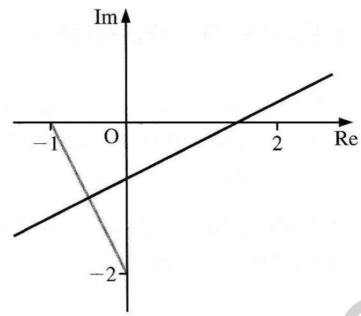
Applying this to $|z| = 1$, we obtain

$$|wi - 2| = |1 + w|$$

Now $|wi - 2| = |i|\,|w + 2i|$, thus we have

$$|w + 2i| = |1 + w| \quad (\text{since } |i| = 1)$$

Therefore, the locus is the perpendicular bisector of the line joining $-2i$ to -1 (see page 13).



Example 25 Find the image of a circle, centre O, radius 1, under the transformation $w = \dfrac{1}{1 - z}$.

SOLUTION

The general point on the original circle is $z = e^{i\theta}$ or $z = \cos\theta + i\sin\theta$. Hence, we have

$$w = \frac{1}{1 - e^{i\theta}} = \frac{1}{1 - \cos\theta - i\sin\theta}$$

Note Do **not** use

$$w = \frac{1}{1 - e^{i\theta}} = \frac{1 - e^{-i\theta}}{(1 - e^{i\theta})(1 - e^{-i\theta})}$$

as $1 - e^{-i\theta}$ is **not** the complex conjugate of $1 - e^{i\theta}$.

Multiplying both the numerator and the denominator by $1 - \cos\theta + i\sin\theta$, we obtain

$$w = \frac{1}{1 - \cos\theta - i\sin\theta} = \frac{1 - \cos\theta + i\sin\theta}{(1 - \cos\theta - i\sin\theta)(1 - \cos\theta + i\sin\theta)}$$

$$= \frac{1 - \cos\theta + i\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$

Using $\cos^2\theta + \sin^2\theta = 1$, we have

$$w = \frac{1 - \cos\theta + i\sin\theta}{2 - 2\cos\theta}$$

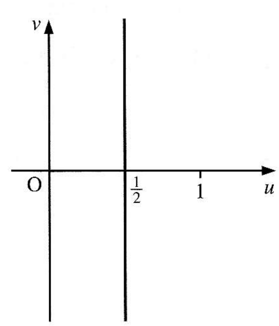
$$= \frac{1}{2} + \frac{i\sin\theta}{2 - 2\cos\theta}$$

Using the half-angle identities for $\sin\theta$ and $\cos\theta$, we obtain

$$w = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right)$$

which gives $u = \frac{1}{2}$, since $w = u + iv$.

Therefore, the locus of w is the straight line, $u = \frac{1}{2}$.



Example 26 Find the image of $|z| = 2$ under the transformation $w = 2z - \dfrac{3}{z}$.

SOLUTION

The general point on the original circle is $z = 2e^{i\theta}$, or $z = 2\cos\theta + 2i\sin\theta$. Hence, we have

$$w = 4\cos\theta + 4i\sin\theta - \frac{3}{2(\cos\theta + i\sin\theta)}$$
$$= 4\cos\theta + 4i\sin\theta - \frac{3}{2}(\cos\theta - i\sin\theta)$$

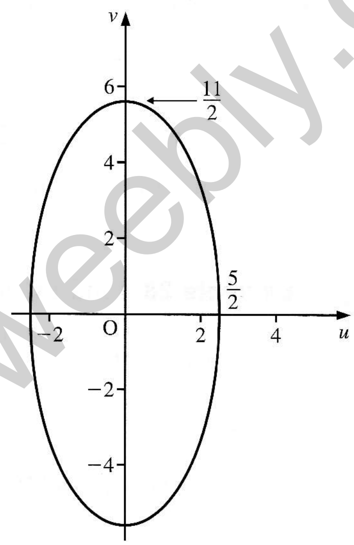
which gives

$$u + iv = \frac{5}{2}\cos\theta + \frac{11}{2}i\sin\theta$$
$$\Rightarrow \quad u = \frac{5}{2}\cos\theta \quad \text{and} \quad v = \frac{11}{2}\sin\theta$$

Eliminating $\cos\theta$ and $\sin\theta$, we obtain

$$\left(\frac{2u}{5}\right)^2 + \left(\frac{2v}{11}\right)^2 = 1$$
$$\Rightarrow \quad \frac{4u^2}{25} + \frac{4v^2}{121} = 1$$

Therefore, the image is an ellipse with the above equation.



Example 27 Find the image of $|z - 7| = 7$ under the transformation $w = \dfrac{28}{z}$ $(z \neq 0)$.

SOLUTION

The general point of $|z - 7| = 7$ is $z = 7 + 7\cos\theta + 7i\sin\theta$. Hence, we have

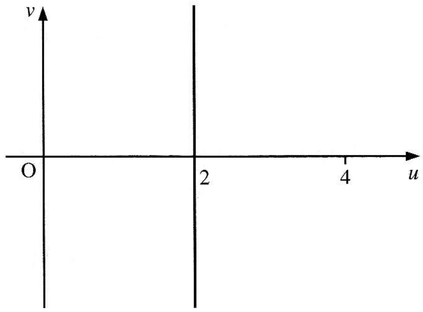
$$w = \frac{28}{7 + 7\cos\theta + 7i\sin\theta}$$
$$= \frac{4}{1 + \cos\theta + i\sin\theta}$$
$$= \frac{4(1 + \cos\theta - i\sin\theta)}{(1 + \cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta)}$$
$$= \frac{4(1 + \cos\theta - i\sin\theta)}{(1 + \cos\theta)^2 + \sin^2\theta}$$
$$= \frac{4(1 + \cos\theta - i\sin\theta)}{2 + 2\cos\theta} = 2 - \frac{4i\sin\theta}{2 + 2\cos\theta}$$

Using the half-angle identities for $\sin\theta$ and $\cos\theta$, we obtain

$$u + \mathrm{i}v = 2 - 2\mathrm{i}\tan\left(\frac{\theta}{2}\right)$$

which gives $u = 2$.

This line, $u = 2$, is $|w - 4| = |w|$, which is the perpendicular bisector of the line joining 0 and 4.



Note We could have found the image from

$$|z - 7| = 7 \quad \Rightarrow \quad \left|\frac{28}{w} - 7\right| = 7$$
$$|28 - 7w| = 7|w|$$

which gives $|w| = |w - 4|$, as required.

Example 28 Find the image of the straight line $3x + 2y = 8$ under the transformation $w = \dfrac{1}{2 - z}$.

