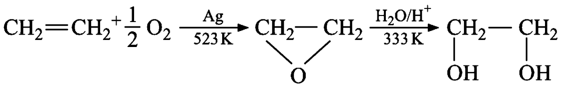
Ethylene glycol (Ethane –1, 2– diol)

It is commercially obtained by the hydroxylation of ethene with cold dilute, alkaline potassium permanganate solution



$$CH_2{=}CH_2 \xrightarrow{KMnO_4/OH^-} \underset{OH}{\underset{|}{CH_2}}-\underset{OH}{\underset{|}{CH_2}}$$

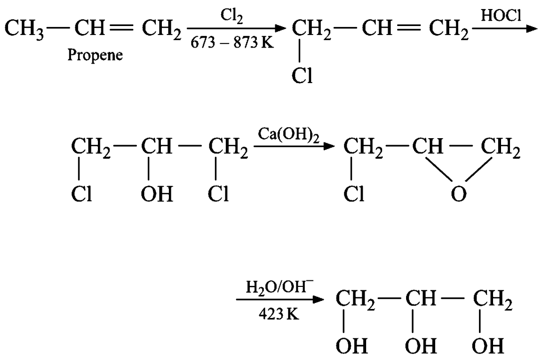
It is also prepared be the oxidation of ethene in presence of silver catalyst to form ethylene oxide followed by its hydrolysis.



Ethylene glycol is used as antifreeze for motor car radiators and as cooling liquid in aeroplane motors. It is also used for the synthesis of various important compoounds *e.g.* terylene, nitroglycol (used as explosive), diglycol oleate (used as rubber) etc.

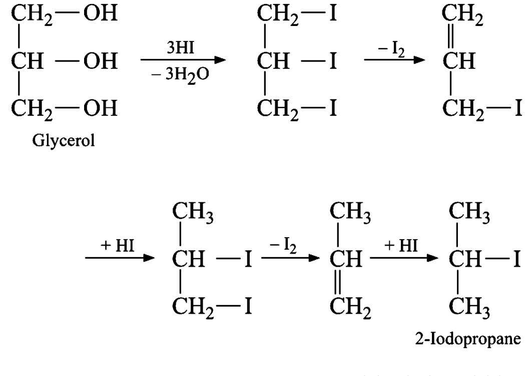
Glycerol (Propane – 1, 2, 3 – triol)

It is the byproduct obtained during the manufacture of soap from oils. Commercially it is obtained from propene through a series of reactions given below.

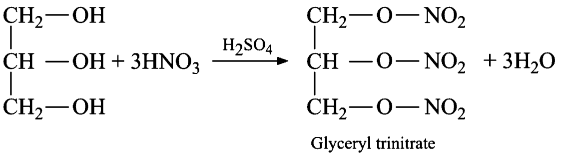


Some important reactions of glycerol are:

(i) **Reaction with hydrogen iodide:** When glycerol is heated with excess of hydrogen iodide, it undergoes a sequence of reactions to yield 2-iodopropane ultimately.



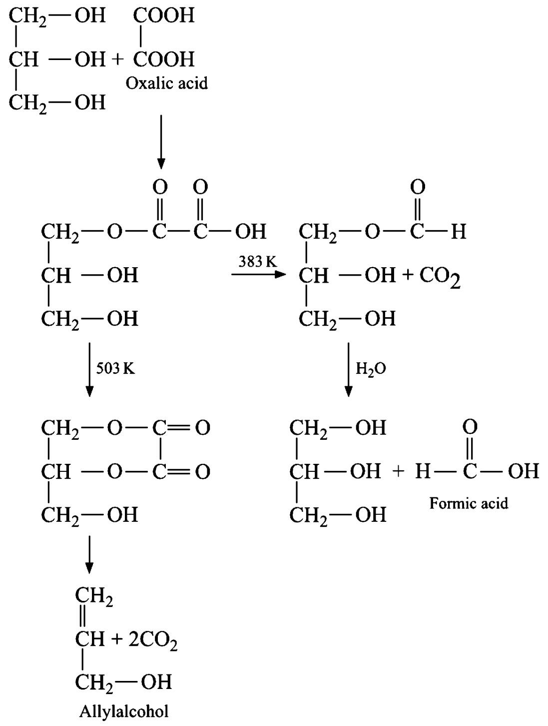
(ii) **Reaction with nitric acid:** It reacts with nitric acid in presence of sulphuric acid to from glyceryl trinitrate, commonly known as nitroglycerine which is used in dynamite.



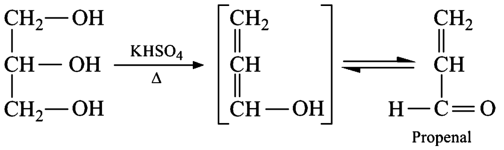
$$\begin{array}{l}CH_2-OH\\ |\\ CH-OH\\ |\\ CH_2-OH\end{array} + 3HNO_3 \xrightarrow{H_2SO_4} \begin{array}{l}CH_2-O-NO_2\\ |\\ CH-O-NO_2\\ |\\ CH_2-O-NO_2\end{array} + 3H_2O$$

Glyceryl trinitrate

(iii) **Reaction with oxalic acid:** It reacts with oxalic acid to give formic acid or allyl alcohol depending upon the temperature.



(iv) **Dehydration** When glycerol is heated with potassium hydrogen sulphate, it forms propenal (acrolein)



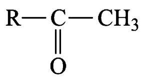
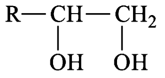
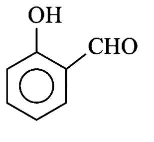
$$\begin{array}{l}CH_2-OH\\ |\\ CH-OH\\ |\\ CH_2-OH\end{array} \xrightarrow[\Delta]{KHSO_4} \left[\begin{array}{l}CH_2\\ \|\\ CH\\ \|\\ CH-OH\end{array}\right] \rightleftharpoons \begin{array}{l}CH_2\\ \|\\ CH\\ |\\ H-C{=}O\end{array}$$

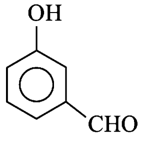
Propenal

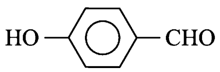
Glycerol can retain humidity due to which it is also called as humectant. It is mainly used

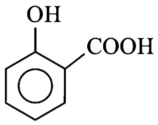
(i) as a sweetening agent
(ii) in creams and other cosmetics
(iii) as a preservative for fruits etc
(iv) in toothpaste, shaving creams and soaps etc.
(v) in making ink for printing
(vi) for the preparation of nitroglycerine which is used in dynamite
(vii) as anti-freeze
(viii) as lubricant in fine machinery such as watches etc.

Problems for Practice

1. Which of the following has maximum hydrogen bonding?
   (a) Ethyl amine (b) Ammonia
   (c) Ethyl alcohol (d) Diethyl ether
2. Which of the following is the most suitable method for removing the traces of water from ethanol?
   (a) Heating with Na metal
   (b) Passing dry HCl through it
   (c) Distilling it
   (d) Reacting with Mg
3. Which of the following is most acidic?
   (a) Phenol (b) Benzyl alcohol
   (c) m-chlorophenol (d) Cyclohexanol
4. R—CH_2—CH_2—OH can be converted into RCH_2CH_2COOH by the following sequence of steps
   (a) PBr_3, KCN, H_3O^+ (b) PBr_3, KCN, H_2/Pt
   (c) KCN, H_3O^+ (d) HCN, PBr_3, H_3O^+
5. In glycerine,
   (a) One primary —OH group is present only
   (b) One tertiary —OH Group is present
   (c) Two secondary —OH groups are present
   (d) One secondary —OH group is present
6. Which of the following process is employed to convert alkyl halide into alcohol?
   (a) Addition
   (b) Substitution
   (c) Dehydrohalogenation
   (d) Molecular rearrangement
7. $R—OH + HX \longrightarrow RX + H_2O$
   In the above reaction, the reactivity of alcohols is
   (a) Tertiary > secondary > primary
   (b) Tertiary < secondary < primary
   (c) Tertiary > primary > secondary
   (d) Secondary > primary > tertiary
8. Which of the following isomer of butanol has a chiral structure?
   (a) $(CH_3)_3COH$
   (b) $(CH_3)_2CH.CH_2OH$
   (c) $CH_3CH(OH)CH_2CH_3$
   (d) $CH_3(CH_2)_3OH$
9. Intramolecular hydrogen bonding is found in:
   (a) p-Nitrophenol (b) o-Nitrophenol
   (c) m-Nitrophenol (d) Phenol
10. Which one of the following compounds gives characteristic deep colour with $FeCl_3$ solution?
    (a) acetone (b) ethanol
    (c) phenol (d) acetic acid
11. Sodium phenoxide reacts with CO_2 at 400Kand 14 atm pressure to give
    (a) Sodium salicylate (b) Salicylaldehyde
    (c) Catechol (d) Benzoic acid
12. Which of the following does not give yellow ppt. with I_2/NaOH?
    (a) C_2H_5OH (b) CH_3CHO
    (c) CH_3COCH_3 (d) HCHO
13. The order of reactivity of halogen acids for reaction with C_2H_5OH is
    (a) HCl > HBr > HI (b) HI > HBr >HCl
    (c) HBr > HI > HCl (d) HBr > HCl > HI
14. Which of the following compound does not react with metallic sodium?
    (a) $CH_3CH(OH)CH_3$ (b) CH_3—O—CH_3
    (c) CH_3COOH (d) C_2H_5OH
15. Which of the following reagents is useful in separating benzoic acid from phenol?
    (a) Dil. HCl
    (b) Dil. H_2SO_4
    (c) 5% aqueous NaOH
    (d) 5% aqueous $NaHCO_3$.
16. Alkene R—CH═CH_2 reacts with B_2H_6 in the presence of H_2O_2 to give
    (a) 
    $R-\underset{\underset{O}{\|}}{C}-CH_3$
    (b) 
    $R-\underset{\underset{OH}{|}}{CH}-\underset{\underset{OH}{|}}{CH_2}$
    (c) R—CH_2—CHO (d) R—CH_2—CH_2—OH
17. Grignard reagent on reaction with acetone forms
    (a) Tertiary alcohol (b) Secondary alcohol
    (c) Acetic acid (d) Acetaldehyde
18. Phenol, *p*-Methylphenol, *m*-Nitrophenol and *p*-Nitrophenol follow order of increasing acidic strength
    (a) Phenol, *p*-Methylphenol, *p*-Nitrophenol, *m*-Nitrophenol
    (b) *p*-Methylphenol, Phenol, m-Nitrophenol, *p*-Nitrophenol
    (c) *p*-Methylphenol, *m*-Nitrophenol, Phenol, *p*-Nitrophenol
    (d) *m*-Nitrophenol, *p*-Nitrophenol, phenol and *p*-Methylphenol
19. What is the major product obtained when phenol is treated with chloroform and aqueous alkali?
    (a) 

    (b) 

(c) 

(d) 

20. An organic compound A reacts with methyl magnesium iodide to form an addition product which on hydrolysis forms the compound B. Compound B gives blue colour salt in Victor Meyer's test. The compounds A and B are respectively.

(a) Acetaldehyde, tertiary butyl alcohol
(b) Acetaldehyde, ethyl alcohol
(c) Acetaldehyde, isopropyl alcohol
(d) Acetone, isopropyl alcohol

21. The compound 'A' when treated with methyl alcohol and few drops of H_2SO_4 gave smell of wintergreen. The compound 'A' is

(a) Succinic acid (b) Salicylic acid
(c) Tartaric acid (d) Oxalic acid

22. The ether that undergoes electrophilic substitution reaction is

(a) $CH_3OC_2H_5$ (b) $C_6H_5OCH_3$
(c) CH_3OCH_3 (d) $C_2H_5OC_2H_5$

23. Which of the following has highest solubility in water?

(a) n—C_4H_{10} (b) $(CH_3)_2CH—CH_3$
(c) $C_2H_5—O—C_2H_5$ (d) benzene

24. An organic compound of molecular formula $C_4H_{10}O$ does not react with sodium. With excess of HI, it gives only one type of alkyl halide. The compound is

(a) Ethoxyethane (b) 2-Methoxypropane
(c) 1-Methoxypropane (d) 1-Butanol

25. For the reaction

$$C_2H_5OH + HX \xrightarrow{ZnCl_2} C_2H_5X$$

The order of reactivity is:

(a) HBr > HI > HCl (b) HI > HCl > HBr
(c) HI > HBr > HCl (d) HCl > HBr > HI

26. When ether is exposed in air for sometime an explosive substance produced is

(a) Peroxide (b) TNT
(c) Oxide (d) Superoxide

27. When an alkyl halide reacts with an alkoxide, the product is

(a) hydrocarbon
(b) ether
(c) unsaturated hydrocarbon
(d) alcohol

28. Carbolic acid is the name for

(a) phenol (b) phenyl benzoate
(c) phenyl acetate (d) salol

29. Which of the following compound is known as *oil of winter green*?

(a) Phenyl benzoate (b) Phenyl acetate
(c) Phenyl acetate (d) Methyl salicylate

30. Glycerol, on heating with oxalic acid at 383 K gives

(a) Ethanol (b) Methanoic acid
(c) Ether (d) Acetone

31. The wrong statement about glycerol is

(a) It is a trihydric alcohol
(b) It is generally obtained by hydrolysis of oils/fats
(c) It is a tertiary alcohol
(d) Used in the manufacture of explosives

32. Propanone is the product obtained by dehydrogenation of

(a) 2-Propanol (b) 1-Propanol
(c) Isobutyl alcohol (d) Propanethiol

33. Phenol is treated with bromine water and shaken well to get white precipitate. The white precipitate is

(a) 1-Bromophenol
(b) 2,4, 6-Tribromophenol
(c) 2, 4-Dibromophenol
(d) Mixture of *o*-, *p*-bromophenol

34. Heating of methyl phenyl ether with HI gives

(a) Methanol + Iodobenzene
(b) Methyl iodide + Iodobenzene
(c) Methyl alcohol + Benzyl alcohol
(d) Methyl iodide + Phenol

35. Organic acid without a carboxylic acid group is

(a) ascorbic acid (b) vinegar
(c) oxalic acid (d) picric acid

36. Which of the following statements is correct?

(a) Phenol is less acidic than ethyl alcohol
(b) Phenol is more acidic than ethanol
(c) Phenol is more acidic than carbonic acid
(d) Phenol is more acidic than CH_3COOH

37. Ethyl alcohol gives ethyl chloride with the help of

(a) $SOCl_2$ (b) NaCl
(c) Cl_2 (d) KCl

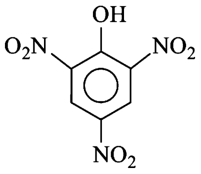
38. Isopropyl alcohol on oxidation gives

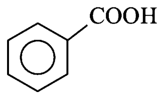
(a) Acetone (b) Ether
(c) Ethylene (d) Acetaldehyde

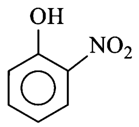
39. The reaction of ethanol with H_2SO_4 does not give

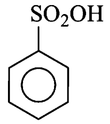
(a) Ethylene
(b) Diethyl ether
(c) Acetylene
(d) Ethyl hydrogen sulphate

40. Which of the following will not be soluble in sodium carbonate solution?

(a) 

(b) 

(c) 

(d) 

41. Which of the following reagent will convert Propanoic acid into Propan-1-ol?

(a) $KMnO_4$
(b) MnO_2
(c) Cr_2O_3
(d) $LiAlH_4$

42. HBr reacts faster with

(a) 2-methylpropanol-2
(b) Propanol-1
(c) Propanol-2
(d) 2-methylpropanol-1

43. 3 moles of ethanol react with one mole of phosphorus tribromide to form 3 moles of bromoethane and one mole of X. Which of the following is X?

(a) H_3PO_4
(b) H_3PO_2
(c) HPO_3
(d) H_3PO_3

44. C_2H_5OH can be differentiated from CH_3OH by

(a) reaction with HCl
(b) reaction with NH_3
(c) Iodoform test
(d) Solubility is water

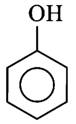
45. The reaction between alcohol and carboxylic acids is called

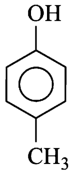
(a) Esterification
(b) Hydrolysis
(c) Saponification
(d) Hydrogenation

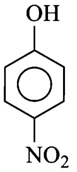
46. Which of the following is not formed when glycerol reacts with HI?

(a) $CH_2 = CH—CH_2I$
(b) $CH_2OH—CHI—CH_2OH$
(c) $CH_3—CH = CH_2$
(d) $CH_3—CHI—CH_3$

47. The correct acidic order of the following is

I. 

II. 

III. 

(a) I > II > III
(b) III > I > II
(c) II > III > I
(d) I > III > II

48. A compound of the formula $C_4H_{10}O$ reacts with sodium and undergoes oxidation to give a carbonyl compound which does not reduce Tollen's reagent, the original compound is

(a) Diethyl ether
(b) *n*-Butyl alcohol
(c) Isobutyl alcohol
(d) *sec*-Butyl alcohol

49. Zinc powder + Ph—OH $\longrightarrow$ X. In the above reaction the product X will be

(a) Benzaldehyde
(b) Benzene
(c) Anisole
(d) Phenyl acetate

50. Phenol is more readily soluble in

(a) dil.HCl
(b) Both NaOH and HCl
(c) NaOH solution
(d) Sodium bicarbonate solution

51. Lucas Test is performed for

(a) Amines
(b) Alcohols
(c) Ethers
(d) Alkyl halides

52. The correct order of acidic strength of CH_3CH_2OH, $(CH_3)_2CHOH$, $(CH_3)_3COH$ and H_2O is

(a) $H_2O > (CH_3)_3C—OH > (CH_3)_2CHOH > CH_3\ CH_2OH$
(b) $H_2O > CH_3CH_2OH > (CH_3)_2\ CHOH > (CH_3)_3C.OH$
(c) $(CH_3)_3C.OH > (CH_3)_2CHOH > CH_3CH_2OH > H_2O$
(d) None of these

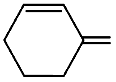
53. In Victor-Meyer test, blue colouration is shown by

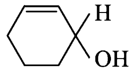
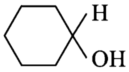
(a) Primary alcohol
(b) Diol
(c) Secondary alcohol
(d) Tertiary alcohol

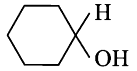
54. Identify the final product

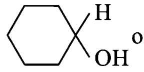
$$CH_2 = CH_2 \xrightarrow{O_2, Ag} (X) \xrightarrow[\text{Steam}]{473\text{ K}} Y$$

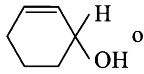
(a) Ethanol
(b) Ethanal
(c) Epoxyethane
(d) Ethylene glycol

55. Products expected from the reaction of  with (a) $NaBH_4$ and (b)H_2/Pt. are

(a)  and  respectively

(b)  and  respectively

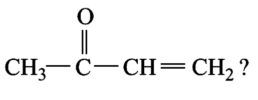
(c)  only

(d)  only

56. Which reagent will you use to convert

$$CH_3—CHOH—CH= CH_2$$

into



$$CH_3—\overset{\overset{\displaystyle O}{\|}}{C}—CH= CH_2\ ?$$

(a) Acidic $KMnO_4$
(b) Alkaline $KMnO_4$
(c) MnO_2
(d) None of these

57. Which of the following reagent will convert glycerol to acrolein?

(a) P_2O_5
(b) Conc. H_2SO_4
(c) $KHSO_4$
(d) All the three

58. Electrophilic substitution in phenol generally occurs at

(a) *o*- and *p*-positions
(b) meta positions
(c) only at ortho positions
(d) only at para position

59. Phenols are more acidic than alcohols because
(a) phenoxide ion is stabilised by resonance
(b) phenols are more soluble in polar solvents
(c) phenoxide ions do not exhibit resonance
(d) Alcohols do not lose H atoms

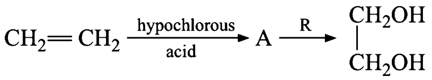
60. Conversion of ethyl alcohol into acetaldehyde is an example of
(a) Reduction
(b) Molecular rearrangement
(c) Hydrolysis
(d) Oxidation

61. To prepare 3-ethylpentan-3-ol, the reagents needed are
(a) $CH_3CH_2MgBr + CH_3COCH_2CH_3$
(b) $CH_3MgBr + CH_3CH_2CH_2COOCH_2CH_3$
(c) $CH_3CH_2MgBr + CH_3CH_2COCH_2CH_3$
(d) $CH_3CH_2CH_2MgBr + CH_3COCH_2CH_3$

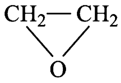
62. Which of the following alcohols is wood spirit?
(a) Methanol (b) 2-Methylpropanol-1
(c) propanol-1 (d) Butanol-1

63. When wine is put in air, it becomes sour due to
(a) Bacteria
(b) Oxidation of C_2H_5OH into CH_3COOH
(c) Virus
(d) Formic acid formation

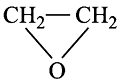
64. In the reaction sequence,



$$CH_2{=}CH_2 \xrightarrow[\text{acid}]{\text{hypochlorous}} A \xrightarrow{R} \begin{array}{c} CH_2OH \\ | \\ CH_2OH \end{array}$$

A and R are respectively
(a)  and heat
(b) CH_3CH_2Cl and NaOH
(c) CH_3CH_2OH and H_2SO_4
(d) $CH_2Cl.CH_2OH$ and $NaHCO_3$

65. $(CH_3)_2CHOH \xrightarrow{\text{Mild oxid}} X \xrightarrow[(ii)\ \text{HOH}]{(i)\ CH_3MgBr} Y$ Here Y is
(a) Isobutyl alcohol (b) *tert*-Butyl alcohol
(c) Isobutylene (d) *sec*-Butyl alcohol

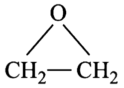
66. Reaction of  with R MgX followed by hydrolysis will lead to produce
(a) RCHOHR (b) $RCHOHCH_3$
(c) R_2CHCH_2OH (d) RCH_2CH_2OH

67. The reaction of aromatic acyl chloride and phenol in the presence of a base NaOH or pyridine is called
(a) Kolbe's reaction
(b) Perkin's reaction
(c) Sandmeyer's reaction
(d) Schotten Baumann reaction

68. Action of nitrous acid on methyl amine could give
(a) CH_4 (b) CH_3OH
(c) NH_3 (d) Nitromethane

69. Propan-1-ol can be obtained from propene by treating with
(a) B_2H_6 + NaOH (b) $B_2H_6 + H_2O_2/OH^-$
(c) $LiAlH_4 + H_2O/H^+$ (d) $HCl + H_2O/H^+$

70. Which is used as an anti-freeze?
(a) Glycol (b) Ethyl alcohol
(c) Water (d) Methanol

71. 2-Phenylethanol may be prepared by the reaction of phenyl magnesium bromide with
(a) HCHO (b) CH_3CHO
(c) CH_3COCH_3 (d) 

72. Which of the following statements is correct?
(a) C_2H_5 Br reacts with alcoholic KOH to from C_2H_5OH
(b) C_2H_5Br when treated with metallic sodium forms diethyl ether
(c) C_2H_5 Br when treated with sodium ethoxide forms diethyl ether
(d) Ethers have zero dipole moment

73. Diethyl ether absorbs oxygen to form
(a) Red coloured sweet smelling compound
(b) Acetic acid
(c) Ether suboxide
(d) Ether peroxide

74. The compound which reacts fastest with Lucas reagent at room temperature is
(a) Butan-1-ol
(b) Butan-2-ol
(c) 2-Methylpropan-1-ol
(d) 2-Methylpropan-2-ol

75. Which of the following ethers can not be synthesized by Williamson's synthesis?
(a) $C_2H_5OC_2H_5$
(b) $CH_3\ CH_2\ CH_2\ CH_2O\ CH_2\ CH_3$
(c) $CH_3O\ CH_2\ CH_2\ CH_2\ CH_2\ CH_3$
(d) $(CH_3)_3CO\ C(CH_3)_3$

76. Ethyl chloride is converted into diethyl ether by
(a) Wurtz synthesis
(b) Grignard reaction
(c) Perkin's reaction
(d) Williamson's synthesis

ANSWERS

1. (c)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (c)
11. (a)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (d)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (c)
21. (b)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (b)
31. (c)	**32.** (a)	**33.** (b)	**34.** (d)	**35.** (d)	**36.** (b)	**37.** (a)	**38.** (a)	**39.** (c)	**40.** (c)
41. (d)	**42.** (a)	**43.** (d)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (b)	**48.** (d)	**49.** (b)	**50.** (c)
51. (b)	**52.** (b)	**53.** (c)	**54.** (d)	**55.** (a)	**56.** (c)	**57.** (d)	**58.** (a)	**59.** (a)	**60.** (d)
61. (c)	**62.** (a)	**63.** (b)	**64.** (d)	**65.** (b)	**66.** (d)	**67.** (d)	**68.** (b)	**69.** (b)	**70.** (a)
71. (d)	**72.** (c)	**73.** (d)	**74.** (d)	**75.** (d)	**76.** (d)				

26

Aldehydes and Ketones

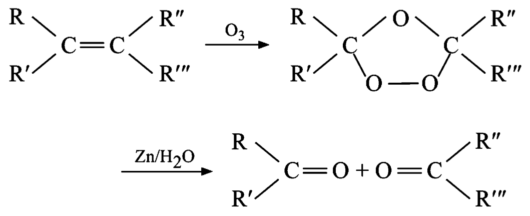
BRIEF REVIEW OF THE CONCEPTS

Aldehydes and Ketones both contain $>C=O$ group which is known as carbonyl group. In aldehydes the carbonyl carbon is attached to at least one hydrogen atom, while the other can be hydrogen, alkyl or aryl group. In ketones both the groups are alkyl or aryl which can be same or different. The general formula of aldehydes and ketones are as follows:

$$\underset{\text{(Aldehydes)}}{R-\overset{\overset{\displaystyle O}{\|}}{C}-H} \qquad \underset{\text{(Ketones)}}{R-\overset{\overset{\displaystyle O}{\|}}{C}-R'}$$

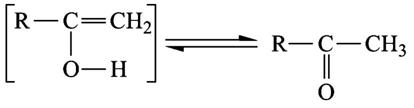
Methods of preparations of aldehydes and ketones

(i) **By the ozonolysis of alkenes** Alkenes on reaction with ozone yield ozonides which on further treatment with water in presence of zinc metal (also called reductive hydrolysis) give aldehydes and ketones.



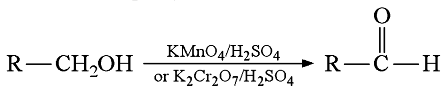
(ii) **By hydration of alkynes** Alkynes add up a molecule of water in presence of 40% H_2SO_4 and 1% $HgSO_4$ to give an enol which undergo ketoenol tautomerism to give aldehyde or ketone. The addition of water molecule proceeds according to Markownikov's rule.

$$RC\equiv CH + H_2O \xrightarrow[1\%\ HgSO_4]{40\%\ H_2SO_4}$$



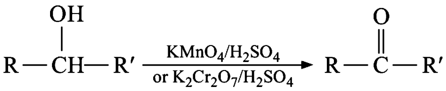
$$\left[\begin{array}{c}R-C=CH_2\\ \quad\ |\\ \quad O-H\end{array}\right] \rightleftharpoons R-\underset{\underset{\displaystyle O}{\|}}{C}-CH_3$$

(iii) **By oxidation of alcohols** Primary alcohols on controlled oxidation give aldehydes whereas ketones are obtained by the oxidation of secondary alcohols. The most commonly used oxidising agents are potassium permanganate or potassium dichromate and H_2SO_4.



$$\underset{(1^\circ\ \text{alcohol})}{R-CH_2OH} \xrightarrow[\text{or } K_2Cr_2O_7/H_2SO_4]{KMnO_4/H_2SO_4} \underset{\text{(aldehydes)}}{R-\overset{\overset{\displaystyle O}{\|}}{C}-H}$$

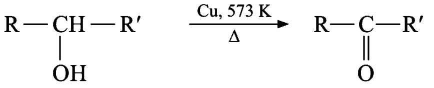
Aldehyde formed is distilled out as soon as it is formed to prevent it from oxidation to corresponding acid.



$$\underset{(2^\circ\ \text{alcohol})}{R-\overset{\overset{\displaystyle OH}{|}}{C}H-R'} \xrightarrow[\text{or } K_2Cr_2O_7/H_2SO_4]{KMnO_4/H_2SO_4} \underset{\text{(ketones)}}{R-\overset{\overset{\displaystyle O}{\|}}{C}-R'}$$

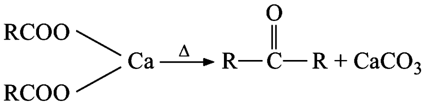
(iv) **From alcohols by dehydrogenation**

$$\underset{(1^\circ\ \text{alcohol})}{R-CH_2-OH} \xrightarrow[\Delta]{Cu,\ 573\ K} \underset{\text{(aldehydes)}}{RCHO}$$

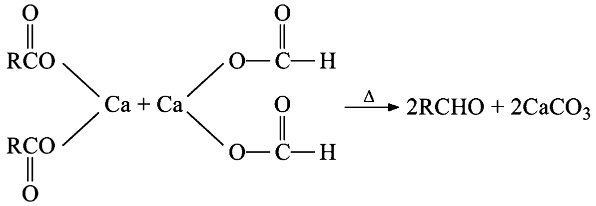


$$\underset{(2^\circ\ \text{alcohol})}{R-\underset{\underset{\displaystyle OH}{|}}{C}H-R'} \xrightarrow[\Delta]{Cu,\ 573\ K} \underset{\text{(ketones)}}{R-\underset{\underset{\displaystyle O}{\|}}{C}-R'}$$

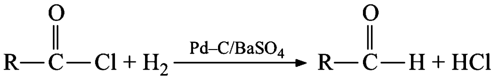
(v) **From carboxylic acids** Calcium salts of carboxylic acids on heating yield ketones.



An aldehyde can be prepared by heating Ca salt of a fatly acid with calcium formate.

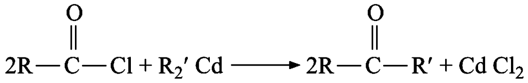


(vi) **From acid chlorides** Aldehydes can be obtained by the *Rosenmund's reduction* of acid chlorides where the reduction is carried out by hydrogen in presence of palladium spread over charcoal containing some $BaSO_4$.



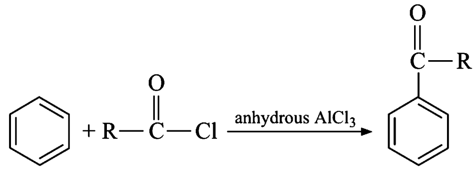
$$R-\overset{\overset{\displaystyle O}{\|}}{C}-Cl + H_2 \xrightarrow{Pd\text{–}C/BaSO_4} R-\overset{\overset{\displaystyle O}{\|}}{C}-H + HCl$$

Ketones are obtained by the reaction of acid chlorides with dialkyl cadmium.

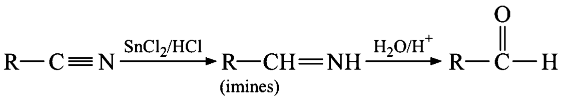


$$2R-\overset{\overset{\displaystyle O}{\|}}{C}-Cl + R_2'\,Cd \longrightarrow 2R-\overset{\overset{\displaystyle O}{\|}}{C}-R' + Cd\,Cl_2$$

Acid chlorides are also used in the preparation of aromatic ketones by Friedel Crafts acylation.

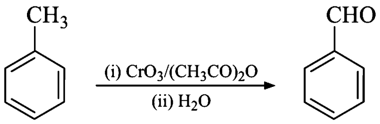


(vii) **From cyanides** Aldehydes are obtained from cyanides by partial reduction followed by hydrolysis. The method is known as **Stephen's method.**



$$R-C\equiv N \xrightarrow{SnCl_2/HCl} \underset{\text{(imines)}}{R-CH=NH} \xrightarrow{H_2O/H^+} R-\overset{\overset{O}{\|}}{C}-H$$

(viii) **Special methods for benzaldehyde** It is obtained by the oxidation of toluene with chromic oxide in acetic anhydride.

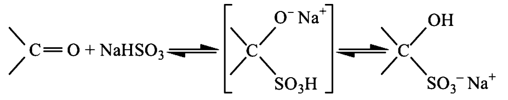


Use of acetic anhydride prevents the oxidation of benzaldehyde to benzoic acid by forming an oxidation resistant intermediate.

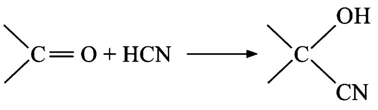
Benzaldehyde can also be obtained from toluene by **Etard's reaction.** In this method chromyl chloride (CrO_2CI_2) is used as the oxidizing agent.

Properties of aldehydes and ketones Since both aldehydes and ketones have the same functional group (i.e., C = O group) so most of their properties are also common. They undergo mainly nucleophilic addition reactions. Aldehydes are more reactive towards these reactions than ketones. Aromatic aldehydes and ketones are generally less reactive than aliphatic aldehydes and ketones. These reactions are briefly discussed below.

(i) **Addition of sodium bisulphite ($NaHSO_3$)** Sodium bisulphite adds on to most of the aldehydes and methyl ketones to give solid products which are called bisulphite addition products and are generally used to purify these compounds.

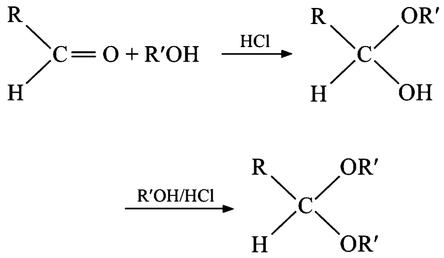


(ii) **Addition of HCN** Both aldehydes and ketones react with HCN to give addition products called *cyanohydrins*



(iii) **Addition of Grignard's reagents** Grignard reagents add to aldehydes and ketones to give alcohols.

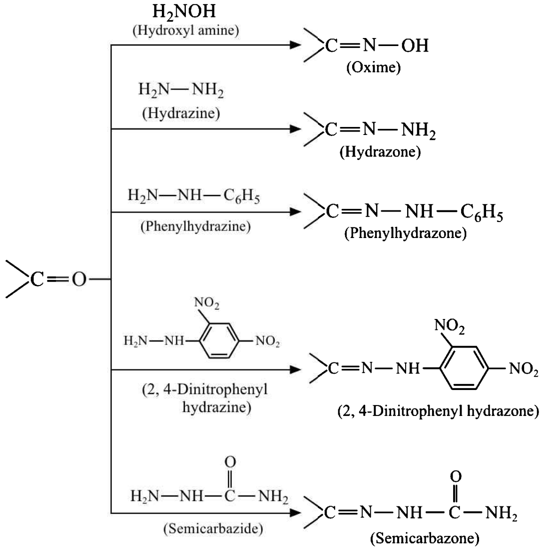
(iv) **Addition of alcohols to aldehydes** Aldehydes react with alcohols in the presence of dry hydrogen chloride. A *hemiacetal* is first formed which reacts with another molecule of alcohol to form a stable acetal. Ketones do not react with alcohols directly but ketals can be prepared by alternate methods.



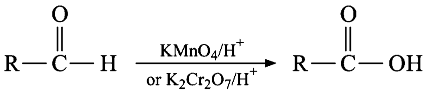
(v) **Addition of ammonia and its derivatives** Both aldehydes and ketones undergo condensation reaction with ammonia and many of its derivatives to give different compounds. Formaldehyde condenses with ammonia to give a compound called *hexamethylenetetraamine* which is used as a medicine called urotropine.

$$6HCHO + 4NH_3 \longrightarrow (CH_2)_6N_4 + 6H_2O$$

Some important ammonia derivatives used and the products obtained are summarized below.

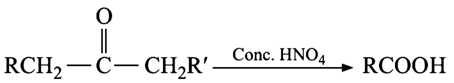


(vi) **Oxidation** Aldehydes are easily oxidized by $KMnO_4$ or $K_2Cr_2O_7$ to the corresponding carboxylic acids.



$$R-\overset{\overset{O}{\|}}{C}-H \xrightarrow[\text{or } K_2Cr_2O_7/H^+]{KMnO_4/H^+} R-\overset{\overset{O}{\|}}{C}-OH$$

Ketones are oxidised under vigorous conditions such as boiling with conc. HNO_3. Their oxidation involves the cleavage of carbon-carbon bond yielding a mixture of carboxylic acids.



$$RCH_2-\overset{\overset{O}{\|}}{C}-CH_2R' \xrightarrow{\text{Conc. } HNO_4} RCOOH$$

$$+ R'\,CH_2COOH + RCH_2COOH + R'\,COOH$$

(vii) **Reduction** $LiAlH_4$ or $NaBH_4$ reduce aldehydes and ketones to corresponding primary or secondary alcohols. Reduction can be carried to hydrocarbons by the following two reactions.

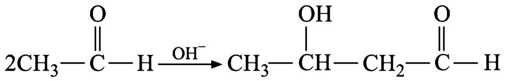
Clemmensen reduction The reaction involves the reduction with zinc amalgam and concentrated hydrochloric acid

$$>C = O \xrightarrow{Zn/Hg-HCl} >CH_2$$

Wolff-Kishner reduction This method involves the treatment with hydrazine followed by boiling with a strong base like KOH

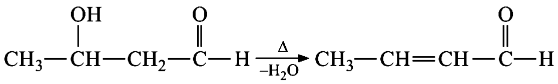
$$>C=O \xrightarrow[\text{(ii) KOH boil in glycol}]{\text{(i) } H_2N-NH_2} >CH_2$$

(viii) **Aldol condensation** This reaction is given by those aldehydes and ketones which have a α-hydrogen atom. The reaction involves the addition of two molecules in presence of dilute alkali to give β-hydroxy-carbonyl compounds (commonly called aldols) e.g.



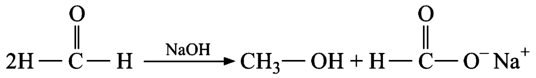
$$2CH_3-\overset{\overset{O}{\|}}{C}-H \xrightarrow{OH^-} CH_3-\overset{\overset{OH}{|}}{CH}-CH_2-\overset{\overset{O}{\|}}{C}-H$$

Such compounds undergo easy dehydration to give α, β-unsaturated compounds.



$$CH_3-\overset{\overset{OH}{|}}{CH}-CH_2-\overset{\overset{O}{\|}}{C}-H \xrightarrow[-H_2O]{\Delta} CH_3-CH=CH-\overset{\overset{O}{\|}}{C}-H$$

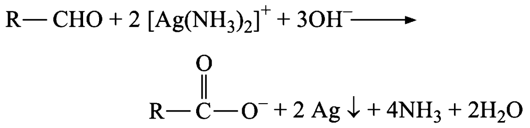
(ix) **Cannizzaro reaction** Aldehydes which do not contain any α-hydrogen atom undergo self oxidation-reduction reaction in presence of conc. Alkali to give corresponding alcohol and salt of the acid.



$$2H-\overset{\overset{O}{\|}}{C}-H \xrightarrow{NaOH} CH_3-OH + H-\overset{\overset{O}{\|}}{C}-O^-Na^+$$

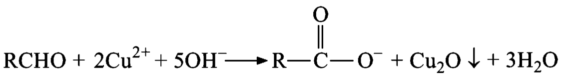
(x) Tests used to distinguish between aldehydes and ketones Due to the easy oxidation of aldehydes, certain reagents have been developed which are used to distinguish between aldehydes and ketones.

(a) **Tollen's test** Tollen's reagent is an ammoniacal solution of silver nitrate. When it is added to an aldehyde, the reduction of silver ions produces metallic silver which is deposited on the inner wall of the test tube and it looks like a mirror, hence it is called silver mirror test.



$$R-CHO + 2[Ag(NH_3)_2]^+ + 3OH^- \longrightarrow R-\overset{\overset{O}{\|}}{C}-O^- + 2Ag\downarrow + 4NH_3 + 2H_2O$$

(b) **Fehling's test** Fehling's solution contain Cu^{2+} ions in the form of a complex with tartrate ions and the medium is made alkaline by adding NaOH. Aldehydes reduce Cu^{2+} to Cu^+ which separates as a red precipitate of Cu_2O.



$$RCHO + 2Cu^{2+} + 5OH^- \longrightarrow R-\overset{\overset{O}{\|}}{C}-O^- + Cu_2O\downarrow + 3H_2O$$

(c) **Iodoform test** Aldehydes and ketones containing $CH_3-\overset{\overset{O}{\|}}{C}-$ group, when heated with I_2 and aqueous alkali form a yellow precipitate of iodoform, which has a characteristic smell.

$$CH_3CHO + 3I_2 + 4NaOH \longrightarrow CHI_3 + HCOONa + 3NaI + 3H_2O$$

$$CH_3COCH_3 + 3I_2 + 4KOH \longrightarrow CHI_3 + CH_3COOK + 3KI + 3H_2O$$

(d) **Action on Schiff's reagent** Schiff's reagent is an aqueous solution of a pink coloured dye called pararosaniline which is decolourized by passing SO_2 gas through it. When aldehydes are treated with this reagent, the pink colour is restored whereas it remains colourless with ketones.

Problems for Practice

1. Paraaldehyde is a trimer of

(a) Formaldehyde (b) Acetaldehyde

(c) Benzaldehyde (d) Propionaldehyde.

2. Which of the following products is formed when benzaldehyde is treated with CH_3MgBr and the additaion product so obtained is subjected to hydrolysis?

(a) Secondary alcohol (b) A primary alcohol

(c) Phenol (d) tert-butyl alcohol

3. The addition of HCN to carbonyl compounds is an example of

(a) Nucleophilic substitution

(b) Electrophilic addition

(c) Nucleophilic addition

(d) Electrophilic substitution

4. The end product in the following sequence of reactions is

$$HC\equiv CH \xrightarrow[40\%H_2SO_4]{1\%HgSO_4} A \xrightarrow{CH_3MgX} B \xrightarrow{[O]}$$

(a) Acetic acid (b) Isopropyl alcohol

(c) Acetone (d) Ethanol

5. Schiff's reagent is

(a) Dilute solution of HBr in water

(b) Dilute solution of HI in water

(c) Solution of pararosaniline hydrochloride in water whose colour is discharged by SO_2

(d) None of these

6. Which of the following does not turn Schiff's reagent pink?

(a) Formaldehyde (b) Benzaldehyde

(c) Acetophenone (d) Acetaldehyde

7. Formaldehyde when reacted with methyl magnesium bromide gives

(a) C_2H_5OH (b) CH_3COOH

(c) HCHO (d) CH_3CHO

8. In which reaction, aromatic aldehyde is treated with acid anhydride in the presence of corresponding salt of the acid to give unsaturated aromatic acid?

(a) Friedel-Crafts reaction

(b) Perkin's reaction

(c) Wurtz reaction

(d) None of these

9. Oxidation of toluene to benzaldehyde by the use of chromyl chloride is called

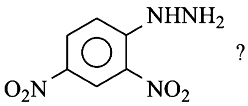
(a) Wurtz reaction

(b) Etard's reaction
(c) Fittig reaction
(d) Rosenmund's reaction

10. Which of the following is a method of converting a ketone into hydrocarbon?
(a) Aldol condensation
(b) Reimer Tiemann reaction
(c) Cannizzaro's reaction
(d) Wolf-Kishner reduction

11. Reductive ozonolysis of benzene produces
(a) Acetone (b) Maleic anhydride
(c) Phthalic acid (d) Glyoxal

12. Which of the following compounds containing carbonyl group will give coloured crystalline compound with



(a) CH_3COCI (b) CH_3COCH_3
(c) $CH_3CO(OC_2H_5)$ (d) CH_3CONH_2

13. Which of the following organic compounds exhibits positive Fehling test as well as Iodoform test?
(a) Methanal (b) Ethanol
(c) Propanone (d) Ethanal

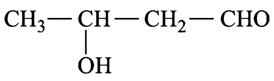
14. Identify the product Y in the sequence

$CH_3CHO + CH_3MgI \xrightarrow{Ether} X \xrightarrow{H_2O/H^+} Y.$

(a) CH_3OH (b) CH_3CH_2OH
(c) $(CH_3)_2\ CHOH$ (d) $(CH_3)_3COH$

15. Ketones are first oxidation products of
(a) Primary alcohols (b) Secondary alcohols
(c) Tertiary alcohols (d) None of these

16. Acetaldehyde when treated with dil. NaOH gives
(a) CH_3CH_2OH
(b) CH_3COOH
(c)

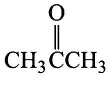


$CH_3—\underset{\displaystyle OH}{\underset{|}{CH}}—CH_2—CHO$

(d) $CH_3—CH_3$

17. An organic compound X on treatment with acidified $K_2Cr_2O_7$ gives compound Y which reacts with I_2 and sodium carbonate to form Triiodomethane. The compound X can be

(a) CH_3OH (b)



$CH_3\overset{\overset{\displaystyle O}{\|}}{C}CH_3$

(c) CH_3CHO (d) $CH_3CH(OH)CH_3$

18. Aldol condensation will not occur in
(a) HCHO (b) CH_3CH_2CHO
(c) CH_3COCH_3 (d) CH_3CHO

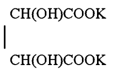
19. Chromyl chloride and toluene react to produce
(a) *p*-chlorotoluene (b) Benzaldehyde
(c) Benzyl chloride (d) Benzoic acid

20. The most reactive compound towards formation of cyanohydrin on treatment with HCN
(a) Benzaldehyde
(b) *p*-Nitrobenzaldehyde
(c) Phenylacetaldehyde
(d) *p*-Hydroxybenzaldehyde

21. In nucleophilic addition reactions the reactivity of carbonyl compounds follows order.
(a) $H_2C = O > R_2C = O > Ar_2C = O > RCHO > ArCHO.$
(b) $H_2C = O > RCHO > ArCHO > R_2CO > Ar_2C = O$
(c) $Ar_2C = O > R_2C = O > ArCHO > RCHO > H_2C = O$
(d) $ArCHO > Ar_2C = O > RCHO > R_2C = H_2C = O$

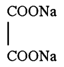
22. Fehling's solution is is a mixture of two solutions. While one solution contains $CuSO_4$, the other contains

(a)



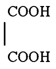
$\begin{array}{l} CH(OH)COOK \\ | \\ CH(OH)COOK \end{array}$

(b)



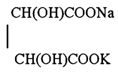
$\begin{array}{l} COONa \\ | \\ COONa \end{array}$

(c)



$\begin{array}{l} COOH \\ | \\ COOH \end{array}$

(d)



$\begin{array}{l} CH(OH)COONa \\ | \\ CH(OH)COOK \end{array}$

23. From which of the following, tertiary butyl alcohol is obtained by the action of methyl magnesium iodide?
(a) HCHO (b) CH_3CHO
(c) CH_3COCH_3 (d) CO_2

24. Which of the following is incorrect?
(a) $FeCI_3$ is used in the detection of phenols
(b) Fehling solution is used in the detection of glucose.
(c) Tollen's reagent is used in detection of unsaturation.
(d) $NaHSO_3$ is used in the detection of carbonyl compounds.

25. Aromatic aldehydes undergo disproportionation in presence of sodium or potassium hydroxide to give corresponding alcohol and acid. The reaction is known as:
(a) Wurtz reaction
(b) Cannizzaro reaction
(c) Friedel Craft reaction
(d) Claisen reaction

26. Which of the following compounds gives a ketone with a Grignard's reagent?
(a) Formaldehyde (b) Ethyl alcohol
(c) Methyl cyanide (d) Methyl iodide

27. If acetyl chloride is reduced in the presence of $BaSO_4$ and Pd, then
(a) CH_3CHO is formed
(b) CH_3CH_2OH is formed
(c) CH_3COOH is formed
(d) CH_3COCH_3 is formed.

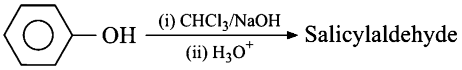
28. C_2H_5CHO and $(CH_3)_2CO$ can be distinguished by testing with
(a) Phenyl hydrazine (b) Hydroxylamine
(c) F ehling solution (d) Sodium bisulphite.

29. Which of the following compounds does not react with $NaHSO_3$?

(a) C_6H_5CHO (b) Acetophenone
(c) Acetone (d) Acetaldehyde.

30. Which of the following is most reactive towards nucleophilic addition reactions?
(a) HCHO (b) CH_3CHO
(c) C_2H_5CHO (d) CH_3COCH_3.

31.



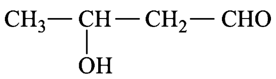
The above reaction is called
(a) Gattermann reaction
(b) Perkin's reaction
(c) Duff's reaction
(d) Reimer Tiemann reaction

32. Which of the following statements regarding chemical properties of acetophenone are wrong?
(I) It is reduced to methyl phenyl carbinol by sodium and ethanol
(II) It is oxidized to benzoic acid with acidified $KMnO_4$
(III) It does not undergo electrophilic substitution like nitration at meta position.
(IV) It does not undergo iodoform reaction with iodine and alkali.
(a) I and II (b) II and IV
(c) III and IV (d) I and II

33. Predict the product 'B' in the sequence of reaction

$$HC \equiv CH \xrightarrow[HgSO_4]{40\%\ H_2SO_4} A \xrightarrow{NaOH} B$$

(a) CH_3COONa
(b) CH_3COOH
(c) CH_3CHO
(d)

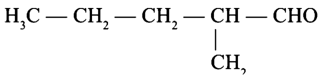


$CH_3-\underset{OH}{CH}-CH_2-CHO$

34. Clemmensen's reduction of ketones is carried out in
(a) H_2 with Pd catalyst
(b) Glycol with KOH
(c) $LiAlH_4$ in water
(d) Zn-Hg with HCl

35. Benzaldehyde reacts with Methyl amine to give
(a) $C_6H_5NH_2$ (b) $C_6H_5CH_2NH_2$
(c) $C_6H_5CH = NCH_3$ (d) $C_6H_5CONH_2$

36. Which of the following aldehyde can give Cannizzaro reaction with base?
(a) $CH_3—CH_2—CH_2—CHO$
(b)



$H_3C-CH_2-CH_2-\underset{CH_3}{CH}-CHO$

(c) $(H_3C)_2CH—CH_2—CHO$
(d) $(CH_3)_3C—CHO$

37. Formalin is an aqueous solution of
(a) Furfural (b) Fluorescein
(c) Formaldehyde (d) Formic acid

38. Propanal on treatment with dilute sodium hydroxide forms
(a) $CH_3CH_2CH_2CH_2CH_2CHO$
(b) $CH_3CH_2CH(OH)CH_2CH_2CHO$
(c) $CH_3CH_2CH(OH)CH(CH_3)CHO$
(d) CH_3CH_2COONa

39. Which is not true about acetophenone?
(a) Reacts to form 2, 4-dinitrophenylhydrazone
(b) Reacts with Tollen's reagent to form silver mirror
(c) Reacts with I_2/NaOH to form iodoform
(d) On oxidation with alkaline $KMnO_4$ followed by hydrolysis gives benzoic acid.

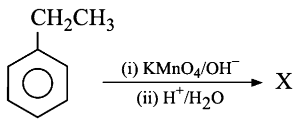
40. Which of the following reactions can produce R—CO—Ar?
(a) $ArCOCl + H—Ar \xrightarrow{AlCl_3}$
(b) $COCl_2 + RMgX \longrightarrow$
(c) $R + CrO_3 \longrightarrow$
(d) $RCOCl + H—Ar \xrightarrow{AlCl_3}$.

41. The reduction of benzoyl chloride with H_2/Pd—$BaSO_4$ produces
(a) Benzyl alcohol (b) Benzaldehyde
(c) Benzoic acid (d) Benzoyl cyanide

42. Acetaldehyde cannot exhibit
(a) Iodoform test (b) Lucas test
(c) Benedict's test (d) Tollen's test

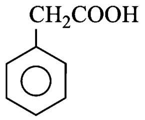
43. Formaldehyde reacts with ammonia to give
(a) Hexamethylene tetramine
(b) Formaldehyde ammonia
(c) Formalin
(d) Hydrobenzamide

44.

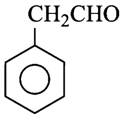


Predict X in above reaction.

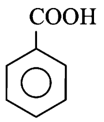
(a)



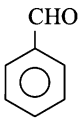
(b)



(c)



(d)



45. Cyanohydrin of which of the following compounds forms lactic acid on hydrolysis
(a) HCHO (b) CH_3COCH_3
(c) CH_3CHO (d) CH_3CH_2CHO

46. Aromatic aldehydes in the presence of cyanide ion as catalyst, are converted to acylions as intermediate. This reaction is called
(a) Perkin reaction
(b) Cannizzaro reaction
(c) Benzoin condensation
(d) Claisen condensation

47. Acetal is produced by reacting alcohol in the presence of dry HCI with

(a) Acetaldehyde (b) Ketone

(c) Ether (d) Carboxylic acid

48. Calcium acetate on heating yields

(a) acetone (b) acetaldehyde

(c) acetic acid (d) butanone

49. A substance $C_4H_{10}O$ yields on oxidation a compound C_4H_8O which gives an oxime and a positive iodoform test. The original substance on treatment with conc. H_2SO_4 gives C_4H_8. The structure of the compound is

(a) $CH_3CH_2CH_2CH_2OH$

(b) $CH_3CH(OH)CH_2CH_3$

(c) $(CH_3)_3COH$

(d) $CH_3CH_2–O–CH_2CH_3$

50. Ketones (R_1COR_2); R_1 and R_2 = alkyl group, can be obtained in one step by

(a) Hydrolysis of esters

(b) Oxidation of primary alcohols

(c) Oxidation of secondary alcohols

(d) Reaction of acid halides and alcohols

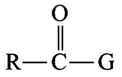
ANSWERS

1. (b)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (d)
11. (d)	**12.** (b)	**13.** (d)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (a)	**19.** (b)	**20.** (b)
21. (b)	**22.** (d)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (a)
31. (d)	**32.** (c)	**33.** (d)	**34.** (d)	**35.** (c)	**36.** (d)	**37.** (c)	**38.** (c)	**39.** (b)	**40.** (d)
41. (b)	**42.** (b)	**43.** (a)	**44.** (c)	**45.** (c)	**46.** (c)	**47.** (a)	**48.** (a)	**49.** (b)	**50.** (c)

27 Carboxylic Acids and Their Derivatives

BRIEF REVIEW OF THE CONCEPTS

Carboxylic acids are the compounds which contain a hydroxyl group attached of the carbon of the carbonyl group $\left(-\overset{\overset{\displaystyle O}{\|}}{C}-OH\right)$. This functional group is called carboxyl group. Carboxylic acids can be aliphatic or aromatic depending upon whether the carboxyl group is attached to hydrogen, alkyl or an aryl group. When the —OH group of $-\overset{\overset{\displaystyle O}{\|}}{C}-OH$ is replaced by another atom or group, the compounds so formed are known as functional derivatives of carboxylic acids. The common derivatives are acid chlorides, esters, acid anhydrides and acid amides.



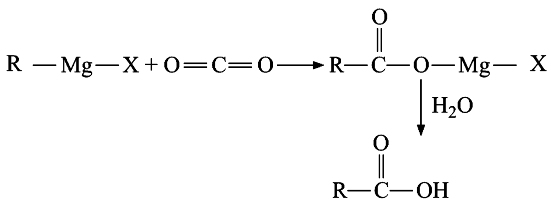
$$R-\overset{\overset{\displaystyle O}{\|}}{C}-G$$

G = –OH	(Carboxylic acids)
G = –Cl	(Acid chlorides)
G = –OR¹	(Esters)
G = $-O-\overset{\overset{\displaystyle O}{\|}}{C}-R'$	(Acid anhydrides)
G = $-NH_2$	(Acid amides)

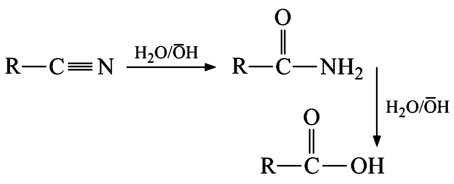
Methods for the Preparation of Carboxylic acids

Carboxylic acids are obtained by the oxidation of primary alcohols and aldehydes. Other methods are briefly discussed below.

(i) **By the addition of Grignard reagent to carbon dioxide** Grignard reagents add on to CO_2, followed by hydrolysis to give carboxylic acids.



(ii) **From Cyanides** Compounds containing cyanide group on hydrolysis with dilute mineral acids or alkali give carboxylic acids.

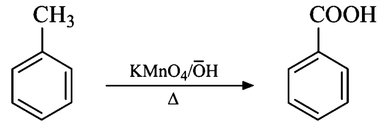


Formic acid can be obtained by heating NaOH with CO under pressure.

$$NaOH + CO \xrightarrow{\Delta} HCOO^-Na^+$$

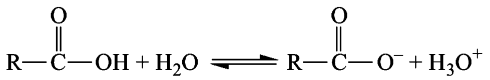
$$\xrightarrow{H^+} HCOOH + Na^+$$

(iii) **From alkyl benzenes.** An alkyl group attached to an aromatic ring when heated with alkaline $KMnO_4$ is oxidized to carboxylic group irrespective of the size of the alkyl group.



