

We can express this result as

$$\frac{1}{2} \ln \left(\frac{x-1}{\sqrt{5}} + \sqrt{\frac{(x-1)^2}{5} - 1} \right) + c$$

which gives

$$\begin{aligned} \int \frac{1}{\sqrt{4x^2 - 8x - 16}} dx &= \frac{1}{2} \ln \left(\sqrt{(x-1)^2 - 5} + x - 1 \right) - \frac{1}{2} \ln \sqrt{5} + c \\ &= \frac{1}{2} \ln \left(\sqrt{x^2 - 2x - 4} + x - 1 \right) + c' \end{aligned}$$

Exercise 10B

1 Differentiate each of the following with respect to x .

a) $\sinh^{-1} 5x$

b) $\cosh^{-1} 3x$

c) $\sinh^{-1} \sqrt{2}x$

d) $\cosh^{-1} \frac{3}{4}x$

e) $\sinh^{-1} x^2$

f) $\operatorname{sech}^{-1} x$

g) $\coth^{-1} x$

2 Find each of the following integrals.

a) $\int \frac{dx}{\sqrt{x^2 - 4}}$

b) $\int \frac{dx}{\sqrt{x^2 - 9}}$

c) $\int \frac{dx}{\sqrt{4x^2 - 25}}$

d) $\int \frac{dx}{\sqrt{9x^2 - 16}}$

e) $\int \frac{dx}{\sqrt{9 + x^2}}$

f) $\int \frac{dx}{\sqrt{16 + x^2}}$

g) $\int \frac{dx}{\sqrt{25 + 16x^2}}$

h) $\int \frac{dx}{\sqrt{9 + 25x^2}}$

3 Evaluate each of the following definite integrals, giving the exact value of your answer.

a) $\int_0^1 \frac{dx}{\sqrt{1 + x^2}}$

b) $\int_0^2 \frac{dx}{\sqrt{4 + x^2}}$

c) $\int_4^8 \frac{dx}{\sqrt{x^2 - 16}}$

d) $\int_0^2 \frac{dx}{\sqrt{4 + 3x^2}}$

e) $\int_{\frac{1}{5}}^1 \frac{dx}{\sqrt{25x^2 - 1}}$

4 Evaluate each of the following integrals, giving your answer in terms of logarithms.

a) $\int_1^2 \frac{dx}{\sqrt{25x^2 - 4}}$

b) $\int_1^2 \frac{dx}{\sqrt{4 + 9x^2}}$

c) $\int_3^4 \frac{dx}{\sqrt{(x-1)^2 - 3}}$

d) $\int_0^1 \frac{dx}{\sqrt{4(x+1)^2 + 5}}$

e) $\int_0^2 \frac{dx}{\sqrt{4 + 8x + x^2}}$

f) $\int_0^1 \frac{dx}{\sqrt{16x^2 + 20x + 35}}$

5 Given that $f(x) \equiv \frac{1}{\sqrt{(x^2 + 4x - 12)}}$,

a) find $\int f(x) dx$.

b) Hence find the exact value of $\int_6^{10} f(x) dx$, giving your answer as a single logarithm.

(EDEXCEL)

6 a) Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

b) Evaluate $\int_{-1}^0 \frac{dx}{\sqrt{x^2 + 2x + 2}}$. (WJEC)

7 a) Find $\int x \operatorname{sech}^2 x dx$.

b) Find the general solution of the differential equation

$$\cosh x \frac{dy}{dx} - y \sinh x = x$$

giving your answer in the form $y = f(x)$. (EDEXCEL)

8 $4x^2 + 4x + 5 \equiv (px + q)^2 + r$

a) Find the values of the constants p , q and r .

b) Hence, or otherwise, find $\int \frac{1}{4x^2 + 4x + 5} dx$.

c) Show that

$$\int \frac{2}{\sqrt{(4x^2 + 4x + 5)}} dx = \ln[(2x + 1) + \sqrt{(4x^2 + 4x + 5)}] + k$$

where k is an arbitrary constant. (EDEXCEL)

9 a) Show that $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$.

b) Evaluate $\int_0^1 \frac{dx}{\sqrt{x^2 + 6x + 10}}$, giving your answer correct to four decimal places. (WJEC)

10 a) Express $4x^2 + 4x + 26$ in the form $(px + q)^2 + r$, where p , q and r are constants.

b) Hence determine $\int \frac{1}{\sqrt{(4x^2 + 4x + 26)}} dx$. (EDEXCEL)

11 i) Find A , B and C such that

$$3x^2 + 24x + 23 \equiv A(x + B)^2 + C$$

ii) Show that

$$\int \frac{dx}{\sqrt{3x^2 + 24x + 23}} = \frac{1}{\sqrt{3}} \cosh^{-1} \left(\frac{\sqrt{3}(x + 4)}{5} \right) + c \quad (\text{NICCEA})$$

12 Express $x^2 - 6x + 8$ in the form $(x - p)^2 - q^2$, for positive integers p and q .

Hence evaluate $\int_4^5 \frac{dx}{\sqrt{(x^2 - 6x + 8)}}$ giving your answer in terms of natural logarithms.

(AEB 97)

13 a) Simplify $(e^x + e^{-x})^2 - (e^x - e^{-x})^2$ and hence deduce that $\cosh^2 x - \sinh^2 x = 1$.

b) Given that $y = \operatorname{arsinh} x$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{(x^2 + 1)}}$.

c) Find $\int \operatorname{arsinh} x \, dx$. (EDEXCEL)

14 A curve has equation $y = x \sinh^{-1} x$.

i) Show that

$$\frac{d^2 y}{dx^2} = \frac{2 + x^2}{(1 + x^2)^{\frac{3}{2}}}$$

ii) Deduce that the curve has no point of inflexion. (OCR)

15 Starting from the definition of \cosh in terms of exponentials, show that

$$\cosh^{-1} x = \ln[x + \sqrt{(x^2 - 1)}]$$

Show that

$$\int_1^2 \frac{1}{\sqrt{(4x^2 - 1)}} \, dx = \frac{1}{2} \ln \left(\frac{4 + \sqrt{15}}{2 + \sqrt{3}} \right) \quad (\text{OCR})$$

16 Given that $y = \tanh^{-1} x$, derive the result $\frac{dy}{dx} = \frac{1}{1 - x^2}$.

[No credit will be given for merely quoting the result from the *List of Formulae*.]

Show that $\int_0^{\frac{1}{2}} \tanh^{-1} 2x \, dx = \frac{1}{8} \ln \frac{27}{16}$. (OCR)

17 i) Let $x = \sinh u$. By first expressing x in terms of exponentials, show that

$$\sinh^{-1} x = \ln[x + \sqrt{(x^2 + 1)}]$$

ii) By using an appropriate substitution, show that

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} \, dx = \sinh^{-1} \left(\frac{x}{a} \right) + c$$

where a and c are constants ($a > 0$).

iii) Evaluate

$$\int_0^4 \frac{1}{\sqrt{(9x^2 + 4)}} \, dx$$

giving your answer in terms of a natural logarithm. (OCR)

18 a) State the values of x for which $\cosh^{-1} x$ is defined.

b) A curve C is defined for these values of x by the equation $y = x - \cosh^{-1} x$.

i) Show that C has just one stationary point.

ii) Evaluate y at the stationary point, giving your answer in the form $p - \ln q$, where p and q are numbers to be determined. (NEAB)

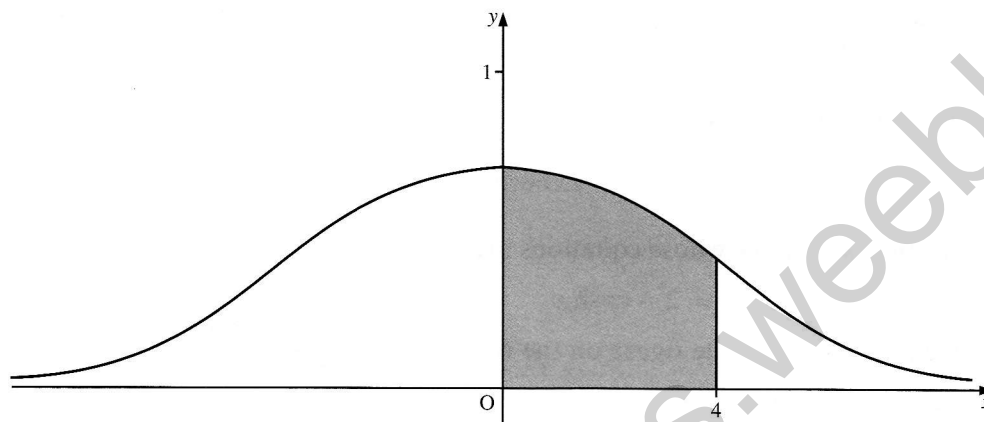
19 a) Using the substitution $u = e^x$, find $\int \operatorname{sech} x \, dx$.

b) Sketch the curve with equation $y = \operatorname{sech} x$.

