Integration of fractions

Example 4 Find $\int \frac{x^2+5}{x^2+9}\,dx$.

SOLUTION

When the power on the numerator is higher than, or equal to, the power on the denominator, we first divide the numerator by the denominator.

Hence, we have

$$\int \frac{x^2+5}{x^2+9}\,dx = \int\left(1-\frac{4}{x^2+9}\right)dx$$
$$= x-\frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

Example 5 Find $\int \frac{x^2+3x+7}{x^2+2x+4}\,dx$.

SOLUTION

Dividing, we obtain

$$\frac{x^2+3x+7}{x^2+2x+4} = 1+\frac{x+3}{x^2+2x+4}$$

To integrate $\frac{x+3}{x^2+2x+4}$, we use

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)+c$$

(derived on page 422 of *Introducing Pure Mathematics*).

So, we need to obtain in the numerator a multiple of the differential of x^2+2x+4. Hence, we convert

$$\frac{x+3}{x^2+2x+4} \quad \text{to} \quad \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+4}$$

which gives

$$\int \frac{x^2+3x+7}{x^2+2x+4}\,dx = \int\left(1+\frac{x+3}{x^2+2x+4}\right)dx$$
$$= \int\left(1+\frac{\frac{1}{2}(2x+2)}{x^2+2x+4}+\frac{2}{x^2+2x+4}\right)dx$$
$$= \int 1\,dx+\int\frac{\frac{1}{2}(2x+2)}{x^2+2x+4}\,dx+2\int\frac{dx}{(x+1)^2+3}$$

Therefore, we have

$$\int \frac{x^2+3x+7}{x^2+2x+4}\,dx = x+\frac{1}{2}\ln(x^2+2x+4)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

Note When the denominator is the square root of a linear, or quadratic, function, a similar method is used.

Example 6 Find $\int \frac{x+3}{\sqrt{x^2+9}}\,\mathrm{d}x$.

SOLUTION

Ignoring the square-root sign, we note that the differential of x^2+9 is $2x$. Hence, we split the integral to obtain

$$\int\left(\frac{x}{\sqrt{x^2+9}}+\frac{3}{\sqrt{x^2+9}}\right)\mathrm{d}x$$

which gives

$$\sqrt{x^2+9}+3\sinh^{-1}\left(\frac{x}{3}\right)+c$$

Therefore, we have

$$\int\frac{x+3}{\sqrt{x^2+9}}\,\mathrm{d}x=\sqrt{x^2+9}+3\sinh^{-1}\left(\frac{x}{3}\right)+c$$

Example 7 Find $\int \frac{3x+8}{\sqrt{x^2+4x+9}}\,\mathrm{d}x$.

SOLUTION

Ignoring the square-root sign, we note that the differential of x^2+4x+9 is $2x+4$. Hence, we express the integral as

$$\begin{aligned}\int\frac{3x+8}{\sqrt{x^2+4x+9}}\,\mathrm{d}x &= \frac{3}{2}\int\frac{2x+4}{\sqrt{x^2+4x+9}}\,\mathrm{d}x+\int\frac{2}{\sqrt{(x+2)^2+5}}\,\mathrm{d}x\\ &= 3\sqrt{x^2+4x+9}+2\sinh^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+c\\ &= 3\sqrt{x^2+4x+9}+2\left[\ln(\sqrt{x^2+4x+9}+x+2)-\ln\sqrt{5}\right]+c\end{aligned}$$

which gives

$$\int\frac{3x+8}{\sqrt{x^2+4x+9}}=3\sqrt{x^2+4x+9}+2\ln(\sqrt{x^2+4x+9}+x+2)+c'$$

Example 8 Find $\int \frac{x^2+3x+7}{\sqrt{x^2+2x+9}}\,\mathrm{d}x$.

SOLUTION

Proceeding as in Examples 4 and 6, we obtain

$$\begin{aligned}\int\frac{x^2+3x+7}{\sqrt{x^2+2x+9}}\,\mathrm{d}x &= \int\left(\frac{x^2+2x+9}{\sqrt{x^2+2x+9}}+\frac{x-2}{\sqrt{x^2+2x+9}}\right)\mathrm{d}x\\ &= \int\sqrt{x^2+2x+9}\,\mathrm{d}x+\int\frac{x-2}{\sqrt{x^2+2x+9}}\,\mathrm{d}x\end{aligned}$$

Taking the first integral on the RHS, we have

$$\int \sqrt{x^2+2x+9}\,dx = \int 1 \times \sqrt{x^2+2x+9}\,dx$$

$$= x\sqrt{x^2+2x+9} - \int x \times \frac{\frac{1}{2}(2x+2)}{\sqrt{x^2+2x+9}}\,dx$$

$$= x\sqrt{x^2+2x+9} - \int \frac{x^2+x}{\sqrt{x^2+2x+9}}\,dx$$

$$= x\sqrt{x^2+2x+9} - \int \frac{x^2+2x+9-x-9}{\sqrt{x^2+2x+9}}\,dx$$

$$= x\sqrt{x^2+2x+9} - \int \sqrt{x^2+2x+9}\,dx + \int \frac{x+9}{\sqrt{x^2+2x+9}}\,dx$$

which gives

$$2\int \sqrt{x^2+2x+9}\,dx = x\sqrt{x^2+2x+9} + \int \frac{x+9}{\sqrt{x^2+2x+9}}\,dx$$

$$\Rightarrow \quad \int \sqrt{x^2+2x+9}\,dx = \frac{1}{2}x\sqrt{x^2+2x+9} + \frac{1}{2}\int \frac{x+9}{\sqrt{x^2+2x+9}}\,dx$$

Hence, we obtain

$$\int \frac{x^2+3x+7}{\sqrt{x^2+2x+9}}\,dx = \frac{1}{2}x\sqrt{x^2+2x+9} + \frac{1}{2}\int\left(\frac{x+9}{\sqrt{x^2+2x+9}} + \frac{2(x-2)}{\sqrt{x^2+2x+9}}\right)dx$$

$$= \frac{1}{2}x\sqrt{x^2+2x+9} + \frac{1}{2}\int\left(\frac{\frac{3}{2}(2x+2)}{\sqrt{x^2+2x+9}} + \frac{2}{\sqrt{x^2+2x+9}}\right)dx$$

$$= \frac{1}{2}x\sqrt{x^2+2x+9} + \frac{3}{2}\sqrt{x^2+2x+9} + \sinh^{-1}\left(\frac{x+1}{2\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{x^2+2x+9}}{2}(3+x) + \ln\left(\frac{\sqrt{x^2+2x+9}}{2\sqrt{2}} + \frac{x+1}{2\sqrt{2}}\right) + c$$

which gives

$$\int \frac{x^2+3x+7}{\sqrt{x^2+2x+9}}\,dx = \frac{1}{2}(3+x)\sqrt{x^2+2x+9} + \ln(\sqrt{x^2+2x+9}+x+1) + c'$$

Exercise 12A

1 Find each of these integrals.

a) $\int 2x(x^2+1)^5\,dx$ **b)** $\int x(x^2-1)^4\,dx$ **c)** $\int x^3(x^4-1)^7\,dx$

d) $\int x^2(1-x^3)^4\,dx$ **e)** $\int \sin x\cos^5 x\,dx$ **f)** $\int \cosh x\sinh^4 x\,dx$

g) $\int \sinh 3x\cosh^4 3x\,dx$ **h)** $\int \sin^5 2x\cos 2x\,dx$

2 Find each of these integrals.

a) $\int e^x \cos x \, dx$ **b)** $\int e^x \sin 2x \, dx$ **c)** $\int e^{2x} \cos x \, dx$ **d)** $\int e^{3x} \cos 5x \, dx$

e) $\int e^{4x} \cosh 2x \, dx$ **f)** $\int e^{-7x} \sinh 3x \, dx$

3 Integrate each of the following with respect to x.

a) $\dfrac{x^2}{1+x^2}$ **b)** $\dfrac{x^2-4}{x^2+16}$ **c)** $\dfrac{2x-5}{8x+3}$ **d)** $\dfrac{3+7x}{5-4x}$

e) $\dfrac{2x-1}{x^2+2x+3}$ **f)** $\dfrac{x+1}{x^2+x+1}$ **g)** $\dfrac{x-1}{\sqrt{x^2+x-1}}$ **h)** $\dfrac{2x-7}{\sqrt{2x^2-4x+5}}$

i) $\dfrac{2x+5}{\sqrt{1-4x-x^2}}$ **j)** $\dfrac{3x-7}{\sqrt{2-5x-3x^2}}$

4 a) Find $\displaystyle\int \frac{x+1}{\sqrt{(1-x^2)}} \, dx$.

b) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{x+1}{\sqrt{(1-x^2)}} \, dx$, giving your answer in the form $p + q\pi$, where $p, q \in \mathbb{R}$. (EDEXCEL)

5 If $x = 5\sin\theta - 3$, show that $16 - 6x - x^2 = 25\cos^2\theta$.

Hence, or otherwise, find

$$\int \frac{1}{\sqrt{(16-6x-x^2)}} \, dx \qquad \text{(OCR)}$$

6 i) Express

$$f(x) \equiv \frac{x^3 + 3x^2 + 8x + 26}{(x+1)(x^2+9)}$$

in partial fractions of the form

$$a + \frac{b}{x+1} + \frac{cx+d}{x^2+9}$$

ii) Hence show that

$$\int_0^3 f(x)dx = 3 + 4\ln 2 - \frac{\pi}{12} \qquad \text{(OCR)}$$

7 Express $y = \dfrac{7x^2 + 11x + 13}{(3x+4)(x^2+9)}$ in partial fractions.

Hence show that

$$\int_0^3 y \, dx = \frac{1}{3}\ln 26 + \frac{\pi}{12} \qquad \text{(OCR)}$$

Reduction formulae

We need reduction formulae to facilitate the integration of functions whose integrals cannot otherwise be found directly.

An example of a reduction formula is

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$$

which enables us to convert, for example, $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$ into $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$.

This we may further reduce to $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$, and hence to $\int_0^{\frac{\pi}{2}} \sin^0 x \, dx$, which is $\int_0^{\frac{\pi}{2}} 1 \, dx$,which we can integrate easily.

We usually obtain a reduction formula by changing the form of the integrand into a product which can be integrated by parts. But we must exercise discretion. For example, a possible product of $\int \sin^n x \, dx$ is $\int 1 \times \sin^n x \, dx$.

