Exercise 9C

1 Verify the identity

$$\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)} \equiv \frac{2}{(r-1)(r+1)}$$

Hence, using the method of differences, prove that

$$\sum_{r=2}^{n} \frac{2}{(r-1)(r+1)} = \frac{3}{2} - \frac{2n+1}{n(n+1)}$$

Deduce the sum of the infinite series

$$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \ldots \frac{1}{(n-1)(n+1)} + \ldots \qquad \text{(AEB 98)}$$

2 Show that

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \equiv \frac{2}{r(r+1)(r+1)}$$

Hence, or otherwise, find a simplified expression for

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+1)} \qquad \text{(WJEC)}$$

3 **a)** Express $\dfrac{1}{(2r-1)(2r+1)}$ in partial fractions.

b) Hence, or otherwise, show that

$$\sum_{r=n}^{r=2n} \frac{1}{(2r-1)(2r+1)} = \frac{an+b}{(2n-1)(4n+1)}$$

where a and b are integers to be found.

c) Determine the limit as $n \to \infty$ of $\displaystyle\sum_{r=n}^{r=2n} \frac{1}{(2r-1)(2r+1)}$. (NEAB)

4 Find the value of the constant A for which $(2r+1)^2 - (2r-1)^2 \equiv Ar$.

Use this result, and the method of differences, to prove that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \qquad \text{(AEB 96)}$$

5 Express $\dfrac{1}{(2r+1)(2r+3)}$ in partial fractions.

Hence find the sum of the series

$$\frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n+1)(2n+3)}$$

Show that the series

$$\frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n+1)(2n+3)} + \ldots$$

is convergent and state the sum to infinity. (OCR)

6 Verify that

$$\frac{1}{1+(n-1)x} - \frac{1}{1+nx} = \frac{x}{\{1+(n-1)x\}(1+nx)}$$

Hence show that, for $x \neq 0$,

$$\sum_{n=1}^{N} \frac{1}{\{1+(n-1)x\}(1+nx)} = \frac{N}{1+Nx}$$

Deduce that the infinite series

$$\frac{1}{1 \times \frac{3}{2}} + \frac{1}{\frac{3}{2} \times 2} + \frac{1}{2 \times \frac{5}{2}} + \dots$$

is convergent and find its sum to infinity. (OCR)

7 Let $a_n = e^{-(n-1)x} - e^{-nx}$, where $x \neq 0$.

i) Find $\sum_{n=1}^{N} a_n$ in terms of N and x.

ii) Find the set of values of x for which the infinite series

$$a_1 + a_2 + a_3 + \dots$$

converges, and state the sum to infinity. (OCR)

8 Given that

$$u_n = \frac{1}{\sqrt{(2n-1)}} - \frac{1}{\sqrt{(2n+1)}}$$

express $\sum_{n=25}^{N} u_n$ in terms of N.

Deduce the value of $\sum_{n=25}^{\infty} u_n$. (OCR)

9 Show that

$$\frac{r}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!} \quad (r \in \mathbb{N})$$

Hence or otherwise, evaluate

i) $\displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!}$ **ii)** $\displaystyle\sum_{r=1}^{\infty} \frac{r+2}{(r+1)!}$

giving your answer to part **ii** in the terms of e. (NEAB)

10 a) Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} \equiv \frac{1}{(r+1)(r+2)} \qquad r \in \mathbb{Z}^+$$

b) Hence, or otherwise, find

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$$

giving your answer as a single fraction in terms of n. (EDEXCEL)

Convergence

As we found in geometric progressions, an infinite series is the sum of an infinite sequence of numbers (see *Introducing Pure Mathematics*, pages 248–50). For example, we have the infinite geometric progression

$$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k}+\ldots$$

When we state that an infinite series $\sum_{k=0}^{\infty} a_k$ **converges**, we mean that the sums $S_n = \sum_{k=0}^{n} a_k$ have a **limit** as $n \to \infty$.

We say that an infinite series **diverges** if it does not converge.

When a series diverges, it could behave in one of the following ways.

- Diverge to $+\infty$; for example: $1+2+4+8+16+\ldots$
- Diverge to $-\infty$; for example: $-1-2-4-8-16-\ldots$
- Oscillate finitely; for example: $1-1+1-1+1-\ldots$
- Oscillate infinitely: for example: $1-2+4-8+16-\ldots$

D'Alembert's ratio test

D'Alembert's ratio test states that a series of the form $\sum_{n=0}^{\infty} a_n$ converges when

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

The test also states when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is greater than 1, the series diverges.

It does **not** imply anything when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Example 15 Prove that the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all real values of x.

SOLUTION

First, we find the ratio $\left|\frac{a_{n+1}}{a_n}\right|$. Then we find its limit as $n \to \infty$.

Hence, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|$$

$$= \left|\frac{x}{n+1}\right|$$

As $n \to \infty$, this ratio has a limit of zero regardless of the (real) value of x.

Therefore, the ratio test implies that the series converges for all real values of x.

Note The series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is Maclaurin's expansion for e^x (see page 178) and is therefore known as the **exponential series**. That is,

$$e^x \equiv \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 16 Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

SOLUTION

Applying d'Alembert's ratio test, we obtain

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \frac{n}{n+1}$$

which gives

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

Thus, in this case, d'Alembert's ratio test **fails**, because it does **not** establish whether the series converges or diverges.

To prove that the series does not converge, we write out its first few terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Now, the first term is greater than $\frac{1}{2}$.

The second term is $\frac{1}{2}$.

The sum of the next two terms is greater than $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

The sum of the next four terms is greater than $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

Similarly, the sum of the next eight terms is greater than eight times $\frac{1}{16}$, which is $\frac{1}{2}$.

This pattern keeps repeating. We can always increase the sum by more than $\frac{1}{2}$ by adding the next 2^k terms. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ exceeds any pre-assigned real number L. Hence, it cannot converge to L, and so it diverges.

Even though each term is less than the preceding term, and the terms tend to zero, the sum is not finite.

Maclaurin's series

Assuming that $f(x)$ can be expanded as a series in ascending positive integral powers of x, we can deduce the terms of the series, as shown below for $\sin x$, $\cos x$, e^x and $\ln(1+x)$. These four expansions are needed frequently and therefore should be known.

Power series for $\sin x$

Let $\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, where the a's are constants.

When $x = 0$, $\sin 0 = a_0$. But $\sin 0 = 0$, therefore $a_0 = 0$.

Differentiating $\sin x = a_1x + a_2x^2 + a_3x^3 + \ldots$, we obtain

$$\cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

When $x = 0$, $\cos 0 = a_1$. But $\cos 0 = 1$, therefore $a_1 = 1$.

Differentiating again, we obtain

$$-\sin x = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + 5 \times 4a_5x^3 + \ldots$$

When $x = 0$, $\sin 0 = 2a_2 \quad \Rightarrow \quad a_2 = 0$.

Differentiating yet again, we obtain

$$-\cos x = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + 5 \times 4 \times 3a_5x^2 + \ldots$$

When $x = 0$, $-\cos 0 = 3 \times 2a_3 \quad \Rightarrow \quad a_3 = -\dfrac{1}{3 \times 2 \times 1} = -\dfrac{1}{3!}$.

Repeating the differentiation, we obtain

$$a_4 = 0 \qquad a_5 = \frac{1}{5!} \qquad a_6 = 0 \qquad a_7 = -\frac{1}{7!}$$

Therefore, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$

By d'Alembert's ratio test, this series converges for all real x.

Power series for $\cos x$

We can use the procedure for $\sin x$ to find the power series for $\cos x$. However, it is much easier to start from the expansion for $\sin x$. Hence, we have

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

which gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

By d'Alembert's ratio test, this series is convergent for all real x.

Power series for e^x

Let $e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, where the a's are constants.

When $x = 0$, $e^0 = a_0$. But $e^0 = 1$, therefore $a_0 = 1$.