SOLUTION

We start by expressing $z \equiv x + \mathrm{i}y$ in terms of $w = u + \mathrm{i}v$:

$$w = \frac{1}{2 - z} \quad \Rightarrow \quad z = 2 - \frac{1}{w}$$

$$\Rightarrow \quad x + \mathrm{i}y = 2 - \frac{1}{u + \mathrm{i}v}$$

$$= 2 - \frac{u - \mathrm{i}v}{u^2 + v^2}$$

Hence, we have

$$x = 2 - \frac{u}{u^2 + v^2} \qquad y = \frac{v}{u^2 + v^2}$$

Using these values in the equation of the line, $3x + 2y = 8$, we obtain

$$3\left(2 - \frac{u}{u^2 + v^2}\right) + 2\left(\frac{v}{u^2 + v^2}\right) = 8$$

$$\Rightarrow \quad 6(u^2 + v^2) - 3u + 2v = 8(u^2 + v^2)$$

$$\Rightarrow \quad u^2 + v^2 + \frac{3}{2}u - v = 0$$

which gives

$$\left(u + \frac{3}{4}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{13}{16}$$

This is the equation of a circle, centre $(-\frac{3}{4}, \frac{1}{2})$ or $-\frac{3}{4} + \frac{1}{2}\mathrm{i}$, radius $\frac{1}{4}\sqrt{13}$.

Exercise 15D

1 For the transformation $w = z^2$, find the locus of w when

a) z lies on a circle centre O, radius 5
b) z lies on the real axis
c) z lies on the imaginary axis.

2 For the transformation $w^2 = z$, find the locus of w when

a) z lies on a circle centre O, radius 5
b) z lies on a circle centre O, radius 2
c) z lies on the imaginary axis.

3 For the transformation $w = z^2$, show that the locus of w, when z moves along a line $y = k$, is a parabola. Find its equation.

4 For the transformation $w = \dfrac{z + \mathrm{i}}{\mathrm{i}z + 2}$, find

a) the locus of w when z lies on the real axis
b) the locus of w when z lies on the imaginary axis
c) any invariant points.

5 For the transformation $w = 3z + 2\mathrm{i} - 5$, find the locus of w for $|z| = 4$.

6 a) For the transformation $w = \dfrac{az + b}{z + c}$, where $a, b, c \in \mathbb{R}$, find a, b and c given that $w = 3\mathrm{i}$ when $z = -3\mathrm{i}$, and $w = 1 - 4\mathrm{i}$ when $z = 1 + 4\mathrm{i}$.
b) Show that the points for which $w = \bar{z}$ lie on a circle. Find its centre and radius.

7 Find the image under the transformation $w = \dfrac{3\mathrm{i} + z}{2 - z}$, where z is the circle $|z| = 3$.

8 Find the image of $|z| = 3$ under the transformation $w = 3z + \dfrac{4}{z}$.

9 Find the image of $|z - 5| = 5$ under the transformation $w = \dfrac{30}{z}$ $(z \neq 0)$.

10 The point P in the Argand diagram represents the complex number z.

a) Given that $|z| = 1$, sketch the locus of P.

The point Q is the image of P under the transformation

$$w = \frac{1}{z - 1}$$

b) Given that $z = \mathrm{e}^{\mathrm{i}\theta}$, $0 < \theta < 2\pi$, show that $w = -\frac{1}{2} - \frac{1}{2}\mathrm{i}\cot\frac{1}{2}\theta$
c) Make a separate sketch of the locus of Q. (EDEXCEL)

11 i) Solve the equation $z^3 + 8\mathrm{i} = 0$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$.

ii) The point P represents the complex number z in an Argand diagram. Given that $|z - 3\mathrm{i}| = 2$,

a) sketch the locus of P in an Argand diagram.

Transformations T_1, T_2 and T_3 from the z-plane to the w-plane are given by

$$T_1: \quad w = \mathrm{i}z$$
$$T_2: \quad w = 3z$$
$$T_3: \quad w = z^*$$

b) Describe precisely the locus of the image of P under each of these transformations.
(EDEXCEL)

12 A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+1}{z-1} \qquad z \neq 1$$

Find the image in the w-plane of the circle $|z| = 1$, $z \neq 1$, under the transformation T.
(EDEXCEL)

13 The transformation, T, from the z-plane to the w-plane is given by

$$w = \frac{1}{z-2} \qquad z \neq 2$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

Show that under T the straight line with equation $2x + y = 5$ is transformed to a circle in the w-plane with centre $(1, -\frac{1}{2})$ and radius $\frac{1}{2}\sqrt{5}$. (EDEXCEL)

14 The complex numbers z and w are defined by

$$z = \mathrm{e}^{(1+2\mathrm{i})\phi} \quad \text{and} \quad w = \frac{z}{1+\mathrm{i}}$$

where ϕ is real.

a) i) Show that $|z| = \mathrm{e}^{\phi}$ and $\arg z = 2\phi$.

ii) In an Argand diagram, z is represented by the point P. Sketch the locus of P when ϕ varies from 0 to π.

b) i) Show that the imaginary part of w is

$$\frac{1}{2}\mathrm{e}^{\phi}(\sin 2\phi - \cos 2\phi)$$

ii) Determine the values of ϕ in the interval $0 \leqslant \phi \leqslant \pi$ for which w is real. (NEAB)

15 Given that $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are complex numbers related by $w = \dfrac{1}{z} + 1$, obtain expressions for u and v in terms of x and y.

The complex numbers z and w are represented by the points P and Q respectively in the Argand diagram. Given that P moves along the line $y = 2x$, show that Q moves along the line $2u + v - 2 = 0$. (WJEC)

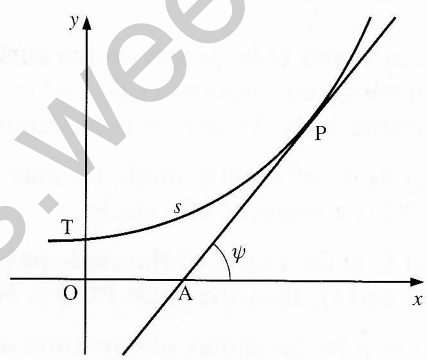
16 Intrinsic coordinates

It is no paradox to say that in our most theoretical moods we may be nearest to our practical applications.
ALFRED NORTH WHITEHEAD

We have already seen that the position of a point on a curve (and hence the curve's equation) may be given in terms of:

- cartesian coordinates (x, y), or
- polar coordinates (r, θ) (see pages 43–56).

We can also define the position of a point on a curve by means of **intrinsic coordinates** (s, ψ), where s is the length of the arc from a fixed point to the given point, and ψ is the angle which the tangent to the curve at that point makes with the x-axis.



