

**Example 16** Let  $T$  be reflection in the  $x$ -axis. Find the transformation matrix of  $T$  with respect to these bases:

a)  $\mathbf{e}_1 = \mathbf{i}$  and  $\mathbf{e}_2 = \mathbf{j}$

b)  $\mathbf{e}_1 = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$  and  $\mathbf{e}_2 = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

**SOLUTION**

a) When  $\mathbf{e}_1 = \mathbf{i}$  and  $\mathbf{e}_2 = \mathbf{j}$ ,  $\mathbf{e}_1$  is unchanged by reflection in the line  $y = 0$ , and  $\mathbf{e}_2$  is moved to  $-\mathbf{e}_2$ . So, we have

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b) When  $\mathbf{e}_1 = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ ,  $\mathbf{e}_1$  is reflected to  $\mathbf{e}'_1 = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ . We have to write this in terms of  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , so we solve the equation

$$\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} = a\mathbf{e}_1 + b\mathbf{e}_2 = a\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) + b\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$$

Equating components of  $\mathbf{i}$  and  $\mathbf{j}$ , we find that

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}a - \frac{1}{2}b$$

$$-\frac{1}{2} = \frac{1}{2}a + \frac{\sqrt{3}}{2}b$$

Solving these, we obtain  $a = \frac{1}{2}$ ,  $b = -\frac{\sqrt{3}}{2}$ .

Therefore, we have

$$\mathbf{T}(\mathbf{e}_1) = \frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2$$

Similarly, we can find

$$\mathbf{T}(\mathbf{e}_2) = -\frac{\sqrt{3}}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2$$

Therefore, the transformation matrix is

$$\mathbf{T} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

### Linear transformations of the set of all complex numbers

The complex conjugate mapping  $z \rightarrow z^*$  is a linear transformation of the set of all complex numbers regarded as a real vector space. This is because  $z \rightarrow z^*$  corresponds to a reflection in the real axis of an Argand diagram, which corresponds to a reflection in the line  $y = 0$  in  $\mathbb{R}^2$ .

**Example 17** Find the transformation matrix for the mapping

$$f : z \rightarrow (1+i)z + (2-i)z^*$$

with respect to the basis  $\{1, i\}$ .

**SOLUTION**

We put  $e_1 = 1$  and  $e_2 = i$ . (This reminds us to think of 1 and  $i$  as **basis vectors**.)

We now calculate the effect of the transformation on each vector.

$$f(e_1) = f(1) = (1+i) + (2-i) = 3 = 3e_1$$

$$f(e_2) = f(i) = (1+i)i + (2-i)(-i) = -2 - i = -2e_1 - e_2$$

Therefore, with respect to the basis  $\{1, i\}$ , the transformation matrix is

$$\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix}$$

## Exercise 17C

1 The vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -2 \\ -8 \end{pmatrix}$$

Prove that  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are linearly dependent.

2 What is the span of the three vectors  $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$  in  $\mathbb{R}^3$ ?

Find two vectors with the same span.

3 The vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 \\ 9 \\ 13 \end{pmatrix}$$

Prove that  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are linearly independent.

4 The vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  in  $\mathbb{R}^3$  are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Prove that  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are linearly dependent. Find a vector  $\mathbf{d}$  which, together with  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ , spans  $\mathbb{R}^3$ .

5 Find the transformation matrix (with respect to the standard basis) of the linear transformation which takes  $(1, 0)$  to  $(2, 1)$  and  $(1, 1)$  to  $(3, 4)$ .

Now find the transformation matrix with respect to the basis  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

- 6** The linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- a) Describe all the points in the image of  $T$ .  
 b) Write down the dimension of the image of  $T$ .  
 c) Find a spanning set for the image of  $T$ .

- 7** Certain sets of functions can be viewed as real vector spaces. For example, consider the set

$$T := \left\{ \text{functions } | f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = a_0 + \sum_{n=0}^{\infty} [b_n \sin nx + a_n \cos nx] \right\}$$

'Vectors' in the vector space  $T$  are really functions.

- a) Prove that  $T$  forms a real vector space.  
 b) Prove that the vectors  $\sin x$  and  $\sin 2x$  are linearly independent.

- 8** Let  $V$  be a real vector space with operations  $+$  and  $\cdot$ . A set  $U$ , with operations  $+$  and  $\cdot$ , is said to be a subspace of  $V$  if the following conditions hold:

- $U$  contains 0.
- $U$  is closed under addition.
- $U$  is closed under scalar multiplication.

- a) Prove that if  $U$  is a subspace of  $V$ , then  $U$  is also a vector space.  
 b) What is the smallest possible subspace of  $\mathbb{R}^3$ ?

- 9** The set  $\mathbb{C}$  of complex numbers can be regarded as a real vector space, where addition of complex numbers, and multiplication of a complex number by a real scalar, are defined in the usual way.

- i) Show that  $\{1, j\}$  is a basis for this vector space.

Let  $u = a + bj$  and  $v = c + dj$  (where  $a, b, c$  and  $d$  are real) be fixed complex numbers.

A mapping  $T : \mathbb{C} \rightarrow \mathbb{C}$  is defined by  $T(z) = uz + vz^*$  (where  $z^*$  is the complex conjugate of  $z$ ).

- ii) Show that  $T$  is a linear mapping.  
 iii) Find the matrix  $\mathbf{M}$  associated with  $T$  and the basis  $\{1, j\}$ .

- iv) Given that  $\mathbf{M} = \begin{pmatrix} 6 & 1 \\ 11 & 2 \end{pmatrix}$ , find the complex number  $z$  for which  $uz + vz^* = 1 + 4j$ .

(MEI)

# Answers

## Exercise 1A

- 1 a)  $-i$  b) 1 c)  $-1$  d)  $i$  2 a)  $3 + 2i$  b)  $6 - 3i$  c)  $-4 + 3i$  d)  $-2 + 2\sqrt{2}i$  e)  $2i$  3 a)  $3 - 4i$  b)  $2 + 6i$   
 c)  $-4 + 3i$  d)  $-8 - 5i$  4 a)  $-1 \pm \sqrt{3}i$  b)  $\frac{3}{2} \pm \frac{\sqrt{15}}{2}i$  c)  $\frac{1}{4}(-1 \pm \sqrt{7}i)$  d)  $1 \pm \frac{\sqrt{2}}{2}i$  5 a)  $10 - 2i$  b)  $1 - i$  c)  $-1 + 2i$   
 d)  $2 + 55i$  e)  $1 + i$  f)  $3 + 8i$  g)  $10 + 18i$  h)  $18 + 13i$  6 a)  $3 + 11i$  b)  $26 + 2i$  c)  $74 + 7i$  d)  $42 - 24i$  e)  $10 + 11i$   
 f)  $11 - 29i$  7 a)  $\frac{1}{17}(5 + 14i)$  b)  $\frac{1}{26}(23 + 11i)$  c)  $1 - 2i$  d)  $\frac{1}{13}(4 - 19i)$  8 a)  $x = 4, y = -2$  b)  $x = -11, y = 22$   
 c)  $x = 8, y = -1$  d)  $x = 17, y = -17$  e)  $x = \frac{13}{5}, y = \frac{9}{5}$  f)  $x = -5, y = -12$  9  $3.3 + 0.9i$  10 a)  $-2 + \sqrt{3}i$  b)  $-1 \pm \sqrt{5}i$   
 c)  $\frac{1}{2}(-3 \pm 3i)$  d)  $\frac{5}{2}(1 \pm \sqrt{3}i)$

## Exercise 1B

- 2 a)  $2\sqrt{2}, \pi/4$  b)  $3\sqrt{2}, 3\pi/4$  c)  $4, 2\pi/3$  d)  $\sqrt{2}, -3\pi/4$  e)  $4, \pi/2$  f)  $13, \tan^{-1}\left(\frac{12}{5}\right)$  g)  $4, \pi$  h)  $7, \tan^{-1}\left(\frac{\sqrt{13}}{6}\right)$   
 3 a) i)  $-7 + 24i$  ii)  $-117 + 44i$  b) i) 5 ii) 25 iii) 125 c) i) 0.9273 ii) 1.8546 iii) 2.7819 4 a)  $1 + \sqrt{3}i$  b)  $2\sqrt{2} + 2\sqrt{2}i$   
 4 c)  $-i$  d)  $-2\sqrt{2} + 2\sqrt{2}i$  e)  $-\sqrt{3} + i$  f)  $-3\sqrt{3} - 3i$  5 a)  $-\frac{1}{5} + \frac{2}{5}i$  b)  $13, \pi - \tan^{-1}\left(\frac{12}{5}\right)$  or 1.9656 6  $\frac{5}{13}, 2.1033$   
 7 a)  $-\frac{1}{5} + \frac{3}{5}i$  b)  $\frac{1}{5}\sqrt{10}, \pi - \tan^{-1} 3$  or 1.8925 8 i) b)  $3\pi/4$  ii)  $\pm(5 - 3i)$  9 i)  $\sec \alpha$  ii)  $4 \sec \alpha$  iii)  $\pi/2 - \alpha$  iv)  $2\pi/5 - \alpha$   
 10 i)  $\sin \alpha$  ii)  $\pi/2 - \alpha$  11 a)  $3 - 4i$  c) 1.95 d)  $6 + 14i$  12 a)  $z = -2 \pm \sqrt{3}i$  c) i) 2.65 ii)  $\pm 2.43$  13  $2 - 3i$  14  $\frac{6}{5} + \frac{12}{5}i$   
 15 b) 5 c)  $\frac{1}{13}(7 + 17i)$  d) 1.18 e)  $-7, -5$  16 a)  $-11 - 2i, 1 + 2i$  c) 2.2 d)  $-1, -2$  17 a)  $2\sqrt{2}, 3\pi/4$  b)  $1/2\sqrt{2}, -3\pi/4$   
 17 d)  $90^\circ$  18 a) 2.68 rad b) 6, 4 d) 28 19 i)  $\pm(3 + 2i)$  ii)  $-\frac{119}{169}$  20 a)  $14 + 2i, 1 + i$  b) i)  $2\sqrt{5}, 0.464$  rad;  $\sqrt{10}, -0.3218$  rad  
 20 b) ii)  $\sqrt{10}$

## Exercise 1C

- 6 a)  $-1$  b) 2 c) 0 7 a) 0 or 3 b) 2 c)  $w^2$  d)  $-1$  or 2 8  $3, \frac{3}{2} + \frac{1}{2}\sqrt{3}i, \frac{3}{2} - \frac{1}{2}\sqrt{3}i$  10 b)  $5i, 3\sqrt{2}$  11 i) 2 ii)  $5\pi/6$   
 14 2 15 i) a)  $\frac{1}{2}(1 + i)$  b)  $\frac{1}{2}(1 - i)$  ii)  $(1 - w)/w$  16 b) i)  $3/2, \sqrt{3}/2$  ii)  $\pi/6$  17 a)  $\pi/4, -4(1 + a)^4$  b)  $y = x$

## Exercise 2A

- 1  $n\pi + (-1)^n \frac{\pi}{4}$ ,  $180n^\circ + (-1)^n 45^\circ$  2  $2n\pi \pm \frac{2\pi}{3}$ ,  $360n^\circ \pm 120^\circ$  3  $n\frac{\pi}{2} + (-1)^n \frac{\pi}{12}$ ,  $90n^\circ + (-1)^n 15^\circ$  4  $n\frac{\pi}{3} + \frac{\pi}{12}$ ,  $60n^\circ + 15^\circ$   
 5  $n\pi + \frac{\pi}{8}$  6  $\frac{2}{3}n\pi + \frac{2\pi}{9}$ ,  $\frac{2}{3}n\pi$  7  $n\frac{\pi}{2} - \frac{\pi}{24}$  8  $n\frac{\pi}{4} + \frac{\pi}{16}$ ,  $45n^\circ + 11.25^\circ$  9  $\frac{2}{3}n\pi + \frac{\pi}{3}$ ,  $120n^\circ + 60^\circ$   
 10  $n\pi + \frac{\pi}{2}$ ,  $2n\pi \pm \frac{\pi}{3}$ ;  $180n^\circ + 90^\circ$ ,  $360n^\circ \pm 60^\circ$  11  $n\pi + \frac{\pi}{2}$ ,  $180n^\circ + 90^\circ$  12  $n\frac{\pi}{2}, \frac{2}{3}n\pi \pm \frac{\pi}{9}$ ;  $90n^\circ, 120n^\circ \pm 20^\circ$  13  $n\pi, 180n^\circ$   
 14  $2n\pi + \frac{3\pi}{2}$ ,  $n\pi + (-1)^n 0.848$ ;  $360n^\circ + 270^\circ$ ,  $180n^\circ + (-1)^n 48.59^\circ$  15  $n\frac{\pi}{2} - \frac{\pi}{8}, n\frac{\pi}{5} - \frac{\pi}{20}$ ;  $90n^\circ - 22.5^\circ, 36n^\circ - 9^\circ$  16  $n\pi, n\pi \pm \frac{\pi}{6}$   
 17  $n\pi + \frac{\pi}{2}$ ,  $2n\pi + (-1)^n \frac{\pi}{6}$  18  $\frac{\pi}{3} + 2n\pi, \frac{2}{3}n\pi - \frac{\pi}{9}$  19  $\frac{2t}{1-t^2}; n\pi, n\pi \pm \frac{\pi}{3}$