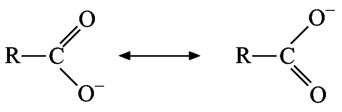
Properties of carboxylic acids

(i) **Acidic character** Among the organic compounds, carboxylic acids are the most acidic, but they are less acidic than the mineral acids



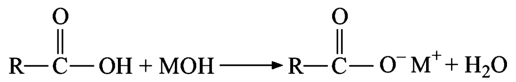
$$R-\overset{\overset{\displaystyle O}{\|}}{C}-OH + H_2O \rightleftharpoons R-\overset{\overset{\displaystyle O}{\|}}{C}-O^- + H_3O^+$$

Carboxylate ion is resonance stabilized



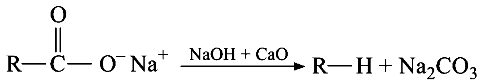
Acid strength increases by the presence of groups with – I effect as substituents in the side chain and decreases by the attachment of groups with + I effect.

(ii) **Reaction with metals and alkali** The carboxylic acids react with alkalies to form salts.



$$R-\overset{\overset{O}{\|}}{C}-OH + MOH \longrightarrow R-\overset{\overset{O}{\|}}{C}-O^-M^+ + H_2O$$

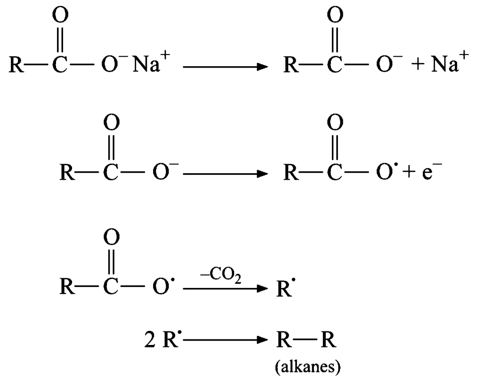
(iii) **Decarboxylation** Sodium salts of carboxylic acids on heating with soda-lime undergo a loss of CO_2 molecule and give alkanes with one carbon less than the parent acid.



$$R-\overset{\overset{O}{\|}}{C}-O^-Na^+ \xrightarrow{NaOH + CaO} R-H + Na_2CO_3$$

(iv) **Kolbe's electrolysis** The aqueous solutions of sodium or potassium salts of carboxylic acids on electrolysis undergo a series of steps and ultimately give alkanes at anode, as described below:

At Anode



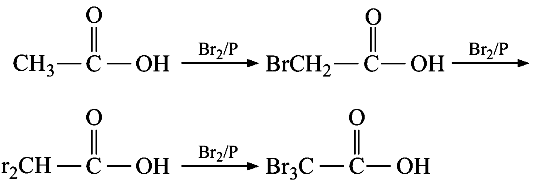
$$R-\overset{\overset{O}{\|}}{C}-O^-Na^+ \longrightarrow R-\overset{\overset{O}{\|}}{C}-O^- + Na^+$$

$$R-\overset{\overset{O}{\|}}{C}-O^- \longrightarrow R-\overset{\overset{O}{\|}}{C}-O^{\bullet} + e^-$$

$$R-\overset{\overset{O}{\|}}{C}-O^{\bullet} \xrightarrow{-CO_2} R^{\bullet}$$

$$2\,R^{\bullet} \longrightarrow \underset{\text{(alkanes)}}{R-R}$$

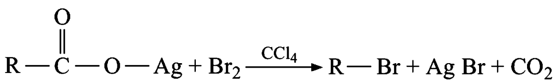
(v) **Hell-Volhard-Zelinsky reaction** Carboxyl acids having α-hydrogen atoms react with chlorine or bromine in presence of red phosphorus to yield α-halogenated acids.



$$CH_3-\overset{\overset{O}{\|}}{C}-OH \xrightarrow{Br_2/P} BrCH_2-\overset{\overset{O}{\|}}{C}-OH \xrightarrow{Br_2/P}$$

$$Br_2CH-\overset{\overset{O}{\|}}{C}-OH \xrightarrow{Br_2/P} Br_3C-\overset{\overset{O}{\|}}{C}-OH$$

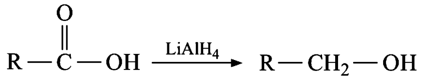
(vi) **Hunsdiecker reaction** The silver salts of fatty acids react with bromine, and produce alkyl bromides by losing a molecule of carbon dioxide.



$$R-\overset{\overset{O}{\|}}{C}-O-Ag + Br_2 \xrightarrow{CCl_4} R-Br + Ag\,Br + CO_2$$

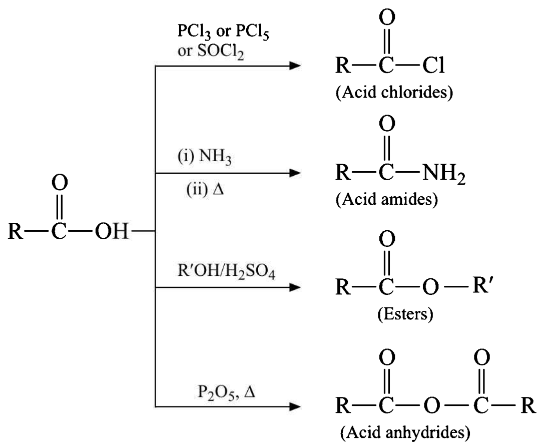
(vii) **Effect of heating the calcium salts** The calcium salts of carboxylic acids on heating give aldehydes or ketones as discussed earlier.

(viii) **Reduction** Carboxylic acids are reduced to corresponding primary alcohols on treatment with $LiAlH_4$?

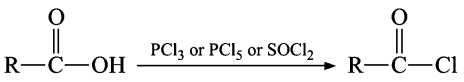


$$R-\overset{\overset{O}{\|}}{C}-OH \xrightarrow{LiAlH_4} R-CH_2-OH$$

(ix) **Replacement of the —OH group-Formation of acid derivatives**

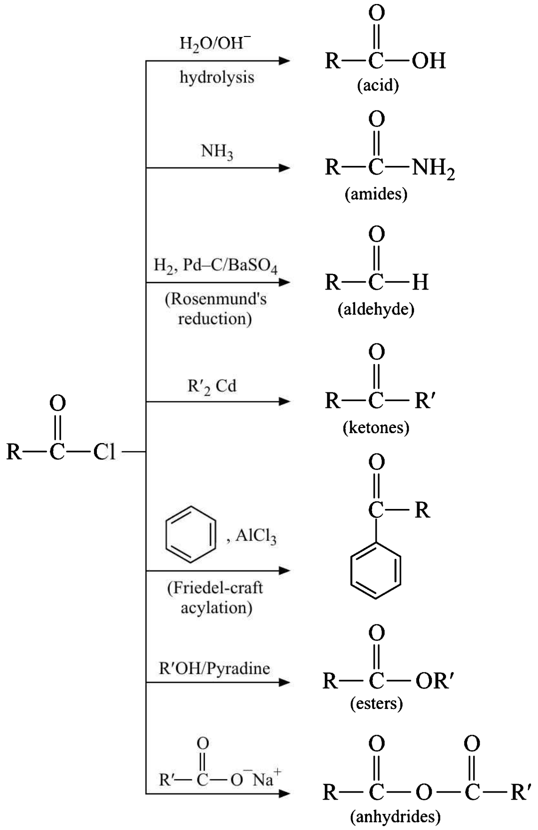


Preparation of acid chlorides Acid chlorides are obtained mainly from carboxylic acids



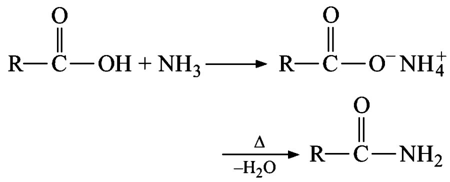
$$R-\overset{\overset{O}{\|}}{C}-OH \xrightarrow{PCl_3 \text{ or } PCl_5 \text{ or } SOCl_2} R-\overset{\overset{O}{\|}}{C}-Cl$$

Properties of acid chlorides Properties of acid chlorides are summarized below



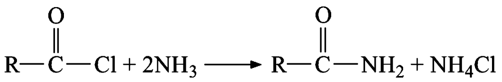
Preparation of acid amides

(i) **From acids**



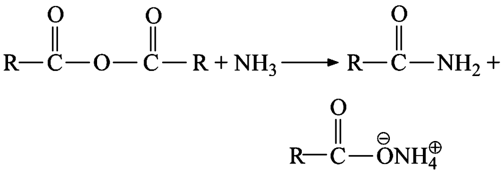
$$R{-}\overset{\overset{O}{\|}}{C}{-}OH + NH_3 \longrightarrow R{-}\overset{\overset{O}{\|}}{C}{-}O^-NH_4^+ \xrightarrow[-H_2O]{\Delta} R{-}\overset{\overset{O}{\|}}{C}{-}NH_2$$

(ii) **From acid chlorides**



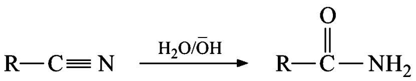
$$R{-}\overset{\overset{O}{\|}}{C}{-}Cl + 2NH_3 \longrightarrow R{-}\overset{\overset{O}{\|}}{C}{-}NH_2 + NH_4Cl$$

(iii) **From anhydrides**



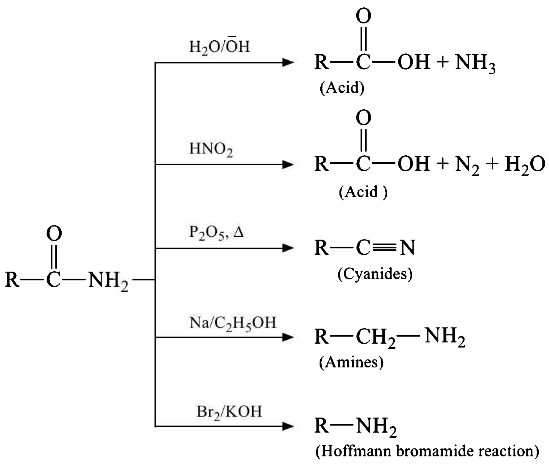
$$R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R + NH_3 \longrightarrow R{-}\overset{\overset{O}{\|}}{C}{-}NH_2 + R{-}\overset{\overset{O}{\|}}{C}{-}\overset{\ominus}{O}NH_4^{\oplus}$$

(iv) **From cyanides by partial hydrolysis**



$$R{-}C{\equiv}N \xrightarrow{H_2O/\bar{O}H} R{-}\overset{\overset{O}{\|}}{C}{-}NH_2$$

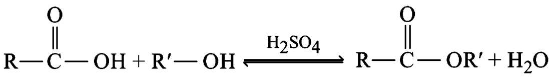
Properties of Acid amides



$$\begin{aligned}
R{-}\overset{\overset{O}{\|}}{C}{-}NH_2 &\xrightarrow{H_2O/\bar{O}H} \underset{\text{(Acid)}}{R{-}\overset{\overset{O}{\|}}{C}{-}OH} + NH_3\\
&\xrightarrow{HNO_2} \underset{\text{(Acid)}}{R{-}\overset{\overset{O}{\|}}{C}{-}OH} + N_2 + H_2O\\
&\xrightarrow{P_2O_5,\ \Delta} \underset{\text{(Cyanides)}}{R{-}C{\equiv}N}\\
&\xrightarrow{Na/C_2H_5OH} \underset{\text{(Amines)}}{R{-}CH_2{-}NH_2}\\
&\xrightarrow{Br_2/KOH} \underset{\text{(Hoffmann bromamide reaction)}}{R{-}NH_2}
\end{aligned}$$

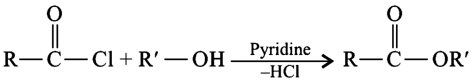
Preparation of Esters

(i) **From acids**



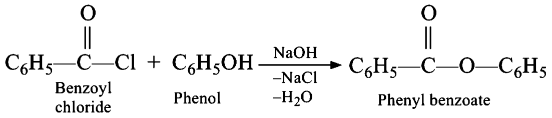
$$R{-}\overset{\overset{O}{\|}}{C}{-}OH + R'{-}OH \xrightleftharpoons{H_2SO_4} R{-}\overset{\overset{O}{\|}}{C}{-}OR' + H_2O$$

(ii) **From acid chlorides**



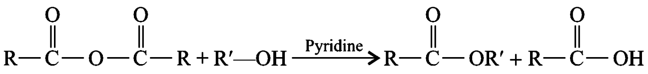
$$R{-}\overset{\overset{O}{\|}}{C}{-}Cl + R'{-}OH \xrightarrow[-HCl]{Pyridine} R{-}\overset{\overset{O}{\|}}{C}{-}OR'$$

If the reaction of an aromatic acid chloride is carried with a phenol is presence of NaOH, it is known as *Schotten-Baumann Reaction.*



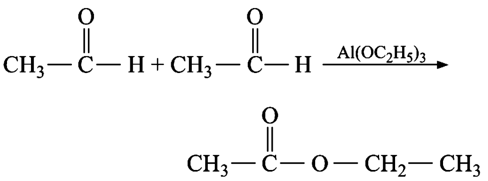
$$\underset{\text{Benzoyl chloride}}{C_6H_5{-}\overset{\overset{O}{\|}}{C}{-}Cl} + \underset{\text{Phenol}}{C_6H_5OH} \xrightarrow[-NaCl,\ -H_2O]{NaOH} \underset{\text{Phenyl benzoate}}{C_6H_5{-}\overset{\overset{O}{\|}}{C}{-}O{-}C_6H_5}$$

(iii) **From acid anhydrides**



$$R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R + R'{-}OH \xrightarrow{Pyridine} R{-}\overset{\overset{O}{\|}}{C}{-}OR' + R{-}\overset{\overset{O}{\|}}{C}{-}OH$$

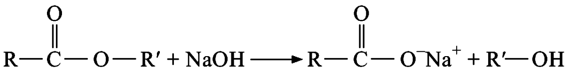
(iv) **By Tischenko reaction.** Aldehydes on treatment with aluminium alkoxides gives esters.



$$CH_3{-}\overset{\overset{O}{\|}}{C}{-}H + CH_3{-}\overset{\overset{O}{\|}}{C}{-}H \xrightarrow{Al(OC_2H_5)_3} CH_3{-}\overset{\overset{O}{\|}}{C}{-}O{-}CH_2{-}CH_3$$

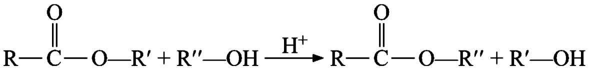
Properties of esters

(i) **Hydrolysis**



$$R{-}\overset{\overset{O}{\|}}{C}{-}O{-}R' + NaOH \longrightarrow R{-}\overset{\overset{O}{\|}}{C}{-}O^-Na^+ + R'{-}OH$$

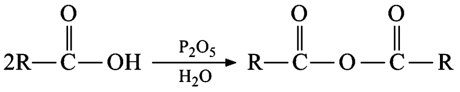
(ii) **Transesterification.** When an ester is treated with a different alcohol having simpler alkyl group, in the presence of an acid (e.g., H_2SO_4) as catalyst, the alcohol part gets exchanged



$$R{-}\overset{\overset{O}{\|}}{C}{-}O{-}R' + R''{-}OH \xrightarrow{H^+} R{-}\overset{\overset{O}{\|}}{C}{-}O{-}R'' + R'{-}OH$$

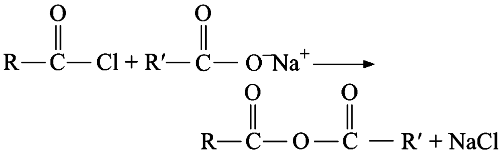
Preparation of acid Anhydrides

(i) **By dehydration of acids**



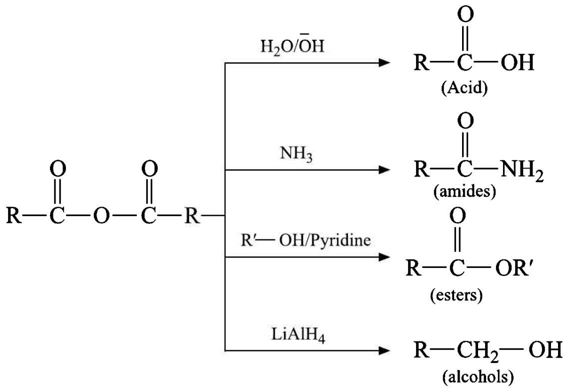
$$2R{-}\overset{\overset{O}{\|}}{C}{-}OH \xrightarrow[H_2O]{P_2O_5} R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R$$

(ii) **From acid chlorides**



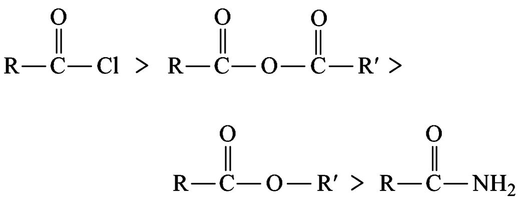
$$R{-}\overset{\overset{O}{\|}}{C}{-}Cl + R'{-}\overset{\overset{O}{\|}}{C}{-}O^-Na^+ \longrightarrow R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R' + NaCl$$

Properties of acid anhydrides



$$\begin{aligned}
R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R &\xrightarrow{H_2O/\bar{O}H} \underset{\text{(Acid)}}{R{-}\overset{\overset{O}{\|}}{C}{-}OH}\\
&\xrightarrow{NH_3} \underset{\text{(amides)}}{R{-}\overset{\overset{O}{\|}}{C}{-}NH_2}\\
&\xrightarrow{R'{-}OH/Pyridine} \underset{\text{(esters)}}{R{-}\overset{\overset{O}{\|}}{C}{-}OR'}\\
&\xrightarrow{LiAlH_4} \underset{\text{(alcohols)}}{R{-}CH_2{-}OH}
\end{aligned}$$

The order of reactivity of different acid derivatives towards hydrolysis or any other nucleophilic acyl substitution follows the sequence



$$R{-}\overset{\overset{O}{\|}}{C}{-}Cl > R{-}\overset{\overset{O}{\|}}{C}{-}O{-}\overset{\overset{O}{\|}}{C}{-}R' > R{-}\overset{\overset{O}{\|}}{C}{-}O{-}R' > R{-}\overset{\overset{O}{\|}}{C}{-}NH_2$$

Problems for Practice

1. Carboxylic acids are functional isomers of
   (a) Ethers (b) Esters
   (c) Both (a) and (b) (d) None of these

2. Ethyl alcohol on oxidation with $K_2Cr_2O_7$ gives
   (a) Acetic acid (b) Acetaldehyde
   (c) Formaldehyde (d) Formic acid

3. The acid formed when propyl magnesium bromide is treated with carbon dioxide is
   (a) C_3H_7COOH (b) C_2H_5COOH
   (c) Both (a) and (b) (d) None of these

4. Electrolysis of aqueous solution of CH_3COOK gives
   (a) CH_4 (b) C_2H_4
   (c) C_2H_6 (d) C_2H_2

5. Propanoic acid on warming with Cl_2 in presence of red P gives
   (a) CH_3CH_2COCl (b) CH_3CH_2Cl
   (c) $CH_3CHClCOOH$ (d) CH_2ClCH_2COOH

6. Which of the following reactions is expected to give readily a hydrocarbon product in good yields?
   (a) $RCOOK \xrightarrow{\text{Electrolyis}}$
   (b) $RCOOAg \xrightarrow{I_2}$
   (c) $CH_3CH_3 \xrightarrow[hv]{Cl_2}$
   (d) $(CH_3)_2CCl_2 \xrightarrow{C_2H_5OH}$

7. Identify the product C in the series
   $CH_3CN \xrightarrow{Na/C_2H_5OH} A \xrightarrow{HNO_2} B \xrightarrow{\text{Tollen's reagent}} C.$
   (a) CH_3COOH (b) CH_3CH_2NHOH
   (c) CH_3CONH_2 (d) CH_3CHO

8. Lower carboxylic acids are soluble in water due to
   (a) Low molecular mass
   (b) Hydrogen bonding
   (c) Dissociation into ions
   (d) Easy hydrolysis

9. Ethyl acetate reacts with CH_3MgBr to form
   (a) Secondary alcohol
   (b) Tertiary alcohol
   (c) Primary alcohol and acid
   (d) Carboxylic acid

10. Identify the final product B in the following reactions:
    $CH_3COOH \xrightarrow{Cl_2/P} A \xrightarrow{KCN} B.$
    (a) CH_3CN (b) CH_3CONH_2
    (c) $ClCH_2COOH$ (d) $CH_2(CN)COOH$

11. Which reaction is suitable for preparing α-Chloroacetic acid?
    (a) Hell-Volhard Zelinsky reaction
    (b) Stephen's reaction
    (c) Perkin's reaction
    (d) None of these

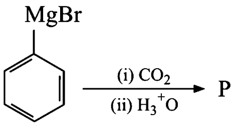
12. Which of the following reagents is used to distinguish between methanoic acid and ethanoic acid?
    (a) Ammoniacal silver nitrate solution
    (b) Neutral ferric chloride
    (c) Sodium hydroxide solution
    (d) Phenolphthalein

13. When propanamide reacts with $Br_2$and NaOH then which of the following compounds is formed?
    (a) Ethyl alcohol (b) propyl alcohol
    (c) Propyl amine (d) Ethylamine

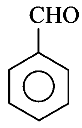
14. The order of decreasing ease of reaction with ammonia is
    (a) Anhydrides, esters, ethers
    (b) Anhydrides, ethers, esters
    (c) Ethers, anhydrides, esters
    (d) Esters, ethers, anhydrides

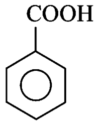
15. Acyl halide is formed by reacting PCl_5 with
    (a) Alcohol
    (b) Ester
    (c) Amide
    (d) carboxylic acids

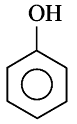
16. The refluxing of $(CH_3)_2NCOCH_3$ with NaOH solution gives
    (a) $2CH_3NH_2 + CH_3COOH$
    (b) $2CH_3OH + CH_3CONH_2$
    (c) $(CH_3)_2NH + CH_3COOH$
    (d) $(CH_3)_2NCOOH + CH_4$

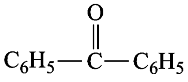
17.


In the above reaction, product P is

(a)


(b)


(c)


(d)

$C_6H_5-\overset{\overset{\Large O}{\|}}{C}-C_6H_5$

18. Which of the following is the strongest acid?
    (a) CF_3COOH (b) CBr_3COOH
    (c) CH_3COOH (d) CCl_3COOH

19. Carboxylic acids are more acidic than phenol and alcohol because of
    (a) intermolecular hydrogen bonding
    (b) formation of dimers
    (c) highly acidic hydrogen
    (d) resonance stabilization of their conjugate base

20. Which of the following compounds can be produced by the reaction of carbon monoxide and caustic soda (NaOH)?
    (a) CH_3COOH (b) $C_2H_2O_4$
    (c) HCOONa (d) HCHO

21. When acetamide is treated with NaOBr, the product formed is

(a) CH_3CN (b) $CH_3CH_2NH_2$

(c) CH_3NH_2 (d) None of the above

22. Hydrolytic reaction of fats by caustic soda is known as

(a) Acetylation (b) Carboxylation

(c) Esterification (d) Saponification

23. The fatty acid which shows reducing property is

(a) Acetic acid (b) Ethanoic acid

(c) Oxalic acid (d) Formic acid

24. What will happen if $LiAlH_4$ is added to an ester?

(a) Two units of alcohol are obtained

(b) One unit of alcohol and one unit of acid is obtained

(c) Two units of acids are obtained

(d) None of these

25. Which reagent will bring about the conversion of carboxylic acids into esters?

(a) C_2H_5OH

(b) Dry HCl + C_2H_5OH

(c) $LiAlH_4$

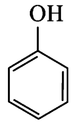
(d) $Al(OC_2H_5)_3$

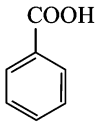
26. $CH_3CH_2COOH \xrightarrow[\text{Red P}]{Cl_2} A \xrightarrow{\text{Alc.KOH}} B$ what is B?

(a) CH_3CH_2COCl (b) CH_3CH_2CHO

(c) $CH_2 = CHCOOH$ (d) $ClCH_2CH_2COOH$

27. Which of the following is the weakest acid?

(a)  (b) CH_3COOH

(c) HCOOH (d) 

28. A fruity smell is produced by the reaction of C_2H_5OH with

(a) PCl_5 (b) CH_3COCH_3

(c) CH_3COOH (d) None of these

29. Which of the following orders of relative strength of acids is correct?

(a) $FCH_2COOH > ClCH_2COOH > BrCH_2COOH$

(b) $ClCH_2COOH > BrCH_2COOH > FCH_2COOH$

(c) $BrCH_2COOH > ClCH_2COOH > FCH_2COOH$

(d) $ClCH_2COOH > FCH_2COOH > BrCH_2COOH$

30. Identify Z

$$CH_3COONH_4 \xrightarrow{\Delta} X \xrightarrow{P_2O_5} Y \xrightarrow{H_2O/H^+} Z$$

(a) $CH_3CH_2CONH_2$ (b) CH_3CN

(c) $(CH_3CO)_2O$ (d) CH_3COOH

31. The reagent used to convert Ethanoic acid to Ethanol is

(a) $LiAlH_4$ (b) BH_3

(c) PCl_3 (d) $K_2Cr_2O_7/H^+$

32. In a reaction involving ring substitution of C_6H_5Y, the major product is *meta* isomer. The group Y can be

(a) $—NH_2$ (b) —COOH

(c) $—CH_3$ (d) —Cl

33. Saponification of ethyl benzoate with caustic soda as alkali gives

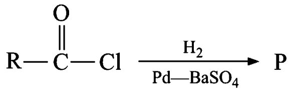
(a) benzyl alcohol and ethanoic acid

(b) sodium benzoate and ethanol

(c) benzoic acid and sodium ethoxide

(d) phenol and ethanoic acid

34. In the following reaction, product P is



$$R—\overset{\overset{\large O}{||}}{C}—Cl \xrightarrow[\text{Pd—BaSO}_4]{H_2} P$$

(a) R CH_2OH (b) R COOH

(c) R CHO (d) R CH_3

35. Phenol $\xrightarrow{\text{Zn dust}} X \xrightarrow[\text{anhyd. AlCl}_3]{CH_3Cl} Y \xrightarrow[\text{KMnO}_4]{\text{Alkaline}} Z$

(a) Toluene (b) Benzene

(c) Benzoic acid (d) Benzaldehyde

36. In the presence of aluminium ethoxide, aldehydes get converted into esters. The reaction is known as

(a) Schmidt reaction

(b) Aldol condensation

(c) Beckmann's rearrangement reaction

(d) Tischenko reaction

37. Between CH_3COOH and HCOOH, the later is

(a) Less acidic (b) More acidic

(c) Equally acidic (d) None of these

38. A compound of molecular formula $C_6H_{12}O_6$ on acetylation with actic anhydride and pyridine gives a compound of molecular formula $C_{16}H_{22}O_{11}$. How many hydroxyl groups are present in the compound?

(a) Five (b) Four

(c) Six (d) Three

39. Saponification of phenyl benzoate with caustic soda as alkali, gives

(a) Benzyl alcohol, benzoic acid

(b) Sodium benzoate, sodium phenoxide

(c) Benzoic acid, sodium ethoxide

(d) Phenol, benzoic acid

40. Hydrolysis of trichloromethane with aqueous KOH gives

(a) Potassium formate

(b) Acetylene

(c) Chloral

(d) Methanol

41. Formic acid is obtained when:

(a) Calcium acetate is heated with conc. H_2SO_4

(b) Calcium formate is heated with calcium acetate

(c) Glycerol is heated with oxalic acid at 383 K

(d) Acetaldehyde is oxidised with $K_2Cr_2O_7$ and H_2SO_4

42. Acetic anhydride is obtained from acetyl chloride by the reaction of

(a) P_2O_5 (b) H_2SO_4

(c) CH_3COONa (d) CH_3CH_2OH

43. The organic compounds A and B react with sodium metal and release H_2 gas. A and B react with each other to give ethyl acetate. The A and B are

(a) CH_3COOH and C_2H_5OH
(b) $HCOOH$ and C_2H_5OH
(c) CH_3COOH and CH_3OH
(d) CH_3COOH and $HCOOH$

44. Acetic acid reacts with PCl_5 to from

(a) $CH_2ClCOOH$ (b) $CHCl_2COOH$
(c) CH_3COCl (d) CH_3COOCl

45. The product A of the following reaction, is

$CH_3COOH \xrightarrow{P_2O_5} A$

(a) CO and H_2O
(b) Formic acid
(c) Ethanoic anhydride
(d) Ethyl ethanoate

ANSWERS

1. (b)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (c)	**6.** (a)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (d)
11. (a)	**12.** (a)	**13.** (d)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (d)	**20.** (c)
21. (c)	**22.** (d)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (d)
31. (a)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (d)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (a)
41. (c)	**42.** (c)	**43.** (a)	**44.** (c)	**45.** (c)					

28 Organic Compounds with Functional Groups Containing Nitrogen

BRIEF REVIEW OF THE CONCEPTS

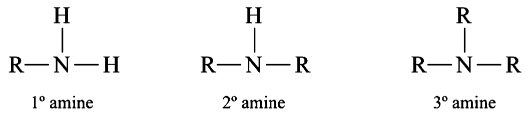
Cyanides and **isocyanides** are considered as derivatives of hydrocyanic acid which exists in the following tautomeric forms :

$$\underset{\text{Hydrogen cyanide}}{\underset{\text{(I)}}{H-C\equiv N}} \longrightarrow \underset{\text{Hydrogen isocyanide}}{\underset{\text{(II)}}{H-N\overrightarrow{\equiv}C}}$$

The compounds obtained by the replacement of hydrogen atom from **I** are called **cyanides** or **nitriles** and those obtained from **II** are called **isocyanides** or **carbylamines.**

Nitro compounds are characterised by the presence of a nitro group $-N\begin{smallmatrix}{/\!\!/}O\\ \searrow O\end{smallmatrix}$ on an alkyl chain or an aromatic ring.

Amines may be regarded as the derivatives of ammonia in which one or more hydrogen atoms of ammonia have been replaced by alkyl or aryl groups. They are classified as *primary, secondary or tertiary amines* depending upon whether *one, two or three hydrogens of ammonia have been replaced.*



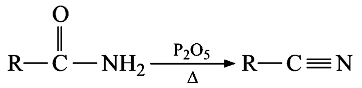
$$\underset{1^\circ\text{ amine}}{R-\overset{\overset{H}{|}}{N}-H} \qquad \underset{2^\circ\text{ amine}}{R-\overset{\overset{H}{|}}{N}-R} \qquad \underset{3^\circ\text{ amine}}{R-\overset{\overset{R}{|}}{N}-R}$$

Methods for the preparation of cyanides

(i) From alkyl halides

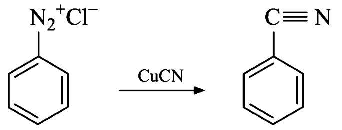
$$R-X + NaCN\ (\text{or } KCN) \longrightarrow R-C\equiv N + NaX$$

(ii) By dehydration of amides

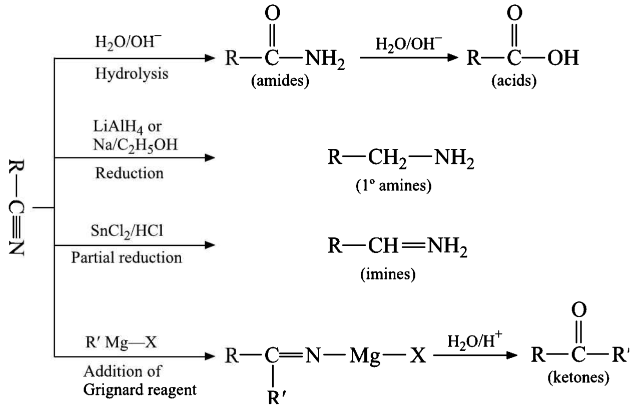


$$R-\overset{\overset{O}{\|}}{C}-NH_2 \xrightarrow[\Delta]{P_2O_5} R-C\equiv N$$

(iii) From diazonium salts (Sandmeyer's reaction)



Reactions of cyanides



Methods for the preparation of isocyanides

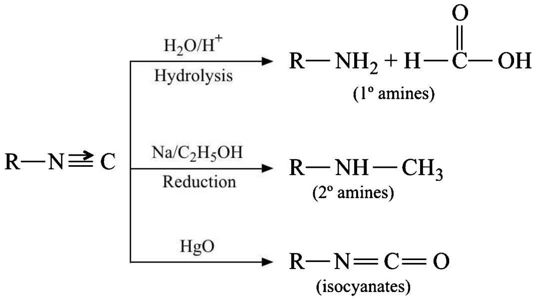
(i) From alkyl halides

$$R-X + AgCN \longrightarrow R-N\overrightarrow{\equiv}C + AgX$$

(ii) From primary amines (carbylamine reaction)

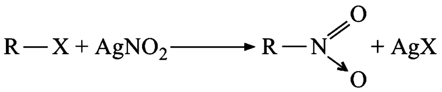
$$R-NH_2 + CHCl_3 + KOH \longrightarrow R-N\overrightarrow{\equiv}C$$

Reactions of isocyanides



Methods for the preparation of Nitro compounds

(i) From alkyl halides



$$R-X + AgNO_2 \longrightarrow R-N\begin{smallmatrix}{/\!\!/}O\\ \searrow O\end{smallmatrix} + AgX$$

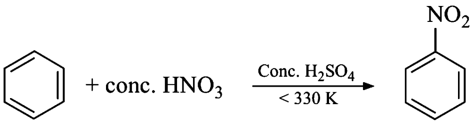
(ii) By vapour phase nitration of alkanes

$$CH_3-CH_3 + HNO_3 \xrightarrow{700\ K} CH_3-CH_2-NO_2$$

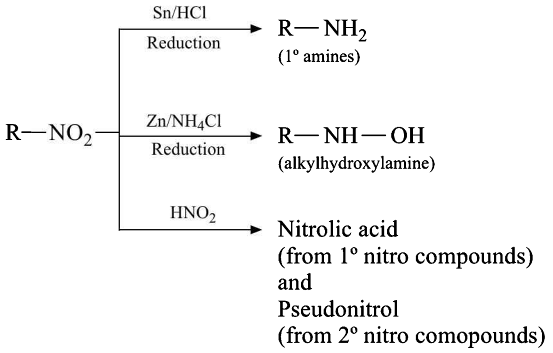
$$+$$

Some oxidation products

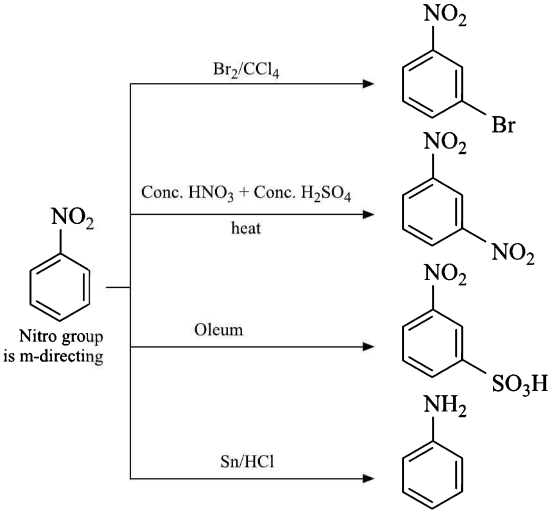
(iii) By Nitration of aromatic compounds



Reactions of aliphatic nitro compounds

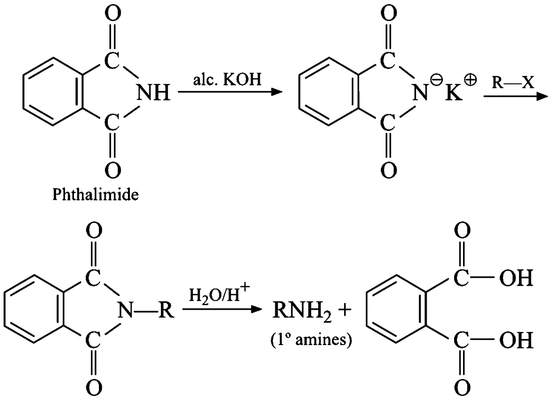


Reactions of aromatic nitro compounds



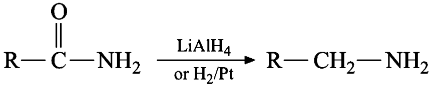
Methods for preparation of primary amines

(i) Gabriel's phthalimide synthesis



(ii) By the reduction of cyanides, amides and nitro compounds

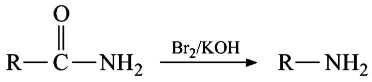
$$R-C{\equiv}N \xrightarrow[\text{or } H_2/Pt]{LiAlH_4} R-CH_2-NH_2$$



$$R-\overset{\overset{\Large O}{\|}}{C}-NH_2 \xrightarrow[\text{or } H_2/Pt]{LiAlH_4} R-CH_2-NH_2$$

$$R-NO_2 \xrightarrow{Sn/HCl} R-NH_2$$

(iii) By Hofmann Bromamide degradation of amides



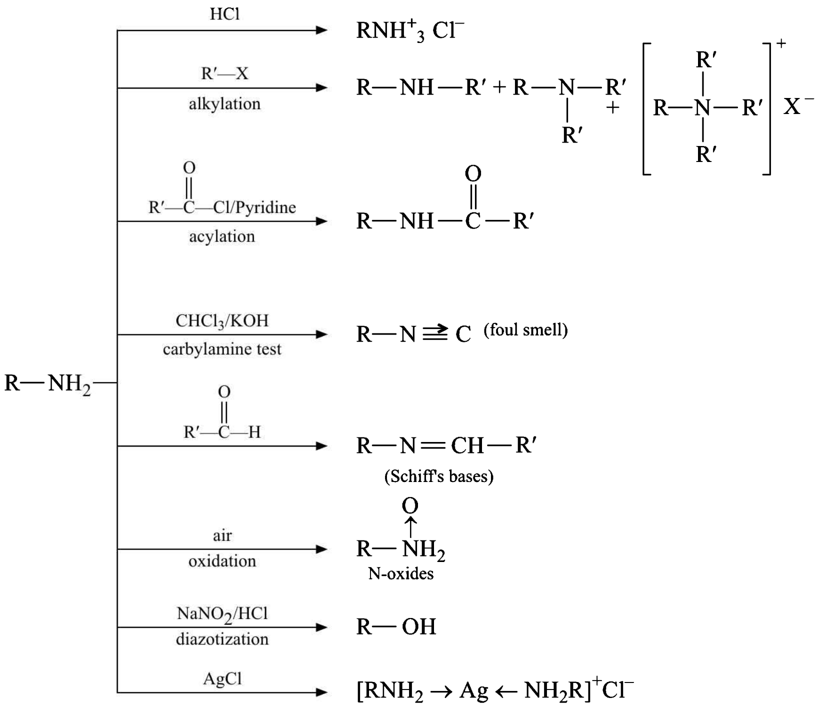
$$R-\overset{\overset{\Large O}{\|}}{C}-NH_2 \xrightarrow{Br_2/KOH} R-NH_2$$

(iv) From alcohols

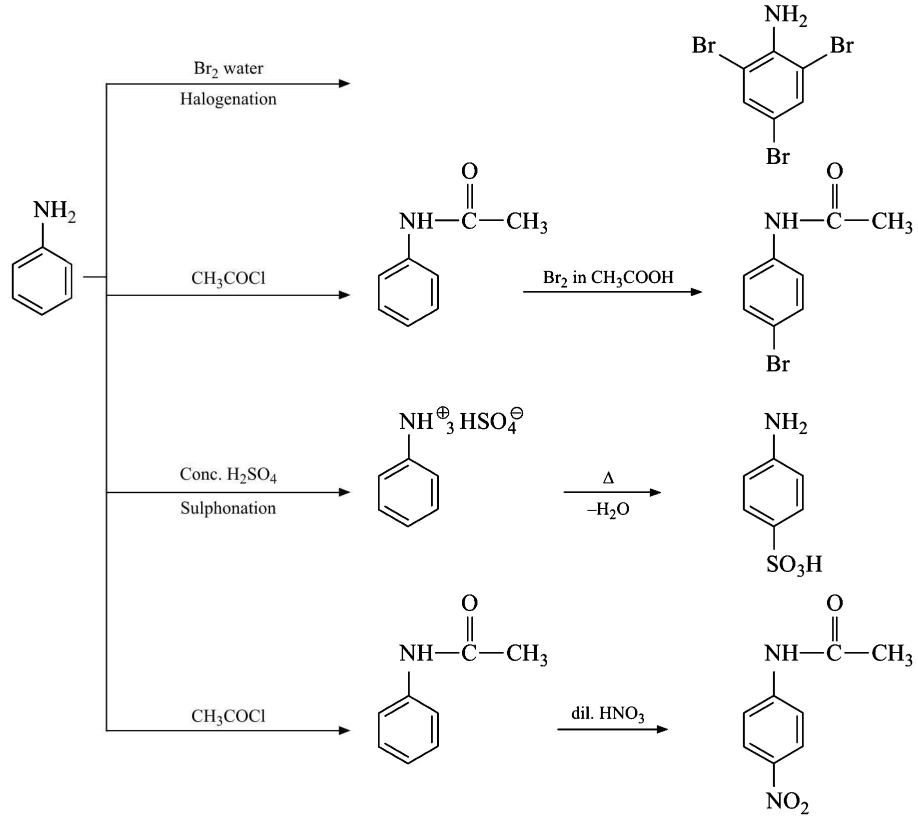
$$R{-}OH + NH_3 \xrightarrow[573\ K]{Al_2O_3} R-NH_2$$

+ 2° and 3° amines are also formed

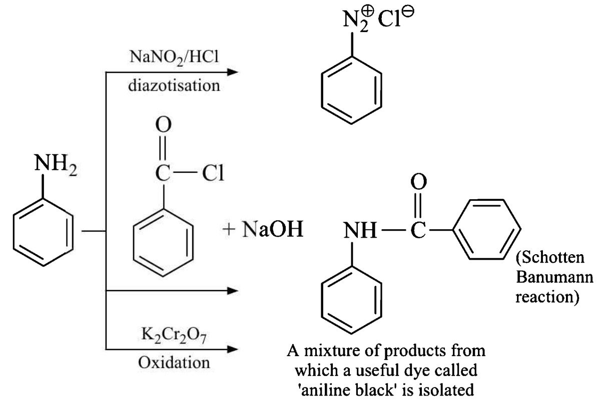
Reactions of primary aliphatic amines



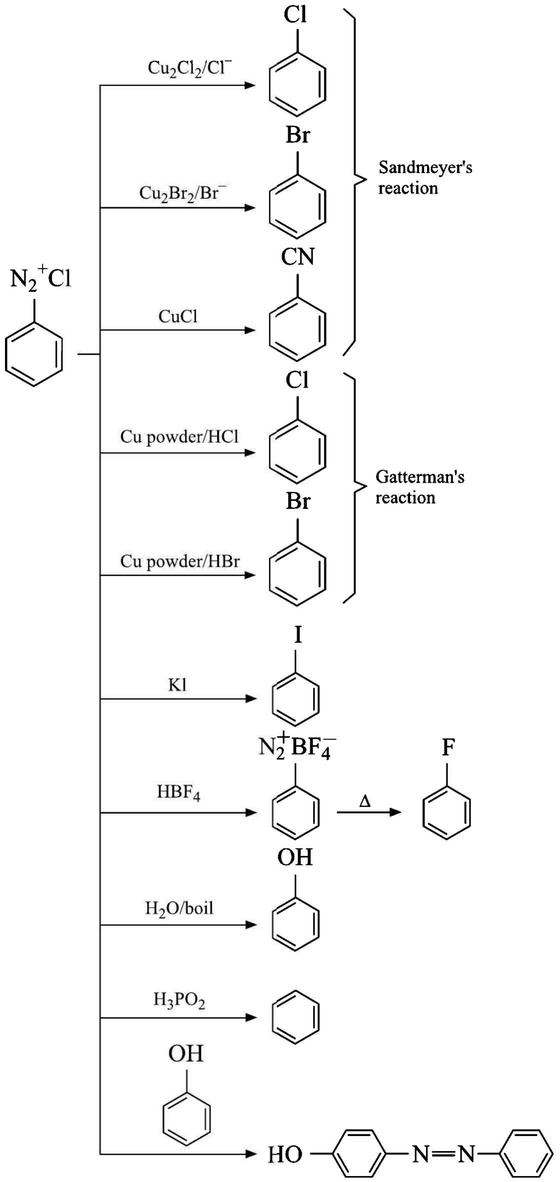
Ring substitution reactions of aniline



Some other reactions of aniline

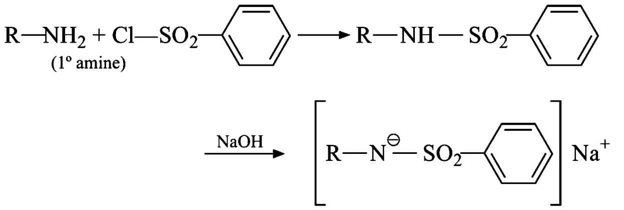


Reactions and synthetic uses of aryl diazonium salts

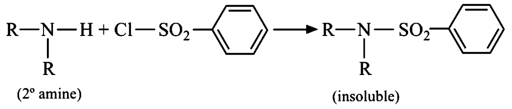


Hinsberg test This test is used to distinguish between primary, secondary and tertiary amines. In this, the amine is treated with benzenesulphonyl chloride in the presence of an excess of aqueous sodium hydroxide. The reaction with different amines is explained as below:

(a) A primary amine gives *N*-alkyl benzenesulphonamide which is soluble in aq. NaOH and we obtain a clear solution.



(b) A secondary amine reacts to give *N,N*-dialkylbenzene sulphonamide which is insoluble in aq. NaOH, so a precipitate is obtained.



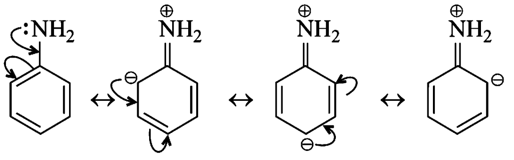
(c) A tertiary amine does not react with benzenesulphonyl chloride and remains unchanged.

Relative basic strength of amines The basicity of amines depends upon the availability of electron pair on nitrogen atom. As such, any factor that increases the availability of electron pair on nitrogen will increase the basicity of amine. The greater basicity of aliphatic amines as compared to ammonia can be explained in terms of the electron releasing inductive effect of alkyl groups. Aliphatic amines contain one or more alkyl groups (in place of hydrogen atoms of ammonia) which have electron releasing inductive effect and thus make the electron pair on nitrogen more available for sharing. On this consideration, a tertiary amine should be more basic than a secondary amine, which in turn, should be more basic than a primary amine.

This order has been found to be correct in gaseous state but in aqueous medium, it has been observed that 3° amines are always weaker base than 2° amines. It means some other factor is also playing the role in determining the basicities. The other factor has been found to be the steric factor. Three alkyl groups in tertiary amine create so much crowding that it cannot accept a proton. Generally the order of basicity has been found to be

1° amines < 2° amines > 3° amines.

Aromatic amines such as aniline are less basic than ammonia. It can be attributed to the electron withdrawing nature of the aromatic ring due to resonance.



As a result of the resonance, the lone pair of electrons on nitrogen gets delocalized over the benzene ring. This decreases the availability of electron pair for protonation and hence makes it a weaker base.

In general, electron with drawing groups like —NO_2, —COOH,—SO_3H,—X decrease the basic character and electron releasing groups like — OCH_3,—CH_3, etc. increase the basic character of aromatic amines.

Problems for Practice

1. The correct decreasing order of basic strength of gaseous NH_3, RNH_2, R_2NH, R_3N is

(a) $R_3N > R_2NH > RNH_2 > NH_3$

(b) $NH_3 > RNH_2 > R_2NH > R_3N$

(c) $R_2NH > RNH_2 > R_3N > NH_3$

(d) $R\,NH_2 > R_2NH > R_3N > NH_3$.

2. Reaction of primary amines with aldehyde yields

(a) Amides (b) Aldimines

(c) Nitriles (d) Nitro compounds

3. The reagent that reacts with nitromethane to form methyl hydroxylamine is

(a) Zn/HCl (b) Zn/NH_4Cl

(c) Zn/NaOH (d) Sn/HCl

4. The product formed when benzene is nitrated by fuming nitric acid is

(a) *m*-Dinitrobenzene

(b) Nitrobenzene

(c) Sym-trinitrobenzene

(d) None of these

5. The nitro group in nitrobenzene is a

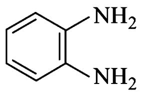
(a) Ortho directing

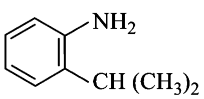
(b) Meta directing

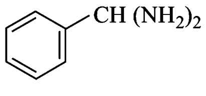
(c) Para directing

(d) Ortho and para directing

6. Orthophenylene diamine has which structure?

(a) 

(b) 

(c) 

(d) None of these

7. Reaction of aniline with benzaldehyde is

(a) Substitution (b) Addition

(c) Condensation (d) Polymerization

8. The indicator that is obtained by coupling the diazonium salt of sulphanilic acid with *N,N*-dimethylaniline is

(a) Phenanthroline (b) Methyl orange

(c) methyl read (d) Phenolphthalein.

9. Nitrobenzene on reduction in acidic medium gives:

(a) Aniline (b) Nitrosobenzene

(c) Azobenzene (d) Phenylhydroxyl amine

10. Ethyl amine can be obtained by the:

(a) Action of NH_3on ethyl iodide

(b) Action of NH_3 on ethyl alcohol

(c) Both (a) and (b)

(d) None of these

11. Consider the following reaction

$C_6H_5NO_2 \xrightarrow{Sn/HCl} X \xrightarrow{C_6H_5COCl} Y + HCl$. What is *Y*?

(a) Acetanilide (b) Benzanilide

(c) Azobenzene (d) Hydrazobenzene

12. Aniline is weaker base than ethyl amine because

(a) It is insoluble in water while ethyl amine is soluble

(b) Its hydrogen atoms are easily removable

(c) Its hydrogen are not easily removable

(d) The lone pair of electrons on nitrogen in aniline is not freely available for protonation.

13. Acetamide is treated separately with the following reagents. Which one of these would give methylamine?

(a) PCl_5 (b) NaOH + Br_2

(c) Soda lime (d) Hot conc. H_2SO_4

14. Aniline when acetylated, the product on nitration followed by hydrolysis gave

(a) o-Nitroacetanilide (b) p-Nitroaniline

(c) p-Benzoquinone (d) Acetanilide

15. When benzene diazonuim chloride undergoes hydrolysis, it forms

(a) Chlorobenzene (b) Phenol

(c) Benzyl alcohol (d) Benzene

16. In the following reaction ;

$C_6H_5NH_2 + CHCl_3 + 3NaOH \longrightarrow A + 3B + 3C$. The product A is

(a) Phenylisocyanide

(b) Phenylcyanide

(c) Benzyl Chloride

(d) Chlorobenzene

17. Diazo coupling is useful to prepare

(a) Pesticides (b) Dyes

(c) Proteins (d) Vitamins

18. Which of the following statements is correct?

(a) Methylamine is less basic than ammonia

(b) Methylamine is more basic than NH_3

(c) Methylamine is acidic towards litmus

(d) Methylamine forms salt with NaOH

19. Tertiary nitro compounds cannot show tautomerism because

(a) They are very stable

(b) Isomerize to give *sec.* nitro compounds

(c) Do not have labile hydrogen atom

(d) They are highly reactive

20. Methyl cyanide on treatment with methyl magnesium bromide followed by subsequent hydrolysis gives

(a) Propanone (b) Ethanone

(c) Ethanal (d) Propanal

21. Acetaldoxime reacts with P_2O_5 to give

(a) Methyl cyanide (b) Ethyl cyanate

(c) Ethyl cyanide (d) Mixture of all these

22. Arrange the following:

CH_3NH_2(I); $(CH_3)_2NH$(II) ; $C_6H_5NH_2$(III) and $(CH_3)_3N$ (IV) in increasing order of basicity in aqueous medium

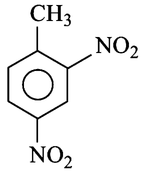
(a) II < I < IV < III (b) III < IV < I < II
(c) I < II < III < IV (d) II < III < I < IV

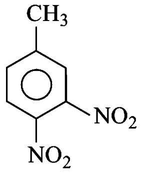
23. The amine which does not react with acetyl chloride is
(a) CH_3NH_2 (b) $(CH_3)_2$ NH
(c) $(CH_3)_3N$ (d) None of these.

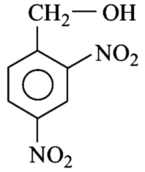
24. A compound *A* has a molecular formula C_7H_7NO. On treatment with Br_2 and KOH, *A* gives an amine *B* which gives carbylamine test. *B* upon diazotisation and coupling with phenol gives an azodye. *A* can be
(a) $C_6H_5CONHCH_3$
(b) $C_6H_5CONH_2$
(c) C_6H_5NO
(d) *o, m*-or *p*-C_6H_4 (NH_2) CHO.

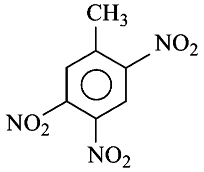
25. The reaction
$C_6H_5NO_2 + KOH(s) \xrightarrow{\text{Heat}}$ *o*-and *p*-nitrophenol is an example of
(a) Nucleophilic substitution
(b) Electrophilic substitution
(c) Free radical substitution
(d) Electrophilic addition

26. *p*-Nitrotoluene on further nitration gives

(a) 

(b) 

(c) 

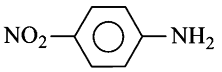
(d) 

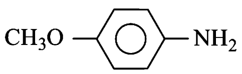
27. On reduction secondary amine is given by
(a) Nitrobenzene (b) Methyl cyanide
(c) Nitroethane (d) Methyl isocyanide

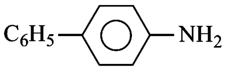
28. The one which is least basic is
(a) NH_3 (b) $C_6H_5NH_2$
(c) $(C_6H_5)_3N$ (d) $(C_6H_5)_2NH$

29. If *N* and *S* elements are present in organic compound, then during Lassaigne's test both may change into
(a) Na_2S and NaCN
(b) NaSCN
(c) Na_2SO_3 and Na_2CO_3
(d) Na_2S and NaCNO

30. Which of the following is least basic?

(a) 

(b) 

(c) 

(d) All are equally basic

31. In amines, the hybridization state of N is
(a) *sp* (b) sp^2
(c) sp^3 (d) sp^2d

32. On reaction with HNO_2, $C_2H_5NH_2$ produces
(a) C_2H_5OH (b) $C_2H_5NO_2$
(c) CH_3CHO (d) CH_3COOH

33. Reaction of aniline with acetyl chloride in the presence of pyridine gives
(a) Acetanilide (b) *p*-chloroaniline
(c) A red dye (d) Aniline hydrochloride

34. 'A' is subjected to reduction with Zn-Hg/HCl and product formed is N-methylmethanamine. A can be
(a) Ethane nitrile (b) Nitroethane
(c) Carbylaminoethane (d) Carbylaminomethane

35. Which reagent should be employed to get ethyl carbylamine from ethyl iodide as major product
(a) HCN (b) AgCN
(c) KCN
(d) $AgNO_2$ followed by reduction

36. In the following reaction, X is
$X \xrightarrow{\text{Bromination}} Y \xrightarrow[+\,HCl]{NaNO_2} Z \xrightarrow{H_3PO_2}$
1,3,5–Tribromo Benzene
(a) Benzoic acid (b) Salicylic acid
(c) Phenol (d) Aniline

37. Which of the following reactions will not give primary amine?
(a) $CH_3CONH_2 \xrightarrow{KOH, Br_2}$
(b) $CH_3CN \xrightarrow{LiAlH_4}$
(c) $CH_3NC \xrightarrow{LiAlH_4}$
(d) $CH_3CONH_2 \xrightarrow{LiAlH_4}$

38. The electrophile involved in the nitration of benzene is
(a) NO_2 (b) NO_2^+
(c) HNO_3 (d) NO_2^-

39. The correct decreasing order of basic strength of aq. soln of NH_3, RNH_2, R_2NH, R_3N is
(a) $R_2NH > RNH_2 > R_3N > NH_3$
(b) $R_3N > R_2NH > RNH_2 > NH_3$
(c) $NH_3 > RNH_2 > R_2NH > R_3N$
(d) $NH_3 > R_3N > R_2NH > RNH_2$

40. Dye test can be used to distinguish between
(a) Ethylamine and acetamide
(b) Ethylamine and aniline
(c) Urea and acetamide
(d) Methylamine and ethylamine

41. Aniline on heating with conc. HNO_3 + conc. H_2SO_4 mixture yields
(a) *o*- and *p*-Nitroanilines
(b) *m*-Nitroaniline
(c) A black tarry matter
(d) No reaction

42. Which of the following compounds cannot be identified by carbylamine test?
(a) $C_6H_5NHC_6H_5$ (b) $CH_3CH_2NH_2$
(c) $CHCl_3$ (d) $C_6H_5NH_2$

43. Hinsberg reagent is

(a) $C_6H_5SO_3H$ (b) C_2H_5NO
(c) $C_6H_5SO_2Cl$ (d) $C_6H_5N_2Cl$

44. Reaction of nitrous acid with aliphatic 1° amine in cold gives
(a) A diazonium salt (b) An alcohol
(c) A nitrile (d) A dye

45. Which of the following reagents can be used to convert benzene diazonium chloride into benzene?
(a) CH_3OH (b) H_3PO_2
(c) $Br_2—H_2O$ (d) $LiAlH_4$

46. Gabriel synthesis is used for the preparation of
(a) 1° Aromatic amines
(b) 1° Aliphatic amines
(c) 2° Amines
(d) Tertiary amines

47. Which of the following is most basic in nature?
(a) NH_3 (b) CH_3NH_2
(c) $(CH_3)_2NH$ (d) $C_6H_5N(CH_3)_2$

48. An organic compound (A) having molecular formula C_2H_3N on reduction gave another compound B. Upon treatment with nitrous acid B gave ethyl alcohol. On warming with chloroform and alcoholic KOH, it formed an offensive smelling compound (C). The compound C is
(a) $CH_3CH_2NH_2$ (b) $CH_3CH_2N \rightleftarrows C$
(c) $CH_3N \equiv N$ (d) CH_3CH_2OH

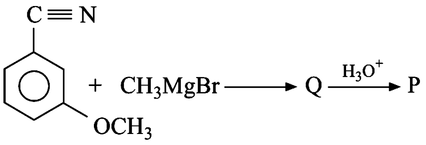
49. Aniline is reacted with bromine water and the resulting product is treated with an aqueous solution of sodium nitrite in the presence of dilute HCl. The compound so formed is converted into tetrafluoroborate which is subsequently heated dry. The final product is
(a) *p*-bromoflurorobenzene
(b) *p*-bromoaniline
(c) 2, 4, 6-tribromofluorobenzene
(d) 1, 3, 5-tribromobenene

50. Reduction of nitrobenzene in the presence of Zn/NH_4Cl gives
(a) Hydrazobenzene
(b) N-Phenyl hydroxylamine
(c) Aniline
(d) Azobenzene

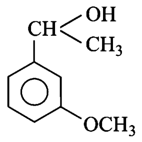
51. Indicate which nitrogen compound amongst the following would undergo Hoffmann reaction (i.e., reaction with Br_2 and strong KOH) to furnish the primary amine ($R—NH_2$)
(a) $R\ CONHCH_3$ (b) $R\ COONH_4$
(c) $R\ CONH_2$ (d) $R—CO—NHOH$

52. Hydrolysis of *benzonitrile* gives
(a) Benzylamine (b) Aniline
(c) Benzoic acid (d) Benzene.

53. Which of the following will give primary amine on hydrolysis?
(a) Nitroparaffins (b) Alkycyanide
(c) Oxime (d) Alkylisocyanide

54.