Thus, referring to the figure on the right, intrinsic coordinates would give the position of point P in terms of the arc length PT and the angle which PA makes with Ox.

We must stress, however, that the majority of the equations of curves cannot realistically be given in intrinsic form. Also, only in rare cases is it sensible to try to convert the cartesian, parametric or polar equation of a curve to its intrinsic form.

But two curves in particular are more readily treated in their intrinsic forms. They are the **catenary** (see Example 2, on pages 365–6) and the **cycloid** (see Example 3, on pages 366–7).

Trigonometric functions of ψ

Considering the gradient of a tangent, we have

$$\frac{dy}{dx} = \tan\psi$$

When we derived the length of the arc of a curve (see pages 250–3), we found that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \tan^2\psi}$$

Using the identity $1 + \tan^2 \psi \equiv \sec^2 \psi$, we obtain

$$\frac{ds}{dx} = \sec \psi$$

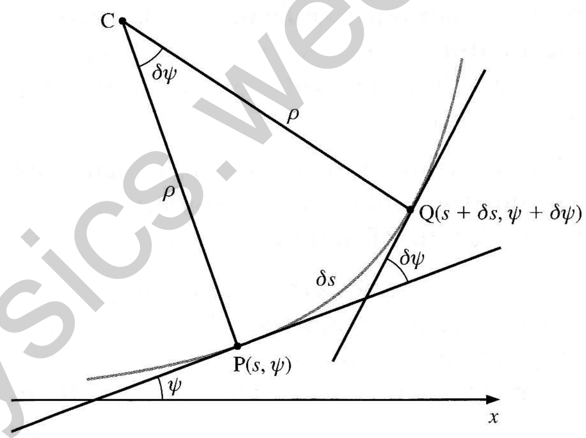
$$\Rightarrow \quad \cos \psi = \frac{dx}{ds}$$

Using $\sin \psi = \tan \psi \cos \psi$, we have

$$\sin \psi = \frac{dy}{dx}\frac{dx}{ds}$$

$$\Rightarrow \quad \sin \psi = \frac{dy}{ds}$$

Radius of curvature



Let P and Q be points on the curve with intrinsic coordinates (s, ψ) and $(s + \delta s, \psi + \delta\psi)$ respectively. Hence, δs is the length of PQ.

If δs is sufficiently small, we may assume that PQ is a segment of a circle.

If C is the centre of the circle passing through P and Q, then the angle PCQ is $\delta\psi$.

Let ρ be the radius of curvature at P. Hence, the length of PQ is $\rho\delta\psi$. That is,

$$\delta s = \rho\delta\psi \quad \Rightarrow \quad \rho = \frac{\delta s}{\delta\psi}$$

As $\delta s \to 0$, this gives

$$\text{Radius of curvature} = \rho = \frac{ds}{d\psi}$$

To find the radius of curvature in terms of x and y, we need to differentiate $\frac{dy}{dx} = \tan \psi$ with respect to x, which gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan \psi)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{d\psi}(\tan \psi)\frac{d\psi}{dx} = (\sec^2 \psi)\frac{d\psi}{dx}$$

$$\Rightarrow \quad \frac{dx}{d\psi} = \frac{\sec^2 \psi}{\dfrac{d^2y}{dx^2}}$$

Using $\rho = \dfrac{\mathrm{d}s}{\mathrm{d}\psi} = \dfrac{\mathrm{d}s}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}\psi}$ and substituting for $\dfrac{\mathrm{d}s}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}\psi}$, we have

$$\rho = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\left(\frac{\sec^2\psi}{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}}\right)$$

$$\Rightarrow \quad \rho = \frac{\sqrt{1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}(1 + \tan^2\psi)}{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}} = \frac{\sqrt{1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\left[1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]}{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}}$$

which gives

$$\rho = \frac{\left[1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}} \qquad [1]$$

When x and y are given in terms of a parameter t, we can find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}$, and hence $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, in terms of t.

We have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}$$

Using $\dot{y}$ and $\dot{x}$ to indicate that we have differentiated with respect to t (see page 252), we have

$$\dot{y} \equiv \frac{\mathrm{d}y}{\mathrm{d}t} \quad \text{and} \quad \ddot{y} \equiv \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{and} \quad \ddot{x} \equiv \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

which give

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{1}{\dot{x}}$$

Remembering that $\dfrac{\dot{y}}{\dot{x}}$ is a quotient, we find

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right)\frac{1}{\dot{x}} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3}$$

Substituting this expression in [1], we obtain

$$\rho = \frac{\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{3}{2}}}{\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3}} \quad \Rightarrow \quad \rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\ddot{y}\dot{x} - \ddot{x}\dot{y}}$$

Thus, the radius of curvature is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \text{or} \quad \rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\ddot{y}\dot{x} - \ddot{x}\dot{y}} \quad \text{or} \quad \rho = \frac{ds}{d\psi}$$

Example 1 Find the radius of curvature of the rectangular hyperbola $y = \frac{16}{x}$, given that its parametric coordinates are $x = 4t$, $y = \frac{4}{t}$.

SOLUTION

Method 1

We use the recommended method of staying in the parametric form throughout. Hence, we have

$$\dot{x} = 4 \quad \Rightarrow \quad \ddot{x} = 0 \qquad \dot{y} = -\frac{4}{t^2} \quad \Rightarrow \quad \ddot{y} = \frac{8}{t^3}$$

Substituting for $\dot{x}$, $\ddot{x}$, $\dot{y}$ and $\ddot{y}$ in

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\ddot{y}\dot{x} - \ddot{x}\dot{y}}$$

where ρ is the radius of curvature, we obtain

$$\rho = \frac{t^3\left(16 + \frac{16}{t^4}\right)^{\frac{3}{2}}}{32}$$

$$\Rightarrow \quad \rho = 2t^3\left(1 + \frac{1}{t^4}\right)^{\frac{3}{2}}$$

Method 2

We could use the cartesian form, which readily gives $\frac{dy}{dx}$ but from which $\frac{d^2y}{dx^2}$ is rather more difficult to obtain, as the following shows.