Differentiating $e^x = a_1x + a_2x^2 + a_3x^3 + \dots$, we obtain

$$e^x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

When $x = 0$, $e^0 = a_1 \quad \Rightarrow \quad a_1 = 1$.

Differentiating again, we obtain

$$e^x = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + 5 \times 4a_5x^3 + \dots$$

When $x = 0$, $e^0 = 2a_2 \quad \Rightarrow \quad a_2 = \frac{1}{2}$.

Differentiating yet again, we obtain

$$e^x = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + 5 \times 4 \times 3a_5x^2 + \dots$$

When $x = 0$, $e^0 = 3 \times 2a_3 \quad \Rightarrow \quad a_3 = \frac{1}{3 \times 2 \times 1} = \frac{1}{3!}$.

Repeating the differentiation, we obtain

$$a_4 = \frac{1}{4!} \qquad a_5 = \frac{1}{5!} \qquad a_6 = \frac{1}{6!} \qquad a_7 = \frac{1}{7!}$$

Therefore, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

By d'Alembert's ratio test, this series converges for all real x.

Power series for $\ln(1 + x)$

Since $\ln 0$ is not finite, we cannot have a power series for $\ln x$. Instead, we use a power series for $\ln(1 + x)$.

Let $\ln(1 + x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

When $x = 0$, $\ln 1 = a_0$. But $\log 1 = 0$, therefore $a_0 = 0$.

Differentiating $\ln(1 + x) = a_1x + a_2x^2 + a_3x^3 + \dots$, we obtain

$$\frac{1}{1 + x} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

However, using the binomial theorem, we can expand $\frac{1}{1 + x}$ as $(1 + x)^{-1}$ to give $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$. Hence, we have

$$1 - x + x^2 - x^3 + x^4 - x^5 + \dots \equiv a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

Equating coefficients, we obtain $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = -\frac{1}{4}$, ...

Therefore, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

Using d'Alembert's ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

Thus, when $|x| < 1$, the series converges. By inspection, we notice that the expansion is valid when $x = 1$, but not when $x = -1$. Hence, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \quad \text{for } -1 < x \leqslant 1$$

Similarly, we have

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots \quad \text{for } -1 \leqslant x < 1$$

Summary

The general result of this method for obtaining the power series of functions is known as **Maclaurin's series**, and is expressed as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

Exercise 9D

1 a) Show that the first two non-zero terms in the Maclaurin expansion of $\sin^{-1}x$ are given by

$$\sin^{-1}x = x + \frac{x^3}{6} + \ldots$$

b) By writing $x = \frac{1}{2}$, deduce an approximation to π as a rational fraction in its lowest terms.

c) The equation $\sin^{-1}x = 1.002x$ is satisfied by a small positive value of x. Find an approximation to this value, giving your answer correct to three decimal places. (WJEC)

2 i) Use Maclaurin's theorem to derive the series expansion for $\log_e(1+x)$, where $-1 < x \leqslant 1$, giving the first three non-zero terms.

ii) If $\log_e(1+x) \approx x(1+ax)^b$ for small x, find the values of a and b so that the first three non-zero terms of the series expansions of the two sides agree. (NICCEA)

3 a) Find the first three derivatives of $(1+x)^2 \cos x$.

b) Hence, or otherwise, find the expansion of $(1+x)^2 \cos x$ in ascending powers of x up to and including the term in x^3. (EDEXCEL)

4 i) Use Maclaurin's theorem to derive the first three non-zero terms of the series expansion for $\sin x$.

ii) Show that, for sufficiently small x,

$$\sin x \approx x\left(1 - \frac{x^2}{15}\right)^{\frac{5}{2}}$$

iii) Show that when $x = \frac{\pi}{2}$ the error in using the approximation in part **ii** is about 0.2%.

(NICCEA)

5 Show that the first two non-zero terms of the Maclaurin series for $\ln(1 + x)$ are given by

$$\ln(1 + x) = x - \frac{x^2}{2} + \ldots$$

a) Use the series to show that the equation $3\ln(1 + x) = 100x^2$ has an approximate solution $x = 0.03$.

b) Taking $x = 0.03$ as a first approximation, obtain an improved value of the root by two applications of the Newton–Raphson method. Give your answer correct to six decimal places. (WJEC)

6 Given that $y = (1 + \sin x)e^x$, find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = (1 + 2\cos x)e^x$.

Hence, or otherwise, prove that the Maclaurin series for y, in ascending powers of x, up to and including the term in x^2 is

$$1 + 2x + \frac{3}{2}x^2$$

The binomial expansion of $(1 + ax)^n$ also begins $1 + 2x + \frac{3}{2}x^2$. Find the value of the constants a and n. (AEB 97)

7 i) Use Maclaurin's theorem to find the values of A, B, C and D in the series expansion

$$\tan^{-1}x = A + Bx + Cx^2 + Dx^3 + \frac{x^5}{5} - \frac{x^7}{7}\ldots$$

where $-1 < x < 1$.

ii) Find, using the binomial expansion, the first three non-zero terms of the series expansion, in ascending powers of u, for $\frac{1}{1 + u^2}$.

iii) Using the series in part **ii**, evaluate

$$\int_0^x \frac{1}{1 + u^2}\,du$$

as a series expansion in ascending powers of x.

iv) Explain briefly how the series expansion in part **i** can be derived from the result in part **iii**.

(NICCEA)

8 Given that

$$y^2 = \sec x + \tan x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0$$

show that

a) $\dfrac{dy}{dx} = \dfrac{1}{2} y \sec x$

b) $\dfrac{d^2y}{dx^2} = \dfrac{1}{4} y \sec x (\sec x + 2\tan x)$

Given that x is small and that terms in x^3 and higher powers of x may be neglected, use Maclaurin's expansion to express y in the form $A + Bx + Cx^2$, stating the values of A, B and C. (EDEXCEL)

9 Given that $f(x) = (1 + x) \ln (1 + x)$,

a) find the fifth derivative of $f(x)$

b) show that the first five non-zero terms in the Maclaurin expansion for $f(x)$ are

$$x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20}$$

c) find, in terms of r, an expression for the rth term ($r \geqslant 2$) of the Maclaurin expansion for $f(x)$. (WJEC)

10 a) i) Given that $y = \ln(2 + x^2)$, find $\dfrac{dy}{dx}$ and show that

$$\frac{d^2y}{dx^2} = \frac{4 - 2x^2}{(2 + x^2)^2}$$

ii) Deduce the Maclaurin series for $\ln(2 + x^2)$ in ascending powers of x, up to and including the term in x^2.

b) By writing $2 + x^2$ as $2(1 + \frac{1}{2}x^2)$ and using the series expansion

$$\ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots$$

verify your result from part **a** and determine the next non-zero term in the series for $\ln(2 + x^2)$. (AEB 97)

11 i) Use Maclaurin's theorem to derive the first five terms of the series expansion for $(1 + x)^r$, where $-1 < x < 1$.

ii) Assuming that the series, obtained above, continues with the same pattern, sum the following infinite series

$$1 + \frac{1}{6} - \frac{1.2}{6.12} + \frac{1.2.5}{6.12.18} - \frac{1.2.5.8}{6.12.18.24} + \ldots \qquad \text{(NICCEA)}$$

12 i) Use Maclaurin's theorem to derive the first five terms of the series expansion for e^x.

Consider the infinite series

$$\frac{1}{1!} + \frac{4}{2!} + \frac{7}{3!} + \frac{10}{4!} + \ldots$$

ii) If the series continues with the same pattern, find an expression for the nth term.

iii) Sum the infinite series. (NICCEA)

Using power series

The series studied on pages 177–9 are used in a number of situations, including the two which are discussed below

Finding the limit of $\dfrac{f(x)}{g(x)}$ as $x \to 0$, when $f(0) = g(0) = 0$

If we simply insert $x = 0$, we obtain $\dfrac{f(0)}{g(0)} = \dfrac{0}{0}$, which means that we have proceeded incorrectly.