The finite region R is bounded by the curve with equation $y = \operatorname{sech} x$, the lines $x = 2$, $x = -2$ and the x -axis.

c) Using your result from part a, find the area of R , giving your answer to three decimal places. (EDEXCEL)

20 The diagram shows the curve with equation $y = \frac{1}{(x^2 + 4)^{\frac{1}{4}}}$.



The finite region bounded by the curve, the x -axis, the y -axis and the line $x = 4$ is rotated through one full turn about the x -axis to form a solid of revolution.

Use integration to determine the volume of this solid, giving your answer in terms of π and a natural logarithm. (AEB 98)

21 a) Use the definition of $\coth x$ in terms of exponential functions to prove that

$$\operatorname{arcoth} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

The function f is defined by $f(x) = \operatorname{arcoth} \left(\frac{x}{3} \right)$, $x^2 > 9$.

b) Show that f is odd.

c) Find $f'(x)$.

d) Expand $f(x)$ in a series of ascending powers of $\frac{1}{x}$ as far as the term in $\frac{1}{x^7}$ and state the coefficient of $\frac{1}{x^{2n+1}}$.

e) Hence, or otherwise, derive the expansion of $\frac{1}{9-x^2}$ in a series of ascending powers of $\frac{1}{x}$ as far as the term in $\frac{1}{x^8}$ and state the coefficient of $\frac{1}{x^{2n}}$. (EDEXCEL)

- 22** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that for $|x| < 1$,

$$\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

- a) Expand $\operatorname{artanh} x$ as a series in ascending powers of x , as far as the term in x^5 and state the coefficient of x^{2n+1} in this expansion.
b) Solve the equation

$$3 \operatorname{sech}^2 x + 4 \tanh x + 1 = 0$$

giving any answers in terms of natural logarithms.

- c) Sketch the graph of $y = \operatorname{artanh} x$ and evaluate the area of the finite region bounded by the curve with equation $y = \operatorname{artanh} x$ and the lines $x = \frac{1}{2}$ and $y = 0$. (EDEXCEL)

- 23 a)** Use integration by parts to show

$$\int x^2 \cosh x \, dx = x^2 \sinh x - 2x \cosh x + 2 \sinh x + c$$

- b) Consider the two curves whose equations are

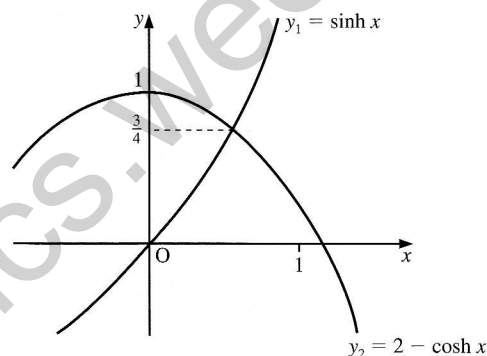
$$y_1 = \sinh x \quad y_2 = 2 - \cosh x$$

and which are shown in the figure on the right.

- i) Show that they cross at the point $(\log_e 2, \frac{3}{4})$.
ii) Find the area bounded by the y -axis, the curve y_1 and the curve y_2 .
iii) The area bounded by the y -axis, the line $y = \frac{3}{4}$ and the curve y_1 is rotated about the y -axis to form a solid of revolution. Show that its exact volume is

$$\frac{\pi}{4} [3(\log_e 2)^2 - 10 \log_e 2 + 6]$$

$$\left[\text{The volume of revolution about the } y\text{-axis is given by } \pi \int x^2 \, dy. \right] \quad (\text{NICCEA})$$



'Double-angle' formulae

To integrate $\cosh^2 x$ and $\sinh^2 x$, we must express each in a form which contains $\cosh 2x$, in a similar manner to integrating $\cos^2 x$ and $\sin^2 x$ (see *Introducing Pure Mathematics*, pages 451–2).

To obtain the identity relating $\cosh 2x$ to $\cosh^2 x$, we have

$$\begin{aligned} \cosh 2x &= \frac{1}{2} (e^{2x} + e^{-2x}) = \frac{1}{2} [(e^x + e^{-x})^2 - 2] \\ &= \frac{1}{2} [4 \cosh^2 x - 2] \end{aligned}$$

which gives

$$\cosh 2x = 2 \cosh^2 x - 1$$

To obtain the identity relating $\cosh 2x$ to $\sinh^2 x$, we take

$$\cosh 2x = 2 \cosh^2 x - 1$$

and make the substitution $\cosh^2 x = 1 + \sinh^2 x$ to obtain

$$\cosh 2x = 2(1 + \sinh^2 x) - 1$$

which gives

$$\cosh 2x = 2 \sinh^2 x + 1$$

Similarly, we have

$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

which gives

$$\sinh 2x = 2 \sinh x \cosh x$$

Hence, we see that $\int \cosh^2 ax \, dx$ is given by

$$\int \cosh^2 ax \, dx = \int \frac{1}{2}(\cosh 2ax + 1) \, dx$$

which gives

$$\int \cosh^2 ax \, dx = \frac{1}{4a} \sinh 2ax + \frac{x}{2} + c$$

Example 14 Using the substitution $x = 3 \sinh u$, find the value of

$$\int \sqrt{9 + x^2} \, dx.$$

SOLUTION

Differentiating the substitution $x = 3 \sinh u$, we obtain

$$\frac{dx}{du} = 3 \cosh u$$

$$\Rightarrow dx = 3 \cosh u \, du$$

Substituting for x and for dx in $\int \sqrt{9 + x^2} \, dx$, we have

$$\begin{aligned} \int \sqrt{9 + x^2} \, dx &= \int \sqrt{9 + 9 \sinh^2 u} (3 \cosh u) \, du \\ &= \int 9 \cosh^2 u \, du \\ &= \frac{9}{2} \int (\cosh 2u + 1) \, du \\ &= \frac{9}{2} \left(\frac{1}{2} \sinh 2u + u \right) + c \end{aligned}$$

Using $\sinh 2u = 2 \sinh u \cosh u$, we obtain

$$\int \sqrt{9+x^2} dx = \frac{9}{2} \sinh u \cosh u + \frac{9}{2} u + c$$

As the question involves an integral in terms of x , the answer must be given in terms of x .

Using $\cosh u = \sqrt{\sinh^2 u + 1}$ and $\sinh u = \frac{x}{3}$, we obtain

$$\begin{aligned} \int \sqrt{9+x^2} dx &= \frac{9}{2} \frac{x}{3} \sqrt{1+\frac{x^2}{9}} + \frac{9}{2} \sinh^{-1}\left(\frac{x}{3}\right) + c \\ &= \frac{1}{2} x \sqrt{9+x^2} + \frac{9}{2} \ln\left(\frac{x}{3} + \sqrt{\frac{x^2}{9} + 1}\right) + c \end{aligned}$$

Therefore, we have

$$\int \sqrt{9+x^2} dx = \frac{x}{2} \sqrt{x^2+9} + \frac{9}{2} \ln(\sqrt{x^2+9}+x) + c'$$

Power series

On page 177, we used Maclaurin's series to find the power series for $\sin x$ and $\cos x$.

In a similar way, we can find the power series for $\sinh x$ and $\cosh x$.

Power series for $\sinh x$

Let $\sinh x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$, where the a 's are constants.

When $x = 0$, $\sinh 0 = a_0$. But $\sinh 0 = 0$, therefore $a_0 = 0$.

Differentiating $\sinh x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$, we obtain

$$\cosh x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

When $x = 0$, $\cosh 0 = a_1$. But $\cosh 0 = 1$, therefore $a_1 = 1$.

Differentiating again, we obtain

$$\sinh x = 2a_2 + 3 \times 2a_3 x + 4 \times 3a_4 x^2 + 5 \times 4a_5 x^3 + \dots$$

When $x = 0$, $\sinh 0 = 2a_2 \Rightarrow a_2 = 0$.

Differentiating yet again, we obtain

$$\cosh x = 3 \times 2a_3 + 4 \times 3 \times 2a_4 x + 5 \times 4 \times 3a_5 x^2 + \dots$$

When $x = 0$, $\cosh 0 = 3 \times 2a_3 \Rightarrow a_3 = \frac{1}{3!}$.

Repeating the differentiation, we obtain

$$a_4 = 0 \quad a_5 = \frac{1}{5!} \quad a_6 = 0 \quad a_7 = \frac{1}{7!}$$

Hence, we have

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots$$

By d'Alembert's ratio test, this series converges for all real x .

Power series for cosh x

We can use the procedure for $\sinh x$ to find the power series for $\cosh x$. However, it is much easier to start from the expansion for $\sinh x$. Hence, we have

$$\cosh x = \frac{d}{dx} \sinh x = \frac{d}{dx} \left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots \right)$$

which gives

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots$$

By d'Alembert's ratio test, this series is convergent for all real x .