But this will not be helpful, as $\int 1 \, dx$ is x and $x \frac{d}{dx} \sin^n x$ is an awkward integrand. Thus, we must use

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx$$

because we can integrate $\sin x$ easily.

Hence, we have

$$\int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx = \Big[-\cos x \sin^{n-1} x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x \times (n-1) \sin^{n-2} x \cos x \, dx$$

$$= 0 + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= (n-1) \left(\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx \right)$$

We usually obtain the integral with which we started as one of the terms on the right-hand side. So, we take this integral to the left-hand side, which gives

$$n \int_0^{\frac{\pi}{2}} \sin^n x \, dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx$$

Denoting $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$ by I_n, we can express this reduction formula as

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2}$$

Example 9 Use the reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ to evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$.

SOLUTION

In the reduction formula $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$, we put $n = 7$, which gives

$$I_7 = \int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \frac{6}{7}\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$$

Using the formula again with $n = 5$, we obtain

$$I_5 = \frac{4}{5}\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$$

which gives

$$I_7 = \frac{6}{7} \times \frac{4}{5}\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$$

Repeating the procedure with $n = 3$, we have

$$\begin{aligned} I_7 &= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx \\ &= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}\Big[-\cos x\Big]_0^{\frac{\pi}{2}} \\ &= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} = \frac{16}{35} \end{aligned}$$

Hence, we find $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \frac{16}{35}$.

Similarly, we can find the reduction formula for $\int_0^{\frac{\pi}{2}} e^{ax}\cos^n x\,dx$, when a is not equal to 0. In this case, the integrand is already a product, and e^{ax} is a term which can be readily integrated. Therefore, we differentiate the term $\cos^n x$, which gives

$$\int_0^{\frac{\pi}{2}} e^{ax}\cos^n x\,dx = \left[\frac{1}{a}e^{ax}\cos^n x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{1}{a}e^{ax} n\cos^{n-1} x\sin x\,dx$$

The new integrand is not in the form of $\int_0^{\frac{\pi}{2}} e^{ax}\cos^n x$, and therefore we must repeat the integration by parts, which gives

$$\begin{aligned} \int_0^{\frac{\pi}{2}} e^{ax}\cos^n x\,dx &= \left[\frac{1}{a}e^{ax}\cos^n x\right]_0^{\frac{\pi}{2}} - \frac{n}{a}\int_0^{\frac{\pi}{2}} -e^{ax}\cos^{n-1} x\sin x\,dx \\ &= -\frac{1}{a} + \frac{n}{a}\left\{0 - \frac{1}{a}\int_0^{\frac{\pi}{2}}\Big[-(n-1)e^{ax}\cos^{n-2}x(1-\cos^2 x) + e^{ax}\cos^n x\Big]dx\right\} \\ &= -\frac{1}{a} - \frac{n}{a^2}\int_0^{\frac{\pi}{2}}[-(n-1)e^{ax}(\cos^{n-2}x - \cos^n x) + e^{ax}\cos^n x]\,dx \\ &= -\frac{1}{a} - \frac{n}{a^2}\int_0^{\frac{\pi}{2}}[ne^{ax}\cos^n x - (n-1)e^{ax}\cos^{n-2}x]\,dx \end{aligned}$$

Hence, we have

$$(n^2 + a^2)\int_0^{\frac{\pi}{2}} e^{ax}\cos^n x\,dx = -a + n(n-1)\int_0^{\frac{\pi}{2}} e^{ax}\cos^{n-2} x\,dx$$

Example 10 If $I_n = \int_0^2 x^n e^{4x}\,dx$, show that

$$I_n = 2^{n-2}e^8 - \frac{n}{4}I_{n-1}$$

Hence find $\int_0^2 x^3 e^{4x}\,dx$.

SOLUTION

The reduction formula requires that the power of x is reduced. Therefore, we differentiate the term in x^n and integrate the term e^{4x}, obtaining

$$\int_0^2 x^n e^{4x}\,dx = \left[x^n \times \frac{e^{4x}}{4}\right]_0^2 - \int_0^2 nx^{n-1} \times \frac{e^{4x}}{4}\,dx$$

$$= \frac{2^n e^8}{4} - \frac{n}{4}\int_0^2 x^{n-1}e^{4x}\,dx$$

That is, we have

$$I_n = 2^{n-2}e^8 - \frac{n}{4}I_{n-1}$$

as required.

To find $\int_0^2 x^3 e^{4x}\,dx$, we first put $n = 3$, which gives

$$\int_0^2 x^3 e^{4x}\,dx = 2e^8 - \frac{3}{4}I_2$$

Then we put $n = 2$, which gives

$$\int_0^2 x^3 e^{4x}\,dx = 2e^8 - \frac{3}{4}\left(e^8 - \frac{2}{4}I_1\right) = 2e^8 - \frac{3}{4}e^8 + \frac{3}{8}I_1$$

Finally, we put $n = 1$, which gives

$$\int_0^2 x^3 e^{4x}\,dx = \frac{5}{4}e^8 + \frac{3}{8}\left(\frac{1}{2}e^8 - \frac{1}{4}I_0\right)$$

$$= \frac{5}{4}e^8 + \frac{3}{16}e^8 - \frac{3}{32}\int_0^2 e^{4x}\,dx$$

$$= \frac{5}{4}e^8 + \frac{3}{16}e^8 - \frac{3}{128}\left[e^{4x}\right]_0^2$$

Hence, we find

$$\int_0^2 x^3 e^{4x}\,dx = \frac{181}{128}e^8 - \frac{3}{128}$$

Example 11 If $I_n = \int_0^1 \frac{x^n}{\sqrt{x^2-1}}\,dx$, show that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$.

SOLUTION

When we separate I_n into a part to be integrated and a part to be differentiated, we must take account of the following:

- $\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x$ This result is unlikely to be helpful.
- $\frac{d}{dx}\left(\frac{1}{\sqrt{x^2-1}}\right) = \frac{x}{(x^2-1)^{\frac{3}{2}}}$ This result increases the power of the denominator, and so it also is unlikely to be helpful.

Hence, we avoid having to integrate or differentiate $\frac{1}{\sqrt{x^2-1}}$ by itself. We therefore separate $\frac{x^n}{\sqrt{x^2-1}}$ into $\frac{x}{\sqrt{x^2-1}} \times x^{n-1}$ because we can integrate $\frac{x}{\sqrt{x^2-1}}$ to give $\sqrt{x^2-1}$.

So, we have

$$\begin{aligned} I_n = \int_0^1 \frac{x^n}{\sqrt{x^2-1}}\,dx &= \int_0^1 \frac{x}{\sqrt{x^2-1}} \times x^{n-1}\,dx \\ &= \left[\sqrt{x^2-1} \times x^{n-1}\right]_0^1 - \int_0^1 \sqrt{x^2-1}(n-1)x^{n-2}\,dx \\ &= -(n-1)\int_0^1 \frac{(x^2-1) \times x^{n-2}}{\sqrt{x^2-1}}\,dx \\ &= -(n-1)\int_0^1 \left(\frac{x^n}{\sqrt{x^2-1}} - \frac{x^{n-2}}{\sqrt{x^2-1}}\right)dx \\ &= -(n-1)(I_n - I_{n-2}) \end{aligned}$$

which gives

$$nI_n = (n-1)I_{n-2} \quad \Rightarrow \quad I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

as required.

Exercise 12B

1 If $\int_0^{\frac{\pi}{2}} \cos^n x\,dx = I_n$, prove that $nI_n = (n-1)I_{n-2}$, $(n > 1)$.

Evaluate

a) $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ **b)** $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$

2 Find the reduction formula for $\int x^n e^x \, dx$.

3 If $I_n = \int_0^1 x^n e^{-x} \, dx$, prove that

$$I_n = nI_{n-1} - e^{-1} \quad (n \geqslant 1)$$

Hence evaluate $\int_0^1 x^5 e^{-x} \, dx$.

4 If $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta$, prove that

$$I_n = \frac{1}{n-1} - I_{n-2} \quad (n > 1)$$

Hence evaluate

a) $\int_0^{\frac{\pi}{4}} \tan^4\theta \, d\theta$ **b)** $\int_0^{\frac{\pi}{4}} \tan^7\theta \, d\theta$

5 If $I_n = \int_0^1 (\ln x)^n \, dx$, **a)** prove that $I_n = -nI_{n-1}$, and **b)** deduce that $I_n = (-1)^n n!$

6 Prove that

$$n \int \cosh^n x \, dx = \cosh^{n-1} x \sinh x + (n-1) \int \cosh^{n-2} x \, dx$$

Hence find $\int_0^1 \cosh^5 x \, dx$.

7 If $I_n = \int_0^1 x^n e^{x^2} \, dx$, prove that $I_n = \frac{1}{2}e - \frac{1}{2}(n-1)I_{n-2}$.

8 If $I_{m,n} = \int \frac{x^m}{(\ln x)^n} \, dx$, prove that

$$(n-1)I_{m,n} = -\frac{x^{m+1}}{(\ln x)^{n-1}} + (m+1)I_{m,n-1}$$

9 If $I_{m,n} = \int x^m (\ln x)^n \, dx$, prove that

$$(m+1)I_{m,n} = x^{(m+1)}(\ln x)^n - nI_{m,n-1}$$

10 Given that $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, show that

$$I_n = \frac{1}{n-1} - I_{n-2} \quad (n \geqslant 2)$$

Hence show that $I_4 = \dfrac{3\pi - 8}{12}$. (WJEC)

11 a) Given that $I_n = \int \cosh^n x \, dx \ (n \geqslant 0)$, show that

$$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2} \quad (n \geqslant 2)$$

b) Prove the identity

$$\cosh x - \operatorname{sech} x \equiv \sinh x \tanh x$$

Hence evaluate

$$\int_0^{\ln 3} (\operatorname{sech} x + \sinh x \tanh x)^3 \, dx \qquad \text{(AEB 98)}$$

12 Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$, show that, for $n \geqslant 2$,

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

Hence find the area of the region enclosed by the curve $y = x^4 \cos x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$. (WJEC)

13 Given that $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$, show that, for $n \geqslant 2$,

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta$. (WJEC)

14 $$I_n = \int_0^1 x^{\frac{1}{2}n} e^{-\frac{1}{2}x} \, dx \quad n \geqslant 0$$

a) Show that $I_n = nI_{n-2} - e^{-\frac{1}{2}}$, $n \geqslant 2$.
b) Evaluate I_0 in terms of e.
c) Find, using the results of parts **a** and **b**, the value of I_4 in terms of e.
d) Show that the approximate value for I using Simpson's rule with three equally spaced ordinates is

$$\frac{1}{6}(2\sqrt{2}e^{-\frac{1}{4}} + e^{-\frac{1}{2}}) \qquad \text{(EDEXCEL)}$$

15 Consider $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta}{\sin \theta} \, d\theta$, where n is a non-negative integer.

i) Using $\sin A - \sin B \equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$, derive the reduction formula

$$I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$$

ii) Find $\int_0^{\frac{\pi}{2}} \frac{\sin 6\theta}{\sin \theta} \, d\theta$. (NICCEA)

16 Assuming the reduction formula

$$\int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \, dx$$

where $n \geqslant 2$, find the exact value of $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$. (NICCEA)

17 Given that $I_n = \int \sec^n x \, dx$,

a) show that

$$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2} \qquad n \geqslant 2$$

b) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sec^3 x \, dx$, giving your answer in terms of natural logarithms and surds. (EDEXCEL)

18 Find the value of each of the constants A, B and C for which

$$\frac{1}{1+x^3} \equiv \frac{A}{(1+x)} + \frac{Bx+C}{(1-x+x^2)}$$

Hence evaluate $\int_0^1 \frac{1}{(1+x^3)} \, dx$.