We have

$$x = 4t \quad \Rightarrow \quad \frac{dx}{dt} = 4 \qquad y = \frac{4}{t} \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{4}{t^2}$$

which give

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{4}{t^2} \times \frac{1}{4} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{t^2}$$

Differentiating again, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{t^2}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{1}{t^2}\right)\frac{dt}{dx} = \frac{2}{t^3} \times \frac{1}{4} = \frac{1}{2t^3}$$

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

where ρ is the radius of curvature, we have

$$\rho = 2t^3\left(1 + \frac{1}{t^4}\right)^{\frac{3}{2}}$$

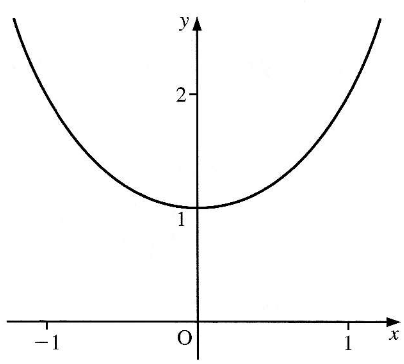
Finding intrinsic equations

Example 2 $y = \cosh x$ passes through the point $(0, 1)$. Find the intrinsic equation of the curve.

SOLUTION

We know that

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



which gives

$$s = \int \sqrt{1 + \sinh^2 x}\, dx$$

$$\Rightarrow \quad s = \int \cosh x\, dx = \sinh x + c$$

When $x = 0$, $s = 0$, which gives $c = 0$. Hence, we find

$$s = \sinh x$$

Now $\frac{dy}{dx} = \sinh x$. So, using $\tan\psi = \frac{dy}{dx}$ (see page 361), we obtain

$$\sinh x = \tan\psi$$

Since $s = \sinh x$, we have

$$s = \tan\psi$$

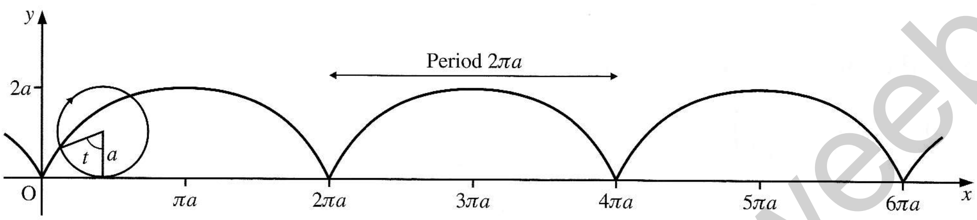
This is an equation with s and ψ as the only variables. Therefore, the intrinsic equation of $y = \cosh x$ is $s = \tan\psi$.

The curve $y = \cosh x$ (which we met on pages 189 and 190) is a **catenary**.

The catenary is the form assumed by a uniform, heavy and flexible cable hanging freely between two points. An example is a slack mooring line between a ship and a quay. In large suspension bridges, where heavy cables are used, the curve assumed by the cables is sometimes close to a catenary.

The standard intrinsic equation of the catenary is $s = a\tan\psi$, where a is the y-intercept, corresponding to the standard cartesian equation $y = a\cosh\left(\frac{x}{a}\right)$.

Another curve of practical interest (for example, as the flank profile of the teeth of certain gear wheels) is the **cycloid**. This is the locus of a point fixed on the circumference of a circle which is rolling along a stationary, straight base-line, as shown below.



Note that the distance between successive cusps is $2\pi a$, where a is the radius of the rolling circle. Hence, the catenary is **periodic** with period $2\pi a$.

The cartesian equation of the cycloid is difficult to derive, hence we normally work with its parametric equations

$$x = a(t - \sin t) \quad \text{and} \quad y = a(1 - \cos t)$$

where t is the central angle of the circle, as shown in the figure.

Example 3 Find the intrinsic equation of the cycloid.

SOLUTION

We know that

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Differentiating the parametric equations for the cycloid and substituting them in the above, we obtain

$$s = \int \sqrt{a^2(1 - \cos t)^2 + a^2\sin^2 t}\, dt$$

$$\Rightarrow \quad s = a\int \sqrt{2 - 2\cos t}\, dt$$

Using $\cos t \equiv 1 - 2\sin^2\left(\frac{t}{2}\right)$, we have

$$s = a\int \sqrt{2 - 2\left[1 - 2\sin^2\left(\frac{t}{2}\right)\right]}\, dt = a\int 2\sin\left(\frac{t}{2}\right) dt$$

which gives

$$s = -4a\cos\left(\frac{t}{2}\right) + c \qquad [1]$$

Using

$$\tan\psi = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

we obtain

$$\tan\psi = \frac{\sin t}{1 - \cos t}$$

Using $\sin t \equiv 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)$ and $\cos t \equiv 1 - 2\sin^2\left(\frac{t}{2}\right)$, we have

$$\tan\psi = \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{1 - \left[1 - 2\sin^2\left(\frac{t}{2}\right)\right]} = \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}$$

which gives

$$\tan\psi = \cot\left(\frac{t}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{t}{2}\right)$$

$$\Rightarrow \quad \psi = \frac{\pi}{2} - \frac{t}{2} \quad \Rightarrow \quad \frac{t}{2} = \frac{\pi}{2} - \psi$$

Substituting for $\frac{t}{2}$ in [1], we have

$$s = c - 4a\cos\left(\frac{\pi}{2} - \psi\right)$$

Therefore, the intrinsic equation of the cycloid is

$$s = c - 4a\sin\psi$$

The value of c will be different for each arch of the cycloid.

Exercise 16

In Questions **1** to **8**, find the radius of curvature of each curve at the point specified.

1 $y^2 = x^3 + 3$, at $(1, 2)$.

2 $y = e^x$, at $(1, e)$.

3 $y = \sin x$, when $x = \frac{\pi}{3}$.

4 $y = x\ln x$, at $(1, 0)$.

5 $x = t^3$, $y = t^2$, when $t = 1$.

6 $x = ct$, $y = \frac{c}{t}$, when $t = 2$.

7 $x = \cos^2 t$, $y = \sin^2 t$, when $t = \frac{\pi}{4}$.

8 $x = a\cos^3 t$, $y = a\sin^3 t$, when $t = \frac{\pi}{3}$.

9 Find the radius of curvature, in terms of ψ, for

a) $s = \psi^3 + \cos\psi$ **b)** $s = 3\psi + 4\psi\sin\psi$ **c)** $s = \psi\cos\psi + \psi^2$

10 Find the intrinsic equation of the curve $y = \ln \sec x$, where s is the distance from the origin.

11 A curve has intrinsic equation $s = a \cos \psi$.

a) Calculate the radius of curvature of the curve in terms of ψ.
b) Show that the tangent to the curve at the point where $s = 0$ is parallel to the y-axis.

(EDEXCEL)

12 The curve C has equation $y = 3\cosh\left(\dfrac{x}{3}\right)$.

a) Show that the radius of curvature, at the point on C where $x = t$, is $3\cosh^2\left(\dfrac{t}{3}\right)$.

b) Find the radius of curvature at the point where $t = 1.5$, giving your answer to three significant figures.
c) Find the area of the surface generated when the arc of C between $x = -3$ and $x = 3$ is rotated through 2π radians about the x-axis, giving your answer in terms of e and π.