Osborn's rule

Taking the power series for $\cos ix$, we have

$$\begin{aligned} \cos ix &= 1 - \frac{1}{2!}(ix)^2 + \frac{1}{4!}(ix)^4 - \dots \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \end{aligned}$$

which is the power series for $\cosh x$. Hence, we have

$$\cos ix \equiv \cosh x$$

For $\sin ix$, we have

$$\begin{aligned} \sin ix &= (ix) - \frac{1}{3!}(ix)^3 + \frac{1}{5!}(ix)^5 - \dots \\ &= i \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \end{aligned}$$

which is the power series for $i \sinh x$. Hence, we have

$$\sin ix \equiv i \sinh x$$

Since $\cos^2 \theta + \sin^2 \theta \equiv 1$ for any angle θ , we know that

$$\cos^2 ix + \sin^2 ix \equiv 1$$

which gives

$$\cosh^2 x + (i \sinh x)^2 \equiv 1$$

Therefore, we have

$$\cosh^2 x - \sinh^2 x \equiv 1$$

The two identities

$$\cos^2 \theta + \sin^2 \theta \equiv 1 \quad \text{and} \quad \cosh^2 x - \sinh^2 x \equiv 1$$

are typical of the similarity between the standard ordinary trigonometric identities and the standard hyperbolic identities (see pages 191–2). Osborn's rule gives guidance on the similarity between such identities based on $\sin ix$ being equivalent to $i \sinh x$.

Osborn's rule states that to change a standard ordinary trigonometric identity into the equivalent standard hyperbolic identity, **change the sign** of the term which is the **product of two sines**, and **substitute the corresponding hyperbolic functions**.

Thus, for example,

$$\cos 2x \equiv 1 - 2 \sin^2 x \quad \text{gives} \quad \cosh 2x \equiv 1 + 2 \sinh^2 x$$

When applying the rule to $1 + \tan^2 x = \sec^2 x$, we treat $\tan^2 x$ as $\frac{\sin^2 x}{\cos^2 x}$. Hence, the equivalent hyperbolic identity is

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

Exercise 10C

1 Expand each of the following expressions up to and including the term in x^4 .

- | | |
|-----------------------|------------------------|
| a) $\cosh 2x$ | b) $\sinh 3x$ |
| c) $(1 + x) \cosh 5x$ | d) $(1 + 2x) \sinh 6x$ |

2 By means of the substitution $x = 3 \cosh \theta$, find $\int \sqrt{x^2 - 9} \, dx$.

3 Using the substitution $x = 4 \sinh \theta$, find $\int \sqrt{16 + x^2} \, dx$.

4 Find $\int \sqrt{25 + x^2} \, dx$.

5 Find $\int \sqrt{x^2 - 25} \, dx$.

6 Find $\int \frac{x^2}{\sqrt{x^2 - 4}} \, dx$.

7 Find $\int \frac{x^2 + 2}{\sqrt{x^2 + 9}} \, dx$.

8 Use the substitution $x = 2 \sinh u$ to find $\int \sqrt{(x^2 + 4)} \, dx$. (OCR)

9 Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to prove that

$$\cosh A + \cosh B = 2 \cosh \frac{1}{2}(A + B) \cosh \frac{1}{2}(A - B)$$

For $n \geq 0$, the function P_n is defined by

$$P_n(x) = 1 - (n + 1) \cosh nx + n \cosh(n + 1)x$$

i) Evaluate $P_0(x)$.

ii) Show that

$$P_r(x) - P_{r-1}(x) = 2r \cosh rx (\cosh x - 1)$$

where $r \geq 1$.

Hence, or otherwise, find $\sum_{r=1}^n r \cosh rx$ for $x \neq 0$. (NEAB)

10 i) Prove that

$$\sinh \{\log_e(x + \sqrt{x^2 + 1})\} \equiv x$$

ii) Show that

$$\int \frac{dx}{\sqrt{16x^2 + 9}} = \frac{1}{4} \log_e \{4x + \sqrt{16x^2 + 9}\} + c$$

iii) Show that

$$\frac{d}{dx}(x\sqrt{16x^2 + 9}) \equiv 2\sqrt{16x^2 + 9} - \frac{9}{\sqrt{16x^2 + 9}}$$

iv) Hence show that

$$\int_0^1 \sqrt{16x^2 + 9} dx = \frac{5}{2} + \frac{9}{8} \log_e 3 \quad (\text{NICCEA})$$

11 i) Show that

$$\int \frac{x^2}{\sqrt{16 - 9x^2}} dx = \frac{16}{27} \int \sin^2 \theta d\theta$$

ii) If $g(x)$ is a continuous function, show that

$$\int_0^{12} \frac{g(x)}{\sqrt{49 + 4x^2}} dx = \frac{1}{2} \int_0^{\log_e 7} g\left(\frac{7}{2} \sinh u\right) du \quad (\text{NICCEA})$$

12 Find the general solution of the differential equation

$$(\sinh x) \frac{dy}{dx} + (2 \cosh x)y = \sinh x \quad (\text{NEAB})$$

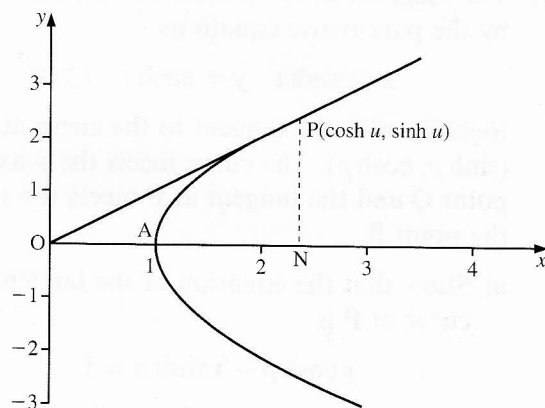
13 a) The locus of a point (x, y) defined by the parametric equations

$$\begin{aligned} x &= \cosh v \\ y &= \sinh v \end{aligned}$$

together with the point P at which $v = u$, where $u > 0$, is shown in the figure.

i) Show that the area bounded by the curve AP, the ordinate line PN and the x-axis is given by

$$\int_0^u \sinh^2 v dv$$



ii) Show that the area bounded by the curve AP, the straight line OP and the x-axis is $\frac{1}{2}u$.

- b) Sketch the curve defined by $x = \cos \theta$, $y = \sin \theta$.

If the co-ordinates of P' are $(\cos \phi, \sin \phi)$, shade a region whose area is $\frac{1}{2}\phi$. Comment on the similarities between the figure on page 215 and your sketch. (NICCEA)

- 14 Differentiate $\sqrt{(x^2 - 1)}$.

Show that

$$\int_1^{\frac{5}{4}} \cosh^{-1} x \, dx = a \ln 2 + b$$

where a and b are rational numbers to be determined. (NEAB)

- 15 The diagram on the right shows a region R in the x - y plane bounded by the curve $y = \sinh x$, the x -axis and the line AB which is perpendicular to the x -axis.

- a) Given that $AB = \frac{4}{3}$, show that $OB = \ln 3$.
b) i) Show that

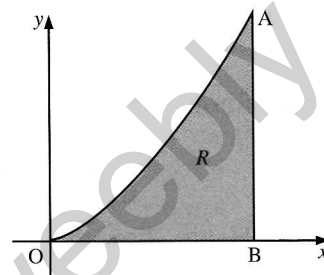
$$\cosh(\ln k) = \frac{k^2 + 1}{2k}$$

- ii) Show that the area of the region R is $\frac{2}{3}$.

- c) i) Show that

$$\int_0^{\ln 3} \sinh^2 x \, dx = \frac{1}{4} [\sinh(\ln 9) - \ln 9]$$

- ii) Hence find, correct to three significant figures, the volume swept out when the region R is rotated through an angle of 2π radians about the x -axis. (NEAB)



- 16 a) Given that $u = \frac{1}{2}(e^y - e^{-y})$, prove that $y = \ln(u + \sqrt{u^2 + 1})$.
b) Using the substitution $x = \sinh \theta$, show that

$$\int \frac{x^2}{\sqrt{1+x^2}} \, dx = \frac{1}{2} [x\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2})] + k$$

where k is an arbitrary constant. (EDEXCEL)

- 17 The diagram shows a sketch of the curve defined by the parametric equations

$$x = \sinh t, \quad y = \cosh t \quad t \geq 0$$

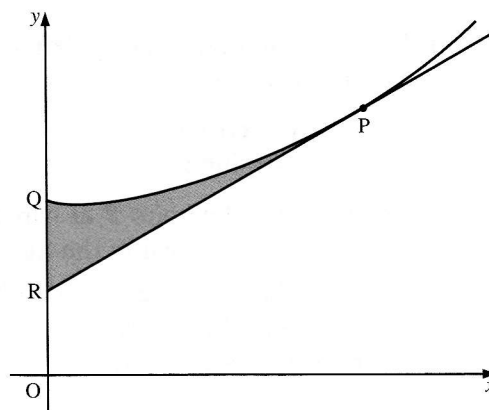
together with the tangent to the curve at the point P ($\sinh p$, $\cosh p$). The curve meets the y -axis at the point Q and the tangent at P meets the y -axis at the point R .