Given that $I_n = \int_0^1 (1+x^3)^n \, dx$, where n is an integer, show that

$$(3n+1)I_n = 2^n + 3nI_{n-1}$$

Hence evaluate $\int_0^1 (1+x^3)^{-2} \, dx$. (EDEXCEL)

19 **i)** If $I_n = \int_0^1 x^n(1-x)^{\frac{1}{2}} \, dx$, prove that $(2n+3)I_n = 2nI_{n-1}$, where n is a positive integer.

ii) Show that $\int_0^1 x^3(1-x)^{\frac{1}{2}} \, dx = \frac{32}{315}$. (NICCEA)

20 Given that $I_n = \int_0^1 x^n \cos \pi x \, dx$ for $n \geqslant 0$, show that

$$\pi^2 I_n + n(n-1)I_{n-2} + n = 0 \qquad \text{for } n \geqslant 2$$

Hence show that

$$\int_0^1 x^4 \cos \pi x \, dx = \frac{4(6-\pi^2)}{\pi^4} \qquad \text{(OCR)}$$

21 Show that

$$\frac{d}{dx}[x^{n-1}\sqrt{(16-x^2)}] = \frac{16(n-1)x^{n-2}}{\sqrt{(16-x^2)}} - \frac{nx^n}{\sqrt{(16-x^2)}}$$

Deduce, or prove otherwise, that if

$$I_n = \int_0^2 \frac{x^n}{\sqrt{(16-x^2)}} \, dx$$

then, for $n \geqslant 2$,

$$nI_n = 16(n-1)I_{n-2} - 2^n\sqrt{3}$$

Hence find the exact value of I_2. (OCR)

22 Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}[t(1+t^4)^n] = (4n+1)(1+t^4)^n - 4n(1+t^4)^{n-1}$$

The integral I_n is defined by $I_n = \int_0^1 (1+t^4)^n \,\mathrm{d}t$.

Show that $(4n+1)I_n = 4nI_{n-1} + 2^n$. (OCR)

23 Let $I_n = \int_0^1 \cosh^n x \,\mathrm{d}x$.

i) By considering

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x \cosh^{n-1} x)$$

or otherwise, show that

$$nI_n = ab^{n-1} + (n-1)I_{n-2}$$

where $a = \sinh(1)$ and $b = \cosh(1)$.

ii) Show that $I_4 = \frac{1}{8}(2ab^3 + 3ab + 3)$. (OCR)

24 It is given that

$$I_n = \int_1^{\mathrm{e}} x(\ln x)^n \,\mathrm{d}x \qquad (n \geqslant 0)$$

By considering $\frac{\mathrm{d}}{\mathrm{d}x}[x^2(\ln x)^n]$, or otherwise, show that, for $n \geqslant 1$,

$$I_n = \frac{1}{2}\mathrm{e}^2 - \frac{1}{2}nI_{n-1}$$

Hence find I_3, leaving your answer in terms of e. (OCR)

25 For each non-negative integer n, let $I_n = \int \cos^n \theta \,\mathrm{d}\theta$.

i) Show that if $n \geqslant 2$, then

$$nI_n = \sin\theta \cos^{n-1}\theta + (n-1)I_{n-2}$$

ii) Show that $\int_0^{\frac{\pi}{3}} \cos^3\theta \,\mathrm{d}\theta = \frac{3\sqrt{3}}{8}$. (NICCEA)

26 $$I_n = \int \frac{\sin nx}{\sin x}\,\mathrm{d}x \qquad n > 0,\ n \in \mathbb{Z}$$

a) By considering $I_{n+2} - I_n$ or otherwise, show that

$$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n$$

b) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,\mathrm{d}x$, giving your answer in the form $p\sqrt{2} + q\sqrt{3}$, where p and q are rational numbers to be found. (EDEXCEL)

27 a) Write down the values of cosh(ln2) and sinh(ln2).

b) For $n \geqslant 0$, the integral I_n is given by $I_n = \int_0^{\ln 2} \cosh^n x \,\mathrm{d}x$.

i) By writing $\cosh^n x$ as $\cosh^{n-1} x \cosh x$, prove that, for $n \geqslant 2$,

$$nI_n = \frac{3 \times 5^{n-1}}{4^n} + (n-1)I_{n-2}$$

ii) Evaluate I_3. (AEB 96)

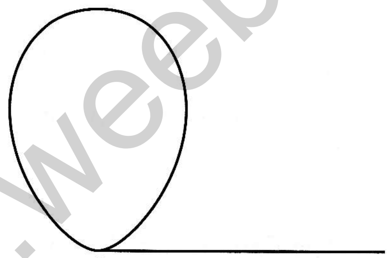
28 $$I_n = \int_0^{\pi} \sin^{2n} x \,\mathrm{d}x \qquad n \in \mathbb{N}$$

a) Calculate I_0 in terms of π.

b) Show that $I_n = \dfrac{(2n-1)}{2n} I_{n-1},\ n \geqslant 1$.

c) Find I_3 in terms of π.

The figure on the right shows the curve with polar equation $r = a\sin^3\theta,\ 0 \leqslant \theta \leqslant \pi$, where a is a positive constant.



d) Using your answer to part **c**, or otherwise, calculate exactly the area bounded by this curve. (EDEXCEL)

29 a) Assuming the derivatives of $\sinh\theta$ and $\cosh\theta$, prove that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\tanh\theta) = \operatorname{sech}^2\theta$$

b) Let I_r denote the integral $\int_0^{\ln 2} \tanh^{2r}\theta \,\mathrm{d}\theta$ for integers $r \geqslant 0$.

i) Evaluate I_0.

ii) Show that $I_{r-1} - I_r = \dfrac{1}{2r-1}\left(\dfrac{3}{5}\right)^{2r-1}$.

iii) Hence prove that

$$\int_0^{\ln 2} \tanh^{2n}\theta \,\mathrm{d}\theta = \ln 2 - \frac{5}{3}\sum_{r=1}^{n} \frac{1}{(2r-1)}\left(\frac{9}{25}\right)^r$$

iv) Deduce the sum of the infinite series

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)}\left(\frac{9}{25}\right)^r \qquad \text{(AEB 98)}$$

30 Let m and n be non-negative integers.

i) Determine $\int \sin\theta \cos^n\theta \,\mathrm{d}\theta$.

ii) Show that

$$\int \sin^m\theta \cos^n\theta \,\mathrm{d}\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}\theta \cos^{n+2}\theta \,\mathrm{d}\theta$$

where $m \geqslant 2$.

iii) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta$, show that

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

where $m \geqslant 2$.

iv) Using the result in part **iii** and the similar result,

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

where $n \geqslant 2$, show that

$$\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^4\theta \, d\theta = \frac{3\pi}{512} \qquad \text{(NICCEA)}$$

31 $$I_n = \int \frac{x^n}{\sqrt{(1+x^2)}} \, dx$$

a) Show that $nI_n = x^{n-1}\sqrt{(1+x^2)} - (n-1)I_{n-2}, \; n \geqslant 2$.

The curve C has equation

$$y^2 = \frac{x^2}{\sqrt{(1+x^2)}} \qquad y \geqslant 0$$

The finite region R is bounded by C, the x-axis and the lines with equations $x = 0$ and $x = 2$. The region R is rotated through 2π radians about the x-axis.

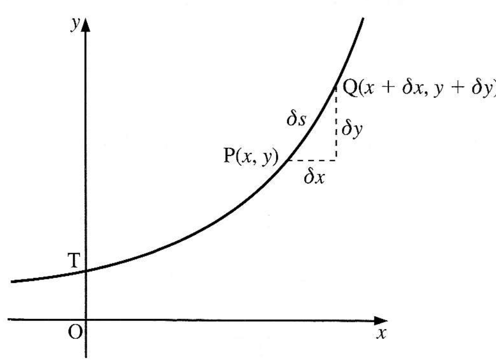
b) Find the volume of the solid so formed, giving your answer in terms of π, surds and natural logarithms.

An estimate for the volume obtained in part **b** is found using Simpson's rule with three ordinates.

c) Find the percentage error resulting from using this estimate, giving your answer to three decimal places. (EDEXCEL)

Arc length

Cartesian form



Consider two points, P and Q, on a curve. P is the point (x, y) and Q is the point $(x + \delta x, \; y + \delta y)$.

Let s be the length of the arc from a point T, and δs the length of the arc PQ.

Since δs is very small, we can approximate the arc PQ to a straight line. Hence, using Pythagoras's theorem, we have

$$(\delta x)^2 + (\delta y)^2 = (\delta s)^2$$

Dividing by $(\delta x)^2$, we obtain

$$1 + \left(\frac{\delta y}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta x}\right)^2$$

As $\delta x \to 0$, this gives

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Therefore, we have

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 12 Find the length of the arc of the curve $x^3 = 3y^2$ from $x = 1$ to $x = 4$.