## Exercise 2B

- 1 a)  $13, 67.3^\circ$  b)  $5, 53.13^\circ$  c)  $5, 53.13^\circ$  d)  $\sqrt{2}, 45^\circ$  e)  $10, 53.13^\circ$  2 a) i)  $15, -15$  ii)  $306.87^\circ, 143.13^\circ$   
 2 b) i)  $10, -10$  ii)  $18.43^\circ, 108.43^\circ$  c) i)  $\frac{4}{3}, \frac{4}{13}$  ii)  $53.13^\circ, 233, 13^\circ$  d) i)  $\frac{3}{4 - \sqrt{5}}$ ,  $\frac{3}{4 + \sqrt{5}}$  ii)  $296.56^\circ, 116.56^\circ$

**2 e) i)**  $-\frac{1}{2}$  (max),  $\frac{1}{3}$  (min)   **ii)**  $233.13^\circ, 53.13^\circ$    **f) i)**  $-\frac{3}{2}$  (max),  $\frac{1}{6}$  (min)   **ii)**  $216.87^\circ, 36.87^\circ$    **3 a)**  $360n^\circ + 53.13^\circ \pm 60^\circ$

**3 b)**  $360n^\circ - 22.62^\circ \pm 60^\circ$    **c)**  $180n^\circ - 7.5^\circ, 180n^\circ + 52.5^\circ$    **d)**  $60n^\circ + 15^\circ + (-1)^n 10^\circ$    **e)**  $60n^\circ, 60n^\circ - 17.7^\circ$

### Exercise 2C

**1 a)**  $-\pi/6$    **b)**  $\pi/6$    **c)**  $5\pi/6$    **d)**  $\pi/4$    **e)**  $\pi/4$    **f)**  $18.4^\circ$  or  $0.3218$  rad   **3**  $\pi/10$    **5**  $360n^\circ - 66.8^\circ \pm 142.0^\circ$    **6 a)** 13   **b)**  $67.4^\circ$

**6 c)**  $360n^\circ - 67.4^\circ \pm 72.1^\circ$    **7 i)**  $\sqrt{58} \cos(\theta + 23.2^\circ)$    **ii)**  $\sqrt{58}, -\sqrt{58}$    **iii)**  $360n^\circ - 23.2^\circ \pm 82.5^\circ$    **8 a)**  $126.9^\circ, 270^\circ$

**8 b)**  $180n^\circ + 90^\circ, 60n^\circ + 15^\circ$    **9 a)**  $25, 73.7^\circ$    **b)**  $360n^\circ + 20.6^\circ$  or  $360n^\circ + 126.8^\circ$    **c)**  $\frac{1}{630} \leq f(x) \leq \frac{1}{5}$    **10**  $60^\circ; 195^\circ, 345^\circ$

**11**  $2 \cos(\theta - 60^\circ), 360n^\circ + 60^\circ \pm 40^\circ$    **12 a)**  $85, 0.154$  rad   **b)**  $2\pi n - 0.154 \pm 1.369$    **13 a)** 25   **b)**  $-25$    **c)**  $1.85$  rad

**13 d)**  $3.84$  rad,  $6.16$  rad   **14 i)** **a)**  $15, 53.13^\circ$    **b)**  $156.9^\circ, 276.9^\circ$    **ii)**  $n\pi + \pi/3$

### Exercise 2D

**1 (a)**  $\frac{5}{\sqrt{1-25x^2}}$    **b)**  $\frac{3}{1+9x^2}$    **c)**  $\frac{\sqrt{2}}{\sqrt{1-2x^2}}$    **d)**  $\frac{12}{16+9x^2}$    **e)**  $\frac{2x}{\sqrt{1-x^4}}$    **f)**  $\frac{1-x^2}{1+3x^2+x^4}$    **g)**  $\frac{6(\sin^{-1}2x)^2}{\sqrt{1-4x^2}}$    **h)**  $\frac{1620(\tan^{-1}5x)^3}{1+25x^2}$

**1 i)**  $\frac{1}{x\sqrt{x^2-1}}$    **j)**  $-\frac{1}{x^2+1}$    **2 a)**  $\sin^{-1}\left(\frac{x}{2}\right) + c$    **b)**  $\sin^{-1}\left(\frac{x}{3}\right) + c$    **c)**  $\frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right) + c$    **d)**  $\frac{1}{9}\sin^{-1}\left(\frac{3x}{4}\right) + c$    **e)**  $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

**2 f)**  $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$    **g)**  $\frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right) + c$    **h)**  $\frac{1}{15}\tan^{-1}\left(\frac{5x}{3}\right) + c$    **3 a)**  $\pi/2$    **b)**  $\pi/8$    **c)**  $\pi/2$    **d)**  $\pi/6\sqrt{3}$    **e)**  $\pi/5$

**4 a)** 0.0505   **b)** 0.0444   **c)** 0.615   **d)** 0.0741   **e)** 0.841   **f)** 0.0207   **5**  $\pi/24$    **6**  $9 - (x-2)^2, \pi/3$    **7**  $2 + \frac{1}{1+x} + \frac{2x+1}{x^2+4}$

**8**  $\frac{\sqrt{3}}{3} - \frac{\pi}{12}$    **9**  $\frac{y+x}{1+x^2y^2} \frac{dy}{dx}$    **10 ii)**  $-\frac{x}{\sqrt{1-x^2}}$    **iii)**  $\frac{\pi}{2} - 1$    **11**  $-\sin^{-1}\left(\frac{1}{x}\right) + c$    **12 ii)**  $\frac{1}{2}t\sqrt{1-t^2}$

### Exercise 3B

**1 a)**  $x^2 + y^2 = 16$    **b)**  $x = 3$    **c)**  $y = 7$    **d)**  $x^2 + y^2 = ax + a\sqrt{x^2 + y^2}$    **e)**  $x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$    **f)**  $y^2 = 4 - 4x$

**2 a)**  $r = 3$    **b)**  $r^2 \sin 2\theta = 32$    **c)**  $\frac{r^2 \cos^2 \theta}{9} + \frac{r^2 \sin^2 \theta}{16} = 1$    **d)**  $r = 6 \cos \theta$    **e)**  $r^2 + 8r \sin \theta = 16$    **f)**  $r^2 = \cos 2\theta$

### Exercise 3D

**1**  $\frac{7\pi^3 a^2}{48}$    **2 a)**  $\frac{\pi a^2}{8}$    **b)**  $\frac{\pi a^2}{8}$    **c)**  $\frac{\pi a^2}{16}$    **3**  $\frac{\pi a^2}{4}$    **4**  $5\sqrt{5} + \frac{17}{2} \cos^{-1}\left(-\frac{2}{3}\right)$    **5 a)**  $r^2 = \sin^4 \theta$    **b) ii)**  $\frac{3\pi}{8}$

**6**  $\left(1, \frac{\pi}{6}\right), \left(1, \frac{5\pi}{6}\right); \frac{7\pi}{3} - 4\sqrt{3}$    **7 b)**  $\frac{\pi a^2}{12}$    **8 b)**  $\frac{9\pi a^2}{2}$

### Exercise 3E

**1**  $y = 0.185a, y = \pm 0.88a$    **2 a)**  $y = \frac{e^{-\pi/4}}{\sqrt{2}}, y = \frac{e^{3\pi/4}}{\sqrt{2}}, \dots$    **b)**  $x = \frac{e^{-3\pi/4}}{\sqrt{2}}, x = \frac{e^{\pi/4}}{\sqrt{2}}, \dots$    **3 a)**  $y = \pm \frac{2}{3\sqrt{6}}$    **b)**  $x = \pm a$

**4 b)**  $\frac{\pi}{16}$    **c) iii)**  $\frac{16}{27}$    **5**  $\sqrt{3} \cos \theta - \cos^2 \theta; 0, \frac{\pi}{6}, -\frac{\pi}{6}, \pi$    **6 b)**  $(0.667, 0.421), (0.667, -0.421)$    **7 b)**  $a^2 \left(2 - \frac{\pi}{4}\right)$

**8 b)**  $(0, 0), \left(3, \frac{\pi}{3}\right)$    **c)** 1.33   **9 a)**  $r = 2$    **b)**  $r \cos \theta = 3$    **c)**  $r \sin\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3}$

### Exercise 4A

**1 a)**  $x^2$    **b)**  $\sqrt{x^2+1}$    **c)**  $\frac{1}{x^3}$    **d)**  $\sec x$    **e)**  $\sqrt{x^2-1}$    **f)**  $2^{3x}$    **2**  $y = \frac{1}{3}x - \frac{1}{9} + ce^{-3x}$    **3**  $y = -\frac{1}{3}e^{2x} + ce^{5x}$    **4**  $y = \frac{x^2}{3} + \frac{c}{x}$

**5**  $y = x^3 + cx^2$    **6**  $y = 5(x-1)^4 \ln(x-1) + c(x-1)^4$    **7**  $y \sin x = \frac{2}{5}e^{2x} \sin x - \frac{1}{5}e^{2x} \cos x + c$    **8 a)**  $ye^x = x + c$

**8 b) i)**  $y = (x-1)e^{-x}$    **ii)**  $y = 0$    **9**  $y = (c + \frac{1}{2}x^2)e^{x^3}, y = (1 + \frac{1}{2}x^2)e^{x^3}$    **10**  $y = x \cos x + c \cos x$

**11**  $y = \frac{1}{5}x + cx^{-4}, y = \frac{1}{5}(x+4x^{-4})$    **12**  $y = x^2 - x + cx^{-4}$    **13 i)**  $y = \frac{1}{\alpha+\beta}e^{\beta t} + ce^{\alpha t}$    **14**  $v = 200 \tan 0.2 + \sec 0.2$

- 15 i)**  $y \sec x = x + c$    **ii)**  $y = (x+2)\cos x$    **16**  $y = (3x^4 + 5)e^{-x^2-x}$    **17**  $N = \frac{\mu}{\lambda} t + \frac{\mu}{\lambda^2} + \left( N_0 - \frac{\mu}{\lambda^2} \right) e^{\lambda t}$    **18 i)**  $y = ce^{kx} - x - \frac{1}{k}$   
**18 ii)** **a)**  $y = -x - \frac{1}{4} - \frac{3}{4}e^{4(x-1)}$    **19 ii)**  $y \sin^2 x = -\frac{1}{2} \cos 2x + c$    **20**  $s = \frac{1}{2} \tan^{-1} \left( \frac{t}{2} \right) + c, s = \frac{1}{2} \tan^{-1} \left( \frac{t}{2} \right) - \frac{\pi}{8}$   
**21 a)**  $y = cx - xe^{-x}$    **b)** **i)**  $y = -xe^{-x}$    **ii)** 0

**Exercise 4B**

- 1**  $y = e^{(3+\sqrt{17})x} + Be^{(3-\sqrt{17})x}$    **2**  $y = Ae^{-x} + Be^{-2x}$    **3**  $y = Ae^{2x} + Be^{-3/2x}$    **4**  $y = Ae^x + Be^{-7x/3}$    **5**  $x = Ae^{8t} + Be^{-t}$   
**6**  $x = Ae^{7t} + Be^{4t}$    **7**  $y = (A+Bx)e^{2x}$    **8**  $y = (A+Bx)e^{-3x}$    **9**  $y = e^{-x/2} \left[ A \cos \left( \frac{\sqrt{3}}{2}x \right) + B \sin \left( \frac{\sqrt{3}}{2}x \right) \right]$   
**10**  $y = e^{-2x}(A \cos 2x + B \sin 2x)$    **11**  $x = Ae^{(3+\sqrt{2})t} + Be^{(3-\sqrt{2})t}$    **12**  $x = e^{-t}(A \cos 2\sqrt{3}t + B \sin 2\sqrt{3}t)$