- a) Show that the equation of the tangent to the curve at P is

$$y \cosh p - x \sinh p = 1$$

- b) Given that R is the point $(0, \frac{1}{2})$, show that

$$p = \ln(2 + \sqrt{3})$$



- c) Show that the area A (shown shaded in the diagram) bounded by QR, RP and the arc PQ is given by

$$A = \int_0^p \cosh^2 t \, dt - \frac{5\sqrt{3}}{4}$$

- d) Hence find the value of A in the form

$$a \ln(2 + \sqrt{3}) + b\sqrt{3}$$

where a and b are rational numbers to be determined. (NEAB)

- 18 a) Use the power series for $\sin x$ to show that, for small values of x ,

$$\sin^3 x \approx x^3 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} \right)^3$$

- b) Hence, or otherwise, find the constants a , b , c in the approximation

$$\sin^3 x \approx ax^3 + bx^5 + cx^7$$

- c) Find a similar approximation for $x^2 \sinh x$ for small values of x .

- d) Show that

$$\lim_{x \rightarrow 0} \frac{x^2 \sinh x - \sin^3 x}{x^5} = \frac{2}{3} \quad (\text{NEAB})$$

11 Conics

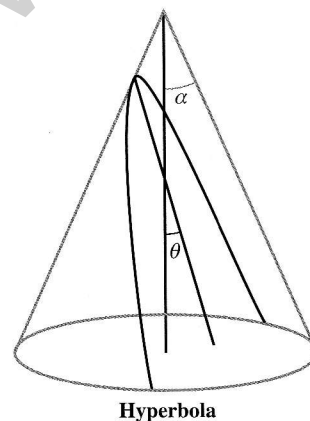
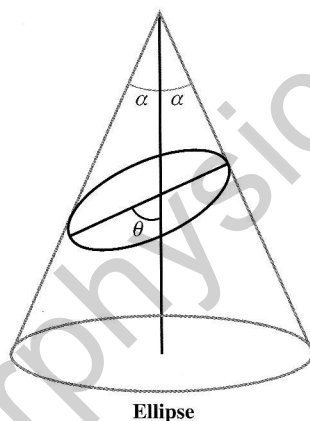
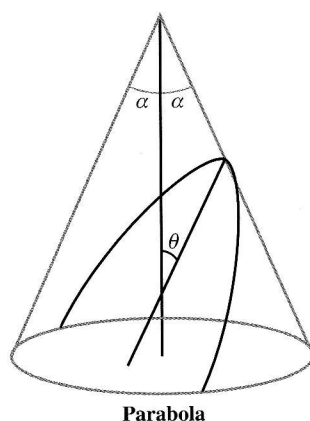
That an extensive theory of the conics was obtained is eloquent testimony to the brilliance of Archimedes and Apollonius.

JEREMY J. GRAY

Generating conics

If we take a solid, right circular cone and, in any direction, cut a plane section through it, we obtain a curve which is a member of the class of curves known as **conics** or **conic sections**.

It follows that the shape of the curve so obtained is determined by the direction in which we make the cut: that is, on the inclination, θ , of the plane section to the axis, as the figure below shows.



Hence, with the cone standing on a horizontal plane, if we cut in a direction parallel to the slant height of the cone, whereby $\theta = \alpha$, we obtain a **parabola**.

If we cut in a direction for which $\alpha < \theta < \frac{\pi}{2}$, we obtain an **ellipse**.

If we cut in a direction, not through the vertex, for which $\theta < \alpha$, we obtain a **hyperbola**.

If we cut horizontally through the cone (that is, $\theta = \frac{\pi}{2}$), we obtain a **circle**.

The study of the parabola, the ellipse and the hyperbola as sections of the same cone originated with the Greek geometer Apollonius, who flourished about 280 BC. They were not defined analytically as loci until the seventeenth century, largely due to the work of the renowned French mathematician René Descartes (1596–1650), and of the English mathematician John Wallis (1616–1703).

Conics as loci

Analytically we define a conic as the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is constant.

The fixed point is called the **focus**, and the fixed line the **directrix**. The constant ratio is known as the **eccentricity** of the conic and is denoted by e .

Hence, in the figure on the right, where the point P is describing a conic, we have

$$PF = ePT$$

When $e = 1$, the conic is a **parabola**.

When $0 < e < 1$, the conic is an **ellipse**.

When $e > 1$, the conic is a **hyperbola**.

The circle (which we met in *Introducing Pure Mathematics*, pages 220–7) may be treated as the limiting case of an ellipse, in which $e = 0$ (see pages 222–6).

Parabola

Let the focus, F, be $(a, 0)$ and the directrix be $x = -a$. Then for the point P(x, y), we have

$$PT = x + a \quad PF = \sqrt{(x - a)^2 + y^2}$$

But $PT = PF$, since for a parabola $e = 1$. Hence, we obtain

$$(x - a)^2 + y^2 = (x + a)^2$$

which gives

$$y^2 = 4ax$$

This is the **standard equation** for a parabola, an example of which is shown at bottom right.

Common parametric equations for the parabola $y^2 = 4ax$ are

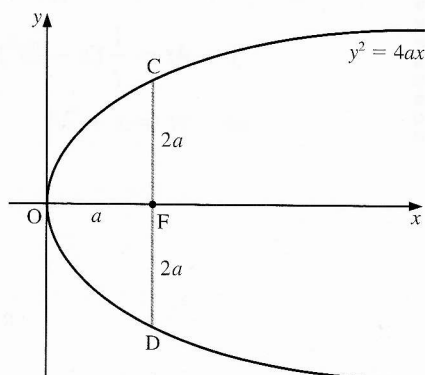
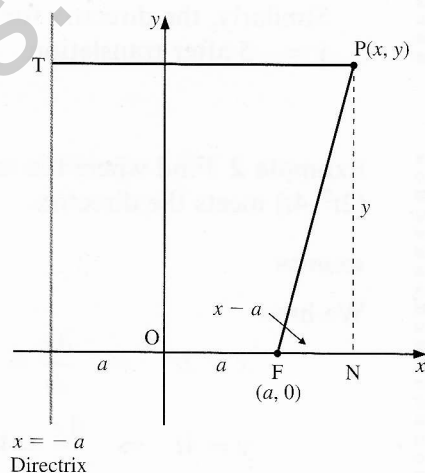
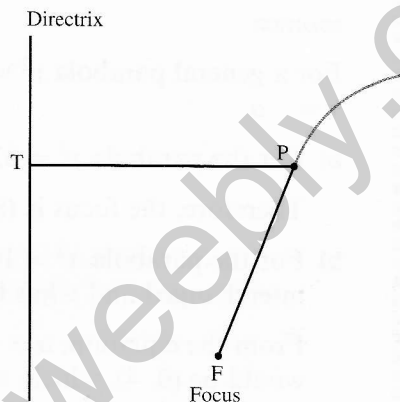
$$x = at^2 \quad \text{and} \quad y = 2at$$

where t is the parameter.

The chord of a parabola through its focus, and perpendicular to its axis, is called the **latus rectum**. Thus, in the diagram on the right, CD is the latus rectum.

Half the length of this chord (FC or FD) is known as the **semi latus rectum**.

From the equation $y^2 = 4ax$, we see that the coordinates of C are $(a, 2a)$ and those of D are $(a, -2a)$. Hence, the length of the latus rectum is $4a$ and that of the semi latus rectum is $2a$. (See also pages 223 and 231.)



Note

- All quadratic curves are parabolas.
- All quadratic curves are similar.

Example 1 Find the focus and directrix of each of the parabolas

a) $y^2 = 32x$ b) $x^2 = 16(y + 1)$

SOLUTION

For a general parabola $y^2 = 4ax$, the focus is at $(a, 0)$ and the directrix is $x = -a$.

a) For the parabola $y^2 = 32x$, $a = 8$.

Therefore, the focus is $(8, 0)$, and the directrix is $x = -8$.

b) For the parabola $x^2 = 16(y + 1)$, the x - and y -axes have been interchanged and y has been translated to $y + 1$.

From the equation, $a = 4$. Thus the focus for the parabola $x^2 = 16y$ would be $(0, 4)$, which, after translation, becomes $(0, 3)$ for the parabola $x^2 = 16(y + 1)$.

Similarly, the directrix for $x^2 = 16y$, which would be $y = -4$, becomes $y = -5$ after translation.