SOLUTION

Differentiating with respect to x, we have

$$3x^2 = 6y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2y}$$

which gives

$$\text{Arc length} = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \left(\frac{x^2}{2y}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \frac{x^4}{4y^2}}\, dx$$

Substituting $y^2 = \dfrac{x^3}{3}$, we obtain

$$\text{Arc length} = \int_1^4 \sqrt{1 + \frac{3x^4}{4x^3}}\, dx = \int_1^4 \sqrt{1 + \frac{3x}{4}}\, dx$$

Putting $1 + \dfrac{3x}{4} = u^2$ and differentiating, we have

$$2u\frac{du}{dx} = \frac{3}{4} \quad \Rightarrow \quad \frac{8}{3}u\, du = dx$$

Substituting these in the original integral and changing the limits to $u = 2$ (from $x = 4$) and $u = \sqrt{\dfrac{7}{4}}$ (from $x = 1$), we obtain

$$\text{Arc length} = \int_{\sqrt{\frac{7}{4}}}^{2} \frac{8}{3}u^2\, du = \left[\frac{8}{9}u^3\right]_{\sqrt{\frac{7}{4}}}^{2}$$

which gives

$$\text{Arc length} = \frac{64}{9} - \frac{8}{9}\left(\frac{7}{4}\right)^{\frac{3}{2}} = \frac{1}{9}(64 - 7\sqrt{7})$$

Parametric form

To obtain the parametric form, we divide $(\delta x)^2 + (\delta y)^2 = (\delta s)^2$ by $(\delta t)^2$, where t is the parameter, which gives

$$\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2 = \left(\frac{\delta s}{\delta t}\right)^2$$

As $\delta x \to 0$, and thus $\delta t \to 0$, we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

which gives

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We sometimes express this as

$$s = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

where $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$

Note We use the dot notation only when the independent variable is t, and mostly when t represents time. Thus, $\dot{x}$ usually expresses speed and $\ddot{x}$ acceleration.

Example 13 Find the circumference of the circle $x^2 + y^2 = r^2$.

SOLUTION

The parametric equations for a circle are $x = r\cos\theta$, $y = r\sin\theta$. Therefore, we have

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Using just that part of the circle in the first quadrant and then multiplying by 4, we obtain

$$s = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

which gives

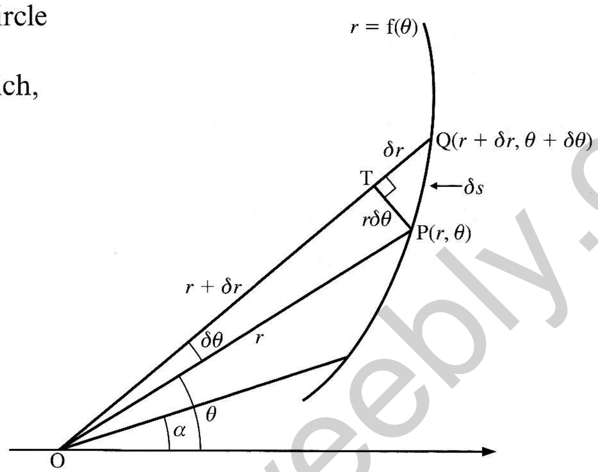
$$s = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} r\, d\theta = 4\Big[r\theta\Big]_0^{\frac{\pi}{2}} = 2\pi r$$

Polar form

To obtain the polar form, we consider two points, P and Q, on a curve which is expressed in its polar equation. P is the point (r, θ) and Q is the point $(r+\delta r, \theta+\delta\theta)$.

As $\delta\theta \to 0$, we can approximate TP to an arc of a circle of radius r, and hence of length $r\delta\theta$. Also, we can approximate TPQ to a right-angled triangle, for which, by Pythagoras's theorem,



$$PQ^2 = TP^2 + TQ^2$$

Thus, we have

$$(r\delta\theta)^2 + (\delta r)^2 = (\delta s)^2$$

Dividing through by $(\delta\theta)^2$, we obtain

$$\left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r^2$$

Therefore, as $\delta\theta \to 0$, we have

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2}$$

Hence, the length of the arc of a curve between the half-lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Example 14 Find the length of the arc of the curve $r = ae^{2\theta}$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

SOLUTION

Differentiating $r = ae^{2\theta}$ with respect to θ, we have

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2ae^{2\theta}$$

Hence, the required arc length is given by

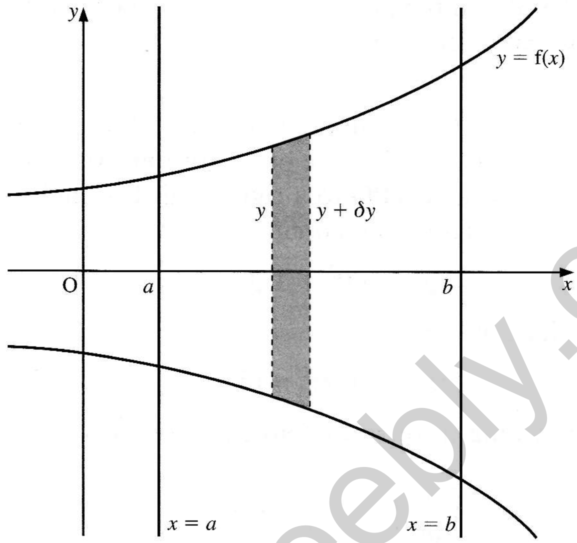
$$s = \int_0^{\frac{\pi}{2}} \sqrt{(ae^{2\theta})^2 + (2ae^{2\theta})^2}\,\mathrm{d}\theta$$

$$= \sqrt{5}a \int_0^{\frac{\pi}{2}} e^{2\theta}\,\mathrm{d}\theta$$

$$= \sqrt{5}a \left[\frac{1}{2}e^{2\theta}\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad s = \frac{\sqrt{5}a}{2}(e^\pi - 1)$$

Area of a surface of revolution



Let A be the area of the surface formed by rotating the curve $y = \mathrm{f}(x)$, between the lines $x = a$ and $x = b$, about the x-axis.

Let the curved surface area of the strip shown shaded be δA.

Treating the strip as being bounded by two cylinders, we have

$$2\pi y \delta s \leqslant \delta A \leqslant 2\pi(y + \delta y)\delta s$$

As $\delta x \to 0$, $\delta s \to 0$, so we have

$$\frac{\mathrm{d}A}{\mathrm{d}s} = 2\pi y$$

$$\Rightarrow \quad A = \int 2\pi y \,\mathrm{d}s$$

$$\Rightarrow \quad A = \int 2\pi y \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x$$

which gives

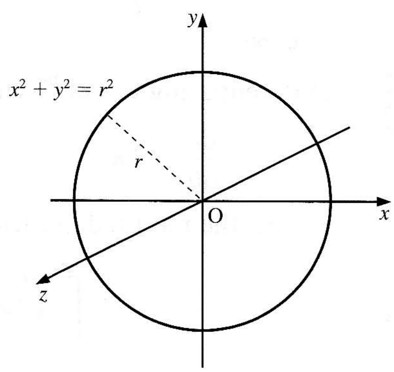
$$A = \int 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

or, in parametric form,

$$A = \int 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \quad \text{or} \quad A = \int 2\pi y \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t$$

Example 15 Find the surface area, A, of the sphere $x^2 + y^2 + z^2 = r^2$.

SOLUTION



The sphere $x^2 + y^2 + z^2 = r^2$ is obtained by rotating the circle $x^2 + y^2 = r^2$ about the x-axis. Hence, we have

$$A = \int 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Differentiating $x^2 + y^2 = r^2$, we obtain

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

which gives

$$A = \int_{-r}^{r} 2\pi\sqrt{r^2 - x^2} \times \sqrt{1 + \frac{x^2}{y^2}}\,\mathrm{d}x$$

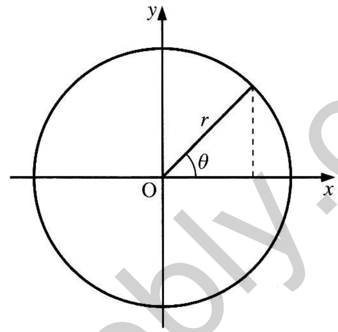
$$= \int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\sqrt{\frac{y^2 + x^2}{y^2}}\,\mathrm{d}x = \int_{-r}^{r} 2\pi r\,\mathrm{d}x$$

By symmetry, the integral from $x = -r$ to $x = r$ is twice the integral from $x = 0$ to $x = r$. Therefore, we have

$$A = 2\int_0^r 2\pi r\,\mathrm{d}x = \Big[4\pi r x\Big]_0^r = 4\pi r^2$$

Hence, the surface area of a sphere is $4\pi r^2$.

Using the parametric form, $x = r\cos\theta$, $y = r\sin\theta$, for the rotated circle, we have



$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r\sin\theta\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 2\pi r^2\sin\theta\,\mathrm{d}\theta = -4\pi r^2\Big[\cos\theta\Big]_0^{\frac{\pi}{2}} = 4\pi r^2$$

Hence, the surface area of the sphere is $4\pi r^2$.

Exercise 12C

1 Find the length of the arc of $x^3 = y^2$ from $x = 0$ to $x = 3$.

2 Find the length of the arc of $x^3 = 6y^2$ from $x = 1$ to $x = 2$.

3 Find the length of the arc of the parabola $x = at^2$, $y = 2at$, between the points $(0, 0)$ and $(ap^2, 2ap)$.

4 Find the length of the arc of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$, between the points $t = 0$ and $t = \pi$.

5 Find the length of the arc of the catenary $y = c\cosh\left(\dfrac{x}{c}\right)$, between the points where $x = 0$ and $x = c$.

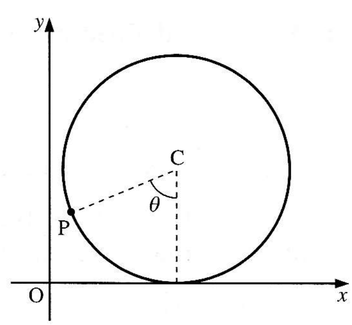
6 Find the area of the surface generated by rotating about the x-axis each of the following.

a) Arc of the curve $x = 2t^3$, $y = 3t^2$, between the points where $t = 0$ and $t = 4$.
b) Arc of the curve $x = t^2$, $y = 2t$, between the points where $t = 0$ and $t = 2$.
c) Part of the asteroid $x = a\cos^3 t$, $y = a\sin^3 t$, which is above the x-axis.
d) Curve $y = 5x^{\frac{1}{2}}$, from $x = 4$ to $x = 9$.
e) Curve $y = \cosh x$, between $x = 0$ and $x = 1$.
f) Curve $y = \mathrm{e}^{3x}$, from $x = 1$ to $x = 4$.