**Exercise 4C**

- 1**  $y = Ae^{-8x} + Be^x - 2x - \frac{7}{4}$    **2**  $y = Ae^{-x} + Be^{-3x} - 4e^{-2x}$    **3**  $y = Ae^{-x} + Be^{5x/2} - 2x^2 + \frac{12}{5}x - \frac{81}{25}$   
**4**  $y = Ae^{-x} + Be^{x/3} - \frac{304}{5876} \sin 5x - \frac{40}{5876} \cos 5x$    **5**  $x = Ae^{5t} + Be^{-t} - \frac{3}{8}e^{3t}$   
**6**  $s = Ae^{5t} + Be^{3t} + \frac{55}{377} \cos 2t - \frac{80}{377} \sin 2t$    **7**  $y = Ae^{-4x} + Be^{-x} + \frac{2}{3}xe^{-x}$    **8**  $y = (A+Bx + \frac{5}{2}x^2)e^{3x}$   
**9**  $y = e^x(A \cos \sqrt{2}x + B \sin \sqrt{2}x) + 2e^{4x}$    **10**  $y = e^{-3x}(A \cos x + B \sin x) + \frac{3}{2}e^{-4x}$    **11**  $x = (A+Bt+2t^2)e^t$   
**12**  $x = A \cos 4t + B \sin 4t + \frac{3}{8}t \sin 4t$    **13**  $x = (2 \sin 2t - 3 \cos 2t)e^t$    **14 a)**  $y = Ae^{2x} + Be^{-2x} + 2e^{3x}$    **b)**  $y = e^{-2x} - 5e^{2x} + 2e^{3x}$   
**15**  $y = A+Bx + \frac{1}{4}e^{2x} - 4 \cos \frac{1}{2}x; y(0)$  and  $y'(0)$ , or  $y$  for two values of  $x$    **16 i)**  $y = e^{-2x}(4 \cos 3x + 3 \sin 3x)$    **ii)**  $\frac{3}{2} - \frac{\pi}{24}$   
**17**  $y = e^{2x}(A \cos x + B \sin x) + \frac{1}{65}(\sin 2x + 8 \cos 2x)$    **18 i)**  $x = A \cos 4t + B \sin 4t$    **iii)** 1.424   **19**  $x = 2t + 3 + 4e^{-t/10} \sin \left( \frac{t}{5} \right)$   
**20**  $x = Ae^{-4t} + Be^{-t} + \sin 3t$    **21**  $y = Ae^{4x} + Be^{-x} + 3 \cos 2x - 4 \sin 2x, y = 3 \cos 2x - 4 \sin 2x - 3e^{-x}$   
**22 i)**  $x = e^{2t}(A \cos 5t + B \sin 5t) + 2 \sin 2t$    **ii)**  $x = 3e^{2t} \cos 5t + 2 \sin 2t$    **23 a)**  $\ln(1+y) = c + \frac{1}{2}x^2$   
**23 b)**  $y = Ce^{3x}; y = -\frac{2}{9} - \frac{2}{3}x + e^{4x}, y = Ce^{3x} - \frac{2}{9} - \frac{2}{3}x + e^{4x}$    **24 a)**  $y = e^{-2x}(A \cos 3x + B \sin 3x)$    **b)**  $\frac{3}{4}, \frac{1}{4}$   
**24 d)**  $y = \frac{3}{4} \cos 3x + \frac{1}{4} \sin 3x - \frac{3}{4}e^{-2x}(\cos 3x + \sin 3x)$    **25 a)**  $y = Ae^{4x} + Be^{-x/2} + \sin x + 2 \cos x$    **b)**  $y = 2e^{4x} - 4e^{-x/2} + \sin x + 2 \cos x$   
**26 i)**  $x = e^{-t} \left[ A \cos \left( \frac{t}{2} \right) + B \sin \left( \frac{t}{2} \right) \right] + 2 \cos t$    **ii)** \$9 550 000   **27**  $p = 0, q = -\frac{1}{4}, y = A \sin 2x + B \cos 2x - \frac{1}{4}x \cos 2x, y \approx \frac{1}{4}(n + \frac{1}{2})\pi$   
**28 a)**  $\frac{1}{2}$    **b)**  $y = (1+t)e^{-t} + \frac{1}{2}t^2e^{-t}$

**Exercise 4D**

- 1 a)**  $x^3(x-4y) = c$    **b)**  $y^2 = 2x^2 \ln x + cx^2$    **c)**  $\ln \left[ \frac{x^4(x-y)}{x+y} \right] = \frac{2x}{x+y} + c$    **d)**  $\frac{3}{8} \ln \left[ \frac{(y-2x)(y+2x)}{y^2} \right] = \ln x + c$   
**2**  $x + y - \frac{1}{4} \ln(4x+4y+5) = 4x+c$    **3**  $2x + 3y + \frac{21}{8} \ln(16x+24y-13) = 8x+c$    **4 a)**  $y = Ax + \frac{B}{x^2}$    **b)**  $y = Ax^{3+\sqrt{15}} + Bx^{3-\sqrt{15}}$   
**4 c)**  $y = Ax^2 + Bx^2 \ln x$    **d)**  $y = x^{-\frac{1}{2}} \left[ A \cos \left( \frac{\sqrt{3}}{2} \ln x \right) + B \sin \left( \frac{\sqrt{3}}{2} \ln x \right) \right]$    **5**  $\frac{dy}{dx} = 2\sqrt{t} \frac{dy}{dt}, y = Ae^{x^2} + Be^{-4x^2} + \frac{1}{6}e^{2x^2}$   
**6 a)**  $z = \frac{1}{2}e^x + ce^{-x}$    **b)**  $y = \frac{1}{2}xe^x + cxe^{-x}$    **7 b)** 2   **c)**  $xy = 2e^4(1-x)e^{-2x} + 2e^{2x}$    **d)** Infinite  
**8**  $x = (A+Bt)e^{-t}, y = (A+B+Bt)e^{-t}; x = (1-t)e^{-t}, y = -te^{-t}$    **9 a)**  $y = \frac{(\cosh^{-1}x+c)}{\sqrt{x^2-1}}$    **b)** **ii)**  $y = \frac{1}{x}(A \sin 5x + B \cos 5x)$

**Exercise 5A**

- 1 a)** 177   **b)** 15   **c)** 0   **d)** -255   **2 a)**  $(a+b+c)(a-b)(b-c)(c-a)$    **b)** -24 pqr   **c)**  $(ab+bc+ca)(b-c)(c-a)$   
**d)** 0   **3**  $(a-b)(b-c)(c-a)(a+b+c+1), -1$

**Exercise 5B**

- 1**  $(a-b)(b-c)(c-a)(a+b+c); 1, 2, -3; (t, 0, t)$    **2 ii)**  $(5-t, t-1, t)$    **3 i)**  $\left( \frac{7t-19}{5}, \frac{37-11t}{5}, t \right)$    **4 a)**  $a=b=c$   
**4 b)**  $x = a+b, y = b-a+t, z = t$    **i)** Planes intersect in a line   **ii)** Two planes parallel   **5**  $\frac{1}{2}, -2$ ; planes form triangular prism  
**6** 10   **7**  $1-k^2, k \neq 1, -1$    **i)** 2, 1, 0

- 7 iii)  $2 + 3t, -1 - 2t, t$  Three planes meet at point; planes form triangular prism; two planes coincident  
 8 a)  $(a-b)(b-c)(c-a)(a+b+c)$  b)  $-5$  9 ii)  $x = -17, y = \frac{17}{2}, z = -1$  iv) Two parallel planes, intersecting third plane  
 10 i)  $(p+1)(q-2)$  iii) 7 iv) Sheaf of planes 11  $(-\frac{1}{3}t, \frac{2}{3}t, t)$ ; three planes intersect in line  $\frac{x}{1} = \frac{y}{-2} = \frac{z}{-3}$  12 i)  $q \neq 2$   
 12 ii)  $p = -4, q = 2$  iii)  $q = 2, p \neq -4$

**Exercise 6A**

- 1 a)  $\mathbf{r} = \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$  b)  $\mathbf{r} = \begin{pmatrix} 4 \\ 8 \\ -6 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$  c)  $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$  d)  $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$   
 2 a)  $\mathbf{r} = \begin{pmatrix} 4 \\ 8 \\ -2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 11 \\ -6 \end{pmatrix}$  b)  $\mathbf{r} = \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -11 \\ 6 \end{pmatrix}$  c)  $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ -10 \end{pmatrix}$  d)  $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -9 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 16 \end{pmatrix}$   
 3 a)  $\frac{x-2}{3} = \frac{y+7}{4} = \frac{z-5}{-7}$  b)  $\frac{x-4}{-2} = \frac{y-8}{3} = \frac{z+6}{6}$  c)  $\frac{x-7}{2} = \frac{y-4}{-1} = \frac{z+1}{-3}$  d)  $\frac{x+8}{1} = \frac{y-1}{3} = \frac{z+3}{-7}$   
 4 a)  $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}$  b)  $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} 5 \\ -7 \\ -2 \end{pmatrix}$  c)  $\mathbf{r} = \begin{pmatrix} -5 \\ 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$  d)  $\mathbf{r} = \begin{pmatrix} \frac{3}{2} \\ 5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$   
 4 e)  $\mathbf{r} = \begin{pmatrix} \frac{5}{3} \\ -2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$  5 a)  $109^\circ$  b)  $93.3^\circ$  6 a)  $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$  b)  $\mathbf{r} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 12 \\ 1 \end{pmatrix}$   
 6 c)  $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$  7  $2\sqrt{2}$  8  $19/5\sqrt{2}$  9 a) i)  $1:2:-2$  ii)  $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$  b) i)  $3:-4:-5$  ii)  $3/5\sqrt{2}, -4/5\sqrt{2}, -1/\sqrt{2}$   
 9 c) i)  $3:2:-5$  ii)  $3/\sqrt{38}, 2/\sqrt{38}, -5/\sqrt{38}$  d) i)  $1:-2:-3$  ii)  $1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14}$   
 10 a)  $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$  d)  $18.7^\circ$  11 b)  $\mathbf{r} = -9\mathbf{j} + 13\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$  c)  $(5, 1, -2)$  e)  $43^\circ$  f)  $(4.5, 0, -0.5)$   
 12 a)  $(4, -1, -3)$  b)  $71.4^\circ$  13 a)  $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$  b)  $21^\circ$  14  $47\mathbf{i} - 9\mathbf{j} + 62\mathbf{k}, a = -5$  15 a)  $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$  c)  $95.2^\circ$   
 16 i)  $(2, 3, 5)$  ii)  $40.9^\circ$  17 a)  $0.148$  rad 18  $(1, 1, -12)$  19 i)  $\mathbf{r} = \begin{pmatrix} 7 \\ -8 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$  ii)  $5\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$

**Exercise 6B**

- 1 a)  $-5\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$  b)  $31\mathbf{i} + 22\mathbf{j} + \mathbf{k}$  c)  $22\mathbf{i} + 14\mathbf{j} + 16\mathbf{k}$  d)  $-32\mathbf{i} + 23\mathbf{j} - 10\mathbf{k}$  2 a)  $\mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = -13$  b)  $\mathbf{r} \cdot \begin{pmatrix} 9 \\ 7 \\ -2 \end{pmatrix} = 47$   
 2 c)  $\mathbf{r} \cdot \begin{pmatrix} 28 \\ -17 \\ 18 \end{pmatrix} = 41$  3 a)  $3x + y + 7z = 4$  b)  $2x + 4y + 3z = 8$  c)  $-x + 5y + 3z + 7 = 0$  4 a)  $68.5^\circ$  b)  $34.1^\circ$  c)  $28.1^\circ$   
 4 d)  $48.5^\circ$  5  $29.1^\circ$  6  $0^\circ$  7  $\mathbf{r} \cdot \begin{pmatrix} 3/5\sqrt{2} \\ 4/5\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = 2\sqrt{2}, 2\sqrt{2}$  8 a) i)  $-13$  ii)  $-12\mathbf{i} + 8\mathbf{j} + 8\mathbf{k}$  b) i)  $+\frac{13}{21}$  ii)  $2\sqrt{17}$   
 8 b) iii)  $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = 3$  9 a)  $-\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$  b)  $3\mathbf{i} + \mathbf{j} - \mathbf{k}$  d)  $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 13 \\ 25 \end{pmatrix}$  10 a)  $\mathbf{r} = 24\mathbf{i} + 6\mathbf{j} + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$   
 10 c)  $(35, 17, 32)$  d)  $18\sqrt{101}$  11 a)  $36\mathbf{i} + 12\mathbf{j} + 9\mathbf{k}$  b)  $\mathbf{r} \cdot \begin{pmatrix} 36 \\ 12 \\ 9 \end{pmatrix} = 9$  c)  $(1, -3, 1)$  12 i)  $-\mathbf{i} + \mathbf{j} + \mathbf{k}$  13  $\mathbf{r} \cdot \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix} = 14$   
 14  $\mathbf{r} = \begin{pmatrix} 1 \\ -8 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  i)  $-x + 5y + 3z = 46$  iii)  $12\mathbf{i} + 14\mathbf{j} - 4\mathbf{k}$  15 i)  $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  iv)  $31.8^\circ$  16  $3\mathbf{i} + 8\mathbf{k} - 15\mathbf{j}, 14.4^\circ$

17 ii)  $3x + y + 2z = 15$  iii)  $5 - x = y = z$  18 a)  $2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$  b)  $\frac{1}{2}\sqrt{17}$  c)  $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = -7$  d)  $2x - 3y - 2z = -7$

18 e)  $\frac{7}{\sqrt{17}}$  f)  $3.2^\circ$  19 i)  $3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$  20 i)  $(2, 1, 1)$  ii)  $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$  iii)  $8\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$  iv)  $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -47 \\ 22 \end{pmatrix}$

21  $-2\mathbf{a} \times \mathbf{b}; 0, \pi$  22 b)  $7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$  c)  $72.8^\circ$  d)  $(-3, \frac{1}{2}, 4\frac{1}{2})$  e)  $\frac{1}{2}\sqrt{26}$  23 a)  $\frac{x}{1} = \frac{y-2}{-1} = \frac{z+3}{-1}$  b)  $x+z=0$

