a) Find the eigenvalues of **P**.
b) Find an eigenvector corresponding to each eigenvalue.
c) Verify that these eigenvectors are orthogonal. (NEAB)

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

a) i) Find the eigenvalues of **A**.
ii) For each eigenvalue find a corresponding eigenvector.

b) Given that $\mathbf{U} = \begin{pmatrix} a & 5 \\ -3 & b \end{pmatrix}$, write down the values of a and b such that

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$ (NEAB)

8 The eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$.

a) Show that $\lambda_1 = 6$ is an eigenvalue and find the other two eigenvalues λ_2 and λ_3.
b) Verify that $\det \mathbf{A} = \lambda_1 \lambda_2 \lambda_3$.
c) Find an eigenvector corresponding to the eigenvalue $\lambda_1 = 6$.

Given that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are eigenvectors of **A** corresponding to λ_2 and λ_3,

d) write down a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is a diagonal matrix. (EDEXCEL)

9 $$\mathbf{A} = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

a) Show that 3 is an eigenvalue of **A** and find the other two eigenvalues.
b) Find an eigenvector corresponding to the eigenvalue 3.

Given that the vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors corresponding to the other two eigenvalues,

c) write down a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is a diagonal matrix. (EDEXCEL)

10 The matrix **A** is given by $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$. The plane transformation **T** is such that **T**: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$.

a) i) Show that **A** has only one eigenvalue. Find this eigenvalue and a corresponding eigenvector.
ii) Hence, or otherwise, determine a cartesian equation of the fixed line of **T**.

b) Under **T**, a square with area $1\,\text{cm}^2$ is transformed into a parallelogram with area $d\,\text{cm}^2$. Find the value of d. (AEB 96)

11 The matrix **P** is defined by

$$\mathbf{P} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

a) Show that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are eigenvectors of $\mathbf{P}$ and find the two corresponding eigenvalues.

b) Given that the third eigenvalue of $\mathbf{P}$ is 4, find the corresponding eigenvector, $\mathbf{v}_3$.

c) Show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

d) Express the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ with coefficients in terms of the constants a, b and c. (NEAB)

12 Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

b) i) Show that $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = \mathbf{Z}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

ii) The matrix $\mathbf{B} = \mathbf{A}^{-1}$. By multiplying the matrix equation $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = \mathbf{Z}$ by $\mathbf{B}$, or otherwise, find the values of the scalars α and β for which $\mathbf{B} = \alpha\mathbf{A} + \beta\mathbf{I}$. (AEB 97)

13 a) Determine the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 6 \\ 0 & 2 & -8 \\ 0 & 0 & -2 \end{pmatrix}$$

b) Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$.

$$\mathbf{B} = \begin{pmatrix} 7 & -6 & 2 \\ 1 & 2 & 3 \\ 1 & -3 & 2 \end{pmatrix}$$

c) Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{B}$ and write down the corresponding eigenvalue.

d) Hence, or otherwise, write down an eigenvector of the matrix $\mathbf{AB}$, and state the corresponding eigenvalue. (EDEXCEL)

14 The transformation $\mathbf{T}$ maps points (x, y) of the plane into image points (x', y') such that

$$x' = 4x + 2y + 14$$
$$y' = 2x + 7y + 42$$

a) i) Find the coordinates of the invariant point of $\mathbf{T}$.

ii) Hence express $\mathbf{T}$ in the form

$$\begin{bmatrix} x' \\ y' + k \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y + k \end{bmatrix}$$

where k is a positive integer and $\mathbf{A}$ is a 2×2 matrix.

b) i) Determine the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$.

ii) Deduce the cartesian equations of the invariant lines of **T**, and prove that they are perpendicular.

c) Give a full geometrical description of **T**. (AEB 98)

15 i) Given that $\mathbf{P} = \begin{pmatrix} 4 & -1 & 0 \\ 1 & 5 & 3 \\ 2 & 1 & 1 \end{pmatrix}$, find $\det \mathbf{P}$ and $\mathbf{P}^{-1}$.

