Hence, the values of $\theta$ at the points where the tangent is perpendicular to the initial line are

$$\theta = 0 \text{ at A} \qquad \theta = \pi - 0.912 \text{ at B}$$

$$\theta = -\pi + 0.912 \text{ at C} \qquad \theta = \frac{\pi}{2} \text{ at D}$$

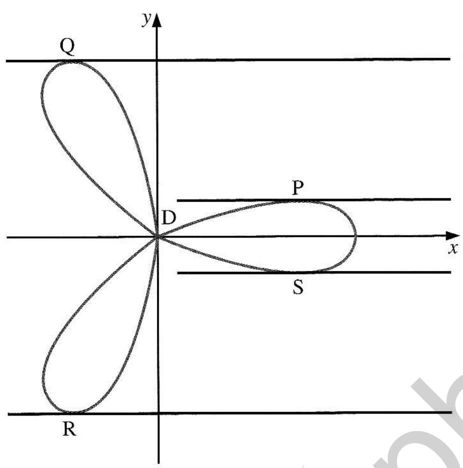
So, the equations of the tangents **perpendicular** to the initial line are

$$x = a$$
$$x = 0$$

and

$$x = -0.9858a \quad \text{or} \quad a\cos\frac{3}{2}\left[\cos^{-1}\left(-\frac{1}{4}\right)\right]$$

The tangents to $r = a\cos 3\theta$ **parallel** to the initial line are shown below. These are at the points P, Q, R and S. To find these points, we find the maximum and minimum values of $r\sin\theta$ in a similar way to that shown above.




## Exercise 3E

**1** Find the equation of each tangent to the curve $r = a\cos 3\theta$ which is parallel to the initial line.

**2** Find the equation of the tangent to the curve $r = e^{\theta}$ which is

**a)** parallel to the initial line
**b)** perpendicular to the initial line.

**3** Give in polar coordinates the points on the curve $r = a\cos 2\theta$ where the tangents are

**a)** parallel to the initial line
**b)** perpendicular to the initial line.

**4** The diagram (top of page 55) shows a square PQRS with sides parallel to the axes O$x$ and O$y$. The square circumscribes a curve $C$ whose cartesian equation is $(x^2 + y^2)^{\frac{3}{2}} = xy$.

**a)** Show that, in terms of polar coordinates $(r, \theta)$, the equation of $C$ is $r = \frac{1}{2}\sin 2\theta$.

**b)** Find the area bounded by $C$.

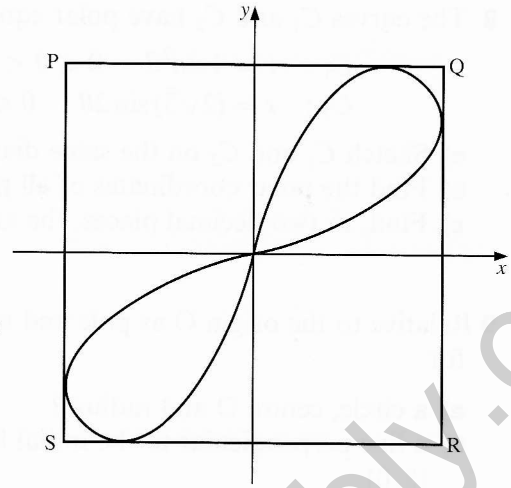
**c)** The coordinates of a variable point on $C$ are $(x, y)$.

**i)** Show that $x = \sin\theta - \sin^3\theta$.

**ii)** Show that, as $\theta$ varies, the maximum value of $x$ occurs when $\sin\theta = \dfrac{1}{\sqrt{3}}$.

**iii)** Calculate the area of the square PQRS.

(NEAB)




**5** The diagram shows a sketch of the curve $C$ whose polar equation is

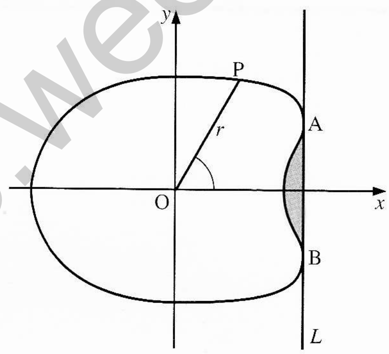
$$r = \sqrt{3} - \cos\theta \quad (-\pi < \theta \leqslant \pi)$$

The line $L$ touches the curve at A and B. Express in terms of $\theta$ the $x$-coordinate of a general point, P, on $C$ and determine the values of $\theta$ for which this coordinate has a stationary value.

Deduce that at A, $\theta = \dfrac{\pi}{6}$.

Show that the area of the region bounded by $C$ and $L$, shown shaded in the diagram, is

$$\frac{17\sqrt{3}}{16} - \frac{7\pi}{12} \qquad \text{(NEAB)}$$




**6** **a)** Sketch the curve with polar equation

$$r = \cos 2\theta \qquad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

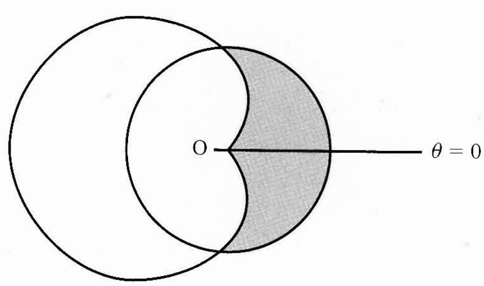
At the distinct points A and B on this curve, the tangents to the curve are parallel to the initial line, $\theta = 0$.

**b)** Determine the polar coordinates of A and B, giving your answers to three significant figures. (EDEXCEL)

**7** The figure on the right shows a sketch of the circle with polar equation $r = a$ and the cardioid with polar equation $r = a(1 - \cos\theta)$, where $a$ is a positive constant.

**a)** Verify that the curves intersect where $\theta = \pm\dfrac{\pi}{2}$.

**b)** Find the area of the shaded region, giving your answer in terms of $a$ and $\pi$. (EDEXCEL)



**8** The curves $C_1$ and $C_2$ have polar equations

$$C_1: \quad r = 4\sin^2\theta \qquad 0 \leqslant \theta < 2\pi$$
$$C_2: \quad r = (2\sqrt{3})\sin 2\theta \qquad 0 \leqslant \theta < \tfrac{1}{2}\pi$$

**a)** Sketch $C_1$ and $C_2$ on the same diagram.
**b)** Find the polar coordinates of all points of intersection of $C_1$ and $C_2$.
**c)** Find, to two decimal places, the area of the region $R$ which is inside both $C_1$ and $C_2$.

(EDEXCEL)

**9** Relative to the origin O as pole and initial line $\theta = 0$, find an equation in polar coordinate form for

**a)** a circle, centre O and radius 2
**b)** a line perpendicular to the initial line and passing through the point with polar coordinates $(3, 0)$
**c)** a straight line through the points with polar coordinates $(4, 0)$ and $\left(4, \dfrac{\pi}{3}\right)$. (EDEXCEL)

# 4 Differential equations

*Change and decay in all around I see.*
H. F. LYTE

We have already solved first-order differential equations in which the variables are separable (see pages 457–60 in *Introducing Pure Mathematics.*)

We will now consider three other main types of differential equation.