Example 17 Find the limit of $\dfrac{x - \sin x}{x^2(e^x - 1)}$ as $x \to 0$.

SOLUTION

To find such a limit, we expand the numerator and the denominator of the expression each as a power series in x and divide both by the lowest power of x present. Then we put $x = 0$.

Hence, we have

$$\frac{x - \sin x}{x^2(e^x - 1)} = \frac{x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots\right)}{x^2\left(1 + x + \dfrac{x^2}{2!} + \dots - 1\right)} = \frac{\dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \dots}{x^3 + \dfrac{x^4}{2!} + \dots}$$

Dividing the numerator and the denominator by x^3, we obtain

$$\frac{x - \sin x}{x^2(e^x - 1)} = \frac{\dfrac{1}{3!} - \dfrac{x^2}{5!} + \dots}{1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} \dots}$$

Therefore, we have

$$\lim_{x \to 0} \frac{x - \sin x}{x^2(e^x - 1)} = \frac{\dfrac{1}{3!}}{1} = \frac{1}{6}$$

Example 18 Find the limit of $\dfrac{1 - \cos x}{\sin^2 x}$ as $x \to 0$.

SOLUTION

Expanding the numerator and the denominator each as a power series, we obtain

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1 - \left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots\right)}{\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots\right)^2} = \frac{\dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dots}{x^2 - \dfrac{2x^4}{3!} + \dots}$$

Dividing the numerator and the denominator by x^2, we obtain

$$\frac{1-\cos x}{\sin^2 x} = \frac{\frac{1}{2!} - \frac{x^2}{4!} + \dots}{1 - \frac{2x^2}{3!} + \dots}$$

Therefore, we have

$$\lim_{x \to 0} \frac{1-\cos x}{\sin^2 x} = \frac{\frac{1}{2!}}{1} = \frac{1}{2}$$

L'Hôpital's rule

When evaluating the limits of some forms of $\frac{f(x)}{g(x)}$, the use of power series is not appropriate and so we apply l'Hôpital's rule, which states that if $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

If $g'(a) = 0$, we repeat the procedure until we find a derivative of $g(x)$ which is not zero when $x = a$.

Thus, if $f(a) = g(a) = 0$ and $g'(a) = 0$, but $g''(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f''(a)}{g''(a)}$$

Example 19 Find $\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6}$.

SOLUTION

We notice that both the numerator and the denominator are zero when $x = 1$. Hence, we have, after differentiating both the numerator and the denominator,

$$\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} = \lim_{x \to 1} \frac{4x^3 - 21x^2 + 16x}{3x^2 + 5}$$

$$\Rightarrow \lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} = -\frac{1}{8} = -0.125$$

Finding f(x) for small x

Example 20 Expand $\tan x$ as a power series in x as far as a term in x^5. Hence find the value of $\tan 0.001$ to 15 decimal places.

SOLUTION

We express $\tan x$ in terms of $\sin x$ and $\cos x$, and expand each as a power series. Hence, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots}$$

$$= \frac{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \ldots}{1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \ldots}$$

We rearrange the above to give

$$\tan x = \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right]\left[1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)\right]^{-1}$$

and then we expand the second bracket, using the binomial theorem and ignoring terms in x^5 and higher, to obtain

$$\tan x = \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right]\left[1 + \frac{x^2}{2} - \frac{x^4}{24} + \ldots + \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)^2 + \ldots\right]$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} - \ldots\right)$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots\right)$$

$$= x + \frac{x^3}{2} + \frac{5x^5}{24} - \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^5}{120} + \ldots$$

Therefore, we have

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$$

Hence, $\tan 0.001$ is given by

$$\tan 0.001 = 0.001 + \frac{1}{3} \times 0.000\,000\,001 + \frac{2}{15} \times 0.000\,000\,000\,000\,001 + \ldots$$

That is,

$$\tan 0.001 = 0.001\,000\,000\,333\,333 \quad \text{to 15 dp}$$

Dividing the numerator and the denominator by x^2, we obtain

$$\frac{1-\cos x}{\sin^2 x} = \frac{\dfrac{1}{2!} - \dfrac{x^2}{4!} + \dots}{1 - \dfrac{2x^2}{3!} + \dots}$$

Therefore, we have

$$\lim_{x \to 0} \frac{1-\cos x}{\sin^2 x} = \frac{\dfrac{1}{2!}}{1} = \frac{1}{2}$$

L'Hôpital's rule

When evaluating the limits of some forms of $\dfrac{f(x)}{g(x)}$, the use of power series is not appropriate and so we apply l'Hôpital's rule, which states that if $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

If $g'(a) = 0$, we repeat the procedure until we find a derivative of $g(x)$ which is not zero when $x = a$.

Thus, if $f(a) = g(a) = 0$ and $g'(a) = 0$, but $g''(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f''(a)}{g''(a)}$$

Example 19 Find $\lim\limits_{x \to 1} \dfrac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6}$.

SOLUTION

We notice that both the numerator and the denominator are zero when $x = 1$. Hence, we have, after differentiating both the numerator and the denominator,

$$\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} = \lim_{x \to 1} \frac{4x^3 - 21x^2 + 16x}{3x^2 + 5}$$

$$\Rightarrow \quad \lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} = -\frac{1}{8} = -0.125$$

Finding f(x) for small x

Example 20 Expand $\tan x$ as a power series in x as far as a term in x^5. Hence find the value of $\tan 0.001$ to 15 decimal places.

SOLUTION

We express $\tan x$ in terms of $\sin x$ and $\cos x$, and expand each as a power series. Hence, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots}$$

$$= \frac{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dots}{1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dots}$$

We rearrange the above to give

$$\tan x = \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right]\left[1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)\right]^{-1}$$

and then we expand the second bracket, using the binomial theorem and ignoring terms in x^5 and higher, to obtain

$$\tan x = \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right]\left[1 + \frac{x^2}{2} - \frac{x^4}{24} + \dots + \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)^2 + \dots\right]$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} - \dots\right)$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots\right)$$

$$= x + \frac{x^3}{2} + \frac{5x^5}{24} - \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^5}{120} + \dots$$

Therefore, we have

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

Hence, $\tan 0.001$ is given by

$$\tan 0.001 = 0.001 + \frac{1}{3} \times 0.000\,000\,001 + \frac{2}{15} \times 0.000\,000\,000\,000\,001 + \dots$$

That is,

$$\tan 0.001 = 0.001\,000\,000\,333\,333 \quad \text{to 15 dp}$$

Power series for more complicated functions

We can combine power series for simple functions to make power series for more complicated functions, as demonstrated in Examples 21 to 24.

Example 21 Find the power series for $\cos x^2$.

SOLUTION

The power series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + \ldots$$

To obtain the power series for $\cos x^2$, we replace every x in the above series with x^2 to obtain

$$\cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \ldots + \frac{(-1)^n}{(2n)!}(x^2)^{2n} + \ldots$$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots + \frac{(-1)^n}{(2n)!}x^{4n} + \ldots$$

Since the power series for $\cos x$ is valid for all real values of x, we know that the power series for $\cos x^2$ is valid for all values of x^2, i.e. for all real values of x.

Example 22 Find the power series for $\ln(1+3x)$, stating when the expansion is valid.

SOLUTION

In the expansion for $\ln(1+x)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

we substitute $3x$ for x, which gives

$$\ln(1+3x) = (3x) - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} -$$

$$= 3x - \frac{9}{2}x^2 + 9x^3 - \ldots$$

Since the expansion for $\ln(1+x)$ is valid for $-1 < x \leqslant 1$, the expansion for $\ln(1+3x)$ is valid for $-1 < 3x \leqslant 1$, i.e. $-\frac{1}{3} < x \leqslant \frac{1}{3}$.