(EDEXCEL)

13 A curve has parametric equations $x = 4t - \frac{1}{3}t^3$, $y = 2t^2 - 8$.

i) Show that the radius of curvature at a general point $(4t - \frac{1}{3}t^3, 2t^2 - 8)$ on the curve is $\frac{1}{4}(4 + t^2)^2$.
ii) Find the centre of curvature corresponding to the point on the curve given by $t = 3$.

The arc of the curve given by $0 \leqslant t \leqslant 2\sqrt{3}$ is denoted by C.

iii) Find the length of the arc C.
iv) Find the area of the curved surface generated when the arc C is rotated about the y-axis.

(MEI)

14 A curve is given parametrically by $x = e^{\theta}(2\sin 2\theta + \cos 2\theta)$, $y = e^{\theta}(\sin 2\theta - 2\cos 2\theta)$. P is the point corresponding to $\theta = 0$, and Q is the point corresponding to $\theta = \alpha$ (where $\alpha > 0$).

i) Show that the gradient of the curve at Q is $\tan 2\alpha$, and find the length of the arc of the curve between P and Q.
ii) Using intrinsic coordinates (s, ψ), where s is the arc length of the curve measured from P and $\tan \psi = \dfrac{dy}{dx}$, show that $s = 5(e^{\frac{1}{2}\psi} - 1)$.
iii) Find the radius of curvature at the point Q.
iv) Show that the centre of curvature corresponding to the point Q is

$$\left(\tfrac{1}{2}e^{\alpha}(2\cos 2\alpha - \sin 2\alpha), \tfrac{1}{2}e^{\alpha}(2\sin 2\alpha + \cos 2\alpha)\right) \qquad \text{(MEI)}$$

17 Groups

Before the word 'group' appeared in the mathematical literature, there had been a longer period of development in which mathematicians applied group-theoretical results without the concept of a group being explicitly defined.

WALTER PURKERT AND HANS WUSSING

Binary and unary operations

A **binary operation**, usually denoted by $*$, is a rule which takes an ordered pair of elements, a and b, and gives a uniquely defined third element, c, so that $a * b = c$. (Other symbols used to represent a binary operation include $\bigcirc$, $\otimes$ and $\oplus$.)

For example, multiplication is a binary operation. If we represent $*$ by multiplication, then

$$4 * 3 = 4 \times 3 = 12$$

Addition is also a binary operation. If we represent $*$ by addition, then

$$6 * 3 = 6 + 3 = 9$$

Likewise for division, where we have

$$6 * 3 = \frac{6}{3} = 2$$

But note that in the case of division, the operation is **not commutative**. Hence, we have

$$3 * 6 = \frac{3}{6} = \frac{1}{2}$$

That is,

$$6 * 3 \neq 3 * 6$$

In general, we have

$$a * b \neq b * a$$

Thus, for some binary operations, the order in which we enter the elements does matter.

Unary operations

A **unary operation** is one which uses only one element. For example, $a \to a^2$ is a unary operation.

Modular arithmetic

We can perform arithmetical operations in different **moduli**. To indicate the use of a particular modulo, say n, we add (mod n) after we have completed the calculation.

Take, for example, the **multiplication** of two integers in modulo 6. We multiply the two integers normally and then subtract 6 repeatedly until the answer is between 0 and 5.

Hence, we have for $3 \times 3 = 9$

$$3 \times 3 = 3 \pmod{6} \quad \text{since } 9 - 6 = 3$$

Similarly, for $5 \times 4 = 20$, we have

$$5 \times 4 = 2 \pmod{6} \quad \text{since } 20 - 6 - 6 - 6 = 2$$

And for $4 \times 3 = 12$, we have

$$4 \times 3 = 0 \pmod{6} \quad \text{since } 12 - 6 - 6 = 0$$

Modular **addition** is similar to multiplication. Suppose we want to add two integers in modulo 4. We add them normally and then subtract 4 repeatedly until the answer is between 0 and 3.

For example, we have

$$2 + 3 = 5 = 1 \pmod{4} \qquad 2 + 0 = 2 \pmod{4}$$

$$1 + 3 = 4 = 0 \pmod{4} \qquad 3 + 3 = 2 \pmod{4}$$

Example 1 Express 9×11 in modulo 17.

SOLUTION

We have $9 \times 11 = 99$, which becomes

$$9 \times 11 = 14 \pmod{17} \quad \text{since } 99 - 17 - 17 - 17 - 17 - 17 = 14$$

Definition of a group

A **group** comprises

- a set of elements (or members), G, together with
- a binary operation $*$ on this set.

To be a group, G must satisfy the following **four properties** (sometimes referred to as **axioms**).

- **Closure** G must be closed. This means that if a and b are members of G, then $a * b$ must also be a member of G. This is written as

$$a * b \in G, \text{ for all } a \text{ and } b \in G$$

- **Associativity** Provided their original order is preserved, the result of combining a, b and c does not depend on which two adjacent elements are combined first. This is written as

$$(a * b) * c = a * (b * c), \text{ for all } a, b \text{ and } c \in G$$

- **Identity** There is an element e in G for which $a * e = e * a = a$ for every a in G. That is, there is an **identity element** e in G which does not change any other element.
- **Inverses** For any element a in G, there is an **inverse element** of a in G, denoted by a^{-1}. This is written as

$$\text{For any } a \in G, \text{ there exists } a^{-1} \in G, \text{ for which } a * a^{-1} = a^{-1} * a = e$$

To confirm that a set of elements, together with an operation on the set, forms a group, we have to verify that the set possesses **every one** of these four properties. This can be difficult, since we need to check **each property** for **every element** or **pairs of elements** of the set.

Note When you are given a question involving a group, you will also always be given a binary operation (which is usually multiplication or addition). It is essential that you recognise which binary operation is being used.

Example 2 Prove that the set $G = \{1, \mathrm{i}, -1, -\mathrm{i}\}$ under multiplication is a group (where $\mathrm{i}^2 = -1$).

SOLUTION

To prove that this is a group, we need to verify each of the four properties in turn. It is essential to confirm that **all** the properties are satisfied.

Closure We have to verify that, for any a and $b \in G$, $a * b \in G$.