24 a)  $(1, -2, 3)$  b)  $2x+y-3z=-9$  c)  $70.9^\circ$  25 ii)  $7 + \frac{2}{3}k$  iii)  $-\frac{21}{2}$  iv)  $\mathbf{r} = \begin{pmatrix} 4 \\ 12 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 10 \\ 11 \end{pmatrix}$  26 ii)  $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

26 ii)  $x-y-z=12$  27 a)  $(-3, 0, 1)$  b)  $-3, -1$  c)  $x-3y-z+4=0$  28 i)  $-2:4, 2x-y-z=0$  29 a) i)  $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

29 a) iii)  $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 14$  b) i)  $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  iii) Opposite side of plane to origin; distance 1 from  $\Pi$

30 a) ii)  $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1$  b) ii)  $3x+5y+3z+1=0$  31 a)  $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$  b)  $(3, 3, 4)$  c)  $5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$  d)  $\sqrt{\frac{35}{34}}$

31 e)  $(5, 4, 7)$  32 i)  $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \frac{\sqrt{65}}{3}$  ii)  $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, 50.8^\circ$  iii)  $\frac{3}{\sqrt{10}}$  33 i) 3 ii)  $\begin{pmatrix} -17 \\ -10 \\ 14 \end{pmatrix}$  iii)  $\frac{21}{\sqrt{65}}$

34  $3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}, 3x-5y-7z=1$  35  $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, 3x-2y-2z=-1$  36 b) i)  $(4, 0, 0)$  iv)  $(2, 1, 4)$  v) 8

### Exercise 6C

1  $-73$  2  $177$  3  $21$  4  $8$  5 i)  $0$  ii)  $\overrightarrow{EF}$  iii)  $\sqrt{\frac{2}{3}}$  6 a)  $-30\mathbf{i} - 15\mathbf{j} + 45\mathbf{k}$  b)  $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$  d)  $35$

7 a)  $5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$  b)  $100$  c)  $50$  8 a)  $-6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$  b)  $\frac{1}{2}\sqrt{65}, \frac{1}{6}$  c)  $\mathbf{r} \cdot (6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 1$  d)  $(\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$  e)  $9.52^\circ$

### Exercise 7A

7 a)  $y \leq 1, y \geq \frac{49}{25}$  b)  $y \geq 1, y \leq \frac{5}{13}$  8  $x = -\frac{2}{3}, x = \frac{5}{7}, y = \frac{4}{21}$  9  $x = -1, y = x - 2$  10 i)  $x = -2$  ii) 7

11 i)  $x = 2, x = -2, y = 10$  ii)  $\frac{16}{(x-2)^3} - \frac{54}{(x+2)^3}$  iii)  $(10, 8\frac{3}{4})$  12 a)  $y = 1, x = 2\sqrt{3} - 1, x = -1 - 2\sqrt{3}$  b)  $(5, \frac{5}{6}), (1, \frac{1}{2})$

13 a) i)  $x = -\frac{1}{2}, y = \frac{x}{2}$  ii)  $(0, 0), (-1, -1)$  14 a)  $\frac{5}{4}$  b) i)  $x = 2$  ii)  $y = 2x + 1$  c)  $(3, 8)$  min 15 i)  $y = 2 + \frac{3}{x-1} + \frac{1}{x+2}$

15 iii)  $y = 2, x = 1, x = -2$  iv)  $(-\frac{5}{4}, 2)$  16 a)  $\frac{2t}{(1+t)^2}$  c)  $y = \frac{x}{2}, x = 0$  d) ii)  $(1, 1), (-1, -1)$

### Exercise 7B

1 a)  $x > -1, x < -2$  b)  $x > 3$  c)  $-10 < x < -3$  d)  $x > 5, x < -14$  e)  $-\frac{3}{10} > x > -\frac{1}{2}$  f)  $\frac{1}{5} < x < \frac{6}{11}$

2 a)  $x < -2, -1 < x < 0$  b)  $x > 8, 2 < x < 3$  c)  $5 < x < \frac{1}{2}(3 + \sqrt{65}), \frac{1}{2}(3 - \sqrt{65}) < x < -1$  d)  $-7 < x, 3 < x < \frac{44}{13}$

2 e)  $-2 < x < -1\frac{1}{2}$  3 a)  $-\frac{5}{2} < x (x \neq -2)$  b)  $-5 < x < -1 (x \neq -2)$  c)  $\frac{5}{3} < x < 11 (x \neq 4)$  d)  $x > 6, -\frac{4}{3} > x (x \neq -5)$

3 e)  $x > 5, x < -\frac{3}{5} (x \neq -2)$  f)  $x > -2\frac{1}{2} (x \neq -3)$  4 a)  $1 > x > -2$  b)  $x > 1, x < -\frac{3}{2}$  c)  $x < -2 (x \neq -1)$

5  $x > 4, -3 < x < 0$  6  $x > 1, x < -1$  7  $x > 2, -1 < x < 1$  8  $-2 < x < -1, 1 < x < 4$  9  $2 < x < 5, x < -1$

11 a)  $f(x) \leq \frac{1}{6}, f(x) \geq \frac{3}{2}; (-4, \frac{3}{2})(6, \frac{1}{6})$  b) One 12  $x < -\frac{3}{2}, x > \frac{7}{4}$  13  $x > -1, x < -\frac{5}{2}$  14 1, 2, 4;  $y = x + 2$

15 a)  $(0, 3), (-\frac{3}{2}, 0)$  b)  $x > -\frac{13}{6}$

### Exercise 8A

1 a)  $-3, -7$  b)  $11, 5$  c)  $-5, -4$  d)  $-\frac{11}{3}, \frac{2}{3}$  e)  $-2, -5$  f)  $-2, -\frac{7}{2}$  2 a)  $x^2 - 7x + 15 = 0$  b)  $x^2 + 3x + 5 = 0$

2 c)  $x^2 + 2x - 4 = 0$  d)  $x^2 + 5x - 11 = 0$  3 a) 0 b)  $-10$  c)  $-9$  4 i)  $9+i$  ii)  $\frac{5}{2} - 2i$  5 i)  $-10 - 37i$  ii)  $-\frac{8}{3} - \frac{7}{3}i$

6  $x^3 - 4x^2 - 4x - 25 = 0$  7  $6, -6$  8  $3x^2 - 11x + 10 = 0$  9 a) 15

**Exercise 8B**

- 1  $x^2 + 14x + 44 = 0$    2  $x^2 - 45x + 63 = 0$    3  $3x^3 - 8x^2 + 32x - 56 = 0$    4  $8x^3 - 12x^2 - 22x + 5 = 0$    5  $4x^2 + 59x + 289 = 0$   
 6  $9x^2 + 41x + 225 = 0$    7 a)  $x^2 + 7x + 6 = 0$    b)  $6x^2 + 7x + 1 = 0$    c)  $4x^2 - 37x + 9 = 0$    d)  $2x^2 - x - 3 = 0$   
 8 a)  $3x^2 + 36x - 32 = 0$    b)  $6x^2 + 9x - 1 = 0$    c)  $9x^2 - 93x + 4 = 0$    d)  $3x^2 + 27x + 52 = 0$    9 a)  $x^3 + 9x^2 + 45x + 189 = 0$   
 9 b)  $x^3 + x^2 - 17x - 49 = 0$    c)  $x^3 - 6x^2 + 14x - 8 = 0$    10  $x^4 + 9x^3 + 63x^2 - 297x + 81 = 0$    11  $x^2 + 10x + 75 = 0$   
 12  $4x^2 - 3x + 1 = 0$

**Exercise 8C**

- 1 i,  $-\frac{5}{2} \pm \sqrt{\frac{21}{2}}$    2  $2i, -2i, 2, -\frac{5}{3}$    3 1   4 3   5  $0 < k < 4$    6  $1 - i, \frac{3}{2} \pm \frac{i\sqrt{11}}{2}$    7 b)  $3 - i, -11$    8 b) i)  $2 + 3i, -1$  (twice)  
 8 b) ii)  $(z^2 - 4z + 13)(z^2 + 2z + 1)$    9 b)  $-i, 1+i, 1-i$    10  $2 - i, \frac{2}{3}$    11 i)  $1+3i$    ii)  $5, -\frac{1}{2}$    12  $-3i, \frac{5}{3}$   
 13 b)  $(z-2)(z^2 - 6z + 10); 2, 3+i, 3-i$    14 i)  $1-3i$    ii)  $z^3 - 4z^2 + 14z - 20 = 0$    15  $1-i; 2, 18$    16 a)  $\frac{8}{7}$    b)  $1-2i, -\frac{6}{7}$   
 17 i)  $3p^2$    iii)  $z^3 - 3pz^2 + 3p^2z - (p^3 + q^3) = 0$    18  $5+4i, 5-4i, \sqrt{2}(1+i), \sqrt{2}(1-i), -\sqrt{2}(1+i), -\sqrt{2}(1-i)$

**Exercise 9A**

6 4   11  $\frac{n}{3}(4n^2 - 1)$    14 a)  $\frac{n^2}{(2n-1)(2n+1)}$    16 a)  $\frac{n}{4}(n+1)(n+2)(n-1)$

18 Counter-example:  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$    23  $\frac{n}{6}(n+1)(2n+7)\ln 2$    25 b) Not convergent.  $(-2)^n$  does not tend to 0

27 Yes   28  $\frac{1}{6}$    29 i) Not true: for example,  $u = 2, v = 3, w = 6$    ii) True

29 iii) True. Converse of ii is 'If  $u$  divides  $v+w$ , then  $u$  divides both  $u$  and  $w$ '. This is not true.

**Exercise 9B**

- 1  $\frac{2n}{3}(n+1)(n+2)$    2  $\frac{n}{2}(n+1)(n^2+n+1)$    3  $\frac{n}{3}(n^2-7)$    4  $\frac{n}{6}(4n^2+33n-1)$    5 b) 61907   6  $n^2(2n^2-1)$    7 18760  
 8  $N^2(2N+1)^2, -N^2(4N+3)$

**Exercise 9C**

- 1  $\frac{3}{4}$    2  $\frac{n^2+3n}{4(n+1)(n+2)}$    3 a)  $\frac{1}{2} \left( \frac{1}{2r-1} - \frac{1}{2r+1} \right)$    b)  $a = b = 1$    c) 0   4 8   5  $\frac{1}{2} \left( \frac{1}{2r+1} - \frac{1}{2r+3} \right), \frac{1}{6} - \frac{1}{2(2n+3)}, \frac{1}{6}$   
 6 2   7 i)  $1 - e^{Nx}$    ii)  $x > 0, 1$    8  $\frac{1}{7} - \frac{1}{\sqrt{2N+1}}, \frac{1}{7}$    9 i)  $1 - \frac{1}{(n+1)!}$    ii)  $2e - 3$    10 b)  $\frac{n}{2(n+2)}$

**Exercise 9D**

- 1 b)  $\frac{25}{8}$    c) 0.110   2 i)  $x - \frac{x^2}{2} + \frac{x^3}{3}$    ii)  $\frac{5}{6}, -\frac{3}{5}$   
 3 a)  $2(1+x)\cos x - (1+x)^2 \sin x, 2\cos x - 4(1+x)\sin x - (1+x)^2 \cos x; -6\sin x - 6(1+x)\cos x + (1+x)^2 \sin x$   
 3 b)  $1+2x+\frac{x^2}{2}-x^3$    4 i)  $x - \frac{x^3}{3!} + \frac{x^5}{5!}$    5 b) 0.029565   6  $(1+\sin x+\cos x)e^x; \frac{1}{2}, 4$    7 i)  $A=0, B=1, C=0, D=-\frac{1}{3}$   
 7 ii)  $1-u^2+u^4$    iii)  $x - \frac{x^3}{2} + \frac{x^5}{5}$    iv)  $\tan^{-1} x$    8 1,  $\frac{1}{2}, \frac{1}{8}$    9 a)  $-\frac{6}{(1+x)^4}$    c)  $\frac{(-1)^r x^r}{r(r-1)}$    10 a) i)  $\frac{2x}{2+x^2}$    ii)  $\ln 2 + \frac{x^2}{2}$   
 10 b)  $-\frac{x^4}{8}$    11 i)  $1+rx+\frac{r(r-1)}{2!}x^2+\frac{r(r-1)(r-2)}{3!}x^3+\frac{r(r-1)(r-2)(r-3)}{4!}x^4$    ii)  $\sqrt[3]{\frac{3}{2}}$    12 i)  $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$   
 12 ii)  $\frac{(3n-2)}{n!}$    iii)  $e+2$

**Exercise 9E**

1 a)  $2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \dots$  b)  $5x - \frac{125}{6}x^3 + \frac{625}{24}x^5$  c)  $1 + 8x + 32x^2 + \frac{256}{3}x^3 + \dots$  d)  $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$

1 e)  $-\left(2x + 2x^2 + \frac{8}{3}x^3 + \dots\right)$  2 a)  $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots + (-1)^n \frac{x^{(4n+2)}}{(2n+1)!} + \dots$  b)  $1 + 4x + \frac{15}{2}x^2 + 9x^3 + \frac{63}{8}x^4$