## First-order equations requiring an integrating factor

This is the other main type of first-order differential equation.

Equations of this type are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

where $P$ and $Q$ are functions of $x$.

Such an equation can be solved by first multiplying both sides by the **integrating factor** $\mathrm{e}^{\int P\,\mathrm{d}x}$.

Multiplying $\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$ by $\mathrm{e}^{\int P\,\mathrm{d}x}$, we get

$$\mathrm{e}^{\int P\,\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + P\mathrm{e}^{\int P\,\mathrm{d}x}y = Q\mathrm{e}^{\int P\,\mathrm{d}x}$$

Since the left-hand side is the differential of $y\mathrm{e}^{\int P\,\mathrm{d}x}$, we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int P\,\mathrm{d}x}\right) = Q\mathrm{e}^{\int P\,\mathrm{d}x}$$

which gives

$$y\mathrm{e}^{\int P\,\mathrm{d}x} = \int Q\mathrm{e}^{\int P\,\mathrm{d}x}\,\mathrm{d}x$$

The right-hand side is often integrated by parts.

**Example 1** If $\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x$, find $y$.

**SOLUTION**

The integrating factor is $\mathrm{e}^{\int 3\,\mathrm{d}x}$, which is $\mathrm{e}^{3x}$.

Multiplying both sides by $\mathrm{e}^{3x}$, we obtain

$$\mathrm{e}^{3x}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{3x}3y = x\mathrm{e}^{3x}$$

$$\Rightarrow \quad \frac{d}{dx}(ye^{3x}) = xe^{3x}$$

Integrating by parts, we have

$$ye^{3x} = \int xe^{3x}\,dx$$

$$= \frac{1}{3}e^{3x} \times x - \int \frac{1}{3}e^{3x}\,dx$$

which gives

$$ye^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

Multiplying both sides by $e^{-3x}$, **including** $c$, we obtain

$$y = \frac{1}{3}x - \frac{1}{9} + ce^{-3x}$$

**Note** The constant term, $c$, has now become a function of $x$.

**Example 2** Solve the differential equation $x\dfrac{dy}{dx} - 2y = x^4$.

SOLUTION

Dividing both sides by $x$ to make the first term $\dfrac{dy}{dx}$, we obtain

$$\frac{dy}{dx} - \frac{2y}{x} = x^3$$

The integrating factor is

$$e^{\int -(2/x)\,dx} = e^{-2\ln x} = e^{\ln x^{-2}}.$$

Applying the result $e^{\ln u} = u$, we have $e^{\ln x^{-2}} = \dfrac{1}{x^2}$.

We now multiply the differential equation by the integrating factor, $\dfrac{1}{x^2}$, to obtain

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = x$$

which we express as

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = x$$

$$\Rightarrow \quad \frac{1}{x^2}y = \int x\,dx$$

$$\Rightarrow \quad \frac{1}{x^2}y = \frac{x^2}{2} + c$$

Multiplying both sides by $x^2$, we obtain the **general solution**

$$y = \tfrac{1}{2}x^4 + cx^2$$

**Note** To obtain a **particular solution**, we need to be given a specific point which lies on the curve. Hence, we can find the value of $c$. This extra fact is called a **boundary condition**. Example 3 illustrates such a situation.

**Example 3** Solve the differential equation $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$, given that $y = 3$ when $x = 2$.

SOLUTION

The integrating factor is $e^{\int (1/x)\,dx} = e^{\ln x} = x$.

Multiplying the differential equation by the integrating factor, $x$, we have

$$x\frac{dy}{dx} + y = x^3$$

which we express as

$$\frac{d}{dx}(xy) = x^3$$

$$\Rightarrow \quad xy = \frac{1}{4}x^4 + c$$

When $x = 2$, $y = 3$, which gives

$$6 = 4 + c \quad \Rightarrow \quad c = 2$$

Therefore, the solution is

$$xy = \frac{1}{4}x^4 + 2 \quad \text{or} \quad y = \frac{1}{4}x^3 + \frac{2}{x}$$

## Exercise 4A

**1** Simplify each of the following.

**a)** $e^{\ln x^2}$ **b)** $e^{\frac{1}{2}\ln(x^2+1)}$ **c)** $e^{-3\ln x}$ **d)** $e^{\int \tan x\,dx}$

**e)** $e^{\int x/(x^2-1)\,dx}$ **f)** $e^{3x\ln 2}$

In each of Questions **2** to **7**, find the general solution.

**2** $\dfrac{dy}{dx} + 3y = x$ **3** $\dfrac{dy}{dx} - 5y = e^{2x}$ **4** $x\dfrac{dy}{dx} + y = x^2$

**5** $x\dfrac{dy}{dx} - 2y = x^3$ **6** $\dfrac{dy}{dx} - \dfrac{4y}{x-1} = 5(x-1)^3$ **7** $\tan x\dfrac{dy}{dx} + y = e^{2x}\tan x$

**8** A curve $C$ in the $x$–$y$ plane passes through the point $(1, 0)$. At any point $(x, y)$ on $C$,

$$\frac{dy}{dx} + y = e^{-x}$$

**a)** Find the general solution of this differential equation.

**b) i)** Hence find the equation of $C$, giving your answer in the form $y = f(x)$.

**ii)** Write down the equation of the asymptote of $C$. (NEAB)

**9** Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2y = xe^{x^3}$$

giving $y$ explicitly in terms of $x$ in your answer.

Find also the particular solution for which $y = 1$ when $x = 0$. (OCR)

**10** Find the general solution of the differential equation

$$(\cos x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x)y = \cos^2 x$$

expressing $y$ in terms of $x$. (OCR)

**11** Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x$$

giving $y$ explicitly in terms of $x$ in your answer.

Find also the particular solution for which $y = 1$ when $x = 1$. (OCR)

**12** Find, in the form $y = \mathrm{f}(x)$, the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{x}y = 6x - 5 \qquad x > 0$$ (EDEXCEL)

**13** A car moves from rest along a straight road. After $t$ seconds the velocity is $v$ metres per second. The motion is modelled by

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \alpha v = e^{\beta t}$$

where $\alpha$ and $\beta$ are positive constants.

**i)** Find $v$ in terms of $\alpha$, $\beta$ and $t$.
**ii)** Show that, as long as the above model applies, the car does not come to rest. (OCR)

**14** The variables $v$ and $t$ are related by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 20 + \tfrac{1}{10}v\tan\left(\tfrac{1}{10}t\right)$$

Given that $v = 1$ when $t = 0$, find $v$ when $t = 2$. (OCR)

**15 i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \cos x$$

**ii)** If $y = 2$ when $x = 0$, find the particular solution. (NICCEA)

**16** Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (2x + 1)y = 12x^3e^{-x^2 - x}$$

and that $y = 5$ when $x = 0$, find $y$ in terms of $x$. (OCR)

**17** The number, $N$, of animals of a certain species at time $t$ years increases at a rate of $\lambda N$ per year by births, but decreases at a rate of $\mu t$ per year by deaths, where $\lambda$ and $\mu$ are positive constants.