The 3×3 matrix **M** has eigenvalues $-1, 2, 5$ with corresponding eigenvectors

$$\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

respectively.

ii) By considering **MP**, or otherwise, find the matrix **M**.

iii) Find the characteristic equation for **M**.

iv) Find p, q and r such that $\mathbf{M}^{-1} = p\mathbf{M}^2 + q\mathbf{M} + r\mathbf{I}$. (MEI)

16 A linear transformation of three-dimensional space is defined by $\mathbf{r}' = \mathbf{Mr}$, where

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 3 \\ 2 & k & 4 \end{pmatrix}$$

a) Show that the transformation is singular if and only if $k = 2$.

b) In the case when $k = 2$, show that **M** represents a transformation of three-dimensional space onto a plane and find a cartesian equation of this plane. (NEAB)

17 The vectors **a**, **b** and **c**, given below, are linearly independent.

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Find α, β and γ such that the vector

$$\mathbf{d} = \begin{pmatrix} 7 \\ 5 \\ -14 \end{pmatrix}$$

can be expressed as a linear combination of **a**, **b** and **c**, in the form

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$ (NEAB)

18 The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$$

a) Find the determinant of **A** in terms of k.

b) The matrix **A** corresponds to a linear transformation T in three-dimensional space. When a region in three-dimensional space is transformed by T its volume, V, is increased by a factor of four to $4V$. Find the possible values of k. (NEAB)

19 A linear transformation T of three-dimensional space is defined by $\mathbf{r}' = \mathbf{Mr}$, where

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

a) Show that every point on the line $x = y$, $z = 0$ is invariant under T.
b) Find $\mathbf{M}^2$ and hence show that $\mathbf{M}^4 = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 unit matrix.
c) Given that T is a rotation, state
  i) the axis of the rotation
  ii) the angle of the rotation.
d) Write down the image under T of the unit vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and hence indicate by means of a diagram the sense of the rotation. (NEAB)

20 a) The matrix $\mathbf{A}$ and a non-singular matrix $\mathbf{M}$ are defined by

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & -2 \\ 2 & 3 & 6 \end{pmatrix}$$

Show that $\mathbf{M}^{\mathrm{T}}\mathbf{AM} = 4\mathbf{I}$, where $\mathbf{M}^{\mathrm{T}}$, the transpose of the matrix $\mathbf{M}$, is given by

$$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 2 \\ -1 & -1 & 3 \\ 0 & -2 & 6 \end{pmatrix}$$

and $\mathbf{I}$ denotes the 3×3 unit matrix.

b) A closed surface S in three-dimensional space is defined by the equation

$$5x^2 + 10y^2 + z^2 - 2xy + 6yz = 4$$

Verify that this equation can be obtained from the equation

$$\mathbf{r}^{\mathrm{T}}\mathbf{Ar} = 4 \qquad (^*)$$

where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{r}^{\mathrm{T}} = (x\ y\ z)$ and $\mathbf{A}$ is the matrix defined in part **a**.

c) A linear transformation L is defined by $\mathbf{R} = \mathbf{M}^{-1}\mathbf{r}$, where $\mathbf{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ and $\mathbf{M}$ is the matrix defined in part **a**.

  i) By using the relationships

$$\mathbf{r} = \mathbf{MR} \quad \text{and} \quad \mathbf{r}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}$$

  where $\mathbf{R}^{\mathrm{T}} = (X\ Y\ Z)$, in equation $(^*)$, or otherwise, show that L maps the surface S on to the surface of a sphere of unit radius centred at the origin which has the equation

$$X^2 + Y^2 + Z^2 = 1$$

  ii) Show that $\det \mathbf{M}^{-1} = \frac{1}{4}$.

iii) Given that the volume enclosed by a sphere of unit radius is $\frac{4}{3}\pi$, find the volume of the region enclosed by S. (NEAB)

21 A transformation T of three-dimensional space is defined by $\mathbf{r}' = \mathbf{Mr}$, where

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

i) Find the image P′ of the point P$(2, -3, 1)$ under T.
ii) Show that there is a line L such that all points on L are invariant under T, and find the cartesian equations of this line.
iii) Obtain the equation of the plane Π through the origin O perpendicular to L and verify that P and P′ lie in Π.
iv) Given that T represents a rotation about the line L, find the magnitude of the angle of rotation.