Example 2 Find where the tangent to the parabola $y^2 = 8x$ at the point $(2t^2, 4t)$ meets the directrix.

SOLUTION

We have

$$x = 2t^2 \Rightarrow \frac{dx}{dt} = 4t$$

$$y = 4t \Rightarrow \frac{dy}{dt} = 4$$

which give

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{t}$$

Using $y - y_1 = m(x - x_1)$, we find the equation of the tangent at $(2t^2, 4t)$:

$$y - 4t = \frac{1}{t}(x - 2t^2)$$

$$\Rightarrow yt = x + 2t^2 \quad [1]$$

The directrix of the standard parabola $y^2 = 4ax$ is $x = -a$. Therefore, for the parabola $y^2 = 8x$, $a = 2$. Hence, the directrix is $x = -2$.

Substituting $x = -2$ in [1], we obtain

$$y = 2t - \frac{2}{t}$$

Therefore, the tangent meets the directrix at $\left(-2, 2t - \frac{2}{t}\right)$.

Example 3

- Find the equation of the normal to the parabola $y^2 = 8x$ at the point $T(2t^2, 4t)$.
- Find where the normals to the parabola at the points $P(2p^2, 4p)$ and $Q(2q^2, 4q)$ intersect.

Note When given the parametric equations for a parabola, it is much easier to stay with these equations. So, do **not** revert to the cartesian equation.

SOLUTION

- We have

$$x = 2t^2 \Rightarrow \frac{dx}{dt} = 4t$$

$$y = 4t \Rightarrow \frac{dy}{dt} = 4$$

which give

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{t}$$

Therefore, the gradient of the normal is $-t$.

Using $y - y_1 = m(x - x_1)$, we find the equation of the normal at $(2t^2, 4t)$:

$$\begin{aligned} y - 4t &= -t(x - 2t^2) \\ \Rightarrow y + tx &= 4t + 2t^3 \end{aligned}$$

- To find the equation of the normal at the point P , we just substitute p for t . Therefore, the equation of the normal at the point P is

$$y + px = 4p + 2p^3 \quad [1]$$

Similarly, the equation of the normal at Q is

$$y + qx = 4q + 2q^3 \quad [2]$$

Subtracting [2] from [1], we find that these normals intersect when

$$px - qx = 4p + 2p^3 - 4q - 2q^3$$

Note In all such situations, where p and q are similarly considered, $(p - q)$ will be a factor. Therefore, we look for this factor, remove it and check that the answer is symmetrical in p and q .

Using $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$, we obtain

$$(p - q)x = 4(p - q) + 2(p - q)(p^2 + pq + q^2)$$

$$\Rightarrow x = 2(p^2 + pq + q^2 + 2) \quad [3]$$

Substituting [3] into [1], we have

$$y = 4p + 2p^3 - 2p(p^2 + pq + q^2 + 2)$$

$$\Rightarrow y = -2pq(p + q)$$

Therefore, the normals intersect at $(2(p^2 + pq + q^2 + 2), -2pq(p + q))$.

Exercise 11A

1 Find the focus and directrix of each of the following parabolas.

a) $y^2 = 16x$

b) $y^2 = 28x$

c) $x^2 = 8y$

d) $x^2 = -16y$

e) $y^2 + 12x = 0$

f) $(y + 1)^2 = 32x$

g) $(y - 2)^2 - 8(x - 3) = 0$

2 Find in cartesian form an equation of the parabola whose focus and directrix are respectively

a) $(3, 0)$, $x + 3 = 0$

b) $(4, 0)$, $x = -4$

c) $(0, 2)$, $y = -2$

d) $(0, -5)$, $y = 5$

3 Find the equation of the tangent to the parabola $y^2 = 20x$ at

a) the point $T(5t^2, 10t)$

b) the point $P(5p^2, 10p)$

c) the point $S(5, 10)$

d) the point $R(20, 20)$

4 a) Find the equation of the normal to the parabola $y^2 = 8x$ at the point $(2, 4)$.

b) Find where this normal meets the parabola again.

Ellipse

Let the focus be $(ae, 0)$ and the directrix be $x = \frac{a}{e}$. Since, for an ellipse, e is less than 1, the directrix is further from the origin than the focus.

For the point $P(x, y)$, we have

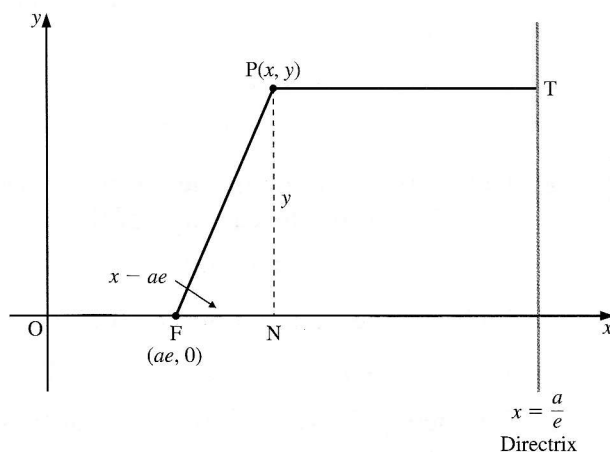
$$PF = \sqrt{(x - ae)^2 + y^2} \quad PT = \frac{a}{e} - x$$

Since the ratio of the distance of P from the focus to the distance of P from the directrix is e , we have

$$\frac{PF}{PT} = e \Rightarrow PF = ePT$$

which gives

$$\sqrt{(x - ae)^2 + y^2} = a - ex$$



Squaring both sides, we obtain

$$\begin{aligned}(x - ae)^2 + y^2 &= (a - ex)^2 \\ \Rightarrow x^2 - 2aex + a^2e^2 + y^2 &= a^2 - 2aex + e^2x^2 \\ \Rightarrow x^2(1 - e^2) + y^2 &= a^2(1 - e^2) \\ \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} &= 1\end{aligned}$$

We express this in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$b^2 = a^2(1 - e^2) \Rightarrow e^2 = 1 - \frac{b^2}{a^2}$$

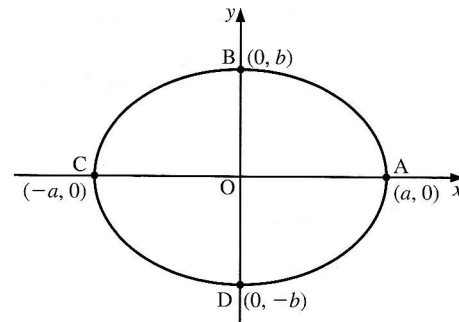
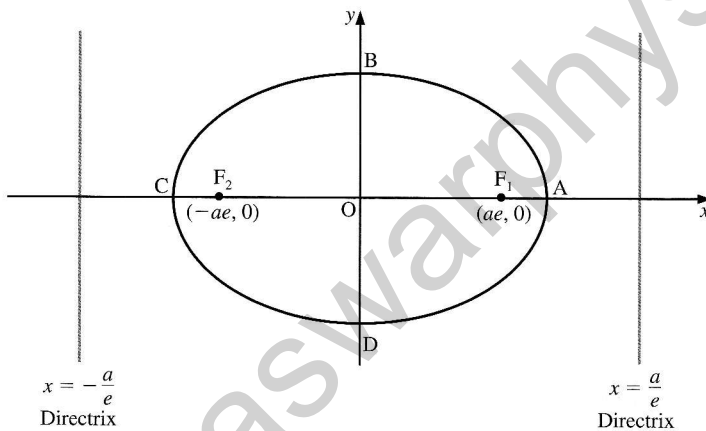
Hence, the **standard equation** for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$e^2 = 1 - \frac{b^2}{a^2}.$$

Note An ellipse is symmetrical with respect to its axes. Hence, it has two foci, one at $(ae, 0)$ and the other at $(-ae, 0)$, and two directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.



The longer axis, AC, is called the **major axis**, and the shorter axis, BD, is called the **minor axis**. We see that the length of the major axis is $2a$ and that of minor axis is $2b$.

A chord which passes through either focus, and which is perpendicular to the major axis, is called a **latus rectum**. Half the length of this chord is known as a **semi latus rectum**. (See also pages 219 and 231.)

The parametric equations for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$$x = a \cos \theta \quad \text{and} \quad y = b \sin \theta$$

where θ is the **eccentric angle** of the ellipse. (Further discussion of this parameter and its use in projective geometry is beyond the scope of this book.)

Example 4

a) Find the eccentricity of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

b) State the coordinates of its foci.

c) State the equations of its directrices.

SOLUTION

a) The general equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence, for the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, we have $a = 3$, $b = 2$.