Modelled as continuous variables, $N$ and $t$ are related by the differential equation

$$\frac{dN}{dt} = \lambda N - \mu t$$

Given that $N = N_0$ when $t = 0$, find $N$ in terms of $t$, $\lambda$, $\mu$ and $N_0$. (OCR)

**18** **i)** Find the general solution of the differential equation

$$\frac{dy}{dx} = k(x + y)$$

where $k$ is a constant, giving your answer in the form $y = f(x)$.

**ii)** The gradient at any point $P(x, y)$ of a curve is proportional to the sum of the coordinates of P. The curve passes through the point $(1, -2)$ and its gradient at $(1, -2)$ is $-4$.

**a)** Find the equation of the curve.

**b)** Show that the line $y = -x - \frac{1}{4}$ is an asymptote to the curve. (OCR)

**19** **i)** Show that the appropriate integrating factor for

$$\frac{dy}{dx} + (2\cot x)y = f(x)$$

is $\sin^2 x$.

**ii)** Hence find the general solution of the differential equation

$$\sin x \frac{dy}{dx} + 2y\cos x = \cos x \qquad \text{(NICCEA)}$$

**20** Find the general solution of the differential equation

$$(4 + t^2)\frac{ds}{dt} = 1$$

Given that $s = 0$ when $t = 2$, express $s$ in terms of $t$. (EDEXCEL)

**21** **a)** Find the general solution of the differential equation

$$x\frac{dy}{dx} - y = x^2 e^{-x}$$

giving your answer in the form $y = f(x)$.

**b)** **i)** Verify that the graphs of all solutions of the differential equation pass through the origin O, and find the particular solution which is such that $\frac{dy}{dx} = -1$ at O.

**ii)** For this particular solution, state the limiting value of $y$ as $x \to \infty$. (NEAB)

# Second-order differential equations

An equation is termed **second order** when it contains the second derivative, $\frac{d^2y}{dx^2}$.

Initially, we will consider equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where $a$, $b$ and $c$ are constants.

To solve the equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$, we make the substitution $y = Ae^{nx}$. Hence, we have

$$\frac{dy}{dx} = nAe^{nx} \quad \text{and} \quad \frac{d^2y}{dx^2} = n^2Ae^{nx}$$

which give

$$an^2Ae^{nx} + bnAe^{nx} + cAe^{nx} = 0$$

That is,

$$an^2 + bn + c = 0$$

This quadratic equation is called the **auxiliary equation**.

The solution of a second-order differential equation depends on the type of solution which satisfies its auxiliary equation. There are three types of solution of a quadratic equation:

**1** Two real and different roots
**2** Two real and equal roots.
**3** Two complex roots.

## Type 1 solution.

The auxiliary equation has two **real, different roots**, $n_1$ and $n_2$. So, the solution of $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ is

$$y = Ae^{n_1x} + Be^{n_2x}$$

where $A$ and $B$ are arbitrary constants.

To verify that this is the full solution, we need to confirm that the following two conditions obtain:

- There are two arbitrary constants, as it is a second-order differential equation.
- The solution does satisfy the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

We notice that there are indeed the two required arbitrary constants.

To prove that the solution, $y = Ae^{n_1x} + Be^{n_2x}$, satisfies the differential equation, we substitute it and its derivatives in the LHS of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

which gives

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= a(n_1^2Ae^{n_1x} + n_2^2Be^{n_2x}) + b(n_1Ae^{n_1x} + n_2Be^{n_2x}) + c(Ae^{n_1x} + Be^{n_2x}) \\ &= Ae^{n_1x}(an_1^2 + bn_1 + c) + Be^{n_2x}(an_2^2 + bn_2 + c) \\ &= 0 \end{aligned}$$

since $n_1$ and $n_2$ are roots of the equation $an^2 + bn + c = 0$.

To find the values of $A$ and $B$, we need **two boundary conditions**. Usually, these are either

- the values of $y$ at two different values of $x$, or
- the value of $y$ and that of $\frac{dy}{dx}$ for one value of $x$.

**Example 4** Find $y$ when $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0$, given that $x = 0$ when $y = 2$ and $y$ is finite as $x$ tends to infinity.

SOLUTION

Substituting $y = Ae^{nx}$ and its derivatives in $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0$, we get

$$2n^2 - n - 3 = 0$$

$$\Rightarrow \quad (2n-3)(n+1) = 0$$

$$\Rightarrow \quad n = \frac{3}{2} \quad \text{and} \quad -1$$

Therefore, we have

$$y = Ae^{\frac{3}{2}x} + Be^{-x}$$

When $x = 0$, $y = 2$, which gives

$$2 = A + B$$

We know that as $x$ tends to infinity, $y$ is finite. Therefore, $A = 0$ because the limit of $e^{\frac{3}{2}x}$ as $x$ tends to infinity is not finite.

Hence, $B = 2$, which gives $y = 2e^{-x}$.

## Type 2 solution

The auxiliary equation has two **real**, **equal roots**, $n$. In this case, we cannot, as in Type 1, use just $y = Ae^{nx} + Be^{nx}$, since this simplifies to $y = (A + B)e^{nx}$ or $y = Ce^{nx}$, which has only **one** arbitrary constant. The solution is, therefore,

$$y = (A + Bx)e^{nx}$$

To prove this is the solution, we must show that it satisfies the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Differentiating $y = (A + Bx)e^{nx}$ twice, we get

$$\frac{dy}{dx} = Be^{nx} + ne^{nx}(A + Bx)$$

$$\frac{d^2y}{dx^2} = Bne^{nx} + n^2e^{nx}(A + Bx) + ne^{nx}B$$

$$= n^2(A + Bx)e^{nx} + 2nBe^{nx}$$

Substituting these in the LHS of $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$, we have

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=a[n^2(A+Bx)e^{nx}+2nBe^{nx}]+b[Be^{nx}+ne^{nx}(A+Bx)]+c(A+Bx)e^{nx}$$
$$=(A+Bx)e^{nx}(an^2+bn+c)+(2na+b)Be^{nx}$$

Since $n$ is a root of $an^2+bn+c=0$, the first term is zero.

Consider now the quadratic formula, $n=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. When its roots are coincident, $b^2-4ac=0$. Therefore, we have

$$n=-\frac{b}{2a} \quad\Rightarrow\quad 2na+b=0$$

So, the second term is also zero.

Hence, $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy$ does equal zero, and $y=(A+Bx)e^{nx}$ is indeed the required solution.

**Example 5** Solve $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=0$.

**SOLUTION**

Substituting $y=Ae^{nx}$ and its derivatives in $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=0$, we get

$$n^2+6n+9=0$$
$$\Rightarrow \quad (n+3)(n+3)=0$$
$$\Rightarrow \quad n=-3$$

Therefore, the general solution is

$$y=(A+Bx)e^{-3x}$$

## Type 3 solution

The auxiliary equation has two **complex roots**, $n_1\pm in_2$.