Therefore, we have

$$\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \ldots \qquad \text{for } -\frac{1}{3} < x \leqslant \frac{1}{3}$$

Example 23 Find the power series for $e^{4x}\sin 3x$, up to and including the term in x^4.

SOLUTION

Since we are asked for terms only up to x^4, we do not need to consider terms in higher powers of x.

The power series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Therefore, the power series for e^{4x} is

$$e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \dots$$

Similarly, using the power series for $\sin x$, and replacing x with $3x$, we obtain the power series expansion for $\sin 3x$:

$$\sin 3x = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots$$

Therefore, the power series for $e^{4x}\sin 3x$ is

$$\begin{aligned} e^{4x}\sin 3x &= \left[1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \dots\right]\left[(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots\right] \\ &= \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \dots\right)\left(3x - \frac{9}{2}x^3 + \dots\right) \end{aligned}$$

Ignoring terms in x^5 and higher powers, we obtain

$$e^{4x}\sin 3x = 3x + 12x^2 + 24x^3 - \frac{9}{2}x^3 + 32x^4 - 18x^4$$

Therefore, we have

$$e^{4x}\sin 3x = 3x + 12x^2 + \frac{39}{2}x^3 + 14x^4$$

Example 24 Find all the terms up to and including x^4 in the power series for $e^{\sin x}$.

SOLUTION

Using the power series for e^x, we obtain

$$e^{\sin x} = 1 + \frac{\sin x}{1!} + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \dots$$

We now apply the power series for $\sin x$. Since we are asked for terms only up to x^4, we can ignore terms in higher powers of x. Therefore, we have

$$e^{\sin x} = 1 + \frac{x - \frac{x^3}{3!} + \dots}{1!} + \frac{\left(x - \frac{x^3}{3!} + \dots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \dots\right)^3}{3!} + \dots$$

$$\Rightarrow \quad e^{\sin x} = 1 + x - \frac{x^3}{3!} + \frac{x^2 - \frac{2x^4}{3!}}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

which gives

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

Exercise 9E

1 Find the power series of each of the following.

a) $\sin 2x$ **b)** $\cos 5x$ **c)** e^{8x}
d) $\ln(1 + x^2)$ **e)** $\ln(1 - 2x)$

2 Find the power series of each of the following, up to and including the term in x^4.

a) $\sin x^2$ **b)** $(1 + x)e^{3x}$ **c)** $(2 + x^2)\cos 3x$
d) $e^{\cos x}$ **e)** $\ln(1 + \cos x)$

3 Find out whether the following infinite series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ **b)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n - 1}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

4 Find the power series expansion of $\cos x^3$. Which values of x is this valid for?

5 Find the power series expansion of e^{2x^2}

6 You are told that $y = \displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{3^n}$. When does this series converge?

7 Given that $|x| < 4$, find, in ascending powers of x up to and including the term in x^3, the series expansion of

a) $(4 - x)^{\frac{1}{2}}$ **b)** $(4 - x)^{\frac{1}{2}} \sin 3x$ (EDEXCEL)

8 a) Find the first four terms of the expansion, in ascending powers of x, of

$$(2 + 3x)^{-1} \qquad |x| < \tfrac{2}{3}$$

b) Hence, or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of

$$\frac{\sin 2x}{2 + 3x} \qquad |x| < \tfrac{2}{3} \qquad \text{(EDEXCEL)}$$

9
$$\cos\left(2x + \frac{\pi}{3}\right) \equiv p \cos 2x + q \sin 2x$$

a) Find the exact values of the constants p and q.

Given that x is so small that terms in x^3 and higher powers of x are negligible,

b) show that $\cos\left(2x + \dfrac{\pi}{3}\right) = \dfrac{1}{2} - \sqrt{3}x - x^2$. (EDEXCEL)

10 The function f is defined by

$$f(x) = e^{ax} - (1 + bx)^{\frac{1}{3}}$$

where a and b are positive constants and $|bx| < 1$.

a) Find, in terms of a and b, the coefficients of x, x^2 and x^3 in the expansion of $f(x)$ in ascending powers of x.

b) Given that the coefficient of x is zero and that the coefficient of x^2 is $\frac{3}{2}$,

i) find the values of a and b

ii) show that the coefficient of x^3 is $-\frac{3}{2}$. (NEAB)

10 Hyperbolic functions

In the 1760s Johann Heinrich Lambert gave a very nice presentation in terms of the parametrization of the hyperbola, by analogy with such a treatment of the sine and cosine on the circle.
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Definitions

The hyperbolic functions, of which there are six, are so named because they are related to the parametric equations for a hyperbola.

We begin with the two functions hyperbolic sine of x and hyperbolic cosine of x, which are written

$$\sinh x \quad \text{and} \quad \cosh x$$

They are defined by the relationships

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

In a similar manner to ordinary trigonometric functions, we have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}$$

By convention, we pronounce sinh as 'shine', tanh as 'than', (co)sech as '(co)sheck' and coth as 'coth'.

Example 1 Find **a)** $\sinh 2$ and **b)** $\operatorname{sech} 3$.

SOLUTION

a) Usually, you would use a calculator to find sinh values. Not all calculators operate in the same way, so you must first consult your calculator instructions to learn the **correct order** in which to press the hyperbolic (hyp) key, the sin key and, in this case, the 2 key. Your answer should be 3.6268...

Otherwise, you would have to evaluate sinh 2 using the relationship

$$\sinh 2 = \frac{1}{2}(e^2 - e^{-2})$$

and putting in the values of e^2 and e^{-2}, which you either obtain from tables or calculate from the exponential series.

b) Again, you would normally use a calculator with the relationship

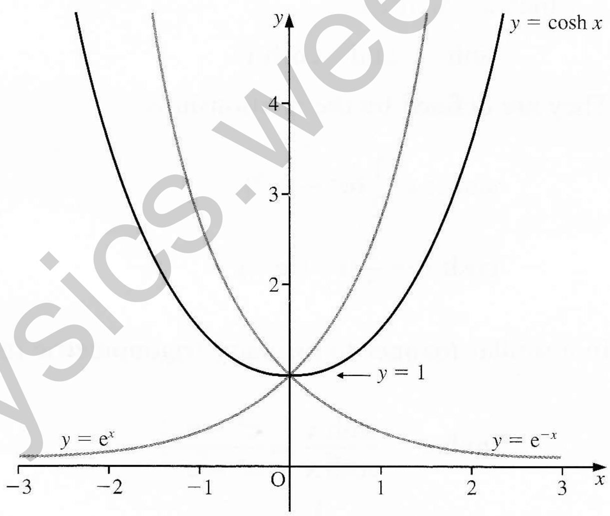
$$\operatorname{sech} 3 = \frac{1}{\cosh 3} = \frac{1}{10.0677} \quad \text{(to 4 dp)}$$

Therefore, sech 3 = 0.0993, to four decimal places.

Graphs of cosh *x*, sinh *x* and tanh *x*

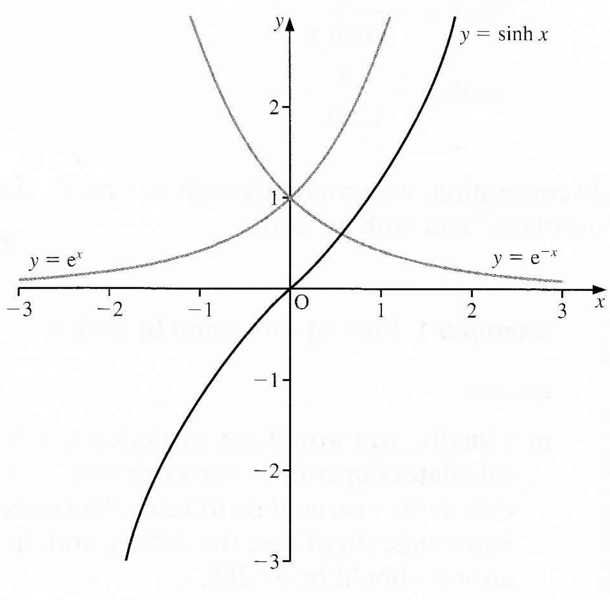
$y = \cosh x$

We obtain the graph of $y = \cosh x$ (shown on the right) by finding the mean values of a few corresponding pairs of values of $y = e^x$ and $y = e^{-x}$, and then plotting these mean values.