The product ‘P’ in the above reaction is

(a)


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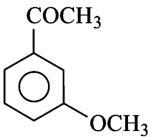

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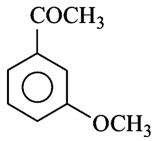

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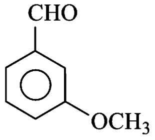
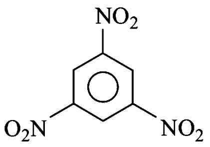


55. Electrolytic reduction of nitrobenzene in acidic medium gives
(a) Azoxybenzene
(b) Aniline
(c) Phenylhydroxylamine
(d) *p*-Aminophenol

56. The reaction of nitrobenzene with zinc and alkali results in the formation of
(a) Aniline
(b) Hydrazobenzene
(c) Nitrosobenzene
(d) Phenyl hydroxylamine

57. The compound obtained by heating a mixture of 1° amine and chloroform with ethanolic potassium hydroxide (KOH) is
(a) An alkyl isocyanide
(b) An alkyl isothiocyanate
(c) An amide
(d) An amide and nitro compound

58. The major product (70% to 80%) of the reaction between *m*-dinitrobenzene with $(NH_4)_2S_x$ is

(a)


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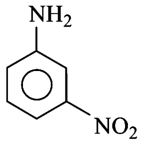

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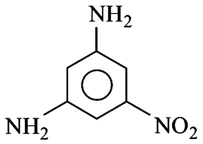
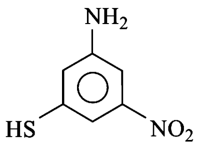


(d)


59. Pick up the correct statement.
(a) Boiling points of alkyl halides are more than corresponding alkane
(b) In water solubility of $CH_3OH > C_2H_5OH > C_6H_5OH$
(c) $C_6H_5NH_2$ is weaker base than NH_3
(d) All the above statements are correct

60. $CH_3CH_2Cl \xrightarrow{NaCN} X \xrightarrow{Ni/H_2} Y \xrightarrow[\text{anhydride}]{\text{Acetic}} Z$ in the above reaction sequence is

(a) $CH_3CH_2CH_2NHCOCH_3$
(b) $CH_3CH_2CH_2NH_2$
(c) $CH_3CH_2CH_2CONHCH_3$
(d) $CH_3CH_2CH_2CONHCOCH_3$

61. Which of the following amine will not respond to carbylamine reaction?

(a) Ethylamine (b) $(CH_3)_2NH$
(c) CH_3NH_2 (d) Benzylamine

ANSWERS

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (c)
11. (b)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (a)
21. (a)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (a)
31. (c)	**32.** (a)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (c)	**38.** (b)	**39.** (a)	**40.** (b)
41. (c)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (b)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (b)
51. (c)	**52.** (c)	**53.** (d)	**54.** (b)	**55.** (d)	**56.** (b)	**57.** (a)	**58.** (b)	**59.** (d)	**60.** (a)
61. (b)									

29

Polymers

BRIEF REVIEW OF THE CONCEPTS

A polymer or a macromolecule is a very big molecule which is made up of a large number of smaller molecules called **monomers.**

Polymers are classified on the basis that they are naturally occurring or synthetic. Some of the natural polymers are vital to our existence such as starch, cellulose, proteins and nucleic acids, etc. Synthetic polymers are also called man-made polymers. The first attempts were aimed at making substitutes for natural macromolecules, rubber and silk. Some of the important synthetic polymers are polyethylene, PVC, teflon, nylon, etc.

Polymers can also be classified on the basis of their structure. On this basis polymers can be split into two classes —

(i) **Homopolymers** When there is only one monomer unit which repeats itself in the polymer and,

(ii) **Copolymers** When a mixture of two or more monomers form a polymer.

For practical use, the polymers are classified on the basis of their physical properties. The important physical properties are crystallinity, tensile strength, elasticity, toughness, etc. Based on these properties, the polymers are divided into following three classes.

(i) **Elastomers** These polymers have an elastic character i.e. They can be stretched and try to regain their former shape when the stress is removed. The polymer chains in such type of polymers are held together by weak intermolecular forces. Rubber (natural and synthetic both) is one of the examples of an elastomer.

The process of heating natural rubber with sulphur is known as *vulcanization*. It improves the tensile strength and resistance to abrasion.

(ii) **Fibres** They are long and thread like materials which are characterised by great tensile strength. Fibres are twisted into threads, which can then be woven into clothes. They have strong intermolecular forces between polymeric chains. Cotton, wool, silk, nylon, orlon, etc. are common fibres.

(iii) **Plastics** These can further be divided into two types—*thermoplastics* and *thermosetting plastics.* **Thermoplastics** soften on heating and become fluid but on cooling they become hard so they can be easily moulded into desired shapes. These are normally linear polymers in which the forces of attraction between the polymer chains are in between those of elastomers and fibres. Polyethylene, polystyrene, PVC, etc., are common examples of thermoplastics. These type of plastics can be reused that is they can be heated and reshaped again.

Thermosetting polymers are space-network polymers. These are highly cross-linked to form a rigid but irregular three-dimensional structure. These cannot be reshaped on heating because of cross-linked polymer chains. Some examples of this type of polymers are bakelite, urea-formaldehyde resin, etc.

All synthetic polymers are usually prepared by two different types of polymerization processes—one is called *addition polymerization* and the other is known as *condensation polymerization.* In **addition polymerization,** the monomers add to one another in such a way that the polymer contains all the atoms of the starting monomers. For example, the polymerization of ethylene to polyethylene

$$nCH_2 = CH_2 \longrightarrow \left(CH_2 - CH_2 \right)_n$$

Addition polymerization is mainly encountered in the case of unsaturated compounds.

Condensation polymers are formed by combination of monomers with the elimination of small molecules like water, methanol, ammonia, etc. For example, the formation of nylon is a condensation poylmerization in which two units, adipic acid and hexamethylenediamine condense to form amide groups with the loss of water molecules.

Many times, it is difficult to find out whether the polymerization has occurred through addition or condensation. Therefore, another classification has been proposed on the basis of the mechanism of combination of monomer units. According to this polymers are classified as *chain growth polymers* and *step growth polymers.*

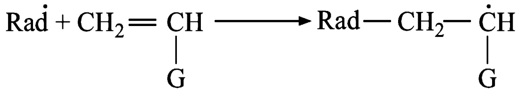
(i) **Chain-growth polymers** These polymers are formed by the successive addition of monomer units to the growing chain carrying a reactive site. The reactive intermediated may be a free-radical, a carbocation or a carbanion.

(ii) **Step-growth polymers** These polymers are formed through a series of independent reactions. Each such reaction involves the condensation between two monomers to produce a dimer which in turn produce tetramers and so on. Hence each monomer has more than one similar or different functional groups. As in this process, the polymer is formed in a step wise manner, so is called step-growth polymer.

Mechanism of addition polymerization As mentioned above that the addition polymerization follows a chain-growth pathway. It may involve free radicals, carbocations or carbanions as intermediates.

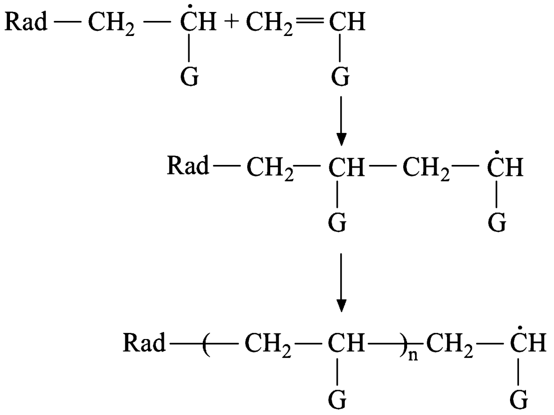
Free radical polymerization is initiated by an initiator. These initiators are molecules which decompose to provide radicals. These radicals add to a monomer molecule to form a new free radical of larger size. This constitutes the chain initiating step.

$$\text{Initiator} \longrightarrow \dot{\text{Rad}}$$



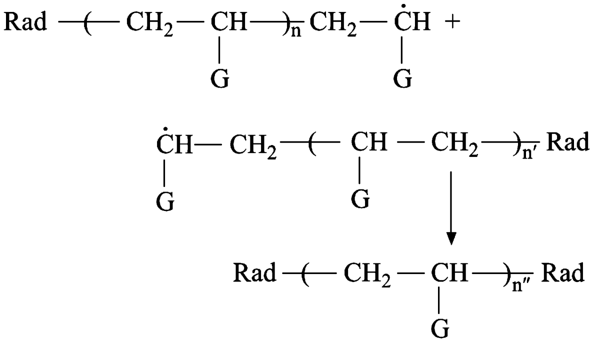
$$\dot{\text{Rad}} + CH_2{=}\underset{G}{CH} \longrightarrow \text{Rad}{-}CH_2{-}\underset{G}{\dot{C}H}$$

The radical so formed combines with another monomer unit to form still bigger radical and chain continues. These steps are called chain propagating steps.



$$\begin{gathered}\text{Rad}{-}CH_2{-}\underset{G}{\dot{C}H} + CH_2{=}\underset{G}{CH}\\ \Big\downarrow\\ \text{Rad}{-}CH_2{-}\underset{G}{CH}{-}CH_2{-}\underset{G}{\dot{C}H}\\ \Big\downarrow\\ \text{Rad}{-}\!\left(CH_2{-}\underset{G}{CH}\right)_{n}\!{-}CH_2{-}\underset{G}{\dot{C}H}\end{gathered}$$

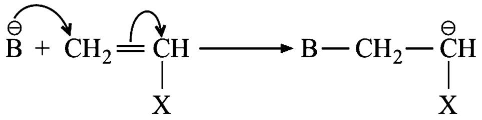
These chains terminate ultimately by combination of two such very large radicals to give the polymer.



$$\begin{gathered}\text{Rad}{-}\!\left(CH_2{-}\underset{G}{CH}\right)_{n}\!CH_2{-}\underset{G}{\dot{C}H} +\\ \underset{G}{\dot{C}H}{-}CH_2{-}\!\left(\underset{G}{CH}{-}CH_2\right)_{n'}\!\text{Rad}\\ \Big\downarrow\\ \text{Rad}{-}\!\left(CH_2{-}\underset{G}{CH}\right)_{n''}\!\text{Rad}\end{gathered}$$

When a molecule like carbon tetrachloride is present during the free radical polymerization, it combines with the radical formed in propagating step. This leads to the formation of polymer of shorter chain length.

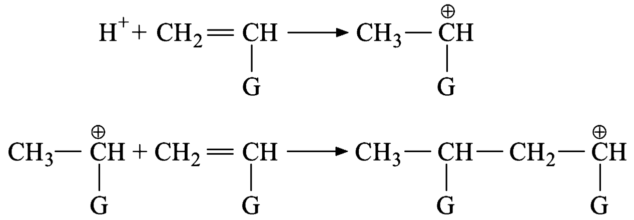
In anionic polymerization, the initiator is a strong base or nucleophile which reacts with alkene to form an anion. The anion so formed combines with other alkene to propagate the chain further.



$$\overset{\ominus}{B} + CH_2{=}\underset{X}{CH} \longrightarrow B{-}CH_2{-}\underset{X}{\overset{\ominus}{C}H}$$

Such an anionic species will be stabilised when '*X*' is an electron withdrawing group. Since in styrene '*X*' is phenyl group which stabilises the anionic species by dispersal of negative charge. That is why styrene undergoes polymerization by anionic pathway.

In cationic polymerization, the initiator is an electrophile. It can be H^+ or a lewis acid like BF_3. On addition to double bond of alkene, they generate a carbocation. This carbocation combines further with alkene molecules and the chain is propagated.



$$H^+ + CH_2{=}\underset{G}{CH} \longrightarrow CH_3{-}\underset{G}{\overset{\oplus}{C}H}$$

$$CH_3{-}\underset{G}{\overset{\oplus}{C}H} + CH_2{=}\underset{G}{CH} \longrightarrow CH_3{-}\underset{G}{CH}{-}CH_2{-}\underset{G}{\overset{\oplus}{C}H}$$

The cationic species are stabilised and are formed easily if, the group '*G*'; is electron releasing. Hence the vinyl monomers containing electron donating groups preferably undergo polymerization by this mechanism.

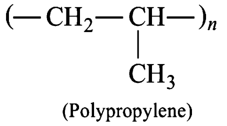
Molecular mass of polymers

Generally, a polymer sample contains chains of varying lengths and, therefore, its molecular mass is always expressed as an average. In contrast, natural polymers such as proteins contain chains of identical length and hence, have definite molecular mass.

Some commercially important addition polymers

Monomer

1. $(-CH_2-CH_2-)_n$ (Polyethylene) — $CH_2{=}CH_2$ (Ethylene)

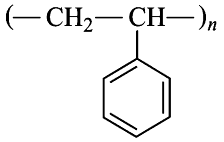
2. 

$(-CH_2-\underset{CH_3}{CH}-)_n$

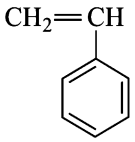
(Polypropylene)

$CH_2{=}CH{-}CH_3$

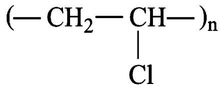
(Propylene)

3. 

(Polystyrene)



(Styrene)

4. 
$(-CH_2-\underset{|}{\underset{Cl}{CH}}-)_n$

(Polyvinyl chloride)
or PVC

$CH_2=CH-Cl$

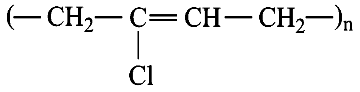
(Vinyl chloride)

5. $(-CF_2-CH-)_n$

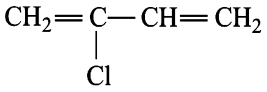
(Teflon)

$CF_2=CF_2$

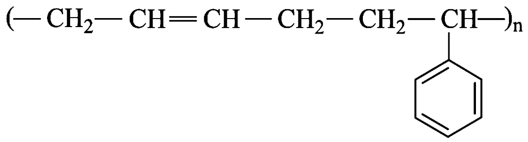
(Tetrafluoroethylene)

6. 
$(-CH_2-\underset{Cl}{\underset{|}{C}}=CH-CH_2-)_n$

(Neoprene)


$CH_2=\underset{Cl}{\underset{|}{C}}-CH=CH_2$

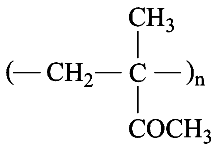
(Chloroprene)

7. 

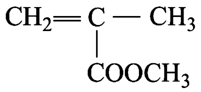
(Buna-S)

$CH_2=CH-CH=CH_2$ and Styrene

1, 3-Butadiene

8. 
$(-CH_2-\overset{CH_3}{\overset{|}{\underset{COCH_3}{\underset{|}{C}}}}-)_n$

(Polymethylmethacrylate
or PMMA)


$CH_2=\underset{COOCH_3}{\underset{|}{C}}-CH_3$

(Methylmethacrylate)

Some commercially important condensation polymers

	Polymer	*Monomers*
1.	Nylon-66	Adipic acid and Hexamethylenediamine
2.	Terylene	Terephthalic acid and ethylene glycol
3.	Phenol-formaldehyde resin (Bakelite)	Phenol and formaldehyde
4.	Urea-formaldehyde resin	Urea and formaldehyde
5.	Melamine-formaldehyde resin	Melamine and formaldehyde

Problems for Practice

1. Copolymer is
(a) Nylon-6 (b) Nylon 66
(c) PMMA (d) Polythene

2. Which of the following contains isoprene units?
(a) Natural rubber (b) Nylon-66
(c) Polyethylene (d) Dacron

3. The widely used PVC is a polymerized product of
(a) $CH_2 = CH_2$ (b) $CH_2 = CCl_2$
(c) $CH_2Cl.CH_2Cl$ (d) $CH_2 = CHCl$

4. Which is an example of thermosetting polymer?
(a) Polythene (b) Neoprene
(c) PVC (d) Bakelite

5. Orlon is a polymer of
(a) Tetrafluoroethylene
(b) Acrylonitrile
(c) Ethanoic acid
(d) Teflon

6. Which of the following fibres are made of polyamides?
(a) Dacron (b) Orlon
(c) Nylon (d) Rayon

7. Terylene is a condensation polymer of ethylene glycol and
(a) Benzoic acid (b) Phthalic acid
(c) Salicylic acid (d) Terephthalic acid

8. Which one of the following is not an example of chain growth polymer?
(a) Neoprene (b) Buna-S
(c) PMMA (d) Glyptal

9. $F_2C = CF_2$ is a monomer of
(a) Teflon (b) Glyptal
(c) Nylon-6 (d) Buna-5

10. Caprolactam polymerises to give
(a) Terylene (b) Teflon
(c) Glyptal (d) Nylon-6

11. The process involving heating of rubber with sulphur is called
(a) Galvanization (b) Vulcanization
(c) Bessemerization (d) Sulphonation

12. Which of the following is build up from single substance with low molecular mass?
(a) Protein (b) Polythene
(c) Nylon (d) Terylene

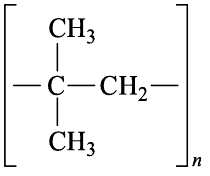
13. Among the following polymers, the strongest molecular forces are present in
(a) Elastomers
(b) Fibers
(c) Thermoplastics
(d) thermosetting polymers

14. Bakelite is obtained from phenol by reacting with
(a) Acetaldehyde (b) Formaldehyde
(c) acrolein (d) benzaldehyde

15. Which of the following is not an example of addition polymer?
(a) Polythene (b) Polystyrene
(c) Neoprene (d) Terylene

16. Which of the following is used to make nonstick cookware?
(a) PVC
(b) Polyethylene
(c) Polystyrene
(d) Polytetrafluoroethylene

17. Monomer of



$$\left[-\underset{\underset{CH_3}{|}}{\overset{\overset{CH_3}{|}}{C}}-CH_2- \right]_n$$

is
(a) 2-Methylpropene (b) Styrene
(c) Propylene (d) Ethene

18. Polyvinyl alcohol can be prepared by
(a) Polymerization of vinyl alcohol
(b) Alkaline hydrolysis of polyvinyl acetate
(c) Polymerization of acetylene
(d) Reaction of acetylene with H_2SO_4 in presence of $HgSO_4$.

19. Tetrafluoroethene is the monomer of
(a) Polyethene (b) PVC
(c) Teflon (d) Nylon-66

20. Of the following which is a step growth polymer
(a) Bakelite (b) Polyethylene
(c) Teflon (d) PVC

21. The weakest interparticle forces are present in
(a) Thermosetting polymers
(b) Thermoplastic polymers
(c) Fibres
(d) Elastomers

22. Natural rubber is which type of polymer?
(a) Condensation polymer
(b) Addition polymer
(c) Coordination polymer
(d) None of these

23. Which of the following pairs is not correctly matched?
(a) **Terylene**-condensation polymer of terephthalic acid and ethylene glycol
(b) **Teflon**-thermally stable cross linked polymer of phenol and formaldehyde
(c) **Perspex**-A homopolymer of methyl methacrylate
(d) **Synthetic rubber**-A copolymer of butadiene and styrene

24. The catalyst used for the polymerization of olefins is
(a) Ziegler-Natta catalyst
(b) Wilkinson's catalyst
(c) Zeolite
(d) Zeise's salt complex

25. Which of the following is not a condensation polymer?
(a) Nylon-66 (b) PTFE
(c) Dacron (d) Glyptal

26. In vulcanization of rubber
(a) Sulphur reacts to form new compound
(b) Sulphur cross-links are introduced
(c) Sulphur forms a very thin protective layer over rubber
(d) All of these

27. Teflon, Polystyrene and neoprene are all
(a) Copolymers
(b) Condensation polymers
(c) Homopolymers
(d) Monomers

28. Synthetic polymer which resembles natural rubber is
(a) Neoprene (b) Chloroprene
(c) Glyptal (d) Nylon

29. Bakelite is a product of the reaction between
(a) Formaldehyde and NaOH
(b) Aniline and urea
(c) Phenol and methanal
(d) Phenol and chloroform

30. One of the raw materials to form nylon is
(a) Adipic acid (b) Butadiene
(c) Isoprene (d) Ethylene

31. Soft drinks and baby feeding bottles are generally made up of
(a) Polyester (b) Polyurethane
(c) Polystyrene (d) Polyamide

32. Which of the following is an example of co-polymer?
(a) Buna-S (b) PAN
(c) Polythene (d) PTFE

33. Buna-S is a polymer of
(a) Butadiene
(b) Styrene

(c) Tetrafluoroethylene
(d) Butadiene and styrene

34. The product of addition polymerization reaction is
(a) PVC (b) Nylon
(c) Terylene (d) Polyamide

35. An example of natural biopolymer is
(a) Teflon (b) Nylon-66
(c) PVC (d) DNA

ANSWERS

1. (b)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (d)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (d)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (a)
21. (d)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (a)	**29.** (c)	**30.** (a)
31. (c)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (d)					

Biomolecules

30

BRIEF REVIEW OF THE CONCEPTS

Our bodies need energy to keep warn, move about and build new tissues. Two of our foods, carbohydrates and lipids provide us virtually all our energy for living. The body takes in food and oxygen and gives off carbon dioxide and water. Energy is generated, transported and consumed by way of chemical reactions.

The compound that stores energy released in the process of oxidation is *ATP* (adenosine triphosphate). Whenever energy is needed, it dissociated into a molecule of *ADP* (Adenosine diphosphate) and a phosphate unit releasing the energy. When energy is produced one molecule of *ADP* takes it up along with a phosphate unit and gets converted into *ATP*.

Photosynthesis The term is generally used to describe a process by which green plants, in the presence of sun light, utilise CO_2 and H_2O to synthesise carbohydrates. The presence of chlorophyll is necessary for this reaction.

It is a complex process which occurs in a sequence of reactions and these reactions can be divided into two stages—light reactions and dark reactions.

In the light reactions, the solar energy and CO_2 are absorbed while oxygen and *ATP* are produced.

$$CO_2 + H_2O \xrightarrow{hv} O_2 + \text{ATP} + \text{organic molecules}$$

In the dark reaction CO_2 gets converted into glucose. The energy required for the process is obtained form the *ATP* produced in light reaction.

The overall reaction of photosynthesis can be represented as follows.

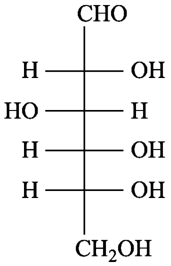
$$6CO_2 + 6H_2O + 2880\text{ kJ} \longrightarrow C_6H_{12}O_6 + 6O_2.$$

Green plants prepare their food in this process and are called *autotrophs.* All animal, fungi and most bacteria are unable to synthesise food and depend on autotrophic organisms directly or indirectly. They are called *heterotrophs.*

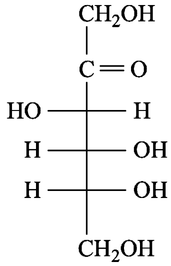
Biomolecules. Organic molecules which built up living organisms are called biomolecules.

Carbohydrates are polyhydroxy aldehydes or ketones or molecules which provide such units on hydrolysis. Based on structure, the carbohydrates have been classified into three main classes.

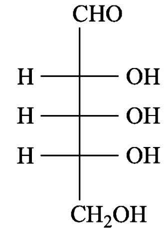
(i) *Monosaccharides.* Carbohydrates which cannot be further hydrolysed to give simpler units of polyhydroxy aldehyde or ketone *e.g.* glucose, fructose, galactose, ribose etc. They are further classified as aldose or ketose depending upon, whether they contain an aldehyde or ketone group. About 20 monosaccharides occur in nature, but most common are glucose, fructose, ribose and 2-deoxyribose.



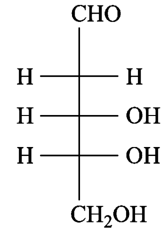
D-Glucose
(an aldohexose)



D-Fructose
(an ketohexose)

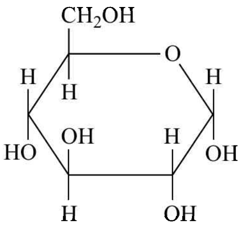


D-Ribose
(an aldopentose)

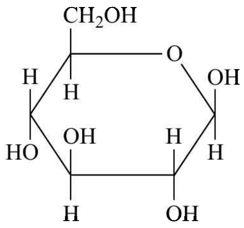


2-Deoxy-D-ribose
(an addopentose)

These mono saccharides generally exist as cyclic compounds in nature. A ring is formed by a reaction between the carbonyl group and one of the hydroxyl group present in the molecule. Glucose preferentially forms the six membered ring which can be in two different isomeric forms called α-and β-forms. The two forms differ only in the configuration of the hydroxyl group at carbon no.1. Such isomers are called **anomers.**

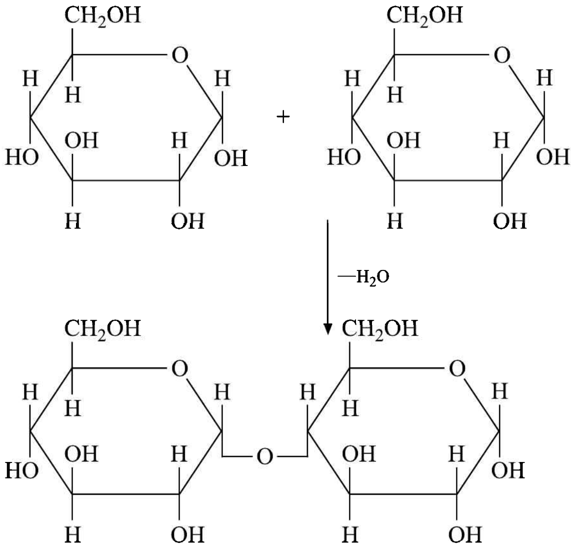


α-*D*-glucose



β-*D*-glucose

(ii) *Disaccharides* are formed by the condensation of two monosaccharide molecules. These monosaccharides join together by the loss of a water molecule between the one hydroxyl group on each monosaccharide. Such a linkage, which holds the monosaccharide units together is called *glycoside linkage*. If two glucose molecules are joined together, the disaccharide, maltose is formed.



Similarly sucrose (the common sugar) consists of one molecule of glucose and one molecule of fructose. Lactose (or milk sugar) is found in milk and contains one molecule of glucose and one molecule of galactose.

(iii) *Polysaccharides* If a large number of monosaccharide units are joined together, we get polysaccharides. These are the most commonly encountered carbohydrates in nature. They have mainly one of the two functions—either food material or structural material. Starch is the main food storage polysaccharide of plants. It is a polymer of α-glucose and consists of two types of chains—known as *amylose* and *amylopectin.*

Amylose is a water soluble fraction and is a linear polymer of α-glucose. Amylopectin is a water insoluble fraction and consists of branched chain polymer of α-glucose.

The carbohydrates are stored in animal body as glycogen which is also a polymer of α-glucose and its structure is similar to amylopectin.

Cellulose is another natural polysaccharide which is the main component of wood and other plant materials. It consists of long chain of β-glucose molecules.

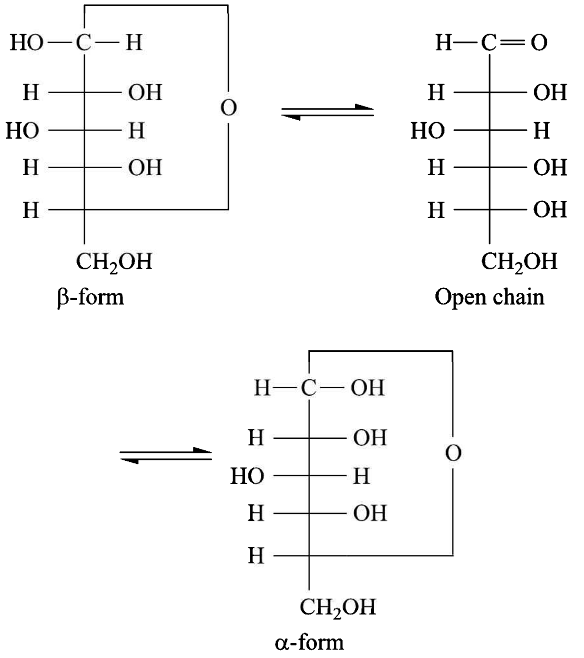
Reducing and Non-reducing sugars

All those carbohydrates which reduce Fehling's solution and Tollen's reagent are called reducing sugars. Other carbohydrates which do not reduce these reagents called nonreducing sugars.

The structural feature which characterises a reducing sugar is the presence of a free aldehyde of ketonic group. All monosaccharides whether aldoses or ketoses are reducing in nature.

Mutarotation

A spontaneous change in the specific rotation of a solution of an optically active compound is called mutarotation. For example *D*-glucose (mp. 146°C) has the specific rotation + 113° but when its solution is allowed to stand for some time, this value drops to + 52.7°. Similarly the specific rotation for the other form of glucose (mp. 150 °C) is + 19° but when its solution is allowed to stand, this value rises to + 52.7°. This is explained on the basis of cyclic structure of glucose. Glucose on cyclization gives two forms α-and β-with specific rotation +113° and + 19° respectively. When the solution of any of these forms in water is allowed to stand, an equilibrium between two forms is obtained through the open chain structure as shown below.



Proteins

These are polymers of aminoacids. A living system contains thousands of different proteins for its various functions. All proteins are the polymers of twenty different α-amino acids. Out of these 10 amino acids cannot be synthesised by our body and hence must form the part of our diet. There are called *essential amino acids.*

Two amino acid molecules join by the loss of water molecule between —NH_2 group of one amino acid and —COOH group of another, leading to the formation of an amide group. This is called a **Peptide bond** and product of two amino acids is called a *dipeptide.* When many amino acids combine in this way, the product is called a polypeptide.

Structure of protein can be discussed at four different levels.

(i) **Primary structure** Information regarding the sequence of amino acids in a protein chain is called its primary structure. The primary structure of a protein determines its functions and biological activity.

(ii) **Secondary structure** The secondary structure arises due to the regular folding of the backbone of the polypeptide chain due to hydrogen bonding between $-\overset{O}{\overset{\|}{C}}-$ and —NH_2 groups. Two types of secondary structure have been reported —α-helix when the chain coils up and β-pleated sheet when hydrogen bonds are formed between the chains.

(iii) **Tertiary structure** It is the three-dimensional structure of proteins. It arises due to folding and superimposition of various secondary structure components.

By different modes of bonding in secondary and tertiary structural levels, a protein molecule appears to have a unique three-dimensional structure called its *native form.* On changing pH, temperature or adding some organic solvents leads to the changes in conformation accompanied by complete loss of activity and functions of protein. This disruption of the native structure of proteins is called denaturation. During denaturation, the protein molecules uncoil from an ordered conformation into a more random conformation. They also lose their biological activity due to denaturation. Chemically, denaturation does not change the primary structure of a protein, it only brings about change in secondary or tertiary structures. The charge in appearance of an egg on heating is due to denaturation. Coagulation of milk on addition of lemon juice is also an example of denaturation.

The denaturation may be reversible or irreversible. The coagulation of egg white on boiling of egg is an example of irreversibly denaturation. In some cases it is found that if the disruptive agents is removed, the protein recovers its original physical, chemical and biological properties. It is called reversible denaturation and the process is called renaturation.

(iv) **Quaternary structure** The quaternary structure refers to the way in which simple protein chains associate with each other resulting in the formation of a complex protein.

Enzymes

Most of the chemical reactions which occur in living systems proceed at very high rate even under mild conditions of temperature and pH. These reactions are catalysed by a group of biomolecules called enzymes. Without enzymes, the living process would be too slow to sustain life. Almost all these enzymes are proteins chemically.

Characteristics of Enzymes

(i) Enzymes speed up biochemical reactions up to ten million times compared to the uncatalysed reaction
(ii) Enzyme catalysed reactions rapidly obtain equilibrium.
(iii) enzymes function in dilute aqueous solutions, at low temperatures and a specific pH.
(iv) They are very specific and selective in their action on substrates.
(v) Even small amounts of enzymes are highly efficient.
(vi) In addition to the protein structure, most active enzymes are associated with some nonprotein component required for their activity, called co-enzymes.

Mechanism of enzyme action First the substrate molecule binds to the active site of the enzyme. There is a particular enzyme for each substrate and they are said to have lock and key arrangement. It results in the formation of an enzyme-substrate complex. In this complex, the substrate is placed in the right orientation to facilitate a given reaction. This complex then breaks to give the molecule of the product and regenerates the enzyme for the next molecule of the substrate.

Till date, more than 3000 enzymes have been identified and they constitute the largest and most highly specialised class of proteins. They differ from ordinary chemical catalysts in two major ways. One is that the enzymes function under proper conditions of temperature, pH and bioenvironment. Secondly they are highly specific in nature for biochemical processes.

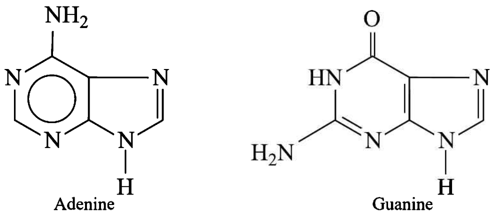
Nucleic acids

They are polymers composed of repeating units of *nucleotides.* Nucleotide is a unit which consists of a phosphate group, a five carbon sugar and an aromatic heterocyclic nitrogen compound called a base. Nucleic acids play a very vital role in the transmission of the hereditary characteristics and the biosynthesis of proteins.

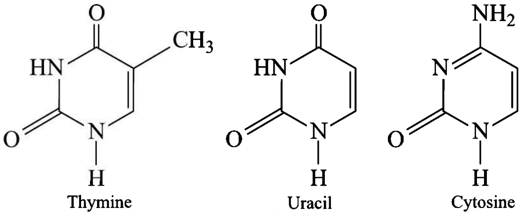
There are two types of nucleic acids, named as DNA and RNA.

The sugar present in DNA is 2-deoxyribose and the bases present are adenine, guanine, cytosine and thymine.

The structures of these bases are given below.



(purine bases)



(pyrimidine bases)

RNA contains ribose sugar and bases present are adenine, guanine, cytosine and uracil. DNA contains the coded message for protein synthesis whereas RNA actually carry out the synthesis of protein. There are three types of RNA molecules which are classified as messenger RNA, ribosomal RNA and transfer RNA on the basis of function they perform. DNA is a double stranded helix in which the two nucleic acid chains are wound about each other and held together by hydrogen bonds between pair of bases. These hydrogen bonds are specific between pair of appropriate bases (guanine with cytosine and adenine with thymine). DNA is capable of duplicating itself by a process called replication. This process occurs before the cell division.

When DNA molecules are hydrolyses, the bases are obtained in a particular ratio because in DNA the bases are present as pairs in two complementary strands. RNA are single strand molecules i.e. they do not have complementary base pairs and hence there is no relationship in the quantities of bases obtained on hydrolysis. This fact also confirms that RNA are single strand molecules.

Replication of DNA

The information contained in the DNA can be transferred to new cells and to the next generation. This transfer is possible because DNA is capable of duplicating itself by a process called replication. In other words it can be said that it is the process by which a single DNA molecule produces two identical copies of itself. The process of replication starts with the partial unwinding of the two strands of DNA double helix. Each strand then acts as the base for the synthesis of two new strands of DNA. The specificity of base pairing ensures that each new strand is complementary to its old base strand. As a result, two identical copies of DNA from the original DNA are produced. Each of these two copies are then passed on to the two new cells resulting from cell division. In this way hereditary characters are preserved and are transmitted to next generation.

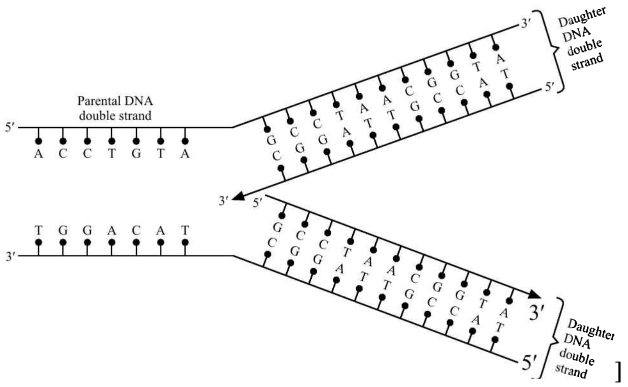


Fig. 30.1

Lipids

Those constituents of the cell which are insoluble in water and soluble in organic solvents of low polarity (such as chloroform, ether and benzene, etc.) are termed as lipids. They are classified into three broad categories on the basis of their molecular structure and the hydrolysis products.

(i) *Simple lipids* They are esters and on hydrolysis yield long chain acids and alcohol such as glycerol. Examples are fats and waxes.

(ii) *Compound lipids* Compound lipids on hydrolysis yield other substance like phosphoric acid and nitrogen containing compounds along with alcohols and fatty acids.

(iii) *Derived lipids* These are the substances derived from simple or compound lipids which are formed during metabolism in the body. They include steroids and fat soluble vitamins etc.

Fats and oils are triesters of glycerol and long chain carboxylic acids, called fatty acids. They are also known as triglycerides. A fat is a triglyceride that is solid or semisolid at room temperature and the oil is liquid at room temperature. Fats and oils are found both in plants and animals.

Waxes are the esters of long chain fatty acids with long chain monohydric alcohol. A thin coating of wax is present on leaves, petals or fruits in plants.

Phospholipids are a class of lipids closely related to triglycerides. They consist of a glycerol esterified with two long chain fatty acids and a phosphate group which is further attached to some aminoalcohol.

Phospholipids are widespread and are found in the brain, liver, egg yolks, yeast, wheat germ etc. The fact that phospholipids have anionic end and a hydrocarbon end, leads to the use of these compounds as emulsifying agents and as membrane forming agents. Phospholipids help in transport of other lipids in the bloodstream by keeping them emulsified.

Hormones. Cells in the body communicate in two different ways—one by sending electrochemical impulses through nerves and other by sending some chemical substances called *hormones.* They are produced by endocrine glands and the secretions are directly poured into blood. Based on the chemical structure they are divided into three classes—steroids, polypeptides and amines.

Vitamins

It has been observed that there are some organic compounds other than carbohydrates, proteins and fats, which are also necessary for normal growth and maintenance of health. These specific organic compounds are known as accessory dietary factors or vitamins. These are required in our system for carrying out various physiological actions in animal bodies and to stimulate the growth of the cells. Their deficiency leads to a number of diseases called deficiency diseases. Many vitamins have now been discovered and they may be divided into two main groups depending upon whether they are fat soluble(vitamin A, D, E and K) or water soluble (vitamin B complex and C).

Deficiency diseases caused due to lack of vitamins are listed below.

Table 30.1

Name of vitamin	*Deficiency disease*
Vitamin A	Xerophthalmia
Vitamin B_1	Beri-beri
Vitamin B_6	Convulsions and dermatitis
Vitamin B_{12}	Anaemia
Vitamin C	Scurvy
Vitamin E	Sterility
Vitamin K	Haemorrhagic conditions

Problems for Practice

1. The energy currency of the cells is
(a) ATP (b) ADP
(c) AMP (d) None of these

2. In polysaccharides, the linkage connecting monosaccharide is called
(a) Glycogen linkage (b) Glycoside linkage
(c) Nucleoside linkage (d) Peptide linkage

3. Which substance chars when heated with conc. H_2SO_4?
(a) Fat (b) Hydrocarbon
(c) Carbohydrate (d) Protein

4. On hydrolysis of starch, we finally get
(a) Glucose (b) Fructose
(c) Both (d) Sucrose.

5. Which of the following is not a reducing sugar?
(a) Glucose (b) Sucrose
(c) Mannose (d) Fructose

6. Sugars like glucose and mannose differ only at C_2. Such type of sugars are called as
(a) Anomers (b) Epimers
(c) Tautomers (d) Enantiomers

7. Enzymers, in the living systems
(a) Provide energy
(b) Provide immunity
(c) Transport oxygen
(d) Catalyse biochemical processes

8. The chemical change in DNA molecule that could lead to synthesis of protein with an altered amino acid sequence is called
(a) Replication (b) Lipid formation
(c) Cellular membrane (d) Mutation

9. Which of the following compounds is responsible for the transmission of heredity characters?
(a) RNA (b) DNA
(c) Glucose (d) Haemoglobin

10. Calorific value is in which order?
(a) Fats > Carbohydrate > Proteins
(b) Carbohydrates > Fats > Proteins
(c) Proteins > Carbohydrates > Fats
(d) Fats > Proteins > Carbohydrates

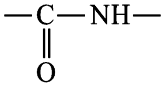
11. α—D-glucopyranose and β—D-glucopyranose are
(a) Anomers (b) Epimers
(c) Enantiomers (d) Tautomers

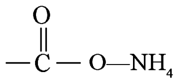
12. A sucrose molecule on hydrolysis yields
(a) Two molecules of glucose only
(b) One molecule of glucose and one of maltose
(c) One molecule of fructose and one of glucose
(d) One molecule of glucose and one of cellobiose

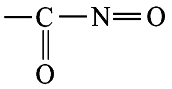
13. Cyanocobalamine is
(a) Vitamin A (b) Vitamin B_{12}
(c) Ergocaliciferol (d) Riboflavin

14. Which of the following represents peptide bond?

(a) $—CONH_2$

(b)

$-\underset{\underset{O}{\|}}{C}-NH-$

(c)

$-\overset{\overset{O}{\|}}{C}-O-NH_4$

(d)

$-\underset{\underset{O}{\|}}{C}-N=O$

15. The change in optical rotation with time of freshly prepared solution of sugar is known as
(a) Specific rotation (b) Inversion
(c) Rotatory motion (d) Mutarotation

16. The disaccharide present in milk is
(a) Sucrose (b) Maltose
(c) Lactose (d) Cellobiose

17. The abbreviation ATP represents:
(a) Atmospheric temperature and pressure
(b) Adenosine triphosphate
(c) An abbreviation of some disease
(d) None of these

18. Enzymes
(a) have optimum activity at body temperature
(b) consists of nucleic acids
(c) are carbohydrates
(d) have all these properties

19. The primary structure of a polypeptide is determined by
(a) The number of disulphide bonds in the polypeptide
(b) The number of amino acids in the polypeptide
(c) The order of amino acids in the polypeptide
(d) The length of the polypeptide

20. Which of the following acts as store house of energy in the body?
(a) Hormones (b) Vitamins
(c) Fats (d) All these

21. Thyroxine is hormone which
(a) is secreted by thyroid glands
(b) does not stimulate metabolism
(c) decreases blood sugar
(d) is secreted by pancreas

22. Amino acids are soluble in
(a) NaOH (b) HCl
(c) Both (d) Ether

23. Purine derivative among the following bases is
(a) Thymine (b) Uracil
(c) Cytosine (d) Guanine

24. Hydrolysis of proteins in the presence of enzymes produces
(a) Hydroxy acids (b) Dicarboxylic acids
(c) Amino acids (d) Aromatic acids

25. Ring structure of glucose is due to formation of hemiacetal and ring formation between
(a) C_1 and C_5 (b) C_1 and C_4
(c) C_1 and C_3 (d) C_2 and C_4

26. On heating with conc. HNO_3, proteins give yellow colour. This test is called
(a) Oxidising test (b) Xanthoprotic test
(c) Hoppe's test (d) Acid base test

27. Which of the following is not present in RNA?
(a) Uracil (b) Thymine
(c) Ribose (d) Phosphate

28. Ascorbic acid is a chemical name of
(a) Vitamin D (b) Vitamin A
(c) Vitamin C (d) Vitamin B_6

29. In DNA, the complementary bases are
(a) Uracil and adenine : Cytosine and guanine
(b) Adenine and thymine : Guanine and cytosine
(c) Adenine and thymine : Guanine and uracil
(d) Adenine and guanine : Thymine and cytosine

30. The simplest amino acid is
(a) Glycine
(b) Alanine
(c) Guanine
(d) All of these

31. The principal buffer present in the blood is
(a) CH_3COONH_4
(b) CH_3COOH/CH_3COONa
(c) CO_2/HCO_3^-
(d) NaH_2PO_4 / Na_2 HPO_4

32. Deficiency of vitamin D causes
(a) Rickets (b) Night blindness
(c) Xerosis (d) Loss of appetite

33. The substances which act as emulsifires in lipid metabolism are

(a) Bile salts
(b) Fatty acids
(c) Sulphonic acids
(d) Amino acids

34. Which among the following statements about DNA is/are correct?

(a) The sugar present in DNA is 2-deoxy ribose.
(b) It contains guanine and adenine as pyrimidine bases
(c) Occurs in cytoplasm of the cell
(d) All of these

35. Which of the following has magnesium?

(a) Carbonic anhydrase
(b) Haemocyanin
(c) Chlorophyll
(d) Vitamin B_{12}

36. Vitamin K

(a) Is phylloquinone
(b) Soluble in oils and fats
(c) Deficiency lengthens the time of blood clotting
(d) All of these

37. Which of the following molecules is capable of forming zwitter ion?

(a) NH_2CH_2COOH
(b) CH_3COOH
(c) $CH_3CH_2NH_2$
(d) $CH_3—CONH_2$

38. On heating glucose with Fehling's solution we get a precipitate whose colour is

(a) Yellow
(b) red
(c) Black
(d) White

39. Saliva contains

(a) Amylase or ptylin
(b) Trypsin
(c) Bile fluid
(d) Vitamin-A

40. Enzymes are

(a) Proteins
(b) Minerals
(c) Fatty acids
(d) lipids

41. Which among the following statements about RNA is/ are correct?

(a) RNA has the unique property of replication
(b) RNA controls the hereditary effects
(c) RNA has single stranded structure
(d) All of these

42. The function of DNA is

(a) To synthesise RNA
(b) To synthesise necessary proteins
(c) To carry hereditary characters
(d) All of these

43. The bond that determines the secondary structure of a protein is

(a) Co-ordinate bond
(b) Covalent bond
(c) Hydrogen bond
(d) Ionic bond

44. An example of water soluble vitamin is

(a) Vitamin D
(b) Vitamin E
(c) Vitamin A
(d) Vitamin C

45. Oils and fats in our food not only provide us energy but also act as carriers of certain vitamins viz.,

(a) A and B
(b) A and D
(c) B and D
(d) A and C

ANSWERS

1. (a)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (b)	**10.** (a)
11. (a)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (c)
21. (a)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (a)
31. (c)	**32.** (a)	**33.** (a)	**34.** (a)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (a)
41. (c)	**42.** (d)	**43.** (c)	**44.** (d)	**45.** (b)					

Chemistry in Everyday Life

31

BRIEF REVIEW OF THE CONCEPTS

Chemicals in Health Care

Chemical substances used for treatment of diseases and for reducing suffering from pain are called *medicines* or *drugs*. Chemicals used are generally classified according to their action, some of which are described below:

(i) **Antipyretics** Those substances which reduce the body temperature under feverish conditions. Some examples are aspirin, paracetamol, analgin and phenacetin.

(ii) **Analgesics** Chemical substances used for relieving pain such as novalgin, aspirin, paracetamol etc. Most of the antipyretics have analgesic action also.

(iii) **Antiseptics** Chemical substances which prevent the growth of micro-organisms and are not harmful to human beings when applied on living tissues. Some common examples are iodine, bithional, chloroxylenol (dettol), mercurochrome and acriflavin etc.

(iv) **Disinfectants** Chemical substances which kill micro-organisms and are applied to objects like instruments, utencils, clothes or public places. Most of them are harmful to living tissues. Some examples are phenol, cresol, H_2O_2 etc.

(v) **Sedatives and hypnotics** Hypnotics are central nervous system depressants that produce sleep whereas sedatives are *CNS* depressants that reduce nervous tension and promote relaxation without producing sleep. They are also called tranquilisers. Some common examples are luminal, seconil, equanil etc.

(vi) **Antimalarials** A substance used in the treatment of malaria is called antimalarial. Some examples are quinine, chloroquine, primaquine etc.

(vii) **Antibiotics** Chemical substances which are produced by micro-organisms and are used to inhibit the growth or to destroy other micro-organisms are called antibiotics. Examples are chloramphenicol, streptomycin, penicillin, ampicillin, cephalexin, tetracycline etc.

(viii) **Sulpha drugs** These were the first effective agents used against bacteria. All such compounds are derivatives of sulphanilic acid. Some examples are sulphadiazine, sulphaguanidine, sulphathiazole etc.

(ix) **Antihistamines** Chemical substance which prevent the release of histamine are called antihistamines. Histamine causes skin rashes and other allergic actions.

Dyes

Dyes are the compounds which can impart colour to fabric, foodstuff, cosmetics and paper etc.

For a substance to be used as dye for fabric, it must have a suitable colour, should be able to fix itself, fast to light and should be resistant to the action of water, soap or other solvents.

The colour of a dye is due to its ability to absorb light of some particular wavelengths from the visible region (400—750 nm). The remaining light which is reflected back gives the colour of the dye. This colour is complementary to the colour absorbed.

Classification of Dyes. Dyes are usually classified on the basis of the method of their application as follows:

(i) **Acid dyes** These are the sodium salts of the sulphonic acids, carboxylic acids or phenols. These dyes are generally used to dye wool, silk and nylon directly. Some example are picric acid, orange-I, methyl orange etc.

(ii) **Basic dyes** These are salts, usually the hydrochloride of amines. The amino group in the form of cation makes them water soluble. These dyes are used for dyeing nylon and polyesters. Some examples are malachite green, aniline yellow etc.

(iii) **Direct dyes** These dyes can be directly applied to the fabric from their aqueous solutions and they attach to the fibre by hydrogen bonds. Such dyes are effective for dyeing cotton, rayon, wool, silk etc. Typical examples are martius yellow and congo red.

(iv) **Disperse dyes** These are usually applied in the form of dispersion of finely divided dye in a soap solution. They are mainly used to dye nylon, dacron and polyesters. Some examples are celliton fast pink *B* and celliton fast blue *B*.

(v) **Fibre reactive dyes** These dyes attach to the fibre by the formation of chemical bonds. Such dyes are 'fast' and the colour is retained for a long time. These are effective for dyeing cotton, wool and silk.

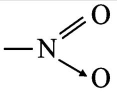
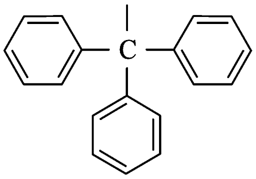
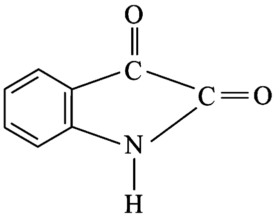
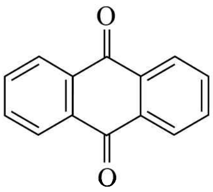
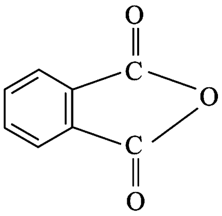
(vi) **Vat dyes** These are the oldest dyes used by mankind. These dyes are water soluble in the reduced form (called leuco base) and also colourless. They are applied on fabric in this form and then colour is developed by oxidation. These are mainly used to dye cotton. A well known example is indigo.

(vii) **Azo dyes** These are insoluble in water and are produced on the fibre itself. The fabric to be dyed is soaked in a solution of a phenolic compound and then treated with a solution of the diazotised amine. The dye is produced by the coupling of the two. These dyes are used in dyeing cotton, silk, polyester and nylon. A common example is nitroaniline red.

(viii) **Mordant dyes** The molecules of these dyes do not attach to the fibre directly and need a binding agent between fibre and dye called mordant. A mordant can be defined as a substance which can be fixed to the fibre and which can be dyed later on. Different mordants used are metal ions, tannins, bifunctional compounds like hydroxy acids. A mordant dye may impart different colour in presence of different mordants. Alizarin is one of the very common examples of this class of dyes.

Dyes can also be classified on the basis of characteristic chemical unit present in them. Some important classes are given below.

Table 31.1

Class of Dyes	Structural unit Present	Examples
1. Nitro dyes		Picric acid, Martius yellow
2. Azo dyes	$—N=N—$	Orange-I
3. Triphenyl methane dyes		Malachite green
4. Indigoid dyes		Indigo
5. Anthraquinone dyes		Alizarin
6. Phthalein dyes		Phenolphthalein

Rocket Propellants

A propellant is a combination of an oxidiser and a fuel which when ignited undergoes combustion to release large quantities of hot gases. The passage of gases through the nozzle of the rocket motor, provides the necessary thrust for the rocket to move forward according to Newton's Third Law of Motion.

Depending upon their physical state, propellants are classified into their types: (i) liquid propellants (ii) solid propellants and (iii) hybrid propellants.

(i) **Liquid Propellants.** Fuels like alcohol, liquid hydrogen, liquid ammonia, kerosene oil, hydrazine and hydrides of boron are employed as liquid propellants. Such fuels also need an oxidiser like liquid oxygen or nitric acid etc.The liquid propellants which consist of two liquids are called **biliquid propellants.** On the other hand, the propellants involving use of single liquid are called **monopropellants** or **monoliquid propellants.** These are chemical compounds like methyl nitrate, nitromethane, hydrogen peroxide etc.

(ii) **Solid propellants.** Solid fuels like polybutadiene or polyurethane are used along with oxidisers such as ammonium perchlorate, nitrate or chlorate. Such propellants are called **composite propellants.** Alongwith a small amount of magnesium or aluminium is also used to burn these solid propellants.

Another type of solid propellant is a **double-base propellant,** which mainly consists of nitroglycerine and nitrocellulose. Nitrocellulose gels in nitroglycerine and sets in as a solid mass. Solid propellants, once ignited burn at a predetermined rate and do not have the start and stop capability.

(iii) **Hybrid propellants.** Hybrid rocket propellants usually consist of a solid fuel and a liquid oxidizer. A common propellant used is a mixture of N_2O_4 and acrylic rubber.

PSLV is a four stage rocket and fuels used in different stages are solid or liquid propellants. In stage-I a solid hydroxyl Terminated Polybutadiene (HTPB) based propellant is used. At the second stage a biliquid propellant is used. It consists of unsymmetrical dimethylhydrazine (UDMH) as fuel and nitrogen tetraoxide (N_2O_4) as oxidiser. The third stage also employs HTPB based solid propellant. Fourth stage again uses a biliquid propellant which consists of monomethyl hydrazine as fuel and mixed oxides of nitrogen as oxidiser.

Some other useful classes of compounds

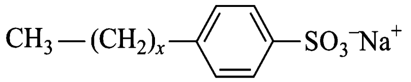
(i) **Preservatives** are the substances used to prevent spoilage of food due to microbial growth during storage. The most common preservatives used are sodium banzoate, sodium metabisulphite etc. Salt, sugar and oils also act as food preservatives and have been used for many years.

(ii) **Artificial sweetners** are the chemical substances which are used to create the sweet taste in food items in place of cane sugar. The first popular artificial sweetner was saccharin but it has some after taste. Most commonly used artificial sweetner these days is aspartame which is almost 200 times sweeter than cane sugar.

(iii) **Antioxidants** retard the action of oxygen on the food and thereby help in its preservation. They also reduce the rate of involvement of free radicals in the aging process. They are generally added to oils and fats etc. The most commonly used antioxidants are butylated hydroxy toluene (BHT) and butylated hydroxy anisole (BHA). Ascorbic acid (Vitamin-C) is also an antioxidant.