Therefore, the solution of $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$ is

$$\begin{aligned}
y&=Ae^{(n_1+in_2)x}+Be^{(n_1-in_2)x}\\
&=e^{n_1x}(Ae^{in_2x}+Be^{-in_2x})\\
&=e^{n_1x}[A\cos n_2x+iA\sin n_2x+B\cos(-n_2x)+iB\sin(-n_2x)]\\
&=e^{n_1x}(A\cos n_2x+iA\sin n_2x+B\cos n_2x-iB\sin n_2x)\\
&=e^{n_1x}[(A+B)\cos n_2x+i(A-B)\sin n_2x]
\end{aligned}$$

Since $A$ and $B$ are arbitrary constants, we can combine $(A+B)$ to give an arbitrary constant $C$, and we can combine $i(A-B)$ to give an arbitrary constant $D$. So, we have

$$y=e^{n_1x}(C\cos n_2x+D\sin n_2x)$$

**Example 6** Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$, given that $y = 0$ and $\frac{dy}{dx} = 6$, when $x = 0$.

**SOLUTION**

Substituting $y = Ae^{nx}$ and its derivatives in $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$, we get

$$n^2 - 2n + 3 = 0$$

$$\Rightarrow \quad n = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \sqrt{2}i$$

Therefore, the general solution is

$$y = e^x(C\cos\sqrt{2}x + D\sin\sqrt{2}x)$$

To find $C$ and $D$, we use the boundary conditions.

When $x = 0$, $y = 0$, which gives

$$0 = C\cos 0 + D\sin 0 \quad \Rightarrow \quad C = 0$$

Hence, we have

$$y = De^x \sin\sqrt{2}x$$

As one boundary condition is given in terms of $\frac{dy}{dx}$, we differentiate the above:

$$\frac{dy}{dx} = De^x \sin\sqrt{2}x + \sqrt{2}De^x \cos\sqrt{2}x$$

When $x = 0$, $\frac{dy}{dx} = 6$, which gives

$$6 = D\sin 0 + \sqrt{2}D\cos 0$$

$$\Rightarrow \quad 6 = \sqrt{2}D \quad \Rightarrow \quad D = 3\sqrt{2}$$

Therefore, the solution is $y = 3\sqrt{2}e^x \sin\sqrt{2}x$.

## Alternative notation for derivatives

Sometimes it is more convenient to denote $\frac{dy}{dx}$ by $y'$ or f$'$, and $\frac{d^2y}{dx^2}$ by $y''$ or f$''$, where $y = f(x)$.

# Exercise 4B

In Questions **1** to **12**, find the general solution of each differential equation.

**1** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} - 8y = 0$

**2** $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

**3** $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

**4** $3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 7y = 0$

**5** $\frac{d^2x}{dt^2} - 7\frac{dx}{dt} - 8x = 0$

**6** $\frac{d^2x}{dt^2} - 11\frac{dx}{dt} + 28x = 0$

**7** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

**8** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

**9** $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

**10** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0$

**11** $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 7x = 0$

**12** $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 13x = 0$

## Second-order differential equations of the type $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)$

If $y = g(x)$ is the solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

and $y = h(x)$ is the solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

then we have

$$y = h(x) + \lambda g(x)$$

as the **general solution** of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

**Proof**

Substituting $y = h + \lambda g$ and its derivatives in the LHS of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

we have

$$\begin{aligned} ay'' + by' + cy &= a(h'' + \lambda g'') + b(h' + \lambda g') + c(h + \lambda g) \\ &= ah'' + bh' + ch + \lambda(ag'' + by' + cy) \\ &= f(x) \end{aligned}$$

since h is a solution of $ah'' + bh' + ch = f(x)$, and g is a solution of $ag'' + bg' + cg = 0$.

Therefore,

$$y = h(x) + \lambda g(x)$$

is the general solution of $ay'' + by' + cy = f(x)$

$g(x)$ is called the **complementary function (CF)**, and $h(x)$ is called the **particular integral (PI)**.

The particular solution is obtained by inserting boundary conditions into the general solution.

## Types of particular integral

The particular integral depends on the function $f(x)$.

We will consider three types of function $f(x)$:

- polynomial
- exponential
- trigonometric

**• f($x$) is a polynomial of degree $n$**

In this case, the particular integral will also be a polynomial of degree $n$.

**Example 7** By finding **a)** the complementary function and **b)** the particular integral, solve the equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 8$$

**SOLUTION**

**a)** For the complementary function, we use

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0$$

Substituting $x = Ae^{nt}$ and its derivatives in the above equation, we get

$$n^2 + 3n - 4 = 0$$

$$\Rightarrow \quad (n+4)(n-1) = 0$$

$$\Rightarrow \quad n = 1 \quad \text{or} \quad -4$$

So, the CF is $x = Ae^t + Be^{-4t}$.

**b)** For the particular integral, f($x$) is a polynomial of degree 0. Hence, we need consider only $x = c$ for the particular integral.

Substituting $x = c$ in $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 8$, we get

$$-4c = 8 \quad \Rightarrow \quad c = -2$$

So, the PI is $x = -2$.

Therefore, the general solution is $x = Ae^t + Be^{-4t} - 2$.

**Example 8** Find the solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 2 + 8x^2$, given that, when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$.

**SOLUTION**

To find the CF, we use

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$$

Substituting $y = Ae^{nx}$ and its derivatives in the above equation, we get

$$n^2 + 3n - 4 = 0$$

$$\Rightarrow \quad (n+4)(n-1) = 0$$

$$\Rightarrow \quad n = 1 \quad \text{or} \quad -4$$

So, the CF is $y = Ae^x + Be^{-4x}$.

To find the PI, we substitute $y = a + bx + cx^2$ and its derivatives in

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 3 + 8x^2$$

which gives

$$2c + 3(b + 2cx) - 4(a + bx + cx^2) = 3 + 8x^2$$

Equating coefficients of $x^2$: $-4c = 8 \Rightarrow c = -2$

Equating coefficients of $x$: $6c - 4b = 0 \Rightarrow b = -3$

Letting $x = 0$ in the above equation, we get

$$2c + 3b - 4a = 3$$
$$\Rightarrow a = -4$$

So, the PI is $y = -4 - 3x - 2x^2$.

Therefore, the general solution is

$$y = Ae^x + Be^{-4x} - 4 - 3x - 2x^2$$

We now need to find values for $A$ and $B$.

When $x = 0$, $y = 0$, which gives

$$0 = A + B - 4$$
$$\Rightarrow A + B = 4 \qquad [1]$$

Differentiating $y = Ae^x + Be^{-4x} - 4 - 3x - 2x^2$, we have

$$\frac{dy}{dx} = Ae^x - 4Be^{-4x} - 3 - 4x$$

When $x = 0$, $\frac{dy}{dx} = 1$, which gives

$$1 = A - 4B - 3$$
$$\Rightarrow A - 4B = 4 \qquad [2]$$

From [1] and [2], we get $A = 4$ and $B = 0$.

Therefore, the general solution is $y = 4e^x - 4 - 3x - 2x^2$.