For an ellipse, $e^2 = 1 - \frac{b^2}{a^2}$, which in this case gives

$$e^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

Therefore, the eccentricity is $\frac{\sqrt{5}}{3}$.

b) The foci are $(\pm ae, 0)$, which in this case gives $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$.

c) The directrices are $x = \pm \frac{a}{e}$, which in this case give

$$x = \pm \frac{3}{\frac{\sqrt{5}}{3}}$$

Therefore, its directrices are

$$x = \frac{9}{\sqrt{5}} \quad \text{and} \quad x = -\frac{9}{\sqrt{5}}$$

Example 5 Find the tangent and the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$.

SOLUTION

We have

$$x = a \cos \theta \Rightarrow \frac{dx}{d\theta} = -a \sin \theta$$

$$y = b \sin \theta \Rightarrow \frac{dy}{d\theta} = b \cos \theta$$

which give

$$\frac{dy}{dx} = \frac{-b \cos \theta}{a \sin \theta}$$

Using $y - y_1 = m(x - x_1)$, we find the equation of the tangent:

$$y - b \sin \theta = \frac{-b \cos \theta}{a \sin \theta} (x - a \cos \theta)$$

$$\Rightarrow a \sin \theta y + b \cos \theta x = ab(\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow a \sin \theta y + b \cos \theta x = ab$$

Note We can check this by taking $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and replacing

x^2 by $(x \times \text{abscissa})$ and y^2 by $(y \times \text{ordinate})$

to obtain the equation of the tangent.

Hence, we have

$$\frac{xa \cos \theta}{a^2} + \frac{yb \sin \theta}{b^2} = 1$$

$$\Rightarrow xb \cos \theta + ya \sin \theta = ab$$

as above.

To find the equation of the normal, we need its gradient, which is given by

$$\begin{aligned} \text{Gradient of normal} &= -\frac{1}{\text{Gradient of tangent}} \\ &= -\frac{1}{\frac{-b \cos \theta}{a \sin \theta}} = \frac{a \sin \theta}{b \cos \theta} \end{aligned}$$

So, the equation of the normal is

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$$

$$\Rightarrow yb \cos \theta = xa \sin \theta - a^2 \sin \theta \cos \theta + b^2 \sin \theta \cos \theta$$

$$\Rightarrow yb \cos \theta = xa \sin \theta + (b^2 - a^2) \sin \theta \cos \theta$$

Example 6 Find the area of the ellipse whose major axis is $2a$ and minor axis is $2b$.

SOLUTION

We have

$$\text{Area} = \int y \, dx$$

To make the integration easier, we use the parametric equation for y , which gives

$$\int y \, dx = \int b \sin \theta \, dx$$

We cannot integrate a function in θ with respect to x . Therefore, we must convert the integration with respect to x to an integration with respect to θ . Hence, we have

$$\text{Area of ellipse} = \int_{-a}^a y \, dx = \int_0^{2\pi} b \sin \theta \frac{dx}{d\theta} d\theta$$

Using $x = a \cos \theta$, this gives

$$\text{Area} = \int_0^{2\pi} b \sin \theta (-a \sin \theta) d\theta$$

Because the ellipse is symmetrical about its axes, we can express the integral as

$$\text{Area} = 4 \int_0^{\frac{\pi}{2}} ab \sin^2 \theta d\theta$$

Using $\cos 2\theta = 1 - 2 \sin^2 \theta$, we obtain

$$\begin{aligned} \text{Area} &= 4ab \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos 2\theta) d\theta \\ &= 4ab \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{2}} \end{aligned}$$

Therefore, the area of an ellipse is πab .

Exercise 11B

1 Find the eccentricity, foci and directrices of each of the following ellipses.

a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

b) $\frac{x^2}{49} + \frac{y^2}{16} = 1$

c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

d) $\frac{x^2}{4} + \frac{y^2}{9} = 4$

e) $\frac{(x-1)^2}{25} + \frac{(y+2)^2}{9} = 1$

2 Find, in cartesian form, the equation of each ellipse with the focus and the directrix as given.

a) $(3, 0)$, $x = 12$

b) $(2, 0)$, $x = 18$

c) $(0, 4)$, $y = 8$

d) $(0, 3)$, $y = 15$

3 Find the equation of a) the tangent and b) the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the point $(5 \cos \theta, 4 \sin \theta)$.

4 Sketch the curve given in polar coordinates by the equation

$$r = \frac{2a}{3 + 2 \cos \theta}$$

Prove that this curve is an ellipse and identify its foci.

Hyperbola

Let the focus be $(ae, 0)$ and the directrix be $x = \frac{a}{e}$.

Since, for a hyperbola, e is greater than 1, the directrix is situated between the origin and the focus.

For the point $P(x, y)$, we have

$$PF = \sqrt{(x - ae)^2 + y^2} \quad PT = x - \frac{a}{e}$$

Since $\frac{PF}{PT} = e \Rightarrow PF = ePT$, we have

$$\begin{aligned} \sqrt{(x - ae)^2 + y^2} &= e \left(x - \frac{a}{e} \right) \\ &= ex - a \end{aligned}$$

Squaring both sides, we obtain

$$\begin{aligned} (x - ae)^2 + y^2 &= (ex - a)^2 \\ \Rightarrow x^2 - 2aex + a^2e^2 + y^2 &= e^2x^2 - 2aex + a^2 \\ \Rightarrow x^2(1 - e^2) + y^2 &= a^2(1 - e^2) \end{aligned}$$

which gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

But $e > 1$, therefore $a^2(1 - e^2)$ is negative.

Hence, the **standard equation** for a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

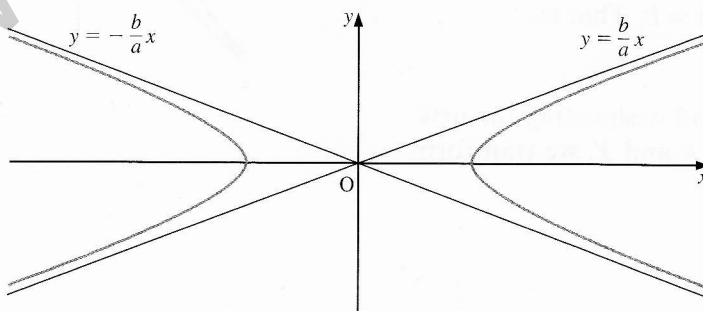
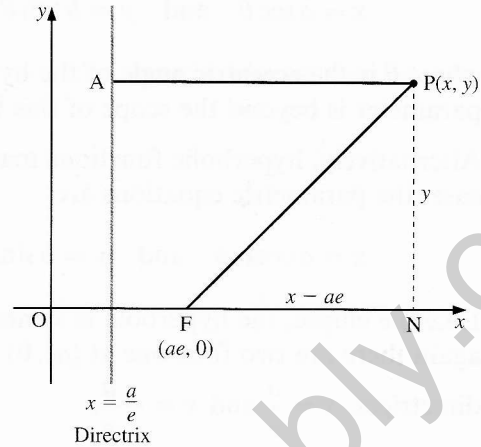
where

$$b^2 = a^2(e^2 - 1) \quad \text{or} \quad e^2 = 1 + \frac{b^2}{a^2}$$

As x and y become large, we have

$$\frac{x^2}{a^2} \rightarrow \frac{y^2}{b^2} \Rightarrow y \rightarrow \pm \frac{bx}{a}$$

Therefore, the asymptotes of a hyperbola are $y = \pm \frac{b}{a}x$



Common parametric equations for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$x = a \sec \theta \quad \text{and} \quad y = b \tan \theta$$

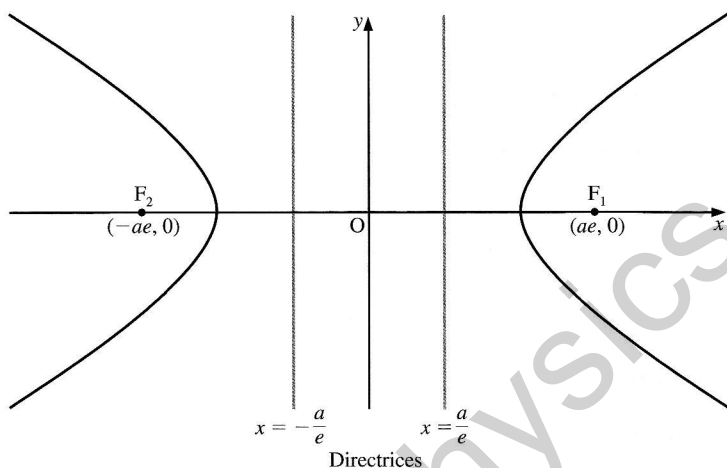
where θ is the **eccentric angle** of the hyperbola. (Further discussion of this parameter is beyond the scope of this book.)

Alternatively, hyperbolic functions may be used (see pages 189–91). In this case, the parametric equations are

$$x = a \cosh \phi \quad \text{and} \quad y = b \sinh \phi$$

Like the ellipse, the hyperbola is symmetrical with respect to its axes. Hence, again there are two foci, one at $(ae, 0)$ and the other at $(-ae, 0)$, and two

directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.