7 The diagram shows a wheel of radius a which rolls along the line Ox. The centre of the wheel is C and P is a point fixed on the rim of the wheel. Initially P is at O. When CP has rotated through an angle θ, show that the coordinates of P are

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$$

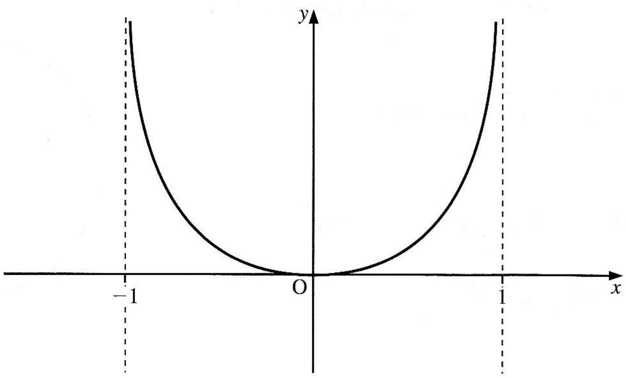
Hence find the length of the path of P when the wheel rolls through one complete revolution. (NEAB)



8 a) Find the values of the constants A and B for which

$$\frac{1+x^2}{1-x^2} \equiv A + \frac{B}{1-x^2}$$

b) The function f is defined for $-1 < x < 1$ by $f(x) = -\ln(1 - x^2)$. The graph of the curve with equation $y = f(x)$ is shown.



i) Find $\frac{dy}{dx}$ in terms of x. Hence show that

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1+x^2}{1-x^2}\right)^2$$

ii) Prove that the length of the arc of the curve from $x = 0$ to $x = \frac{3}{4}$ is $\ln 7 - \frac{3}{4}$. (AEB 96)

9 a) Using the definitions of the hyperbolic functions $\cosh x$ and $\sinh x$ given in the information booklet, show that

i) $\cosh^2 x - \sinh^2 x = 1$ **ii)** $2\cosh^2 x - 1 = \cosh 2x$ **iii)** $2\sinh x \cosh x = \sinh 2x$

b) Show that the length of the arc of the curve $y = x^2$ between the origin and the point $(1, 1)$ is $\frac{1}{4}(2\sqrt{5} + \sinh^{-1} 2)$. (WJEC)

10 A curve C is given parametrically by

$$x = e^t \cos t \qquad y = e^t \sin t \qquad (0 \leqslant t \leqslant \pi)$$

Show that the length of C is $\sqrt{2}(e^{\pi} - 1)$.

Show also that in polar coordinates the equation of C is $r = e^{\theta}$ $(0 \leqslant \theta \leqslant \pi)$, and hence sketch C.

Find the area of the region bounded by C and the x-axis. (NEAB)

11 A curve is defined parametrically by

$$x = \frac{8}{3}t^{\frac{3}{2}} \qquad y = t^2 - 2t + 4$$

The points A and B on the curve are defined by $t = 0$ and $t = 1$ respectively.

i) Find the length of the arc AB.

ii) Show that the area of the surface generated by one complete revolution of the arc AB about the y-axis is $\frac{256}{35}\pi$. (OCR)

12 The parametric equations of a curve are

$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

where a is a positive constant. Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4a^2 \sin^2(\tfrac{1}{2}t)$$

The arc of this curve between $t = 0$ and $t = 2\pi$ is rotated completely about the x-axis. Show that the area of the surface of revolution formed is

$$8\pi a^2 \int_0^{2\pi} [1 - \cos^2(\tfrac{1}{2}t)] \sin(\tfrac{1}{2}t)\, dt$$

and hence find this area. (OCR)

13 The curve C is defined parametrically by

$$x = 3 + e^{-t}(\cos t + \sin t) \qquad y = 4 + e^{-t}(\cos t - \sin t)$$

Find the exact value, in terms of π and e, of the length of the arc of C from the point where $t = 0$ to the point where $t = \frac{1}{2}\pi$.

This arc is rotated about the x-axis through one revolution. Express the area of the surface generated as a definite integral. (You are not required to evaluate this integral.) (OCR)

14 The parametric equations of a curve are

$$x = 3\cos\theta - \cos 3\theta \qquad y = 3\sin\theta - \sin 3\theta$$

Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 36\sin^2\theta$$

Hence find the length of the arc of the curve between the points given by $\theta = 0$ and $\theta = \frac{1}{2}\pi$. (OCR)

15 The arc of the curve $y = e^x$ from the point where $y = \frac{3}{4}$ to the point where $y = \frac{4}{3}$ is rotated through one revolution about the x-axis. Show that the area, S, of the surface generated is given by

$$S = 2\pi \int_{\frac{3}{4}}^{\frac{4}{3}} \sqrt{(1 + y^2)}\, dy$$

By using the substitution $y = \sinh u$, show that

$$S = \pi\left[\frac{185}{144} + \ln\left(\frac{3}{2}\right)\right] \qquad \text{(OCR)}$$

16 A curve C is defined parametrically by

$$x = 2(1 + t)^{\frac{3}{2}} \qquad y = 2(1 - t)^{\frac{3}{2}}$$

where $0 \leqslant t \leqslant 1$. Find

i) the length of C

ii) the area of the surface generated when C is rotated through one revolution about the x-axis. (OCR)

17 The curve C is defined parametrically by the equations $x = \frac{1}{3}t^3 - t$, $y = t^2$, where t is a parameter.

a) Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (t^2+1)^2$.

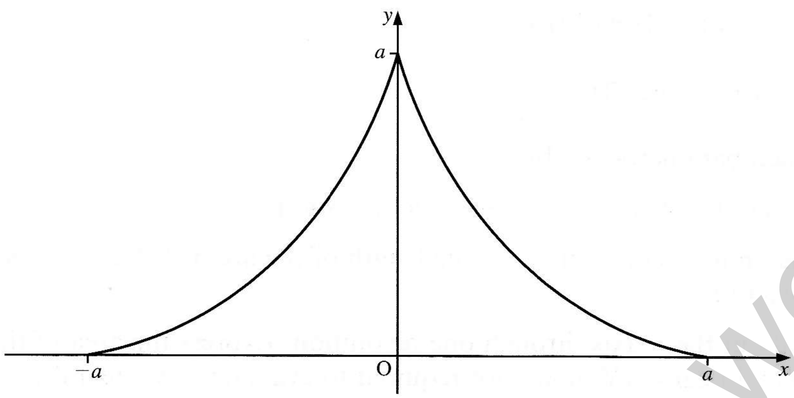
b) The arc of C between the points where $t = 0$ and $t = 3$ is denoted by L. Determine

i) the length of L

ii) the area of the surface generated when L is rotated through 2π radians about the x-axis.

(AEB 98)

18



The figure above shows the curve C with parametric equations

$$x = a\cos^3 t \qquad y = a\sin^3 t \qquad 0 \leqslant t \leqslant \pi$$

where a is a positive constant.

The curve C is rotated through 2π radians about the x-axis. Show that the area of the surface of revolution formed is $\frac{12\pi a^2}{5}$. (EDEXCEL)

19 The arc of the curve $y = x^3$, between $x = 0$ and $x = 1$, is rotated through 2π radians about the x-axis. Determine the exact value of the surface area generated. (AEB 98)

20 The curve C has the parametric equations

$$x = e^{\theta}\sin\theta \qquad y = e^{\theta}\cos\theta \qquad \text{for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

a) Show that the area S of the surface generated when C is rotated through four right angles about the x-axis is given by

$$S = 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\cos\theta\,d\theta$$

b) Find the value of S. (WJEC)

21 a) i) Using only the definitions $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ and $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$, prove the identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

ii) Deduce a relationship between $\operatorname{sech}\theta$ and $\tanh\theta$.

b) A curve C has parametric representation $x = \operatorname{sech}\theta$, $y = \tanh\theta$.

i) Show that $\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \operatorname{sech}^2\theta$.

ii) The arc of the curve between the points where $\theta = 0$ and $\theta = \ln 7$ is rotated through one full turn about the x-axis. Show that the area of the surface generated is $\dfrac{36}{25}\pi$ square units. (AEB 97)

22 a) Find $\displaystyle\int \cosh^2 t \sinh t \,\mathrm{d}t$.

The curve C has parametric equations

$$x = \cosh^2 t \qquad y = 2\sinh t \qquad 0 \leqslant t \leqslant 2$$

The curve C is rotated through 2π radians about the x-axis.

b) Show that the area S of the curved surface generated is given by

$$S = 8\pi \int_0^2 \cosh^2 t \sinh t \,\mathrm{d}t$$

c) Evaluate S to three significant figures. (EDEXCEL)

Improper integrals

An **improper integral** is one which has either

- a limit of integration of $\pm\infty$, or
- an integrand which is infinite at one or other of its limits of integration, or between these limits.

In the first case, we replace $\pm\infty$ with n, say, and then find the limit of the integral as $n \to \pm\infty$. When this limit is **finite**, the integral **can be found** (see Example 16). When this limit is **not finite**, the integral **cannot be found** (see Example 17).

In the second case, we replace one or other of the limits of integration with p, say, and then find the limit of the integral as p tends to the value of the limit it has replaced (see Examples 18 and 19).

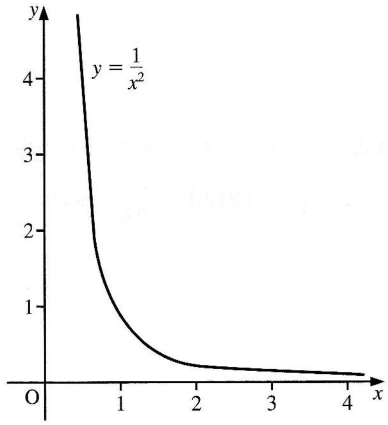
Example 16 Determine $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$.

SOLUTION

The upper limit is ∞, so we replace it with n, which gives

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{n\to\infty}\int_1^n \frac{1}{x^2}\,\mathrm{d}x$$

$$= \lim_{n\to\infty}\left[-\frac{1}{x}\right]_1^n = \lim_{n\to\infty}\left(-\frac{1}{n}+1\right)$$



As $n \to \infty$, $\frac{1}{n} \to 0$, which gives

$$\lim_{n\to\infty}\left(-\frac{1}{n}+1\right) = 1$$

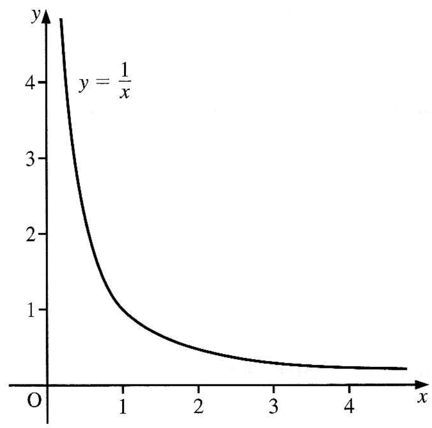
That is, we have

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

This shows that the area under the curve $y = \frac{1}{x^2}$ is finite even though the boundary is of infinite length.