- **f($x$) is an exponential function**

Take, for example, the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 3e^{7x}$$

In this case, $f(x) = 3e^{7x}$. The particular integral will be of the same form: $Ce^{7x}$.

Therefore, the CF is $y = Ae^x + Be^{-4x}$ (see Example 8).

To find the PI, we substitute $y = Ce^{7x}$ and its derivatives in

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 3e^{7x}$$

which gives

$$49Ce^{7x} + 21Ce^{7x} - 4Ce^{7x} = 3e^{7x}$$

$$\Rightarrow \quad 66C = 3 \quad \Rightarrow \quad C = \tfrac{1}{22}$$

So, the PI is $y = \frac{1}{22}e^{7x}$.

Therefore, the general solution is $y = Ae^{x} + Be^{-4x} + \dfrac{e^{7x}}{22}$.

- **f($x$) is a trigonometric function of the form $a \sin nx$**

Take, for example, $f(x) = 4\sin 2x$. The particular integral will be of the form $C\sin 2x + D\cos 2x$

**Example 9** Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 4\sin 2x$.

SOLUTION

The CF is $y = Ae^{x} + Be^{-4x}$ (see Example 8).

**Caution** Suppose we were simply to consider $y = C\sin 2x$ as the PI. Because there is only a $\sin 2x$ term on the right-hand side, we would obtain

$$\frac{dy}{dx} = 2C\cos 2x \quad \text{and} \quad \frac{d^2y}{dx^2} = -4C\sin 2x$$

Substituting these in $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 4\sin 2x$, we would obtain

$$-4C\sin 2x + 3 \times 2C\cos 2x - 4C\sin 2x = 4\sin 2x$$

which includes only one term in $\cos 2x$ $\left(\text{from } \dfrac{dy}{dx}\right)$.

This means that this equation **cannot be solved**.

Hence, the PI used **must** contain **both** $\sin 2x$ **and** $\cos 2x$ terms. That is,

$$y = C\sin 2x + D\cos 2x$$

Differentiating this, we have

$$y' = 2C\cos 2x - 2D\sin 2x$$
$$y'' = -4C\sin 2x - 4D\cos 2x$$

Substituting $y'$ and $y''$ in $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 4\sin 2x$, we get

$$-4C\sin 2x - 4D\cos 2x + 6C\cos 2x - 6D\sin 2x - 4C\sin 2x - 4D\cos 2x = 4\sin 2x$$

Equating coefficients of $\sin 2x$: $\quad -8C - 6D = 4$

$$\Rightarrow \quad -4C - 3D = 2 \qquad [1]$$

Equating coefficients of $\cos 2x$: $\quad -8D + 6C = 0$

$$\Rightarrow \quad -4D + 3C = 0 \qquad [2]$$

Solving the simultaneous equations [1] and[2], we get

$$C = -\frac{8}{25} \quad \text{and} \quad D = -\frac{6}{25}$$

Therefore, the PI is

$$y = -\frac{8}{25}\sin 2x - \frac{6}{25}\cos 2x$$

Hence, the general solution is

$$y = Ae^{x} + Be^{-4x} - \frac{8}{25}\sin 2x - \frac{6}{25}\cos 2x$$

**Example 10** Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$, given that $y = 0$ and $\frac{dy}{dx} = 11$ when $x = 0$.

**SOLUTION**

To find the CF, we use

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Substituting $y = Ae^{nx}$ and its derivatives in the above equation, we get

$$n^2 - n - 2 = 0$$
$$\Rightarrow \quad (n-2)(n+1) = 0$$
$$\Rightarrow \quad n = 2 \quad \text{or} \quad -1$$

So, the CF is $y = Ae^{2x} + Be^{-x}$.

To find the PI, we let $y = Cxe^{2x}$.

(**Note** $xe^{2x}$ is used here because $e^{2x}$ already forms part of the CF.)

Differentiating $y = Cxe^{2x}$, we have

$$\frac{dy}{dx} = Ce^{2x} + 2Cxe^{2x}$$

$$\frac{d^2y}{dx^2} = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x}$$

Substituting these in $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$, we get

$$4Ce^{2x} + 4Cxe^{2x} - Ce^{2x} - 2Cxe^{2x} - 2Cxe^{2x} = 3e^{2x}$$

(**Note** The $x$-terms should cancel at this stage.)

$$3Ce^{2x} = 3e^{2x} \quad \Rightarrow \quad C = 1$$

Therefore, the PI is $y = xe^{2x}$.

Hence, the general solution is $y = Ae^{2x} + Be^{-x} + xe^{2x}$.

At this stage, after adding the CF and the PI, we insert the boundary conditions:

$$y = 0 \text{ when } x = 0 \quad \Rightarrow \quad 0 = A + B$$

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x} + e^{2x} + 2xe^{2x}$$

$$\frac{dy}{dx} = 11 \text{ when } x = 0 \quad \Rightarrow \quad 11 = 2A - B + 1 \quad \Rightarrow \quad 10 = 2A - B$$

Since $0 = A + B$, we have

$$A = \frac{10}{3} \quad \text{and} \quad B = -\frac{10}{3}$$

The solution is, therefore,

$$y = \left(\frac{10}{3} + x\right)e^{2x} - \frac{10}{3}e^{-x}$$

**Example 11** Solve $y'' - 4y' + 4y = 3e^{2x}$.

SOLUTION

To find the CF, we substitute $y = Ae^{nx}$ and its derivatives in $y'' - 4y' + 4y = 0$, which gives

$$n^2 - 4n + 4 = 0$$
$$\Rightarrow \quad (n-2)(n-2) = 0$$
$$\Rightarrow \quad n = 2 \quad \text{(repeated root)}$$

Therefore, the CF is $y = (A + Bx)e^{2x}$.

To find the PI, we need to use a term in $x^2e^{2x}$, since both $e^{2x}$ **and** $xe^{2x}$ already form terms in the CF. Therefore, we let $y = Cx^2e^{2x}$, which gives

$$y' = 2Cx^2e^{2x} + 2Cxe^{2x}$$
$$y'' = 4Cx^2e^{2x} + 4Cxe^{2x} + 2Ce^{2x} + 4Cxe^{2x}$$
$$= 4Cx^2e^{2x} + 8Cxe^{2x} + 2Ce^{2x}$$

Substituting these in $y'' - 4y' + 4y = 3e^{2x}$, we have

$$4Cx^2e^{2x} + 8Cxe^{2x} + 2Ce^{2x} - 4(2Cx^2e^{2x} + 2Cxe^{2x}) + 4Cx^2e^{2x} = 3e^{2x}$$

(**Note** The terms in $x^2$ and $x$ should cancel at this stage.)

$$2Ce^{2x} = 3e^{2x} \quad \Rightarrow \quad C = \tfrac{3}{2}$$

Therefore, the PI is $y = \frac{3}{2}x^2e^{2x}$.

Hence, the general solution is $y = (A + Bx + \frac{3}{2}x^2)e^{2x}$.