Find $\mathbf{M}^2$ and $\mathbf{M}^3$ and state what transformations are represented by these matrices. (NEAB)

22 Determine the eigenvalues and corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} 26 & -5 \\ -5 & 2 \end{bmatrix}$$

The plane transformation **T** is defined by $\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$.

a) Write down a cartesian equation of the line of invariant points of **T**.
b) Show that all lines of the form $y = -\frac{1}{5}x + k$ (where k is an arbitrary constant) are invariant lines of **T**.
c) Evaluate the determinant of **A**, and explain the geometrical significance of this answer in relation to **T**.
d) Give a full geometrical description of **T**. (AEB 98)

23 A transformation T of three-dimensional space is defined by $\mathbf{r}' = \mathbf{Mr}$, where

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 2 & k \end{pmatrix}$$

where k is real.

i) Find $\mathbf{M}^{-1}$ for $k \neq \frac{1}{2}$
ii) In the case when $k = 1$, find the coordinates of the point whose **image** under T is the point $(2, 1, 2)$.
iii) In the case when $k = \frac{1}{2}$, show that the image under T of every point in space lies in the plane

$$3x - 5y - 2z = 0$$

iv) Show that, for one particular value of k, there is a line L such that every point on L is invariant under T. Find the cartesian equations of L. (NEAB)

15 Further complex numbers

In his Miscellanea analytica (1730), Abraham de Moivre presented further analytical trigonometric results (some formulated as early as 1707), making use of complex numbers. Although he did not state what is now known as de Moivre's theorem, it is clear that he was making use of it.

ALBERT C. LEWIS

De Moivre's theorem

On page 8, we found that

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) \equiv \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$$

Hence, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^2 \equiv (\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)$$
$$\equiv \cos 2\theta + \mathrm{i}\sin 2\theta$$

The general case of this result is known as de Moivre's theorem, which states that, for all real values of n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta$$

When n is not an integer, then $\cos n\theta + \mathrm{i}\sin n\theta$ is only one of the possible values.

Proof when *n* is a positive integer

This proof is an example of **proof by induction** (see page 159).

We assume that the statement is true when $n = k$. Hence, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^k \equiv (\cos k\theta + \mathrm{i}\sin k\theta)$$
$$\Rightarrow \quad (\cos\theta + \mathrm{i}\sin\theta)^{k+1} \equiv (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)$$

Using $(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) \equiv \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$, we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} \equiv \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$$

Therefore, statement is true for $n = k+1$.

When $n = 1$, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos\theta + \mathrm{i}\sin\theta$$

and

$$\cos n\theta + \mathrm{i}\sin n\theta \equiv \cos\theta + \mathrm{i}\sin\theta$$

Therefore, the statement is true for $n = 1$.

Therefore, de Moivre's theorem is true for all values of $n \geqslant 1$. That is, for all positive integers.

Proof when n is a negative integer

When n is a negative integer, $n = -p$, where p is a positive integer. Hence, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv (\cos\theta + \mathrm{i}\sin\theta)^{-p}$$
$$\equiv \frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^p}$$

Using de Moivre's theorem for the positive integer p, we obtain

$$\frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^p} \equiv \frac{1}{(\cos p\theta + \mathrm{i}\sin p\theta)}$$
$$\equiv \frac{\cos p\theta - \mathrm{i}\sin p\theta}{(\cos p\theta + \mathrm{i}\sin p\theta)(\cos p\theta - \mathrm{i}\sin p\theta)}$$

which gives

$$\frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^p} \equiv \cos p\theta - \mathrm{i}\sin p\theta$$

But $n = -p$, hence we have

$$\cos p\theta - \mathrm{i}\sin p\theta \equiv \cos(-n\theta) - \mathrm{i}\sin(-n\theta)$$
$$\equiv \cos n\theta + \mathrm{i}\sin n\theta$$

Therefore, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta$$

for all negative integers.