(iv) **Edible colours** used for food are basically dyes. They donot add to the nutritive value of food but they are used just to give an attractive look. Carotene is a natural dye which is safe to health.

(v) **Detergents.** Detergents are the sodium or potassium salts of long chain benzene sulphonic acids. They can be represented as



Detergents are mainly classified into three categories which are (a) anionic, (b) cationic and (c) non-ionic detergents.

The type of detergents mentioned above *i.e.* alkyl benzene sulphonates belong to anionic type of detergents. These are the most commonly used detergents for household and laundary purposes.

Detergents are preferred over soaps due to the following reasons:

(a) They can be used in hard water.
(b) They can be used in acidic medium.
(c) They have a stronger cleansing power than soap.

Detergents which are degraded by bacterial decay in nature are known as biodegradable detergents and which are not degraded by bacterial attack are called non-biodegradable detergents. It has been observed that detergents having branched hydrocarbon side chain are non-biodegradable. Use of such detergents ultimately leads to pollution of water in rivers because they get accumulated. This problem now has been overcome. It has been found that if straight chain hydrocarbon is used in the detergent instead of branched chain hydrocarbon, then the detergent is biodegradable and more detergents of this type are used these days.

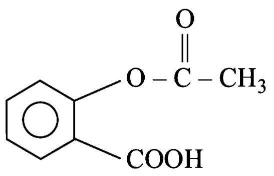
(vi) **Pheromones.** Pheromones are the chemical substances which attract the insects of a particular sex. They actually provide a chemical means of communication between insects of opposite sex.

Problems for Practice

1. The chemical part of the dye that absorbs light and produces colour is called
(a) Photochrome (b) Chromophore
(c) Auxochrome (d) Photosensitizer

2. Which of the following is an azo dye?
(a) Orange-I (b) Phenolphthalein
(c) Malachite green (d) Methylene blue

3. The following compound is used as

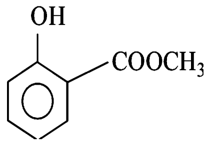


(a) An anti-inflammatory compound
(b) Analgesic
(c) Hypnotic
(d) Antiseptic

4. Paracetamol is
(a) Analgesic
(b) Antipyritic
(c) Both antipyritic and analgesic
(d) Antimalarial

5. Which of the following can possibly be used as analgesic without causing addiction?
(a) Morphine
(b) N-Acetylpara aminophenol
(c) Diazepam
(d) All of these

6. The compound



is used as
(a) antiseptic (b) antibiotic
(c) analgesic (d) pesticide

7. Drug which helps to reduce anxiety and brings about calmness is
(a) Tranquiliser (b) Diuretic
(c) Analgesic (d) Antihistamine

8. Substances which bring body temperature down are known as

(a) Antipyretics (b) Analgesics
(c) Antibiotics (d) Hypnotics

9. Aspirin is an acetylation product of

(a) *p*-Dihydroxybenzene
(b) *o*-Hydroxybenzoic acid
(c) *o*-Dihydroxybenzene
(d) *m*-Hydroxybenzoic acid

10. Which of the following is a basic dye?

(a) Alizarin (b) Phthalein dye
(c) Aniline yellow (d) Orange I

11. Which of the following is an antidiabetic drug?

(a) Insulin (b) Penicillin
(c) Chloroquine (d) Aspirin

12. Aspirin is an

(a) Antibiotic (b) Antipyretic
(c) Antiseptic (d) None of these

13. Amoxycillin is semi-synthetic modification of

(a) Penicillin (b) Streptomycin
(c) Tetracyline (d) Chloramphenicol

14. A broad spectrum antibiotic is

(a) Paracetamol (b) Penicillin
(c) Aspirin (d) Chloramphenicol

15. One of the oxidants used with liquid propellants is

(a) ammonium perchlorate
(b) phosphorus pentoxide
(c) sulphuric acid
(d) dinitrogen tetroxide (N_2O_4)

16. Which of the following is not a pesticide?

(a) BHC (b) DDT
(c) Malathion (d) Seconal

17. Which of these is a hypnotic?

(a) metaldehdye (b) acetaldehyde
(c) paraldehyde (d) None of these

18. An example of a mordant dye is

(a) Congo red (b) Alizarin
(c) Malachite green (d) Martius yellow

19. Soaps are obtained by

(a) Saponification of oils or fats
(b) Alkaline hydrolysis of glycerol
(c) Acidic hydrolysis of esters of fatty acids and glycerol
(d) All of these

20. The antiseptic present in Dettol is

(a) Quodine (b) Chloroxylenol
(c) Bithional (d) None of these

21. Dye which requires a binding agent between the fibre and the dye belongs to the category of

(a) Vat dye
(b) Mordant dye
(c) Disperse dye
(d) None of these

22. Which of the following is/are monoliquid propellants?

(a) H_2O_2 (b) $CH_3O\ NO_2$
(c) CH_3NO_2 (d) All of these

23. Oxidizers used in rocket propellant is/are

(a) Liquid O_2 (b) Liquid N_2O_4
(c) Nitric oxide (d) All of these

24. The dyes which are used in reduced state and then oxidised by air are

(a) Azo dyes (b) Vat Dyes
(c) Disperse dyes (d) Basic dyes

25. Acrylic rubber and liquid N_2O_4 is an example of

(a) Composite propellants
(b) Biliquid propellants
(c) Hybrid propellants
(d) Double base propellants

ANSWERS

1. (b)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (d)	**16.** (d)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (b)
21. (b)	**22.** (d)	**23.** (d)	**24.** (b)	**25.** (c)					

Environmental Chemistry

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BRIEF REVIEW OF THE CONCEPTS

Environmental chemistry is the study of the source, transportation, transformation and effects of chemical pollutants. It determines the adverse consequences which a given chemical species would have on air, water and land resources. In other words it can be said that environmental chemistry involves the study of various chemical phenomenon taking place in the environment.

Pollutants Substances added to natural surroundings due to human activities are called pollutants. There are some natural pollutants also e.g., pollen grains, organic compounds released from trees, volcanic eruptions etc.

Pollution Deterioration in the quality of natural resources such as air, water and earth due to addition of various pollutants, is called pollution.

Air Pollutants

Gaseous Pollutants	**Liquid Pollutants**	**Particulate pollutants**
CO and CO_2	Hydrocarbons	Soot
SO_2	Aersosols	Flyash
H_2S	Mists and fumes	Metals like Hg, Zn, Pb, As, Cd etc.
NO_2 and other oxides of nitrogen		Dust from cement industry
NH_3		Agricultural chemicals
O_3 and other oxidants		

Acid Rain: Rain water normally has a pH of 5.6. This is due to the dissolution of atmospheric carbon dioxide in rainfall. A solution of carbon dioxide in water gives carbonic acid.

$$CO_2 + H_2O \longrightarrow \underset{\text{(Carbonic Acid)}}{H_2CO_3}$$

As a result, the pH of rain water falls in the acidic range. If the pH drops below 5.6, the rainfall is described as acid rain.

Acid rain is caused by the emission of sulphur oxides and nitrogen oxides to the atmosphere. Sulphur oxides are released to the environment mainly because of fossil fuel combustion, but also due to ore smelting. The oxides of nitrogen are emitted as a result of fossil fuel combustion and automobile exhaust. These gases are absorbed by rain water and converted into sulphuric acid and nitric acid.

Acid rain does not remain confined to the region where oxides of sulphur and nitrogen are emitted; it may fall hundreds of kilometers away from the pollution source. The reason being that gaseous sulphur oxides and nitrogen oxides are dispersed to far off locations by tall chimneys, are well as by thunderstorms. Thus, a local environmental hazard manifests itself as a trans regional problem.

Monuments constructed from marble are attacked by acid rain.

$$\underset{\text{(Marble)}}{CaCO_3(s)} + H_2SO_4(aq) \longrightarrow CaSO_4(aq) + H_2O(l) + CO_2(g)$$

Monuments or buildings constructed from sandstone generally have a coating of iron oxide which holds the quartz grains together. The iron oxide too dissolves in acid rain.

Thus persistent acid rain disfigures monuments and buildings. This phenomenon is called stone leprosy.

Green House Effect and Global warming

The glass roof of a greenhouse is permeable to the rays of the sun. Once inside the greenhouse, the radiations strike a surface and are converted into heat. The glass roof of the greenhouse does not allow this heat to pass out. Since heat is trapped, the inside of a greenhouse is warmer than the outside. Like glass, a few atmospheric gases allow the sun's radiation to pass through them and strike the surface of earth. But like the glass roof a greenhouse, these gases prevent the resulting heat from escaping . The net result is that the earth is warming up. This global warming is called greenhouse effect and the gases which cause it are called greenhouse gases. The chief greenhouse gas is carbon dioxide. However, methane, Chlorofluorocarbons, ozone, nitrous oxide and water vapors also contribute, though to lesser extent to the greenhouse phenomena.

Smog Smog is a Combination of fog, smoke and fumes released by factories, homes and automobiles There are tow types of smog-classical smog and photochemical smog.

Classical Smog This type of smog occurs in cool, humid climate, mostly in towns located in the valleys. It is a mixture of smoke, fog and sulphur dioxide. Chemically it is a reducing mixture and is also called a reducing smog. It causes respiratory problems among the residents of the area.

Photochemical Smog Photochemical smog is mainly an urban phenomena. It is formed due to the reaction between nitrogen oxides and hydrocarbons under the influence of sunlight. The precursors that lead to the formation of photochemical smog (that is, nitrogen oxides and hydrocarbons) originate from automobile exhaust, although there is a definite contribution from industrial activity. For this reason, smog is formed during peak hours, when automobile exhaust is maximum. It also occurs on sunny days because the reactions leading to smog formation are catalysed by solar radiations. Photochemical smog contains a lot of organic derivatives, like aldehydes, ketones and peroxyacetyl nitrate (PAN)

Mists and Fumes Mists are formed by dispersion of extremely small liquid droplets in air. Alternately, these may be formed by condensation of vaporised substances into small liquid aerosols, that become suspended in air. Liquid insecticides, for example, parathion, when sprayed in excess contribute to mist formation.

Fume are the vapours excluded by spraying a low boiling liquid in the air. A fumigant is a volatile pesticide, the fumes of which kill the pest. Methyl bromide and Telone are examples of fumigants.

Ozone Depletion A chlorine-catalysed free radical mechanism causes the depletion of stratospheric ozone. The chlorine free radicals are released to the atmosphere by photolysis of chlorofluorocarbons. The latter are industrial chemicals which are so inert that they are not degraded in the troposphere. However, when they reach the stratosphere, they are cleaved by strong solar radiations.

$$\text{CFCs} + h\nu \rightarrow \text{Cl}^\bullet$$

The chlorine radicals then intitate the following chain reactions.

$$\text{Cl}^\bullet + O_3 \rightarrow \text{ClO}^\bullet + O_2$$
$$\text{ClO}^\bullet + \text{ClO}^\bullet \rightarrow (\text{ClO})_2$$
$$(\text{ClO})_2 + h\nu \rightarrow \text{Cl}^\bullet + \text{ClO}^\bullet_2$$
$$\text{ClO}^\bullet_2 \rightarrow \text{Cl}^\bullet + O_2$$
$$\text{Cl}^\bullet + O_3 \rightarrow \text{ClO}^\bullet \rightarrow O_2$$

Net reaction: $2O_3 + h\nu \rightarrow 3O_2$

Hence the final result is that ozone becomes depleted in the stratosphere, creating what has come to be known as ozone hole. Destruction of ozone layer will allow more UV rays to enter the troposphere which may cause many harmful effects, such as:

(i) smog formation would increase
(ii) temperature of earth will increase leading to rise in sea level
(iii) UV rays will directly fall on skin leading to skin cancer.
(iv) Leaves of plants will show chlorosis (loss of chlorophyll)

Biochemical Oxygen Demand (BOD) The amount of dissolved oxygen needed by the microorganisms to cause biodegradation of water pollutants in 5 days at 20°C.

In order to elucidate biochemical oxygen demand, the water sample is first saturated with oxygen. It is then incubated at a constant temperature, usually 20°C, for five days. This allows time for microorganisms to act on pollutants. The remaining amount of dissolved oxygen is determined and biochemical oxygen demand is obtained by subtraction.

$$\text{BOD} = \begin{bmatrix}\text{Saturation value of dissolved oxygen} \\ \text{per litre at 20°C}\end{bmatrix} - \begin{bmatrix}\text{Amount of dissolved oxygen per litre} \\ \text{at 20°C after 5 days}\end{bmatrix}$$

Biochemical oxygen demand indicates the extent of pollution load. If the pollution load is large, then greater amount of dissolved oxygen shall be consumed and according to equation, BOD shall be high. Conversely, a low value of BOD indicates relatively unpolluted water. For drinking water the biochemical oxygen demand should be in the 0.75 – 1.50 ppm range.

Biochemical oxygen demand does not measure the contamination caused by any specific chemical. It is a measure of the contamination by the totality of those compounds which can be oxidised in the presence of oxygen and microorganisms. A large number of organic compounds – for example, detergents – are, however, resistant to microbial oxidation. Some, like cellulose, do not get completely oxidised within the 5 days period required for elucidation of BOD. It is also possible that the water sample may contain some toxic chemicals, like pesticides, which may poison the water-purifying microorganisms. Moreover, the 5 days period for BOD assessment is quite long.

In order to avoid these problems, another analytical method, called Chemical Oxygen Demand or COD, is sometimes preferred.

Chemical Oxygen Demand The amount of oxygen, released by an oxidising agent, to oxidize the water pollutants to non-hazardous compounds.

For determining COD, the water sample is treated with a known quantity of an oxidsing agent. Normally, potassium dichromate in acidic medium is used as the oxidizing agent. The reagent oxides virtually all the pollutants, even those which are resistant to microbial attack. The remaining potassium dichromate is back titrated with Mohr's shalt. The amount of potassium dichromate utilized for oxidation is determined by subtraction.

$$\begin{matrix}\text{Concentration of} \\ \text{potassium} \\ \text{dichromate consumed}\end{matrix} = \left\{\begin{matrix}\text{Concentration of potassium} \\ \text{dichromate initially added.}\end{matrix}\right\}\theta - \left\{\begin{matrix}\text{Concentration of potassium} \\ \text{dichromate determined by} \\ \text{back titration}\end{matrix}\right\}\theta$$

From the concentration of potassium dichromate consumed, the amount of oxygen used in the oxidation may be calculated using the chemical reaction,

$$K_2Cr_2O_7 + 4H_2SO_4 \rightarrow K_2SO_4 + Cr_2(SO_4)_3 + 4H_2O + 3(O)$$

The results of COD are expressed in terms of the amount of oxygen, in ppm, that is necessary to oxidize the contaminants.

Problems for Practice

1. The gas which acts as intermediate in the formation of photochemical smog is

(a) CO_2 (b) CO
(c) CH_4 (d) O_3

2. The hazardous metal present in automobile exhaust is

(a) copper (b) Mercury
(c) Lead (d) Iron

3. Ozone hole is mainly caused by

(a) CO_2 (b) CFC
(c) Cl_2 (d) Hydrocarbons

4. Air pollution is caused by

(a) industries (b) automobiles
(c) pollen grains (d) all of these

5. The chemical responsible for Bhopal tragedy was
(a) Methyl isocyanate
(b) Methyl iodide
(c) Methyl methacrylate
(d) Methyl amine

6. Sulphur dioxide causes pollution by enhancing
(a) alkalinity (b) acidity
(c) basicity (d) rain fall

7. Main water pollutant is
(a) Soap (b) Detergent
(c) Industrial waste (d) Chlorine

8. Biochemical oxygen demand measures
(a) Industrial pollution
(b) oxygen required by fishes
(c) oxygen required by microbes
(d) Dissolved oxygen in water

9. One of the following processes for production of electricity does not create pollution.
(a) Generator
(b) Solar cell
(c) Gas based power station
(d) Coal based power station

10. Inhalation of lead particles inhibits the formation of
(a) DNA
(b) plasma
(c) nucleus in the cell
(d) haemoglobin

11. Which one of the following is responsible for depletion of the ozone layer in upper strata of the atmosphere?
(a) freons (b) ferrocene
(c) polyhalogens (d) fullerenes

12. Choose the wrong statement.
(a) CO_2 is responsible for green house effect
(b) Normally the pH of rain water is about 5.6
(c) Acid rain contains mainly HNO_3
(d) Pollen grains are also considered as air pollutants.

13. Photochemical smog is
(a) formed in winter
(b) a mixture of light and fog
(c) both of these
(d) none of these

14. Ultra-violet radiations
(a) produce vitamin D in skin
(b) may cause skin cancer
(c) enhance the formation of chlorophyll in plants
(d) all of these

15. DDT spray creates pollution of
(a) air (b) air and water
(c) water and soil (d) air, water and soil

16. The main air pollutant which causes the ozone depletion is
(a) fluorine (b) hydrocarbons
(c) chlorofluoro carbons
(d) sulphur dioxide

17. COD stands for
(a) chemical oxygen demand
(b) chemistry of dangerous pollutants
(c) another name for BOD
(d) carbon monoxide demand

18. BOD value for sewage water is expected to be
(a) more than that of drinking water
(b) less then that of drinking water
(c) almost equal to that of drinking water
(d) can not be predicted

19. Which one of the following attaches more strongly to blood haemoglobin than oxygen?
(a) Methane (b) Carbon monoxide
(c) Chlorine (d) Carbon dioxide

20. Which of the following is mainly affecting the beauty of Taj?
(a) Excess of O_3 in atmosphere
(b) Acid rain
(c) U.V radiations
(d) All of these

21. Classical smog occurs in places of
(a) tourist interest (b) historical interest
(c) low temperature (d) high temperature

22. Fishes die in water bodies polluted by sewage due to
(a) foul smell
(b) mosquitoes
(c) clogging of gills by mud
(d) less dissolved oxygen

23. Water pollution is due to
(a) sewage discharge
(b) industrial effluents
(c) agricultural discharge
(d) all of these

24. One of the following is a secondary pollutant
(a) CO (b) PAN
(c) CO_2 (d) CH_4

25. Increased cases of skin cancer are due to
(a) ozone depletion (b) acid rain
(c) use of cosmetics (d) lead particles

ANSWERS

1. (d) **2.** (c) **3.** (b) **4.** (d) **5.** (a) **6.** (b) **7.** (c) **8.** (c) **9.** (b) **10.** (d)
11. (a) **12.** (c) **13.** (d) **14.** (b) **15.** (d) **16.** (c) **17.** (a) **18.** (a) **19.** (b) **20.** (b)
21. (c) **22.** (d) **23.** (d) **24.** (b) **25.** (a)

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Principles Related to Practical Chemistry

BRIEF REVIEW OF THE CONCEPTS

Detection of the functional groups in Orgaure Compound

1. **Carboxylic acid group (–COOH)**

(i) An aqueous solution of the compound turns blue litmus red.

(ii) Add a small amount of the compound to the saturated solution of sodium bicarbonate. Effervescence occurs due to the evolution of carbon dioxide.

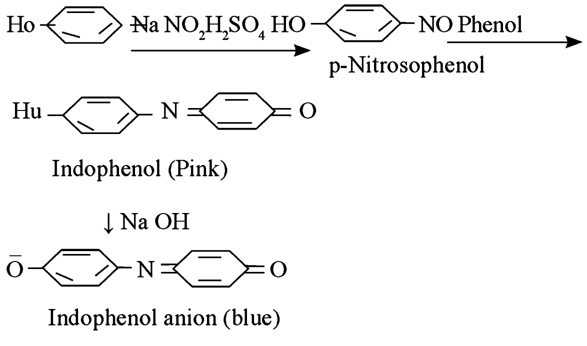
$R\,COOH + Na\,HCO_3 \rightarrow R\,COO\,Na + H_2O + CO_2$

2. **Phenols (–OH group attached to an aromatic system)**

(i) A solution of suspension of phenol in water turns blue litmus red but does not give effervescence with sodium bicarbonate solution.

(ii) **Ferric Chloride Test** Phenols react with ferric ions to form coloured complexes. Simple phenol, resorcinol and cresols produce violet or blue colour where as β–naphthol produces a green colour and α–naphthol produces pink colour. Test involves the addition of few drips of solution of phenol to about 1 ml of neutral ferric chloride solution.

(iii) **Liebermann's Nitroso Test** All phenols compounds having no substituent at para position, show this test. In this test phenolie compound in treated with sodium nitrite and concentrated sulphuric acid. When pink colour is produced. The colour turns blue on addition of sodium hydroxide solution.



(iv) **Phthalein Test:** When phenols are heated with phthalic anhydride in presences of conc. H_2SO_4, phthaleins with characteristic colour is alkaline medium are obtained. For example, phenol gives a compound called phenophthalein which is used as indicator Resorcinol on similar reaction gives fluoresce in which is used as dye.

3. **Alcohols**

(i) **Effervescence with Sodium Metal:** In a dry test tube 2 ml of the given compound is taken and to it about 1 g of anhydrous sodium salphate is added (to absorb any water present in the compound) and filtered. If the filtrate gives effervescences with a small piece of sodium, then the substance may he an alcohol, phenol or an acid. The compound is confirmed as alcohol if absence of phenol or acids is proved by the tests discussed above.

$2ROH + 2Na \rightarrow 2RONa + H_2$

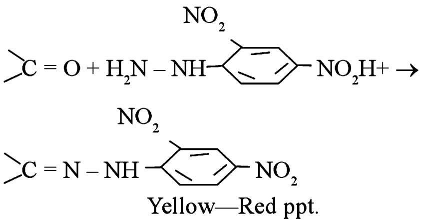
(ii) **Ceric Ammonium Nitrate Test:** Red colour is produced when few drops of the ceric ammonium nitrate solution are added to about $\frac{1}{2}$ mL of alcohol.

$(NH_4)_2\,[Ce\,(NO_3)_6] + 2\,ROH \rightarrow$ Colourless

$[Ce\,(NO_3)_4\,(ROH)_2] + 2NH_4\,NO_3$ Red colour

4. **Aldehydes and Kelones** ($>C=0$)

(i) **2, 4 –DNP Test** All aldehydes and kelones are detected by this test. When 3–4 drops of the solution of the compound are added to 0.5 ml. If 2, 4 dinitrophenyl hydrazine reagent, a yellow to red coloured precipitate formed.



Aldehydes can further he distinguished from ketones by the following tests.

(ii) **Tollen's Test (Silver mirror test):** To about 2ml of Tollen's reagent, 0.5 of the compound is added and heated in a water bath for about 5 minutes. Formation of a silver mirror indicates the presence of aldehyde group. Tollen's reagent is prepared by adding ammonium hydroxide to 1ml of silver nitrate solution till the initially formed white precipitate redissolves to give a clear solution.

$2\,[Ag\,(NH_3)_2]^+ . + RCHO + 3OH^- \rightarrow RCOO^- + 4\,NH_3 + 2H_2\,O + 2H_2\,O + 2Ag \downarrow$

This test is gives by formic acid also but formic acid will not give 2, 4 – DNP test and will give effervescence with $NaHCO_3$ solution.

(iii) Schiff's Test: In this test, to 1ml of Schiff's reagent few drops of the solution of the compound are added. If the given compound is an aldehyde then pink clolour in formed.

(iv) Fehlings Test: Fehlings solution contains Cu^{2+} ions complexed with tartarate ions in alkaline medium. When 2-3 drops of the compound (or 100 mg if solid) are mixed with 1 ml of the Fehling's solution and heated for about 5 minutes, a red precipitate is formed if the compound is an aldehyde.

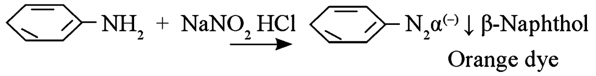
$$R\,CHO + 2\,Cu^{2+} + 5\,OH^- \rightarrow \underset{(Red)}{Cu_2O\downarrow} + RCOO^- + 3H_2O$$

5. Amines ($-NH_2$ group)

(i) Carbylamine Test (Isocyanide test)**:** This test is given by both aliphatic and aromatic primary amines. A small amount of the compound is heated with 1 ml of ethanol and one pallette of solid potassium hydroxide. An unpleasant odour due to the formation of an isocyanide confirms a primary amine.

$$RNH_2 + CHCl_3 + 3\,KOH \rightarrow \underset{(isocyanide)}{RNC} + 3\,KCl + 3H_2O$$

(ii) Azodye Test: This test is given by primary aromatic amines. These on reaction with nitrous acid give diazonium salts which undergo coupling reaction with phenols resulting is the formation of coloured azo dyes. The phenolic compound β-naphthol is generally used in the test.



Chemistry involved in the preparation of following compound

Mohr's salt (Ferrous ammonium sulphate)

Mohr's salt is a double salt and in prepared by dissolving an equimolar mixture of ammonium sulphate and ferrous sulphate is water to make a saturated solution. Few drops of sulphuric acid are also added and then the solution is concentrated. On cooling, light green crystals of ferrous ammonium sulphate separate out.

$$(NH_4)_2\,SO_4 + Fe\,SO_4.7H_2O \rightarrow$$
$$Fe\,SO_4.(NH)_2\,SO_4.6H_2O + H_2O$$

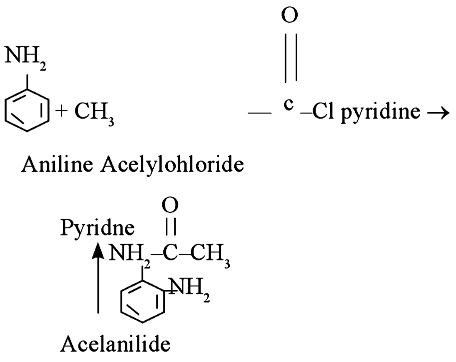
Mohr's Salt.

Potash Alum Potash alum is prepared by dissolving potassium sulphate and aluminum sulphate in equimolar ratio in minimum amount of water containing few drops of sulphuric acid. On cooling potash alum separates as colourless crystals.

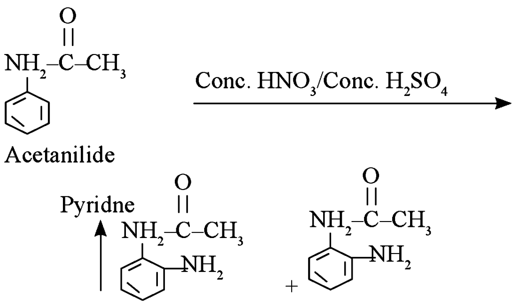
$$K_2SO + Al_2(SO_4)_3.18H_2O + 6H_2O \rightarrow$$
$$K_2SO_4.Al_2(SO_4)_3.\,24H_2O$$

(Potash alum)

Acetanilide Acetanilide is prepared by the acelylation of aniline by acetyl chloride or acetic anhydride. The reaction is carried out in presence of few drops of a base like pyridine.



p-Nitroacetanilide In acetanilide, the amino group of aniline is protected and it is not as strong activating group as $-NH_2$ group but it is still o–and p– directing group. When its nitration is carried out, then the p-nitro product is obtained as the major product because the ortho position faces an steric hinderance. The small amount of o-nitroacetanilide is removed by crystallization.



p–Nitroacetanilide (major) o–Nitroacetanilide (minor)

Chemistry involved in the titrinetric exercises

In titrimetric or volumetric analysis, the amount of a chemical species present is the given unknown solution is determined by measuring the volumes of the solution taking part in the given chemical reaction. The chemical species react in the ratio of their chemical equivalent masses. The main process of this analysis is called titration which means the determination of the volume of a reagent required to bring a definite reaction to completion.

Strength of the various solutions used is generally expressed as **normality**. It is defined as the number of gram equivalents of the substance dissolved per litre of the solution. If one gram equivalent of an acid, base or salt is dissolved in water and the solution is made up to one litre, such a solution is called normal solution or its normality is 1 N.

The equivalent mass of a substance is equal to its molar mass (or formula mass) divided by the number of electron gained by one molecule or ion of the substance in the reaction.

$$\text{Equivalent mass}=\frac{\text{Molar mass or formula mass}}{\text{No. of electron change}}$$

Following table shows the calculation of equivalent masses of some substances

Table 33.1 Calculation of equivalent masses

Substance	*Ionic equation*	*No of electrons involved*	*Molar mass*	*Equivalent mass*
1. $KMnO_4$ (in acidic medium)	$MnO_4^- + 8H^+ + 5e^- \rightarrow Mn^{2+} + 4H_2O$	5	158	$\frac{158}{5} = 31.6$
2. Mohr's salt [$Fe\ SO_4 . (NH)_2SO.6H_2O$]	$Fe^{2+} \rightarrow Fe^{3+} + e^-$	1	392	$\frac{392}{1} = 392$
3. Oxalic acid (crystals) ($C_2H_2O_4 . 2H_2O$)	$C_2O^2{}_4{}^- \rightarrow 2CO_2 + 2e^-$	2	126	$\frac{126}{2} = 63$

Use of indicators Indicators are the substances which show different colour in acidic and alkaline solutions. These are use to detect the end point of acid-alkali titrations and also to measure approximate ph values of solutions. The indicators that change their colour over a narrow range of ph are more useful.

Table 33.2 shows the colour shown by some commonly used indicators in acid-base titrations. In redox titrations involving KM_nO_4 it acts as self indicator. It is pink in colour and its colour disappears when it reacts.

Table 33.2 Colours of some common indicatiors

	Substance	*Colours in acid*	*Colours in alkai*	*pH range*
1.	Litmus	Red	Blue	5.0—8.0
2.	Methyl	Orange	Yellow	3.1—4.4
3.	Phenol phthalein	Colourless	Pink	8.3—10.0
4.	Methyl red	Red	Yellow	4.4—6.3

CHEMICAL PRINCIPLES INVOLVED IN THE QUALITATIVE SALT ANALYSIS

The inorganic qualitative analysis involves the identification of anion and catain present is the given salt sample. A preliminary hint about some anions is obtained by the formation of volatile or gaseous products when the salt is treated with dilute and conc. H_2SO_4 separately. These are summarized in table 33.3 and table 33.4 respectively. These anions are further confirmed by their confirmatory tests.

Table 33.3 Dilut sulphuric acid test

	Observation	*Gas evolved*	*Anion indicated*
1.	Colourless gas with brisk effervescence, turns line water milky	CO_2	CO_3^{2-}
2.	Colourless, pungent gas	SO_2	SO_3^{2-}
3.	Colourless gas with the smell of rotter eggs	H_2S	S^{2-}
4.	Reddish brown gas, turns $FeSO_4$ solution black	NO_2	NO_2^-

Table 33.4 Concentracted sulphuric acid test

	Observation	*Gas evolved*	*Anion indicated*
1.	Colourless gas, Pun--gent Small and gives while fumes with NH_3	HCl	Cl^-
2.	Reddish brown gas turns starch paper yellow and does not effect	Br_2	Br^-
3.	Violet vapors turns starch paper blue	I_2	I^-
4.	Reddish brown gas, turns $FeSO_4$ solution black	NO_2	NO_3^-

CONFIRMATORY TEST FOR VARIOUS ANIONS

1. **Carbonate ($CO_3{}^2$)$^-$** Carbonates on reaction with dil. HCl give CO_2 gas which turns lime water milky due to formation of insoluble compound, $Ca\ CO_3$.

   $$CO_3^{2-} + 2HCl \rightarrow CO_2\uparrow + H_2O + 2Cl^-$$

   $$Ca(OH)_2 + CO_2 \rightarrow Ca\ CO_3\downarrow + H_2O$$

   Lime Water

2. **Sulphite ($SO_3{}^{2-}$)** When few drops of barium chloride solution are added to the solution of the salt, a white ppt soluble in dilute HCl confirms($SO_3{}^{2-}$).

   $$(SO_3{}^{2-}) + BaCl_2 \rightarrow BaSO_3\downarrow + 2Cl^-$$

   White ppt.

   $$BaSO_3 + 2HCl \rightarrow BaCl_2 + SO_2 + H_2O$$

3. **Sulphide (S^{2-})** The solution containing sulphide ions gives purple colour with sodium netroprusside solution.

   $$Na_2S + Na_2[Fe(CN)_5NO] \rightarrow Na_4[fe(CN)_5NOS]$$

   Sodium nitroprusside purple colour

   Sulphide ions can also be confirmed by the formation of black precipitate on treatment with lead acetate solution.

   $$Na_2S + Pb_2(CH_3COO)_2 \rightarrow PbS\downarrow + 2CH_3COONa$$

   Black

4. **Nitrite(NO^-_2)** Nitrite is confirmed by the formation of dark brown colour when dil. acetic acid and ferrous sulphate are added to it.

   $$KNO_2 + CH_3COOH \rightarrow CH_3COOK + HNO_2$$

   $$3HNO_2 \rightarrow HNO_3 + 2NO + H_2O$$

$Fe\,SO_4 + NO \rightarrow Fe\,SO_4\,NO$
(dark brown)

5. **Chloride (Cl^-)** (i) Silver nitrate test: A portion of the aqueous solution is acidified with dilute HNO_3 and $Ag\,NO_3$ solution is added. A white precipitate which is soluble in ammonium hydroxide confirms the presence of Cl^- ions.

$Cl^- + Ag\,NO_3 \rightarrow Ag\,Cl\downarrow + NO_3^-$
white ppt.

$Ag\,Cl + 2NH_4\,OH \rightarrow [Ag(NH_3)_2]Cl + 2H_2O$
soluble complex.

(ii) Chromyl chloride test: A small quantity of the salt is mixed with some $K_2Cr_2O_7$ and few drops of conc. H_2SO_4 and then heated. The red vapours formed are passed through Na OH solution which turns yellow. This yellow solution is acidified with acetic acid followed by the addition of lead acetate solution. Formation of a yellow precipitate confirms Cl ions.

$4Cl^- + K_2Cr_2O_7 + 3H_2SO_4 \rightarrow 2(SO_4^{3-}) + K_2SO_4 + 2\,CrO_2Cl_2 + 3H_2O$
Chromyl chloride

$CrO_2Cl_2 + 4\,NaOH \rightarrow Na_2CrO_4 + 2H_2O + 2NaCl$
Yellow colour

$Na_2CrO_4 + (CH_3\,COO)_2\,Pb \rightarrow Pb\,CrO_4\downarrow + 2CH_3COO\,Na$
Yellow ppt.

6. **Bromide (Br^-)** (i) Silver nitrate test: In silver nitrate test (as for cl^-), a light yellow precipitate is formed which dissolves in excess of ammonium hydroxide—

$Br^- + Ag\,NO_3 \rightarrow Ag\,Br\downarrow + NO_3^-$
Yellow

$Ag\,Br + 2NH_4OH \rightarrow [Ag\,(NH_3)_2]\,Br + 2H_2O$

(ii) Layer test: A portion of the aqueous solution of the salt is acidified with dil HNO_3 and about 1 ml of CS_2 is added. Then to this solution about 2–3 ml of chlorine water is added and it is shaken vigorously. If Br^- is present then carbon disulphide layer acquires orange colour.

$2Br^- + Cl_2 \rightarrow 2Cl^- + Br_2$

This bromine dissolves in CS_2 to give orange colour.

7. **Iodide (I^-)** (i) Silver nitrate test: In silver nitrate test a deep yellow precipitate is formed which is insoluble in ammonium hydroxide.

$I^- + Ag\,NO_3 \rightarrow Ag\,I\downarrow + NO_3^-$
Yellow

(ii) Layer test: In this test, the CS_2 layer acquires a purple colour if I^- is present because it is oxidized to iodine which dissolves in CS_2 layer.

$2I^- + Cl_2 \rightarrow 2Cl^- + I_2$

8. **Nitrate (NO_3^-)** A small quantity of freshly prepared ferrous sulphate solution is added to the aqueous solution of the salt and concentrated sulphuric acid is added from the side of the test tube. Formation of a dark brown ring at the junction of two layers confirms NO_3^-.

$NO_3^- + H_2\,SO_4 \rightarrow HNO_3 + H\,So_4^-$

$6\,Fe\,SO_4 + 3H_2SO_4 + 2HNO_3 \rightarrow 3Fe\,(SO_4)_3 + 4H_2O + 2NO$

$Fe\,SO_4 + NO + 5H_2O \rightarrow [Fe\,(H_2O)_5\,NO]\,SO_4$
(Brown ring)

9. **Sulphate (SO_4^{2-})** To a part of the aqueous solution, few drops of barium chloride solution are added. Formation of white ppt, insoluble in dil HCl confirms SO_4^{2-}.

$SO_4^{2-} + Ba\,Cl_2 \rightarrow Ba\,SO_4\downarrow + 2Cl^-$
White ppt.

10. **Phosphate** (PO_4^{3-}) Whenthe aqueous solution of the salt is boiled with nitric acid and ammonium molybdate solution, formation of a yellow precipitate of ammonium phosphor molybdate confirms phosphate.

$3\,K_3\,Po_4 + 3\,(NH_4)_2\,M_oO_4 \rightarrow 2\,(NH_4)_3PO_4 + {}_3K_2\,M_oO_4$
Ammonium molybdate

$K_2\,M_oO_4 + 2\,HNO_3 \rightarrow H_2M_oO_4$

$H_2\,MoO_4 \rightarrow MoO_3 + H_2O$

$(NH_4)_3\,PO_4 + 12MoO_3 + 6H_2O \rightarrow (NH_4)_3\,PO_4 \,.\, 12\,MoO_3\,.\,6H_2O\downarrow$
Yellow ppt.

A preliminary idea about the cation present in the salt is obtained by flame test. Certain salts on reacting with conc. HCl form their chlorides that are volatile in non-luminous flame. Their vapors impart characteristic colour to the flame. Table 33.6 lists some of the cations and the colour which they impart to the flame.

Table 33.5

	Colour of the flame	*Inference*
1.	Brick red	Ca^{2+}
2.	Crimson red	Sr^{2+}
3.	Grassy green	Ba^{2+}
4.	Bluish-green	Cu^{2+}

Cations are deteted and confirmed through a systematic analysis involving wet tests. For the sake of analysis, the cations are classified into various groups.

Table 33.6

Group	*Cations*
GROUP I	Pb^{2+}, Hg_2^{2+}, Ag^+
GROUP II	Hg^{2+}, Cu^{2+}, Cd^{2+}, As^{2+}, Bt^{3+}
GROUP III	Fe^{3+}, Al^{3+}, Cr^{3+}
GROUP IV	Zn^{2+}, Co^{2+}, Nl^{2+}, Mn^{2+}
GROUP VI	Ba^{2+}, Sr^{2+}, Ca^{2+}
GROUP VI	Mg^{2+}, Nh_4^+

Once the solution of the salt is prepared, radicals from it precipitated groupwise using a particular precipitating reagent.

The first group cations are precipitated in the form of their chlorides and the reagent used is dilute hydrochloric acid. The cations of second group are precipitated in the form of their sulphides using H_2S gas in acidic medium. The third group cations are precipitated in the form of their hydroxides using ammonium hydroxide. The concentration of hydroxide ions is kept low by adding some $NH_4\,Cl$. The fourth group cations are precipitated as their sulphides using H_2S gas in alkaline medium. The fifth group cations are precipitated in the form of their carbonates using ammonium carbonate as the medium. The sixth group cations are tested independently by using specific tests. These cations are further ed by their confirmatory tests.

Chemical principles involved in the following experiments

1. **Enthalpy of solution of $CuSO_4$** Enthalpy of solution is defined as the enthalpy change when 1 mole of the solute is dissolved in such a large amount of solvent that further dilution causes no enthalpy change. Enthalpy of solution is determined by dissolving a known amount of the salt in a measured volume of water in a calorimeter. The rise of fall in temperature is recorded as Δt. Then ΔH is calculated using the formula

   $$\Delta H \frac{(W+w)\times \Delta t\times M}{m}$$

   Where W = water equivalent of calorimeter

   m = mass of the solute taken

   m = mass of the water taken

   M = Molar mass of solute

   $\Delta t = t_{\text{final}} - t_{\text{initial}}$

   Enthalpy of solution comes with a positive sign if it is absorbed and with a negative sign if it is evolved.

2. **Enthalpy of neutralization of strong acid and strong base** Enthalpy of neutralization at a given temperature is defined as the enthalpy change involved when one gram equivalent of an acid is neutralized by are gram equivalent of a base. When both the acid and base are strong. The heat released is 57.32 kJ (13.7 k Cal). In case of HCl and NaOH, it can be shown by a equation.

   $HCl_{(aq)} + NaOH_{(aq)} \rightarrow NaCl_{(aq)} + H_2O_{(l)}$;
   DH = -57.32 kJ mol^{-1}

   As both the reagents are strong and they undergo complete dissociation, the precise reaction can be taken as the formation of water.

   $H^+_{(aq)} + OH^-_{(aq)} \rightarrow H_2O_{(l)}$

   Since the neutralization of strong acid and strong base involves the combination of H^+ and ^-OH ions to form water, hence it is expected that the enthalpy of neutralization of every strong acid by a strong base should be same. The enthalpy of neutralization will be less than this value if any (or both) of the acid or base is weak.

3. **Preparation of lyophilic and lyopholic sols** A colloid in which dispersed phase is a solid and dispersion medium is a liquid is termed as sol. When the dispersed phase has great affinity for the dispersion medium, it is called lyophilic sol. Such sols are quite stable and dispersed phase does not get easily precipitated. If the dispersion medium is water these are called hydrophilic sols. Some examples of this type of sols are sols of gum, gelantine, starch and protein etc. They can be easily prepared by mixing, grinding or heating the two substances together.

   When the dispersed phase has little affinity for the dispersion medium, such sols are less stable and are named as lyophobic sols. They can he easily coagulated by the addition of small amount of any electrolyte. Example of lyophobic sols include sols of metals, metal oxides or metal sulphides etc.

4. **Kinetic study of reaction of iodide ion with hydrogen peroxide at room temperature** Iodide ions are oxidized by hydrogen peroxide in acidic medium to from iodine. The reaction is studied by adding a known amount of sodium thiosulphate solution and a little of starch of solution. Iodine, that is liberated after oxidation reads immediately with sodium thiosulphate and is reduced back to iodide ions.

   $H_2O_2 + 2I^- + 2H^+ \rightarrow 2H_2O + I_2$

   $I_2 + 2S_2O_3^{2-} \rightarrow S_4O_6^{2-} + 2I^-$.

   When thiosulphate ions are completely consumed, the liberated iodine reacts with starch to give blue colour.

   Rate of the reaction is compared by taking various concentrations of iodide ions and time taken for the blue colour to appear.

Problems for Practice

1. Sodium reacts readily with
   (a) $R-NH_2$ (b) $R-CH_2-OH$
   (c) $R-CHO$ (d) $R-O-R$
2. Which of the following reacts fastest with Lucas reagent?
   (a) Propan – 2 – ol
   (b) Butan – 2 – ol
   (c) 2 – Methylpropan – 2 – ol
   (d) Propan – 1 – ol
3. Lucas reagent is
   (a) Conc. HCl + $ZnCl_2$ (b) Dil. HCl + $ZnCl_2$
   (c) $H_2SO_4 + ZnCl_2$ (d) Conc. HCl + Zn
4. Which of the following does not give colour in Victor Meyer test?
   (a) $CH_3 . CH_2 . OH$ (b) $(CH_3)_2CH\,OH$
   (c) $(CH_3)_3C\,OH$ (d) All of these
5. The solution of an organic compound in dry benzene evolved hydrogen on reaction with sodium metal. The compound can be
   (a) aldehyde (b) ketone
   (c) alkyl halide (d) alcohol
6. Which of the following gives violet colour with neutral $Fe\,Cl_3$ solution?
   (a) Salicylic acid (b) Benzoic acid
   (c) Toluene (d) Nitrobenzene
7. Phthalein test is given by
   (a) alcohols (b) phenols
   (c) aldehydes (d) ketones
8. Picric acid and benzoic acid can be distinguished by the aqueous solution of
   (a) $NaHCO_3$ (b) Na_2CO_3
   (c) NaOH (d) $FeCl_3$
9. Resorcinol and phthalic anhydride in presence of cone H_2SO_4 produce a compound which is
   (a) a dye (b) an antiseptic
   (c) an explosive (d) a detergent

10. The colour reaction not given by alcohols is
(a) Lieberman's nitro so test
(b) Neutral Fecl$_3$ test
(c) Phthalein test
(d) All of these

11. Which of the following compounds gives brisk effervescence with NaHCO$_3$?
(a) Phenol (b) Acetic acid
(c) Both of these (d) None of these

12. Formic acid and acetic acid can be distinguished by
(a) Fehling's solution (b) Tollen's reagent
(c) Both of these (d) None of these

13. Formaldehyde and formic acid can be distinguished by
(a) Tollen's reagent
(b) Fehling's solution
(c) Neutral FeCl$_3$ solution
(d) NaHCO$_3$ solution

14. Which of the following gives silver mirror test?
(a) Propan-1-ol (b) Propenal
(c) Propanone (d) Propane

15. Tollen's reagent is ammoniacal solution of
(a) cupric chloride (b) copper sulphate
(c) silver nitrate (d) silver nitrite

16. Which of the following gives positive haloform test and positive Fehling solution test?
(a) Acetone (b) Acetaldehyde
(c) Ethanol (d) Formaldehyde

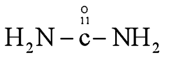
17. Which of the following does not turn Schiff's reagent pink?
(a) HCHO (b) C_6H_5CHO
(c) CH_3COCH_3 (d) CH_3CHO

18. Aldehydes can be distinguished from ketones by using
(a) Tollen's reagent (b) Schiff's reagent
(c) Fehling's reagent (d) All of these

19. Ethanal and propanal can be differentiated by
(a) I_2 in presence of NaOH
(b) Ammoniacal Ag NO$_3$
(c) Na HSO$_3$ solution
(d) Fehling's reagent

20. A compound forms a brilliant red dye on reaction with HNO$_2$ at 0°C followed by treatment with alkaline β-naphthol solution. The compound can be
(a) methylamine (b) aniline
(c) toluene (d) phenol

21. Which of the following can be detected by carbylamines reaction?

(a)  $H_2N-\overset{O}{\overset{||}{C}}-NH_2$ (b) CH_3-CONH_2
(c) $CH_3-CH_2-NH_2$ (d) All of these

22. Carbylamine reaction is not shown by
(a) methylamine (b) ethylamine
(c) ethyl methylamine (d) n-propylamine

23. The acidic solution of a salt produces deep blue colour with starch-iodide solution. The salt may be
(a) chloride (b) nitrite
(c) acetate (d) bromide

24. When concentrated H$_2$SO$_4$ is added to dry KNO$_3$, brown fumes evolve. These fumes are
(a) SO$_2$ (b) SO$_3$
(c) NO (d) NO$_2$

25. The reaction of NaCl and K$_2$ Cr$_2$O$_7$ mixture with concentrated H$_2$SO$_4$ results in
(a) CrO_2Cl (b) $CrOCl_2$
(c) CrO_2Cl_2 (d) $(CrOCl)_2$

26. Which of the following compound is formed when potassium iodide is heated with cone. H$_2$SO$_4$?
(a) HI (b) I_2
(c) HIO_3 (d) HIO_4

27. In the precipitation of the III group cations, ammonium chloride is added before adding ammonium hydroxide to
(a) decrease concentration of OH$^-$ ions
(b) prevent interference by PO_4^{3-} ions
(c) increase concentration of OH$^-$ ions
(d) oxidize Fe^{2+} into Fe^{3+} ions

28. A metal salt solution gives a yellow precipitate with silver nitrate. The precipitate dissolves in dilute nitric acid as well as in ammonium hydroxide. The solution contains
(a) bromide (b) iodide
(c) phosphale (d) chromate

29. The ion which is not precipitated by H$_2$S in the presence of HCl is
(a) Cu^{2+} (b) Ag$^+$
(c) Pb^{2+} (d) Al^{3+}

30. The presence of magnesium is confirmed in the qualitative analysis by the formation of a white crystalline precipitate is due to
(a) $Mg(HCO_3)_2$ (b) $MgNH_4PO_4$
(c) $MgCO_3$ (d) $MgNH(HCO_3)_3$

31. With potassium thiocyanate solution, ferric chloride yields
(a) pink colour (b) prussian blue colour
(c) blood-red colour (d) green colour

32. Nesslar's reagent is used to detect
(a) PO_4^{3-} (b) Cl$^-$
(c) SO_4^{2-} (d) NH_4^+

33. In the second group of qualitative analysis H$_2$S is passed through a solution acidified with HCl in order to
(a) Limit the concentration of S^{2-} ions
(b) Increase the solubility of H$_2$S
(c) Increase the concentration of S^{2-} ions
(d) Add the Cl$^-$ ions

34. When ammonia is added to the solution of salt in water, a red precipitate appears which is soluble in dilute HCl. The cation presenting the salt is
(a) Al^{3+} (b) Zn^{2+}
(c) Fe^{3+} (d) Cd^{2+}

35. The salt which is not decomposed by dilute or concentrated H$_2$SO$_4$ contains
(a) SO_3^{2-} ion (b) Cl$^-$ ion
(c) NO_3^- ion (d) SO_4^{2-} ion

36. When dimethyl glyoxime solution is added to an aqueous solution of nickel (II) chloride followed by ammonium hydroxide:
(a) No change occurs
(b) A pink precipitate is formed
(c) A black precipitate is formed
(d) Yellow vapours are obtained

37. To avoid the precipitation of hydroxides of Ni^{2+}, Co^{2+}, Zn^{2+} and Mn^{2+} along with those of Fe^{3+}, Al^{3+} and Cr^{3+}, the third group solution should be
(a) heated with a few drops of conc. HNO_3
(b) treated with excess of NH_4Cl
(c) boiled to make it free from H_2S
(d) concentrated

38. Which one of the following does not dissolve in concentrated ammonium hydroxide solution?
(a) Ag F (b) Ag Cl
(c) Ag Br (d) Ag I

39. The reagent used in chromyl chloride test is
(a) K_2CrO_4 (b) CrO_3
(c) $K_2Cr_2O_7$ (d) $(NH_4)_2 Cr_2O_7$

40. Starch-iodide paper is used to test for the presence of
(a) Oxidizing agents (b) Reducing agents
(c) Iodine (d) Iodide ion

41. Methyl orange gives red colour in the solution of
(a) Na_2CO_3 (b) NaCl
(c) HCl (d) KOH

42. Which of the following statements is correct?
(a) ΔH is positive for exothermic reactions
(b) ΔH is negative for endothermic reactions
(c) The enthalpy of neutralization is positive
(d) The enthalpy of neutralization of strong acid with a strong base is always the same

43. Concentration of HCl is 10 N. 1000 mL of 1 N HCl can be obtained by diluting
(a) 1 mL to 1000 mL (b) 10 mL to 1000 mL
(c) 100 mL to 1000 mL (d) 50 mL to 1000 mL

44. Molar mass of oxalic acid is 126. The weight of oxalic acid required to neutralize 100ml of In NaOH is
(a) 6.3g (b) 12.6g
(c) 126g (d) 63g

45. To neutralize 10 ml of M/5 NaOH, the volume of $\frac{N}{20}$ HCl required is
(a) 10 ml (b) 20 ml
(c) 40 ml (d) 50 ml

46. In alkaline medium, $KMnO_4$ (158) reacts as $2\,KMnO_4 + 2KOH \rightarrow 2K_2MnO_4 + H_2O + [O]$.
What will be its equivalent mass for this reaction?
(a) 31.5 (b) 52.7
(c) 72.0 (d) 158

47. What is the normality of 1m solution of H_3Po_4?
(a) 0.5 N (b) 1 N
(c) 2 N (d) 3 N

48. When 10 ml of 10 M solution of H_2SO_4 and 100 ml of 1 M solution of Na OH are mixed. The resulting solution will be
(a) acidic (b) neutral
(c) alkaline (d) can not be predicted

49. How much of 1 M H_2SO_4 solution is required to neutralize 10 mL of 1 M Na OH solution?
(a) 2.5 mL (b) 5.0 mL
(c) 10.0 mL (d) 20.0 mL

50. Volume of 0.1 M H_2SO_4 required to neutralize 200 ml of $Ba(OH)_2$ solution is 100 mL. The normality of $Ba(OH)_2$ solution is
(a) 0.1 N (b) 0.2 N
(c) 0.05 N (d) 0.5 N

ANSWERS

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (d)
11. (b)	**12.** (c)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (b)	**24.** (d)	**25.**(c)	**26.** (b)	**27.** (a)	**28.** (a)	**29.** (d)	**30.** (b)
31. (c)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (d)	**36.** (b)	**37.** (b)	**38.** (d)	**39.** (c)	**40.** (a)
41. (c)	**42.** (d)	**43.** (c)	**44.** (a)	**45.** (c)	**46.** (d)	**47.** (d)	**48.** (a)	**49.** (b)	**50.** (a)

PART D ORIGINAL QUESTION PAPERS

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*In 2011, the exam was rescheduled, and therefore, two question papers have been provided as Part I and Part II.

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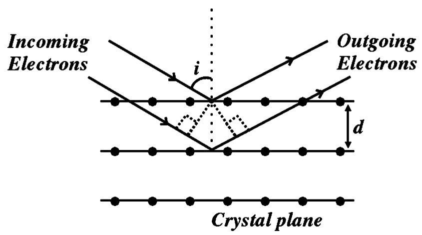
AIEEE Paper 2008

PHYSICS

Direction (1–3) These are based on the following paragraph.

Wave property of electrons implies that they will show diffraction effects. Davisson and Germer demonstrated this by diffracting electrons from crystals. The laws governing the diffraction from a crystal is obtained by requiring that electron waves reflected from the planes of atoms in a crystal interfere constructively (see in figure).



1. Electrons accelerated by potential V are diffracted from a crystal. If d = 1Å and i = 30°, V should be about ($h = 6.6 \times 10^{-34}$ Js, $m_e = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C)

   (a) 2000 V (b) 50 V
   (c) 500 V (d) 1000 V

2. If a strong diffraction peak is observed when electrons are incident at an angle 'i' from the normal to the crystal planes with distance 'd' between them (see figure), de Broglie wavelength λ_{dB} of electrons can be calculated by the relationship (n is an integer)

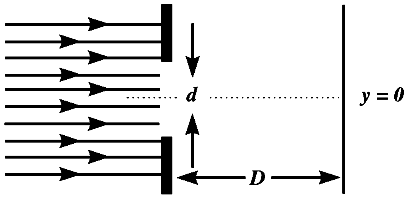
   (a) $d \sin i = n\lambda_{dB}$

   (b) $2d \cos i = n\lambda_{dB}$

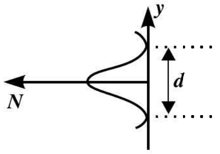
   (c) $2d \sin i = n\lambda_{dB}$

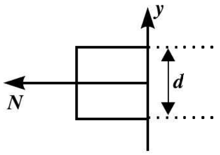
   (d) $d \cos i = n\lambda_{dB}$

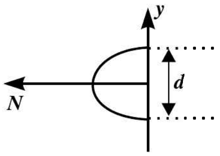
3. In an experiment, electrons are made to pass through a narrow slit of width 'd' comparable to their de Broglie wavelength. They are detected on a screen at a distance 'D' from the slit (see figure).

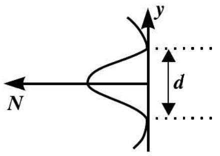


Which of the following graph can be expected to represent the number of electrons 'N' detected as a function of the detector position 'y' (y = 0) corresponds to the middle of the slit)?

(a)


(b)


(c)


(d)


4. A planet in a distant solar system is 10 times more massive than the earth and its radius is 10 times smaller. Given that the escape velocity from the earth is 11 kms^{-1}, the escape velocity from the surface of the planet would be

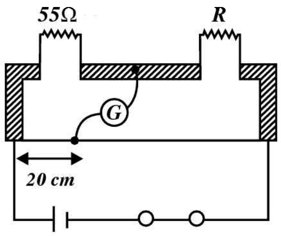
   (a) 1.1 kms^{-1} (b) 11 kms^{-1}
   (c) 110 kms^{-1} (d) 0.11 kms^{-1}

5. A spherical solid ball of volume V is made of a material of density ρ_1. It is falling through a liquid of density $\rho_2 (\rho_2 < \rho_1)$. Assume that the liquid applies a viscous force on the ball that is proportional to the square of its speed v, i.e., $F_{viscous} = -kv^2$ (k>0). The terminal speed of the ball is

   (a) $\sqrt{\dfrac{Vg(\rho_1 - \rho_2)}{k}}$ (b) $\dfrac{Vg\rho_1}{k}$

   (c) $\sqrt{\dfrac{Vg\rho_1}{k}}$ (d) $\dfrac{Vg(\rho_1 - \rho_2)}{k}$

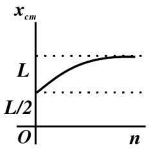
6. Shown in the given figure is a metre-bridge set up with null deflection in the galvanometer.

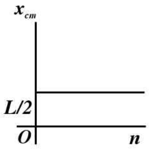


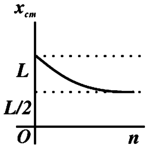
The value of the unknown resistor R is

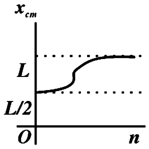
(a) 13.75Ω (b) 220Ω
(c) 110Ω (d) 55Ω

7. A thin rod of length 'L' is lying along the x-axis with its ends at $x = 0$ and $x = L$. Its linear density (mass/length) varies with x as $k\left(\frac{x}{L}\right)^n$, where n can be zero or any positive number. If the position x_{cm} of the centre of mass of the rod is plotted against 'n', which of the following graphs best approximates the dependence of x_{cm} on n?