2 c)  $2 - 8x^2 + \frac{9}{4}x^4$  d)  $e\left(1 - \frac{x^2}{2} + \frac{x^4}{6}\right)$  e)  $\ln 2 - \frac{x^2}{4} - \frac{x^4}{96}$  3 a) Converge b) Converge c) Converge

4  $1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \dots + (-1)^n \frac{x^{6n}}{(2n)!} + \dots$ , all values of  $x$  5  $1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \dots$  6  $|x| < 3$  7 a)  $2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512}$

7 b)  $6x - \frac{3}{4}x^2 - \frac{579}{64}x^3$  8 a)  $\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3$  b)  $x - \frac{3}{2}x^2 + \frac{19}{12}x^3 - \frac{19}{8}x^4$  9 a)  $\frac{1}{2}, -\frac{1}{2}\sqrt{3}$

10 a)  $a - \frac{1}{3}b, \frac{1}{2}a^2 + \frac{1}{9}b^2, \frac{1}{6}a^3 - \frac{5}{81}b^3$  b) i) 1, 3

**Exercise 10A**

1 a) i)  $\frac{1}{2}(e^2 + e^{-2})$  ii) 3.76 b) i)  $\frac{1}{2}(e^3 - e^{-3})$  ii) 10.0 c) i)  $\frac{e^8 - 1}{e^8 + 1}$  ii) 0.999 3 a)  $2 \sinh 2x$  b)  $5 \cosh 5x$  c)  $3 \operatorname{sech}^2 3x$

3 d)  $8 \sinh 4x - 15 \cosh 3x$  e)  $6 \sinh 2x + 30 \cosh 5x$  f)  $-\operatorname{sech}^2 x$  g)  $-\operatorname{sech} x \tanh x$  h)  $45 \sinh 3x \cosh^4 3x$

3 i)  $64 \cosh 8x \sinh^3 8x$  j)  $\tanh x$  k)  $2 \cosh 2x e^{\sinh 2x}$  l)  $5 \operatorname{cosech} 5x \operatorname{sech} 5x$  4 a)  $\frac{1}{3} \cosh 3x$  b)  $\frac{1}{4} \sinh 4x$  c)  $3 \cosh\left(\frac{x}{3}\right)$

4 d)  $10 \sinh\left(\frac{x}{5}\right)$  e)  $\frac{3}{5} \sinh 5x - 4 \cosh\left(\frac{x}{2}\right)$  f)  $\frac{1}{4} \ln \cosh 4x$  5 a) 0.540 b) 0.693 c) 0.457 d) 0.191 e) 1.10, -0.625

5 f) 0.514 6  $\ln \frac{1}{2}, \ln \frac{2}{3}$  7 a) i)  $\frac{e^x - e^{-x}}{e^x + e^{-x}}$  b)  $\ln 2$  8  $(-\ln 2, 4)$ , min 10 i)  $x \sinh x + 4 \cosh x$  ii), iii)  $x \sinh x + 2n \cosh x$

11  $-\ln 3, 0$  16 b) i) (1.32, 2.15) ii) 3.3000 c) -1, 4

**Exercise 10B**

1 a)  $\frac{5}{\sqrt{1+25x^2}}$  b)  $\frac{3}{\sqrt{9x^2-1}}$  c)  $\frac{\sqrt{2}}{\sqrt{1+2x^2}}$  d)  $\frac{3}{\sqrt{9x^2-16}}$  e)  $\frac{2x}{\sqrt{1+x^4}}$  f)  $-\frac{1}{x\sqrt{1-x^2}}$  g)  $\frac{1}{1-x^2}$  2 a)  $\cosh^{-1}\left(\frac{x}{2}\right) + c$

2 b)  $\cosh^{-1}\left(\frac{x}{3}\right) + c$  c)  $\frac{1}{2} \cosh^{-1}\left(\frac{2x}{5}\right) + c$  d)  $\frac{1}{3} \cosh^{-1}\left(\frac{3x}{4}\right) + c$  e)  $\sinh^{-1}\left(\frac{x}{3}\right) + c$  f)  $\sinh^{-1}\left(\frac{x}{4}\right) + c$  g)  $\frac{1}{4} \sinh^{-1}\left(\frac{4x}{5}\right) + c$

2 h)  $\frac{1}{5} \sinh^{-1}\left(\frac{5x}{3}\right) + c$  3 a)  $\ln(1+\sqrt{2})$  b)  $\ln(1+\sqrt{2})$  c)  $\ln(2+\sqrt{3})$  d)  $\frac{1}{\sqrt{3}} \ln(2+\sqrt{3})$  e)  $\frac{1}{5} \ln(5+\sqrt{24})$

4 a)  $\frac{1}{5} \ln\left(\frac{10+4\sqrt{6}}{5+\sqrt{21}}\right)$  b)  $\frac{1}{3} \ln\left(\frac{6+2\sqrt{10}}{3+\sqrt{13}}\right)$  c)  $\ln\left(1+\sqrt{\frac{2}{3}}\right)$  d)  $\frac{1}{2} \ln\left(\frac{4+\sqrt{21}}{5}\right)$  e)  $\ln\left(1+\sqrt{\frac{2}{3}}\right)$  f)  $\frac{1}{4} \ln\left(\frac{13+2\sqrt{71}}{5+2\sqrt{35}}\right)$

5 a)  $\cosh^{-1}\left(\frac{x+2}{4}\right) + c$  b)  $\ln\left(\frac{3+\sqrt{8}}{2+\sqrt{3}}\right)$  6 b)  $\ln(1+\sqrt{2})$  7 a)  $x \tanh x - \ln \cosh x + c$

7 b)  $y = x \sinh x - \cosh x \ln \cosh x + c \cosh x$  8 a) 2, 1, 4 b)  $\frac{1}{4} \tanh^{-1}\left(\frac{2x+1}{2}\right) + c$  9 b) 0.2763 10 a)  $(2x+1)^2 + 25$

10 b)  $\frac{1}{2} \sinh\left(\frac{2x+1}{5}\right) + c$  11 i) 3, 4, -25 12  $(x-3)^2 - 1^2, \ln(2+\sqrt{3})$  13 c)  $x \sinh^{-1} x - \sqrt{1+x^2} + c$

17 iii)  $\frac{1}{3} \ln(6+\sqrt{37})$  18 a)  $x \geq 1$  b) ii)  $\sqrt{2} - \ln(1+\sqrt{2}), \sqrt{2}, 1+\sqrt{2}$  19 a)  $2 \tan^{-1} e^x + c$  c) 2.604 20  $\pi \ln(2+\sqrt{5})$

21 c)  $\frac{3}{x^2-9}$  d)  $-\frac{3}{x} - \frac{9}{x^3} - \frac{243}{5x^5} - \frac{2187}{7x^7}, -\frac{3^{2n+1}}{2n+1}$  e)  $\frac{3}{x^2} + \frac{27}{x^4} + \frac{243}{x^6} + \frac{2187}{x^8}, 3^{2n-1}$  22 a)  $x + \frac{x^3}{3} + \frac{x^5}{5}, \frac{1}{2n+1}$

22 b)  $\frac{1}{2} \ln\left(\frac{1}{5}\right)$  c)  $\frac{1}{4} \ln\left(\frac{27}{2}\right)$  23 b) ii)  $2 \ln 2 - 1$

**Exercise 10C**

1 a)  $1 + 2x + \frac{2}{3}x^4$  b)  $3x + \frac{9}{32}x^3$  c)  $1 + x + \frac{25}{2}x^2 + \frac{25}{2}x^3 + \frac{625}{24}x^4$  d)  $6x + 12x^2 + 36x^3 + 72x^4$

2  $\frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2} \cosh^{-1}\left(\frac{x}{3}\right) + c$  3  $\frac{1}{2}x\sqrt{x^2+16} + 8 \sinh^{-1}\left(\frac{x}{4}\right) + c$  4  $\frac{1}{2}x\sqrt{x^2+25} + \frac{25}{2} \sinh^{-1}\left(\frac{x}{5}\right) + c$

**5**  $\frac{1}{2}x\sqrt{x^2 - 25} - \frac{25}{2}\cosh^{-1}\left(\frac{x}{5}\right) + c$     **6**  $\frac{1}{2}x\sqrt{x^2 - 4} + 2\cosh^{-1}\left(\frac{x}{2}\right) + c$     **7**  $\frac{1}{8}x\sqrt{x^2 + 9} - \frac{5}{2}\sinh^{-1}\left(\frac{x}{3}\right) + c$

**8**  $\frac{x}{2}\sqrt{x^2 + 4} + 2\sinh^{-1}\left(\frac{x}{2}\right) + c$     **9 i)**  $0; \frac{1}{2}\left[\frac{1 - (n+1)\cosh nx + n\cosh(n+1)x}{\cosh x - 1}\right]$     **12**  $y = \frac{1}{2}\coth x - \frac{1}{2}x\operatorname{cosech} x + c\operatorname{cosech}^2 x$

**14**  $\frac{x}{\sqrt{x^2 - 1}}; \frac{5}{4}, -\frac{3}{4}$     **15 c) ii)**  $1.76$     **17 d)**  $\frac{1}{2}\ln(2 + \sqrt{3}) - \frac{1}{4}\sqrt{3}; \frac{1}{2}, -\frac{1}{4}$     **18 b)**  $1, -\frac{1}{2}, \frac{13}{120}$     **c)**  $x^3 + \frac{1}{6}x^5$

**Exercise 11A**

**1 a)**  $(4, 0), x = -4$     **b)**  $(7, 0), x = -7$     **c)**  $(0, 2), y = -2$     **d)**  $(0, -4), y = 4$     **e)**  $(-3, 0), x = 3$     **f)**  $(8, -1), x = -8$     **g)**  $(5, 2), x = 1$

**2 a)**  $y^2 = 12x$     **b)**  $y^2 = 16x$     **c)**  $x^2 = 8y$     **d)**  $x^2 = -20y$     **3 a)**  $ty = x + 5t^2$     **b)**  $py = x + 5p^2$     **c)**  $y = 5 + x$     **d)**  $2y = x + 20$

**4 a)**  $x + y = 6$     **b)**  $(18, -12)$

**Exercise 11B**

**1 a)**  $\frac{\sqrt{7}}{4}, (\pm\sqrt{7}, 0), x = \pm\frac{16}{\sqrt{7}}$     **b)**  $\frac{\sqrt{33}}{7}, (\pm\sqrt{33}, 0), x = \pm\frac{49}{\sqrt{33}}$     **c)**  $\frac{3}{5}; (\pm 3, 0), x = \pm\frac{25}{3}$     **d)**  $\frac{\sqrt{5}}{3}, (0, \pm 2\sqrt{5}), y = \pm\frac{18}{5}$

**1 e)**  $\frac{4}{5}, (5, -2), x = \frac{29}{4}, x = -\frac{21}{4}$     **2 a)**  $\frac{x^2}{36} + \frac{y^2}{27} = 1$     **b)**  $\frac{x^2}{36} + \frac{y^2}{32} = 1$     **c)**  $\frac{x^2}{16} + \frac{y^2}{32} = 1$     **d)**  $\frac{x^2}{36} + \frac{y^2}{45} = 1$

**3 a)**  $4x\cos\theta + 5y\sin\theta = 20$     **b)**  $4y\cos\theta = 5x\sin\theta - 9\sin\theta\cos\theta$     **4**  $(-\frac{12}{5}a, 0), (-\frac{6}{5}a, 0)$

**Exercise 11C**

**1 a)**  $\frac{5}{4}, (\pm 5, 0), x = \pm\frac{16}{5}$     **b)**  $\frac{\sqrt{65}}{7}, (\pm\sqrt{65}, 0), x = \pm\frac{49}{\sqrt{65}}$     **c)**  $\frac{\sqrt{41}}{5}, (\pm\sqrt{41}, 0), x = \pm\frac{25}{\sqrt{41}}$     **d)**  $\frac{\sqrt{13}}{2}, (\pm\sqrt{13}, 0), x = \pm\frac{4}{\sqrt{13}}$

**1 e)**  $\frac{\sqrt{34}}{5}, (1 \pm \sqrt{34}, -2), x = 1 \pm \frac{25}{\sqrt{34}}$     **2 a)**  $\frac{x^2}{36} - \frac{y^2}{108} = 1$     **b)**  $\frac{x^2}{36} - \frac{y^2}{288} = 1$     **c)**  $\frac{y^2}{32} - \frac{x^2}{32} = 1$     **d)**  $\frac{y^2}{45} - \frac{x^2}{180} = 1$

**3 a)**  $4x\sec\theta - 5y\tan\theta = 20$     **b)**  $5x\sin\theta + 4y = 41\tan\theta$

**Exercise 11D**

**2 a)**  $(0, 2t)$     **b)**  $y^2 = 9x$     **3 b)**  $(apq, a(p+q))$     **c)**  $pq = -1$     **4 b)**  $3x + 7y = 37, y + 3x = 13$     **5 a)**  $\frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$

**5 b)**  $ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$     **c)**  $\left(\frac{a^2 - b^2}{2a}\right)\cos t, \frac{b}{2\sin t}$     **6 a)**  $3x\cos\theta + 2y\sin\theta = 6$

**6 c)**  $\theta = 0^\circ, 233.1^\circ, 360^\circ; (-1.2, -2.4)$     **8 c)**  $1$     **9 a)**  $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1, \frac{\sqrt{5}}{3}$     **b)**  $(\pm\sqrt{5}, 0)$     **d)**  $33.84$

**10 c)**  $C_1: (\pm 3a, 0), y = \pm\frac{x}{3}; C_2: \left(\pm\frac{10a}{3}, 0\right), y = \pm 3x$     **d)**  $C_1: \frac{\sqrt{10}}{3}, (\pm\sqrt{10}a, 0), x = \pm\frac{9}{10}$ ;  $C_2: \sqrt{10}, \left(\pm\frac{100a}{3}, 0\right), x = \pm\frac{a}{3}$

**12 iv)**  $(-1, 2t)$

**Exercise 12A**

The constant of integration is omitted from these answers.