Example 17 Determine $\int_1^{\infty} \frac{1}{x}\,dx$.

SOLUTION



We have

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{n\to\infty}\int_1^{n} \frac{1}{x}\,dx$$

$$= \lim_{n\to\infty}\Big[\ln x\Big]_1^n$$

$$= \lim_{n\to\infty}(\ln n - \ln 1)$$

which is not finite since $\lim\limits_{n\to\infty} \ln n$ is ∞.

This shows that the area under $y = \frac{1}{x}$ is not finite although the curve looks very similar to $y = \frac{1}{x^2}$, which has a finite area.

Example 18 Determine $\int_0^{1} \frac{1}{\sqrt{x}}\,dx$.

SOLUTION

This is an improper integral since the integrand, $\frac{1}{\sqrt{x}}$, is infinite when $x = 0$. So, we replace the lower limit with p, which gives

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{p\to 0}\int_p^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{p\to 0}\Big[2x^{\frac{1}{2}}\Big]_p^1 = \lim_{p\to 0}(2 - 2\sqrt{p})$$

Since $\lim\limits_{p\to 0} 2\sqrt{p} = 0$, we have

$$\lim_{p\to 0}(2 - 2\sqrt{p}) = 2$$

That is, we have

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

Example 19 Determine $\displaystyle\int_b^a \frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)}}$.

SOLUTION

At both limits the integrand is infinite, so we replace the upper limit with p and the lower limit with q and find the limit of the integral as $p \to a$ and $q \to b$. Hence, we have

$$\int_b^a \frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)}} = \lim_{p\to a}\lim_{q\to b}\int_q^p \frac{\mathrm{d}x}{\sqrt{-ab+(a+b)x-x^2}}$$

$$= \lim_{p\to a}\lim_{q\to b}\int_q^p \frac{\mathrm{d}x}{\sqrt{-[ab-(a+b)x+x^2]}}$$

$$= \lim_{p\to a}\lim_{q\to b}\int_q^p \frac{\mathrm{d}x}{\sqrt{-\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\right]}}$$

$$= \lim_{p\to a}\lim_{q\to b}\int_q^p \frac{\mathrm{d}x}{\sqrt{\left(\frac{a-b}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2}}$$

which gives

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)}} = \lim_{p\to a}\lim_{q\to b}\left[\sin^{-1}\frac{x-\frac{a+b}{2}}{\frac{a-b}{2}}\right]_q^p$$

$$= \lim_{p\to a}\lim_{q\to b}\left[\sin^{-1}\frac{2x-(a+b)}{a-b}\right]_q^p$$

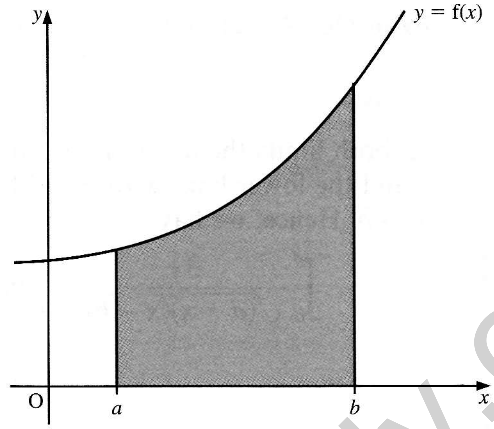
$$= \lim_{p\to a}\lim_{q\to b}\left\{\sin^{-1}\left[\frac{2p-(a+b)}{a-b}\right]-\sin^{-1}\left[\frac{2q-(a+b)}{a-b}\right]\right\}$$

$$= \sin^{-1}1-\sin^{-1}(-1) = \frac{\pi}{2}+\frac{\pi}{2}$$

That is, we have

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)}} = \pi$$

Summation of series



On pages 196–7 of *Introducing Pure Mathematics*, we noted that the definite integral $\int_a^b f(x)\,dx$ is an area bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

We arrived at this result by dividing the given area into a series of infinitesimally narrow 'rectangular' strips of equal width, δx, and summing their areas, $y\delta x$: that is, $f(x)\delta x$.

If we divide the interval $a \leqslant x \leqslant b$ into n equal strips, the x-coordinates of the strips are

$$a, \quad a + \frac{b-a}{n}, \quad a + 2\left(\frac{b-a}{n}\right), \quad a + 3\left(\frac{b-a}{n}\right), \quad \ldots, \quad b$$

Hence, the sum of the areas of all n rectangles 'inside' the integral, shown on Figure A, is

$$f(a)\frac{b-a}{n} + f\left(a + \frac{b-a}{n}\right)\frac{b-a}{n} + f\left(a + 2\frac{b-a}{n}\right)\frac{b-a}{n} + \ldots$$
$$+ f\left(a + [n-1]\frac{b-a}{n}\right)\frac{b-a}{n}$$

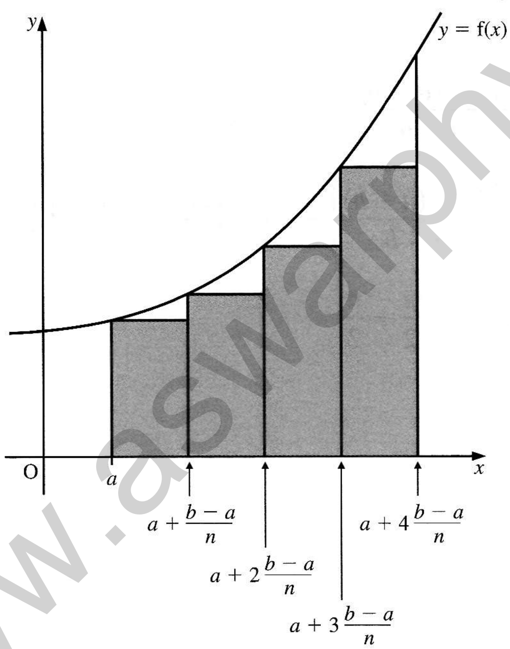


Figure A

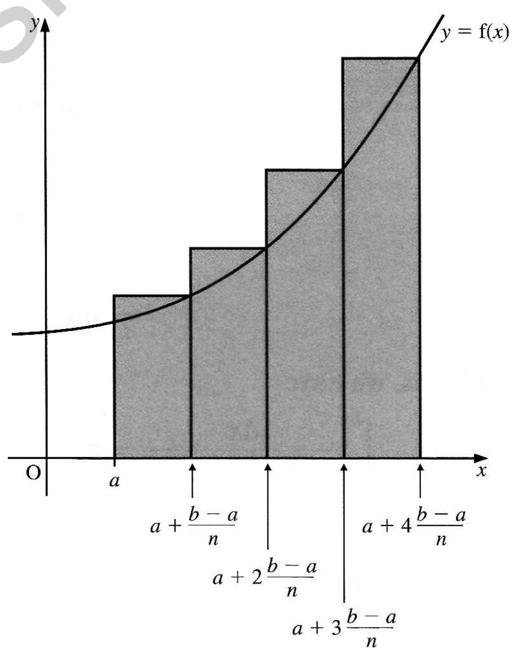


Figure B

And the sum of the areas of all n rectangles 'outside' the integral, shown in Figure B, is

$$\mathrm{f}\left(a+\frac{b-a}{n}\right)\frac{b-a}{n}+\mathrm{f}\left(a+2\frac{b-a}{n}\right)\frac{b-a}{n}+\mathrm{f}\left(a+3\frac{b-a}{n}\right)\frac{b-a}{n}+\ldots$$

$$+\,\mathrm{f}\left(a+[n-1]\frac{b-a}{n}\right)\frac{b-a}{n}+\mathrm{f}(b)\frac{b-a}{n}$$

The actual value of the definite integral is between these two values. As n tends to infinity, these two sums tend to the same value.

Hence, we have

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\left[\mathrm{f}(a)\frac{b-a}{n}+\mathrm{f}\left(a+\frac{b-a}{n}\right)\frac{b-a}{n}+f\left(a+2\frac{b-a}{n}\right)\frac{b-a}{n}+\ldots\right]$$

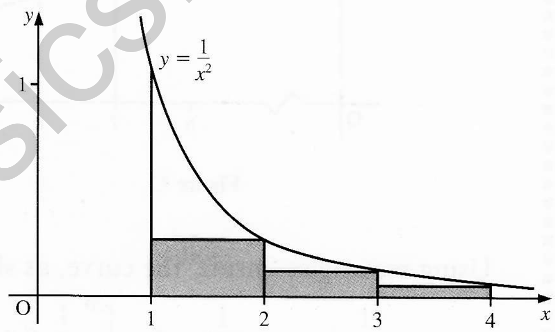
which gives

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{r=0}^{n-1}\mathrm{f}\left(a+r\frac{b-a}{n}\right)\frac{b-a}{n}$$

That is, the integral is the limit as n tends to infinity of the sum of the series.

This method may be used to find **upper** and **lower bounds** of integrals and series.

Consider, for example, the curve $y=\dfrac{1}{x^2}$.

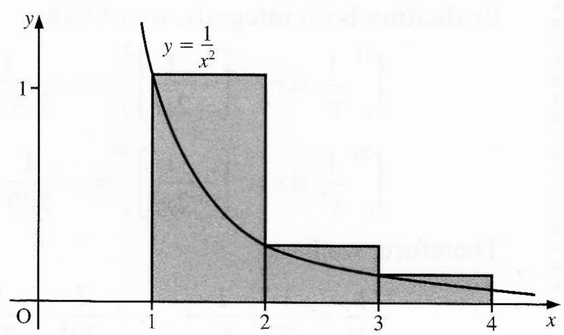


The areas of the rectangles 'inside' $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ are

$$\frac{1}{2^2},\quad \frac{1}{3^2},\quad \frac{1}{4^2},\quad \frac{1}{5^2},\quad \ldots$$

The areas of the rectangles 'outside' the integral are

$$\frac{1}{1^2},\quad \frac{1}{2^2},\quad \frac{1}{3^2},\quad \frac{1}{4^2},\quad \ldots$$



Therefore, we have

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \;>\; \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$$

$$>\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1$ (see page 260), this gives

$$\sum_{r=1}^{\infty}\frac{1}{r^2} > 1 > \sum_{r=2}^{\infty}\frac{1}{r^2}$$

Example 20 Prove that

$$\frac{4}{441} < \frac{1}{7^3} + \frac{1}{8^3} + \ldots + \frac{1}{20^3} < \frac{91}{7200}$$

SOLUTION

Since we require terms $\frac{1}{r^3}$, we take the curve $y = \frac{1}{x^3}$.