Therefore, if we take any element in G and multiply it by any other element in G, the result should be an element in G. One way to check this is to take every pair in turn. (This method is only feasible in this case because G is a small group.) Hence, we have

$1 * 1 = 1$	$1 * \mathrm{i} = \mathrm{i}$	$1 * -1 = -1$	$1 * -\mathrm{i} = -\mathrm{i}$
$\mathrm{i} * 1 = \mathrm{i}$	$\mathrm{i} * \mathrm{i} = -1$	$\mathrm{i} * -1 = -\mathrm{i}$	$\mathrm{i} * -\mathrm{i} = 1$
$-1 * 1 = -1$	$-1 * \mathrm{i} = -\mathrm{i}$	$-1 * -1 = 1$	$-1 * -\mathrm{i} = \mathrm{i}$
$-\mathrm{i} * 1 = -\mathrm{i}$	$-\mathrm{i} * \mathrm{i} = 1$	$-\mathrm{i} * -1 = \mathrm{i}$	$-\mathrm{i} * -\mathrm{i} = -1$

That is, $a * b \in G$.

Associativity We have to verify that $(a * b) * c = a * (b * c)$ for all a, b and c in G.

We have, for example,

$$(\mathrm{i} * -1) * -\mathrm{i} = -\mathrm{i} * -\mathrm{i} = -1 \quad \text{and} \quad \mathrm{i} * (-1 * -\mathrm{i}) = \mathrm{i} * \mathrm{i} = -1$$

This verifies associativity for just this one triple combination. To prove associativity by this method, we would have to check every other triple combination, of which there are 64.

Alternatively, we can simply recall and state the fact that multiplication of complex numbers is associative.

Identity 1 is the identity element of this group. This is because multiplying any number by 1 does not change its value. To confirm that 1 is the identity element, we have to verify that $1 * a = a * 1 = a$ for every a in G.

In this case, it is not too onerous to do all the calculations concerned. Hence, we have

$$1 * 1 = 1 * 1 = 1 \qquad 1 * \mathrm{i} = \mathrm{i} * 1 = \mathrm{i}$$

$$1 * -\mathrm{i} = -\mathrm{i} * 1 = -\mathrm{i} \qquad 1 * -1 = -1 * 1 = -1$$

Alternatively, we can simply state that 1 is the identity, since we know that multiplying any number by 1 does not change its value.

Inverses We need to find the inverse of each of the elements 1, i, -1 and $-\mathrm{i}$ to confirm that each inverse is a member of the group. That is, for each element a in G, we need to find an a^{-1}, and verify that $a * a^{-1} = a^{-1} * a = 1$.

Hence, we have

Inverse of 1 is 1, since $1 * 1 = 1 * 1 = 1$

Inverse of i is $-\mathrm{i}$, since $\mathrm{i} * -\mathrm{i} = -\mathrm{i} * \mathrm{i} = 1$

Inverse of -1 is -1, since $-1 * -1 = -1 * -1 = 1$

Inverse of $-\mathrm{i}$ is i, since $-\mathrm{i} * \mathrm{i} = \mathrm{i} * -\mathrm{i} = 1$

Since all of the properties are satisfied for any choice of elements, we have proved that G is a group.

It can take a long time to prove that a set of elements, together with an operation on the set, forms a group, especially if there are many elements. However, there are short-cuts we can take.

- We can use algebraic rules to prove closure. For example, to prove that the set of integers under addition forms a group, we just state that the sum of any two integers is always an integer.
- Associativity is always difficult to prove. However, we recall that the multiplication and the addition of real numbers, the multiplication and the addition of complex numbers, and the multiplication and the addition of square matrices, are all associative.
- To find the identity of a group, we recall that 0 is the identity for addition (since adding zero to a number does not change the number). We recall also that 1 is the identity for multiplication (since multiplying a number by 1 does not change the number). We must be careful, however, because in some unusual cases of multiplication, such as modulo 14, the identity may not be 1 (see page 375).
- To find inverses, we often just need to give a general formula which identifies all the inverses.

Example 3 Prove that the set $G = \{0, 1, 2, 3\}$ under the binary operation addition (mod 4) forms a group.

SOLUTION

As usual, we must verify that all the group properties are satisfied.

Closure Whenever we add two numbers (mod 4), we always get a number between 0 and 3. Therefore, addition (mod 4) is closed.

Associativity Addition is associative, and so addition (mod 4) must also be associative.

Identity Adding 0 to a number (mod 4) does not change the number. So 0 is the identity of addition (mod 4).

Inverses We have:

Inverse of 0 is 0, since $0 + 0 = 0 \pmod 4$

Inverse of 1 is 3, since $1 + 3 = 3 + 1 = 0 \pmod 4$

Inverse of 2 is 2, since $2 + 2 = 0 \pmod 4$

Inverse of 3 is 1, since $3 + 1 = 1 + 3 = 0 \pmod 4$

Therefore, all four group properties are satisfied.

Hence, the set $G = \{0, 1, 2, 3\}$ under the binary operation addition (mod 4) forms a group.

Group table

A **group table** shows the effect of combining any two elements. (Other descriptions commonly used are Cayley table, composition table, combination table, operation table and multiplication table.) The entry in row a and column b is the composition $a * b$.

The group table for the set $G = \{0, 1, 2, 3\}$ under addition modulo 4 is shown below. As an example, [3] identifies the result $1 * 2 = 3$.

+(mod 4)	0	1	2	3
0	0	1	2	3
1	1	2	[3]	0
2	2	3	0	1
3	3	0	1	2

To complete the table, we need to find each of the 16 results.

We can use the fact that $x * e = x$ and $e * x = x$ to find seven of these results quite simply. All the other entries have to be calculated.

Even though we have to complete all the entries in the table, it is often easier to draw and use a group table to see whether the set under the operation forms a group.

For the group properties, we have:

Closure This can be seen by noting that all the results in the group table are in the original set.

Associativity This **cannot** be seen from the group table.

Identity The column under the identity element and the row across from the identity element contain the elements in the same order as the original set.

The row and the column given below show that 0 is the identity element:

+ (mod 4)	0	1	2	3
0	0	1	2	3
1	1			
2	2			
3	3			

Note The identity element does not have to be 0 or 1. For example, see the group table on page 375 for the set of integers $\{2, 4, 6, 8, 10, 12\}$ under multiplication (mod 14).

Inverses We can find the position of the identity element in each column and each row. For example, $1 * 3 = 3 * 1 = 0$, which is the identity. Therefore, 3 is the inverse of 1.

In fact, the set $G = \{0, 1, 2, \ldots, m-1\}$ under the binary operation, addition (mod m), also forms a group. (You can check this for yourself for various values of m.) Notice that, in general, the inverse of k under addition (mod m) is $m - k$.