(a)


(b)


(c)


(d)


8. While measuring the speed of sound by performing a resonance column experiment, a student gets the first resonance condition at a column length of 18 cm during winter. Repeating the same experiment during summer, she measures the column length to be x cm for the second resonance. Then

(a) $18 > x$ (b) $x > 54$
(c) $54 > x > 36$ (d) $36 > x > 18$

9. The dimension of magnetic field in M, L, T and C (Coulomb) is given as

(a) $MLT^{-1}C^{-1}$ (b) MT^2C^{-2}
(c) $ML^{-1}C^{-1}$ (d) $MT^{-2}C^{-1}$

10. Consider a uniform square plate of side 'a' and mass 'm'. The moment of inertia of this plate about an axis perpendicular to its plane and passing through one of its comers is

(a) $\frac{5}{6}ma^2$ (b) $\frac{1}{12}ma^2$

(c) $\frac{7}{12}ma^2$ (d) $\frac{2}{3}ma^2$

11. A body of mass m = 3.513 kg is moving along the x-axis with a speed of 5.00 ms^{-1}. The magnitude of its momentum is recorded as

(a) 17.6 kg ms^{-1} (b) 17.565 kg ms^{-1}
(c) 17.56 kg ms^{-1} (d) 17.57 kg ms^{-1}

12. An athlete in the olympic games covers a distance of 100 m in 10 s. His kinetic energy can by estimated to be in the range

(a) 200 J – 500 J (b) 2×10^5 J – 3×10^5 J
(c) 20,000 J – 50,000 J (d) 2,000 J – 5,000 J

13. A parallel plate capacitor with air between the plates has a capacitance of 9 pF. The separation between its plates is 'd'. The space between the plates is now filled with two dielectrics. One of the dielectrics has dielectric constant $k_1 = 3$ and thickness $\frac{d}{3}$ while the other one has dielectric constant $k_2 = 6$ and thickness $\frac{2d}{3}$. Capacitance of the capacitor is now

(a) 1.8 pF (b) 45 pF
(c) 40.5 pF (d) 20.25 pF

14. The speed of sound in oxygen (O_2) at a certain temperature is 460 ms^{-1}. The speed of sound in helium (He) at the same temperature will be (assumed both gases to be ideal)

(a) 460 ms^{-1} (b) 500 ms^{-1}
(c) 650 ms^{-1} (d) 330 ms^{-1}

15. This question contains Statement I and Statement II. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement I:
Energy is released when heavy nuclei undergo fission or light nuclei undergo fusion.
and

Statement II:
For heavy nuclei, binding energy per nucleon increases with increasing Z while for light nuclei it decreases with increasing Z.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is correct explanation for Statement-I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II false

16. This question contains Statement I and Statement II. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement I:
For a mass M kept at the centre of a cube of side 'a', the flux of gravitational field passing though its sides is 4π GM.
and

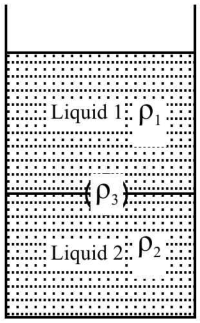
Statement II:
If the direction of a field due to a point source is radial and its dependence on the distance 'r' for the source is given as $1/r^2$, its flux through a closed surface depends only on the strength of the source enclosed by the surface and not on the size or shape of the surface.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II false

17. A jar filled with two non mixing liquids 1 and 2 having densities ρ_1 and ρ_2 respectively. A solid ball, made of a material of density ρ_3, is dropped in the jar. It comes to equilibrium in the position shown in the figure.

Which of the following is true for ρ_1, ρ_2, and ρ_3?

(a) $\rho_3 < \rho_1 < \rho_2$ (b) $\rho_1 < \rho_3 < \rho_2$
(c) $\rho_1 < \rho_2 < \rho_3$ (d) $\rho_1 < \rho_3 < \rho_2$



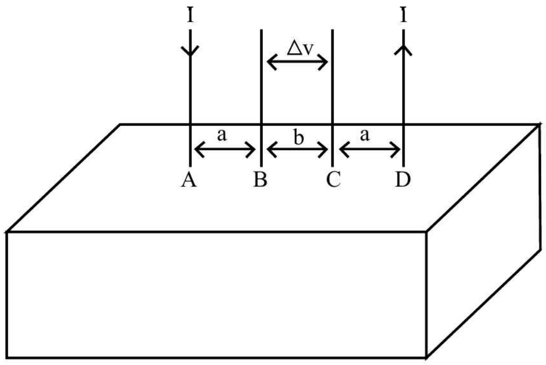
18. A working transistor with its three legs marked P, Q and R is tested using a multimeter. No conduction is found between P and Q. By connecting the common (negative) terminal of the multimeter to R and the other (positive) terminal to P or Q, some resistance is seen on the multimeter. Which of the following is true for the transistor?

(a) It is an npn transistor with R as base
(b) It is a pnp transistor with R as collector
(c) It is a pnp transistor with R as emitter
(d) It is an npn transistor with r as collector

Direction (19–20) These are based on the following paragraph:

Consider a block of conducting material of resistivity ‘ρ’ shown in the figure. Current ‘I’ enters at ‘A’ and leaves from ‘D’. We apply superposition principle to find voltage ‘ΔV’ developed between ‘B’ and ‘C’. The calculation is done in the following steps:

(a) Take current ‘I’ entering from ‘A’ and assume it to spread over a hemispherical surface in the block.
(b) Calculate field E(r) at distance ‘r’ from A by using Ohm’s law E = ρj, where j is current per unit area at ‘r’.
(c) From the ‘r’ dependence of E(r), obtain the potential V (r) at r.
(d) Repeat (a), (b), and (c) for current ‘I’ leaving ‘D’ and superpose result for ‘A’ and ‘D’.



19. ΔV measured between B and C is

(a) $\frac{\rho l}{\pi a}-\frac{\rho l}{\pi(a+b)}$ (b) $\frac{\rho l}{a}-\frac{\rho l}{(a+b)}$

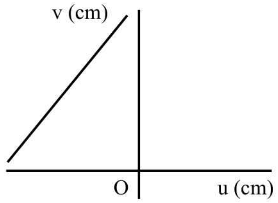
(c) $\frac{\rho l}{2\pi a}-\frac{\rho l}{2\pi(a+b)}$ (d) $\frac{\rho l}{2\pi(a-b)}$

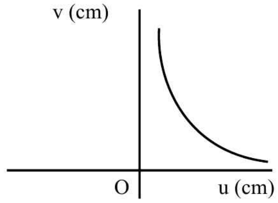
20. For current entering at A, the electric field at a distance ‘r’ from A is

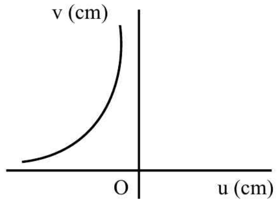
(a) $\frac{\rho l}{8\pi r^2}$ (b) $\frac{\rho l}{8\pi r^2}$

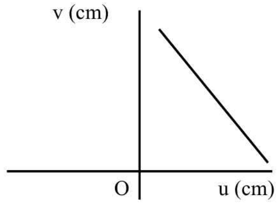
(c) $\frac{\rho l}{2\pi r^2}$ (d) $\frac{\rho l}{4\pi r^2}$

21. A student measures the focal length of convex lens by putting an object pin at a distance ‘u’ from the lens and measuring the distance ‘v’ of the image pin. The graph between ‘u’ and ‘v’ plotted by the student should look like

(a)


(b)


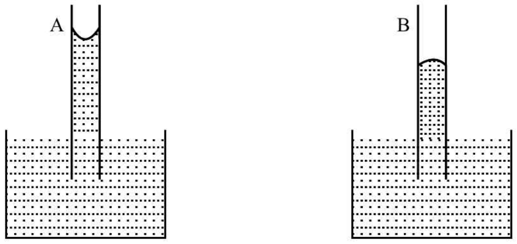
(c)


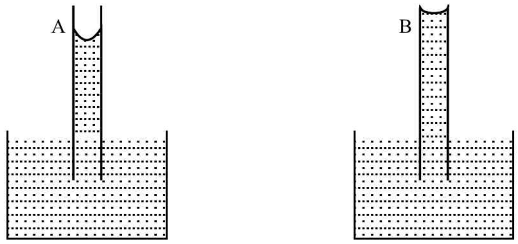
(d)


22. A block of mass 0.50 kg is moving with a speed of 2.00 m/s on a smooth surface. It strikes another mass of 1.00 kg and then they move together as a single body. The energy loss during the collision is

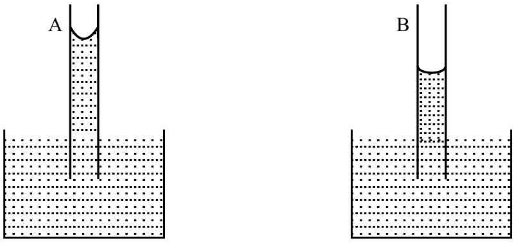
(a) 0.16 J (b) 1.00 J
(c) 0.67 J (d) 0.34 J

23. A capillary tube (A) is dropped in water. Another identical tube (B) is dipped in a soap water solution. Which of the following shows the relative nature of the liquid columns in the two tubes?

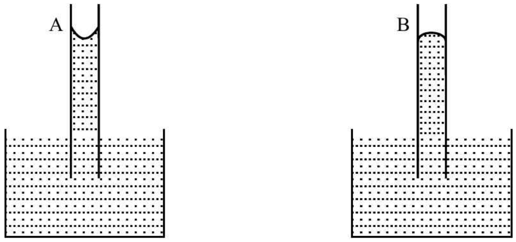
(a)


(b)


(c)



(d)



24. Suppose an electron is attracted towards the origin by a force k/r where 'k' is a constant and 'r' is the distance of the electron from the origin. By applying Bohr model to this system, the radius of the nth orbital of the electron is found to be 'r_n' and kinetic energy of the electron to be T_n. Then which of the following is true?

(a) $T_n \infty 1/n^2, r_n \infty n^2$

(b) T_n independent of $n_1, r_n \infty n$

(c) $T_n \infty 1/n, r_n \infty n$

(d) $T_n \infty 1/n, r_n \infty n^2$

25. A wave travelling along the x-axis is described by the equation y (x, t) = 0.005 cos $(\infty x - \beta t)$. If the wavelength and the time period of the wave are 0.08 m and 2.0 s, respectively, then α and β in appropriate units are

(a) $\alpha = 25.00\pi, \beta = \pi$ (b) $\alpha = \frac{0.08}{\pi}, \beta = \frac{2.0}{\pi}$

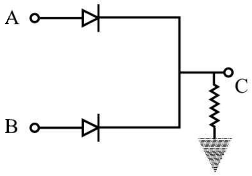
(c) $\alpha = \frac{0.04}{\pi}, \beta = \frac{1.0}{\pi}$ (d) $\alpha = 12.50\pi, \beta = \frac{\pi}{2.0}$

26. Two coaxial solenoids are made by winding thin insulated wire over a pipe of cross sectional area A = 10 cm^2 and length 20 cm. If one of the solenoids has 300 turns and the other 400 turns, their mutual inductance is $(\mu_0 = 4\pi \times 10^{-7} Tm\,A^{-1})$

(a) $2.4\pi \times 10^{-5}\,H$ (b) $4.8\pi \times 10^{-4}\,H$

(c) $4.8\pi \times 10^{-5}\,H$ (d) $2.4\pi \times 10^{-4}\,H$

27. In the circuit below, A and B represent two inputs and C represents the output.



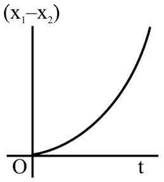
The circuit represents

(a) NOR gate (b) AND gate

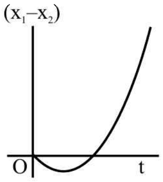
(c) NAND gate (d) OR gate

28. A body is at rest at x = 0. At t = 0, it starts moving in the positive x-direction with a constant acceleration. At the same instant another body passes through x = 0 moving in the positive x-direction with a constant speed. The position of the first body is given by $x_1(t)$ after time 't' and that of the second body by $x_2(t)$ after the same time interval. Which of the following graphs correctly describes $(x_1 - x_2)$ as a function of times 't'?

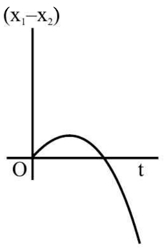
(a)



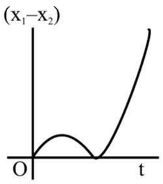
(b)



(c)



(d)

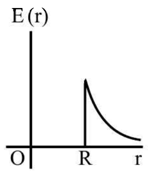


29. An experiment is performed to find the refractive index of glass using a travelling microscope. In this experiment distance are measured by

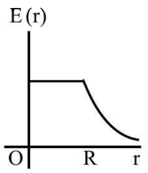
(a) a vernier scale provided on the microscope
(b) a standard laboratory scale
(c) a metre scale provided on the microscope
(d) a screw gauage provided on the microscope

30. A thin spherical shell of radius R has charge Q spread uniformly over its surface. Which of the following graphs most closely represents the electric fields E(r) produced by the shell in the range $0 \le r < \infty$, when r is the distance from the centre of the shell?

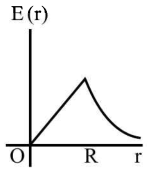
(a)



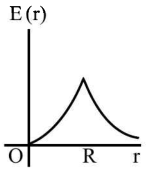
(b)



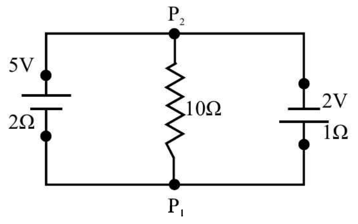
(c)



(d)



31. A 5V battery with internal resistance 2Ω and a 2V battery with internal resistance 1Ω are connected to a 10Ω resistor as shown in the figure. The current in the 10Ω resistor is



(a) 0.27 A P_2 to P_1 (b) 0.03 A P_1 to P_2

(c) 0.03 A P_2 to P_1 (d) 0.27 A P_1 to P_2

32. A horizontal overhead power line is at a height of 4m from the ground and carries a current 100 A from east to west. The magnetic field directly below it on the ground is $(\mu_0 = 4\pi \times 10^{-7}\,\text{T m A}^{-1})$

(a) 2.5×10^{-7} T southward
(b) 5×10^{-6} T northward
(c) 5×10^{-6} T southward
(d) 2.5×10^{-7} northward

33. Relative permittivity and permeability of a material are ε_r and μ_r, respectively. Which of the following values of these quantities are allowed for a diamagnetic material?
(a) $\varepsilon_r = 0.5,\ \mu_r = 1.5$ (b) $\varepsilon_r = 1.5,\ \mu_r = 0.5$
(c) $\varepsilon_r = 0.5,\ \mu_r = 0.5$ (d) $\varepsilon_r = 1.5,\ \mu_r = 1.5$

34. Two full turns of the circular scale of a screw gauge cover a distance of 1 mm on its main scale. The total number of divisions on the circular scale is 50. Further, it is found that the screw gauge has a zero error of – 0.03 mm while measuring the diameter of a thin wire, a student notes the main scale reading of 3 mm and the number of circular scale divisions in line with the main scale as 35. The diameter of the wire is
(a) 3.32 mm
(b) 3.73 mm
(c) 3.67 mm
(d) 3.38 mm

35. An insulated container of gas has two chambers separated by an insulating partition. One of the chambers has volume V_1 and contains ideal gas at pressure P_1 and temperature T_1. The other chamber has volume V_2 and contains ideal gas at pressure P_2 and temperature T_2. If the partition is removed without doing any work on the gas, the final equilibrium temperature of the gas in the container will be

(a) $\dfrac{T_1T_2(P_1V_1+P_2V_2)}{P_1V_1T_2+P_2V_2T_1}$ (b) $\dfrac{P_1V_1T_1+P_2V_2T_2}{P_1V_1+P_2V_2}$

(c) $\dfrac{P_1V_1T_2+P_2V_2T_1}{P_1V_1+P_2V_2}$ (d) $\dfrac{T_1T_2(P_1V_1+P_2V_2)}{P_1V_1T_1+P_2V_2T_2}$

ANSWERS

1. (b)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (d)
11. (a)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (d)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (a)
31. (c)	**32.** (c)	**33.** (b)	**34.** (d)	**35.** (c)					

MATHEMATICS

1. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point d such that CD = 7 m. From d the angle of elevation of the point A is 45°. Then the height of the pole is

(a) $\dfrac{7\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{3}-1}$ m (b) $\dfrac{7\sqrt{3}}{2}\cdot\left(\sqrt{3}+1\right)$ m

(c) m (d) $\dfrac{7\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{3}-1}$ m

2. It is given that the events A and B are such that P (A) = $\dfrac{1}{4}$, $P\left(\dfrac{A}{B}\right) = \dfrac{1}{2}$ and $P\left(\dfrac{B}{A}\right) = \dfrac{2}{3}$. Then P (B) is

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{3}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{1}{2}$

3. A die is thrown. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

(a) $\dfrac{3}{5}$ (b) 0

(c) 1 (d) $\dfrac{2}{5}$

4. A focus of an ellipse is at the origin. The directrix is the line x = 4 and the eccentricity is 1/2. Then the length of the semi-major axis is

(a) $\dfrac{8}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{4}{3}$ (d) $\dfrac{5}{3}$

5. A parabola has the origin as its focus and the line x = 2 as the directrix. Then the vertex of the parabola is at

(a) (0, 2) (b) (1, 0)
(c) (0, 1) (d) (2, 0)

6. The point diametrically opposite to the point P (1, 0) on the circle $x^2 + y^2 + 2x\ 4y - 3 = 0$ is

(a) (3, –4) (b) (–3, 4)
(c) (–3, –4) (d) (3, 4)

7. Let f: N→ Y be a function defined as f(x) = 4x + 3, where Y = {y ∈ N : y = 4x + 3 for some x ∈ N}. Show that f is invertible and its inverse is

(a) $g(y) = \dfrac{3y+4}{3}$ (b) $g(y) = 4 + \dfrac{y+3}{4}$

(c) $g(y) = \dfrac{y+3}{4}$ (d) $g(y) = \dfrac{y-3}{4}$

8. The conjugate of a complex number is $\frac{1}{i-1}$. Then the complex number is

(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$

(c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$

9. Let R be the real line. Consider the following subsets of the plane $R \times R$.

$S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$, $T = \{(x, y): x - y \text{ is an integer}\}$.
Which one of the following is true?

(a) neither S nor T is an equivalence relation on R
(b) both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

10. The perpendicular bisector of the line segment joining P (1, 4) and Q (k, 3) has y –intercept –4. Then a possible value of k is

(a) 1 (b) 2
(c) –2 (d) –4

11. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition y(1)= 1 is

(a) $y = \text{In}\, x + x$ (b) $y = x\, \text{In}\, x = x^2$
(c) $y = xe^{(x-1)}$ (d) $y = x\, \text{In}\, x + x$

12. The mean of the numbers, a, b, 8, 5, 10 is 6 and the variance is 6.80. Then which one of the following gives possible values of a and b?

(a) a = 0, b = 7 (b) a = 5, b = 2
(c) a = 1, b = 6 (d) a = 3, b = 4

13. The vector $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$ lies in the plane of the vectors $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{j} + \hat{k}$ and bisects the angle between $\vec{b}$ and $\vec{c}$. Then which one of the following gives possible values of α and β?

(a) $\alpha = 2, \beta = 2$ (b) $\alpha = 1, \beta = 2$
(c) $\alpha = 2, \beta = 1$ (d) $\alpha = 1, \beta = 1$

14. The non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are related by $\vec{a} = 8\vec{b}$ and $\vec{c} = -7\vec{b}$. Then the angle between $\vec{a}$ and $\vec{c}$ is

(a) 0 (b) $\pi/4$
(c) $\pi/2$ (d) π

15. The line passing through the points (5, 1, a) and (3, b, 1) crosses the yz–plane at the point $\left(0, \frac{17}{2}, \frac{13}{2}\right)$. Then

(a) a = 2, b = 8
(b) a = 4, b = 6
(c) a = 6, b = 4
(d) a = 8, b = 2

16. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to

(a) – 5 (b) 5
(c) 2 (d) – 2

Direction (17–21) are Assertion–Reason type questions. Each of these questions contains two statements: **Statement I** (Assertion) and **Statement II** (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice.

17. **Statement I:** For every natural number $n \geq 2\; n \geq 2, \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

Statement II: For every natural number $n \geq 2, \sqrt{n(n+1)} < n+1$

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

18. Let A be a 2 × 2 matrix with real entries. Let I be the 2 × 2 identity matrix. Denote by tr (A), the sum of diagonal entries of A. Assume that $A^2 = 1$.

Statement I: If $A \neq I$ and $A \neq -I$ then det $A = -I$.

Statement II: If $A \neq I$ and $A \neq -I$, then tr (A) $\neq 0$.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

19. **Statement I:** $\sum_{r=0}^{n} (r+1)^n C_r = (n+2)2^{n-1}$.

Statement II: $\sum_{r=0}^{n} (r+1)^n C_r x^r = (1+x)^n + nx(1+x)^{n-1}$.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

20. Let p the statement "x is an irrational number", q be the statement "y is a transcendental number", and r be the statement "x is a rational number if y is a transcendental number".

Statement I: r is equivalent to either q or p.

Statement II: r is equivalent to $\sim(p \leftrightarrow \sim q)$.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

21. In a shop there are five types of ice-creams available. A child buys six ice-creams.

Statement I: The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.

Statement II: The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4B's in a row.

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(d) Statement I is true, Statement II is false

22. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0_1 & \text{if } x = 1 \end{cases}$. Then which one of the following is true?

(a) f is neither differentiable at x = 0 nor at x = 1

(b) f is differentiable at x = 0 and at x = 1

(c) f is differentiable at x = 0 but not at x = 1

(d) f is differentiable at x = 1 but not at x = 0

23. The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is

(a) – 4 (b) – 12

(c) 12 (d) 4

24. Suppose the cube $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds?

(a) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

25. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have?

(a) 7 (b) 1

(c) 3 (d) 5

26. The statement $p \to (q \to p)$ is equivalent to

(a) $p \to (p \to q)$ (b) $p \to (p \vee q)$

(c) $p \to (p \wedge q)$ (d) $p \to (p \leftrightarrow q)$

27. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is

(a) $\frac{6}{17}$ (b) $\frac{3}{17}$

(c) $\frac{4}{17}$ (d) $\frac{5}{17}$

28. The differential equation of the family of circles with fixed radius 5 units and centre on the line y = 2 is

(a) $(x-2)y'^2 = 25 - (y-2)^2$

(b) $(y-2)y'^2 = 25 - (y-2)^2$

(c) $(y-2)^2 y'^2 = 25 - (y-2)^2$

(d) $(x-2)^2 y'^2 = 25 - (y-2)^2$

29. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$. Then which one of the following is true?

(a) $I > \frac{2}{3}$ and $J > 2$ (b) $I < \frac{2}{3}$ and $J < 2$

(c) $I < \frac{2}{3}$ and $J > 2$ (d) $I > \frac{2}{3}$ and $J < 2$

30. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$

(c) $\frac{2}{3}$ (d) $\frac{4}{3}$

31. The value of $\sqrt{2}\int \frac{\sin x dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is

(a) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$

(b) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$

(c) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$

(d) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$

32. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent?

(a) $8 \,.\, {}^6C_4 \,.\, {}^7C_4$ (b) $6 \,.\, 7 \,.\, {}^8C_4$

(c) $6 \,.\, 8 \,.\, {}^7C_4$ (d) $7 \,.\, {}^8C_4 \,.\, {}^8C_4$

33. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that x = cy + bz, y = az + cx and z = bx + ay. Then $a^2 + b^2 + c^2 + 2abc$ is equal to

(a) 2 (b) –1

(c) 0 (d) 1

34. Let A be a square matrix all of whose entries are integers. Then which one of the following is true?

(a) If $\det A = \pm 1$, then A^{-1} exists but all entries are not necessarily integers

(b) If $\det A = \pm 1$, then A^{-1} exists and all its entries are non-integers

(c) If $\det A = \pm 1$, then A^{-1} exists and all its entries are integers

(d) If $\det A = \pm 1$, then A^{-1} need not exist

35. The quadratic equation $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equation are integers in the ratio 4 : 3. Then the common root is

(a) 1 (b) 4

(c) 3 (d) 2

ANSWERS

1. (b)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (d)
11. (d)	**12.** (d)	**13.** (d)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (b)	**20.** (d)
21. (a)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (d)
31. (c)	**32.** (d)	**33.** (d)	**34.** (c)	**35.** (d)					

CHEMISTRY

1. The organic chloro compound, which shows complete astrochemical inversion during a S_N2 reaction, is

(a) $(C_2H_5)^2CHCl$ (b) $(CH3)^3CCl$
(c) $(CH_3)^2CHCl$ (d) CH_3Cl

2. Toluene is nitrated and the resulting product is reduced with tin and hydrochloric acid. The product so obtained is diazotised and then heated with cuprous bromide. The reaction mixture so formed contains

(a) mixture of o– and p–bromotoluenes
(b) mixture of o– and p–dibromobenzenes
(c) mixture of o– and p–bromoanilines
(d) mixture of o– and m–bromotoluenes

3. The coordination number and the oxidation state of the element 'E' in the complex $[E(en)_2(C_2O_4)]NO_2$ (where (en) is ethylene diamine) are, respectively,

(a) 6 and 2 (b) 4 and 2
(c) 4 and 3 (d) 6 and 3

4. Identify the wrong statements in the following:

(a) Chlorofluorocarbons are responsible for ozone layer depletion.
(b) Greenhouse effect is responsible for global warming.
(c) Ozone layer does not permit infrared radiation from the sun to reach the earth.
(d) Acid rains is mostly because of oxides of nitrogen and sulphur.

5. Phenol, when it first reacts with concentrated sulphuric acid and then with concentrated nitric acid, gives

(a) 2,4,6-trinitrobenzene
(b) o-nitrophenol
(c) p-nitrophenol
(d) nitrobenzene

6. In the following sequence of reactions, the alkene affords the compound 'B' ,

$$CH_3CH = CHCH_3 \xrightarrow{O_3} A \xrightarrow[Zn]{H_2O} B.$$

The compound B is

(a) CH_3CH_2CHO (b) CH_3COCH_3
(c) $CH_3CH_2COCH_3$ (d) CH_3CHO

7. Larger number of oxidation states are exhibited by the actinoids than those by the lanthanoids, the main reason being

(a) 4f orbitals more diffused than the 5f orbitals
(b) lesser energy difference between 5f and 6d than between 4f and 5d orbitals
(c) more energy difference between 5f and 6d than between 4f and 5d orbitals
(d) more reactive nature of the actinoids than the lanthanoids

8. In which of the following octahedral complexes of Co (at. no. 27), will the magnitude of Δ_o be the highest?

(a) $[Co(CN)_6]^{3-}$ (b) $[Co(C_2O_4)_3]^{3-}$
(c) $[Co(H_2O)_6]^{3+}$ (d) $[Co(NH_3)_6]^{3+}$

9. At 80°C, the vapour pressure of pure liquid 'A' is 520 mm Hg and that of pure liquid 'B' is 1000 mm Hg. If a mixture solution of 'A' and 'B' boils at 80°C and 1 atm pressure, the amount of 'A' in the mixture is (1 atm = 760 mm Hg)

(a) 52 mol per cent (b) 34 mol per cent
(c) 48 mol per cent (d) 50 mol per cent

10. For a reaction $\frac{1}{2}A \rightarrow 2B$, rate of disappearance of 'A' is related to the rate of appearance of 'B' by the expression

(a) $-\frac{d[A]}{dt} = \frac{1}{2}\frac{d[B]}{dt}$ (b) $-\frac{d[A]}{dt} = \frac{1}{4}\frac{d[B]}{dt}$

(c) $-\frac{d[A]}{dt} = \frac{d[B]}{dt}$ (d) $-\frac{d[A]}{dt} = 4\frac{d[B]}{dt}$

11. The equilibrium constants K_{P_1} and K_{P_2} for the reactions $X \rightleftharpoons 2Y$ and $Z \rightleftharpoons P + Q$ respectively are in the ratio 1 : 9. If the degree of dissociation of X and Z be equal then the ratio of total pressure at these equilibria is

(a) 1 : 36 (b) 1 : 1
(c) 1 : 3 (d) 1 : 9

12. Oxidizing power of chlorine in aqueous solution can be determined by the parameters indicated below:

$$\frac{1}{2}CI_2(g) \xrightarrow{\frac{1}{2}\Delta_{diss}H^{\ominus}} CI(g) \xrightarrow{\frac{1}{2}\Delta_{hyd}H^{\ominus}} CI^-(aq).$$

The energy involved in the conversion of $\frac{1}{2}CI_2(g)$ to $CI^-(g)$

(using the data, $\Delta_{diss}H^{\ominus}_{Cl_2} = 240\ kJmol^{-1}$, $\Delta_{eg}H^{\ominus}_{Cl} = -349\ kJmol^{-1}$,

$\Delta_{hyd}H^{\ominus}_{Cl} = -381\ kJmol^{-1}$)

(a) $+152\ kJmol^{-1}$ (b) $-610\ kJmol^{-1}$
(c) $-850\ kJmol^{-1}$ (d) $+120\ kJmol^{-1}$

13. Which of the following factors is of **no significance** for roasting sulphide ores to the oxides and not subjecting the sulphide ores to carbon reduction directly?

(a) Metal sulphides are thermodynamically more stable than CS_2
(b) CO_2 is thermodynamically more stable than CS_2
(c) Metal sulphides are less stable than the corresponding oxides
(d) CO_2 is more volatile than CS_2

14. Bakelite is obtained from phenol by reacting with

(a) $(CH_2OH)_2$ (b) CH_3CHO
(c) CH_3COCH_3 (d) HCHO

15. For the following three reactions a, b and c, equilibrium constants ate given:

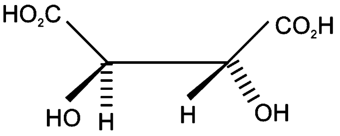
A. $CO(g)+H_2O(g) \rightleftharpoons CO_2(g)+H_2(g);\ K_1$

B. $CO_4(g)+H_2O(g) \rightleftharpoons CO(g)+3H_2(g);\ K_2$

C. $CH_4(g)+2H_2O(g) \rightleftharpoons CO_2(g)+4H_2(g);\ K_3$

(a) $K_1\sqrt{K_2}=K_3$ (b) $K_2K_3=K_1$
(c) $K_3=K_1K_2$ (d) $K_1.K_2^3=K_1^2$

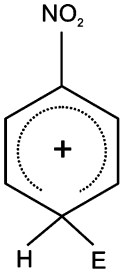
16. The absolute configuration of



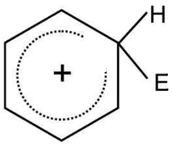
(a) S, S (b) R, R
(c) R, S (d) S, R

17. The electrophile, $E^{\oplus}$ attacks the benzene ring to generate the intermediate σ – complex. Of the following, which σ – complex is of lowest energy?

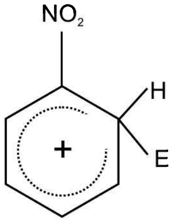
(a)



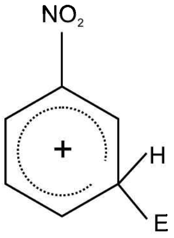
(b)



(c)



(d)



18. α-D-(+)- glucose and β-D-(+)- glucose are

(a) conformers (b) epimers
(c) anomers (d) enantionmers

19. Standard entropy of X_2, Y_2 and XY_3 are 60, 40 and 50 $JK^{-1}mol^{-1}$, respectively. For the reaction, $\frac{1}{2}X_2+\frac{3}{2}Y_2 \rightarrow XY_3, \Delta H = - -30\text{ kJ}$, to be at equilibrium, the temperature will be

(a) 1250 K (b) 500 K
(c) 750 K (d) 1000 K

20. Four species are listed below

i. HCO_3^- ii. H_3O^+
iii. HSO_4^- iv. HSO_3F

(a) iv < ii < i (b) ii < iii < iv
(c) i < iii < ii < iv (d) iii < i < iv < ii

21. Which one of the following constitutes a group of the isoelectronic species?

(a) C_2^{2-}, O_2^-, CO, NO
(b) $NO^+, C_2^{2-}, CN^-, N_2$
(c) $CN^-, N_2, O_2^{2-}, C_2^{2-}$
(d) N_2, O_2^-, NO^+, CO

22. Which one of the following pairs of species have the same bond order?

(a) CN^- and NO^+ (b) CN^- and CN^+
(c) O_2^- and CN^- (d) NO^+ and CN^+

23. The ionization enthalpy of hydrogen atom is $1.312 \times 10^6\ Jmol^{-1}$. The energy required to excite the electron in the atom from n = 1 to n = 2 is

(a) $8.51 \times 10^5\ Jmol^{-1}$
(b) $8.56 \times 10^5\ Jmol^{-1}$
(c) $7.56 \times 10^5\ Jmol^{-1}$
(d) $9.84 \times 10^5\ Jmol^{-1}$

24. Which one of the following is the correct statement?

(a) Boric acid is a protonic acid
(b) Beryllium exhibits coordination number of six
(c) Cholorides of both beryllium and aluminum have bridged chloride structures in solid phase
(d) $B_2H_6.2NH_3$ is known as 'inorganic benzene'

25. Given $E^{\circ}_{Cr^{3+}/Cr} = -0.72V, E^{\circ}_{Fe^{2+}/Fe} = -0.42V$. The potential for the cell

$Cr|Cr^{3+}(0.1M)||Fe^{2+}(0.01M)|Fe$ is

(a) 0.26 V (b) 0.399 V
(c) – 0.399 V (d) – 0.23 V

26. Amount of oxalic acid present in a solution can be determined by its titration with $KMnO_4$ solution in the presence of H_2SO_4. The titration gives unsatisfactory result when carried out in the presence of HCl, because HCl

(a) gets oxidized by oxalic acid to chlorine
(b) furnishes H^+ ions in addition to those from oxalic acid
(c) reduces permanganate to Mn^{2+}
(d) oxidizes oxalic acid to carbon dioxide and water

27. The vapour pressure of water at 20°C is 17.5 mm Hg. If 18 g of glucose ($C_6H_{12}O_6$) is added to 178.2 g of water at 20°C, the vapour pressure of the resulting solution will be

(a) 17.675 mm Hg (b) 15.750 mm Hg
(c) 16.500 mm Hg (d) 17.325 mm Hg

28. Among the following substituted silanes the one which will give rise to cross linked silicone polymer on hydrolysis is

(a) R_4Si (b) $RSiCI_3$
(c) R_2SiCI_2 (d) R_3SiCI

29. In context with the industrial preparation of hydrogen from water gas ($CO + H_2$), which of the following is the correct statement?

(a) CO and H_2 are fractionally separated using differences in their densities.
(b) CO is removed by absorption in aqueous Cu_2CI_2 solution.
(c) H_2 is removed through occlusion with Pd.
(d) CO is oxidized to CO_2 with steam in the presence of a catalyst followed by absorption of CO_2 in alkali.

30. In a compound atoms of element Y from ccp lattice and those of element X occupy 2/3rd of tetrahedral voids. The formula of the compound will be

(a) X_4Y_3 (b) X_2Y_3
(c) X_2Y (d) X_3Y_4

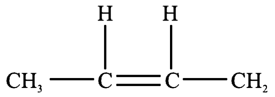
31. Gold number of protective colloids A, B, C and D are 0.50, 0.01, 0.005, respectively. The correct order of their protective powers is

(a) D < A < C < B (b) C < B < D < A
(c) A < C < B < D (d) B < D < A < C

32. The hydrocarbon which can react with sodium in liquid ammonia is

(a) $CH_3CH_2CH_2C\equiv CCH_2CH_2CH_3$
(b) $CH_3CH_2C\equiv CH$
(c) $CH_3CH\equiv CHCH_3$
(d) $CH_3CH_2C\equiv CCH_2CH_3$

33. The treatment of CH_3MgX with $CH_3C\equiv C–H$ produces

(a) $CH_3–CH=CH_2$
(b) $CH_3C\equiv C–CH_3$
(c)

$CH_3–\overset{H}{C}=\overset{H}{C}–CH_2$
(d) CH_4

34. The correct decreasing order of priority for the functional groups of organic compounds in the IUPAC system of nomenclature is

(a) $–COOH, –SO_3H, –CONH_2, –CHO$
(b) $–SO_3H, –COOH, –CONH_2, –CHO$
(c) $–CHO, –COOH, –SO_3H, –CONH_2$
(d) $–CONH_2, –CHO, –SO_3H, –COOH$

35. The pK_a of a weak acid, HA, is 4.80. The pK_b of a weak base, BOH, is 4.78. the pH of an aqueous solution of the corresponding slat, BA, will be

(a) 9.58 (b) 4.79
(c) 7.01 (d) 9.22

ANSWERS

1. (d)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (d)	**7.** (b)	**8.** (a)	**9.** (d)	**10.** (b)
11. (a)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (c)
21. (b)	**22.** (a)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (d)	**30.** (a)
31. (c)	**32.** (b)	**33.** (d)	**34.** (b)	**35.** (c)					

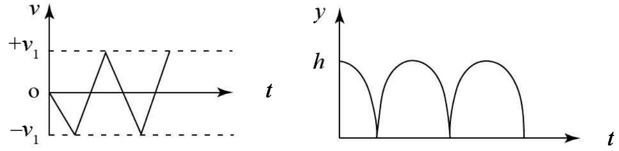
AIEEE Paper 2009

PHYSICS

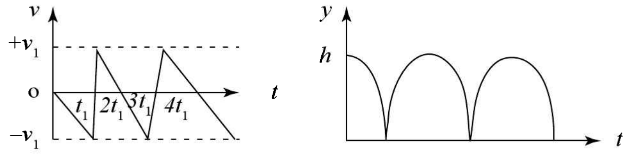
1. Consider a rubber ball freely falling from a height h = 4.9 m onto a horizontal elastic plate. Assume that the duration of collision is negligible and the collision with the plate is totally elastic.

Then the velocity as a function of time and the height as a function of time will be:

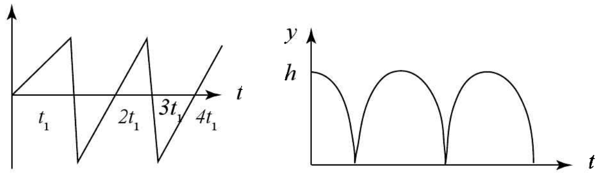
(a)



(b)



(c)



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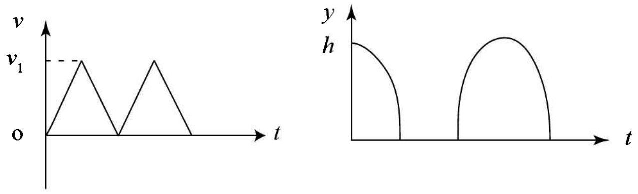


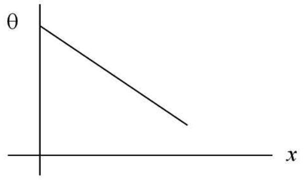
2. The height at which the acceleration due to gravity becomes $\frac{g}{9}$ (where g = the acceleration due to gravity on the surface of the earth) in terms of R, the radius of the earth, is

(a) $\frac{R}{\sqrt{2}}$ (b) $\frac{R}{2}$

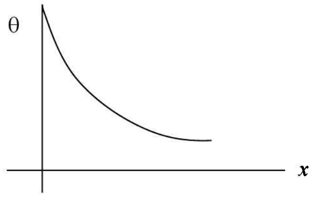
(c) $\sqrt{2}\,R$ (d) 2R

3. A long metallic bar is carrying heat from one of its ends to the other end under steady-state. The variation of temperature θ along the length x of the bar from its hot end is best described by which of the following figures

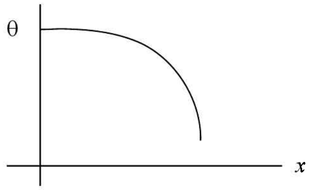
(a)



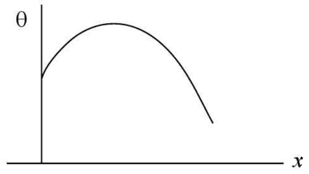
(b)



(c)



(d)



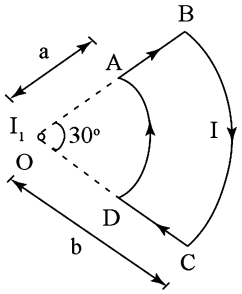
4. Two points P and Q are maintained at the potentials of 10 V and –4 V respectively. The work done in moving 100 electrons from P to Q is:

(a) 9.60×10^{-17} J (b) -2.24×10^{-16} J

(c) 2.24×10^{-16} J (d) -9.60×10^{-17} J

Direction: Question numbers 5 and 6 are based on the following paragraph:

A current loop ABCD is held fixed on the plane of the paper as shown in the figure. The arcs BC (radius = b) and DA (radius = a) of the loop are joined by two straight wires AB and. CD. A steady current I is flowing in the loop. Angle made by AB and CD at the origin O is 30°. Another straight thin wire with steady current I_1 flowing out of the plane of the paper is kept at the origin.



5. The magnitude of the magnetic field (B) due to the loop ABCD at the origin (O) is:

(a) $\frac{\mu_0 I(b-a)}{24ab}$ (b) $\frac{\mu_0 I}{4\pi}\left[\frac{b-a}{ab}\right]$

(c) $\frac{\mu_0 I}{4\pi}\left[2(b-a)+\frac{\pi}{3}(a+b)\right]$ (d) zero

6. Due to the presence of the current I_1 at the origin:

(a) The forces on AD and BC are zero.

(b) The magnitude of the net force on the loop is given by $\frac{\mu_0 I_1 I}{4\pi}\left[2(b-a)+\frac{\pi}{3}(a+b)\right]$

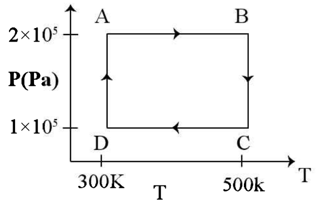
(c) The magnitude of the net force on the loop is given by $\frac{\mu_0 I I_1}{24ab}(b-a)$.

(d) The forces on AB and DC are zero.

Directions: Question number 7, 8 and 9 are based on the following paragraph.

Paragraph:

Two moles of helium gas are teken over the cycle ABCDA, as shown in the P-T diagram.



7. Assuming the gas to be ideal the work done on the gas in taking it from A to B is:

(a) 300 R (b) 400 R
(c) 500 R (d) 200 R

8. The work done on the gas in taking it from D to A is

(a) +414 R (b) –690 R
(c) +690 R (d) –414 R

9. The net work done on the gas in the cycle ABCDA is:

(a) 276 R (b) 1076 R
(c) 1904 R (d) Zero

10. In an experiment the angles are required to be measured using an instrument. 29 divisions of the main scale exactly coincide with the 30 divisions of the vernier scale. If the smallest division of the main scale is half-a-degree (0.5°), then the least count of the instniment is:

(a) half minute (b) one degree
(c) half degree (d) one minute

11. A charge Q is placed at each of the opposite corners of a square. A charge q is placed at each of the other two corners. If the net electrical force on Q is zero, then Q/q equals

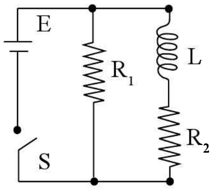
(a) –1 (b) 1

(c) $-\frac{1}{\sqrt{2}}$ (d) $-2\sqrt{2}$

12. One kg of a diatomic gas is at a pressure of 8×10^4 N/m². The density of the gas is 4 kg/m³. What is the energy of the gas due to its thermal rnotion?

(a) 5×10^4 J (b) 6×10^4 J
(c) 7×10^4 J (d) 3×10^4 J

13.



An inductor of inductance L = 400 mH and resistors of resistances $R_1 = 2\ \Omega$ and $R_2 = 2\ \Omega$ are connected to a battery of emf 12 V as shown in the figure. The internal resistance of the battery is negligible. The switch S is closed at t = 0. The potential drop across L as a funcution of time is:

(a) $\frac{12}{t}e^{-3t}$ V (b) $6(1 - e^{-t/0.2})$ V
(c) $12\,e^{-5t}$ V (d) $6\,e^{-5t}$ V

This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

14. **Statement 1:** The temperature dependence of resistance is usually given as $R = R_0(1 + \alpha\,\Delta t)$. The resistance of a wire changes from 100 Ω to 150 Ω when its temperature is increased from 27°C to 227°C. This implies that $\alpha = 2.5 \times 10^{-3}$/°C.

Statement 2: $R = R_0(1 + \alpha\,\Delta t)$ is valid only when the change in the temperature ΔT is small and $\Delta R = (R-R_0) << R_0$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is false.

15. The transition from the state n = 4 to n = 3 in a hydrogen like atom results in ultraviolet radiation. Infrared radiation will be obtained in the transition from:

(a) 3 → 2 (b) 4 → 2
(c) 5 → 4 (d) 2 → 1

16. A mixture of light consisting of wavelength 590 nm and an unknown wavelength, illuminates Young's double slit and gives rise to two overlapping interference patterns on the screen. The central maximum of both lights coincide. Further, it is observed that the third bright fringe of known light coincides with the 4th bright fringe of the unknown light. From this data, the wavelength of the unknown light is:

(a) 885.0 nm (b) 442.5 nm
(c) 776.8 nm (d) 393.4 nm

17. A particle has an initial velocity of $3\hat{i}+4\hat{j}$ and an acceleration of $0.4\hat{i}+0.3\hat{j}$. Its speed after 10 s is:

(a) $7\sqrt{2}$ units (b) 7 units
(c) 8.5 units (d) 10 units

18. The surface of a metal is illuminated with the light of 400 nm. The kinetic energy of the ejected photoelectrons was found to be 1.68 eV. The work fuction of the metal is: (hc = 1240 eV.nm)

(a) 1.41 eV (b) 1.51 eV
(c) 1.68 eV (d) 3.09 eV

19. Three sound waves of equal amplitudes have frequencies $(v-1)$, v, $(v+1)$. They superpose to give beats. The number of beats produced per second will be:

(a) 3 (b) 2
(c) 1 (d) 4

20. A motor cycle starts from rest and accelerates along a straight path at 2 m/s^2. At the starting point of the motor cycle there is a stationary electric siren. How far has the motor cycle gone when the driver hears the frequency of the siren at 94% of its value when the motor cycle was at rest? (Speed of sound = 330 ms^{-1})

(a) 98 m (b) 147 m
(c) 196 m (d) 49 m

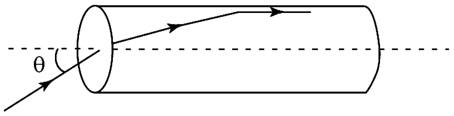
21. The above is a plot of binding energy per nucleon E_b, against the nuclear mass M; A, B, C, D, E, correspond to different nuclei. Consider four reactions:

(i) $A+B\rightarrow C+\varepsilon$
(ii) $C\rightarrow A+B+\varepsilon$
(iii) $D+E\rightarrow F+\varepsilon$ and
(iv) $F\rightarrow D+E+\varepsilon$,

where ε is the ener y released? In which reactions is ε positive?

(a) (i) and (iii)
(b) (ii) and (iv)
(c) (ii) and (iii)
(d) (i) and (iv)

22. A transparent solid cylindrical rod has a refractive index of $\frac{2}{\sqrt{3}}$. It is surrounded by air. A light ray is incident at the mid-point of one end of the rod as shown in the figure.



The incident angle (θ) for which the light ray grazes along the wall of the rod is:

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (b) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$
(c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\sin^{-1}\left(\frac{1}{2}\right)$

23. Two wires are made of the same material and have the same volume. However wire 1 has cross-sectional area A and wire 2 has cross-sectional area 3A. If the length of wire 1 increases by Δx on applying force F, how much force is needed to stretch wire 2 by the same amount?

(a) 4F (b) 6F
(c) 9F (d) F

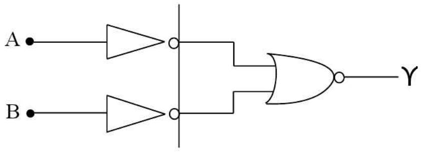
This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

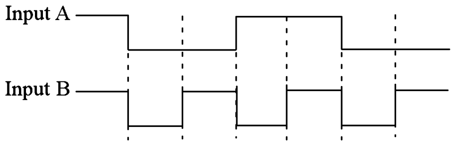
24. **Statement 1:** For a charged particle moving from point P to point Q, the net work done by an electrostatic field on the particle is independent of the path connecting point P to point Q.

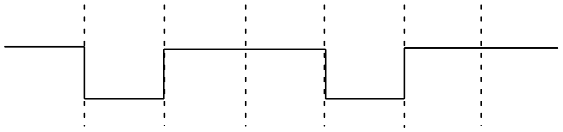
Statement 2: The net work done by a conservative force on an object moving along a closed loop is zero.

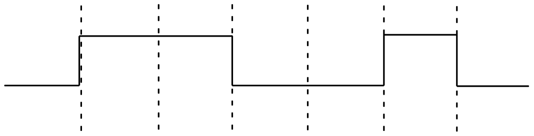
(a) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is false.

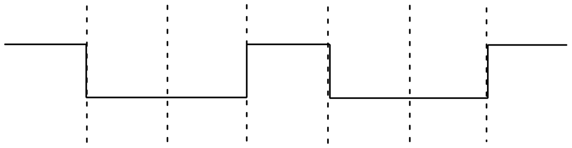
25. The logic circuit shown below has the input waveforms 'A' and 'B' as shown. Pick out the correct out put waveform.

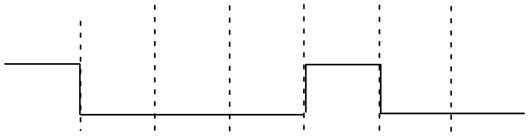




(a)


(b)


(c)


(d)


26. If x, v and a denote the displacement, the velocity and the acceleration of a particle executing simple harmonic motion of time period T, then, which of the following does not change with time?

(a) $\frac{aT}{x}$ (b) $aT+2\pi v$
(c) $\frac{aT}{x}$ (d) $a^2T^2+4\pi^2v^2$

27. A thin uniform rod of length ℓ and mass m is swinging freely about a horizontal axis passing through its end. Its maximum angular speed is ω. Its centre of mass rises to a maximum height of:

(a) $\frac{1}{6}\frac{\ell\omega}{g}$ (b) $\frac{1}{2}\frac{\ell^2\omega^2}{g}$

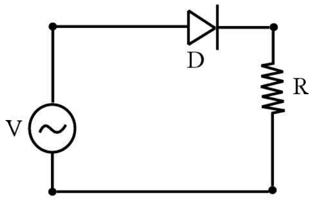
(c) $\frac{1}{6}\frac{\ell^2\omega^2}{g}$ (d) $\frac{1}{3}\frac{\ell^2\omega^2}{g}$

28. In an optics experiment, with the position of the object fixed, a student varies the position of a convex lens and for each position, the screen is adjusted to get a clear image of the object. A graph between the object distance u and the image distance v, from the lens, is plotted using the same scale for the two axes. A straight line passing through the origin and making an angle of 45° with the x-axis meets the experimental curve at P. The coordinates of P will be:

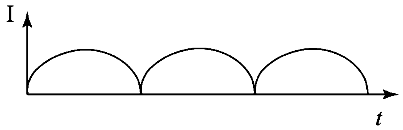
(a) $\left(\frac{f}{2},\frac{f}{2}\right)$ (b) (f, f)

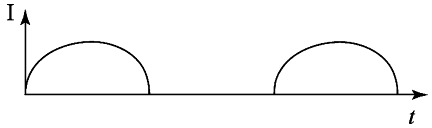
(c) (4f, 4f) (d) (2f, 2f)

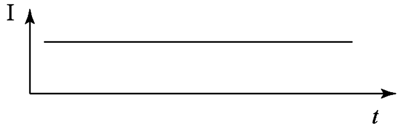
29. A p-n junction (D) shown in the figure can act as a rectifier. An alternating current source (V) is connected in the circuit.

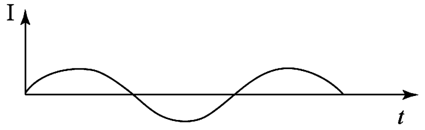


The current (I) in the resistor (R) can be shown by:

(a)


(b)


(c)


(d)


30. Let $P(r) = \frac{Q}{\pi R^4} r$ be the charge density distribution for a solid sphere of radius R and total charge Q. For a point 'p' inside the sphere at distance r_1 from the centre of sphere, the magnitude of electric field is:

(a) $\frac{Q}{4\pi\varepsilon_0 r_1^2}$ (b) $\frac{Qr_1^2}{4\pi\varepsilon_0 R^4}$

(c) $\frac{Qr_1^2}{3\pi\varepsilon_0 R^4}$ (d) 0

ANSWERS

1. (b)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (a)	**10.** (d)
11. (d)	**12.** (a)	**13.** (c)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (a)
21. (d)	**22.** (c)	**23.** (c)	**24.** (a)	**25.** (d)	**26.** (a or b)	**27.** (c)	**28.** (d)	**29.** (b)	**30.** (b)

MATHEMATICS

1. Let a, b, c be such that $b(a + c) \neq 0$.

$$\text{If } \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0,$$

then the value of 'n' is

(a) zero (b) any even integer

(c) any odd integer (d) any integer

2. If the mean deviation of number 1, 1 + d, 1 + 2d, ….. , 1 + 100d from their mean is 255, then the d is equal to

(a) 10.0 (b) 20.0

(c) 10.1 (d) 20.2

3. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is

(a) greater than 4ab (b) less than 4ab

(c) greater than – 4ab (d) less than – 4ab

4. Let A and B denote the statements

A: $\cos\alpha + \cos\beta + \cos\gamma = 0$

B: $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then

(a) A is true and B is false (b) A is false and B is true

(c) both A and B are true (d) both A and B are false

5. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for

(a) no value of p (b) exactly one value of p

(c) exactly two values of p (d) more than two values of p

6. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $A = B$ (b) $A = C$
(c) $B = C$ (d) $A \cap B = \phi$

7. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\vec{u}\ \ p\vec{v}\ \ p\vec{w}] - [p\vec{v}\ \ \vec{w}\ \ q\vec{u}] - [2\vec{w}\ \ q\vec{v}\ \ q\vec{u}] = 0$ holds for

(a) exactly one value of (p, q)
(b) exactly two values of (p, q)
(c) more than two but not all values of (p, q)
(d) all values of (p, q)

8. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals

(a) (6, –17)
(b) (–6, 7)
(c) (5, –15)
(d) (–5, 15)

9. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(a) less than 500
(b) at least 500 but less than 750
(c) at least 750 but less than 1000
(d) at least 1000

10. $\int_0^{\pi} [\cot x]\, dx$, [•] denotes the greatest integer function, is equal to

(a) $\frac{\pi}{2}$ (b) 1
(c) – 1 (d) $-\frac{\pi}{2}$

11. For real x, let $f(x) = x^3 + 5x + 1$, then

(a) f is one-one but not onto R
(b) f is onto R but not one-one
(c) f is one-one and onto R
(d) f is neither one-one nor onto R

12. In a binomial distribution $B\left(n,\ p = \frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than

(a) $\frac{1}{\log_{10}^{4} - \log_{10}^{3}}$ (b) $\frac{1}{\log_{10}^{4} + \log_{10}^{3}}$
(c) $\frac{9}{\log_{10}^{4} - \log_{10}^{3}}$ (d) $\frac{4}{\log_{10}^{4} - \log_{10}^{3}}$

13. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for

(a) all values of p
(b) all except one value of p
(c) all except two values of p
(d) exactly one value of p

14. The projections of a vector on the three coordinate axis are 6, – 3, 2 respectively. The direction cosines of the vector are

(a) 6, –3, 2 (b) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$
(c) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (d) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

15. If $\left|Z - \frac{4}{z}\right| = 2$, , then the maximum value of $|Z|$ is equal to

(a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$
(c) 2 (d) $2 + \sqrt{2}$

16. Three distinct points A, B and C are given in the 2 – dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point

(a) (0, 0) (b) $\left(\frac{5}{4}, 0\right)$
(c) $\left(\frac{5}{2}, 0\right)$ (d) $\left(\frac{5}{3}, 0\right)$

17. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is

(a) 0 (b) 2
(c) 7 (d) 8

18. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn in inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is

(a) $x^2 + 16y^2 = 16$ (b) $x^2 + 12y^2 = 16$
(c) $4x^2 + 48y^2 = 48$ (d) $4x^2 + 64y^2 = 48$

19. The sum to the infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...$ is

(a) 2 (b) 3
(c) 4 (d) 6

20. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$, where c_1 and c_2 are arbitrary constants is

(a) $y' = y^2$ (b) $y'' = y'y$
(c) $yy'' = y'$ (d) $yy'' = (y')^2$

21. One ticket is selected at random from 50 tickets numbered 00, 01, 02, …., 49. Then the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals

(a) $\frac{1}{14}$ (b) $\frac{1}{7}$
(c) $\frac{5}{14}$ (d) $\frac{1}{50}$

22. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(a) –1 (b) 1
(c) log 2 (d) –log 2

23. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point (2, 3) and the x-axis is

(a) 3 (b) 6
(c) 9 (d) 12

24. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$. then in the interval $[-1, 1]$

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

25. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{2\sqrt{3}}{8}$

(c) $\frac{3\sqrt{2}}{5}$ (d) $\frac{\sqrt{3}}{4}$

Directions: Question number 26 to 30 are Assertion – Reason type questions. Each of these questions contains two statements

Statement-1 (Assertion) and Statement-2 (Reason).