**1 a)**  $\frac{1}{6}(x^2 + 1)^6$     **b)**  $\frac{1}{10}(x^2 - 1)^5$     **c)**  $\frac{1}{32}(x^4 - 1)^8$     **d)**  $-\frac{1}{15}(1 - x^3)^5$     **e)**  $-\frac{1}{6}\cos^6 x$     **f)**  $\frac{1}{5}\sinh^5 x$     **g)**  $\frac{1}{15}\cosh^5 3x$     **h)**  $\frac{1}{12}\sin^6 2x$

**2 a)**  $\frac{1}{2}e^x(\cos x + \sin x)$     **b)**  $\frac{1}{5}e^x(\cos 2x + 2\sin 2x)$     **c)**  $\frac{1}{5}e^{2x}(\sin x + 2\cos x)$     **d)**  $\frac{1}{34}e^{3x}(5\sin 5x + 3\cos 5x)$

**2 e)**  $\frac{1}{10}e^{4x}(2\cosh 2x - \sinh 2x)$     **f)**  $-\frac{1}{40}e^{-7x}(3\cosh 3x + 7\sinh 3x)$     **3 a)**  $x - \tan^{-1} x$     **b)**  $x - 5\tan^{-1}\left(\frac{x}{4}\right)$     **c)**  $\frac{1}{4}x - \frac{23}{32}\ln(8x + 3)$

**3 d)**  $-\frac{7}{4}x + \frac{47}{16}\ln(5 - 4x)$     **e)**  $\ln(x^2 + 2x + 3) - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$     **f)**  $\frac{1}{2}\ln(x^2 + x + 1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

**3 g)**  $\sqrt{x^2 + x - 1} - \frac{3}{2}\cosh^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$     **h)**  $\sqrt{2x^2 - 4x + 5} - \frac{5}{\sqrt{2}}\sinh^{-1}\left(\sqrt{\frac{2}{3}}(x-1)\right)$     **i)**  $-2\sqrt{1 - 4x - x^2} + \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)$

**3 i)**  $-\sqrt{2 - 5x - 3x^2} - \frac{19}{2\sqrt{3}}\sin^{-1}\left(\frac{6x+5}{7}\right)$     **4 a)**  $-\sqrt{1 - x^2} + \sin^{-1} x$     **b)**  $1 - \frac{\sqrt{3}}{2} + \frac{\pi}{6}$     **5**  $\sin^{-1}\left(\frac{x+3}{5}\right)$

**6 i)**  $1 + \frac{2}{1+x} - \frac{1}{9+x^2}$     **7**  $\frac{1}{3x+4} + \frac{2x+1}{x^2+9}$

**Exercise 12B**

- 1 a)**  $5\pi/32$  **b)**  $\frac{16}{35}$  **2**  $I_n = x^n e^n - n I_{n-1}$  **3**  $120 - 326/e$  **4 a)**  $\frac{1}{4}\pi - \frac{2}{3}$  **b)**  $\frac{5}{12} - \frac{1}{2}\ln 2$   
**6**  $\frac{1}{5}\cosh^4 1 \sinh 1 + \frac{4}{15}\cosh^2 1 \sinh 1 + \frac{8}{15}\sinh 1$  **11 b)**  $\frac{316}{81}$  **12**  $\frac{1}{16}\pi^4 - 3\pi^2 + 24$  **13**  $\frac{8}{15}$  **14 b)**  $2 - \frac{2}{\sqrt{e}}$  **c)**  $16 - \frac{26}{\sqrt{e}}$   
**15 ii)**  $\frac{26}{15}$  **16**  $\frac{1}{2}\ln 2 - \frac{1}{4}$  **17 b)**  $\sqrt{3} + \frac{1}{2}\ln(2 + \sqrt{3})$  **18**  $\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}; \frac{1}{3}\ln 2 + \frac{\pi}{3\sqrt{3}}; \frac{1}{6} + \frac{2\pi}{9\sqrt{3}} + \frac{2}{9}\ln 2$  **21**  $\frac{4\pi}{3} - 2\sqrt{3}$  **24**  $\frac{1}{8}e^2 + \frac{3}{8}$   
**26 b)**  $-\frac{17}{15}\sqrt{2} + \frac{4}{5}\sqrt{3}$  **27 a)**  $\frac{5}{4}, \frac{3}{4}$  **b)** ii)  $\frac{57}{64}$  **28 a)**  $\pi$  **c)**  $\frac{5}{16}\pi$  **d)**  $\frac{5a^2\pi}{32}$  **29 b)** i)  $\ln 2$  iv)  $\frac{3}{5}\ln 2$  **30 i)**  $-\frac{1}{n+1} \cos^{n+1} \theta$   
**31 b)**  $\pi\sqrt{5} - \frac{\pi}{2}\ln(2 + \sqrt{5})$  **c)** 1.641%

**Exercise 12C**

- 1**  $\frac{1}{27}(31\sqrt{31} - 8)$  **2**  $\frac{(14\sqrt{14} - 11\sqrt{11})}{9\sqrt{2}}$  **3**  $\frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln(t + \sqrt{1+t^2})$  **4** 4a **5**  $c \sinh 1$  **6 a)**  $\frac{12\pi}{5}(2 + 782\sqrt{17})$   
**6 b)**  $\frac{8\pi}{3}(5\sqrt{5} - 1)$  **c)**  $\frac{12\pi a^2}{5}$  **d)**  $\frac{5\pi}{6}(61^{3/2} - 41^{3/2})$  **e)**  $\pi(\frac{1}{2}\sinh 2 + 1)$   
**6 f)**  $\frac{\pi}{3}e^{12}\sqrt{1+9e^{24}} + \frac{\pi}{9}\sinh^{-1}(3e^{12}) - \frac{\pi}{3}e^3\sqrt{1+9e^6} - \frac{\pi}{9}\sinh^{-1}(3e^3)$  **7** 8a **8 a)**  $-1, 2$  **b)** i)  $\frac{2x}{1-x^2}$  **10**  $\frac{1}{4}(e^{2\pi} - 1)$  **11 i)** 3  
**12**  $\frac{64\pi a^2}{3}$  **13**  $2 - 2e^{-\pi/2}, 2 \int_0^{\pi/2} [4 + e^{-t}(\cos t - \sin t)]2e^{-t} dt$  **14** 6 **16 i)**  $3\sqrt{2}$  ii)  $\frac{24\sqrt{2}}{5}\pi$  **17 b)** i) 12 ii)  $\frac{576}{5}\pi$   
**19**  $\frac{\pi}{27}(10\sqrt{10} - 1)$  **20 b)**  $(\frac{1}{5}e^\pi - \frac{2}{5})2\sqrt{2}\pi$  **22 a)**  $\frac{1}{3}\cosh^3 t + c$  **c)** 438

**Exercise 12D**

- 1**  $\frac{3}{2}$  **2** Does not exist **3** Does not exist **4** Does not exist **5** Does not exist **6** Does not exist **7** Does not exist **8**  $\pi/2a$   
**9** Does not exist **10** Does not exist **11 a)**  $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$  **b)**  $-\frac{1}{4}$  **12 a)**  $\frac{1}{2}$  **b)**  $\ln\left(\frac{3}{2}\right)$  **13 a)**  $-\frac{k\pi}{2}$  **b)** ii)  $\frac{1}{2}\ln 2$   
**14 i)**  $\ln 2$  iii)  $\ln 2$  iv)  $\pi/4$  **15**  $\frac{1}{4}$  **16 iii)**  $-4$

**Exercise 13A**

- 1** 1.12 **2** 0.87 **3** 2.46 **4** 0.95 **5** 0.93 **6** 0.60 **7** 1.41 **8 a)**  $-3.8125, -3.7936, -3.7915, -3.7913, -3.7912, -3.7912, -3.791$   
**8 b)**  $-3.791$  **c)** Five **9** [0.375, 0.5], fourteen **10 b)** 1.432 **d)** 0.669 **11 b)** 1.373  
**12** Starting with  $x_0 = 0$ :  $x_1 = 1$ ,  $x_2 = 0.8068242641$ ,  $x_3 = 0.7921349597$ ,  $x_4 = 0.7920599704$ ,  $x_5 = 0.7920599684$ ,  $x_6 = 0.7920599684$   
**13 a)** 0.109,  $-0.402$  **b)** 2.11 **c)** 2.13 **14** 0.53 Tangent parallel to  $x$ -axis **15** 4.026 **16 b)** 6.135 **17 b)** 1.54 **c)**  $-0.54$   
**18 a)** i) 1.4973 ii) 1.4973043 **19 c)** 1.07 **20 a)** 0.2443, 0.2553, 0.2582, 0.2589 **b)** 2.544 **21 a)** 0.337609 **22 a)**  $\frac{1}{2}, \frac{3}{2}$   
**22 b)** i)  $4x^3 - 12x^2 + 9x + 3 = 0$  ii)  $-0.2460$  **23**  $-0.670, 0.78, [1.2, 1.20625]$

**Exercise 13B**

- 1 a)** i) 42.0000 ii) 41.3333 **b)** i) 328.0000 ii) 320.0000 **c)** i) 3.6734 ii) 3.7175 **d)** i) 5.0898 ii) 5.1795 **2 i)** 1255.81  
**2 ii)** 1403.734 **3 b)** 3.6281 **c)**  $\sinh 2$  **d)** 0.03% **4 b)**  $2 - \frac{2}{\sqrt{e}}$  **c)**  $16 - \frac{26}{\sqrt{e}}$  **5 a)** i) 0.4285 ii) 0.4217 **b)** 0.4207  
**6 b)**  $\frac{\pi}{4}[2\sqrt{5} - \ln(2 + \sqrt{5})]$  **c)** 1.641% **7** 0.579 **8** 2.53, 1.26 **9** 0.52, 0.52 **10** 0.74988 **11** 0.1026  
**12** 0.82 i) Underestimate ii) Larger estimate and better approximation

**Exercise 13C**

- 1**  $y = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{35x^4}{8}$  **2**  $y = 2 + 2x^2 + \frac{2x^3}{3} + 2x^4$  **3**  $y = x - \frac{5x^3}{6}$  **4**  $y = 1 - \frac{x^4}{6}, 0.999999998$   
**5**  $y = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \dots$  **a)** 0.1055 **b)**  $-0.0955$  **6**  $y = 2x + \frac{8x^3}{3}$

## ANSWERS

**7**  $\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left( x - \frac{\pi}{4} \right) - \frac{1}{2\sqrt{2}} \left( x - \frac{\pi}{4} \right)^2, 0.719448$

**8**  $f(0.5 + h) = -2 - 28h - 784h^2$  Sensitive because derivative large near  $x = 0.5$     **9**  $y = 1 + 2x - \frac{3x^2}{2} - \frac{4x^3}{3}$

**10 a)**  $3 \left( 1 + \frac{dy}{dx} \right) (x+y)^2, 3 \frac{d^2y}{dx^2} (x+y)^2 + 6 \left( 1 + \frac{dy}{dx} \right)^2 (x+y)$     **b)**  $y = 1 + x + 3x^2 + 7x^3$     **c)** 0.9    **11**  $y = 3 + 2x + 3x^2 - \frac{7x^3}{3}$

**12 a)** 0.9    **b)** 0.8362    **c)** 0.7607    **13 a)** 1.1    **b)** 1.221    **14 a)**  $x = 2.27, y = -0.33$     **b)**  $\frac{d^2x}{dt^2} + 16x = 20$     **15 a)** 1.628

**15 b)**  $y = x(2 \ln x + 1)$     **c)** 0.6%    **16** 0.0025, 0.0051, 0.0078    **17 a) i)**  $\frac{dy}{dt} = 4 - \frac{3x}{v}$     **ii)** 0.22    **b) i)**  $x = Ae^{3t} + Be^t$

**17 b) ii)**  $x = e^{3t} - e^t, 0.245$     **18** 2.21, 0.64    **19** 2.0766, 2.0743    **20** 0.049,  $\frac{1}{5}$

### Exercise 14A

**1**  $\begin{pmatrix} 14 & 0 \\ 8 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 18 & 17 \end{pmatrix}, \mathbf{PQ} \neq \mathbf{QP}$