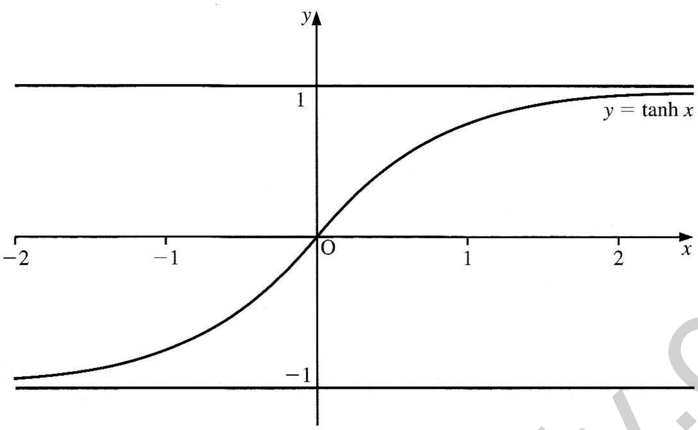


$y = \sinh x$

To produce the graph of $y = \sinh x$ (shown on the right), we find half the difference between a few corresponding pairs of values of $y = e^x$ and $y = e^{-x}$, and then plot these values.



$y = \tanh x$



We have $\tanh x = \dfrac{\sinh x}{\cosh x}$, which gives

$$\tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

$$\Rightarrow \quad \tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Therefore, $\tanh x < 1$ for all values of x, and as $x \to +\infty$, $\tanh x \to 1$.

Since $\tanh x = -\dfrac{1 - e^{2x}}{1 + e^{2x}}$, $\tanh x > -1$ for all values of x, and as $x \to -\infty$, $\tanh x \to -1$.

Hence, the graph of $y = \tanh x$ lies between the asymptotes $y = 1$ and $y = -1$.

Standard hyperbolic identities

From the exponential definitions for $\cosh x$ and $\sinh x$, we have

$$\cosh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) \qquad [1]$$

and

$$\sinh^2 x = \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \qquad [2]$$

Hence, subtracting [2] from [1], we obtain

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$$

Therefore, we have

$$\cosh^2 x - \sinh^2 x \equiv 1$$

Notice the similarity of this hyperbolic identity with the usual trigonometric identity $\cos^2 x + \sin^2 x \equiv 1$. See page 213 for Osborn's rule, which will help you to recall the standard hyperbolic identities.

Dividing $\cosh^2 x - \sinh^2 x \equiv 1$ by $\sinh^2 x$, we obtain

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} \equiv \frac{1}{\sinh^2 x}$$

which gives

$$\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$$

Similarly, dividing $\cosh^2 x - \sinh^2 x \equiv 1$ by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x}$$

which gives

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

Differentiation of hyperbolic functions

To differentiate $\sinh x$ and $\cosh x$, we use their exponential definitions. Hence, for $\sinh x$, we have

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right] = \frac{1}{2}(e^x + e^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(e^x + e^{-x}) = \cosh x$$

Therefore, we have

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right] = \frac{1}{2}(e^x - e^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Therefore, we have

$$\frac{d}{dx}\cosh x = \sinh x$$

To differentiate $\tanh x$, we use the identity

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

which gives

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{d}{dx}\frac{\sinh x}{\cosh x} \\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \qquad \text{(using the quotient rule)} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\end{aligned}$$

$$\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$$

Similarly, dividing $\cosh^2 x - \sinh^2 x \equiv 1$ by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x}$$

which gives

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

Differentiation of hyperbolic functions

To differentiate $\sinh x$ and $\cosh x$, we use their exponential definitions. Hence, for $\sinh x$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})\right] = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \cosh x$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\right] = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \sinh x$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$$

To differentiate $\tanh x$, we use the identity

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

which gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \tanh x &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sinh x}{\cosh x} \\
&= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \qquad \text{(using the quotient rule)} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x
\end{aligned}$$

$$\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$$

Similarly, dividing $\cosh^2 x - \sinh^2 x \equiv 1$ by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x}$$

which gives

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

Differentiation of hyperbolic functions

To differentiate $\sinh x$ and $\cosh x$, we use their exponential definitions. Hence, for $\sinh x$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})\right] = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \cosh x$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\right] = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$$

From the definitions, we know that

$$\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \sinh x$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$

To differentiate $\tanh x$, we use the identity

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

which gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tanh x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sinh x}{\cosh x}\\ &= \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x} \qquad \text{(using the quotient rule)}\\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\end{aligned}$$

Therefore, we have

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

To differentiate functions such as $\cosh ax$, again we use the exponential definitions. Hence, we have

$$\frac{d}{dx}\cosh ax = \frac{d}{dx}\left[\frac{1}{2}(e^{ax} + e^{-ax})\right]$$
$$= \frac{1}{2}(ae^{ax} - ae^{-ax})$$

From the exponential definitions, we note that

$$a\left[\frac{1}{2}(e^{ax} - e^{-ax})\right] = a\sinh ax$$

Therefore, we have

$$\frac{d}{dx}\cosh ax = a\sinh ax$$

Similarly, we have

$$\frac{d}{dx}\sinh ax = a\cosh ax$$
$$\frac{d}{dx}\tanh ax = a\operatorname{sech}^2 ax$$

Example 2 Find $\frac{dy}{dx}$ when $y = 3\cosh 3x + 5\sinh 4x + 2\cosh^4 7x$.

SOLUTION

To differentiate $\cosh^4 7x$, we express it as $(\cosh 7x)^4$ and apply the chain rule. Hence, we have

$$\frac{dy}{dx} = 9\sinh 3x + 20\cosh 4x + 2 \times 4 \times 7\sinh 7x\cosh^3 7x$$
$$= 9\sinh 3x + 20\cosh 4x + 56\sinh 7x\cosh^3 7x$$

Integration of hyperbolic functions

From the differentiation formulae given on pages 192–3, we deduce that

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + c$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + c$$

Example 3 Find $\int (2\sinh 4x + 9\,\text{sech}^2 3x)\,dx$.

SOLUTION

Splitting the given integral into two parts, we obtain

$$\int 2\sinh 4x\,dx + \int 9\,\text{sech}^2 3x\,dx = \frac{2}{4}\cosh 4x + \frac{9}{3}\tanh 3x + c$$
$$= \tfrac{1}{2}\cosh 4x + 3\tanh 3x + c$$

Inverse hyperbolic functions

We define the inverses of the hyperbolic functions in a similar way to the inverses of the ordinary trigonometric functions. Hence, for example, if $y = \sinh^{-1} x$, then $\sinh y = x$. Likewise for $\cosh^{-1} x$, $\tanh^{-1} x$, $\text{cosech}^{-1} x$, $\text{sech}^{-1} x$ and $\coth^{-1} x$.

Sometimes, these functions are written as arsinh x, arcosh x etc.

Sketching inverse hyperbolic functions

The curve of $y = \sinh^{-1} x$ is obtained by reflecting the curve of $y = \sinh x$ in the line $y = x$.