Each of these questions also have four alternative choices, only one of which is the correct answer. You have to select the correct choice

26. Let $f(x) = (x + 1)^2 - 1, x \geq -1$

Statement-1: The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Statement-2: f is a bijection.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

27. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement-1: gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement-2: gof is twice differentiable at $x = 0$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

28. **Statement-1:** The variance of first n even natural numbers is $\frac{n^2 - 1}{4}$

Statement-2: The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

29. **Statement-1:** $\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Statement-2: $\sim (p \leftrightarrow \sim q)$ is a tautology.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

30. Let A be a 2×2 matrix

Statement-1: adj(adj A) = A

Statement-2: $|\text{adj } A| = |A|$

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

ANSWERS

1. (c)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (c)	7 (a)	**8.** (b)	**9.** (d)	**10.** (d)
11. (c)	**12.** (a)	**13.** (a)	**14** (c)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (d)
21. (a)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (b)

CHEMISTRY

1. Copper crystallizes in fcc with a unit cell length of 361 pm. What is the radius of copper atom?

(a) 127 pm (b) 157 pm

(c) 181 pm (d) 108 pm

2. Calculate the wavelength (in monometer) associated with a proton moving at $1.0 \times 10^3 ms^{-1}$. (Mass of proton $= 1.67 \times 10^{-27}$ kg and $h = 6.63 \times 10^{-34}$ Js).

(a) 0.40 nm (b) 2.5 nm

(c) 14.0 nm (d) 0.032 nm

3. In an atom, an electron is moving with a speed of 600 m/s with an accuracy of 0.005%. Certainty with which the position of the electron can be located is ($h = 6.6 \times 10^{-34}$ kg m^2 s^{-1}, mass of electron, $e_m = 9.1 \times 10^{-31}$ kg):

(a) 5.10×10^{-3} m (b) 1.92×10^{-3} m

(c) 3.84×10^{-3} m (d) 1.52×10^{-3} m

4. Two liquids X and Y form an ideal solution. At 300K, vapour pressure of the solutions containing 1 mol of X and 3 mol of Y is 550 mmHg. At the same temperature, if 1 mol of Y is further added to this solution, vapour pressure of the solution increases by 10 mmHg. Vapour pressure (in mmHg) of X and Y in their pure states will be, respectively

(a) 300 and 400 (b) 400 and 600
(c) 500 and 600 (d) 200 and 300

5. A binary liquid solution is prepared by mixing n-heptane and ethanol. Which of the following statement is correct regarding the behaviour of the solution?

(a) The solution is non-ideal, showing +ve deviation from Raoult's Law.
(b) The solution is non-ideal, showing-ve deviation from Raoult's Law.
(c) n-heptane shows +ve deviation while ethanol shows-ve deviation from Raoult's Law.
(d) The solution formed is an ideal solution.

6. On the basis of the following thermochemical data:
$(\Delta f\ G^{o}H^{+}_{(aq)} = 0)$

$H_2O(l) \rightarrow H^+\ (aq) + OH^-(aq);$
$\Delta H = 57.32$ kJ

$H_2(g) + \frac{1}{2}\ O_2(g) \rightarrow H_2O(l);$
$\Delta H = -286.20$ kJ

The value of enthalpy of formation of OH^- ion at 25°C is:

(a) –228.88 kJ (b) +228.88 kJ
(c) –343.52 kJ (d) –22.88 kJ

7. Solid $Ba(NO_3)_2$ is gradually dissolved in a 1.0×10^{-4} M Na_2CO_3 solution. At what concentration of Ba^{2+} will a precipitate begin to form?
(K_{sp} for $Ba\ CO_3 = 5.1 \times 10^{-9}$):

(a) 5.1×10^{-5} M (b) 8.1×10^{-8} M
(c) 8.1×10^{-7} M (d) 4.1×10^{-5} M

8. In a fuel cell, methanol is used as fuel and oxygen gas is used as an oxidizer. The reaction is

$CH_3OH(l) + 3/2\ O_2(g) \rightarrow CO_2(g) + 2H_2O(l)$

At 298 K standard Gibb's energies of formation for $CH_3OH(l)$, H_2O (l) and $CO_2(g)$ are –166.2, –237.2 and –394.4 kJmol⁻¹ respectively. If standard enthalpy of combustion of methanol is –7269 kJ mol⁻¹, efficiency of the fuel cell will be

(a) 87 % (b) 90 %
(c) 97 % (d) 80%

9. Given:
$E^0_{Fe^{3+}/Fe} = -0.036V$, $E^0_{Fe^{2+}/Fe} = -0.439$ V. The value of standard electrode potential for the change, $Fe^{3+}_{(aq)} + e^- \rightarrow Fe^{2+}$ (aq) will be

(a) 0.385 V (b) 0.770 V
(c) –0.270 V (d) –0.072V

10. The half-life period of a first order chemical reaction is 6.93 minutes. The time required for the completion of 99% of the chemical reaction will be (log 2 = 0.301)

(a) 23.03 minutes (b) 46.06 minutes
(c) 460.6 minutes (d) 230.3 minutes

11. Which of the following statement is incorrect regarding physissorptions?

(a) More easily liquefiable gases are adsorbed readily.
(b) Under high pressure, it results into multi-molecular layer on adsorbent surface.
(c) Enthalpy of adsorption ($\Delta H_{adsorption}$) is low and positive.
(d) It occurs because of van der Waal's forces.

12. The set representing the correct order of ionic radius is

(a) $Na^+ > Li^+ > Mg^{2+} > Be^{2+}$
(b) $Li^+ > Na^+ > Mg^{2+} > Be^{2+}$
(c) $Mg^{2+} > Be^{2+} > Li^+ > Na^+$
(d) $Li^+ > Be^{2+} > Na^+ > Mg^{2+}$

13. Using MO theory predict which of the following species has the shortest bond length?

(a) O_2^+ (b) O_2^-
(c) O_2^{2-} (d) O_2^{2+}

14. The bond dissociation energy of B– F in BF_3 is 646 kJ mol⁻¹ whereas that of C– F in CF_4 is 515 kJ mol⁻¹. The correct reason for higher B–F bond dissociation energy as compared to that of C–F is

(a) stronger σ bond between B and F in BF_3 as compared to that between C and F in CF_4
(b) significant pπ –pπ interaction between B and F in BF_3 whereas there is no possibility of such interaction between C and F in CF_4
(c) lower degree of pπ –pπ interaction between B and F BF_3 than that between C and F in CF_4
(d) smaller size of B-atom as compared to that of C-atom

15. Which one of the following reactions of Xenon compound is not feasible?

(a) $3Xe\ F_4 + 6H_2O \rightarrow 2Xe + XeO_3 + 12HF + 1.5O_2$
(b) $2XeF_2 + 2H_2O \rightarrow 2Xe + 4HF + O_2$.
(c) $XeF_6 + RbF \rightarrow Rb[XeF_7]$
(d) $XeO_3 + 6HF \rightarrow XeF_6 + 3H_2O$

16. In which of the following arrangements , the sequence is not strictly according to the property written against it?

(a) $HF < HCl < HBr < HI$: increasing acid strength
(b) $NH_3 < PH_3 < AsH_3 < SbH_3$: increasing basic strength
(c) $B < C < O < N$: increasing first ionization enthalpy
(d) $CO_2 < SiO_2 < SnO_2 < PbO_2$: increasing oxidising power

17. In context with the transition elements, which of the following statement is incorrect?

(a) In the highest oxidation states, the transition metal show basic character and cationic complex.
(b) In the highest oxidation states of the first five transition element (Sc to Mn), all the 4s and 3d electrons are used for bonding.
(c) Once the d^5 configuration is exceeded, the tendency to involve all the 3d electrons in bonding decreases.
(d) In addition to the normal oxidation states, the zero oxidation state is also shown by these elements in complex.

18. Knowing that the chemistry of lanthanoids (Ln) is dominated by its +3 oxidation state, which of the following statements is incorrect?

(a) The ionic sizes of Ln (III) decrease in general with increasing atomic number.
(b) Ln (III) compounds are generally colourless.
(c) Ln (III) hydroxides are mainly basic in character.
(d) Because of the large size of the Ln (III) ions the bonding in its compounds is predominantly ionic in character.

19. Which of the following pairs represents linkage isomers?

(a) $[Pd\ (P\ Ph_3)_2\ (NCS)_2]$ and
$[Pd\ (P\ Ph_3)_2\ (SCN)_2]$

(b) $[CO\ (NH_3)_5\ NO_3]\ SO_4$ and
$[CO\ (\ NH_3)_5\ SO_4]\ NO_3$

(c) $[Pt\,Cl_2(NH_3)_4]\,Br_2$ and
$[Pt\,B_2(NH_3)_4]\,Cl_2$

(d) $[Cu(NH_3)_4]\,[Pt\,Cl_4]$ and
$[Pt\,(NH_3)_4]\,[CuCl_4]$

20. Which of the following has an optical isomer?

(a) $[CO(en)\,(NH_3)_2]^{2+}$
(b) $[CO\,((H_2O)_4\,(en)]^{3+}$
(c) $[CO\,(en)_2\,(NH_3)_2]^{3+}$
(d) $[CO\,(NH_3)_3\,Cl]^{+}$

21. The IUPAC name of neopentane is:

(a) 2,2-dimethylpropane (b) 2-mehylpropane
(c) 2, 2-dimethylbutane (d) 2-methylbytane

22. The number of stereo isomers possible for a compound of the molecular formula

CH_3 —CH=CH —CH(OH) —Me is

(a) 2 (b) 4
(c) 6 (d) 3

23. Arrange the carbanions, $(CH_3)_3C$, $C\,Cl_3$, $(CH_3)_2\,\bar{C}H$, $C_6H_5\bar{C}H_2$, in order of their decreasing stability.

(a) $(CH_3)_2\bar{C}H > \bar{C}Cl_3 > C_6H_5\,\bar{C}H_2 > (CH_3)_3\bar{C}$

(b) $\bar{C}Cl_3 > C_6H_5\,\bar{C}H_2 > (CH_3)_2\,\bar{C}H > (CH_3)_3\,\bar{C}$

(c) $(CH_3)_3\,\bar{C} > (CH_3)_2\,\bar{C}H > C_6H_5\bar{C}H_2 > \bar{C}Cl_3$

(d) $C_6H_5\,\bar{C}H_2 > \bar{C}Cl_3 > (CH_3)_3\,\bar{C} > (CH_3)_2\,\bar{C}H$

24. The alkene that exhibits geometrical isomerism is
(a) 2-methyl propene
(b) 2-butene
(c) 2-methyl-2-butene
(d) propene

25. The major product obtained on interaction of phenol with sodium hydroxide and carbon dioxide is
(a) salicylaldehyde
(b) salicylic acid
(c) phthalic acid
(d) benzoic acid

26. A liquid was mixed with ethanol and a drop of concentrated H_2SO_4 was added. a compound with a fruity smell was formed. The liquid was

(a) HCHO (b) CH_3COCH_3
(c) CH_3COOH (d) CH_3OH

27. Which of the following on heating with aqueous KOH produces acetaldehyde?

(a) CH_3CH_2Cl (b) CH_2ClCH_2Cl
(c) CH_3CHCl_2 (d) CH_3COCl

28. In Cannizaro reaction given below

$$2\,Ph\,CHO \xrightarrow{:\overset{\ominus}{O}H} Ph\,CH_2OH + PhC\overset{\cdot\cdot\ominus}{}_2$$

the slowest step is

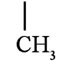
(a) the transfer of hydride to the carbonyl group
(b) the abstraction of proton from the carboxylic group
(c) the deprotonation of Ph CH_2OH
(d) the attack of :OH at the carboxyl group

29. Buna-N synthetic rubber is a copolymer of:

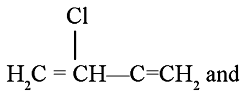
(a) $H_2C = CH—CH = CH_2$ and
$H_5C_6—CH = CH_2$

(b) $H_2C = CH—CN$ and
$H_2C = CH—CH = CH_2$

(c) $H_2C = CH—CN$ and



$H_2C = CH—\underset{\displaystyle CH_3}{\underset{|}{C}}=CH_2$

(d)  and

$H_2C = CH—\overset{\displaystyle Cl}{\overset{|}{C}}=CH_2$ and
$H_2C = CH—CH =CH_2$

30. The two functional groups present in a typical carbohydrate are

(a) —CHO and —COOH
(b) >C=O and —OH
(c) —OH and —CHO
(d) —OH and —COOH

ANSWERS

1. (a)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (b)
11. (c)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (c)
21. (a)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (c)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (b)

AIEEE Paper 2010

PHYSICS

Directions: *Questions number 1–3 are based on the following paragraph.*

An initially parallel cylindrical beam travels in a medium of refractive index $\mu(I) = \mu_0 + \mu_2 I$, where μ_0 and μ_2 are positive constants and I is the intensity of the light beam. The intensity of the beam is decreasing with increasing radius.

1. The initial shape of the wavefront of the beam is
   (1) planar
   (2) convex
   (3) concave
   (4) convex near the axis and concave near the periphery

2. The speed of light in the medium is
   (1) maximum on the axis of the beam
   (2) minimum on the axis of the beam
   (3) the same everywhere in the beam
   (4) directly proportional to the intensity I

3. As the beam enters the medium, it will
   (1) travel as a cylindrical beam
   (2) diverge
   (3) converge
   (4) diverge near the axis and converge near the periphery

Directions: *Questions number 4–5 are based on the following paragraph.*

A nucleus of mass $M + \Delta m$ is at rest and decays into two daughter nuclei of equal mass $\frac{M}{2}$ each. Speed of light is c.

4. The speed of daughter nuclei is
   (1) $c\sqrt{\frac{\Delta m}{M + \Delta m}}$
   (2) $c\frac{\Delta m}{M + \Delta m}$
   (3) $c\sqrt{\frac{2\Delta m}{M}}$
   (4) $c\sqrt{\frac{\Delta m}{M}}$

5. The binding energy per nucleon for the parent nucleus is E_1 and that for the daughter nuclei is E_2. Then
   (1) $E_1 = 2E_2$
   (2) $E_2 = 2E_1$
   (3) $E_1 > E_2$
   (4) $E_2 > E_1$

Directions: *Questions number 6–7 contain Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.*

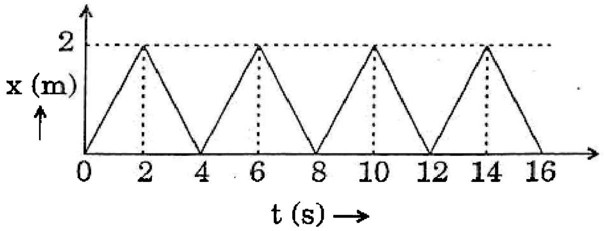
6. **Statement-I:** When ultraviolet light is incident on a photocell, its stopping potential is V_0 and the maximum kinetic energy of the photoelectrons is K_{max}. When the ultraviolet light is replaced by X-rays, both V_0 and K_{max} increase.

Statement-2: Photoelectrons are emitted with speeds ranging from zero to a maximum value because of the range of frequencies present in the incident light.
   (1) Statement-1 is true, Statement-2 is false.
   (2) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.
   (3) Statement-1 is true, Statement-2 is true; Statement-2 is ***not*** the correct explanation of Statement-1.
   (4) Statement-1 is false, Statement-2 is true.

7. **Statement-1:** Two particles moving in the same direction do not lose all their energy in a completely inelastic collision.

Statement-2: Principle of conservation of momentum holds true for all kinds of collisions.
   (1) Statement-1 is true, Statement-2 is false.
   (2) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.
   (3) Statement-1 is true, Statement-2 is true; Statement-2 is ***not*** the correct explanation of Statement-1.
   (4) Statement-1 is false, Statement-2 is true.

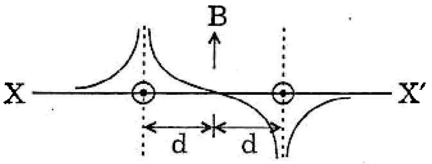
8. The figure shows the position – time (x – t) graph of one-dimensional motion of a body of mass 0.4 kg. The magnitude of each impulse is



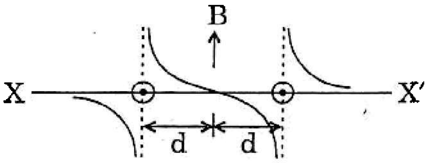
(1) 0.2 Ns
(2) 0.4 Ns
(3) 0.8 Ns
(4) 1.6 Ns

9. Two long parallel wires are at a distance 2d apart. They carry steady equal currents flowing out of the plane of the paper as shown. The variation of the magnetic field B along the line XX′ is given by

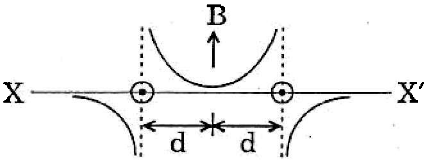
(1)



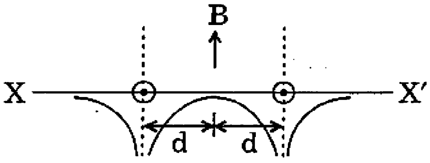
(2)



(3)

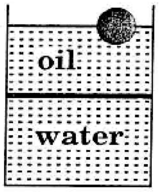


(4)

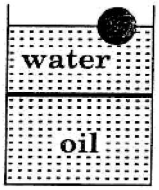


10. A ball is made of a material of density ρ where $\rho_{\text{oil}} < \rho < \rho_{\text{water}}$ with ρ_{oil} and ρ_{water} representing the densities of oil and water, respectively. The oil and water are immiscible. If the above ball is in equilibrium in a mixture of this oil and water, which of the following pictures represents its equilibrium position?

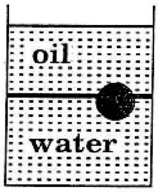
(1)



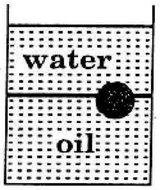
(2)



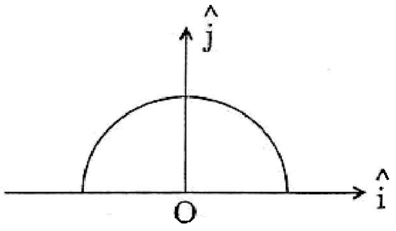
(3)



(4)



11. A thin semi-circular ring of radius r has a positive charge q distributed uniformly over it. The net field $\vec{E}$ at the centre O is



(1) $\dfrac{q}{2\pi^2\varepsilon_0 r^2}\hat{j}$

(2) $\dfrac{q}{4\pi^2\varepsilon_0 r^2}\hat{j}$

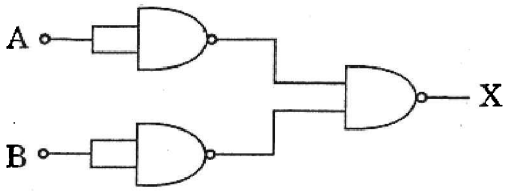
(3) $-\dfrac{q}{4\pi^2\varepsilon_0 r^2}\hat{j}$

(4) $-\dfrac{q}{2\pi^2\varepsilon_0 r^2}\hat{j}$

12. A diatomic ideal gas is used in a Carnot engine as the working substance. If during the adiabatic expansion part of the cycle the volume of the gas increases from V to 32 V, the efficiency of the engine is
(1) 0.25
(2) 0.5
(3) 0.75
(4) 0.99

13. The respective number of significant figures for the numbers 23.023, 0.0003 and 2.1×10^{-3} are
(1) 4, 4, 2
(2) 5, 1, 2
(3) 5, 1, 5
(4) 5, 5, 2

14. The combination of gates shown below yields



(1) NAND gate
(1) OR gate
(1) NOT gate
(1) XOR gate

15. If a source of power 4 kW produces 10^{20} photons/second, the radiation belongs to a part of the spectrum called
(1) γ-rays
(2) X-rays
(3) ultraviolet rays
(4) microwaves

16. A radioactive nucleus (initial mass number A and atomic number Z) emits 3 α-particles and 2 positrons. The ratio of number of neutrons to that of protons in the final nucleus will be

(1) $\frac{A-Z-4}{Z-2}$

(2) $\frac{A-Z-8}{Z-4}$

(3) $\frac{A-Z-4}{Z-8}$

(4) $\frac{A-Z-12}{Z-4}$

17. Let there be a spherically symmetric charge distribution with charge density varying as $\rho(r) = \rho_o\left(\frac{5}{4}-\frac{r}{R}\right)$ upto r = R, and $\rho(r) = 0$ for r > R, where r is the distance from the origin. The electric field at a distance r(r < R) from the origin is given by

(1) $\frac{\rho_o r}{3\varepsilon_o}\left(\frac{5}{4}-\frac{r}{R}\right)$

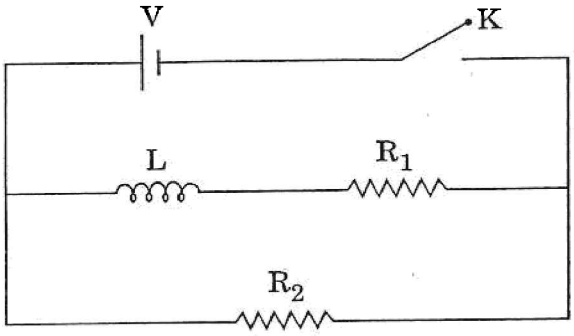
(2) $\frac{4\pi\rho_o r}{3\varepsilon_o}\left(\frac{5}{3}-\frac{r}{R}\right)$

(3) $\frac{\rho_o r}{4\varepsilon_o}\left(\frac{5}{3}-\frac{r}{R}\right)$

(4) $\frac{4\rho_o r}{3\varepsilon_o}\left(\frac{5}{3}-\frac{r}{R}\right)$

18. In a series LCR circuit R = 200 Ω and the voltage and the frequency of the main supply is 220 V and 50 Hz respectively. On taking out the capacitance from the circuit the current lags behind the voltage by 30°. On taking out the inductor from the circuit the current leads the voltage by 30°. The power dissipated in the LCR circuit is
(1) 242 W
(2) 305 W
(3) 210 W
(4) Zero W

19. In the circuit shown below, the key K is closed at t = 0. The current through the battery is



(1) $\frac{V(R_1+R_2)}{R_1R_2}$ at t = 0 and $\frac{V}{R_2}$ at t = ∞

(2) $\frac{VR_1R_2}{\sqrt{R_1^2+R_2^2}}$ at t = 0 and $\frac{V}{R_2}$ at t = ∞

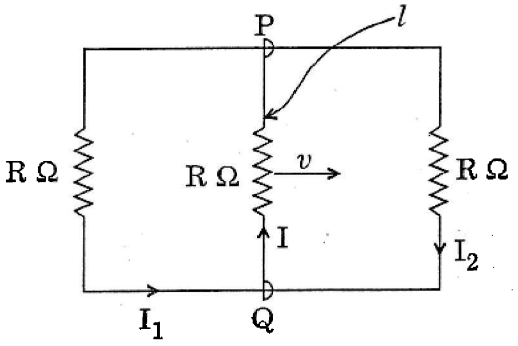
(3) $\frac{V}{R_2}$ at t = 0 and $\frac{V(R_1+R_2)}{R_1R_2}$ at t = ∞

(4) $\frac{V}{R_2}$ at t = 0 and $\frac{VR_1R_2}{\sqrt{R_1^2+R_2^2}}$ at t = ∞

20. A particle is moving with velocity $\vec{v} = K(y\hat{i}+\hat{j})$, where K is a constant. The general equation for its path is
    (1) $y^2 = x^2 +$ constant
    (2) $y = x^2 +$ constant
    (3) $y^2 = x +$ constant
    (4) $yx =$ constant

21. Let C be the capacitance of a capacitor discharging through a resistor R. Suppose t_1 is the time taken for the energy stored in the capacitor to reduce to half its initial value and t_2 is the time taken for the charge to reduce to one-fourth its initial value. Then the ratio t_1/t_2 will be
    (1) 2
    (2) 1
    (3) $\frac{1}{2}$
    (4) $\frac{1}{4}$

22. A rectangular loop has a sliding connector PQ of length l and resistance R Ω and it is moving with a speed v as shown. The set-up is placed in a uniform magnetic field going into the plane of the paper. The three currents I_1, I_2 and I are



(1) $I_1 = I_2 = \frac{Blv}{6R}, \quad I = \frac{Blv}{3R}$

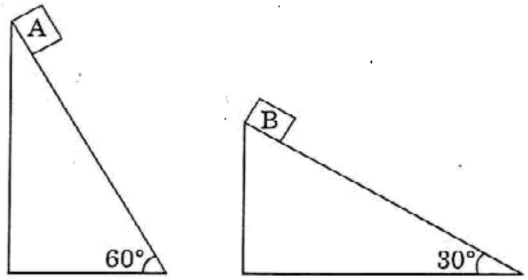
(2) $I_1 = -I_2 = \frac{Blv}{R}, \quad I = \frac{2Blv}{R}$

(3) $I_1 = I_2 = \frac{Blv}{3R}, \quad I = \frac{2Blv}{3R}$

(4) $I_1 = I_2 = I = \frac{Blv}{R}$

23. The equation of a wave on a string of linear mass density 0.04 kg m^{-1} is given by $y = 0.02(\text{m}) \sin\left[2\pi\left(\frac{t}{0.04(\text{s})} - \frac{x}{0.50\text{m}}\right)\right]$. The tension in the string is
    (1) 6.25 N
    (2) 4.0 N
    (3) 12.5 N
    (4) 0.5 N

24. Two fixed frictionless inclined planes making an angle 30° and 60° with the vertical are shown in the figure. Two blocks A and B are placed on the two planes. What is the relative vertical acceleration of A with respect to B?



(1) 4.9 ms^{-2} in vertical direction
(2) 4.9 ms^{-2} in horizontal direction
(3) 9.8 ms^{-2} in vertical direction
(4) Zero

25. For a particle in uniform circular motion, the acceleration $\vec{a}$ at a point P (R, θ) on the circle of radius R is (Here θ is measured from the x-axis)

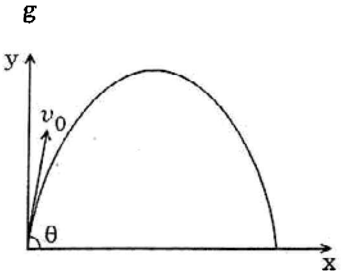
(1) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$

(2) $-\frac{v^2}{R}\cos\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$

(3) $-\frac{v^2}{R}\sin\theta\,\hat{i} + \frac{v^2}{R}\cos\theta\,\hat{j}$

(4) $-\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$

26. A small particle of mass m is projected at an angle 0 with the x-axis with an initial velocity v_0 in the x-y plane as shown in the figure. At a time $t < \frac{v_0 \sin\theta}{g}$, the angular momentum of the particle is



(1) $\frac{1}{2}$ mg v_0 t^2 cos $\theta\hat{i}$

(2) $-$ mg v_0 t^2 cos $\theta\hat{j}$

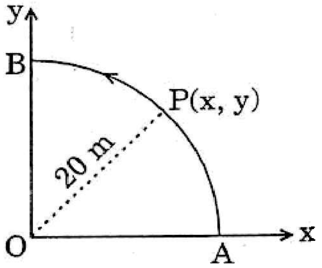
(3) mg v_0 t cos $\theta\hat{k}$

(4) $-\frac{1}{2}$ mg v_0 t^2 cos $\theta\hat{k}$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y and z-axis respectively.

27. Two identical charged spheres are suspended by strings of equal lengths. The strings make an angle of 30° with each other. When suspended in a liquid of density 0.8 g cm^{-3}, the angle remains the same. If density of the material of the sphere is 1.6 g cm^{-3}, the dielectric constant of the liquid is
(1) 1
(2) 4
(3) 3
(4) 2

28. A point P moves in counter-clockwise direction on a circular path as shown in the figure. The movement of 'P' is such that it sweeps out a length $s = t^3 + 5$, where s is in metres and t is in seconds. The radius of the path is 20 m. The acceleration of P when t = 2 s is nearly



(1) 14 m/s^2
(2) 13 m/s^2
(3) 12 m/s^2
(4) 7.2 m/s^2

29. The potential energy function for the force between two atoms in a diatomic molecule is approximately give by $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are constants and x is the distance between the atoms. If the dissociation energy of the molecule is $D = [U(x = \infty) - U_{\text{at equilibrium}}]$, D is

(1) $\frac{b^2}{6a}$

(2) $\frac{b^2}{2a}$

(3) $\frac{b^2}{12a}$

(4) $\frac{b^2}{4a}$

30. Two conductors have the same resistance at 0°C but their temperature coefficients of resistance are α_1 and α_2. The respective temperature coefficients of their series and parallel combinations are nearly

(1) $\frac{\alpha_1 + \alpha_2}{2}$, $\frac{\alpha_1 + \alpha_2}{2}$

(2) $\frac{\alpha_1 + \alpha_2}{2}$, $\alpha_1 + \alpha_2$

(3) $\alpha_1 + \alpha_2$, $\frac{\alpha_1 + \alpha_2}{2}$

(4) $\alpha_1 + \alpha_2$, $\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$

ANSWERS

1. (1)	**2.** (2)	**3.** (2)	**4.** (3)	**5.** (4)	**6.** (1)	**7** (2)	**8.** (3)	**9.** (2)	**10.** (3)
11. (4)	**12.** (3)	**13.** (2)	**14.** (4)	**15.** (2)	**16.** (2)	**17.** (3)	**18.** (1)	**19.** (3)	**20.** (1)
21. (4)	**22.** (3)	**23.** (1)	**24.** (2)	**25.** (4)	**26.** (3)	**27.** (1)	**28.** (1)	**29.** (4)	**30.** (1)

CHEMISTRY

31. The standard enthalpy of formation of NH_3 is – 46.0 kJ mol^{-1}. If the enthalpy of formation of H_2 from its atoms is –436 kJ mol^{-1} and that of N_2 is –712 kJ mol^{-1}, the average bond enthalpy of N – H bond in NH_3 is
    (A) – 964 kJ mol^{-1} (B) +352 kJ mol^{-1}
    (C) + 1056 kJ mol^{-1} (D) – 1102 kJ mol^{-1}

32. For a particular reversible reaction at temperature T, ΔH and ΔS were found to be both +ve. If T_e is the temperature at equilibrium, the reaction would be spontaneous when
    (A) $T_e > T$ (B) $T > T_e$
    (C) T_e is 5 times T (D) $T = T_e$

33. The time for half life period of a certain reaction A → products is 1 hour. When the initial concentration of the reactant 'A', is 2.0 mol L^{-1}, how much time does it take for its concentration to come from 0.50 to 0.25 mol L^{-1} if it is a zero order reaction ?
    (A) 4 h (B) 0.5 h
    (C) 0.25 h (D) 1 h

34. Consider the reaction :

$$Cl_2\,(aq) + H_2S\,(aq) \rightarrow S\,(s) + 2H^+\,(aq) + 2Cl^-\,(aq)$$

The rate equation for this reaction is

$$\text{Rate} = k[Cl_2][H_2S]$$

Which of these mechanisms is/are consistent with this rate equation ?

(1) $Cl_2 + H_2S \longrightarrow H^+ + Cl^- + Cl^+ + HS^-$ (slow)
$Cl^+ + HS^- \longrightarrow H^+ + Cl^- + S$ (fast)

(2) $H_2S \Leftrightarrow H^+ + HS^-$ (fast equilibrium)
$Cl_2 + HS^- \longrightarrow 2Cl^- + H^+ + S$ (slow)
(A) 2 only (B) Both 1 and 2
(C) Neither 1 nor 2 (D) 1 only

35. If 10^{-4} dm^3 of water is introduced into 1.0 dm^3 flask at 300 K, how many moles of water are in the vapour phase when equilibrium is established? (Given: Vapour pressure of H_2O at 300 K is 3170 Pa; R = 8.314 JK^{-1} mol^{-1})
    (A) 5.56×10^{-3} mol (B) 1.53×10^{-2} mol
    (C) 4.46×10^{-2} mol (D) 1.27×10^{-3} mol

36. If sodium sulphate is considered to be completely dissociated into cations and anions in aqueous solution, the change in freezing point of water (ΔT_f), when 0.01 mol of sodium sulphate is dissolved in 1 Kg of water, is (K_f = 1.86 K kg mol^{-1})
    (A) 0.0372 K (B) 0.0558 K
    (C) 0.0744 K (D) 0.0186 K

37. On mixing, heptane and octane form an ideal solution. At 373 K, the vapour pressures of the two liquid components (heptane and octane) are 105 kPa and 45 kPa respectively. Vapour pressure of the solution obtained by mixing 25.0 g of heptane and 35 g of octane will be (molar mass of heptane = 100 g mol^{-1} and of octane = 114 g mol^{-1})
    (A) 72.0 kPa (B) 36.1 kPa
    (C) 96.2 kPa (D) 144.5 kPa

38. The energy required to break one mole of Cl – Cl bonds in Cl_2 is 242 kJ mol^{-1}. The longest wavelength of light capable of breaking a single Cl – Cl bond is (c = 3×10^8 ms^{-1} and $N_A = 6.02 \times 10^{23}$ mol^{-1})
    (A) 594 nm (B) 640 nm
    (C) 700 nm (D) 494 nm

39. Ionization energy of He^+ is 19.6×10^{-18} J atom^{-1}. The energy of the first stationary state (n =1) of Li^{2+} is
    (A) 4.41×10^{-16} J atom^{-1}
    (B) -4.41×10^{-17} J atom^{-1}
    (C) -2.2×10^{-15} J atom^{-1}
    (D) 8.82×10^{-17} J atom^{-1}

40. Three reactions involving $H_2PO_4^-$ are given below:-

(i) $H_3PO_4 + H_2O \longrightarrow H_3O^+ + H_2PO_4^-$
(ii) $H_2PO_4^- + H_2O \longrightarrow HPO_4^{2-} + H_3O^+$
(iii) $H_2PO_4^- + OH^- \longrightarrow H_3PO_4 + O^{2-}$

In which of the above does $H_2PO_4^-$ act as an acid ?
(A) (ii) only (B) (i) and (ii)
(C) (iii) only (D) (i) only

41. In aqueous solution the ionization constants for carbonic acid are

$$K_f = 4.2 \times 10^{-7} \text{ and } K_2 = 4.8 \times 10^{-11}$$

Select the correct statement for a saturated for carbonic acid are
(A) The concentration of CO_3^{2-} is 0.034 M.
(B) The concentration of CO_3^{2-} is greater than that of HCO_3^-.
(C) The concentrations of H^+ and HCO_3^- are approximately equal.
(D) The concentration of H^+ and HCO_3^- are approximately equal.

42. Solubility product of silver bromide is 5.0×10^{-13}. The quantity of potassium bromide (molar mass taken as 120 g mol^{-1}) to be added to 1 litre of 0.05 M solution nitrate to start the precipitation of AgBr is
    (A) 1.2×10^{-10} g (B) 1.2×10^{-9} g
    (C) 6.2×10^{-5} g (D) 5.0×10^{-8} g

43. At 25°C, the solubility product of $Mg(OH)_2$ is 1.0×10^{-11}. At which pH, will Mg^{2+} ions start precipitating in the form of $Mg(OH)_2$ from a solution of 0.001 M Mg^{2+} ions ?

(A) 9	(B) 10
(C) 11	(D) 8

44. The edge length of a face centred cubic cell of an ionic substance is 508 pm. If the radius of the cation is 10 pm, the radius of the anion is

(A) 288 pm	(B) 398 pm
(C) 618 pm	(D) 144 pm

45. Percentage of free space in cubic close packed structure and in body centered packed structure are respectively
(A) 30 % and 26 %
(B) 26 % and 32 %
(C) 32 % and 48 %
(D) 48 % and 26 %

46. The Gibbs energy for the decomposition of Al_2O_3 at 500°C is as follows:

$$\frac{2}{3}Al_2O_3 \rightarrow \frac{4}{3}\ Al + O_2, \Delta_r G = +966 \text{ kJ mol}^{-1}$$

The potential difference needed for electrolytic reduction of Al_2O_3 at 500°C is at least

(A) 5.0 V	(B) 4.5 V
(C) 3.0 V	(D) 2.5 V

47. The correct order of $E^{o}_{M^{2+}/M}$ values with negative sign for the four successive elements Cr, Mn, Fe and Co is
(A) Mn > Cr > Fe > Co
(B) Cr > Fe > Mn > Co
(C) Fe > Mn > Cr > Co
(D) Cr > Mn > Fe > Co

Inorganic Chemistry

48. A solution containing 2.675 g of $CoCl_3.6NH_3$ (molar mass = 267.5 g mol^{-1}) is passed through a cation exchanger. The chloride ions obtained in solution were treated with excess of $AgNO_3$ to give 4.78 g of AgCl (molar mass = 143.5 g mol^{-1}). The formula of the complex is (Atomic mass of Ag = 108 u)
(A) $[Co(NH_3)_6]Cl_3$
(B) $[CoCl_2(NH_3)_4]Cl$
(C) $[CoCl_3(NH_3)_3]$
(D) $[CoCl(NH_3)_5]Cl_2$

49. Which one of the following has an optical isomer ?
(A) $[Zn(en)(NH_3)_2]^{2+}$
(B) $[Co(en)_3]^{3+}$
(C) $[Co(H_2O)_4(en)]^{3+}$
(D) $[Zn(en)_2]^{2+}$
(en = ethylenediamine)

50. The correct sequence which shows decreasing order of the ionic radii of the elements is
(A) $Al^{3+} > Mg^{2+} > Na^{+} > F^{-} > O^{2-}$
(B) $Na^{+} > Mg^{2+} > Al^{3+} > O^{2-} > F^{-}$
(C) $Na^{+} > F^{-} > Mg^{2+} > O^{2-} > Al^{3+}$
(D) $O^{2-} > F^{-} > Na^{+} > Mg^{2+} > Al^{3+}$

Organic Chemistry

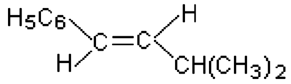
51. One mole of a symmetrical alkene on ozonolysis gives two moles of an aldehyde having a molecular mass of 44u. The alkene is

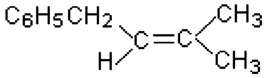
(A) Propene	(B) 1- butene
(C) 2- butane	(D) ethane

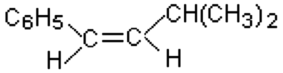
52. From amongst the following alcohols the one that would react fastest with conc. HCl and anhydrous $ZnCl_2$, is
(A) 2- Butanol
(B) 2- Methylpropan -2-ol
(C) 2- Methylpropanol
(D) 1- Butanol

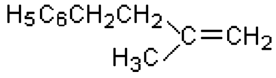
53. The main product of the following reaction is

$$C_6H_5CH_2CH(OH)CH(CH_3)_2 \xrightarrow{conc.\ H_2SO_4}$$

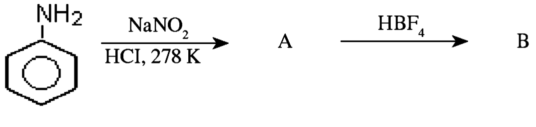
(A)


(B)


(C)


(D)


54. In the chemical reactions.



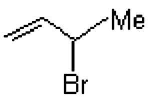
The compounds 'A' and 'B' respectively are
(A) nitrobenzene and fluorobenzene
(B) phenol and benzene
(C) benzene diazonium chloride and fluorobenzene
(D) nitrobenzene and chlorobenzene

55. Burst test is not given by
(A) carbohydrates
(B) polypeptides
(C) urea
(D) Proteins

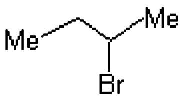
56. 29.5 mg of an organic compound containing nitrogen was digested according to Kjeldahl's method and the evolved ammonia was absorbed in 20 mL of 0.1 M HCl solution. The excess of the acid required 15 mL of 0.1 M NaOH solution for complete neutralization. The percentage of nitrogen in the compound is
(A) 59.0 (B) 47.4
(C) 23.7 (D) 29.5

57. Consider the following bromides:

(I)



(II)



(III)

The correct order of S_N1 reactivity is
(A) II > III > I (B) II > I > III
(C) III > II > I (D) I > II > III

58. Out of the following, the alkene that exhibits optical isomerism is
(A) 3- methyl -2- pentene
(B) 4- methyl -1- pentene
(C) 3- methyl -1- pentene
(D) 2- methyl -2- pentene

69. The correct order of increasing basicity of the given conjugate bases ($R = CH_3$) is

(A) $RCO\bar{O} < HC\equiv\bar{C} < \bar{R} < \bar{N}H_2$

(B) $\bar{R} < HC\equiv\bar{C} < RCO\bar{O} < \bar{N}H_2$

(C) $RCO\bar{O} < \bar{N}H_2 < HC\equiv\bar{C} < \bar{R}$

(D) $RCO\bar{O} < HC\equiv\bar{C} < \bar{N}H_2 < \bar{R}$

60. The polymer containing strong intermolecular forces e.g. hydrogen bonding is
(A) Teflon (B) nylon 6,6
(C) polystyrene (D) natural rubber

ANSWERS

31. (B)	**32.** (B)	**33.** (C)	**34.** (D)	**35.** (D)	**36.** (B)	**37** (A)	**38.** (D)	**39.** (B)	**40.** (A)
41. (C)	**42.** (B)	**43.** (B)	**44.** (D)	**45.** (B)	**46.** (D)	**47.** (A)	**48.** (A)	**49.** (B)	**50.** (D)
51. (C)	**52.** (B)	**53.** (A)	**54.** (C)	**55.** (A)	**56.** (C)	**57.** (A)	**58.** (C)	**59.** (D)	**60.** (B)

MATHEMATICS

1. Consider the following relations
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$; $S = \{(\frac{m}{n}, \frac{p}{q}) \mid m, n, p$ and q are integers such that $n, q \neq 0$ and $qm = pn\}$. Then
   (1) R is an equivalence relation but S is not an equivalence relation
   (2) Neither R nor S is an equivalence relation
   (3) S is an equivalence relation but R is not an equivalence relation
   (4) R and S both are equivalence relations

2. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals
   (1) 0
   (2) 1

3. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$
   (1) -2
   (2) -1
   (3) 1
   (4) 2

4. Consider the system of linear equations:
$x_1 + 2x_2 + x_3 = 3$
$2x_1 + 3x_2 + x_3 = 3$
$3x_1 + 5x_2 + x_3 = 1$
The system has
   (1) Infinite number of solutions
   (2) Exactly 3 solutions
   (3) a unique solution
   (4) No solution

5. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is
   (1) 3
   (2) 36
   (3) 66
   (4) 108

6. Let f: $(-1, 1) \rightarrow$ R be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$,. Then $g'(0) =$
   (1) 4
   (2) -4
   (3) 0
   (4) -2

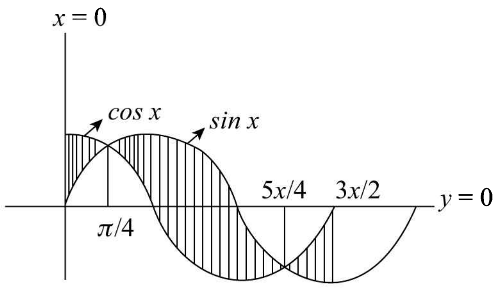
7. Let f: R $\rightarrow$ R be a positive increasing function with $\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty} \frac{f(2x)}{f(x)} =$
   (1) 1
   (2) $\frac{2}{3}$
   (3) $\frac{3}{2}$
   (4) 3

8. Let p(x) be a function defined on R such that $p'(x) = p'(1 - x)$ for all $x \in [0, 1]$, p(0) and p(1) $= 41$. Then $\int_0^1 p(x)ds$ equals
   (1) $\sqrt{41}$
   (2) 21
   (3) 41
   (4) 42

9. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = ... = a_{10} = 150$ and $a_{10}, a_{11}, ...$ are in an AP with common difference -2, then the time taken by him to count all notes is
   (1) 24 minutes
   (2) 34 minutes
   (3) 125 minutes
   (4) 135 minutes

10. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x– axis, is
   (1) $y = 0$
   (2) $y = 1$
   (3) $y = 2$
   (4) $y = 3$

11. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is
   (1) $4\sqrt{2} - 2$
   (2) $4\sqrt{2} + 2$
   (3) $4\sqrt{2} - 1$
   (4) $4\sqrt{2} + 1$



12. Solution of the differential equation cos x dy = y (sin x − y)dx, $0 < x < \frac{\pi}{2}$ is
(1) sec x = (tan x + c)y
(2) y sec x = tan x + c
(3) y tan x = sec x + c
(4) tan x = (sec x + c)y

13. Let $\vec{a} = \hat{j} - \hat{k}$ *and* $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}.\vec{b} = 3$ is
(1) $-\hat{i} + \hat{j} - 2\hat{k}$
(2) $2\hat{i} - \hat{j} + 2\hat{k}$
(3) $\hat{i} - \hat{j} - 2\hat{k}$
(4) $\hat{i} + \hat{j} - 2\hat{k}$

14. If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then $(\lambda, \mu) =$
(1) (−3, 2)
(2) (2, −3)
(3) (−2, 3)
(4) (3, −2)

15. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is
(1) x = 1
(2) 2x + 1 = 0
(3) x = −1
(4) 2x − 1 = 0

16. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is
(1) $\frac{23}{\sqrt{15}}$
(2) $\sqrt{17}$
(3) $\frac{17}{\sqrt{15}}$
(4) $\frac{23}{\sqrt{17}}$

17. A line AB in three dimensional space makes angles 45° and 120° with the positive x − axis and the positive y − axis respectively. If AB makes an acute angle θ with the positive z − axis, then θ equals
(1) 30°
(2) 45°
(3) 60°
(4) 75°

18. Let S be a non-empty subset of R. Consider the following statement:
P: There is a rational number x ∈ S such that x > 0
Which of the following statements is the negation of the statement P?
(1) There is a rational number x ∈ S such that x ≤ 0.
(2) There is no rational number x ∈ S such that x ≤ 0.
(3) Every rational number x ∈ S satisfies x ≤ 0.
(4) x ∈ S and x ≤ 0 ⇒ x is not rational

19. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and let $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then $\tan 2\alpha =$
(1) $\frac{25}{16}$
(2) $\frac{56}{33}$
(3) $\frac{19}{12}$
(4) $\frac{20}{7}$

20. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line 3x − 4y = m at two distinct points if
(1) −85 < m < −35
(2) −35 < m < 15
(3) 15 < m < 65
(4) 35 < m < 85

21. For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is
(1) $\frac{5}{2}$
(2) $\frac{11}{2}$
(3) 6
(4) $\frac{13}{2}$

22. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours is
(1) $\frac{1}{3}$
(2) $\frac{2}{7}$
(3) $\frac{1}{21}$
(4) $\frac{2}{23}$

23. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is
(1) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
(2) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
(3) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
(4) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

24. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is
    (1) less than 4
    (2) 5
    (3) 6
    (4) at least 7

25. Let $f : R \rightarrow R$ be defined by $f(x) = \begin{cases} k - 2x, & if\ x \leq -1 \\ 2x + 3, & if\ x > -1 \end{cases}$ If f has a local minimum at $x = -1$, then a possible value of k is
    (1) 1
    (2) 0
    (3) $-\frac{1}{2}$
    (4) -1

26. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \ldots\ldots, 20\}$.
    Statement - 1:
    The probability that the chosen numbers when arranged in some order will form an AP is $\frac{1}{85}$.
    Statement - 2:
    If the four chosen numbers form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$
    (1) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
    (2) Statement -1 is true, Statement -2 is true; Statement -2 is ***not*** a correct explanation for Statement -1.
    (3) Statement -1 is true, Statement -2 is false.
    (4) Statement -1 is false, Statement -2 is true.

27. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$
    Statement - 1: $S_3 = 55 \times 2^9$.
    Statement - 2: $S_1 = 90 \times 2^8$. and $S_2 = 10 \times 2^8$
    (1) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
    (2) Statement -1 is true, Statement -2 is true; Statement -2 is ***not*** a correct explanation for Statement -1.
    (3) Statement -1 is true, Statement -2 is false.
    (4) Statement -1 is false, Statement -2 is ture.

28. **Statement - 1:** The point A(3, 1, 6) is the mirror image of the point B(1, 3, 4) in the plane $x - y + z = 5$.
    Statement - 2: The plane $x - y + z = 5$ bisects the line segment joining A(3, 1, 6) and B(1, 3, 4).
    (1) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
    (2) Statement -1 is true, Statement -2 is true; Statement -2 is ***not*** a correct explanation for Statement -1.
    (3) Statement -1 is true, Statement -2 is false.
    (4) Statement -1 is false, Statement -2 is true.

29. Let $f : R \rightarrow R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$
    Statement - 1: $f(c) = \frac{1}{3}$, for some $c \in R$.
    Statement - 2: $0 < f(x) \leq \frac{1}{2\sqrt{2}}$, for all $x \in R$.
    (1) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
    (2) Statement -1 is true, Statement -2 is true; Statement -2 is ***not*** a correct explanation for Statement -1.
    (3) Statement -1 is true, Statement -2 is false.
    (4) Statement -1 is false, Statement -2 is true.

30. Let A be a 2×2 matrix with non zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define Tr(A) = sum of diagonal elements of A and $|A|$ = determinant of matrix A.
    Statement - 1: $Tr(A) = 0$
    Statement - 2: $|A| = 1$
    (1) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
    (2) Statement -1 is true, Statement -2 is true; Statement -2 is ***not*** a correct explanation for Statement -1.
    (3) Statement -1 is true, Statement -2 is false.
    (4) Statement -1 is false, Statement -2 is true.

ANSWERS

1. (3)	**2.** (2)	**3.** (3)	**4.** (4)	**5.** (4)	**6.** (2)	**7** (1)	**8.** (2)	**9.** (2)	**10.** (4)
11. (1)	**12.** (1)	**13.** (1)	**14.** (1)	**15.** (3)	**16.** (4)	**17.** (3)	**18.** (3)	**19.** (2)	**20.** (2)
21. (2)	**22.** (2)	**23.** (3)	**24.** (4)	**25.** (4)	**26.** (3)	**27.** (3)	**28.** (1)	**29.** (1)	**30.** (3)

2011 AIEEE Paper

PART I

PHYSICS

1. The transverse displacement $y\ (x, t)$ of a wave on a string is given by $y\ (x, t) = e^{-\left(ax^2 + bt^2 + 2\sqrt{ab}xt\right)}$. This represents a
   (a) wave moving in $-x$ direction with speed $\sqrt{\frac{b}{a}}$
   (b) standing wave of frequency $\sqrt{b}$
   (c) standing wave of frequency $\frac{1}{\sqrt{b}}$
   (d) wave moving in $+x$ direction with $\sqrt{\frac{a}{b}}$

2. A screw gauge gives the following reading when used to measure the diameter of a wire.
   Main scale reading : 0 mm
   Circular scale reading : 52 divisions
   Given that 1 mm on main scale corresponds to 100 divisions of the circular scale. The diameter of wire from the above date is:
   (a) 0.052 cm (b) 0.026 cm
   (c) 0.005 cm (d) 0.52 cm

3. A mass m hangs with the help of a string wrapped around a pulley on a frictionless bearing. The pulley has mass m and radius R. Assuming pulley to be a perfect uniform circular disc, the acceleration of the mass m, if the string does not slip on the pulley, is
   (a) g (b) $\frac{2}{3}g$
   (c) $\frac{g}{3}$ (d) $\frac{3}{2}g$

4. Work done in increasing the size of a soap bubble from a radius of 3 cm to 5 cm is nearly (Surface tension of soap solution = 0.03 Nm^{-1}):
   (a) 0.2π mJ (b) 2π mJ
   (c) 0.4π mJ (d) 4π mJ

5. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect is at rest at a point near the rim of the disc. The insect now moves along a diameter of the disc to reach its other end. During the journey of the insect, the angular speed of the disc:
   (a) continuously decreases
   (b) continuously increases
   (c) first increases and then decreases
   (d) remains unchanged

6. Two particles are executing simple harmonic motion of the same amplitude A and frequency ω along the x-axis. Their mean position is separated by distance X_0 ($X_0 > A$). If the maximum separation between them is $(X_0 + A)$, the phase difference between their motion is:
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
   (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

7. Two bodies of masses m and 4 m are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is:
   (a) $-\frac{4Gm}{r}$ (b) $-\frac{6Gm}{r}$
   (c) $-\frac{9Gm}{r}$ (d) zero

8. Two identical charged spheres suspended from a common point by two massless strings of length l are initially a distance d ($d << 1$) apart because of their mutual repulsion. The charge begins to leak from both the spheres at a constant rate. As a result the charges approach each other with a velocity v. Then as a function of distance x between them,
   (a) $v \mu x^{-1}$ (b) $v \mu x^{1/2}$
   (c) $v \mu x$ (d) $v \mu x^{-1/2}$

9. A boat is moving due east in a region where the earth's magnetic field is $5.0 \times 10^{-5}\ NA^{-1}m^{-1}$ due north and horizontal. The boat carries a vertical aerial 2m long. If the speed of the boat is 1.50 ms^{-1}, the magnitude of the induced emf in the wire of aerial is:
   (a) 0.75mV (b) 0.50mV
   (c) 0.15mV (d) 1 mV

10. An object, moving with a speed of 6.25 m/s, is decelerated at a rate given by:
    $$\frac{dv}{dt} = -2.5\sqrt{v}$$
    where v is the instantaneous speed. The time taken by the object, to come to rest, would be:
    (a) 2 s (b) 4 s
    (c) 8 s (d) 1 s

11. A fully charged capacitor C with initial charge q_0 is connected to a coil of self inductance L at $t = 0$. The time at which the energy is stored equally between the electric and the magnetic field is:

(a) $\frac{\pi}{4}\sqrt{LC}$ (b) $2\pi\sqrt{LC}$

(c) $\sqrt{LC}$ (d) $\pi\sqrt{LC}$

12. Let the $x - z$ plane be the boundary between two transparent media. Medium 1 in $z \geq 0$ has a refractive index of $\sqrt{2}$ and medium 2 with $z < 0$ has a refractive index of $\sqrt{3}$. A ray of light in medium 1 given by the vector $\vec{A} = 6\sqrt{3}\hat{i} + 8\sqrt{3}\hat{j} - 10\hat{k}$ is incident on the plane of separation. The angle of refraction in medium 2 is

(a) 45° (b) 60°

(c) 75° (d) 30°

13. A current I flows in an infinitely long wire with cross section in the form of a semicircular ring of radius R. The magnitude of the magnetic induction along its axis is

(a) $\frac{\mu_0 I}{2\pi^2 R}$ (b) $\frac{\mu_0 I}{2\pi R}$

(c) $\frac{\mu_0 I}{4\pi^2 R}$ (d) $\frac{\mu_0 I}{\pi^2 R}$

14. A thermally insulated vessel contains an ideal gas of molecular mass M and ratio of specific heats γ. It is moving with speed υ and is suddenly brought to rest. Assuming no heat is lost to the surroundings, its temperature increases by:

(a) $\frac{(\gamma - 1)}{2\gamma R} M\upsilon^2 K$ (b) $\frac{\gamma M v^2}{2R} K$

(c) $\frac{(\gamma - 1)}{2R} M\upsilon^2 K$ (d) $\frac{(\gamma - 1)}{2(\gamma + 1)R} M\upsilon^2 K$

15. A mass M, attached to a horizontal spring, executes S.H.M. with amplitude A_1. when the mass M passes through its mean position then a smaller mass m is placed over it and both of them move together with amplitude A_2. The ratio of $\left(\frac{A_1}{A_2}\right)$ is:

(a) $\frac{M + m}{M}$ (b) $\left(\frac{M}{M + m}\right)^{1/2}$

(c) $\left(\frac{M + m}{M}\right)^{1/2}$ (d) $\frac{M}{M + m}$

16. Water is flowing continuously from a tap having an internal diameter 8×10^{-3}m. The water velocity as it leaves the tap is 0.4 ms^{-1}. The diameter of the water stream at a distance 2×10^{-1}m below the lap is close to:

(a) 7.5×10^{-3}m (b) 9.6×10^{-3}m

(c) 3.6×10^{-3}m (d) 5.0×10^{-3}m

17. This question has Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement-1: Sky wave signals are used for long distance radio communication. These signals are in general, less stable than ground wave signals.

Statement-2: The state of ionosphere varies from hour to hour, day to day and season to season.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation of Statement-1.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is false.