The first two rectangles 'outside' the curve, areas $\frac{1}{7^3}$ and $\frac{1}{8^3}$, are shown in Figure C.

We could continue in a similar manner until we obtain

$$\frac{1}{7^3} + \frac{1}{8^3} + \ldots + \frac{1}{20^3} > \int_7^{21} \frac{1}{x^3}\,dx$$

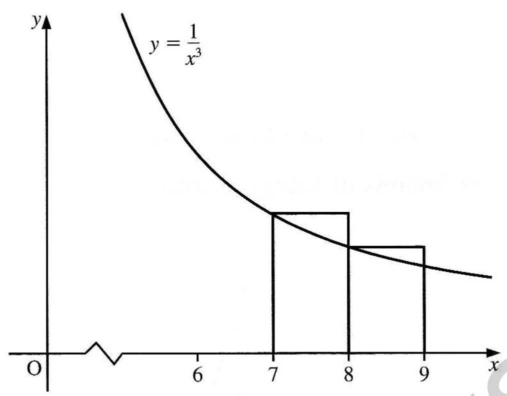


Figure C

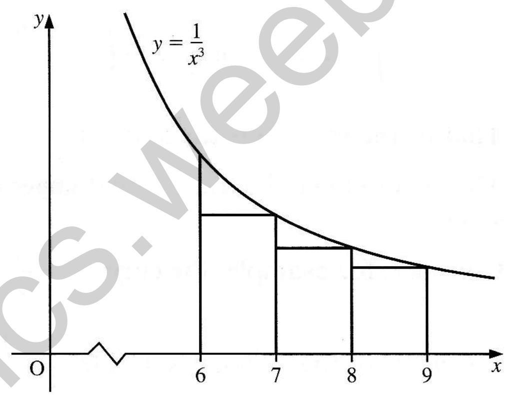


Figure D

Using rectangles 'inside' the curve, as shown in Figure D, we obtain

$$\frac{1}{7^3} + \ldots + \frac{1}{20^3} < \int_6^{20} \frac{1}{x^3}\,dx$$

Evaluating both integrals, we obtain

$$\int_7^{21} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_7^{21} = -\frac{1}{2 \times 21^2} + \frac{1}{98} = \frac{4}{441}$$

$$\int_6^{20} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_6^{20} = -\frac{1}{800} + \frac{1}{72} = \frac{91}{7200}$$

Therefore, we have

$$\frac{4}{441} < \frac{1}{7^3} + \frac{1}{8^3} + \ldots + \frac{1}{20^3} < \frac{91}{7200}$$

as required.

Exercise 12D

Find the value, where it exists, of each of the following.

1 $\displaystyle\int_0^1 \frac{1}{x^{\frac{1}{3}}}\,\mathrm{d}x$

2 $\displaystyle\int_0^1 \frac{1}{x^{\frac{3}{2}}}\,\mathrm{d}x$

3 $\displaystyle\int_1^2 \frac{1}{(1-x)^2}\,\mathrm{d}x$

4 $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,\mathrm{d}x$

5 $\displaystyle\int_0^\infty \frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x$

6 $\displaystyle\int_0^\infty \frac{x}{x^{\frac{4}{3}}}\,\mathrm{d}x$

7 $\displaystyle\int_{-a}^\infty \frac{1}{x^2-a^2}\,\mathrm{d}x$

8 $\displaystyle\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x$

9 $\displaystyle\int_{-2}^2 \frac{1}{x+2}\,\mathrm{d}x$

10 $\displaystyle\int_0^{\frac{\pi}{2}} \tan x\,\mathrm{d}x$

11 **a)** Use integration by parts to find $\displaystyle\int x\ln x\,\mathrm{d}x$.

b) Explain why $\displaystyle\int_0^1 x\ln x\,\mathrm{d}x$ exists, and obtain the value of this integral. (NEAB)

12 **a)** Write down the value of $\displaystyle\lim_{x\to\infty}\frac{x}{2x+1}$.

b) Evaluate

$$\int_1^\infty \left(\frac{1}{x}-\frac{2}{2x+1}\right)\mathrm{d}x$$

giving your answer in the form $\ln k$, where k is a constant to be determined. (NEAB)

13 **a)** Find (in terms of the constant k) the limit of $\dfrac{\cos\left(\frac{1}{2}\pi x^k\right)}{\ln x}$ as $x \to 1$.

b) **i)** Explain in detail how $\displaystyle\sum_{r=1}^{n}\frac{r}{n^2+r^2}$ is related to the area under the curve $y=\dfrac{x}{1+x^2}$ between $x=0$ and $x=1$.

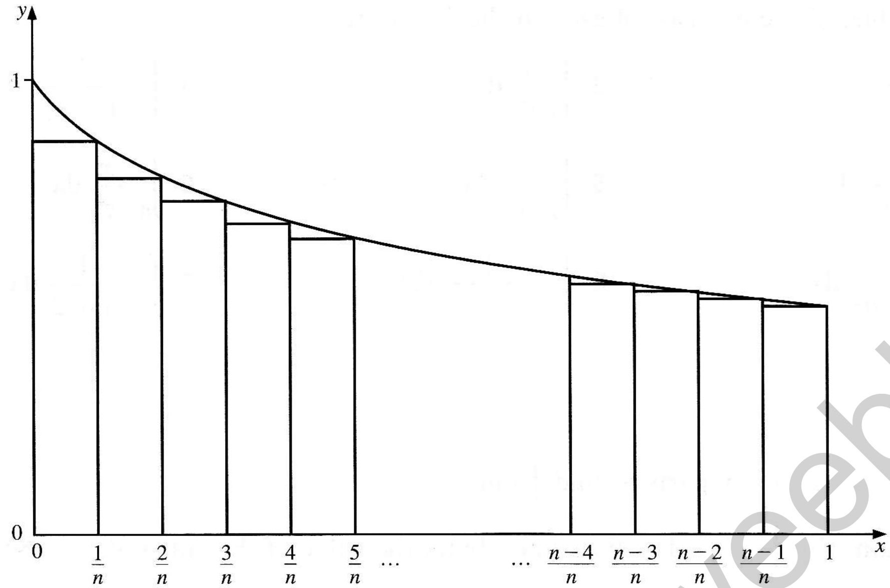
(You should include a diagram. You may assume that $\dfrac{x}{1+x^2}$ is an increasing function for $0 \leqslant x \leqslant 1$.)

ii) Evaluate the limit $L=\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2+r^2}$.

iii) Show that $L<\displaystyle\sum_{r=1}^{n}\frac{r}{n^2+r^2}<L+\frac{1}{2n}$. (MEI)

14 i) Find the exact value of $\displaystyle\int_0^1 \frac{1}{1+x}\,dx$.

ii)



The graph of $y = \dfrac{1}{1+x}$, for $0 \leqslant x \leqslant 1$, is shown in the diagram together with n rectangles, each of width $\dfrac{1}{n}$. Show that the total area of all n rectangles is

$$\frac{1}{n}\left\{\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \ldots + \frac{1}{2}\right\}$$

iii) State the limit, as $n \to \infty$, of the expression in part **ii**.

iv) By considering an appropriate graph, find the limit, as $n \to \infty$, of

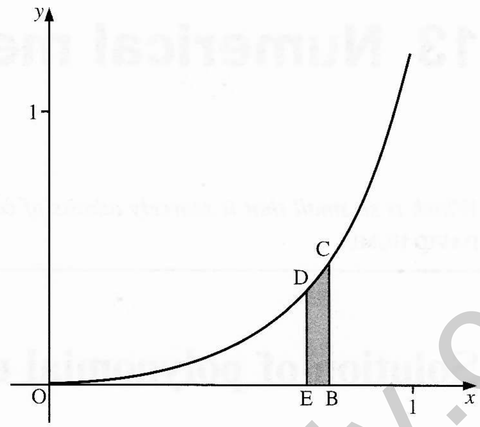
$$\frac{1}{n}\left\{\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{2}\right\} \qquad \text{(OCR)}$$

15 Prove by induction, or otherwise, that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

The diagram shows a sketch of the graph of

$$y = x^3 \quad (0 \leqslant x \leqslant 1)$$



The area of the region between the curve and the x-axis is divided up into n strips, each of width $\frac{1}{n}$, by lines drawn parallel to the y-axis. Show that the area A of the rth strip BCDE, shown in the diagram, satisfies the inequalities

$$\frac{(r-1)^3}{n^4} < A_r < \frac{r^3}{n^4}$$

Hence show that the sum S of the areas of all n strips satisfies

$$\frac{1}{4}\left(\frac{n-1}{n}\right)^2 < S < \frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

Deduce the value of the integral $\int_0^1 x^3\,dx$. (NEAB)

16 In this question, you may assume the following three results:

A) $\frac{\ln w}{w} \to 0$ as $w \to \infty$.

B) $y = \frac{\ln x}{x^{\frac{3}{2}}}$ is a decreasing function for $x > 2$.

C) $\int_a^b x^k \ln x\,dx = \frac{b^{k+1}\ln b - a^{k+1}\ln a}{k+1} - \frac{b^{k+1} - a^{k+1}}{(k+1)^2}$, where $0 < a < b$ and $k \neq -1$.

i) By substituting $w = \frac{1}{\sqrt{x}}$ into the result in A, show that $\sqrt{x}\ln x \to 0$ as $x \to 0$.

ii) Show that $\frac{\ln x}{\sqrt{x}} \to 0$ as $x \to \infty$.

iii) Explain why $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ is an improper integral, and evaluate this integral.

iv) Draw a diagram to show that $\sum_{r=3}^{n} \frac{\ln r}{r^{\frac{3}{2}}} < \int_2^n \frac{\ln x}{x^{\frac{3}{2}}}\,dx$, and write down a similar integral I for which $\sum_{r=3}^{n} \frac{\ln r}{r^{\frac{3}{2}}} > I$.

v) Deduce that the infinite series $\sum_{r=3}^{\infty} \frac{\ln r}{r^{\frac{3}{2}}}$ is convergent, and show that

$$3.57 < \sum_{r=3}^{\infty} \frac{\ln r}{r^{\frac{3}{2}}} < 3.81 \qquad \text{(MEI)}$$

13 Numerical methods

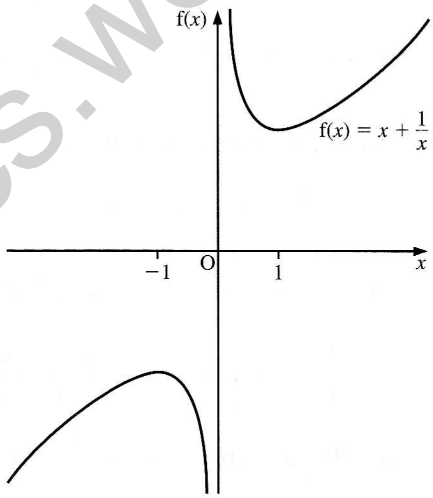
Which is so small that it scarcely admits of calculation.
DAVID HUME

Solution of polynomial equations

Most equations cannot be solved using algebraic procedures which give exact solutions, and so we have to turn to numerical methods to solve them.