Example 4 Find whether the set $\{1, 3\}$ under multiplication (mod 11) forms a group.

SOLUTION

We can find the answer by checking each of the group properties in turn, until we find one which does **not** work. We recall that for G to be a group, we need to check that **all** four group properties are satisfied. So, to check that G is **not** a group, we need only to find **one** property which is not satisfied.

In this case, since

$$3 * 3 = 3 \times 3 = 9 \pmod{11}$$

and 9 is **not** a member of the original set, **closure does not hold.**

Since the set $\{1, 3\}$ is not closed under multiplication (mod 11), it does **not** form a group.

Note If in Example 4 we were to consider the other group properties, we would find:

- The group is associative, since multiplication is associative.
- There is an identity element, 1, since 1 is the identity under multiplication.
- There is, however, no element a for which $3 * a \equiv 1 \pmod{11}$, and so the property of possessing an inverse element is not satisfied either.

Example 5 Prove that the set of integers $\{2, 4, 6, 8, 10, 12\}$ under multiplication (mod 14) forms a group.

SOLUTION

Again, we need to check that all the group properties hold. However, the last two are difficult to prove, and so we have to use a group table to work out how all the elements combine.

Closure If we multiply two even integers together, we obtain an even integer, which is also even (mod 14). Hence, the set is closed.

Associativity Multiplication is associative.

Identity There is no obvious identity element. The identity element we would naturally look for, 1, is missing from the group. To overcome this problem, we draw the group table, which shows the effect of combining any two elements.

$\times$ (mod 14)	2	4	6	8	10	12
2	4	8	12	2	6	10
4	8	2	10	4	12	6
6	12	10	8	6	4	2
8	2	4	6	8	10	12
10	6	12	4	10	2	8
12	10	6	2	12	8	4

From this table, we can see the column under 8, or the row across from 8, is 2, 4, 6, 8, 10, 12, which is the same as the original set. Thus, multiplication by 8 changes none of the elements of the group, and so 8 is the identity element of this group.

Inverses As with most problems involving multiplication (mod n), there is no easy way to prove that every element has an inverse. However, as in Example 4, we can use the group table. To find the inverse of 2, we need an element a for which $2 * a = 8$. (Remember that 8 is the identity element of this group.)

Inverse of 2 is 4, since $2 * 4 = 4 * 2 = 8$ (mod 14)

Inverse of 4 is 2, since $4 * 2 = 2 * 4 = 8$ (mod 14)

Inverse of 6 is 6, since $6 * 6 = 6 * 6 = 8$ (mod 14)

Inverse of 8 is 8, since $8 * 8 = 8 * 8 = 8$ (mod 14)

Inverse of 10 is 12, since $10 * 12 = 12 * 10 = 8$ (mod 14)

Inverse of 12 is 10, since $12 * 10 = 10 * 12 = 8$ (mod 14)

Thus, we have checked that all the group properties hold.

Therefore, the set of integers $\{2, 4, 6, 8, 10, 12\}$ under multiplication (mod 14) forms a group.

Note We say that the number 6 in Example 5 is **self inverse**, since its inverse is itself. (See also page 393.)

Exercise 17A

In Questions **1** to **4**, prove that each set under the given operation satisfies all the group properties and hence forms a group.

1 The set $\{1, 5\}$ under ($\times$, mod 12).

2 The set $\{1, 2, 3, 4\}$ under ($\times$, mod 5).

3 The set $\{0, 1, 2, 3, 4, 5\}$ under ($+$, mod 6).

4 The set $\{1, 2, 3, 4, 5, 6\}$ under ($\times$, mod 7).

5 Show that the set $\{1, 3\}$ under ($\times$, mod 12) does not form a group.

6 Show that the set of positive integers under addition is not a group.

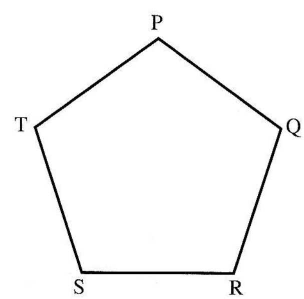
Symmetries of a regular *n*-sided polygon

The set of symmetries of a regular polygon forms a group under the composition of symmetries. Hence, this is true of the set of symmetries of, for example, a square, a regular hexagon and a regular heptagon.

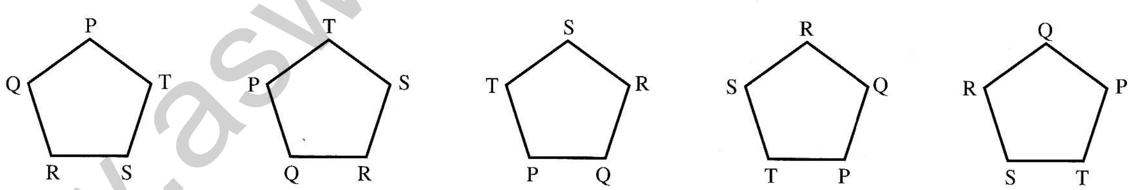
Example 6 Prove that the set of symmetries of a regular pentagon under composition forms a group.

SOLUTION

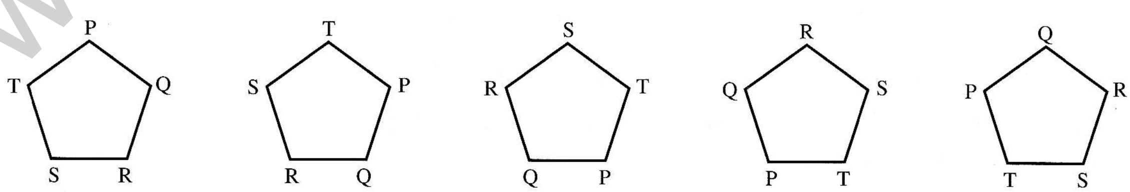
It is easier to specify this group geometrically than to write down all the elements. The symmetries of a pentagon, PQRST, shown on the right, are the five reflections (top row) and the five rotations (bottom row) drawn below.



Reflections



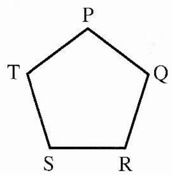
Rotations



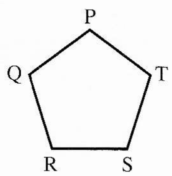
The binary operation in this case is the composition of symmetries. For example, the composition of a clockwise rotation through 72° and a clockwise rotation through 216° is a clockwise rotation through 288°.

To prove that the set of symmetries forms a group, we must check each of the four group properties.

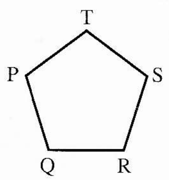
Closure The composition of two rotations is another rotation. The composition of a reflection and a rotation is a reflection, as shown in the example below.