To draw the curve with reasonable accuracy, we need to find the gradient of $y = \sinh x$ at the origin. Accordingly, we differentiate $y = \sinh x$, to obtain

$$\frac{dy}{dx} = \cosh x$$

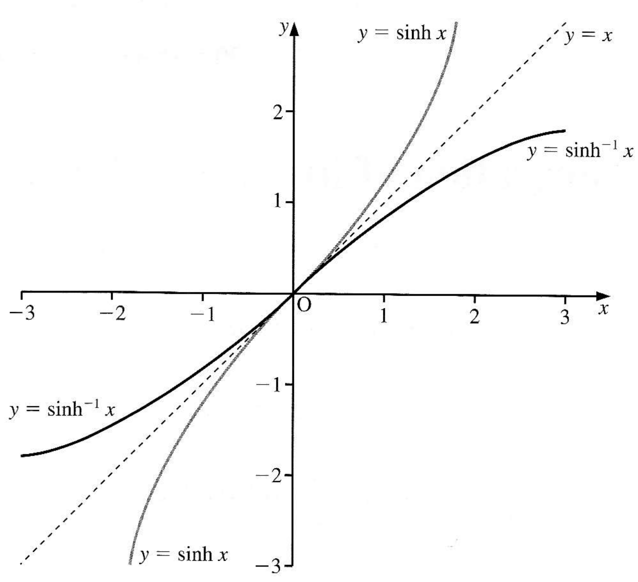
Thus, at the origin, where $x = 0$, we have

$$\frac{dy}{dx} = \cosh 0 = 1$$

That is, the gradient of $y = \sinh x$ at the origin is 1.

We now proceed as follows:

- Draw the line $y = x$ as a dashed line.
- Sketch carefully the graph of $y = \sinh x$, remembering that $y = x$ is a tangent to $y = \sinh x$ at the origin.
- Reflect this sinh curve in the line $y = x$.



Similarly, we can sketch any other inverse hyperbolic function: that is, by reflecting the curve of the relevant hyperbolic function in the line $y = x$. In each case, we must find the gradient of the hyperbolic curve at the origin.

Take, for example, $y = \tanh x$, which gives

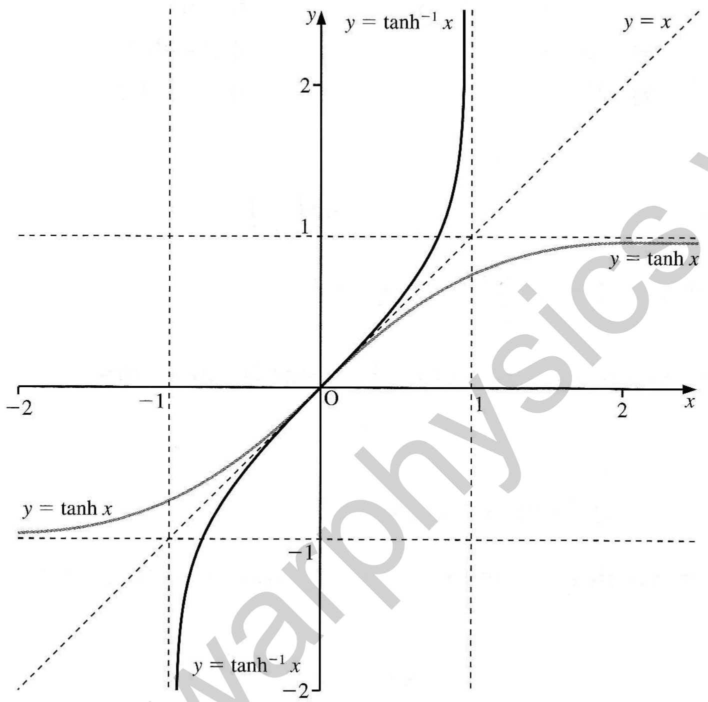
$$\frac{dy}{dx} = \operatorname{sech}^2 x$$

At the origin, where $x = 0$, we have

$$\frac{dy}{dx} = \operatorname{sech}^2 0 = \frac{1}{\cosh^2 0} = 1$$

That is, the gradient of $y = \tanh x$ at the origin is 1.

Also, we know that $y = \tanh x$ has asymptotes $y = 1$ and $y = -1$. Therefore, because $y = \tanh^{-1} x$ is the reflection of $y = \tanh x$ in $y = x$, $y = \tanh^{-1} x$ has asymptotes $x = 1$ and $x = -1$.



Example 4 Solve the equation $2\cosh^2 x - \sinh x = 3$.

SOLUTION

Using the identity $\cosh^2 x - \sinh^2 x \equiv 1$, we obtain

$$2(1 + \sinh^2 x) - \sinh x - 3 = 0$$
$$\Rightarrow \quad 2\sinh^2 x - \sinh x - 1 = 0$$

We now factorise this to obtain

$$(2\sinh x + 1)(\sinh x - 1) = 0$$
$$\Rightarrow \quad \sinh x = 1 \quad \text{or} \quad -\tfrac{1}{2}$$
$$\Rightarrow \quad x = 0.8814 \quad \text{or} \quad -0.4812$$

Exercise 10A

1 Evaluate each of the following, giving your answer **i)** in terms of e and **ii)** correct to three significant figures.

a) $\cosh 2$ **b)** $\sinh 3$ **c)** $\tanh 4$

2 Starting with the definitions of $\sinh x$ and $\cosh x$, prove each of the following identities.

a) $\cosh(A+B) \equiv \cosh A \cosh B + \sinh A \sinh B$
b) $\sinh(A-B) \equiv \sinh A \cosh B - \cosh A \sinh B$
c) $\sinh A + \sinh B \equiv 2\sinh\left(\dfrac{A+B}{2}\right)\cosh\left(\dfrac{A-B}{2}\right)$

3 Differentiate each of the following.

a) $\cosh 2x$ **b)** $\sinh 5x$ **c)** $\tanh 3x$
d) $2\cosh 4x - 5\sinh 3x$ **e)** $3\cosh 2x + 6\sinh 5x$ **f)** $\coth x$
g) $\operatorname{sech} x$ **h)** $3\cosh^5 3x$ **i)** $2\sinh^4 8x$
j) $\ln\cosh x$ **k)** $e^{\sinh 2x}$ **l)** $\ln\tanh 5x$

4 Integrate, with respect to x, each of the following.

a) $\sinh 3x$ **b)** $\cosh 4x$ **c)** $\sinh\left(\dfrac{x}{3}\right)$
d) $2\cosh\left(\dfrac{x}{5}\right)$ **e)** $3\cosh 5x - 2\sinh\left(\dfrac{x}{2}\right)$ **f)** $\tanh 4x$

5 Solve each of these equations, giving your answer to three significant figures.

a) $3\sinh x + 2\cosh x = 4$ **b)** $4\cosh x - 8\sinh x + 1 = 0$
c) $\cosh x + 4\sinh x = 3$ **d)** $3\operatorname{sech} x - 2 = 5\tanh x$
e) $9\cosh^2 x - 6\sinh x = 17$ **f)** $3\sinh^2 x + \cosh x - 2 = 0$

6 Find the values of x for which $8\cosh x + 4\sinh x = 7$, giving your answers as natural logarithms. (EDEXCEL)

7 a) i) Write down an expression for $\tanh x$ in terms of e^x and e^{-x}.
ii) Hence show that

$$1 - \tanh x = \frac{2e^{-2x}}{1+e^{-2x}}$$

b) Using the result in part **a ii**, evaluate

$$\int_0^{\infty} (1 - \tanh x)\,dx \qquad \text{(NEAB)}$$

8 The curve C has equation $y = 5\cosh x + 3\sinh x$. Find the exact values of the coordinates of the turning point on C and determine its nature. (EDEXCEL)

9 Show that, if x is real, $1 + \frac{1}{2}x^2 > x$.
Deduce that $\cosh x > x$.

The point P on the curve $y = \cosh x$ is such that its perpendicular distance from the line $y = x$ is a minimum. Show that the coordinates of P are $(\ln(1+\sqrt{2}), \sqrt{2})$. (NEAB)