**2 a)**  $\begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}$     **b)**  $\begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}$     **c)**  $\begin{pmatrix} -20 & 4 & -9 \\ -5 & 1 & -2 \\ 11 & -2 & 5 \end{pmatrix}$     **d)**  $\begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}$

**2 e)**  $-\frac{1}{141} \begin{pmatrix} -21 & -12 & 18 \\ -17 & -3 & -19 \\ 5 & -24 & -11 \end{pmatrix}$     **3**  $\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{k}{3} & \frac{k}{3} & -1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$     **5 a)**  $\frac{1}{2x-5} \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$     **b)** 19, -14, -27

**6**  $\frac{1}{20} \begin{pmatrix} -1 & 11 & -4 \\ -4 & 4 & 4 \\ 11 & -21 & 4 \end{pmatrix}$     **7**  $\begin{pmatrix} -1 & 0 & 1 \\ 6\frac{1}{2} & -2\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$     **9 i)**  $\frac{1}{1+a} \begin{pmatrix} 1 & -a & 0 \\ 1 & 1 & 0 \\ -5-a & a^2-5 & 1+a \end{pmatrix}$     **ii)** (-1, 1, 0)

**10**  $\frac{1}{6a-6} \begin{pmatrix} 3a-1 & a+1 & -4 \\ 1 & 2a-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}, \left( \frac{2}{1-a}, \frac{1}{1-a}, \frac{3}{1-a} \right)$     **11 i)**  $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix}$     **ii)**  $\begin{pmatrix} \frac{1}{a} & 1 & -\frac{1}{a} \\ 1-\frac{1}{2a} & -1 & \frac{1}{2a} \\ -\frac{1}{2a} & 0 & \frac{1}{2a} \end{pmatrix}$

**12 i)**  $\frac{1}{10-a} \begin{pmatrix} 5 & 5 & -5 \\ 3a-5 & 5-2a & -1 \\ -a & -a & 2 \end{pmatrix}$     **ii)**  $\frac{2}{5}y_1 - \frac{3}{5}y_2 + \left( \frac{a}{10-5a} \right) y_3$     **13 c)** -A, A, -A;  $(-1)^n \mathbf{A}$

**13 d)**  $x - y - z = 1, -x + y - z = 0, -x - y + z = 0; x = 0, y = z = -\frac{1}{2}$     **e)**  $\frac{1}{2}, -\frac{1}{2}$     **14 a)**  $\begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix}$     **d)**  $\begin{pmatrix} 4 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}$

**15 a)**  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$     **b)**  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 7 \\ 0 & 0 & 1 \end{pmatrix}$     **d)**  $\frac{1}{2^n} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$

### Exercise 14B

**1** Rotation clockwise about O through  $2\pi/3$

**2** Reflection in  $y$ -axis, followed by one-way stretch in  $y$ -direction, scale factor 2     $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

**3** Shear in  $x$ -direction moving (0, 1) to (-1, 1)    **i)**  $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$     **ii)** Rotation clockwise about O through  $\pi/3$

**4** **i)** (0, 0), (3, 0), (2, 4)    **ii)** 6, 150    **iii)** 6    **5** 0, 1    **6 a)** 3, -1    **b)**  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$     **7 a) i)** 5, -1    **ii)**  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$     **b)** 6, 5

**8 a)** 4, -3    **c)**  $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$     **d)**  $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}$     **9 a)** 9, -3    **b)**  $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$     **c)**  $\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$     **10 a) i)** 5,  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$     **ii)**  $x + 2y = 0$

**10 b)** 25    **11 a)** -2, 1    **b)**  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$     **d)**  $\frac{1}{3}(a-2b+c)\mathbf{v}_1 + \frac{1}{3}(a+b-2c)\mathbf{v}_2 + \frac{1}{3}(a+b+c)\mathbf{v}_3$     **12 a)** 4, -2;  $\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$

12 b) ii)  $\frac{1}{8}, -\frac{1}{4}$  13 a) 2, -2, 3 c) 5 d)  $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, 10$  14 a) i)  $(0, -7)$  ii)  $k = 7, A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}$  b) i) 3, 8;  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

14 b) ii)  $x + 2y = 0, y = 2x$  14 c) One-way stretch centred at  $(0, -7)$ , scale factor 3, in direction of  $x + 2y = 0$ , followed by

one-way stretch centred at  $(0, -7)$ , scale factor 8, in direction of  $y = 2x$  15 i) 3,  $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{5}{3} & \frac{4}{3} & -4 \\ -3 & -2 & 7 \end{pmatrix}$  ii)  $\begin{pmatrix} -6 & -4 & 12 \\ -29 & -17 & 66 \\ -13 & -8 & 29 \end{pmatrix}$

15 iii)  $\lambda^3 - 6\lambda^2 + 3\lambda + 10 = 0$  iv)  $-\frac{1}{10}, \frac{3}{5}, -\frac{3}{10}$  16 b)  $2x + 2y - z = 0$  17 2, -3, 5 18 a)  $k^2 - 2k + 1$  b) 3, -1

19 b)  $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$  c) i)  $x = y, z = 0$  ii)  $90^\circ$  d)  $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$  20 c) iii)  $\frac{16\pi}{3}$  21 i)  $(-3, 1, 2)$  ii)  $x = y = z$

21 iii)  $x + y + z = 0$  iv)  $\frac{2\pi}{3}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  Rotation about  $L$  through  $\frac{4\pi}{3}$ ; identity 22 1, 27;  $\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

22 a)  $5x = y$  c) 27 Scale factor of area enlargement d) One-way stretch in  $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$  direction, scale factor 27

23 i)  $\frac{1}{1-2k} \begin{pmatrix} k-2 & 4-3k & 1 \\ -1-k & k+2 & 1 \\ 3 & -5 & -2 \end{pmatrix}$  ii)  $(-1, -1, 3)$  iv)  $\frac{x}{3} = \frac{y}{2} = -\frac{z}{3}$

### Exercise 15A

- 1 a)  $\cos 6\theta + i \sin 6\theta$  b)  $\cos 8\theta + i \sin 8\theta$  c) -1 d) -i e)  $\cos 8\theta - i \sin 8\theta$  f) -1 g) 1 h) -i 2 a)  $\cos 10\theta + i \sin 10\theta$   
 2 b)  $\cos \theta - i \sin \theta$  c) -1 d) -8 3 a) 16 b)  $512i$  c)  $24\sqrt{3}$  d) -4 e)  $2^{12}$  f)  $-512i$  4 a)  $\cos 5\theta - i \sin 5\theta$   
 4 b)  $\cos 4\theta + i \sin 4\theta$  c)  $-\cos 6\theta + i \sin 6\theta$  d) -1 5  $\cos 11x + i \sin 11x$

### Exercise 15B

- 1 a) i)  $\pm\sqrt{2}(1 \pm i)$  ii)  $2e^{i\pi/4}, 2e^{-i\pi/4}, 2e^{3i\pi/4}, 2e^{-3i\pi/4}$  b) i)  $2^{\frac{3}{2}}(1+i), 2^{\frac{7}{6}} \left[ \cos\left(\frac{11\pi}{12}\right) + i \sin\left(\frac{11\pi}{12}\right) \right], 2^{\frac{7}{6}} \left[ \cos\left(\frac{5\pi}{12}\right) - i \sin\left(\frac{5\pi}{12}\right) \right]$   
 1 b) ii)  $2^{\frac{7}{6}} e^{i\pi/4}, 2^{\frac{7}{6}} e^{11i\pi/12}, 2^{\frac{7}{6}} e^{-5i\pi/12}$  c) i)  $\frac{3}{2}(\sqrt{3}+i), \frac{3}{2}(-\sqrt{3}+i), -3i$  ii)  $3e^{i\pi/6}, 3e^{5i\pi/6}, 3e^{-i\pi/2}$  d) i)  $\pm 2\sqrt{2}(1+i)$   
 1 d) ii)  $4e^{i\pi/4}, 4e^{-3i\pi/4}$  e) i)  $\frac{5}{\sqrt{2}}(-1+i), \frac{5}{\sqrt{2}}(1-i)$  ii)  $5e^{3i\pi/4}, 5e^{-i\pi/4}$  f) i)  $-2, 2 \left[ \cos\left(\frac{\pi}{5}\right) \pm i \sin\left(\frac{\pi}{5}\right) \right], 2 \left[ \cos\left(\frac{3\pi}{5}\right) \pm i \sin\left(\frac{3\pi}{5}\right) \right]$   
 1 f) ii)  $2e^{i\pi/5}, 2e^{3i\pi/5}, 2e^{i\pi}, 2e^{-i\pi/5}, 2e^{-3i\pi/5}$  2  $e^{\pm i\pi/3}, e^{\pm 2i\pi/3}, e^{i\pi}, e^{i0}$  or 1, -1,  $\pm \left( \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right)$  3 a)  $2-2i, -2-2i$  b) 3,  $\pm\sqrt{3}i$   
 3 c)  $-\frac{1}{2}$  d)  $1+3i, \frac{1}{3}-i$  e)  $-3-\frac{1}{2}i, -\frac{3}{2}+\frac{1}{4}i, -\frac{21}{10}+\frac{7}{10}i, -\frac{3}{2}-\frac{1}{2}i$  4  $e^{i0}, e^{2i\pi/7}, e^{4i\pi/7}, e^{6i\pi/7}, e^{-2i\pi/7}, e^{-4i\pi/7}, e^{-6i\pi/7}$   
 5  $2e^{i\pi/10}, 2e^{i\pi/2}, 2e^{9i\pi/10}, 2e^{-3i\pi/10}, 2e^{-7i\pi/10}$  9 a)  $\frac{1}{41}(4\cos 5x + 5\sin 5x)e^{4x} + c$  b)  $\frac{1}{58}(3\sin 7x - 7\cos 7x)e^{3x} + c$   
 9 c)  $-\frac{1}{20}(2\sin 4x + 4\cos 4x)e^{-2x} + c$  d)  $\frac{1}{25}(3\sin 3x - 4\cos 3x) + c$  10  $2e^{-i\pi/6}, 2e^{-2i\pi/3}, 2e^{5i\pi/6}, 2e^{i\pi/3}$  11 3 - i, -3 + 3i  
 12 i)  $8, -5\pi/6$  ii)  $2e^{-5i\pi/18}, 2e^{7i\pi/18}, 2e^{-17i\pi/18}$  13  $16 \left[ \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right], 2 \left[ \cos\left(\frac{\pi}{16}\right) + i \sin\left(\frac{\pi}{16}\right) \right],$   
 $2 \left[ \cos\left(\frac{9\pi}{16}\right) + i \sin\left(\frac{9\pi}{16}\right) \right], 2 \left[ \cos\left(-\frac{7\pi}{16}\right) + i \sin\left(-\frac{7\pi}{16}\right) \right], 2 \left[ \cos\left(-\frac{15\pi}{16}\right) + i \sin\left(-\frac{15\pi}{16}\right) \right]$  14  $2e^{-i\pi/3}, 2e^{i\pi/6}$   
 15 b)  $2^{\frac{1}{6}}, \pi/12; 2^{\frac{1}{6}}, 3\pi/4; 2^{\frac{1}{6}}, -7\pi/12$ .

- 16 64,  $\pi; 2\sqrt{2} \left[ \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right], 2\sqrt{2} \left[ \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right], 2\sqrt{2} \left[ \cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right], 2\sqrt{2} \left[ \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right];$   
 $2+2i, -2+2i, -2-2i, 2-2i$  17 a)  $\sqrt{2}e^{i\pi/20}, \sqrt{2}e^{9i\pi/20}, \sqrt{2}e^{17i\pi/20}$  18 ii)  $\frac{1}{13}e^{3ix}(3\cos 2x + 2\sin 2x), \frac{1}{13}e^{3ix}(3\sin 2x - 2\cos 2x)$   
 18 iii)  $\frac{1}{13}e^{3ix}(3\cos 2x + 2\sin 2x) + c, \frac{1}{13}e^{3ix}(3\sin 2x - 2\cos 2x) + c$  19 b)  $1 + e^{3i\pi/5}, 1 + e^{i\pi}$  (or 0),  $1 + e^{-3i\pi/5}, 1 + e^{-i\pi/5}, 1 + e^{i\pi/5}$   
 19 c) (1, 0), 1 d) i)  $\frac{\pi}{10}$  ii)  $2\cos\left(\frac{\pi}{10}\right)$  20 i)  $4-2i, 4+\sqrt{3}+i, 4-\sqrt{3}+i; 3\sqrt{3}$  ii) -i; -11, -50  
 21  $\cos\left(\frac{2r\pi}{5}\right) + i \sin\left(\frac{2r\pi}{5}\right)$  ( $r = 0, 1, 2, 3, 4$ ) i)  $\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{10}\right), \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right), \cos\left(\frac{9\pi}{10}\right) + i \sin\left(\frac{9\pi}{10}\right),$   
 $\cos\left(\frac{13\pi}{10}\right) + i \sin\left(\frac{13\pi}{10}\right), \cos\left(\frac{17\pi}{10}\right) + i \sin\left(\frac{17\pi}{10}\right)$  22 i)  $\sqrt{2}e^{i\pi/4}, \sqrt{2}e^{7i\pi/12}, \sqrt{2}e^{11i\pi/12}, \sqrt{2}e^{-3i\pi/4}, \sqrt{2}e^{-5i\pi/12}, \sqrt{2}e^{-i\pi/12}$