Rectangular hyperbola

When $a = b$, the asymptotes of the hyperbola are $y = x$ and $y = -x$, which are perpendicular to each other. Hence, such a hyperbola (shown on the right) is called a **rectangular hyperbola**.

The general equation of a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, becomes

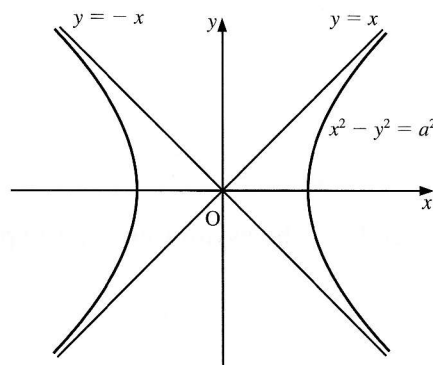
$$x^2 - y^2 = a^2$$

for a rectangular hyperbola, as $a = b$. That is,

$$(x + y)(x - y) = a^2$$

Rotating the axes through 45° and designating the new axes (which are the asymptotes) X and Y , we transform this equation to

$$XY = \frac{a^2}{2}$$



where

$$X = \frac{1}{\sqrt{2}}(x - y) \quad \text{and} \quad Y = \frac{1}{\sqrt{2}}(x + y).$$

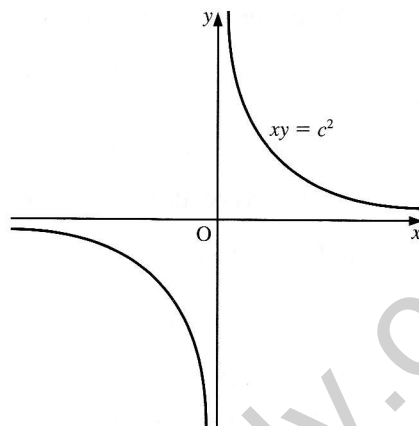
Hence, for a rectangular hyperbola, we have the equation

$$xy = c^2$$

where $c^2 = \frac{a^2}{2}$.

Common parametric equations for the rectangular hyperbola are

$$x = ct \quad \text{and} \quad y = \frac{c}{t}$$



Example 7 Find the eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, and the coordinates of its foci.

SOLUTION

For a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $e^2 = 1 + \frac{b^2}{a^2}$.

Therefore, for the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, we obtain

$$e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

Hence, the eccentricity is $\frac{5}{4}$.

When $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the foci are $(\pm ae, 0)$. Therefore, for $\frac{x^2}{16} - \frac{y^2}{9} = 1$, the

foci are $\left(\pm \frac{5}{4} \times 4, 0\right)$, giving $(\pm 5, 0)$.

Example 8 Find the equation of the tangent to $xy = c^2$ at the point

$\left(ct, \frac{c}{t}\right)$ Hence find the equation of the tangent to $xy = 16$ at the points

a) $(8, 2)$ and **b)** $\left(-12, -\frac{4}{3}\right)$.

SOLUTION

To find the equation of the tangent, we need its gradient. Hence, we have

$$x = ct \Rightarrow \frac{dx}{dt} = c$$

$$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$$

which give

$$\frac{dy}{dx} = -\frac{1}{t^2}$$

Therefore, the equation of the tangent is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$\Rightarrow t^2y + x = 2ct$$

For the hyperbola $xy = 16$, $c = 4$. Therefore, the equation of the tangent

at $\left(4t, \frac{4}{t}\right)$ is

$$t^2y + x = 8t$$

a) At the point $(8, 2)$, we have $4t = 8$, which gives $t = 2$.

Therefore, the equation of the tangent at this point is

$$4y + x = 16$$

b) At the point $\left(-12, -\frac{4}{3}\right)$, we have $4t = -12$, which gives $t = -3$.

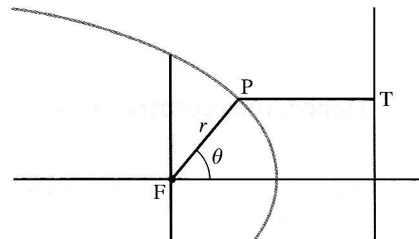
Therefore, the equation of the tangent at this point is

$$9y - x + 24 = 0$$

Polar equation of a conic

The polar equation (see pages 45–7) of a conic is formed by positioning the pole at the focus and keeping the directrix at $x = \frac{a}{e}$.

With reference to the diagram on the right, the locus of a general point, P, expressed in polar coordinates (r, θ) satisfies the condition (see page 219)



We can express this equation in several different forms.

For example, when $x = d$ is used as the equation of the directrix, we have

$$r = \frac{ed}{1 + \cos \theta}$$

When the focus is at $(ae, 0)$ and the directrix is $x = \frac{a}{e}$, we have as the equation of the general conic

$$r = \frac{b^2}{a(1 + e \cos \theta)}$$

which gives

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad \text{for an ellipse}$$

$$r = \frac{a(e^2 - 1)}{1 + e \cos \theta} \quad \text{for a hyperbola}$$

We can also derive a similar polar equation in terms of l , the length of the semi latus rectum (see also pages 219 and 223).

With reference to the diagram on the right, we have

$$PT = ePT \Rightarrow r = e(AB - r \cos \theta)$$

The point A is on the conic, so we have

$$FA = e AB$$

FA is the semi latus rectum, so we obtain

$$l = e AB \Rightarrow AB = \frac{l}{e}$$

which gives

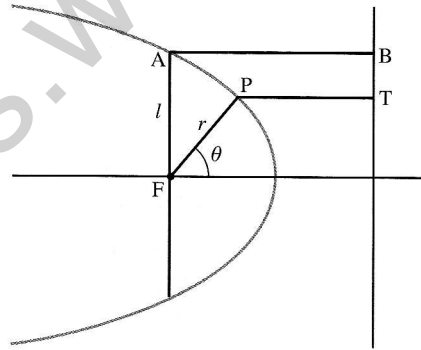
$$r = e \left(\frac{l}{e} - r \cos \theta \right)$$

$$\Rightarrow r(1 + e \cos \theta) = l$$

That is, we have

$$r = \frac{l}{1 + e \cos \theta}$$

Note The distance between the directrix and the focus is $\frac{l}{e}$.



Exercise 11C

1 Find the eccentricity, foci and directrices of each of the following hyperbolas.

a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b) $\frac{x^2}{49} - \frac{y^2}{16} = 1$

c) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

d) $\frac{x^2}{4} - \frac{y^2}{9} = 4$

e) $\frac{(x-1)^2}{25} - \frac{(y+2)^2}{9} = 1$

- 2 Find, in cartesian form, the equation of each hyperbola with the focus and the directrix as given.
- a) (12, 0), $x = 3$ b) (18, 0), $x = 2$ c) (0, 8), $y = 4$ d) (0, 15), $y = 3$
- 3 Find the equation of a) the tangent and b) the normal to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at the point $(5 \sec \theta, 4 \tan \theta)$.

Exercise 11D

- 1 Consider the parabola $y^2 = 4ax$.
- i) Show that the following parametric equations define a point on this parabola
- $$x = at^2 \quad y = 2at$$
- ii) Show that the tangent drawn to the parabola at the point $(at^2, 2at)$ has an equation given by
- $$ty = x + at^2$$
- Consider the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p \neq q$. Let M be the mid-point of PQ, and H be the intersection point of the tangents at P and Q.
- iii) Show that the line MH is parallel to the x -axis. (NICCEA)
- 2 The equation of the curve C is $y^2 = 8x$. The point $P(2t^2, 4t)$ lies on C . The line through the point $(2, 0)$ perpendicular to the tangent to C at P intersects this tangent at the point Q.
- a) Find the coordinates of Q.
- b) Given that R is the mid-point of PQ, find the equation of the locus of R in cartesian form. (WJEC)
- 3 The point P lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.
- a) Show that an equation of the tangent to the parabola at $P(ap^2, 2ap)$ is $py = x + ap^2$.
- The tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ ($p \neq q$, $p \neq 0$, $q \neq 0$) meet at the point N.
- b) Find the coordinates of N.
- Given further that N lies on the directrix of the parabola,
- c) write down a relationship between p and q . (EDEXCEL)
- 4 The line with equation $y = mx + c$ is a tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- a) Show that $c^2 = a^2m^2 + b^2$.
- b) Hence, or otherwise, find the equations of the tangents from the point $(3, 4)$ to the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$. (EDEXCEL)
- 5 An ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are positive constants and $a > b$.
- a) Find an equation of the tangent at the point $P(a \cos t, b \sin t)$.
- b) Find an equation of the normal at the point $P(a \cos t, b \sin t)$.

The normal at P meets the x -axis at the point Q. The tangent at P meets the y -axis at the point R.

- c) Find, in terms of a , b and t , the coordinates of M, the mid-point of QR.