18. Three perfect gases at absolute temperatures T_1, T_2 and T_3 are mixed. The masses of molecules are m_1, m_2 and m_3 and the number of molecules are n_1, n_2 and n_3 respectively. Assuming no loss of energy, the final temperature of the mixture is:

(a) $\frac{n_1T_1 + n_2T_2 + n_3T_3}{n_1 + n_2 + n_3}$ (b) $\frac{n_1T_1 + n_2T_2^2 + n_3T_3^2}{n_1T_1 + n_2T_2 + n_3T_3}$

(c) $\frac{n_1^2T_1^2 + n_2^2T_2^2 + n_3^2T_3^2}{n_1T_1 + n_2T_2 + n_3T_3}$ (d) $\frac{(T_1 + T_2 + T_3)}{3}$

19. A pulley of radius 2 m is rotated about its axis by a force $F = (20t - 5t^2)$ Newton (where t is measured in seconds) applied tangentially. If the moment of inertia of the pulley about its axis of rotation made by the pulley before its direction of motion if reversed, is:

(a) more than 3 but less than 6

(b) more than 6 but less than

(c) more than 9

(d) less than 3

20. A resistor 'R' and $2\mu F$ capacitor in series is connected through a switch to 200 V direct supply. Across the capacitor is a neon bulb that lights up at 120 V. Calculate the value of R to make the bulb light up 5 s after the switch has been closed. ($\log_{10} 2.5 = 0.4$)

(a) $1.7 \times 10^5\Omega$ (b) $2.7 \times 10^6\Omega$

(c) $3.3 \times 10^7\Omega$ (d) $1.3 \times 10^4\Omega$

21. A Carnot engine operating between temperatures T_1 and T_2 has efficiency $\frac{1}{6}$. When T_2 is lowered by 62 K, its efficiency increases to $\frac{1}{3}$. Then T_1 and T_2 are, respectively:

(a) 372 K and 330 K (b) 330 K and 268 K

(c) 310 K and 248 K (d) 372 K and 310 K

22. If a wire is stretched to make it 0.1% longer, its resistance will:

(a) increase by 0.2% (b) decrease by 0.2%

(c) decrease by 0.05% (d) increases by 0.05%

23. ***Direction***: The question has a paragraph followed by two statements, Statement-1 and Statement-2. Of the given four alternatives after the statements, choose the one that describes the statements.

A thin air film is formed by putting the convex surface of a plane – convex lens over a plane glass plate.

With monochromatic light, this film gives an interference pattern due to light reflected from the top (convex) surface and the bottom (glass plate) surface of the film.

Statement-1: When light reflects from the air-glass plate interface, the reflected wave suffers a phase change of π.

Statement-2: The centre of the interference pattern is dark.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation of Statement-1.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is false.

24. A car is fitted with a convex side-view mirror of focal length 20cm. A second car 2.8 m behind the first car is overtaking the first car at relative speed of 15 m/s. The speed of the image of the second car as seen in the mirror of the first one is:

(a) $\frac{1}{15}$ m/s (b) 10m/s
(c) 15m/s (d) $\frac{1}{10}$ m/s

25. Energy required for the electron excitation in Li^{++} from the first to the third Bohr orbit is:

(a) 36.3 eV (b) 108.8 eV
(c) 122.4 eV (d) 12.1 eV

26. The electrostatic potential inside a charged spherical ball is given by $\phi = a\rho^2 + b$ where r is the distance from the centre; a, b are constants. Then the charge density inside ball is

(a) $-6a\varepsilon_0 r$ (b) $-24\pi a\varepsilon_0 r$
(c) $-6a\varepsilon_0$ (d) $-24\pi a\varepsilon_0 r$

27. A water fountain on the ground sprinkles water all around it. If the speed of water coming out of the fountain is v, the total area around the fountain that gets wet is:

(a) $\pi\frac{v^4}{g^2}$ (b) $\frac{\pi}{2}\frac{v^4}{g^2}$
(c) $\pi\frac{v^2}{g^2}$ (d) $\pi\frac{v^4}{g}$

28. 100g of water is heated from 30°C to 50°C. Ignoring the slight expansion of the water, the change in its internal energy is (specific heat of water is 4148 J/kg/K):

(a) 8.4 kJ (b) 84 kJ
(c) 2.1 kJ (d) 4.2 kJ

29. The half life of a radioactive substance is 20 minutes. The approximate time interval $(t_2 - t_1)$ between the time t_2 when $\frac{2}{3}$ of it has decayed and time t_1 and $\frac{1}{3}$ of it had decayed is:

(a) 14 min (b) 20 min
(c) 28 min (d) 7 min

30. This question has Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement-1 : A metallic surface is irradiated by a monochromatic light of frequency $v > v_0$ (the threshold frequency). The maximum kinetic energy and the stopping potential are K_{max} and V_0 respectively. If the frequency incident on the surface doubled, both the K_{max} and V_0 are also doubled.

Statement-2: The maximum kinetic energy and the stopping potential of photoelectrons emitted from a surface are linearly dependent on the frequency of incident light.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation of Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is false.

ANSWERS

1. (a)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (d)	**7** (c)	**8.** (d)	**9.** (c)	**10.** (a)
11. (a)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (a)	**20.** (b)
21. (d)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (c)

MATHEMATICS

1. The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at Π and Q respectively. The bisector of the acute angle between L_1 and L_2 intersect L_3 at R.

Statement-1: The ratio $\Pi R : RQ$ equals $2\sqrt{2} : \sqrt{5}$.
Statement-2: In any triangle, bisector of an angle divides the triangle into two similar triangles.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

2. If $A = \sin^2 x + \cos^4 x$, then for all real x

(a) $\frac{13}{16} \le A \le 1$ (b) $1 \le A \le 2$
(c) $\frac{3}{4} \le A \le \frac{13}{16}$ (d) $\frac{3}{4} \le A \le 1$

3. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is

(a) –132 (b) –144
(c) 132 (d) 144

4. $\lim_{x \to 2}\left(\frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2}\right)$

(a) equals $\sqrt{2}$ (b) equals $-\sqrt{2}$
(c) equals $\frac{1}{\sqrt{2}}$ (d) does not exist

5. Statement-1: The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$

Statement-2: The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

6. $\frac{d^2x}{dy^2}$ equals

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

7. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then y (ln) is equal to

(a) 5 (b) 13
(c) –2 (d) 7

8. Let R be the set of real numbers

Statement-1: $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.
Statement-2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement– 2 is false.
(c) Statement-1 is false, Statement– 2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

9. The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{2}\log 2$
(c) log 2 (d) $\pi \log 2$

10. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $z = 1$, then it is necessary that

(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$

11. Consider 5 independent Bernoulli's trials each with probability of success π. If the probability of at least one failure is greater than or equal to $\frac{31}{32}$, then π lies in the interval

(a) $\left[\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$

(c) $\left[\frac{11}{12}, 1\right]$ (d) $\left[\frac{1}{2}, \frac{3}{4}\right]$

12. A man saves Rs. 200 in each of the first three months of his service. In each of the subsequent months his saving increases by Rs. 40 more than the saving of immediately previous month. His total saving from the start of service will be Rs. 11040 after

(a) 19 months (b) 20 months
(c) 21 months (d) 18 months

13. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is

(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty) - \{0\}$ (d) $(-\infty, \infty)$

14. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals

(a) $\frac{3}{2}$ (b) $\frac{2}{5}$
(c) $\frac{5}{3}$ (d) $\frac{2}{3}$

15. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of $(2\vec{a} - \vec{b}) \cdot \left[(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})\right]$ is

(a) –3 (b) 5
(c) 3 (d) –5

16. Equation of the ellipse whose axes are the axes of co-ordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is

(a) $5x^2 + 3y^2 - 48 = 0$ (b) $3x^2 + 5y^2 - 15 = 0$
(c) $5x^2 + 3y^2 - 32 = 0$ (d) $3x^2 + 5y^2 - 32 = 0$

17. Let l be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dv(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is

(a) $l - \frac{kT^2}{2}$ (b) $l - \frac{k(T-t)^2}{2}$
(c) e^{-kT} (d) $T^2 - \frac{l}{k}$

18. The vector $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying: $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to

(a) $\vec{c} + \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (b) $\vec{b} + \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$

(c) $\vec{c} - \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (d) $\vec{b} - \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$

19. The two circles $x^2 + y^2 = \alpha x$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if

(a) $|\alpha| = c$ (b) $\alpha = 2c$
(c) $|\alpha| = 2c$ (d) $2|\alpha| = c$

10. If C and D are two events such that $C \subset D$ and $\Pi(D) \neq 0$, then the correct statement among the following is

(a) $\Pi(C \mid D) \geq \Pi(C)$ (b) $\Pi(C \mid D) < \Pi(C)$

(c) $\Pi(C \mid D) = \frac{P(D)}{P(C)}$ (d) $\Pi(C \mid D) = \Pi(C)$

21. The number of values of k for which the linear equations
$4x + ky + 2z = 0$; $kx + 4y + z = 0$; $2x + 2y + z = 0$ possess a non-zero solution is
(a) 2 (b) 1
(c) zero (d) c

22. Consider the following statements

P : Suman is brilliant
Q : Suman is rich
R : Suman is honest

The negation of the statement "Suman is brilliant and dishonest if and only if Suman is rich" can be expressed as

(a) $\sim(Q \leftrightarrow (P \wedge \sim R))$ (b) $\sim Q \leftrightarrow \sim P \wedge R$
(c) $\sim(P \wedge \sim R) \leftrightarrow Q$ (d) $\sim P \wedge (Q \leftrightarrow \sim R)$

23. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$
(c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

24. If the mean deviation about the median of the numbers $a, 2a, \ldots, 50a$ is 50, then $|a|$ equals

(a) 3 (b) 4
(c) 5 (d) 2

25. **Statement-1:** The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Statement-2: The line: $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

26. Let A and B be two symmetric matrices of order 3.

Statement-1: $A(BA)$ and $(AB)A$ are symmetric matrices.
Statement-2: AB is symmetric matrix if matrix multiplication of A and B is commutative.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

27. If $\omega\ (\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals

(a) (1, 1) (b) (1, 0)
(c) (–1, 1) (d) (0, 1)

28. The value of π and q for which the function $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , \ x < 0 \\ q & , \ x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & , \ x > 0 \end{cases}$ is continuous for all x in R, is

(a) $\pi = \frac{5}{2}, q = \frac{1}{2}$ (b) $\pi = -\frac{3}{2}, q = \frac{1}{2}$
(c) $\pi = \frac{1}{2}, q = \frac{3}{2}$ (d) $\pi = \frac{1}{2}, q = -\frac{3}{2}$

29. The area of the region enclosed by the curves $y = x, x = e, y = \frac{1}{x}$ and the positive x-axis is

(a) 1 square units (b) $\frac{3}{2}$ square units
(c) $\frac{5}{2}$ square units (d) $\frac{1}{2}$ square units

30. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

ANSWERS

1. (b)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (c)	**7** (d)	**8.** (b)	**9.** (d)	**10.** (c)
11. (b)	**12.** (c)	**13.** (b)	**14.** (d)	**15.** (d)	**16.** (d)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (a)
21. (a)	**22.** (a)	**23.** (a)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (c)

CHEMISTRY

1. Among the following the maximum covalent character is shown by the compound:

(a) $SnCl_2$ (b) $AlCl_3$
(c) $MgCl_2$ (d) $FeCl_2$

2. The presence or absence of hydroxyl group on which carbon atom of sugar differentiates RNA and DNA?

(a) 2nd (b) 3rd
(c) 4th (d) 1st

3. Trichloroacetaldehyde was subjected to Cannizzaro's reaction by using NaOH. The mixture of the product contains sodium trichloroacetate and another compound. The other compound is:

(a) Trichloromethanol (b) 2, 2, 2-Trichloropropanol
(c) Chloroform (d) 2, 2, 2-Trichloroethanol

4. Sodium ethoxide has reacted with ethanoyl chloride. The compound that is produced in the above reaction is:

(a) 2-Butanone (b) Ethyl chloride
(c) Ethyl ethanoate (d) Diethyl ether

5. The reduction potential of hydrogen half cell will be negative if:

(a) $p(H_2) = 1$ atm and $[H^+] = 1.0$ M
(b) $p(H_2) = 2$ atm and $[H^+] = 1.0$ M
(c) $p(H_2) = 2$ atm and $[H^+] = 2.0$ M
(d) $p(H_2) = 1$ atm and $[H^+] = 2.0$ M

6. The strongest acid amongst the following compounds is:

(a) HCOOH
(b) $CH_3CH_2CH(Cl)CO_2H$
(c) $ClCH_2CH_2CH_2COOH$
(d) CH_3COOH

7. The degree of dissociation (α) of a weak electrolyte, A_xB_y is related to van't Hoff factor (i) by the expression:

(a) $\alpha = \dfrac{i-1}{x+y+1}$
(b) $\alpha = \dfrac{x+y-1}{i-1}$
(c) $\alpha = \dfrac{x+y+1}{i-1}$
(d) $\alpha = \dfrac{i-1}{(x+y-1)}$

8. '*a*' and '*b*' are van der Waals' constants for gases. Chlorine is more easily liquefied than ethane because

(a) a and b for $Cl_2 <$ a and b for C_2H_6
(b) a for $Cl_2 <$ a for C_2H_6 but b for $Cl_2 >$ b for C_2H_6
(c) a for $Cl_2 >$ a for C_2H_6 but b for $Cl_2 <$ b for C_2H_6
(d) a and b for $Cl_2 >$ a and b for C_2H_6

9. A vessel at 1000 K contains CO_2 with a pressure of 0.5 atm. Some of the CO_2 is converted into CO on the addition of graphite. If the total pressure at equilibrium is 0.8 atm, the value of K is

(a) 3 atm
(b) 0.3 atm
(c) 0.18 atm
(d) 1.8 atm

10. Boron cannot form which one of the following anions?

(a) BH_4^-
(b) $B(OH)_4^-$
(c) BO_2^-
(d) BF_6^{3-}

11. Which of the following facts about the complex $[Cr(NH_3)_6]Cl_3$ s wrong?

(a) The complex is paramagnetic
(b) The complex is an outer orbital complex
(c) The complex gives white precipitate with silver nitrate solution
(d) The complex involves d^2sp^3 hybridization and is octahedral in shape.

12. Ethylene glycol is used as an antifreeze in a cold climate. Mass of ethylene glycol which should be added to 4 kg of water to prevent it from freezing at –6°C will be:

[K_f for water = 1.86 K kg mol^{-1}, and molar mass of ethylene glycol = 62g mol^{-1})

(a) 204.30 g
(b) 400.00 g
(c) 304.60 g
(d) 804.32 g

13. Which one of the following order represents the correct sequence of the increasing basic nature of the given oxides?

(a) $MgO < K_2O < Al_2O_3 < Na_2O$
(b) $Na_2O < K_2O < MgO < Al_2O_3$
(c) $K_2O < Na_2O > Al_2O_3 < MgO$
(d) $Al_2O_3 < MgO < Na_2O < K_2O$

14. The rate of a chemical reaction doubles for every 10°C rise of temperature. If the temperature is raised by 50°C, the rate of the reaction increases by about:

(a) 24 times
(b) 32 times
(c) 64 times
(d) 10 times

15. The magnetic moment (spin only) of $[NiCl_4]^{2-}$ is

(a) 5.46 BM
(b) 2.83 BM
(c) 1.41 BM
(d) 1.82 BM

16. The hybridization of orbitals of N atom in NO_3^-, NO_2^+ and NH_4^+ are respectively:

(a) sp^2, sp, sp^3
(b) sp, sp^3, sp^2
(c) sp^2, sp^3, sp
(d) sp, sp^2, sp^3

17. In context of the lanthanoids, which of the following statements is not correct?

(a) All the members exhibit +3 oxidation state
(b) Because of similar properties the separation of lanthanoids is not easy.
(c) Availability of 4f electrons results in the formation of compounds in +4 state for all the members of the series.
(d) There is a gradual decrease in the radii of the members with increasing atomic number in the series.

18. A 5.2 molal aqueous solution of methyl alcohol, CH_3OH, is supplied. What is the mole fraction of methyl alcohol in the solution?

(a) 0.190
(b) 0.086
(c) 0.050
(d) 0.100

19. Which of the following statement is wrong?

(a) Nitrogen cannot form $d\pi - p\pi$ bond.
(b) Single N- N bond is weaker than the single P – P bond,
(c) N_2O_4 has two resonance structures
(d) The stability of hydrides increases from NH_3 to BiH_3 in group 15 of the periodic table

20. The outer electron configuration of Gd (Atomic No: 64 is:

(a) $4f^8\,5d^0\,6s^2$
(b) $4f^4\,5d^4\,6s^2$
(c) $4f^7\,5d^1\,6s^2$
(d) $4f^3\,4d^5\,6s^2$

21. Which of the following statements regarding sulphur is incorrect?

(a) The vapour at 200°C consists mostly of S_8 rings
(b) At 600°C the gas mainly consists of S_2 molecules
(c) The oxidation state of sulphur is never less than +4 in its compounds
(d) S_2 molecule is paramagnetic.

22. The structure of IF_7 is:

(a) trigonal bipyramid
(b) octahedral
(c) pentagonal bipyramid
(d) square pyramid

23. Ozonolysis of an organic compound gives formaldehyde as one of the products. This confirms the presence of:

(a) a vinyl group
(b) an isopropyl group
(c) an acetylenic triple bond
(d) two ethylenic double bonds

24. A gas absorbs a photon of 355 nm and emits at two wavelengths. If one of the emissions is at 680 nm, the other is at:

(a) 325 nm
(b) 743 nm
(c) 518 nm
(d) 1035 nm

25. Silver Mirror test is given by which one of the following compounds?

(a) Acetone
(b) Formaldehyde
(c) Benzophenone
(d) Acetaldehyde

26. Which of the following reagents may be used to distinguish between phenol and benzoic acid?

(a) Tollen's reagent
(b) Molisch reagent
(c) Neutral $FeCl_3$
(d) Aqueous NaOH

27. Phenol is heated with a solution of mixture of KBr and $KBrO_3$. The major product obtained in the above reaction is

(a) 3-Bromophenol
(b) 4-Bromophenol
(c) 2, 4, 6-Tribromophenol
(d) 2-Bromophenol

28. In a face centred cubic lattice, atom A occupies the corner positions and atom B occupies the face centre positions. If one atom of B is missing from one of the face centred points, the formula of the compound is

(a) AB_2 (b) A_2B_3
(c) A_2B_5 (d) A_2B

29. The entropy change involved in the isothermal reversible expansion of 2 moles of an ideal gas from a volume of 10 dm³ to a valume of 100 dm³ at 27°C

(a) 35.8J $mol^{-1}K^{-1}$ (b) 32.3J $mol^{-1}K^{-1}$
(c) 42.3J $mol^{-1}K^{-1}$ (d) 38.3J $mol^{-1}K^{-1}$

30. Identify the compound that exhibits tautomerism.

(a) Lactic acid (b) 2-Pentanone
(c) Phenol (d) 2- Butene

ANSWERS

1. (b)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (b)	**7** (d)	**8.** (c)	**9.** (d)	**10.** (d)
11. (b)	**12.** (d)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (c)
21. (c)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (b,d)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (d)	**30.** (b,c)

SOLUTIONS

Physics

1. $y_{(x,\,t)} = e^{-\left(\sqrt{a}\,x+\sqrt{b}\,t\right)^2}\; V = \sqrt{\dfrac{b}{a}}$

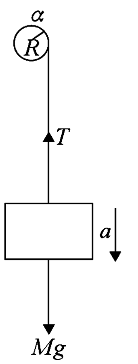
Wave moving in –ve x-direction.

2. Diameter of wire $= \dfrac{1}{100}\times 52 = 0.52$ mm = 0.52 cm

3. Mg – T = Ma .....(1)

$T\times R = I\alpha = \dfrac{1}{2}MR^2\alpha$

$T = \dfrac{1}{2}$ Ma $(a = \alpha R)$ .....(2)



4. $W = T\times\Delta A = T\times 8\pi\,(r_2^{\,2}-r_1^{\,2}) = 0.4\,\pi\,mj$

5. $\tau = 0$

Angular momentum is conserve

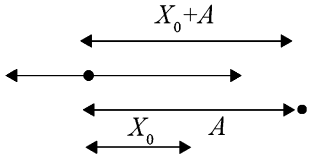
$I_1\omega_1 = I_2\omega_2 \Rightarrow \omega_2 = \dfrac{I_1\omega_1}{I_2}$

I_2 first decreases and then increases

∴ ω first increases and then decreases.

6. $\phi_1 = 0$

$\phi_2 = \dfrac{\pi}{2}$



7. Position of the null point from mass m, $x = \dfrac{r}{1+\sqrt{\dfrac{4m}{m}}} = \dfrac{r}{3}$

$v = -Gm\left(\dfrac{3}{r}+\dfrac{12}{2r}\right) = -9\dfrac{Gm}{r}$

8. At any instant of separation between charges is x.

equilibrium condition $= K\dfrac{Q^2}{x^2} = \omega\dfrac{x}{2\ell}$

$\Rightarrow Q^2 = Cx^3$

$\Rightarrow 2Q\dfrac{dQ}{dt} = C3x^2\dfrac{dx}{dt}$

$\Rightarrow \dfrac{dx}{dt} \propto \dfrac{x^{3/2}}{x^2} \propto x^{-1/2}$

9. $8E = B_H \ell V = 0.15\text{mV}$

10. $\frac{dv}{dt} = -2.5\sqrt{v}$

Integrating the above equation.

$\Rightarrow 2\sqrt{v} = -2.5t + C$

at $t = 0, v = 6.25 \Rightarrow C = 5$

at $v = 0 \Rightarrow t = \frac{5}{2.5} = 2s$

11. Charge oscillates simple harmonic motion $q = q_0 \sin \omega t$,

$U = \frac{1}{2}\frac{q^2}{C}$

$q = \frac{q_0}{\sqrt{2}} \Rightarrow \omega t = \frac{\pi}{4}$

$\Rightarrow t = \frac{T}{8} = \frac{2\pi}{8}\sqrt{LC} = \frac{\pi}{4}\sqrt{LC}$

12. Normal to the plane is z–axis

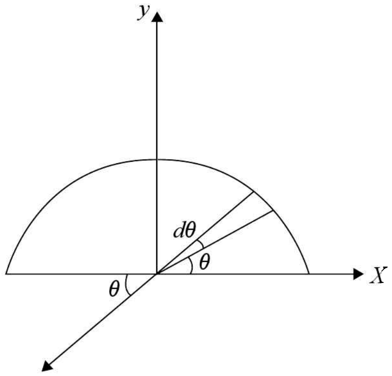
$\cos\theta_1 = \frac{A_z}{A} = \frac{10}{20} = \frac{1}{2}, \theta_1 = 60$

$\mu_1 \sin \theta_1 = \mu_2 \sin \theta_2 \Rightarrow \sqrt{2} \times \frac{\sqrt{3}}{2} = \sqrt{3}\ \sin \theta_2 \Rightarrow \theta_2 = 45°$

13. $d\vec{B} = \frac{\mu_0 di}{2\pi R}\left[-\cos\theta\hat{i} - \sin\theta\hat{i}\right]$

$di = \frac{T}{\pi R} R d\theta$

$= \frac{l}{\pi} d\theta$



$d\vec{B} = \frac{\mu_0 l}{2\pi^2 R}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right)$

$\vec{B} = -\frac{\mu_0 I}{\pi^2 R}\hat{j}$

14. $W = \Delta U$

$\frac{1}{2}mv^2 = nC_v dt$

$= \frac{m}{M}\frac{R}{\gamma - 1}dT$

$dT = \frac{M(\gamma - 1)v^2}{2R}K$

15. Energy of simple harmonic oscillator is constant.

$\Rightarrow \frac{1}{2}M\omega^2 A_1^2 = \frac{1}{2}(m + M)\,\omega^2 A_2^2$

$\frac{A_1^2}{A_2^2} = \frac{M + m}{M}$

$\therefore \frac{A_1}{A_2} = \sqrt{\frac{M + m}{M}}$

16. Equation of continuity

$\Rightarrow (a \times v)$ top $= (a \times v)$ bottom

$v_b^2 - (0.4)^2 = 2 \times 9.8 \times 0.2$ [$v^2 - u^2 = 2gh$ is used]

$v_b = 2\text{m/s}$ (nearly)

$\pi[8 \times 10^{-3}] \times 0.4 = \pi d^2 \times 4$

$d \approx 3.6 \times 10^{-3}$ m

17. Since ionospheric properties change with time, these signals are in general less stable than ground wave signals.

18. Data $\Rightarrow n, k, t_1 + n_2 kT_2 + n_3 kT_3 = (n_1 + n_2 + n_3)$ kT

$\therefore T = \frac{n_1T_1 + n_2T_2 + n_3T_3}{n_1 + n_2 + n_3}$

19. $r \times F = I \times \alpha$

$2(20t - 5t^2) = 10\alpha \Rightarrow \alpha = 4t - t^2$

$\frac{d\omega}{dt} = 4t - t^2$

$d\omega = (4t^2 - t^2)\ dt$

$\omega = 2t^2 - \frac{t^3}{3}$ (on integration)

$\omega = 0 \Rightarrow t = 6s$

$\omega = \frac{d\theta}{dt} = 2t^2 - \frac{t^3}{3}$

$d\theta = \left(2t^2 - \frac{t^3}{3}\right)dt$

$\Rightarrow \theta = \frac{2t^3}{3} - \frac{t^4}{12}$ (on integration)

θ (in 6s) = 36 rad

$\Rightarrow 2\pi n = 36$

$n = \frac{36}{2\pi} =< 6$

20. $V_c = E\,(1 - e^{-t/Rc})$

$1 - e^{-t/Rc} = \frac{120}{200} = \frac{3}{5}$

$\Rightarrow R = \frac{5}{1.84 \times 10^{-6}} = 2.7 \times 10^6 \Omega$

21. $\eta_1 = \frac{T_1 - T_2}{T_1} = \frac{1}{6}$

$\eta_2 = \frac{T_1 - (T_2 - 62)}{T_1} = \frac{1}{3}$

$\Rightarrow \frac{T_1 - T_2}{T_1} + \frac{62}{T_1} = \frac{1}{3}$

$\frac{1}{6} + \frac{62}{T_1} = \frac{1}{3}$

$\frac{62}{T_1} = \frac{1}{6}$

$\therefore T_1 = 62 \times 6 = 372\ K$

$$\frac{T_1 - T_2}{T_1} = \frac{1}{6}$$

$$1 - \frac{T_2}{T_1} = \frac{1}{6}$$

$$\frac{T_2}{372} = \frac{5}{6}$$

$\Rightarrow T_2 = 310\text{K}$

22. $R \propto \ell^2$ (for a given volume)

$$\Rightarrow \frac{\Delta R}{R}\% = \frac{2\Delta \ell}{\ell}\%$$

Thus when wire is stretched by 0.1% resistance increases by 0.2%

23. As light enters from air to glass it suffers a phase change on π and therefore at centre there will be destructive interference.

24. $$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$-\frac{1}{v^2}\frac{dv}{dt} - \frac{1}{u^2}\frac{du}{dt} = 0$$

$$\frac{dv}{dt} = -\frac{v^2}{u^2}\left(\frac{du}{dt}\right)$$

$f = 20$ cm

$$\frac{1}{u} + \frac{1}{-280} = \frac{1}{20}$$

$$\Rightarrow v = \frac{280}{15}\text{ cm}$$

$$v_1 = -\left(\frac{280}{15 \times 280}\right)^2 \times 15$$

$$= \frac{1}{15}\text{ m/s}$$

25. $$E_n = -13.6\frac{Z^2}{n^2}$$

$$E_{Li}{}^{++} = -13.6\ \times \frac{9}{1} = -122.4\text{eV}$$

$$E_{Li}{}^{+++} = -13.6\ \times \frac{9}{9} = -13.6\text{eV}$$

$\Delta E = -13.6 - (-122.4)$
$= 108.8$ eV

26. Potential inside $(\phi) = ar^2 + b$

$$\therefore\ E_r = -\frac{\delta v}{\delta r} = -2ar$$

Electric field inside uniformly charged solid volume varies with 'r'. So charge density is constant $\phi_{\text{net}} = (-2ar)\ 4\pi r^2 = -8\pi ar^3$

$$-8\pi ar^3 = \frac{\sigma \times \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$\therefore\ \sigma = -6a\varepsilon_0$

27. Max. range $= \dfrac{u^2}{g}$ i.e., $\dfrac{v^2}{g}$ (radius of circle)

Area occupied $= \pi\left(\dfrac{v^2}{g}\right)^2 = \dfrac{\pi v^4}{g^2}$

28. $\Delta Q = \Delta U + \Delta W$ (ignoring expansion)

$\Delta U = \text{ms}\Delta T = 0.1 \times 4.184 \times 20 = 8.368$ kJ

29. $t_{\frac{1}{2}} = 20$ minutes

$$N = N_0 e^{-\lambda t_2}\ \ \lambda t_1 = \text{In } 3$$

$$\frac{2}{3}N_0 = N_0 e^{-\lambda t_2}\ \ t_1 = \frac{1}{\lambda}\text{In } 3$$

$$\frac{2}{3}N_0 = N_0 e^{-\lambda t_2}$$

$$t_2 = \frac{1}{\lambda}\text{In }\frac{3}{2}$$

$$t_2 - t_1 = \frac{1}{\lambda}\left[\text{In}\frac{3}{2} - \text{In } 3\right]$$

$$= \frac{1}{\lambda}\text{In}\left[\frac{1}{2}\right] = \frac{0.693}{\lambda}$$

$= 20$ min

30. $KE_{\text{max}} = h\upsilon - h\upsilon_0$

$h\upsilon - h\upsilon_0 = e \times \Delta v$

$$V_0 = \frac{h\upsilon}{e} - \frac{h\upsilon_0}{e}$$

'υ' is doubled

$KE_{\text{max}} = 2h\upsilon - \text{h}\upsilon_0$

$$v_0{}' = (\Delta V)' = \frac{2h\upsilon}{e} - \frac{2h\upsilon_0}{e}$$

$\dfrac{KE_{\text{max}}}{KE_{\text{max}}}$ may not be equal to 2

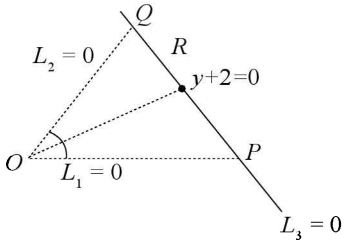
$\Rightarrow \dfrac{V_0'}{V_0}$ may not equal to 2

KE max $= \text{h}\upsilon - \text{h}\upsilon_0$

$$V = \frac{h\upsilon}{e} - \frac{h\upsilon}{e}$$

Mathematics

1.



$\Pi(-2, -2)$; $Q = (1, -2)$

Equation of angular bisector $\overline{OR}$ is $(\sqrt{5} + 2\sqrt{2})x = (\sqrt{5} - \sqrt{2})y$

$\therefore PR : RQ = 2\sqrt{2} : \sqrt{5}$

2. $$A = \sin^2 x + \cos^4 x = \frac{7 + \cos 4x}{8} \Rightarrow \frac{3}{4} \le A \le 1$$

3. $[1 - x - x^2(1-x)]^6 = (1 - x)^6(1 - x^2)^6$

$[{}^6C_0 - {}^6C_1x + {}^6C_2x^2 - {}^6C_3x^3 + {}^6C_4x^4 - {}^6C_5x^5 + {}^6C_6x^6] \times$
$[{}^6\text{C}_0 - {}^6\text{C}_1x^2 + {}^6\text{C}_2x^4 - {}^6\text{C}_3x^6 + \ldots]$

Coefficient of $x^7 = {}^6C_1\,{}^6C_3 - {}^6C_3\,{}^6C_2 + {}^6C_5\,{}^6C_1 = 120 - 300 + 36$
$= -144$

4. $\lim\limits_{x\to 2} \dfrac{\sqrt{2\sin^2(x-2)}}{x-2}$

$\lim\limits_{x\to 2} \dfrac{\sqrt{2}\,|\sin(x-2)|}{x-2}$

R.H.L. $= \sqrt{2}$, L.H.L. $= -\sqrt{2}$

Limit does not exist.

5. ${}^{(n-1)}C_{(r-1)} = {}^{(10-1)}C_{(4-1)} = {}^9C_3$
Statement 1 is correct
Statement 2 is also correct
From 9 we can select 3 in ${}^9C_3$ ways. It is correct explanation.

6. $$\frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d}{dy}\left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{dy}{dx}\right)^{-2}\frac{1}{\left(\frac{dy}{dx}\right)}\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

7. $\dfrac{dy}{dx} = y + 3 \Rightarrow \dfrac{dy}{y+3} = dx$

In $(y + 3) = x + c$

$x = 0 \Rightarrow y = 2$

$\Rightarrow$ In $5 = 0 + c$

$c =$ In 5

In $(y + 3) = x +$ In 5

$y + 3 = e^{x + \text{In}\,5} \Rightarrow y + 3 = e^{\text{In}\,2 + \text{In}\,5}$

$y + 3 = 10 \Rightarrow y = 7$

8. $x - y$ is an integer
$x - x = 0$ is an integer $\Rightarrow A$ is Reflexive
$x - y$ is an integer $\Rightarrow y - x$ is an integer $\Rightarrow A$ is symmetric
$x - y, y - z$ are integers
As sum of two integers is an integer.
$\Rightarrow (x - y) + (y - z) = x - z$ is an integer
$\Rightarrow A$ is transitive. Hence statement-1 is true.

Also $\dfrac{x}{x} = 1$ is a rational number $\Rightarrow B$ is reflexive

$\dfrac{x}{y} = \alpha$ is rational $\Rightarrow \dfrac{y}{x}$ need not be rational

i.e., $\dfrac{0}{1}$ is rational $\Rightarrow \dfrac{1}{0}$ is not rational

Hence B is not symmetric

$\Rightarrow B$ is not an equivalence relation.

9. $$I = 8\int_0^1 \frac{\log(1+x)}{1+x^2}dx$$

$$= 8\int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta \text{ (let } x = \tan\theta)$$

$$= 8\int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \log 2\,d\theta - 8\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$$

$$= 8\log 2\frac{\pi}{4} - 1$$

$2l = 2\pi \log 2$
$1 = \pi \log 2$

10. Suppose roots are $1 + \pi i$, $1 + qi$

Sum of roots $1 + \pi i + 1 + qi = -\alpha$ which is real

$\Rightarrow$ roots of $1 + \pi i$, $1 - \pi i$

Product of roots $= \beta = 1 + \pi^2 \in (1, \infty)$

$\pi \neq 0$ since roots are distinct.

11. $n = 5$

Success $= \pi$

Failure $= q$

Π (at least one failure) $\geq \dfrac{31}{32}$

$1 - \Pi$ (no failure) $\geq \dfrac{31}{32}$

$1 - \Pi\,(x = 5) \geq \dfrac{31}{32}$

$1 - {}^5C_5 p^5 \geq \dfrac{31}{32}$

$-p^5 \geq -\dfrac{1}{32}$

$p^5 \leq \dfrac{1}{32}$

$p \leq \dfrac{1}{2}$

$p \in \left[0, \dfrac{1}{2}\right]$

12.

1	2	3	4	5	6	......
200	200	200	240	280	....	....

Sum = 11040

120 + 80 + 160 + 40 + 200 + 240 + ... = 11040

$\dfrac{n}{2}[2a + (n-1)d] + 80 + 40 = 11040$

$\dfrac{n}{2}[240 + (n-1)40] = 10920$

$n[6 + n - 1] = 546$

$n(n + 5) = 546$
$n = 21$

13. $\dfrac{1}{\sqrt{|x| - x}} \Rightarrow |x| - x > 0 \Rightarrow |x| > x \Rightarrow x$ is negative

$x \in (-\infty, 0)$

14. $\cos\theta = \sqrt{\frac{5}{14}}$

$\sin\theta = \frac{3}{\sqrt{14}}$

$\sin\theta = \frac{1+4+3\lambda}{\sqrt{1+4+\lambda^2}\ \sqrt{1+4+9}}$

$\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{5+\lambda^2}\ \sqrt{14}} \Rightarrow \lambda = \frac{2}{3}$

15. $(2\bar{a}-\bar{b})\cdot\left\{(\bar{a}\times\bar{b})\times(\bar{a}+2\bar{b})\right\}$

$= (2\bar{a}-\bar{b})\cdot\left\{\left[\bar{a}.(\bar{a}+2\bar{b})\right]\bar{b}-\left[\bar{b}\cdot(\bar{a}+2\bar{b})\bar{a}\right]\right\}$

$= -5(\bar{a})^2(\bar{b})^2 + 5(\bar{a}.\bar{b})^2 = -5$

16. $b^2 = a^2(1-e^2) = a^2\left(1-\frac{2}{5}\right) = a^2\frac{3}{5} = \frac{3a^2}{5}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow \frac{9}{a^2}+\frac{5}{3a^2} = 1$

$a^2 = \frac{32}{3}$

$b^2 = \frac{32}{5}$

$\therefore$ Required equation of ellipse $3x^2 + 5y^2 - 32 = 0$

17. $\frac{dV}{dt} = -k(T-t)$

$\Rightarrow dV = -k(T-t)dt$

Integrate

$V = \frac{-k(T-t)^2}{(-2)} + c \Rightarrow V = \frac{k(T-t)^2}{2} + c$

at $t = 0 \Rightarrow V = I$

$I = \frac{kT^2}{2} + c \Rightarrow c = I - \frac{kT^2}{2} \Rightarrow c = V(T) = I - \frac{kT^2}{2}$

18. $\bar{b}\times\bar{c} = \bar{b}\times\bar{d}$

$\Rightarrow \bar{a}\times(\bar{b}\times\bar{c}) = \bar{a}\times(\bar{b}\times\bar{d})$

$\Rightarrow (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} = (\bar{a}.\bar{d})\bar{b} - (\bar{a}.\bar{b})\bar{d}$

$\Rightarrow (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} = -(\bar{a}.\bar{b})\bar{d}$

$\therefore \bar{d} = \bar{c} - \left(\frac{\bar{a}.\bar{c}}{\bar{a}.\bar{b}}\right)\bar{b}$

19. $c_1 = \left(\frac{a}{2}, 0\right); c_2 = (0, 0) \qquad r_1 = \frac{a}{2}; r_2 = c$

$c_1c_2 = r_1 - r_2 \Rightarrow \frac{a}{2} = c - \frac{a}{2} \Rightarrow c = a$

20. $C \cap D = C \Rightarrow P(C \cap D) = P(C)$

$\Rightarrow P\left(\frac{C}{D}\right) = \frac{P(C\cap D)}{P(D)} \geq P(C)$

21. **(a)** $\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \Rightarrow k^2 - 6k + 8 = 0 \Rightarrow k = 4, 2$

22. $\sim\{(\Pi_{\wedge} \sim R) \leftrightarrow Q\} = \sim\{Q \leftrightarrow (\Pi_{\wedge} \sim R)\}$

23. $\Pi = (y^2, y)$

Perpendicular distance from Π to $x - y + 1 = 0$ is $\frac{|y^2-y+1|}{\sqrt{2}}$

$y^2 - y + 1 > 0\ \forall\, y \in R$

$\therefore$ Coefficient $y^2 > 0$

$\therefore$ Min value $= \frac{1}{\sqrt{2}}\left(\frac{4ac-b^2}{4a}\right) = \frac{3}{4\sqrt{2}}$

24. $\frac{1}{n}\Sigma|x_i - A|$

A = Median $= \frac{25a+26a}{2} = 25.5a$

Mean deviation

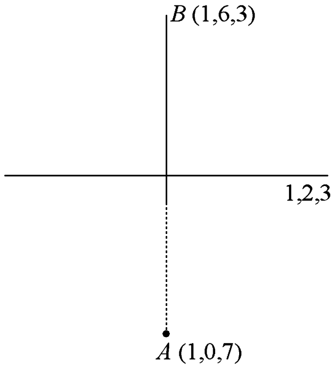
$= \frac{1}{50}\left\{|a-25.5a| + |2a-25.5a|\right\}$

$= \frac{2}{50}\left\{(24.5a+23.5a)+...(0.5a)\right\}$

$= \frac{2}{50}\left\{312.5a\right\} = 50$ (Given)

$\Rightarrow 625a = 2500 \Rightarrow a = 4$

25.



Statement-1: AB is perpendicular to given line and mid point of AB lies on line

Statement-2 is true but it is not correct explanation as it is bisector only.

If it is perpendicular bisector then only statement-2 is correct explanation.

26. $A^T = A$, $B^T = B$

$(A(BA))^T = (BA)^T A^T = (A^T B^T) A = (AB)A = A(BA)$

$((AB)A)^T = A^T(AB)^T = A(B^TA^T) = A(BA) = (AB)A$

$\therefore$ Statement-1 is correct

Statement-2

$(AB)^T = B^TA^T = BA = AB$ ($\because$ AB is commutative)

Statement -2 is also correct but it is not correct explanation of Statement-1

27. $1 + \omega = -\omega^2$

$(1+\omega)^7 = (-\omega^2)^7 = -\omega^{14} = -\omega^2 = 1 + \omega = A + B\omega$

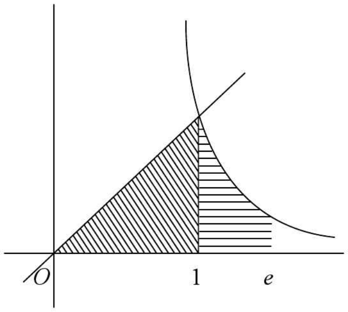
$\Rightarrow (A, B) = (1, 1)$

28. $$\lim_{x\to 0}\frac{\sin(p+1)+\sin x}{x}=q=\lim_{x\to 0}\frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}$$

$$\lim_{x\to 0}(p+1)\cos(p+1)x+\cos x=q=\frac{1}{2}$$

$$\Rightarrow p+1+1=\frac{1}{2}\Rightarrow p=-\frac{3}{2};q=\frac{1}{2}$$

29.



$$\text{Area}=\int_0^1 x\,dx+\int_1^e\frac{1}{x}dx=\frac{1}{2}+1=\frac{3}{2}$$

30. $f'(x)=\sqrt{x}\sin x$

Given $x\in\left(0,\frac{5\pi}{2}\right)$

$f'(x)$ changes sign from +ve to –ve at π

$f'(x)$ changes sign from –ve to +ve at 2π

f has local max at π, local min at 2π

Chemistry

1. Greater charge and small size of cation cause more polarization and more covalent is that compound

2. In RNA, the sugar is β – *D* – Ribose, where as in DNA the sugar is β -*D*-2-deoxy Ribose

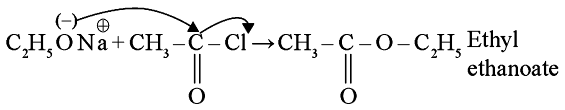
3. $2CCl_3CHO \xrightarrow{OH^{(-)}} CCl_3COONa + CCl_3CH_2OH$

Cannizaro reaction is a disproportionation reaction

One aldehyde molecule is oxidized to salt of the carboxylic Acid, other one is reduced to Alcohol. So the compound is CCl_3CH_2OH

IUPAC Name is 2, 2, 2, - Trichloro ethanol

4.



$$C_2H_5\overset{(-)}{O}\overset{\oplus}{Na}+CH_3-\underset{\underset{O}{\|}}{C}-Cl\rightarrow CH_3-\underset{\underset{O}{\|}}{C}-O-C_2H_5$$ Ethyl ethanoate

5. $2H^+ + 2e^- \rightarrow H_2(g)$

$$E=E^\circ-0.059\log\left(\frac{P_{H_2}}{[H^+]^2}\right)$$

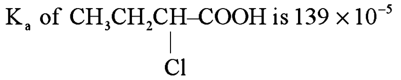
(here E is –ve when $P_{H_2}>[H^+]^2$)

$$=\frac{-0.0591}{2}\log_{10}\left(\frac{2}{1}\right)=\frac{-.0591}{2}\times.3010=\text{negative value}$$

6. Electron releasing groups (Alkyl groups) de stabilizes conjugate base.

The +I effect of C_3H_7 is less than - I effect of Cl

K_a of HCOOH is 17.9×10^{-5}



K_a of $CH_3CH_2\underset{\underset{Cl}{|}}{CH}-COOH$ is 139×10^{-5}

7. $i=1-\alpha+n\alpha=1+\alpha(n-1)$

$$\frac{i-1}{n-1}=\alpha$$

$$A_xB_y\rightarrow xA^{+y}+yB^{-x}$$

$n=x+y$

$$\text{So }\alpha=\frac{i-1}{x+y-1}$$

8. ease of liquefaction $\propto\frac{a}{b}$

for ethane $a=5.49$, $b=0.0638$

for Cl_2 $a=6.49$, $b=0.0562$

9. $CO_2(g)+C\rightarrow 2CO(g)$

Initial moles	p	O
Equilibriumm moles	p-x	$2x$

Total pressure at equilibrium = 0.8 atm ; Total no. of moles = $p+x$.

Therefore $p\ \mu\ n$; $\frac{0.5}{0.8}=\frac{p}{p+x}\Rightarrow x=0.3$

$$K_p=\frac{P^2_{CO}}{P_{CO_2}}=\frac{0.6\times0.6}{0.2}=1.8\text{ atm}$$

10. As Boron has only four orbitals in the valence shall (i.e. $2s$, $2p_x$, $2p_y$ & $2p_z$) it can show a maximum valency of four only.

Therefore $[BF_6]^{3-}$ is not possible

11. $[Cr(NH_3)_6]Cl_3$ involves d^2sp^3 hybridization and it is an inner orbital complex.

12. $$\Delta T_f=K_f\times m=K_f\times\frac{w_2\times1000}{w_1\times m_2}$$

w_1 & w_2 = *wt* of solvent & solute respecting

m_2 = *mw* of solute

$$\Delta T_f=0^\circ-(-6^\circ)=6=1.86\times\frac{w_2\times1000}{4000\times62}$$

Therefore w_2 = 800g

13. Across a period metallic strength decreases & down the group it increases

14. Temperature coefficient μ = 2;

$$\mu^{\frac{\Delta T}{10}}=\frac{k_2}{k_1};$$

$$2^{\frac{50}{10}}=2^5=32=\frac{k_2}{k_1}$$

Therefore $32\,k_1=k_2$

15. In $[NiCl_4]^{2-}$, $n=2$

$$\mu=\sqrt{n(n+2)}\text{ BM}$$

$$=\sqrt{2(2+2)}=2.82\text{BM}$$

16. 5.46 BM

17. The general o.s of lanthanides is +3, only few elements exhibit +4 o.s.

18. Molefraction of solute (X_2) in aqueous solution = $\dfrac{m}{m+\dfrac{1000}{18}}$

$$= \frac{5.2}{5.2+\dfrac{1000}{18}} = 0.09$$

19. Stability of hydrides decreases down the group from NH_3 to BiH_3 as M-H bond energy decreases.

20. $4f^7\,5d^1\,6s^2$

21. 'S' can exhibit a minimum oxidation state of -2 (Ex. H_2S)

22. In IF_7, I undergoes $sp^3\,d^3$ hybridisation

23. Vinyl group

$$\left(CH_2 = CH\_\right)$$

on ozonolosys give formaldehyde

24. $$\Rightarrow \frac{1}{\lambda_{\text{absorbed}}} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

$$\Rightarrow \Rightarrow \frac{1}{355} = \frac{1}{680} + \frac{1}{\lambda_2}$$

$$\Rightarrow \lambda_2 = 742.8 \cong 743 \text{ nm}$$

25. Formaldehyde and Acetaldehyde can be oxidized by tollen's reagent to give silver mirror.

26. Phenol gives violet coloured complex compound with neutral $FeCl_3$, benzoic acid gives pale dull yellow ppt. with neutral $FeCl_3$

27. In acidic medium, KBr + $KBrO_3$ in turn produces Br_2. Phenol reacts with Br_2 (aq) to give 2, 4, 6-trinitrophenol

29. Effective no. of A atoms = $\frac{1}{8} \times 8 = 1$

Effective no. of B atoms = $\frac{1}{2} \times 5$ (One is missing) $= \frac{5}{2}$

Therefore formula is $A_1B_{\frac{5}{2}} = A_2B_5$

28. For an ideal gas, For isothermal reversible process,

$$\Delta S = 2.303\ nR \log\left(\frac{V_2}{V_1}\right)$$

$$= 2.303 \times 2 \times 8.314 \times \log\left(\frac{100}{10}\right)$$

38.3 J $mol^{-1}.k^{-1}$

30. both 2-pentanone, phenol can exhibit tautomerism

PART II

CHEMISTRY

1. Identify the incorrect statements from the following:

(a) Ozone absorbs the intense ultraviolet radiation of the sun.
(b) Depletion of ozone layer is because of its chemical reactions with chlorofluoro alkanes.
(c) Ozone absorbs infrared radiation.
(d) Oxides of nitrogen in the atmosphere can cause the depletion of ozone layer.

2. When r, P and M represent rate of diffusion, pressure and molecular mass, respectively, then the ratio of the rates of diffusion (r_A/r_B) of two gases A and B, is given as:

(a) $(P_A/P_B)\ (M_B/M_A)^{1/2}$ (b) $(P_A/P_B)^{1/2}(M_B/M_A)$
(c) $(P_A/P_B)\ (M_A/M_B)^{1/2}$ (d) $(P_A/P_B)^{1/2}(M_A/M_B)$

3. Consider thiol anion ($RS^{\ominus}$) and alkoxy anion ($RO^{\ominus}$). Which of the following statements is correct?

(a) $RS^{\ominus}$ is less basic but more nucleophilic than $RO^{\ominus}$
(b) $RS^{\ominus}$ is more basic and more nucleophilic than $RO^{\ominus}$
(c) $RS^{\ominus}$ is more basic but less nucleophilic than $RO^{\ominus}$
(d) $RS^{\ominus}$ is less basic and less nucleophilic than $RO^{\ominus}$

4. The change in the optical rotation of freshly prepared solution of glucose is known as:

(a) racemisation (b) specific rotation
(c) mutarotation (d) tautomerism

5. The molality of a urea solution in which 0.0100 g of urea, $[(NH_2)_2CO]$ is added to 0.3000 dm^3 of water at STP is:

(a) 5.55×10^{-4} (b) 33.3 m
(c) 3.33×10^{-2} m (d) 0.555 m

6. The molecular velocity of any gas is:

(a) inversely proportional to absolute temperature.
(b) directly proportional to square of temperature.
(c) directly proportional to square root of temperature.
(d) inversely proportional to the square root of temperature.

7. The correct order of acid strength of the following compounds:

(A) Phenol (B) p-Cresol
(C) m-Nitrophenol (D) p-Nitrophenol
is:
(a) $D > C > A > B$ (b) $B > D > A > C$
(c) $A > B > D > C$ (d) $C > B > A > D$

8. The value of enthalpy change (ΔH) for the reaction

$$C_2H_5OH_{(l)} + 3O_{2(g)} \longrightarrow 2CO_{2(g)} + 3H_2O_{(l)}$$

at 27°C is –1366.5 kJ mol^{-1}. The value of internal energy change for the above reaction at this temperature will be:
(a) –1369.0 kJ (b) –1364.0 kJ
(c) –1361.5 kJ (d) –1371.5 kJ

9. Thermosetting polymer, Bakelite is formed by the reaction of phenol with:

(a) CH_3CHO (b) HCHO
(c) HCOOH (d) CH_3CH_2CHO

10. Ozonolysis of an organic compound 'A' produces acetone and propionaldehyde in equimolar mixture. Identify 'A' from the following compounds:

(a) 1-Pentene (b) 2-Pentene
(c) 2-Methyl-2-pentene (d) 2-Methyl-1-pentene

11. Consider the reaction:

$$4NO_{2(g)} + O_{2(g)} \rightarrow 2N_2O_{5(g)}, \Delta_r H = -111 \text{ kJ.}$$

If $N_2O_{5(s)}$ is formed instead of $N_2O_{5(g)}$ in the above reaction, the $\Delta_r H$ value will be: (given, ΔH of sublimation for N_2O_5 is 54 kJ mol^{-1})
(a) +54 KJ (b) +219 KJ
(c) – 219 KJ (d) – 165 KJ

12. An acid HA ionises as

$$HA \rightleftharpoons H^+ + A^-$$

The pH of 1.0 M solution is 5. Its dissociation constant would be:
(a) 5 (b) 5×10^{-8}
(c) 1×10^{-5} (d) 1×10^{-10}

13. The correct order of electron gain enthalpy with negative sign of F, Cl, Br and I, having atomic number 9, 17, 35 and 53 respectively, is:

(a) F > Cl > Br > I (b) Cl > F> Br > I
(c) Br > Cl > I > F (d) I > Br > Cl > F

14. The frequency of light emitted for the transition $n = 4$ to $n = 2$ of He^+ isequalto the transition in H atom corresponding to which of the following?

(a) $n = 2$ to $n = 1$ (b) $n = 3$ to $n = 2$
(c) $n = 4$ to $n = 3$ (d) $n = 3$ to $n = 1$

15. A 5% solution of cane sugar (molar mass 342) is isotonic with 1% of a solution of an unknown solute. The molar mass of unknown solute in g/mol is:

(a) 171.2 (b) 68.4
(c) 34.2 (d) 136.2

16. In veiw of the sign of $\Delta_r G°$ for the following reactions:

$PbO_2 + Pb \rightarrow 2PbO, \quad \Delta_r G° < 0$
$SnO_2 + Sn \rightarrow 2SnO, \quad \Delta_r G° > 0,$

which oxidation states are more characteristics for lead and tin?
(a) For lead + 2, for tin + 2
(b) For lead + 4, for tin + 4
(c) For lead + 2, for tin + 4
(d) For lead + 4, for tin + 2

17. The K_{sp} for $Cr(OH)_3$ is 1.6×10^{-30}. The molar solubility of this compound in water is:

(a) $4\sqrt{1.6 \times 10^{-30}}$ (b) $4\sqrt{1.6 \times 10^{-30}/27}$
(c) $1.6 \times 10^{-30}/27$ (d) $2\sqrt{1.6 \times 10^{-30}}$

18. The products obtained on heating $LiNO_3$ will be:

(a) $Li_2O + NO_2 + O_2$ (b) $Li_3N + O_2$
(c) $Li_2O + NO + O_2$ (d) $LiNO_3 + O_2$

19. Resistance of 0.2 M solution of an electrolyte is 50 Ω. The specific conductance of the solution is 1.3 S m^{-1}. If resistance of the 0.4 M solution of the same electrolyte is 260 Ω, its molar conductivity is:

(a) 6.25×10^{-4} S m^2 mol^{-1}
(b) 625×10^{-4} S m^2 mol^{-1}
(c) 62.5 S m^2 mol^{-1}
(d) 6250 S m^2 mol^{-1}

20. Among the ligands NH_3, en, CN^- and CO the correct order of their increasing field strength, is:

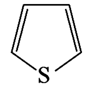
(a) $NH_3 <$ en $< CN^- <$ CO
(b) $CN^- < NH_3 <$ CO $<$ en
(c) en $< CN^- < NH_3 <$ CO
(d) CO $< NH_3 <$ en $< CN^-$

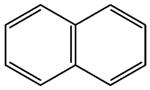
21. Consider the following reaction:

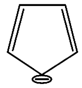
$$C_2H_5OH + H_2SO_4 \rightarrow \text{Product}$$

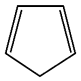
Among the following, which one cannot be formed as a product under any conditions?
(a) Ethylene
(b) Acetylene
(c) Diethyl ether
(d) Ethyl-hydrogen sulphate

22. The non aromatic compound among the following is:

(a) 

(b) 

(c) 

(d) 

23. The number of types of bonds between two carbon atoms in calcium carbide is:

(a) One sigma, one pi (b) Two sigma, one pi
(c) Two sigma, two pi (d) One sigma, one pi

24. A reactant (A) forms two products:

$A \xrightarrow{k_1} B$, Activation Energy Ea_1
$A \xrightarrow{k_2} C$, Activation Energy Ea_2

If $Ea_2 = 2\, Ea_1$, then k_1 and k_2 are related as:
(a) $k_2 = k_1 e^{Ea_1/RT}$ (b) $k_2 = k_1 e^{Ea_2/RT}$
(c) $k_1 = Ak_2 e^{Ea_1/RT}$ (d) $k_1 = 2k_2 e^{Ea_2/RT}$

25. copper crystallises in fcc lattice with a unit cell edge of 361 pm. The radius of copper atom is:

(a) 108 pm (b) 128 pm
(c) 157 pm (d) 181 pm

26. The mass of potassium dichromate crystals required to oxidise 750 cm^3 of 0.6 M Mohr's salt solution is:

(Given molar mass potassium dichromate = 294, Mohr's salt = 392)
(a) 0.45 g (b) 22.05 g
(c) 2.2 g (d) 0.49 g

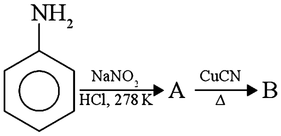
27. What is the best description of the change that occurs when $Na_2O(s)$ is dissolved in water?