**Example 12** Solve $y'' + 16y = 2\cos 4x$.

SOLUTION

To find the CF, we substitute $y = Ae^{nx}$ and its second derivative in $y'' + 16y = 0$, which gives

$$n^2 + 16 = 0 \quad \Rightarrow \quad n = \pm 4\mathrm{i}$$

The CF is, therefore, $y = A\cos 4x + B\sin 4x$.

Note that for the PI we need to use terms in $x\cos 4x$ and $x\sin 4x$, since the CF already contains the terms $\cos 4x$ and $\sin 4x$. Therefore, the PI is given by

$$y = Cx\cos 4x + Dx\sin 4x$$

So, we have

$$y' = C\cos 4x - 4Cx\sin 4x + D\sin 4x + 4Dx\cos 4x$$
$$y'' = -4C\sin 4x - 4C\sin 4x - 16Cx\cos 4x + 4D\cos 4x + 4D\cos 4x - 16D\sin 4x$$

Substituting the above in $y'' + 16y = 2\cos 4x$, we get

$$-8C\sin 4x - 16Cx\cos 4x + 8D\cos 4x - 16Dx\sin 4x + 16Cx\cos 4x + \\ +16Dx\sin 4x = 2\cos 4x$$

Simplifying, equating sin and cos terms, and remembering that the terms in $x$ should cancel, we find

$$C = 0 \quad \text{and} \quad D = \tfrac{1}{4}$$

Therefore, the PI is $y = \frac{1}{4}x\sin 4x$.

Hence, the solution is

$$y = A\sin 4x + B\cos 4x + \frac{x}{4}\sin 4x$$

## Exercise 4C

In Questions **1** to **12**, find the general solution of each differential equation.

**1** $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 8y = 16x$

**2** $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 4e^{-2x}$

**3** $2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 5y = 10x^2 + 1$

**4** $3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - y = 4\sin 5x$

**5** $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} - 5x = 3e^{3t}$

**6** $\dfrac{d^2s}{dt^2} - 8\dfrac{ds}{dt} + 15s = 5\cos 2t$

**7** $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 2e^{-x}$

**8** $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 5e^{3x}$

**9** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = 22e^{4x}$

**10** $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 3e^{-4x}$

**11** $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = 4e^{t}$

**12** $\dfrac{d^2x}{dt^2} + 16x = 3\cos 4t$

**13** Solve the differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0$$

if $x = -3$ and $\dfrac{dx}{dt} = 1$ when $t = 0$. (NICCEA)

**14** **a)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 10e^{3x}$$

**b)** Hence find the solution for which $y = -2$ at $x = 0$, and $\frac{dy}{dx} = -6$ at $x = 0$. (EDEXCEL)

**15** Find the general solution of the equation $\frac{d^2y}{dx^2} = e^{2x} + \cos \frac{1}{2}x$.

State what extra information would be needed to enable a particular solution to be obtained. (NEAB/SMP 16–19)

**16** **i)** Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

for which $y = 4$ and $\frac{dy}{dx} = 1$ at $x = 0$.

**ii)** Given that

$$\cos x \frac{dy}{dx} + 2y \sin x = \cos^3 x + \sin x \quad 0 < x < \tfrac{1}{2}\pi$$

and that $y = 1$ at $x = \frac{1}{3}\pi$, find the value of $y$ at $x = \frac{1}{4}\pi$. (EDEXCEL)

**17** Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \sin 2x$. (EDEXCEL)

**18** **i)** Solve the differential equation $\frac{d^2x}{dt^2} + 16x = 0$ to find its general solution.

**ii)** If $x = 3$ and $\frac{dx}{dt} = -8$ when $t = 0$, show that the particular solution of the differential equation above is

$$x = 3\cos 4t - 2\sin 4t$$

**iii)** By writing the particular solution as $R\cos(4t + \alpha)$, find the first positive value of $t$ for which $x$ is maximum. (NICCEA)

**19** Obtain the solution of the differential equation

$$20\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 2t + 11$$

given that, when $t = 0$, $x = 3$ and $\frac{dx}{dt} = 2.8$. Show that $x \approx 2t + 3$ for large positive $t$. (OCR)

**20** Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 15\cos 3t - 5\sin 3t \qquad \text{(OCR)}$$

**21** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 50\sin 2x$$

Given that $y = 0$ when $x = 0$ and that $y$ remains finite as $x \to \infty$, find $y$ in terms of $x$. (OCR)

**22 i)** Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 29x = -16\cos 2t + 50\sin 2t$$

**ii)** If $x = 3$ and $\frac{dx}{dt} = 10$ when $t = 0$, find the particular solution. (NICCEA)

**23 a)** Solve the equation $\frac{dy}{dx} = x + xy$. You do not need to make $y$ the subject of your solution.

**b)** Find the complementary function and a particular integral for the equation

$$\frac{dy}{dx} - 3y = 2x + e^{4x}$$

Hence write down the general solution of the equation. (NEAB/SMP 16–19)

**24 a)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

**b)** Given that $y = a\cos 3x + b\sin 3x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 6\cos 3x - 8\sin 3x$$

find the values of $a$ and $b$.

**c)** Show that this particular integral has maximum and minimum values of $\frac{\sqrt{10}}{4}$ and $-\frac{\sqrt{10}}{4}$ respectively.

**d)** Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 6\cos 3x - 8\sin 3x$$

for which $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$. (EDEXCEL)

**25 a)** Find the general solution of the differential equation

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 4y = 8\sin x - 19\cos x$$

**b)** Hence find the solution for which $y = 0$ at $x = 0$ and $\frac{dy}{dx} = 11$ at $x = 0$. (EDEXCEL)

**26** The value of the stock held by a large business organisation $t$ years after 1st January 1998 is $(10 + x)$ million dollars. The variation of $x$, which may be regarded as a continuous variable, is modelled by the differential equation

$$4\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 5x = 2\cos t - 16\sin t$$

**i)** Find the general solution for $x$ in terms of $t$.

**ii)** Given that $x = 1$ and $\frac{dx}{dt} = 3$ when $t = 0$, find, correct to four significant figures, the predicted value of the stock held on 1st January 2000. (OCR)

**27** Find the values of the constants $p$ and $q$ for which $y = px \sin 2x + qx \cos 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin 2x$$

Find the general solution of this differential equation.

Show that when $x = n\pi$, where $n$ is a large positive integer, $y \approx -\frac{1}{4}n\pi$, whatever the initial conditions, and find a corresponding approximation for $y$ when $x = (n + \frac{1}{2})\pi$. (OCR)

**28** Given that $x = At^2 e^{-t}$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}$$

**a)** find the value of $A$.

**b)** Hence find the solution of the differential equation for which $x = 1$ and $\dfrac{dx}{dt} = 0$ at $t = 0$.

**c)** Use your solution to prove that for $t \geqslant 0$, $x \leqslant 1$. (EDEXCEL)

# Solution of differential equations by substitution

We can now solve the following three types of differential equation:

- First order in which variables are separable.
- First order requiring an integrating factor.
- Second order of the form $ay'' + by' + cy = f(x)$, where $a$, $b$ and $c$ are constants.