Example 1 Find the value of $(\cos\theta + \mathrm{i}\sin\theta)^5$.

SOLUTION

Applying de Moivre's theorem, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^5 \equiv \cos 5\theta + \mathrm{i}\sin 5\theta$$

Example 2 Find $\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^3$.

SOLUTION

Applying de Moivre's theorem, we have

$$\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^3 \equiv \cos\left(3 \times \frac{\pi}{6}\right) + \mathrm{i}\sin\left(3 \times \frac{\pi}{6}\right)$$
$$\equiv \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)$$

which gives

$$\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^3 = \mathrm{i} \quad \left(\text{since} \cos\left(\frac{\pi}{2}\right) = 0 \text{ and} \sin\left(\frac{\pi}{2}\right) = 1\right)$$

Example 3 Find $\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6$.

SOLUTION

Using $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, we obtain

$$\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6 \equiv \left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^6$$

$$\equiv \cos\pi + \mathrm{i}\sin\pi$$

which gives

$$\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6 = -1 \quad (\text{since } \cos\pi = -1 \text{ and } \sin\pi = 0)$$

Alternatively, we can proceed as follows:

$$\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6 \equiv \left\{\mathrm{i}\left[\cos\left(\frac{\pi}{3}\right) - \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]\right\}^6$$

$$\equiv \left\{\mathrm{i}\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]\right\}^6$$

Applying de Moivre's theorem to the RHS, we obtain

$$\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6 \equiv \mathrm{i}^6[\cos(-2\pi) + \mathrm{i}\sin(-2\pi)] = -1 \times 1 = -1$$

Therefore, we have

$$\left[\sin\left(\frac{\pi}{3}\right) + \mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]^6 = -1$$

as above.

Caution You will have noticed that

$$\left[\cos\left(\frac{\pi}{3}\right) - \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]^6 \equiv \cos 2\pi - \mathrm{i}\sin 2\pi$$

and hence you may have deduced that

$$(\cos\theta - \mathrm{i}\sin\theta)^n \equiv \cos n\theta - \mathrm{i}\sin n\theta$$

However, this **cannot** be used as a correct version of de Moivre's theorem, which is only applicable to $(\cos\theta + \mathrm{i}\sin\theta)^n$

Thus, if you are asked to use de Moivre's theorem to find the value of, say, $\left[\cos\left(\frac{\pi}{3}\right) - \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]^6$, you **must** change this into $\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]^6$, as shown in Example 3.

Example 4 Find the value of $(1+i)^4$.

SOLUTION

Initially, we convert $(1+i)^4$ into its (r, θ) form, and then use de Moivre's theorem. Hence, we have

$$\begin{aligned}(1+i)^4 &= \left\{\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]\right\}^4 \\ &= (\sqrt{2})^4\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]^4 \\ &= 4(\cos\pi + i\sin\pi)\end{aligned}$$

which gives

$$(1+i)^4 = -4$$

Example 5 Find the value of $\dfrac{1}{(4-4i)^3}$.