(a) Oxide ion accepts sharing in a pair of electrons
(b) Oxide ion donates a pair of electrons
(c) Oxidation number of oxygen increases
(d) Oxidation number of sodium decreases

28. Which of the following has maximum number of lone pairs associated with Xe?

(a) XeF_4 (b) XeF_6
(c) XeF_2 (d) XeO_3

29. In the chemical reactions



the compounds A and B respectively are:

(a) Benzene diazonium chloride and benzonitrile
(b) Nitrobenzene and chlorobenzene
(c) Phenol and bromobenzene
(d) Fluorobenzene and phenol

30. Which one of the following complex ions has geometrical isomers?

(a) $[Ni(NH_3)_5Br]^+$
(b) $[Co(NH_3)_2(en)_2]^{3+}$
(c) $[Cr(NH_3)_4(en)]^{3+}$
(d) $[Co(en)_3]^{3+}$(en - ethylenediamine)

PHYSICS

31. At time $t = 0$s a particle starts moving along the x-axis. If its kinetic energy increases uniformly with time 't', the net force acting on it must be proportional to:

(a) constant
(b) t
(c) $\frac{1}{\sqrt{t}}$
(d) $\sqrt{t}$

32. At two points P and Q on a screen in Young's double slit experiment, waves from slits S_1 and S_2 have a path difference of 0 and $\frac{\lambda}{4}$ respectively. The ratio of intensities at P and Q will be:

(a) 2 : 1
(b) $\sqrt{2}:1$
(c) 4 : 1
(d) 3 : 2

33. Two particles of equal mass 'm' go around a circle of radius R under the action of their mutual gravitational attraction. The speed of each partial with respect to their centre of mass is:

(a) $\sqrt{\frac{Gm}{4R}}$
(b) $\sqrt{\frac{Gm}{3R}}$
(c) $\sqrt{\frac{Gm}{2R}}$
(d) $\sqrt{\frac{Gm}{R}}$

34. The minimum force required to start pushing a body up a rough (frictional coefficient μ) inclined plane is F_1 while the minimum force needed to prevent it from sliding down is F_2. If the inclined plane makes an angle θ from the horizontal such that tan θ = 2 μ then the ratio $\frac{F_1}{F_2}$ is:

(a) 1
(b) 2
(c) 3
(d) 4

35. If 400Ω of resistance is made by adding four 100 Ω resistance of tolerance 5%, then the tolerance of the combination is:

(a) 5 %
(b) 10 %
(c) 15 %
(d) 20 %

36. An electric charge + q moves with velocity $\vec{v} = 3\hat{i} + 4\hat{j} + \hat{k}$, in an electromagnetic field given by : $\vec{E} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + 3\hat{k}$. The y - component of the force experienced by + q is:

(a) 11 q
(b) 5 q
(c) 3 q
(d) 2 q

37. The current in the primary circuit of a potentiometer is 0.2A. The specific resistance and cross-section of the potentiometer wire are 4×10^{-7} ohm metre and 8×10^{-7} m^2 respectively. The potential gradient will be equal to:

(a) 1 V/m
(b) 0.5 V/m
(c) 0.1 V/m
(d) 0.2 V/m

38. A particle of mass 'm' is projected with a velocity v making an angle of 30° with the horizontal. The magnitude of angular momentum of the projectile about the point of projection when the particle is at its maximum height 'h' is:

(a) zero
(b) $\frac{mv^3}{\sqrt{2}g}$
(c) $\frac{\sqrt{3}}{16}\frac{mv^3}{g}$
(d) $\frac{\sqrt{3}}{2}\frac{mv^2}{g}$

39. The specific heat capacity of a metal at low temperature (T) is given as:

$$C_p(kjK^{-1}kg^{-1}) = 32\left(\frac{T}{400}\right)^3$$

A 100 gram vessel of this metal is to be cooled from 20°K to 4°K by a special refrigerator operating at room temperature (27°C). The amount of work required to cool the vessel is:

(a) greater than 0.148 kJ
(b) between 0.148 kJ and 0.028 kJ
(c) less than 0.028 kJ
(d) equal to 0.002 kJ

40. A wooden cube (density of wood 'd') of side 'ℓ' floats in a liquid of density 'ρ' with its upper and lower surfaces horizontal. If the cube is pushed slightly down and released, it performs simple harmonic motion of period 'T'. Then, 'T' is equal to:

(a) $2\pi\sqrt{\frac{\ell d}{\rho g}}$
(b) $2\pi\sqrt{\frac{\ell \rho}{d g}}$
(c) $2\pi\sqrt{\frac{\ell d}{(\rho - d)g}}$
(d) $2\pi\sqrt{\frac{\ell \rho}{(\rho - d)g}}$

41. A container with insulating walls is divided into equal parts by a partition fitted with a valve. One part is filled with an ideal gas at a pressure P and temperature T, whereas the other part is completely evacuated. If the valve is suddenly opened, the pressure and temperature of the gas will be:

(a) $\frac{P}{2}, \frac{T}{2}$
(b) P, T
(c) $P, \frac{T}{2}$
(d) $\frac{P}{2}, T$

42. In a Young's double slit experiment, the two slits act as coherent sources of waves of equal amplitude A and wavelength λ. In another experiment with the same arrangement the two slits are made to act as incoherent sources of waves of same amplitude and wavelength. If the intensity at the middle point of the screen in the first case is I_1 and in the second case is I_2, then the ratio $\frac{I_1}{I_2}$ is:

(a) 2
(b) 1
(c) 0.5
(d) 4

43. The output of an OR gate is connected to both the inputs of a NAND gate. The combination will serve as a :

(a) NOT gate
(b) NOR gate
(c) AND gate
(d) OR gate

44. Two positive charges of magnitude 'q' are placed at the ends of a side (side 1) of a square of side '$2a$'. Two negative charges of the same magnitude are kept at the other corners. Starting from rest, if a charge Q moves from the middle of side 1 to the centre of square, its kinetic energy at the centre of square is:

(a) Zero
(b) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1+\frac{1}{\sqrt{5}}\right)$
(c) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1-\frac{2}{\sqrt{5}}\right)$
(d) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1-\frac{1}{\sqrt{5}}\right)$

45. Combination of two identical capacitors, a resistor R and a dc voltage source of voltage $6V$ is use in an experiment on a $(C - R)$ circuit. It is found that for a parallel combination of the capacitor the time in which the voltage of the fully charged combination reduces to half its original voltage is 10 second. For series combination the time needed for reducing the voltage of the fully charged series combination by half is:

(a) 10 second
(b) 5 second
(c) 2.5 second
(d) 20 second

46. A beaker contains water up to a height h_1 and kerosene of height h_2 above water so that the total height of (water + kerosene) is $(h_1 + h_2)$. Refractive index of water is μ_1 and that of kerosene is μ_2. The apparent shift in the position of the bottom of the beaker when viewed from above is:

(a) $\left(1+\frac{1}{\mu_1}\right)h_1-\left(1+\frac{1}{\mu_2}\right)h_2$
(b) $\left(1-\frac{1}{\mu_1}\right)h_1+\left(1-\frac{1}{\mu_2}\right)h_2$
(c) $\left(1+\frac{1}{\mu_1}\right)h_2-\left(1+\frac{1}{\mu_2}\right)h_1$
(d) $\left(1-\frac{1}{\mu_1}\right)h_2+\left(1-\frac{1}{\mu_2}\right)h_1$

47. A metal rod of Young's modulus Y and coefficient of thermal expansion α is held at its two ends such that its length remains invariant. If its temperature is raised by t°C, the linear stress developed in its is:

(a) $\frac{Y}{\alpha t}$
(b) $Y\alpha\, t$
(c) $\frac{1}{(Y\alpha t)}$
(d) $\frac{\alpha t}{Y}$

48. A travelling wave represented by $y = A \sin ((\omega t - kx)$ is superimposed on another wave represented by $y = A \sin (\omega t + kx)$. The resultant is:

(a) A wave travelling along $+ x$ direction
(b) A wave travelling along $- x$ direction
(c) A standing wave having nodes at $x = \frac{n\lambda}{2}$, $n = 0, 1, 2 \ldots$
(d) A standing wave having nodes at $x = \left(n+\frac{1}{2}\right)\frac{\lambda}{2}$; $n = 0, 1, 2 \ldots$

49. A thin circular disk of radius R is uniformly charged with density $\sigma > 0$ per unit area. The disk rotates about its axis with a uniform angular speed ω. The magnetic moment of the disk is:

(a) $\pi R^4\sigma\omega$
(b) $\frac{\pi R^4}{2}\sigma\omega$
(c) $\frac{\pi R^4}{4}\sigma\omega$
(d) $2\pi R^4\sigma\omega$

50. An aluminium sphere of 20 cm diameter is heated from 0°C to 100°C. Its volume changes by (given that coefficient of linear expansion for aluminium $\alpha_{Al} = 23 \times 10^{-6}$/°C)

(a) 2.89 *cc*
(b) 9.28 *cc*
(c) 49.8 *cc*
(d) 28.9 *cc*

51. Two mercury drops (each of radius 'r') merge to from bigger drop. The surface energy of the bigger drop, if T is the surface tension, is:

(a) $4\,\eta\, r^2T$
(b) $2\,\eta\, r^2T$
(c) $2^{8/3}\eta\, r^2T$
(d) $2^{5/3}\eta\, r^2T$

52. If a ball of steel (density $p = 7.8.$ g cm^{-3}) attains a terminal velocity of 10 cm s^{-1} when falling in a water (Coefficient of Viscosity $\eta_{\text{water}} = 8.5 \times 10^{-4}$ Pa.s) then its terminal velocity in glycerine ($p = 1.2.$ g cm^{-3}, $\eta = 13.2$ Pa.s) would be, nearly:

(a) 6.25×10^{-4} cm s^{-1}
(b) 6.45×10^{-4} cm s^{-1}
(c) 1.5×10^{-5} cm s^{-1}
(d) $1.6. \times 10^{-5}$ cm s^{-1}

53. A horizontal straight wire 20 m long extanding from to east to west falling with a speed of 5.0 M/s, at right angles to the horizontal component of the earth's magnetic field 0.30×10^{-4} Wb\m^2. The instantaneous Value of the e.m.f. induced in the wire will be:

(a) 3 mV
(b) 4.5. mV
(c) 1.5 mV
(d) 6.0 mV

54. After absorbing a slowly moving neutron of Mass m_N (momentum ≈ 0) a nucleus of mass M breaks into two nuclei of masses m_1 and $5m_1$ ($6\, m_1 = M + m_N$) respectively. If the de Broglic wavelength of the nucleus with mass m_1 is λ, the de Broglie wevelenght of the nucleus will be:

(a) 5λ
(b) $\lambda\backslash 5$
(c) λ
(d) 25λ

55. Which of the following four alternatives is not correct?

We need modulation:

(a) to reduce the time lag between transmission and reception of the information signal
(b) to reduce the size of antenna
(c) to reduce the e fractional band width, that is the ratio of the signal band width to the centre frequency
(d) to increase the selectivity.

56. If a spring of stiffness 'k' is cut into two parts 'A' and 'B' of length $\ell_A : \ell_B = 2:3$, then the stiffness of spring 'A' is given by :

(a) $\frac{3k}{5}$ (b) $\frac{2k}{5}$

(c) k (d) $\frac{5}{2k}$

57. Statement-1:

A nucleus having energy E_1 decays by β^-emission to daughter nucleus having energy E_2, but the β^- rays are emitted with a continuous energy spectrum having end point energy $E_1 - E_2$.

Statement-2:

To conserve energy and momentum in β-decay at least three particles must take part in the transformation.

(a) Statement-1 is correct but statement-2 is not correct.
(b) Statement-1 and statement-2 both are correct and statement-2 is the correct explanation of statement-1.
(c) Statement-1 is correct, statement-2 is correct and statement-2 is not the correct explanation of statement-1
(d) Statement-1 is incorrect, statement-2 is correct.

58. When monochromatic red light is used instead of blue light in a convex lens, its focal length will:

(a) increase
(b) decrease
(c) remain same
(d) does not depend on colour of light

59. Statement-1:

On viewing the clear blue portion of the sky through a Calcite Crystal, the intensity of transmitted light varies as the crystal is rotated.

Statement-2:

The light coming from the sky is polarized due to scattering of sun light by particles in the atmosphere. The scattering is largest for blue light

(a) Statement-1 is true, statement-2 is false.
(b) Statement-1 is true, statement-2 is true, statement-2 is the correct explanation of statement-1
(c) Statement-1 is true, statement-2 is true, statement-2 is not the correct explanation of statement-1
(d) Statement-1 is false, statement-2 is true.

60. Statement-1:

Two longitudinal waves given by equation : $y_1(x,t) = 2a \sin(\omega t - kx)$ and $y_2(x, t) = a \sin(2\omega t - 2kx)$ will have equal intensity.

Statement-2:

Intensity of waves of given frequency in same medium is proportional to square of amplitude only.

(a) Statement-1 is true, statement-2 is false.
(b) Statement-1 is true, statement-2 is true, statement-2 is the correct explanation of statement-1
(c) Statement-1 is true, statement-2 is true, statement-2 is not the correct explanation of statement-1
(d) Statement-1 is false, statement-2 is true.

MATHEMATICS

61. Let f be a function defined by $f(x) = (x - 1)^2 + 1, (x \geq 1)$.

Statement-1:

The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$.

Statement-2:

f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

62. If $\omega \neq 1$ is the complex cube root of unity and matrix $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then H^{70} is equal to

(a) 0 (b) $-H$
(c) H^2 (d) H

63. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2]\,dx$ is:

(a) 0 (b) $\frac{3}{2}$

(c) $\frac{3}{4}$ (d) $\frac{5}{4}$

64. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact is given by:

(a) $2y - 3x = 0$ (b) $y = \frac{6}{x}$

(c) $x^2 + y^2 = 13$ (d) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$

65. A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 gm and a standard deviation of 2 gm. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2 gm. The correct mean and standard deviation (in gm) of fishes are respectively:

(a) 32, 2 (b) 32, 4
(c) 28, 2 (d) 28, 4

66. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then the set of all possible values of a is the interval:

(a) $(0, \infty)$ (b) $[1, \infty)$
(c) $(-1, \infty)$ (d) $(-1, 1]$

67. If the vector $p\hat{i} + \hat{j} + \hat{k}, \hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is-

(a) 2 (b) 0
(c) -1 (d) -2

68. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along a straight line $x = y = z$ is:

(a) $10\sqrt{3}$ (b) $5\sqrt{3}$

(c) $3\sqrt{10}$ (d) $3\sqrt{5}$

69. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a}, + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}, + 3\vec{b}, + 6\vec{c}$ is:

(a) $\vec{a}$ (b) $\vec{c}$

(c) $\vec{0}$ (d) $\vec{a} + \vec{c}$

70. If $A(2, -3)$ and $B(-2, 1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is:

(a) $x - y = 1$ (b) $2x + 3y = 1$

(c) $2x + 3y = 3$ (d) $2x - 3y = 1$

71. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points then:

(a) $N \le 100$ (b) $100 < N \le 140$

(c) $140 < N \le 190$ (d) $N > 190$

72. Define $F(x)$ as the product of two real functions $f_1(x) = x, x \in R$, and $f_2(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ as follows:

$$F(x) = \begin{cases} f_1(x).f_2(x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Statement-1 : $F(x)$ is continuous on R.

Statement-2 : $f_1(x)$ and $f_2(x)$ are continuous on R.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

73. **Statement-1:** For each natural number n, $(n + 1)^7 - n^7 - 1$ is divisible by 7.

Statement-2: For each natural number n, $n^7 - n$ is divisible by 7.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 false, Statement-2 is true

74. The equation of the circle passing through the point (1, 0) and (0, 1) and having the smallest radius is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$

(b) $x^2 + y^2 - x - y = 0$

(c) $x^2 + y^2 + 2x + 2y - 7 = 0$

(d) $x^2 + y^2 + x + y - 2= 0$

75. The equation of hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by:

(a) $x^2 - 3y^2 = 3$ (b) $3x^2 - y^2 = 3$

(c) $-x^2 + 3y^2 = 3$ (d) $-3x^2 + y^2 = 3$

76. If the trivial solution is the only solution of the system of equations

$$x - ky + z = 0$$
$$kx + 3y - kz = 0$$
$$3x + y - z = 0$$

then the set of all values of k is:

(a) $R - \{2, -3\}$ (b) $R - \{2\}$

(c) $R - \{-3\}$ (d) $\{2, -3\}$

77. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are:

(a) 6, 1 (b) 4, 3

(c) –6, –1 (d) –4, –3

78. Let a_n be the n^{th} term of $A.P.$ If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the $A.P.$ is

(a) $\alpha - \beta$ (b) $\frac{\alpha - \beta}{100}$

(c) $\beta - \alpha$ (d) $\frac{\alpha - \beta}{200}$

79. Consider the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by:

(a) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (b) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$

(c) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$ (d) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$

80. Let f: R → [0, ∞) be such that $\lim_{x \to 5} f(x)$ exists and

$$\lim_{x \to 5} \frac{(f(x))^2 - 9}{\sqrt{|x - 5|}} = 0$$

Then $\lim_{x \to 5} f(x)$ equals:

(a) 0 (b) 1

(c) 2 (d) 3

81. **Statement-1:**

Determinant of a skew-symmetric matrix of order 3 is zero.

Statement-2:

For any matrix A, $\det(A)^T = \det(A)$ and $\det(-A) = -\det(A)$. Where det(2) denotes the determinant of matrix B. Then:

(a) Both statements are true

(b) Both statements are false

(c) Statement-1 is false and Statement-2

(d) Statement-1 is true and Statement-2 is false

82. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are:

(a) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (b) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$

(c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (d) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$

83. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is:

(a) $\frac{32}{3}$ (b) $\frac{16}{3}$

(c) $\frac{8}{3}$ (d) 0

84. Let f be a function defined by

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad x \neq 0$$

Statement-1: $x = 0$ is point of minima of f

Statement-2: $f'(0) = 0$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

85. The only statement among the following that is a tautology is

(a) $A \wedge (A \vee B)$
(b) $A \vee (A \wedge B)$
(c) $[A \wedge (A \to B)] \to B$
(d) $[B \to [A \wedge (A \to B)]$

86. Let A, B, C be pariwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$

Then $P(A^c \cap B^c/C)$.

(a) $P(1) - P(B^c)$ (b) $P(A^c) + P(B^c)$
(c) $P(A^c) - P(B^c)$ (d) $P(A^c) - P(B)$

87. Let for $a \neq a_1 \neq 0$,

$f(x) = ax^2 + bx + c$, g9x) $= a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$.

if $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is:

(a) 3 (b) 9
(c) 6 (d) 18

88. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is:

(a) $\sqrt{29}$ (b) $\sqrt{33}$
(c) $\sqrt{53}$ (d) $\sqrt{66}$

89. Consider the following relation R on the set of real square matrices of order 3.

$R = \{(A, B) \cap A = P^{-1}BP$ for some invertible matrix $P\}$.

Statement-1:
R is equivalence relation.

Statement-2:
For any two invertible 3×3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$.

(a) Statement-1 is true, statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1.
(c) Statement-1 is true, statement-2 is false.
(d) Statement-1 is false, statement-2 is true.

90. If function $f(x)$ is differentiable at $x = a$,

then $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ is:

(a) $-a^2 f'(a)$ (b) $af(a) - a^2 f'(a)$
(c) $2af(a) - a^2 f'(a)0$ (d) $2af(a) + a^2 f'(a)$

ANSWERS

1. (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (b)	**10.** (c)
11. (d)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (a)
21. (b)	**22.** (d)	**23.** (d)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (b)
31. (c)	**32.** (a)	**33.** (a)	**34.** (c)	**35.** (a)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (a)
41. (d)	**42.** (a)	**43.** (b)	**44.** (d)	**45.** (c)	**46.** (b)	**47.** (b)	**48.** (d)	**49.** (c)	**50.** (d)
51. (c)	**52.** (a)	**53.** (a)	**54.** (c)	**55.** (a)	**56.** (d)	**57.** (b)	**58.** (a)	**59.** (b)	**60.** (a)
61. (a)	**62.** (d)	**63.** (d)	**64.** (b)	**65.** (a)	**66.** (b)	**67.** (d)	**68.** (a)	**69.** (c)	**70.** (b)
71. (a)	**72.** (b)	**73.** (a)	**74.** (b)	**75.** (b)	**76.** (a)	**77.** (a)	**78.** (b)	**79.** (c)	**80.** (d)
81. (d)	**82.** (d)	**83.** (b)	**84.** (b)	**85.** (c)	**86.** (d)	**87.** (d)	**88.** (c)	**89.** (b)	**90.** (c)

SOLUTIONS

Chemistry

1. Layer of O_3 in upper atmosphere absorbs UV radiations from sun, thus protects the human being living on the earth.

Nitrogen oxides (particularly nitric oxide) combine very rapidly with ozone and there is, thus, the possibility that nitrogen oxides emitted from the exhaust systems of supersonic jet aeroplanes might be slowly depleting the concentration of the ozone layer in the upper atmosphere.

The use of chlorofluoro carbon in aerosol and refrigerator and there, subsequent escape into the atmosphere is responsible for making holes in the ozone layer. Ozone at all does not absorb infrared radiation.

2. $r \propto \dfrac{p}{\sqrt{M}}$

$$\frac{r_A}{r_B} = \frac{p_A}{p_B}\sqrt{\frac{M_B}{M_A}}$$

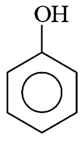
3. $RS^{\ominus}$ is more nucleophilic than $RO^{\ominus}$ due to larger size of orbitals and polarization but $RS^{\ominus}$ is less basic than $R–O^{\ominus}$as the negative charge get stabilized due to larger size of sulphur atom.

4. Freshly prepared glucose solution get equilibriated to an unequal mixture of (+) and (–) glucose. This phenomenon is called mutarotation.

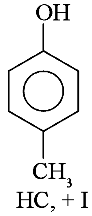
5. Molality $= \dfrac{0.01/60}{0.3} = \dfrac{0.01}{60 \times 0.3}$; $d = 1$ g/ml
$= 5.55 \times 10^{-4}$ m

6. $v = \sqrt{\dfrac{8RT}{\pi M}}$
$v \propto \sqrt{T}$

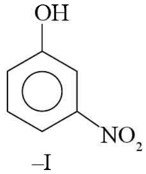
7.



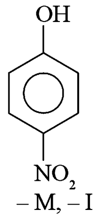
(A)



(B)



(C)



(D)

Therefore acidity order is: $D > C > A > B$

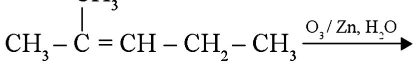
8. $C_2H_5OH(\ell) + 3O_2(g) \longrightarrow 2CO_2(g) + 3H_2O(\ell)$
$\Delta n_g = 2 - 3 = -1$
$\Delta U = \Delta H - \Delta n_g RT$

$$= -1366.5 - (-1) \times \frac{8.314}{10^3} \times 300$$

$= -1366.5 + 0.8314 \times 3 = -1364$ KJ

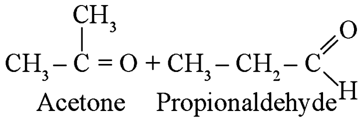
9. Bakelite is a polymer of phenol and HCHO (formaldehyde).

10.



$CH_3-\overset{CH_3}{\overset{|}{C}}=CH-CH_2-CH_3 \xrightarrow{O_3/Zn,\,H_2O}$

(2-Methyl-2-pentene)
(A)



$CH_3-\overset{CH_3}{\overset{|}{C}}=O + CH_3-CH_2-C\genfrac{}{}{0pt}{}{/\!/O}{\diagdown H}$

Acetone Propionaldehyde

11.

$$\begin{tikzcd} 4NO_2(g) + O_2(g) \arrow[r] \arrow[dr, "\Delta H'"'] & 2N_2O_5(g),\ \Delta H = -111\text{ kg} \arrow[d, "-54"] \\ & 2N_2O_5(S) \end{tikzcd}$$

$-111 - 54 = \Delta H'$
$\Delta H' = -165$ KJ

12. $K_a = \dfrac{[H^+)^2}{C-[H^+)}$, $[H^+] \ll C$

$$= \frac{(10^{-5})^2}{1} = 10^{-10}$$

13. As we move in a group from top to bottom, electron gain enthalpy becomes less negative because the size of the atom increase and the added electron would be at larger distance from the nucleus.

Negative electron gain enthalpy of F is less than Cl. This is due to the fact that when an electron is added to F, the added electron goes to the smaller $n = 2$ energy level and experiences significant repulsion from the other electrons present in this level. In Cl, the electron goes to the larger $n = 3$ energy level and consequently occupies a larger region of space leading to much less electron-electron repulsion. So the correct order is Cl > F> Br > I.

14. $h\nu = \Delta E = 13.6z^2\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$

$$v_{He+} = v_H \times z^2\left[\frac{1}{\left(\frac{n_1}{2}\right)^2} - \frac{1}{\left(\frac{n_2}{2}\right)^2}\right]$$

$$= v_H\left[\frac{1}{\left(\frac{2}{2}\right)^2} - \frac{1}{\left(\frac{4}{2}\right)^2}\right]$$

For H-atom
$n_1 = 1,\ n_2 = 2$

15. $\pi_1 = \pi_2$
$C_1 = C_2$

$$\frac{5/342}{0.1} = \frac{1/M}{0.1}$$

$$\frac{5}{342} = \frac{1}{M} \Rightarrow M = \frac{342}{5} = 68.4 \text{ gm/mol}$$

16. Negative $\Delta_r G^\circ$ value indicates that + 2 oxidation state is more stable for Pb^{+2}. Also it is supported by inert pair effect that + 2 oxidation state is more stable for Pb and + 4 oxidation state is more stable for Sn.

17. $Cr(OH)_3(s) \rightleftharpoons Cr^{3+}(aq.) + 3OH^-(aq.)$
S 3S
$27\ S^4 = k_{sp}$

$$S = \left(\frac{K_{sp}}{27}\right)^{1/4} = \left(\frac{1.6 \times 10^{-30}}{27}\right)^{1/4}$$

18. $4LiNO_3 \longrightarrow 2Li_2O + 4NO_2 + O_2$

19. $k = \frac{1}{R} \times \frac{l}{A}$

$$1.3 = \frac{1}{50} \times \frac{l}{A}$$

$$\frac{l}{A} = 65\ m^{-1}$$

$$\Lambda_m = \frac{K}{1000 \times \text{molarity}} \left(\frac{K}{1000 \times (\text{molarity})} \right)$$

$$= \frac{\left(\frac{1}{2604} \times 65 \right)}{1000 \times 0.40} = 6.25 \times 10^{-4} = \frac{1}{4 \times 0.4 \times 1000}$$

$$= \frac{1}{1600} = 6.25 \times 10^{-4} \mathrm{Sm^2 mol^{-1}}$$

20. Ligands can be arranged in a series in the orders of increasing field strength as given below:

$I^- < Br^- < S^{2-} < Cl^- < NO_3^- < F^- < OH^- < EtOH < C_2O_4^{2-} < H_2O <$ $NCS^- < EDTA < NH_3 < en < NO_2^- < CN < CO$

Such a series is termed as spectrochemical series. It is an experimentally determined series based on the absorption of light by complexes with different ligands.

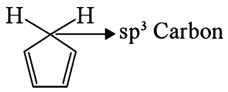
21. $C_2H_5 - OH + H_2SO_4 \xrightarrow{170°C} CH_2 = CH_2$

$\xrightarrow{140°} CH_3 - CH_2 - O - CH_2 - CH_3$

$\xrightarrow{\text{Room temp.}} C_2H_5SO_3H$

Acetylene is not formed under any conditions.

22.



Cyclopentadiene does not obey Huckel's Rule, as it has sp³ carbon in the ring.

23. Calcium carbide exists as Ca^{2+} and C_2^{2-}. According to the molecular orbital model, C_2^{2-} should have a bond order of 3 (configuration $\pi_u^2 \pi_u^2 \sigma_g^2$). MOT configuration suggests that it contains one σ and two π-bonds $[:C \overset{\pi}{\underset{\pi}{\overset{\sigma}{\equiv}}} C:]^{2-}$

24. $k_1 = A_1 e^{-Ea_1/RT}$

$$k_2 = A_2 e^{-Ea_2/RT}$$

$$\frac{K_1}{K_2} = \frac{A_1}{A_2} e^{(Ea_2 - Ea_1)/RT}$$

$$K_1 = K_2 A \times e^{Ea_1/RT}$$

25. FCC lattice

$a = 361$ pm $\qquad a\sqrt{2} = 4r$

$$r = \frac{361 \times \sqrt{2}}{4} = 127.6 \approx 128 \text{ pm.}$$

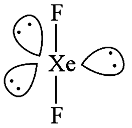
26. $0.6 \times 0.75 \times 1 = \frac{w}{294} \times 6\ w = 22.05$ g

27. O^{2-} (base) + H_2O (acid) $\longrightarrow$ OH^- (C.B) + OH^- (C.A.)

O^{2-} acts as Lewis base.

28.

XeF_2:



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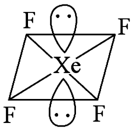
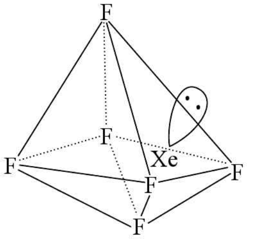
XeF_4:



XeF_6:



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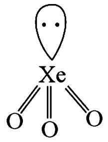
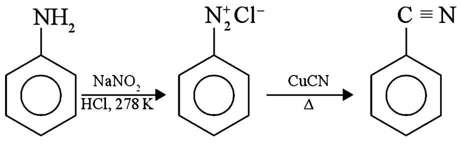
XeO_3:



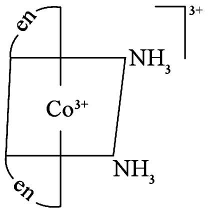
29.



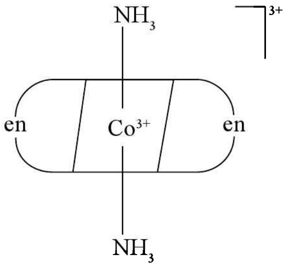
Benzene diazonium chloride (A)

Benzonitrile (B)

30.



cis–



trans–

Physics

31. K.E. = ct

$$\frac{1}{2} mv^2 = ct$$

$$\frac{P^2}{2m} = ct$$

$$P = \sqrt{2ctm}$$

$$F = \frac{dP}{dt} = \sqrt{2\mathrm{cm}}\ \frac{1}{2} \times \frac{1}{\sqrt{t}}$$

$$F \propto \frac{1}{\sqrt{t}}.$$

32. $\Delta x_1 = 0$

$$\Delta\phi = 0°$$

$$I_1 = I_0 + I_0 + 2I_0 \cos 0° = 4I_0$$

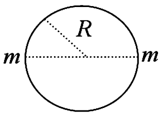
$$\Delta x_2 = \frac{\lambda}{4}$$

$$\Delta\theta = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \left(\frac{\pi}{2} \right)$$

$$I_2 = I_0 + I_0 + 2I_0 \cos \frac{\pi}{2} = 2I_0$$

$$\frac{I_1}{I_2} = \frac{4I_0}{2I_0} = \frac{2}{1}.$$

33. $\frac{Gm^2}{(2R)^2} = m\omega^2 R$



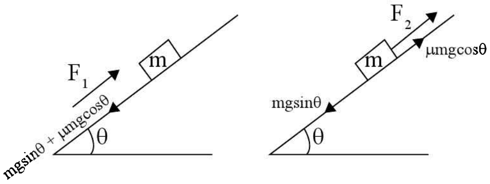
$\frac{Gm^2}{4R^3} = \omega^2$

$\omega = \sqrt{\frac{Gm}{4R^3}}$

$v = \omega R$

$v = \sqrt{\frac{Gm}{4R^3}} \times R = \sqrt{\frac{Gm}{4R}}$

34.



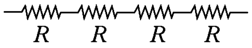
$F_1 = \text{mg}\sin\theta + \mu\,\text{mg}\cos\theta$

$F_2 = \text{mg}\sin\theta - \mu\,\text{mg}\cos\theta$

$\frac{F_1}{F_2} = \frac{\sin\theta + \mu\cos\theta}{\sin\theta - \mu\cos\theta}$

$\frac{\tan\theta + \mu}{\tan\theta - \mu} = \frac{2\mu + \mu}{2\mu - \mu} = \frac{3\mu}{\mu} = 3.$

35.



$R = 100 \pm 5$

$4R = 400 \pm 20$

Tolerance of combination is also 5%.

36. $\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]$

$$= q\left[3\hat{i} + \hat{j} + \hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 1 \\ 1 & 1 & -3 \end{vmatrix}\right]$$

$$= q[3\hat{i} + \hat{j} + 2\hat{k} + \hat{i}) - 12 - 1) - \hat{j}(-9-1) + \hat{k}(3-4)]$$

$$= q[3\hat{i} + \hat{j} + 2\hat{k} - 13\hat{i} + 10\hat{j} - \hat{k}]$$

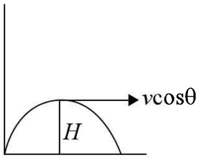
$$= q[-10\hat{i} + 11\hat{j} + \hat{k}]$$

$$= F_y = 11q\hat{j}.$$

37. $X = \frac{V}{\ell} = \frac{IR}{\ell} = \frac{IR}{\ell}\left(\frac{\rho\ell}{A}\right) = \frac{I\rho}{A}$

$x = \frac{0.2 \times 4 \times 10^{-7}}{8 \times 10^{-7}} = \frac{0.8}{8} = 0.1\,V/\text{m}.$

38.



$L_0 = Pr_{\perp}$

$L_0 = \text{m}v\cos\theta\, H$

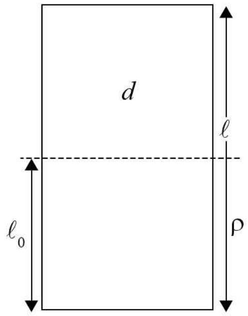
$$= \text{mg}\frac{\sqrt{3}}{2}.\frac{v^2\sin^2 30°}{2g} = \frac{\sqrt{3}mv^3}{16\text{g}}.$$

39. $Q = \int mcdT$

$$= \int_{20}^{4} 0.1 \times 32 \times \left(\frac{T^3}{400^3}\right)dT$$

≈ 0.002 k J.

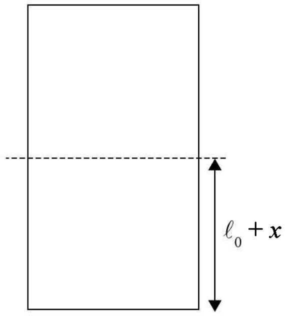
40.



Al equilibrium

$F_b = \text{mg}$

$\rho A\ell_0 g = dA\ell g$ .....(i)



Restoring force,

$F = \text{mg} - F_b'$

$F = \text{mg} - \rho A(\ell_0 + x)g$

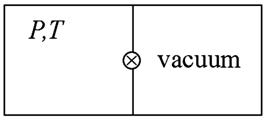
$dA\ell a = dA\ell g - \rho A\ell_0 g = \rho g A x$

$a = \frac{\rho g}{d\ell}x$

$\omega = \sqrt{\frac{\rho g}{d\ell}}$

$T = 2\pi\sqrt{\frac{\ell d}{\rho g}}$ .....(i)

41.



It is the free expansion

So, T remain constant

$P_1V_1 = P_2V_2$

$P\frac{V}{2} = P_2(V)$

$P_2 = \left(\frac{P}{2}\right).$

42. For coherent sources:

$I_1 = 4I_0$

For coherent sources

$I_2 = 2I_0$

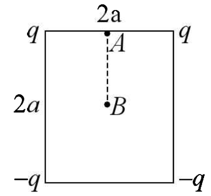
$\frac{I_0}{I_2} = \frac{2}{1}.$

43. $\overline{(A+B)}$ = NOR gate

When both inputs of NAND gate are connected, it behaves as NOT gate OR + NOT = NOR.

44. Potential at point A,

$$V_A = \frac{2Kq}{a} - \frac{2Kq}{a\sqrt{5}}$$



Potential at point B,

$V_B = 0$

∴ Using work energy theroem,

$$W_{AB})_{\text{electric}} = Q(V_A - V_B)$$

$$= \frac{2KqQ}{a}\left[1 - \frac{1}{\sqrt{5}}\right]$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{2Qq}{a}\left[1 - \frac{1}{\sqrt{5}}\right]$$

45. Time constant for parallel combination = $2RC$

Time constant for series combination = $\frac{RC}{2}$

In first case :

$$V = V_0 e^{\frac{t_1}{2RC}} = \frac{V_0}{2} \quad \text{........(i)}$$

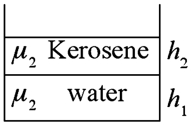
In second case:

$$V = V_0 e^{\frac{t_2}{(RC/2)}} = \frac{V_0}{2} \quad \text{........(ii)}$$

From (i) & (ii),

$$\frac{t_1}{2RC} = \frac{t_2}{(RC/2)} \quad \Rightarrow \quad t_2 = \frac{t_1}{4} = \frac{10}{4} = 2.5 \text{ sec.}$$

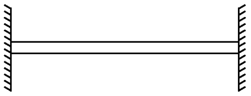
46.



Apparent shift:

$$= h_1\left(1 - \frac{1}{\mu_1}\right) + h_2\left(1 - \frac{1}{\mu_2}\right).$$

47. $\frac{\Delta\ell}{\ell} = \alpha.\Delta T$



and $Y = \frac{F/A}{\Delta\ell/\ell}$

So, $F = AY\alpha\, t$

Thermal stress $\left(\frac{F}{A}\right) = Y\alpha t.$

48. $Y = A\sin(\omega t - kx) + A\sin(\omega t + kx)$

$Y = 2A\sin\omega t\cos kx$ standing wave

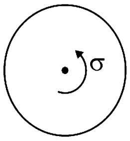
For nodes $\cos kx = 0$

$$\frac{2\pi}{\lambda}.x = (2n+1)\frac{\pi}{2}$$

$$\therefore \quad x = \frac{(2n+1)\lambda}{4},\ n = 0, 1, 2, 3 \text{.........}$$

49. $\frac{q}{2M} = \frac{\text{Magnetic dipole moment}}{\text{Angular momentum}}$

∴ Magnetic dipole moment (M)



$$M = \frac{q}{2M}.\left(\frac{MR^2}{2}\right).\omega$$

$$= \frac{1}{4}\sigma.\pi R^4\omega.$$

50. $\Delta v = v_0(3\alpha)\,\Delta T$

$$= \frac{4}{3}\pi(10)^3 \times 3 \times 23 \times 10^{-6} \times 100$$

$\Delta v = 28.9\ cc$

51. $2.\frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$

$R = 2^{1/3}\, r$

$S.E. = T.4\pi R^2$

$T.4\pi\, 2^{2/3}\, r^2$

$T.\ 2^{8/3}\, \pi r^2.$

52. $V\rho g = 6\pi\eta r v + v\rho_\ell g$

$Vg(\rho - \rho_\ell) = 6\pi\eta r v$

$Vg(\rho - \rho_\ell') = 6\pi\eta' r v'$

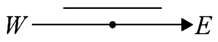
$$V'\eta' = \frac{(\rho - \rho_\ell')}{(\rho - \rho_\ell)} \times v\eta$$

$$V' = \frac{(\rho - \rho_\ell')}{(\rho - \rho_\ell)} \times \frac{v\eta}{\eta'}$$

$$= \frac{(7.8 - 1.2)}{(7.8 - 1)} \times \frac{10 \times 8.5 \times 10^{-4}}{13.2}$$

$v' = 6.25 \times 10^{-4}$ cm/s.

53.



$\varepsilon_{\text{ind}} = Bv\ell$

$= 0.3 \times 10^{-4} \times 5 \times 20$

$= 3 \times 10^{-3}$ v

$= 3$ mv

54. $P_i = 0$

$P_f = P_1 + P_2$

$P_i = P_f$

$0 = P_1 + P_2$

$(P_1 = -P_2)$

$\lambda_1 = \frac{h}{P_1}$

$\lambda_2 = \frac{h}{P_2}$

$|\lambda_1| = |\lambda_2|$

$\lambda_1 = \lambda_2 = \lambda.$

55. Low frequencies cannot be transmitted to long distances. Therefore, they are super imposed on a high frequency carrier signal by a process known as modulation.

Speed of electro–magnetic waves will not change due to modulation. So time lag between transmission and reception of the information signale.

56. $\ell_A = \frac{2\ell}{5}, \ell_B = \left(\frac{3\ell}{5}\right)$

$$K\ell = K_A\ell_A = K_B\ell_B$$

$$K\ell = K_A\left(\frac{2\ell}{5}\right)$$

$$K_A = \frac{5K}{2} \Rightarrow K_B = \frac{5K}{3}$$

57. **Statement-1:** Energy of β^- particle from 0 to maximum so $E_1 - E_2$ is the continuous energy spectrum.

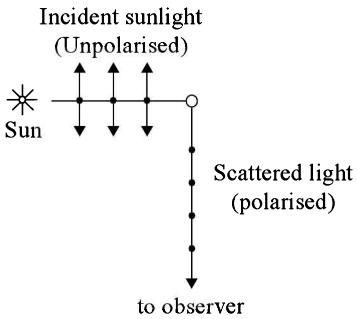
Statement-2: For energy conservation and momentum at least particles daughter nucleus + β^{-1} and antineutron.

58. $\mu_R < \mu_B$

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \frac{1}{f_B} > \frac{1}{f_R}$$

$f_R > f_B$.

59. The light from a clear blue portion of the sky shows a rise and fall of intensity when viewed through a polaroid which in rotated



60. Since, $I \mu A^2\omega^2$

$I_1 \mu (2a)^2 \omega^2$

$I_2 \mu a^2 (2\omega)^2$

$I_1 = I_2$

Intensity depends on frequency also.

Mathematics

61. $f(x) = (x - 1)^2 + 1, x \geq 1$

$f: [1, \infty) \rightarrow [1, \infty)$ is a bijective function

$\Rightarrow y = (x - 1)^2 + 1 \Rightarrow (x - 1)^2 = y - 1$

$\Rightarrow x = 1 \pm \sqrt{y-1} \Rightarrow f^{-1}(y) = 1 \pm \sqrt{y-1}$

$\Rightarrow f^{-1}(x) = 1 + \sqrt{x-1} \; \{\because x \geq 1\}$

so statement-2 is correct

Now $f(x) = f^{-1}(x) \Rightarrow f(x) = x \Rightarrow (x - 1)^2 + 1 = x$

$\Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

so statement-1 is correct

62. $H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$

If $H^k = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix}$, then $H^{k+1} = \begin{bmatrix} \omega^{k+1} & 0 \\ 0 & \omega^{k+1} \end{bmatrix}$

So by mathematical induction,

$$H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix} = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = H$$

63. $\int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$

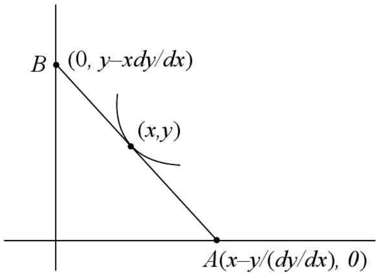
$$\int_0^1 x.0dx + \int_1^{\sqrt{2}} xdx + \int_{\sqrt{2}}^{1.5} 2x\,dx$$

$$0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[x^2\right]_{\sqrt{2}}^{1.5}$$

$$\frac{1}{2}(2 - 1) + (2.25 - 2)$$

$$\frac{1}{2} + .25;\ \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

64. $Y - y = \frac{dy}{dx}(X - x)$



X-intercept is $\left(x - \frac{y}{dy/dx}, 0\right)$

Y-intercept is $\left(0, y - \frac{xdy}{dx}\right)$

According to statement

$$x - \frac{y}{dy/dx} = 2x \text{ and } y - \frac{xdy}{dx} = 2y$$

$$\frac{-y}{\frac{dy}{dx}} = x \qquad \frac{-xdy}{dx} = y$$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\ell ny = -\ell nx + \ell nc$$

$$y = \frac{c}{x}$$

$\Rightarrow c = 6$

Hence $y = \frac{6}{x}$

65. Correct mean = observed mean + 2

30 + 2 = 32

Correct $S.D.$ = observed $S.D.$ = 2

66. $x + y = |a|$

$ax - y = 1$

if $a > 0$

$x + y = a$

$$\frac{ax - y = 1}{x(1+a) = 1 + a \text{ as } x = 1}$$

$y = a - 1$

It is first quadrant

So $a - 1 \geq 0$

$a \geq 1$

$a \geq [1, \infty)$

If $a < 0$

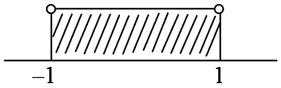
$x + y = -a$

$ax - y = 1$

+ ___________

$x(1 + a) = 1 - a$

$$x = \frac{1-a}{1+a} > 0 \Rightarrow \frac{a-1}{a+1} < 0$$

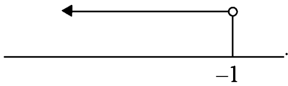


........(1)

$$y = -a - \frac{1-a}{1+a}$$

$$= \frac{-a - a^2 - 1 + a}{1+a} > 0$$

$$-\left(\frac{a^2+1}{a+1}\right) > 0 \Rightarrow \frac{a^2+1}{a+1} < 0$$



........(2)

from (1) and (2) $a \in \{\phi\}$

So correct answer is $a \in [1, \infty)$

67. $$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$\Rightarrow p(qr - 1) + 1(1 - r) + 1(1 - q) = 0$

$\Rightarrow pqr - p + 1 - r + 1 - q = 0$

$\Rightarrow pqr - (p + q + r) = -2$

68. Line through $P(1, -5, 9)$ parallel to $x = y = z$ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda \text{ (say)}$$

$Q(x = 1 + \lambda, y = -5 + \lambda, z = 9 + \lambda)$

Given plane $x - y + z = 5$

$\therefore 1 + \lambda + 5 - \lambda + 9 + \lambda = 5$

$\Rightarrow \lambda = -10$

$\therefore Q(-9, -15, -1)$

$$\therefore PQ = \sqrt{(1+9)^2 + (15-5)^2 + (9+1)^2}$$

$$= \sqrt{300} = 10\sqrt{3}$$

69. $\vec{a} + 3\vec{b} = \lambda\vec{c}$ ....(1)

$\vec{b} + 2\vec{c} = \mu\vec{a}$ ....(2)

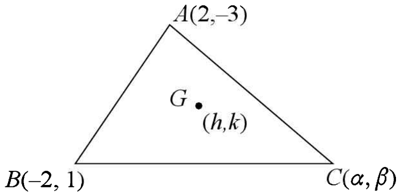
(1) – 3(2) gives $(1 + 3\mu)\,\vec{a} - (\lambda + 6)\,\vec{c} = 0$

As $\vec{a}$ and $\vec{c}$ are non collinear

$\therefore 1 + 3\mu = 0$ and $\lambda + 6 = 0$

From (1) $\vec{a} + 3\vec{b} + 6\vec{c} = 0$

70.



$\alpha = 3h$

$\beta - 2 = 3k$

$\beta = 3k + 2$

third vertex on the line

$2x + 3y = 9$

$2\alpha + 3\beta = 9$

$2(3h) + 3(3k + 2) = 9$

$2h + 3k = 1$

$2x + 3y - 1 = 0$

71. ${}^{10}C_3 - {}^{6}C_3$

$$= \frac{10 \times 9 \times 8}{6} - \frac{6 \times 5 \times 4}{6} = 120 - 20 = 100$$

72. $$f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$

$$LHL = \lim_{h \to 0^+} \left\{-h \sin\left(-\frac{1}{h}\right)\right\}$$

$= 0 \times a$ finite quantity between –1 and 1

$$RHL = \lim_{h \to 0^+} h \sin\frac{1}{h}$$

$= 0$

$f(0) = 0$

$\therefore f(x)$ is continuous on R.

$f_2(x)$ is not continuous at $x = 0$

73. **Statement-2**

$P(n) = n^7 - n$

put $n = 1$ $1 - 1 = 0$ is divisible by 7

Let $n = k$ $P(k) = k^7 - k$ is divisible by 7

Put $n = k + 1$

$\therefore P(k+1) = (k+1)^7 - (k+1)$

$= k^7 + {}^7C_1 k^6 + {}^7C_2 k^5 + \dots + {}^7C_6 k + 1 - k - 1$

$P(k+1) = (k^7 - k)$ + multiple of 7

As 7 is coprime with 1,2,3,4,5,6, so ${}^7C_1, {}^7C_2, \dots {}^7C_6$ are all divisible by 7

$P(k+1)$ is divisible by 7
Hence $P(n) = n^7 - n$ is divisible by 7

Statement-1

$n^7 - n$ is divisible by 7

$\Rightarrow (n+1)^7 - (n+1)$ is divisible by 7

$\Rightarrow (n+1)^7 - n^7 - 1 + (n^7 - n)$ is divisible by 7

$\Rightarrow (n+1)^7 - n^7 - 1$ is divisible by 7

74. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.

$(x-1)(x-0) + (y-0)(y-1) = 0$

$x^2 + y^2 - x - y = 0$

75. $ae = 2$

$e = 2$

$\therefore a = 1$

$b^2 = a^2(e^2 - 1)$

$b^2 = 1(4 - 1)$

$b^2 = 3$

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

$3x^2 - y^2 = 3$

76. $x - ky + z = 0$

$kx + 3y - kz = 0$

$3x + y - z = 0$

this equation will have non trivial solution if

$$\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$1(-3 + k) + k(-k + 3k) + 1(k - 9) = 0$

$k - 3 + 2k^2 + k - 9 = 0$

$2k^2 + 2k - 12 = 0$

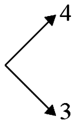
$k^2 + k - 6 = 0$

$k = -3, k = 2$

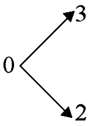
so the equation will have only trivial solution when $k \in R - \{2, -3\}$

77. Let the correct equation be $ax^2 + bx + c = 0$

now sachin's equation $\Rightarrow ax^2 + bx + c' = 0$



Rahul's equation $\Rightarrow ax^2 + b'x + c = 0$



$$-\frac{b}{a} = 7 \quad \dots\dots\dots (i)$$

$$\frac{c}{a} = 6 \quad \dots\dots\dots (ii)$$

from (i) and (ii)

correct equation is

$x^2 - 7x + 6 = 0$

roots are 6 and 1

78. Let $A.P.$ be $a, a + d, a + 2d, \dots\dots$

$a_2 + a_4 + \dots\dots a_{200} = \alpha$

$$\Rightarrow \frac{100}{2}\left[2(a+d) + (100-1)d\right] = \alpha \quad \dots (i)$$

and $a_1 + a_3 + a_5 + \dots\dots a_{199} = \beta$

$$\Rightarrow \frac{100}{2}\left[2a + (100-1)d\right] = \beta \quad \dots (ii)$$

on solving (i) and (ii)

$$d = \frac{\alpha - \beta}{100}$$

79. $$\frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

$$\text{I.F.} = e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$$

$$\text{so} \quad x.e^{-\frac{1}{y}} = \int \frac{1}{y^3} e^{-\frac{1}{y}} dy$$

$$\text{Let } \frac{-1}{y} = t$$

$$\Rightarrow \frac{1}{y^2} dy = dt$$

$$\Rightarrow I = -\int te^t dt = e^t - te^t$$

$$= e^{-\frac{1}{y}} + \frac{1}{y} e^{-\frac{1}{y}} + c$$

$$\Rightarrow xe^{-\frac{1}{y}} = e^{-\frac{1}{y}} + \frac{1}{y} e^{-\frac{1}{y}} + c$$

$$\Rightarrow x = 1 + \frac{1}{y} + c.e^{1/y}$$

since $y(1) = 1$

$$\therefore c = -\frac{1}{e}$$

$$\Rightarrow x = 1 + \frac{1}{y} - \frac{1}{e}.e^{1/y}$$

80. $$\lim_{x \to 5} \frac{(f(x)^2) - 9}{\sqrt{|x-5|}} = 0$$

$$\lim_{x \to 5} \left[(f(x))^2 - 9\right] = 0$$

$$\lim_{x \to 5} f(x) = 3$$

81. **Statement-1:** Determinant of a skew sysmmetric matrix of odd order is zero

Statement-2: det (A^T) = det(A)
det$(-A) = (-1)^n$ det(A) where A is a $n \times n$ order matrix

82. $\sin 4\theta + 2\sin 4\theta \cos 3\theta = 0 \quad \therefore \theta, \in (0, \pi)$

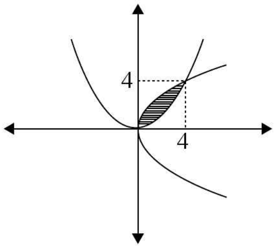
$\sin 4\theta\,(1 + 2\cos 3\theta) = 0$

$\sin 4\theta = 0$ or $\cos 3\theta = -\dfrac{1}{2}$

$4\theta = n\pi\ n \in I$ or $3\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in I$

$\theta = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}$ or $\theta = \dfrac{2\pi}{9}, \dfrac{8\pi}{9}, \dfrac{4\pi}{9}$

83.



$$\text{Area} = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left(2\left(\frac{x^{3/2}}{3/2}\right) - \frac{x^3}{12}\right)_0^4$$

$$= \frac{4}{3} \times 8 - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

84. $f(x) = \begin{cases} \dfrac{\tan x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

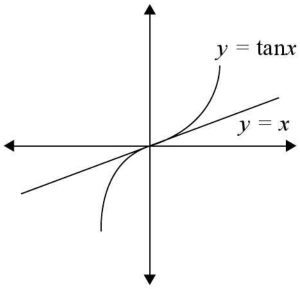
In right neighbourhood of '0'
$\tan x > x$

$\dfrac{\tan x}{x} > 1$

In left neighbourhood of '0'
$\tan x < x$

$\dfrac{\tan x}{x} > 1$ as $(x < 0)$

at $x = 0$, $f(x) = 1$
$\Rightarrow x = 0$ is point of minima
so statement 1 is true.
statement 2 obvious



85.

A	B	A ∨ B	A ∧ B	A ∧ (A ∨ B)	A ∧ (A ∨ B)	A → B	A ∧ (A → B)
T	F	T	F	T	T	F	F
F	T	T	F	F	F	T	F
T	T	T	T	T	T	T	T
F	F	F	F	F	F	T	F

[A ∧ (A → B) → B]	[B → [A ∧ (A → B)]
T	T
T	F
T	T
T	T

86. $P(A^c \cap B^c/C) = \dfrac{P((A^c \cap B^c) \cap C)}{P(C)}$

$$= \frac{P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(C) - P(A).P(C) - P(B)P(C) + 0}{P(C)}$$

$= 1 - P(A) - P(B)$

$= P(A^c) - P(B)$

87. $P(x) = 0$

$\Rightarrow f(x) = g(x)$
$\Rightarrow ax^2 + bx + c = a_1x^2 + b_1x + C$
$\Rightarrow (a - a_1)\,x^2 + (b - b_1)\,x + (c - c_1) = 0.$
It has only one solution $x = -1$
$\Rightarrow b - b_1 = a - a_1 + c - c_1$ ....(1)
vertex $(-1, 0)$

$\Rightarrow \dfrac{b - b_1}{2(a - a_1)} = -1$

$\Rightarrow b - b_1 = 2(a - a_{1)}$ .... (2)
$\Rightarrow f(-2) - g(-2) = 2$
$4a - 2b + c - 4a_1 + 2b_1 - c_1 = 2$
$\Rightarrow 4(a - a_1) - 2\,(b - b_1) + (c - c_1) = 2$ .... (3)

by (1), (2) and (3) $(a - a_1) = (c - c_1) = \dfrac{1}{2}\,(b - b_1) = 2$

Now $P(2) = -g(2)$
$= 4\,(a - a_1) + 2\,(b - b_1) + (c - c_1)$
$= 8 + 8 + 2 = 18$

88. Let feet of perpendicular is

$(2\alpha, 3\alpha + 2, 4\alpha + 3)$
⇒ D′ ratio of the ⊥ line $< 2\alpha - 3, 3\alpha + 3, 4\alpha - 8 >$
and D′ ratio of the line $< 2, 3, 4 >$
⇒ $2\,(2\alpha - 3) + 3\,(3\alpha + 3) + 4\,(4\alpha - 8) = 0$
⇒ $29\,\alpha - 29 = 0$
⇒ $\alpha = 1$
⇒ feet of ⊥ is (2, 5, 7)
⇒ length ⊥ is $\sqrt{1^2 + 6^2 + 4^2}$

$= \sqrt{53}$

89. **for reflexive**

$(A, A) \in R$
⇒ $A = P^{-1}AP$
which .... for $P = I$
∴ refrexive

for symmetry
As $(A, B) \in R$ for matrix P
$A = P^{-1}BP$

$\Rightarrow\quad PAP^{-1} = B$

$\Rightarrow\quad B = PAP^{-1}$

$\Rightarrow\quad B = (P^{-1})\,A\,(P^{-1})$

$\therefore\quad (B, A) \in R$ for matrix P^{-1}

$\therefore\quad R$ is symmetric

for transitivity

$A = P^{-1}BP$

and $B = P^{-1}CP$

$\Rightarrow\quad A = P^{-1}\,(P^{-1}CP)\,P$

$\Rightarrow\quad A = (P^{-1})^2\,CP^2$

$\Rightarrow\quad A = (P^2)^{-1}\,C(P^2)$

$\therefore\quad (A, C) \in R$ for matrix P^2

$\therefore\quad R$ is transitive

so R is equivalence

90. $\ell\text{im}_{x\to a} \dfrac{x^2 f(a) - a^2 f(x)}{x - a}$

$= \ell\lim_{x\to a} \dfrac{2xf(a) - a^2 f'(x)}{1}$

$= 2af(a) - a^2 f'(a)$

Alter $\ell\lim_{x\to a} \dfrac{x^2 f(a) - a^2 f(x)}{x - a}$

$= \ell\lim_{x\to a} \dfrac{x^2 f(a) - a^2 f(a) - a^2 f(x)}{x - a}$

$= \ell\text{im}_{x\to a} \dfrac{(x^2 - a^2) f(a) - a^2 (f(x) - f(a))}{x - a}$

$= \ell\text{im}_{x\to a} (x + a) f(a) - a^2 \left\{ \dfrac{f(x) - f(a)}{(x - a)} \right\}$

$= 2af(a) - a^2 f'(a)$