While there are several, distinct numerical methods available to use, they all have one property in common: if we repeatedly apply any of the methods to a problem, we will normally be able to obtain the solution to any desired degree of accuracy.

Initially, we need to determine an interval in which the root lies. Hence, generally, to find $f(x) = 0$, we find $f(\alpha)$ and $f(\beta)$. If these are of opposite sign, and $f(x)$ is continuous between α and β, then $f(x) = 0$ has a root for some x satisfying $\alpha < x < \beta$.



If $f(x)$ is not continuous, it may be as in the graph on the right, where $f(1)$ and $f(-1)$ are of opposite sign, and $f(x) \neq 0$ for any value between -1 and 1.

Example 1 Find an approximate value for the root of $f(x) \equiv x^3 + 5x - 9 = 0$.

SOLUTION

We have

$$f(1) = 1 + 5 - 9 = -3$$

$$f(2) = 8 + 10 - 9 = 9$$

We know that $f(x)$ is continuous for $1 < x < 2$. Hence, there is a root of $f(x) = 0$ for a value of x between 1 and 2.

To find the value of the root more accurately, we could repeat this method, finding f(1.1), f(1.2), f(1.3), and so on, noting that the values of $f(x)$ change sign between 1.3 and 1.4, and then finding f(1.31), f(1.32), etc.

The method used in Example 1 is time-consuming, although with a sensible choice of values of x it can be reasonably effective in finding a solution without too many unnecessary calculations.

The procedures which are normally used to solve polynomial equations such as that in Example 1 are **interval bisection**, **linear interpolation**, the **Newton–Raphson method**, and **iteration**.

Interval bisection

As the name suggests, if we know that there is a root of $f(x) = 0$ between $x = \alpha$ and $x = \beta$, we try $x = \frac{(\alpha + \beta)}{2}$. The sign of $f(x)$ determines which side of $\frac{(\alpha + \beta)}{2}$ the root lies.

The method is repeated until we obtain the same answer to the degree of accuracy required.

Example 2 Find, by interval bisection, an approximate value for the root of $f(x) \equiv x^3 + 5x - 9 = 0$, correct to two significant figures.

SOLUTION

$$f(1) = 1 + 5 - 9 = -3$$

$$f(2) = 8 + 10 - 9 = 9$$

Therefore, the root lies between $x = 1$ and $x = 2$.

We now put $x = 1.5$, which gives

$$f(1.5) = 1.875$$

We note that $f(1.5)$ and $f(1)$ are of opposite sign. Therefore, the root lies between $x = 1$ and $x = 1.5$.

We continue to bisect the interval in which we know the root lies, until we obtain the required accuracy. Hence, we have the following results.

$f(1.25) = -0.796\,875$
$f(1.25)$ and $f(1.5)$ of opposite sign: root between $x = 1.25$ and 1.5

$f(1.375) = 0.474\,609\,375$
$f(1.25)$ and $f(1.375)$ of opposite sign: root between $x = 1.25$ and 1.375

$f(1.3125) = -0.176\,513\,67$
$f(1.3125)$ and $f(1.375)$ of opposite sign: root between $x = 1.3125$ and 1.375

$f(1.343\,75) = 0.145\,111$
$f(1.343\,75)$ and $f(1.3125)$ of opposite sign: root between $x = 1.3125$ and $1.343\,75$

Only now are we able to state that the solution of $f(x) = 0$ is 1.3 to two significant figures.

Interval bisection is a very long and generally slow method. Also, it fails if the graph of $f(x)$ is not continuous over the interval in question, as in the case of the graph of $f(x) = x + \frac{1}{x}$ on page 268.

(The actual value of the solution is 1.329 744 122 to ten significant figures.)

Linear interpolation

A more efficient method of progressing from $f(1) = -3$ and $f(2) = 9$ is to deduce that the root of $f(x) \equiv x^3 + 5x - 9 = 0$ is likely to be much nearer to 1 than to 2, since $|f(2)| > |f(1)|$.

This intuitive approach is formalised in **linear interpolation**, where the two points $(1, -3)$ and $(2, 9)$ are joined by a straight line and the x-value of the point on this line is calculated.

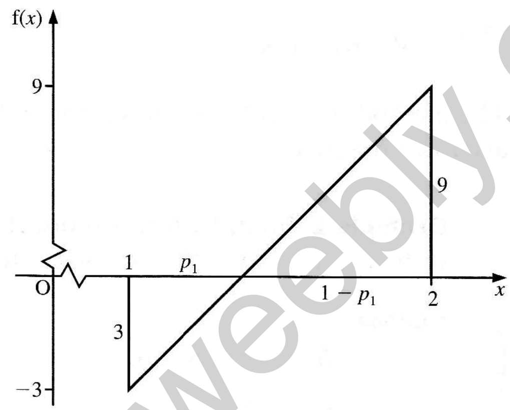
Using similar triangles, with the root at $x = 1 + p_1$, we have

$$\frac{p_1}{3} = \frac{1 - p_1}{9} \quad \Rightarrow \quad p_1 = \frac{1}{4}$$

Therefore, a better approximation to the root of $f(x) = 0$ is 1.25, which gives

$$f(1.25) = -0.796\,875$$

Hence, the root is between 1.25 and 2.



Using similar triangles again, we have

$$\frac{p_2}{0.796\,875} = \frac{0.75 - p_2}{9}$$

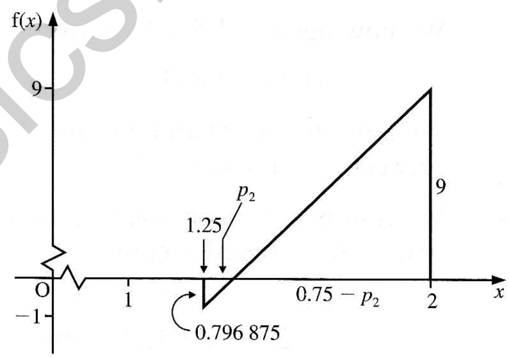
$$\Rightarrow \quad 9.796\,875 p_2 = 0.75 \times 0.796\,875$$

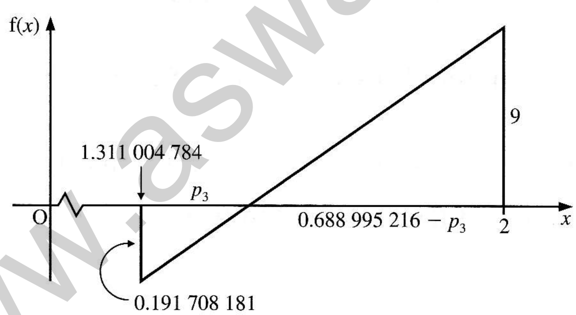
$$\Rightarrow \quad p_2 = 0.061\,004\,784$$

Therefore, the second approximation to the root of $f(x) = 0$ is 1.311 004 784, which gives

$$f(1.311\,004\,784) = -0.191\,708\,181$$

Hence, the root is between 1.311 004 784 and 2.





Repeating the procedure again (see figure above), we obtain

$$\frac{p_3}{0.191\,708\,181} = \frac{0.688\,995\,216 - p_3}{9}$$

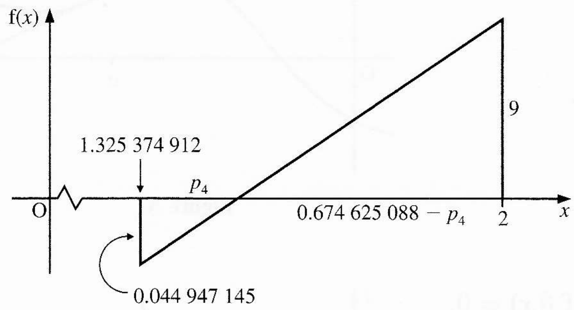
$$\Rightarrow \quad 9.191\,708\,181 p_3 = 0.688\,995\,216 \times 0.191\,708\,181$$

$$\Rightarrow \quad p_3 = 0.014\,370\,127$$

Therefore, the third approximation to the root is 1.325 374 912, which gives

$$f(1.325\,374\,912) = -0.044\,947\,145$$

Hence, the root is between 1.325 374 912 and 2.



Repeating the procedure yet again (see figure above), we obtain

$$\frac{p_4}{0.044\,947\,145} = \frac{0.674\,625\,088 - p_4}{9}$$

$$\Rightarrow \quad 9.044\,947\,145 p_4 = 0.674\,625\,088 \times 0.044\,947\,145$$

$$\Rightarrow \quad p_4 = 0.003\,352\,421\,099$$

Therefore, the fourth approximation 1.328 727 333 is to the root.

Both the fourth and third approximations are 1.33 correct to two decimal places. To check that this is the correct answer to two decimal places, we find f(1.335):

$$f(1.335) = 0.054\,27$$

which has the opposite sign to f(1.326).

Hence, the root is 1.33 correct to two decimal places.

Although linear interpolation is much quicker than interval bisection, it still does not take into account the shape of the graph of f(x) between the starting points.

The procedure which does is the Newton–Raphson method.

Newton–Raphson method

If α is an approximate value for the root of f(x) = 0, then $\alpha - \dfrac{f(\alpha)}{f'(\alpha)}$ is generally a better approximation.

Consider the graph of $y = f(x)$. Draw the tangent at P, where $x = \alpha$, and let the tangent meet the x-axis at T.

We see that the x-value at T is closer than α is to the x-value at N, where the graph cuts the axis.