**Exercise 15C**

- 1 a)  $2i \sin \theta$  b)  $2 \cos 4\theta$  c)  $2 \cos 5\theta$  d)  $2i \sin 2\theta - 2i \sin \theta$
- 2 a)  $\frac{1}{2} \left( z^6 + \frac{1}{z^6} \right)$  b)  $\frac{1}{2i} \left( z^5 - \frac{1}{z^5} \right)$  c)  $\frac{1}{2} \left( z^4 + \frac{1}{z^4} \right)$
- 2 d)  $\frac{1}{2i} \left( z^3 - \frac{1}{z^3} \right)$  e)  $-\frac{1}{4} \left( z^5 - \frac{1}{z^5} \right)^2$  f)  $\frac{1}{16} \left( z^3 + \frac{1}{z^3} \right)^4$
- 3 a)  $32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$  b)  $8 \cos^4 \theta - 8 \cos^2 \theta + 1$
- 3 c)  $8 \cos^3 \theta - 4 \cos \theta$  d)  $32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta$
- 4 a)  $3 \sin \theta - 4 \sin^3 \theta$  b)  $16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$
- 4 c)  $-64 \sin^6 \theta + 80 \sin^4 \theta - 24 \sin^2 \theta + 1$  d)  $16 \sin^4 \theta - 12 \sin^2 \theta + 1$
- 5 a)  $\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$  b)  $\frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$
- 5 c)  $\frac{1}{16} \cos 5\theta + \frac{5}{16} \cos 3\theta + \frac{5}{8} \cos \theta$  d)  $\frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{8} \sin \theta$
- 6 a)  $\frac{1}{32} \cos 6\theta + \frac{3}{16} \cos 4\theta + \frac{15}{32} \cos \theta + \frac{5}{16}$
- 7  $\tan\left(\frac{\pi}{12}\right), \tan\left(\frac{5\pi}{12}\right), \tan\left(\frac{3\pi}{4}\right)$
- 8  $\frac{27}{8}, \frac{7}{8}$
- 9 vi)  $-1 + 2i$
- 13 a)  $4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$  b)  $z^4 = \cos 4\theta + i \sin 4\theta$
- 13 c)  $\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$
- 14 8, 4, 2, 1
- 15 5,  $-10, 1; \pm 2 \cos\left(\frac{\pi}{5}\right), \pm 2 \cos\left(\frac{2\pi}{5}\right)$
- 15 cos  $\theta + i \sin \theta$ ,  $\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$ ,  $\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$ ,  $\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right)$ ,  $\cos\left(\frac{4\pi}{5}\right) - i \sin\left(\frac{4\pi}{5}\right)$
- 15 b)  $(z-1) \left[ z^2 - 2z \cos\left(\frac{2\pi}{5}\right) + 1 \right] \left[ z^2 - 2z \cos\left(\frac{4\pi}{5}\right) + 1 \right]$
- 17  $e^{5i\pi/6}, e^{-i\pi/2}, e^{-i\pi/6}, e^{-5i\pi/6}, e^{i\pi/2}; (z^2 + \sqrt{3}z + 1)(z^2 - \sqrt{3}z + 1)(z^2 + 1)$

**Exercise 15D**

- 1 a)  $w$  lies on circle, centre O, radius 25 b)  $w$  lies on that part of real axis with argument 0
- 1 c)  $w$  lies on that part of real axis with argument  $\pi$
- 2 a)  $w$  lies on circle, centre O, radius  $\sqrt{5}$
- 2 b)  $w$  lies on circle, centre O, radius  $\sqrt{2}$  c)  $w$  lies on line  $u = v$
- 3  $v^2 = 4k^2(u+k^2)$
- 4 a) Circle, centre  $-\frac{1}{4}i$ , radius  $\frac{3}{4}$
- 4 b) Imaginary axis c)  $\pm \frac{1}{2}\sqrt{3} + \frac{1}{2}i$
- 5  $|w+5-2i|=12$  or circle, centre  $(-5, 2)i$ , radius 12
- 6 a) 4, 9, -4 b)  $(4, 0), 5$
- 7 Circle  $|w - \frac{9}{5} - \frac{21}{20}i| = \frac{3}{4}\sqrt{17}$
- 8 Ellipse:  $\left(\frac{4u}{31}\right)^2 + \left(\frac{3v}{23}\right)^2 = 1$
- 9 Straight line:  $w = 3$
- 11 i)  $2e^{i\pi/2}, 2e^{-i\pi/6}, 2e^{-5i\pi/6}$
- 11 ii) Circle, centre  $-3$ , radius 2; circle, centre  $9i$ , radius 6; circle, centre  $-3i$ , radius 2
- 12  $w = 0$
- 14 b) ii)  $\pi/8, 5\pi/8$
- 15  $1 + \frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}$

**Exercise 16**

- 1  $\frac{125}{78}$
- 2  $\frac{(1+e^2)^{3/2}}{e}$
- 3  $\frac{5\sqrt{5}}{4\sqrt{3}}$
- 4  $2\sqrt{2}$
- 5  $\frac{13\sqrt{13}}{6}$
- 6  $\frac{17\sqrt{17}c}{16}$
- 7  $\infty$
- 8  $\frac{\sqrt{3}}{2}a$
- 9 a)  $3\psi^2 - \sin \psi$  b)  $3 + 4 \sin \psi + 4\psi \cos \psi$
- 9 c)  $\cos \psi - \psi \sin \psi + 2\psi$
- 10  $s = \ln |\sec \psi + \tan \psi|$
- 11 a)  $-a \sin \psi$
- 12 b) 3.81
- 12 c)  $\frac{9\pi}{2}(e^2 + 4 - e^{-2})$
- 13 iii)  $16\sqrt{3}$
- 14 iv)  $\left(384 - \frac{192\sqrt{3}}{5}\right)\pi$
- 14 i)  $5e^x - 5$
- 14 iii)  $\frac{5}{2}e^x$

**Exercise 17A**

- 5  $3 \times 3 = 9$ , set is not closed
- 6 No inverse of  $-1$

**Exercise 17B**

- 1 i)  $G_1: 1, G_2: 0$  ii) Because 0 does not have an inverse under  $\times$  iii)  $f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$

G <sub>1</sub>		G <sub>2</sub>				G <sub>3</sub>							
		1	3	7	9	1	5	7	11	1	3	5	7
	1	1	3	7	9		1	5	7	11		1	3
	3	3	9	1	7		5	1	11	7		3	1
	7	7	1	9	3		7	11	1	5		5	7
	9	9	7	3	1		11	7	5	1		7	5
							11	7	5	1		7	5

G<sub>2</sub> and G<sub>3</sub> isomorphic. Isomorphism G<sub>2</sub> to G<sub>3</sub>: 1  $\rightarrow$  1, 5  $\rightarrow$  3, 7  $\rightarrow$  5, 11  $\rightarrow$  7

2 ii)  $G_1: x = 1, 9$ ;  $G_2: x = 1, 5, 7, 11$ ;  $G_3: x = 1, 3, 5, 7$

3 Shear in  $x$ -direction moving  $(0, 1)$  to  $(n, 1)$     4 i)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

ii)  $\{1\}, \{1, 3\}, \{1, 5\}, \{1, 7\}$

	f	g	h	k
f	f	g	h	k
g	g	f	k	h
h	h	k	f	g
k	k	h	g	f

iv) Yes. Composition table of  $\{f, g, h, k\}$  obtained from composition table of  $\{1, 3, 5, 7\}$  by replacing 1 with f, 3 with g, 5 with h, and 7 with k.

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

b) Not closed,  $2 \times 4 = 0 \pmod{8}$     c) Whenever  $n$  is not a prime

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

iii)  $\{I\}, \{I, C\}, \{I, D\}, \{I, E\}, \{I, A, B\}$

iv) a) No    b) Yes    c) No

9 i) 2, 2    ii)  $\{e, a, b, c\}, \{e, a, bc, abc\}, \{e, b, ac, abc\}$  and  $\{e, c, ab, abc\}$

iii) Lagrange's theorem, 8 not divisible by 3

9 iv)

Element	0	1	2	3	4	5	6	7
Order	1	8	4	8	2	8	4	8

v) No. Three elements of order 2 in  $G$ , only one in  $H$ .

Element	e	g	$g^2$	$g^3$	$g^4$	$g^5$
Order	1	6	3	2	3	6

Element	i	$h_1$	$h_2$	$h_3$	$h_4$	$h_5$
Order	1	3	3	2	2	2

Proper subgroups of  $H$ :  $\{i\}, \{i, h_1, h_2\}, \{i, h_3\}, \{i, h_4\}, \{i, h_5\}$

10 iii)  $M$  isomorphic to  $G$ . Correspondence is:  $e \rightarrow 1, g \rightarrow 4, g^2 \rightarrow 3, g^3 \rightarrow 12, g^4 \rightarrow 9, g^5 \rightarrow 10$  or  $e \rightarrow 1, g \rightarrow 10, g^2 \rightarrow 9, g^3 \rightarrow 12, g^4 \rightarrow 3, g^5 \rightarrow 4$     11 i) 4, 4

11 ii)  $\{e, p_1, p_2, p_3\}$  and  $\{e, p_2, q_1, q_2\}$

iii) No, 8 not divisible by 6

v)  $G$  isomorphic to  $K$

12 b) 8    10    14    16

4    14    16    8

2    16    8    4

12 c) 10;  $\{10, 8\}, \{10, 4, 16\}$     d) Yes

13 i)  $E_1$  one-way stretch in  $x$ -direction, scale factor  $\frac{1}{3}$ .  $E_2$  shear

$$13 \text{ ii)} \quad E_1 A = \begin{pmatrix} 1 & \frac{2}{3} \\ 1 & 4 \end{pmatrix}, \quad E_2(E_1 A) = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & \frac{10}{3} \end{pmatrix}, \quad E_3(E_2 E_1 A) = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$$

## ANSWERS

13 iii)  $E_1^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ;  $E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{10}{3} \end{pmatrix}$ ,  $E_4^{-1} = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$  iv)  $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$   
[Http://shop60657810.taobao.com](http://shop60657810.taobao.com)

14 i) 

	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

 6 self-inverse ii)  $-1, \frac{17}{3}$  15 ii) B, I,  $A^2B$

15 iii) I, 1; A, 4;  $A^2$ , 2;  $A^3$ , 4; B, 2;  $AB$ , 2;  $A^2B$ , 2;  $A^3B$ , 2 vi) {I, A,  $A^2$ ,  $A^3$ } and {I,  $AB$ ,  $A^3B$ ,  $A^2$ } vi) Only {I, A,  $A^2$ ,  $A^3$ } cyclic

16 a) By Lagrange's theorem: if X has  $n$  elements, then 2 divides  $n$  and so does 4. Minimum 8

17 b)  $M(\frac{1}{3}i)$  has order 4. Required groups are: group of order 4:  $\{M(\frac{1}{3}i), M(-\frac{1}{3}), M(-\frac{1}{3}i), M(\frac{1}{3})\}$ ; group of order 2:  $M(-\frac{1}{3}), M(\frac{1}{3})\}$

17 d) Inverse of  $M(\frac{2}{3})$  is  $M(\frac{1}{6})$ , which is not in S. Hence, S is not a group, and so cannot be a subgroup of G.

18 a) For A, let  $\theta = 0$ ; for B, let  $\theta = \frac{\pi}{2}$ . Then  $A^2 = B^2 = I$ , but  $(AB)^2 = -I$  d) i)  $D_8$  not abelian since  $q * a \neq a * q$

18 d) ii) Set has 6 elements. Because 8 not divisible by 6, there is no subgroup of  $D_8$  with 6 elements.

19 iii) 

		Second						
		○	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$
First	$\pi_1$		$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$

 iv) Symmetries of equilateral triangle

		Second						
		○	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$
First	$\pi_1$		$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$
	$\pi_2$		$\pi_2$	$\pi_3$	$\pi_1$	$\pi_6$	$\pi_4$	$\pi_5$
	$\pi_3$		$\pi_3$	$\pi_1$	$\pi_2$	$\pi_5$	$\pi_6$	$\pi_4$
	$\pi_4$		$\pi_4$	$\pi_6$	$\pi_5$	$\pi_1$	$\pi_2$	$\pi_3$
	$\pi_5$		$\pi_5$	$\pi_4$	$\pi_6$	$\pi_3$	$\pi_1$	$\pi_2$
	$\pi_6$		$\pi_6$	$\pi_5$	$\pi_4$	$\pi_2$	$\pi_3$	$\pi_1$

20 iii) 0 iv) No inverse of  $-1$  v) Delete  $-1$

### Exercise 17C

2 r)  $\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = 0$ ,  $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  4 For example:  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  5  $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$

6 a) Points on line  $2y = x$  b) 1 c)  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  8 b) Point O 9 iii)  $M = \begin{pmatrix} a+c & d-b \\ b+d & a-c \end{pmatrix}$  iv)  $z = -2 + 13j$

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