10 Let $y = x\sinh x$.

i) Show that $\dfrac{d^2y}{dx^2} = x\sinh x + 2\cosh x$, and find $\dfrac{d^4y}{dx^4}$.

ii) Write down a conjecture for $\dfrac{d^{2n}y}{dx^{2n}}$.

iii) Use induction to establish a formula for $\dfrac{d^{2n}y}{dx^{2n}}$. (OCR)

11 Find the exact solution of the equation $2\cosh x + \sinh x = 2$. (OCR)

12 The curve C is defined parametrically by

$$x = t + \ln(\cosh t) \qquad y = \sinh t$$

i) Show that $\dfrac{dy}{dx} = e^{-t}\cosh^2 t$.

ii) Hence show that $\dfrac{d^2y}{dx^2} = e^{-2t}\cosh^2 t\,(2\sinh t - \cosh t)$.

iii) Deduce that C has a point of inflexion where $t = \frac{1}{2}\ln 3$. (OCR)

13 i) Show that

$$\frac{d}{dy}\left(\frac{1}{2}\sinh 4y + 4\sinh 2y + 6y\right) = 16\cosh^4 y$$

ii) Given that $x = 2\sinh y$, show that

$$\sinh 2y = \frac{1}{2}x\sqrt{(x^2+4)}$$

and also that

$$\sinh 4y = \frac{1}{2}x(x^2+2)\sqrt{(x^2+4)}$$

iii) Use the results of parts **i** and **ii** to show that

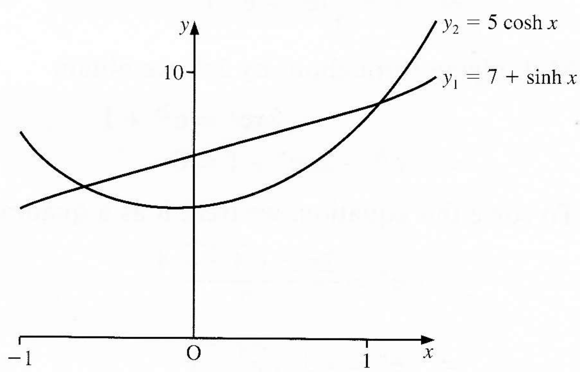
$$\int (x^2+4)^{\frac{3}{2}}\,dx = \frac{1}{4}x(x^2+10)\sqrt{(x^2+4)} + 6\sinh^{-1}\left(\frac{1}{2}x\right) + \text{constant}$$ (OCR)

14 Consider the functions $y_1 = 7 + \sinh x$ and $y_2 = 5\cosh x$ whose graphs are shown in the figure on the right.

i) Show, by solving the equation, that the solutions of $7 + \sinh x = 5\cosh x$ are $-\log_e 2$ and $\log_e 3$.

ii) Show that the area bounded by the two graphs in the figure is $7\log_e 6 - 10$.

(NICCEA)



15 Let $I_n = \int \cosh^n x \, dx$. Show that

$$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2}$$

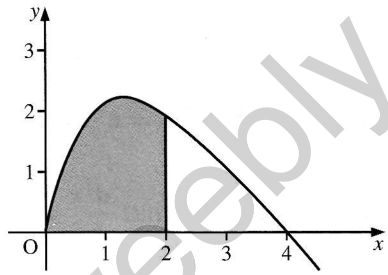
Hence show that

$$\int_0^{\ln 2} \cosh^4 x \, dx = \frac{3}{8}\left(\frac{245}{128} + \ln 2\right) \qquad \text{(OCR)}$$

16 **a)** Show that $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$.

b) The diagram on the right shows a sketch of part of the curve whose equation is

$$y = 4\tanh x - x \qquad x \geqslant 0$$



i) Find, correct to two decimal places, the coordinates of the stationary point on the curve.

ii) Find, correct to four decimal places, the area of the shaded region bounded by the curve, the x-axis and the ordinate $x = 2$.

c) For large values of x, the curve is asymptotic to the line $y = mx + c$, where m and c are constants. State the values of m and c, and give a reason for your answer. (NEAB)

Logarithmic form

The inverse hyperbolic functions $\cosh^{-1} x$, $\sinh^{-1} x$ and $\tanh^{-1} x$ can all be expressed as logarithmic functions.

Expressing $\cosh^{-1} x$ as a logarithmic function

Let $\cosh^{-1} x = y$. We then have

$$x = \cosh y$$

$$\Rightarrow \quad x = \frac{1}{2}(e^y + e^{-y})$$

Multiplying throughout by $2e^y$, we obtain

$$2xe^y = e^{2y} + 1$$

$$\Rightarrow \quad e^{2y} - 2xe^y + 1 = 0$$

To solve this equation, we treat it as a quadratic in e^y, which gives

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$\Rightarrow \quad e^y = x \pm \sqrt{x^2 - 1}$$

Taking the logarithms of both sides, we obtain

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

That is, the **principal value** of $\cosh^{-1} x$ is $\ln(x + \sqrt{x^2 - 1})$.

Expressing the principal value in a different form, we obtain

$$\begin{aligned}\ln(x+\sqrt{x^2-1}) &= \ln\left[\frac{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}{x-\sqrt{x^2-1}}\right]\\ &= \ln\left[\frac{x^2-(x^2-1)}{x-\sqrt{x^2-1}}\right]\\ &= \ln\left[\frac{1}{x-\sqrt{x^2-1}}\right]\\ &= -\ln(x-\sqrt{x^2-1})\end{aligned}$$

Hence, we have

$$\ln(x\pm\sqrt{x^2-1}) = \pm\ln(x+\sqrt{x^2-1})$$

which matches the symmetry of the graph of $\cosh x$.

Example 5 Find the value, in logarithmic form, of $\cosh^{-1}2$.

SOLUTION

Using $\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$, we have

$$\cosh^{-1}2 = \ln(2+\sqrt{3})$$

Example 6 Find the exact coordinates of the points where the line $y = 3$ cuts the graph of $y = \cosh x$.

SOLUTION

When $y = 3$, we have

$$\begin{aligned} & x = \cosh^{-1}3\\ \Rightarrow\quad & x = \ln(3+\sqrt{8}) = \ln(3+2\sqrt{2})\end{aligned}$$

By symmetry, the other value of x is $-\ln(3+2\sqrt{2})$.

Therefore, the two points are

$$(\ln(3+2\sqrt{2}), 3) \quad\text{and}\quad (-\ln(3+2\sqrt{2}), 3)$$

Expressing $\sinh^{-1}x$ as a logarithmic function

Let $y = \sinh^{-1}x$. We then have

$$x = \sinh y \quad\Rightarrow\quad x = \frac{1}{2}(e^y - e^{-y})$$

Multiplying throughout by $2e^y$, we obtain

$$\begin{aligned} & 2xe^y = e^{2y} - 1\\ \Rightarrow\quad & e^{2y} - 2xe^y - 1 = 0\end{aligned}$$

Treating this equation as a quadratic in e^y, we have

$$e^y = \frac{2x\pm\sqrt{4x^2+4}}{2} \quad\Rightarrow\quad e^y = x\pm\sqrt{x^2+1}$$

Taking the logarithms of both sides, we obtain

$$y = \ln(x \pm \sqrt{x^2+1})$$

The value of $\sinh^{-1}x$ can only be $\ln(x + \sqrt{x^2+1})$. We cannot have $\sinh^{-1}x = \ln(x - \sqrt{x^2+1})$, because $x < \sqrt{x^2+1}$, which would give the logarithm of a negative number, which is a complex number.

Hence, we have

$$\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$$

Example 7 Find the value, in logarithmic form, of $\sinh^{-1}3$.