Given that $0 < t < \frac{\pi}{2}$,

- d) show that, as t varies, the locus of M has equation $\left(\frac{2ax}{a^2 - b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$. (EDEXCEL)

- 6 The point $P(2 \cos \theta, 3 \sin \theta)$ lies on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

- a) Find the equation of the tangent to the ellipse at the point $P(2 \cos \theta, 3 \sin \theta)$, where $\theta \neq 0$.
b) Given that the tangent in part a passes through the point $(2, -6)$, show that

$$\cos \theta - 2 \sin \theta = 1$$

- c) Solve the equation in part b for $0^\circ \leq \theta \leq 360^\circ$ and deduce the coordinates of P. (WJEC)

- 7 A curve C has equations

$$x = ct \quad y = \frac{c}{t} \quad t \neq 0$$

where c is a constant and t is a parameter.

- a) Show that an equation of the normal to C at the point where $t = p$ is given by

$$py + cp^4 = p^3x + c$$

- b) Verify that this normal meets C again at the point at which $t = q$, where

$$qp^3 + 1 = 0 \quad (\text{EDEXCEL})$$

- 8 The rectangular hyperbola C has equation $xy = c^2$, where c is a positive constant.

- a) Show that the tangent to C at the point $P\left(cp, \frac{c}{p}\right)$ has equation

$$p^2y = -x + 2cp$$

The point Q has coordinates $Q\left(cq, \frac{c}{q}\right)$, $q \neq p$. The tangents to C at P and Q meet at N.

Given that $p + q \neq 0$,

- b) show that the y -coordinate of N is $\frac{2c}{p+q}$.

The line joining N to the origin O is perpendicular to the chord PQ.

- c) Find the numerical value of p^2q^2 . (EDEXCEL)

- 9 The ellipse C has parametric equations

$$x = 2 + 3 \cos \theta \quad y = 2 \sin \theta$$

- a) Obtain the cartesian equation of C and find the eccentricity of the ellipse.
b) Write down the coordinates of the foci.
c) Sketch C , stating the coordinates of its intersections with the axes.

The arc of the curve C between $\theta = 0$ and $\theta = \frac{1}{2}\pi$ is rotated through 2π about the x -axis.

- d) Show that the area S of the resulting surface of revolution is given by

$$S = 4\pi \int_0^{\frac{\pi}{2}} \sin \theta (9 - 5 \cos^2 \theta)^{\frac{1}{2}} d\theta$$

Using the substitution $(\sqrt{5}) \cos \theta = 3 \sin u$, or otherwise, find the value of S , to two decimal places. (EDEXCEL)

- 10 The curve C_1 is that arc of the hyperbola with equation

$$\frac{x^2}{9a^2} - \frac{y^2}{a^2} = 1 \quad a > 0$$

which contains the point $P(3a \cosh \theta, a \sinh \theta)$.

- a) Show that the equation of the normal to C_1 at the point P can be written in the form

$$y \cosh \theta + 3x \sinh \theta = 10a \sinh \theta \cosh \theta$$

This normal meets the coordinate axes at A and B .

- b) Show that, as θ varies, the locus C_2 of the mid-point of AB , is an arc of a hyperbola.

For each of the arcs C_1 and C_2

- c) give the coordinates of any points of intersection with the coordinate axes and the equations of any asymptotes
d) find the eccentricity of the hyperbola and state the coordinates of the focus and the equation of the corresponding directrix. (EDEXCEL)

- 11 The points $S\left(s, \frac{1}{s}\right)$ and $T\left(t, \frac{1}{t}\right)$ lie on the curve $xy = 1$ and the line ST passes through the point $(1, 2)$.

- a) Show that $s + t = 1 + 2st$.
b) The tangents to the curve at S and T meet at the point P . Show that the locus of P is given by $y = 2 - 2x$. (WJEC)

- 12 The figure on the right shows a parabola and a circle. The circle passes through the parabola's focus S , a point P on the parabola and the intersection point Q of the directrix and the tangent at P .

- i) If the parabola has focus $S(1, 0)$ and directrix $x = -1$, show that its equation is $y^2 = 4x$.

Let the point P be given by $(t^2, 2t)$, where $t \neq 0$.

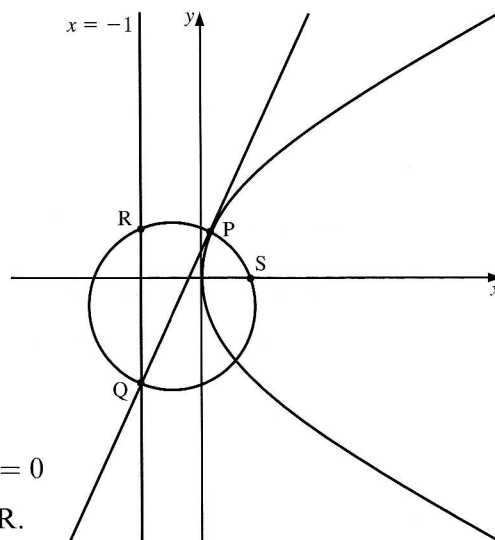
- ii) Show that Q is the point $\left(-1, \frac{t^2 - 1}{t}\right)$.

- iii) Verify that the focus S , the point P and the point Q lie on the circle with equation

$$tx^2 - t(t^2 - 1)x + ty^2 - (3t^2 - 1)y + t^3 - 2t = 0$$

- iv) The circle intersects the directrix again at the point R . Find the coordinates of R .

- v) Show that PR is parallel to the x -axis. (NICCEA)



12 Further integration

Many a smale maketh a grate.

GEOFFREY CHAUCER

In *Introducing Pure Mathematics* (pages 433–8 and 445–7), we met integrals

such as $\int x(x^2 + 1)^7 dx$, where we used the substitution $x^2 + 1 = u$, and

$\int xe^{2x} dx$, where we integrated by parts.

We can extend these methods by using a greater variety of substitutions, including hyperbolic functions, to enable us to find integrals such as

$$\int \sqrt{9 + x^2} dx.$$

To integrate more complicated expressions, we normally use the inverse function of a function rule given on page 294 of *Introducing Pure Mathematics*.

Inverse function of a function rule

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1} [f(x)]^{n+1} + c$$

It is usually quicker to differentiate by inspection the new expression than to obtain $f'(x)$ in the integrand, as shown in Examples 1 and 2.

Example 1 Find the constant k in

$$\int x(x^2 + 1)^7 dx = k(x^2 + 1)^8 + c$$

SOLUTION

Differentiating $(x^2 + 1)^8$, we obtain

$$\frac{d}{dx}(x^2 + 1)^8 = 8 \times 2x(x^2 + 1)^7 = 16x(x^2 + 1)^7$$

Therefore, we have

$$16 \int x(x^2 + 1)^7 dx = (x^2 + 1)^8 + c'$$

$$\Rightarrow \int x(x^2 + 1)^7 dx = \frac{1}{16}(x^2 + 1)^8 + c'$$

which gives $k = \frac{1}{16}$.

Example 2 Find the constant k in

$$\int \sin 2x \cos^7 2x \, dx = k \cos^8 2x + c$$

SOLUTION

Differentiating $\cos^8 2x$, we obtain

$$8 \times -2 \sin 2x \cos^7 2x = -16 \sin 2x \cos^7 2x$$

which gives $k = -\frac{1}{16}$.

Therefore, we have

$$\int \sin 2x \cos^7 2x \, dx = -\frac{1}{16} \cos^8 2x + c$$

Note In those cases where you experience difficulty in spotting the integral, use instead integration by substitution.

Integration by parts

Example 3 Evaluate $\int e^{3x} \cos 4x \, dx$.

SOLUTION

When faced with a product neither term of which will disappear after repeated differentiation, we usually use integration by parts until we obtain the integral with which we started as one of the terms on the right-hand side.

Hence, we have

$$\begin{aligned} \int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x - \int \frac{1}{3} e^{3x} \times -4 \sin 4x \, dx \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \int e^{3x} \sin 4x \, dx \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \left(\frac{1}{3} e^{3x} \sin 4x - \int \frac{1}{3} e^{3x} \times 4 \cos 4x \, dx \right) \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x - \frac{16}{9} \int e^{3x} \cos 4x \, dx \end{aligned}$$

We now move the (original) integral on the RHS to the LHS:

$$\begin{aligned} \int e^{3x} \cos 4x \, dx + \frac{16}{9} \int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x + c \\ \Rightarrow \frac{25}{9} \int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x + c \\ \Rightarrow \int e^{3x} \cos 4x \, dx &= \frac{9}{25} \left(\frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x \right) + c' \end{aligned}$$

Hence, we have

$$\int e^{3x} \cos 4x \, dx = \frac{3}{25} e^{3x} \cos 4x + \frac{4}{25} e^{3x} \sin 4x + c'$$