Reflection in the perpendicular bisector of RS, which passes through P, ...

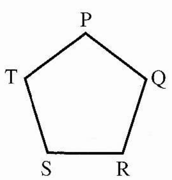


... followed by an anticlockwise rotation through 72° ...

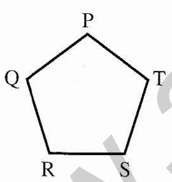


... is the reflection in the perpendicular bisector of PT, which passes through R.

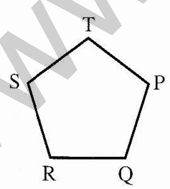
The composition of two reflections is a rotation, as shown in the examples below:



Reflection in the perpendicular bisector of RS, which passes through P, ...



... followed by reflection in the perpendicular bisector of PT, which passes through R, ...



... is an anticlockwise rotation through 288°.

So, the set of symmetries of a pentagon is closed.

Associativity We certainly do **not** want to prove $a * (b * c) = (a * b) * c$ for each of the ten symmetries, giving 1000 possible combinations! Instead, we recall that each symmetry can be represented by a 2×2 matrix (see page 310). Thus, the composition of transformations corresponds to the multiplication of matrices. Since the multiplication of matrices is associative, so is the composition of transformations. Hence, the set of symmetries of a pentagon is associative.

Identity The identity transformation (rotation of $0°$) is in the set of transformations.

Inverses Every symmetry has an inverse. In each case, the inverse is a symmetry which returns the pentagon to its original position. For a clockwise transformation of $n°$, the anticlockwise rotation of $-n°$ is the inverse transformation.

Carrying out the same reflection twice, always returns the pentagon to its original position. Thus, the inverse of a given reflection is the same reflection. Hence, all reflections are self-inverse.

Therefore, we have verified that the symmetries of a pentagon form a group.

Dihedral groups

A group of symmetries is called a **dihedral group**. The group of symmetries of a pentagon contains ten (2×5) elements and is denoted by D_{10}. The symmetries of the other regular polygons also form groups. For example, the symmetries of a regular heptagon form a group. Since a heptagon has seven sides, the group contains 14 (2×7) elements. Hence, the group of symmetries of a regular heptagon is denoted by D_{14}.

Non-finite groups

The groups that we have so far considered are all composed of sets which contain a finite number of elements. We will now consider groups whose sets contain an **infinite** number of elements.

When dealing with a non-finite group, we use a similar approach to that which we use with finite groups, except that we **cannot construct** a group table because there is an infinite number of elements. However, this does not make the verification of the group too much harder; it just means that it has to be done **algebraically**.

Example 7 Prove that the set of integers under addition forms a group.

SOLUTION

Since there is an infinite number of integers, we cannot use a group table. We therefore use algebraic methods to verify that all four properties are satisfied.

Closure If we add together any two integers, we always get an integer. Therefore, if a and b are integers, we know that

$$a * b = a + b = c$$

and hence c is an integer. Therefore, the set of integers under addition is closed.

Associativity We may simply quote the fact that addition is always associative.

Identity As always with addition, 0 is the identity element. For any given integer, a, we have

$$a * 0 = 0 * a = a + 0 = 0 + a = a$$

This proves that 0 is the identity element for the group.

Inverses Given any integer a, its inverse is $-a$. This is because

$$a * -a = a + -a = 0 \quad \text{and} \quad -a * a = -a + a = 0$$

Therefore, we have checked that the four properties are satisfied, and so the set of integers under addition forms a group.

Example 8 Prove that the set of integers under multiplication as the binary operation does not form a group.

SOLUTION

We recall that to prove that a set under an operation does **not** form a group, we just need to check that one of the properties is not satisfied.

In this case, the inverse property does not hold.

The identity element under multiplication would be 1, but the inverse of 2 would be $\frac{1}{2}$, since

$$\tfrac{1}{2} * 2 = \tfrac{1}{2} \times 2 = 1$$

But $\frac{1}{2}$ is not a member of the set of integers, and therefore 2 does not have an inverse in the set.

Since one of the elements does not satisfy one of the properties, the set of integers under multiplication cannot be a group.

Example 9 Prove that the set of real numbers (excluding zero) under the binary operation of multiplication forms a group.

SOLUTION

Again, we need to check that all four properties are satisfied.

Closure The product of any two real numbers which are not zero is also a real number which is not zero. Therefore, the set is closed.

Associativity Multiplication is always associative.

Identity 1 is the identity of multiplication, and it is in this group. Hence, there is an identity element.

Inverses For any real number x, its inverse is $\frac{1}{x}$ since

$$\frac{1}{x} \times x = \frac{1}{x} \times x = 1$$

Hence, every element has an inverse which is a member of the set.

Therefore, the set of real numbers (excluding zero) under the binary operation of multiplication forms a group.

Note The set of real numbers including zero under the binary operation of multiplication is **not** a group. This is because zero does not have an inverse.

To find an inverse of zero, would mean finding $\frac{1}{0}$, which is impossible.

Example 10 Let G be the set of 3×3 matrices with integer elements and determinant 1, under the multiplication of matrices as the binary operation. Prove that G forms a group.

SOLUTION

To find whether a set of matrices under a particular operation forms a group, we have to apply the rules of matrices. But we also still need to check that G satisfies **every** one of the four group properties.

Closure If **A** and **B** are 3×3 matrices, then **AB** will also be a 3×3 matrix.

We also need to check that **AB** has integer elements.

If **A** and **B** have integer elements, then consider how we find **AB**. We multiply the integers in **A** by the integers in **B**, and then add them up. Therefore, the entries in **AB** are all integers.

Finally, we also need to check that the value of the determinant **AB** is 1.

Using $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$, we find that

$$\det(\mathbf{AB}) = 1 \times 1 = 1$$

Hence, **AB** is a member of the set and thus the set under multiplication is closed.

Associativity Multiplication of matrices is associative.

Identity The identity of matrix multiplication is the identity matrix **I.** Since **I** has integer elements and determinant 1, **I** is a member of the set.

Inverse The inverse of a 3×3 matrix is a 3×3 matrix. However, we need to check that the inverse matrix has integer elements and determinant 1.

To verify that the inverse matrix has integer elements, we consider how we would find the inverse (see pages 304–6). To find the inverse of a 3×3 matrix, we need to find the cofactor of each element. In this case, this means finding the determinants of nine 2×2 matrices, each of which has integer elements. This gives integer results. Then we divide each cofactor by the determinant of the original 3×3 matrix, which in this case is 1.