Substitutions can be used to make a differential equation, which is one of these three types, more manageable.

For example, to solve

$$(m + 5kM) + t\frac{dm}{dt} = (m + 5kM)^3$$

we would make the substitution $p = m + 5kM$, which changes this equation into

$$p + t\frac{dp}{dt} = p^3$$

In this form, the equation looks less daunting and is easier to solve.

Substitutions can also be used to convert a more difficult form of differential equation to one of the above three types. (In an A-level examination, these kinds of substitution will normally be given.)

Two such substitutions which you will meet frequently are $y = ux$ and $x = e^u$, where $u$ is a function of $x$. Their application is shown respectively in Examples 13 and 14.

**Example 13** Solve $x^2 \dfrac{dy}{dx} = 4x^2 + xy + y^2$, given that when $x = 1$, $y = 2$.

**SOLUTION**

Notice that in this equation the power of each term, treating $x$ and $y$ as the same, is 2.

Such equations are called **homogeneous equations**, for which the usual substitution is $y = ux$.

Differentiating $y = ux$ with respect to $x$, we have

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Substituting for $\dfrac{dy}{dx}$ and for $y$ in $x^2 \dfrac{dy}{dx} = 4x^2 + xy + y^2$, we get

$$x^2\left(x\frac{du}{dx} + u\right) = 4x^2 + ux^2 + u^2x^2$$

Dividing through by $x^2$ and rearranging the terms, we have

$$x\frac{du}{dx} = 4 + u^2$$

$$\Rightarrow \quad \int \frac{du}{4+u^2} = \int \frac{dx}{x}$$

which gives (see page 36)

$$\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) = \ln x + c$$

$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\left(\frac{y}{2x}\right) = \ln x + c$$

Now when $x = 1$, $y = 2$. Therefore, $c = \dfrac{\pi}{8}$. Hence, we have

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2x}\right) = \ln x + \frac{\pi}{8}$$

$$\Rightarrow \quad \frac{y}{2x} = \tan\left(\frac{\pi}{4} + 2\ln x\right)$$

$$\Rightarrow \quad y = 2x\tan\left(\frac{\pi}{4} + 2\ln x\right)$$

**Example 14** Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 10y = 0$ using the substitution $x = e^u$.

**SOLUTION**

We need to replace $\dfrac{dy}{dx}$ by a term in $\dfrac{dy}{du}$, and $\dfrac{d^2y}{dx^2}$ by a term in $\dfrac{d^2y}{du^2}$.

So, first we differentiate $x = e^u$ with respect to $u$, which gives $\dfrac{dx}{du} = e^u$.

Using $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, we get

$$\frac{dy}{dx} = \frac{1}{e^u}\frac{dy}{du}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{-u}\frac{dy}{du}$$

We now differentiate this equation with respect to $x$, noting that the RHS is differentiated as a product and using

$$\frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} = \frac{d^2y}{du^2}\frac{du}{dx}$$

Hence, we arrive at

$$\frac{d^2y}{dx^2} = -e^{-u}\frac{du}{dx}\frac{dy}{du} + e^{-u}\frac{d^2y}{du^2}\frac{du}{dx}$$

Since $\dfrac{du}{dx} = e^{-u}$, we therefore have

$$\frac{d^2y}{dx^2} = -e^{-2u}\frac{dy}{du} + e^{-2u}\frac{d^2y}{du^2}$$

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 10y = 0$, we get

$$e^{2u}\left(-e^{-2u}\frac{dy}{du} + e^{-2u}\frac{d^2y}{du^2}\right) - 2e^u e^{-u}\frac{dy}{du} - 10y = 0$$

$$\Rightarrow \quad \frac{d^2y}{du^2} - 3\frac{dy}{du} - 10y = 0$$

Substituting $y = Ae^{nu}$ and its derivatives in the above equation, we obtain

$$n^2 - 3n - 10 = 0$$

$$\Rightarrow \quad n = 5 \quad \text{or} \quad -2$$

Therefore, the general solution is

$$y = Ae^{5u} + Be^{-2u}$$

Using $x = e^u$, we have

$$e^{5u} = (e^u)^5 = x^5 \quad \text{and} \quad e^{-2u} = (e^u)^{-2} = x^{-2}$$

which give

$$y = Ax^5 + \frac{B}{x^2}$$

## Exercise 4D

**1** Using the substitution $y = ux$, find the general solution of each of the following.

**a)** $\dfrac{dy}{dx} = \dfrac{x-3y}{x}$ **b)** $xy\dfrac{dy}{dx} = x^2 + y^2$ **c)** $x^2y\dfrac{dy}{dx} = x^3 + x^2y - y^3$

**d)** $3x^3\dfrac{dy}{dx} = y^3 - x^2y$

**2** Using the substitution $p = x + y$, find the general solution of

$$\frac{dy}{dx} = \frac{3x + 3y + 4}{x + y + 1}$$

**3** Use the substitution $p = 2x + 3y$ to find the general solution of

$$\frac{dy}{dx} = \frac{4x + 6y - 5}{2x + 3y + 1}$$

**4** Using the substitution $x = e^u$, find the general solution of

**a)** $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = 0$  **b)** $x^2 \dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} - 6y = 0$

**c)** $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 0$  **d)** $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$

**5** Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$, and $y$ is a function of $x$, find $\dfrac{dy}{dx}$ in terms of $\dfrac{dy}{dt}$ and $t$.

Assuming that $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$, show that the substitution $x = t^{\frac{1}{2}}$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \left(6x - \frac{1}{x}\right)\frac{dy}{dx} - 16x^2y = 4x^2e^{2x^2} \qquad \text{[I]}$$

into the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$$

Hence find the general solution of [I], giving $y$ in terms of $x$. (EDEXCEL)

**6 a)** Find the general solution of the equation

$$\frac{dz}{dx} + z = e^x$$

**b)** Make the substitution $y = xz$ in the equation

$$x\frac{dy}{dx} + (x - 1)y = x^2e^x$$

Hence write down the solution of this equation. (NEAB/SMP 16–19)

**7 a)** Show that the substitution $v = xy$ transforms the differential equation

$$x\frac{d^2y}{dx^2} + 2(1 + 2x)\frac{dy}{dx} + 4(1 + x)y = 32e^{2x} \qquad x \neq 0$$

into the differential equation

$$\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v = 32e^{2x}$$

**b)** Given that $v = ae^{2x}$, where $a$ is a constant, is a particular integral of this transformed equation, find $a$.

**c)** Find the solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} + 4(1+x)y = 32e^{2x}$$

for which $y = 2e^2$ and $\dfrac{dy}{dx} = 0$ at $x = 1$.

**d)** Determine whether or not this solution remains finite as $x \to \infty$. (EDEXCEL)

**8** The variables $x$ and $y$ are functions of $t$, and satisfy the differential equations

$$\frac{dx}{dt} + 2x = y \qquad (*)$$

$$\frac{dy}{dt} + x = 0$$

By eliminating $y$, show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

Find the general solution of this differential equation for $x$ and deduce by substitution in (*) the general solution for $y$.