SOLUTION

Using $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$, we have

$$\sinh^{-1}3 = \ln(3 + \sqrt{10})$$

Expressing $\tanh^{-1}x$ as a logarithmic function

Let $y = \tanh^{-1}x$. We then have

$$x = \tanh y = \frac{\sinh y}{\cosh y}$$

$$\Rightarrow \quad x = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y + e^{-y})}$$

Multiplying the numerator and the denominator by $2e^y$, we obtain

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

$$\Rightarrow \quad e^{2y}x + x = e^{2y} - 1$$

Therefore, we have

$$e^{2y} = \frac{1+x}{1-x} \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Hence, the value of $\tanh^{-1}x$ is $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, where $-1 < x < 1$.

Example 8 Find the value, in logarithmic form, of $\tanh^{-1}\frac{1}{2}$.

SOLUTION

Using $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, we have

$$\tanh^{-1}\tfrac{1}{2} = \frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)$$

which gives $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln 3$.

Example 9 Find the value, in logarithmic form, of $\operatorname{sech}^{-1}\frac{1}{2}$.

SOLUTION

Since $y = \operatorname{sech}^{-1}\frac{1}{2}$, we have

$$\begin{aligned} \operatorname{sech} y &= \tfrac{1}{2} \\ \Rightarrow \quad \frac{1}{\cosh y} &= \frac{1}{2} \\ \Rightarrow \quad \cosh y &= 2 \\ \Rightarrow \quad y &= \cosh^{-1} 2 \end{aligned}$$

Using $\cosh^{-1}x = \ln(x + \sqrt{x^2 - 1})$, we have

$$\cosh^{-1}2 = \ln(2 + \sqrt{3})$$

which gives $\operatorname{sech}^{-1}\frac{1}{2} = \ln(2 + \sqrt{3})$.

Summary

$\cosh^{-1}x = \ln(x \pm \sqrt{x^2 - 1}) \quad x \geqslant 1$ Plus sign gives the **principal value**

$\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$

$\tanh^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) \quad -1 < x < 1$

Differentiation of inverse hyperbolic functions

$\sinh^{-1}x$

We have $y = \sinh^{-1}x$, therefore $\sinh y = x$.

Differentiating $\sinh y = x$, we obtain

$$\cosh y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

which gives

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1 + x^2}}$$

Therefore, we have

$$\int \frac{dx}{\sqrt{1 + x^2}} = \sinh^{-1}x + c$$

We now take $y = \sinh^{-1}\left(\dfrac{x}{a}\right)$, giving $\sinh y = \dfrac{x}{a}$.

Differentiating $\sinh y = \dfrac{x}{a}$, we obtain

$$\cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\cosh y} = \frac{1}{a\sqrt{1+\sinh^2 y}}$$

which gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sqrt{1+\left(\dfrac{x}{a}\right)^2}} = \frac{1}{\sqrt{a^2+x^2}}$$

That is, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2+x^2}}$$

from which it follows that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

$\cosh^{-1}x$

We have $y = \cosh^{-1}x$, therefore $\cosh y = x$.

Differentiating $\cosh y = x$, we obtain

$$\sinh y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

Therefore, we have

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \cosh^{-1}x + c$$

We now take $y = \cosh^{-1}\left(\dfrac{x}{a}\right)$, giving $\cosh y = \dfrac{x}{a}$.

Differentiating $\cosh y = \dfrac{x}{a}$, we obtain

$$\sinh y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sinh y} = \frac{1}{a\sqrt{\cosh^2 y - 1}}$$

which gives

$$\frac{dy}{dx} = \frac{1}{a\sqrt{\left(\frac{x}{a}\right)^2 - 1}} = \frac{1}{\sqrt{x^2 - a^2}}$$

That is, we have

$$\frac{d}{dx}\cosh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 - a^2}}$$

from which it follows that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

$\tanh^{-1} x$

We have $y = \tanh^{-1}\left(\frac{x}{a}\right)$, therefore $\tanh y = \frac{x}{a}$.

Differentiating $\tanh y = \frac{x}{a}$, we obtain

$$\operatorname{sech}^2 y \frac{dy}{dx} = \frac{1}{a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a \operatorname{sech}^2 y} = \frac{1}{a(1 - \tanh^2 y)}$$

which gives

$$\frac{dy}{dx} = \frac{1}{a\left[1 - \left(\frac{x}{a}\right)^2\right]} = \frac{a}{a^2 - x^2}$$

That is, we have

$$\frac{d}{dx}\tanh^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 - x^2}$$

from which it follows that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c$$

Note We can integrate $\frac{1}{a^2 - x^2}$ by partial fractions:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int\left(\frac{1}{a + x} + \frac{1}{a - x}\right)dx = \frac{1}{2a}\ln\left(\frac{a + x}{a - x}\right) + c$$

This result is the logarithmic form of $\tanh^{-1}\left(\frac{x}{a}\right)$. Hence, it is unusual to use a function in $\tanh^{-1} x$ in differentiation or integration.

Example 10 Differentiate

a) i) $\sinh^{-1}\left(\frac{x}{3}\right)$ **ii)** $\sinh^{-1}4x$ **b)** $\cosh^{-1}\left(\frac{x}{5}\right)$

SOLUTION

a) Using $\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2+x^2}}$, we have

i)
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}\left(\frac{x}{3}\right) = \frac{1}{\sqrt{9+x^2}}$$

ii)
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}4x = \frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}\left(\frac{x}{\frac{1}{4}}\right) = \frac{1}{\sqrt{\frac{1}{16}+x^2}}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}4x = \frac{4}{\sqrt{1+16x^2}}$$

b) Using $\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{x^2-a^2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}\left(\frac{x}{5}\right) = \frac{1}{\sqrt{x^2-25}}$$

Example 11 Find

a) $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$ **b)** $\int_0^1 \frac{1}{\sqrt{4+3x^2}}\,\mathrm{d}x$

SOLUTION

a) Using the first integral formula on page 202, we obtain

$$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= \sinh^{-1}1 - \sinh^{-1}0 = \sinh^{-1}1 = \ln(1+\sqrt{2})$$

Therefore, we have

$$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \ln(1+\sqrt{2})$$

b) Before integrating, we must reduce the coefficient of x^2 to unity (as with inverse trigonometric functions), which gives

$$\int_0^1 \frac{1}{\sqrt{4+3x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{3}}\int_0^1 \frac{1}{\sqrt{\frac{4}{3}+x^2}}\,\mathrm{d}x$$
$$= \frac{1}{\sqrt{3}}\int_0^1 \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2+x^2}}$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{4+3x^2}}\,dx = \frac{1}{\sqrt{3}}\left[\sinh^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\sinh^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sinh^{-1} 0\right]$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left(\frac{\sqrt{3}}{2} + \sqrt{\frac{3}{4}+1}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$$

Therefore, we have

$$\int_0^1 \frac{1}{\sqrt{4+3x^2}}\,dx = \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$$

Example 12 Find $\int_3^6 \frac{1}{\sqrt{x^2-9}}\,dx$.

SOLUTION

Using the first integral formula on page 203, we obtain

$$\int_3^6 \frac{1}{\sqrt{x^2-9}}\,dx = \left[\cosh^{-1}\left(\frac{x}{3}\right)\right]_3^6$$

$$= \cosh^{-1}2 - \cosh^{-1}1 = \ln(2+\sqrt{3}) - 0$$

Therefore, we have

$$\int_3^6 \frac{1}{\sqrt{x^2-9}}\,dx = \ln(2+\sqrt{3})$$

Example 13 Find $\int \frac{1}{\sqrt{4x^2-8x-16}}\,dx$.

SOLUTION

Before integrating, we must

- Complete the square (as with inverse trigonometric functions).
- Reduce the coefficient of x^2 to unity.

Hence, we have

$$\sqrt{4x^2-8x-16} = \sqrt{4}\sqrt{x^2-2x-4}$$

$$= 2\sqrt{(x-1)^2-5}$$

which gives

$$\int \frac{1}{\sqrt{4x^2-8x-16}}\,dx = \frac{1}{2}\int \frac{dx}{\sqrt{(x-1)^2-5}}$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + c$$