Hence, or otherwise, find $x$ and $y$ in terms of $t$, given that $x = 1$ and $y = 0$ when $t = 0$.

(NEAB)

**9 a)** Find, in the form $y = f(x)$, the general solution of the equation

$$(x^2 - 1)\frac{dy}{dx} + xy = 1 \quad x > 1$$

**b) i)** Given that $y = \dfrac{u}{x}$, show that

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$$

**ii)** Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + 25y = 0 \quad x > 0 \qquad \text{(EDEXCEL)}$$

# 5 Determinants

*In algebra, to mention only one thing of many, Jacobi cast the theory of determinants into the simple form now familiar to every student.*
E. T. BELL

## Definition of 2 × 2 and 3 × 3 determinants

The $2 \times 2$ determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ represents the expression $ad - bc$.

For example, we have

$$\begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix} = 3 \times 8 - 4 \times 7 = 24 - 28 = -4$$

The $3 \times 3$ determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ represents the expression

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

which is

$$a(ei - fh) - b(di - fg) + c(dh - eg)$$

We see that the determinant of a $3 \times 3$ matrix is found by expanding the matrix along its first row. In turn, we take each **element**, or **entry**, in the first row, cover up its column and the first row, and find the determinant of the $2 \times 2$ matrix which is left. We then combine the three results. Notice the minus sign for the $b$-term, which relates to the fact that $b$ is an **odd** number of places from the first element, $a$.

**Note** It is much easier to learn the method for evaluating a determinant than to remember its formula.

**Example 1** Evaluate $\begin{vmatrix} 3 & 7 & 8 \\ 4 & 2 & 5 \\ 1 & 9 & 15 \end{vmatrix}$

**SOLUTION**

$$\begin{aligned} \begin{vmatrix} 3 & 7 & 8 \\ 4 & 2 & 5 \\ 1 & 9 & 15 \end{vmatrix} &= 3\begin{vmatrix} 2 & 5 \\ 9 & 15 \end{vmatrix} - 7\begin{vmatrix} 4 & 5 \\ 1 & 15 \end{vmatrix} + 8\begin{vmatrix} 4 & 2 \\ 1 & 9 \end{vmatrix} \\ &= 3(30 - 45) - 7(60 - 5) + 8(36 - 2) \\ &= -45 - 385 + 272 \\ &= -158 \end{aligned}$$

Determinants, unlike matrices, **always** consist of a **square array** of elements.

The determinant of the square matrix $\mathbf{A}$ is denoted either by $|\mathbf{A}|$ or by $\det \mathbf{A}$.

Because determinants are always square, the expansion method just described can be applied to determinants of any size. Thus to evaluate the determinant of a $4 \times 4$ matrix, we first expand it along its top row to get an expression involving four $3 \times 3$ matrices, remembering to **alternate the plus and minus signs**. For example,

$$\begin{vmatrix} 1 & 3 & 4 & 2 \\ 5 & -1 & -3 & -4 \\ 2 & -3 & 4 & 7 \\ 1 & 8 & 5 & 6 \end{vmatrix} = 1\begin{vmatrix} -1 & -3 & -4 \\ -3 & 4 & 7 \\ 8 & 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 5 & -3 & -4 \\ 2 & 4 & 7 \\ 1 & 5 & 6 \end{vmatrix} +$$

$$+ 4\begin{vmatrix} 5 & -1 & -4 \\ 2 & -3 & 7 \\ 1 & 8 & 6 \end{vmatrix} - 2\begin{vmatrix} 5 & -1 & -3 \\ 2 & -3 & 4 \\ 1 & 8 & 5 \end{vmatrix}$$

We then proceed to evaluate each $3 \times 3$ matrix as before.

# Rules for the manipulation of determinants

## Changing a determinant without changing its value

We can alter the rows and the columns of a determinant in three ways **without changing its value**. Two are given below.

### Adding any row, or column, to any other row, or column

If we add the corresponding elements in two rows (or columns), the value of the determinant is unaltered. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$$

The rule also applies to the **subtraction** of the corresponding elements in two rows (or columns). So, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d-g & e-h & f-i \\ g & h & i \end{vmatrix}$$

**Example 2** Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$

SOLUTION

The most efficient way to evaluate this determinant is to add the second row to the first row.

**Note** If you cannot quickly spot this simplification, it is better to expand using $2 \times 2$ determinants, rather than to spend time trying various possible simplifications.

So, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix} = \begin{vmatrix} 1+0 & 1-1 & 1-1 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix}$$

Expanding this simplified determinant, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 4 & 6 & 8 \end{vmatrix} = 1 \times \begin{vmatrix} -1 & -1 \\ 6 & 8 \end{vmatrix} = 1 \times (-8+6) = -2$$

**Adding any multiple of any row, or column, to any other row, or column**

If we add the same multiple of the elements of a column (or row) to the corresponding elements of another column (or row), the value of the determinant is unaltered. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+5b & b & c \\ d+5e & e & f \\ g+5h & h & i \end{vmatrix}$$

The rule also applies to negative multiples. So, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ g & h & i \end{vmatrix}$$

**Example 3** Evaluate $\begin{vmatrix} 4 & 6 & 8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix}$

SOLUTION

This determinant is best simplified by subtracting $2 \times$ the third row from the first row.

Again, if you cannot quickly spot this, it is better to expand using $2 \times 2$ determinants.

So, we have

$$\begin{vmatrix} 4 & 6 & 8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 4-2 & 6-6 & 8-8 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix}$$

Expanding this simplified determinant, we get

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 4 \\ 3 & 4 \end{vmatrix} = -16$$

## Two rows or columns can be interchanged by changing the sign of the determinant

For example, by switching columns 1 and 2 in the left-hand determinants, we get

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$$

and

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 3 & 1 & 4 \end{vmatrix}$$

## When any two rows or any two columns are equal, the determinant is zero

Say, for example, the corresponding elements in columns 1 and 3 are equal, as in the determinant below. If we subtract column 3 from column 1, column 1 becomes a column of zeros. Hence, the value of the determinant must be zero.

$$\begin{vmatrix} 4 & 1 & 4 \\ 2 & 3 & 2 \\ 3 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ 0 & 3 & 2 \\ 0 & -5 & 3 \end{vmatrix} = 0$$

**Example 4** Evaluate $\begin{vmatrix} 2 & 0 & 0 & 7 & 0 \\ 0 & 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 3 & 4 \\ 14 & 2 & 3 & 3 & 3 \\ 2 & 0 & 2 & 12 & 2 \end{vmatrix}$

SOLUTION

Evaluating this determinant by normal expansion would be very time consuming. However, we notice that columns 3 and 5 are identical, and so the value of the determinant is 0.

## Multiplying any row, or any column, by $k$, multiplies the value of the determinant by $k$

If we multiply all the elements of one row (or column) by $k$, this is the same as multiplying the value of the determinant by $k$. For example, we have

$$\begin{vmatrix} a & kb & c \\ d & ke & f \\ g & kh & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

If we multiply **every element** in the determinant by $k$, we obtain

$$\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix}$$