SOLUTION

First, we convert $4-4i$ into its (r, θ) form, and then use de Moivre's theorem. Hence, we have.

$$4 - 4i = 4\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad (4-4i)^3 = \left\{4\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]\right\}^3$$

$$\begin{aligned}\Rightarrow \quad \frac{1}{(4-4i)^3} &= \frac{1}{\left\{4\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]\right\}^3} \\ &= \frac{1}{128\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]^3} \\ &= \frac{1}{128\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]^{-3}\end{aligned}$$

Using de Moivre's theorem, we obtain

$$\begin{aligned}\frac{1}{(4-4i)^3} &= \frac{1}{128\sqrt{2}}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] \\ &= \frac{1}{128\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\end{aligned}$$

which gives

$$\frac{1}{(4-4i)^3} = \frac{1}{256}(-1+i)$$

Exercise 15A

1 Using de Moivre's theorem, find the value of each of the following.

a) $(\cos\theta + i\sin\theta)^6$ **b)** $(\cos 2\theta + i\sin 2\theta)^4$ **c)** $\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]^9$

d) $\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]^6$ **e)** $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^4}$ **f)** $\dfrac{1}{\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^6}$

g) $\left[\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right]^{10}$ **h)** $\left[\cos\left(-\frac{\pi}{18}\right) + i\sin\left(-\frac{\pi}{18}\right)\right]^9$

2 Simplify each of the following.

a) $(\cos 3\theta + i\sin 3\theta)(\cos 7\theta + i\sin 7\theta)$ **b)** $(\cos 5\theta + i\sin 5\theta)(\cos 6\theta - i\sin 6\theta)$

c) $\dfrac{\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]^5}{\left[\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right]^4}$ **d)** $(1+i)^4 + (1-i)^4$

3 Simplify each of the following.

a) $(1+i)^8$ **b)** $(2-\sqrt{3}i)^6$ **c)** $(3-\sqrt{3}i)^6$

d) $(1-i)^4$ **e)** $(2+2\sqrt{3}i)^6$ **f)** $(2i-\sqrt{3})^9$

4 Simplify each of the following.

a) $(\cos\theta - i\sin\theta)^5$ **b)** $(\sin\theta - i\cos\theta)^4$

c) $\dfrac{1}{(\sin\theta + i\cos\theta)^6}$ **d)** $\dfrac{1}{\left[\sin\left(\frac{\pi}{5}\right) - i\cos\left(\frac{\pi}{5}\right)\right]^{10}}$

5 Show that

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x}$$

can be expressed in the form $\cos nx + i\sin nx$, where n is an integer to be found. (EDEXCEL)

*n*th roots of unity

When n is not an integer, de Moivre's theorem gives **only one** of the possible values for $(\cos\theta + i\sin\theta)^n$, which is $\cos n\theta + i\sin n\theta$.

However, $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$ can take n different values, as we will now show.

We let

$$(\cos\theta + i\sin\theta)^{\frac{1}{n}} = r(\cos\phi + i\sin\phi)$$

Comparing the moduli of both sides, we have $r = 1$.

Raising both sides to the nth power, and using

$$[(\cos\theta + \mathrm{i}\sin\theta)^{\frac{1}{n}}]^n = \cos\theta + \mathrm{i}\sin\theta$$

we obtain

$$\cos\theta + \mathrm{i}\sin\theta = [(\cos\theta + \mathrm{i}\sin\theta)^{\frac{1}{n}}]^n = (\cos\phi + \mathrm{i}\sin\phi)^n$$

$$\Rightarrow \quad \cos\theta + \mathrm{i}\sin\theta = \cos n\phi + \mathrm{i}\sin n\phi$$

Therefore, we have

$$\cos\theta = \cos n\phi \quad \text{and} \quad \sin\theta = \sin n\phi$$

which give

$$n\phi = \theta,\ \theta + 2\pi,\ \theta + 4\pi,\ \theta + 6\pi,\ \ldots$$

since $\cos(\theta + 2\pi) = \cos\theta$, and $\sin(\theta + 2\pi) = \sin\theta$.

That is, we have

$$\phi = \frac{\theta}{n},\ \frac{\theta + 2\pi}{n},\ \frac{\theta + 4\pi}{n},\ \ldots$$

which means that $(\cos\theta + \mathrm{i}\sin\theta)^{\frac{1}{n}}$ is identical to

$$\cos\left(\frac{\theta}{n}\right) + \mathrm{i}\sin\left(\frac{\theta}{n}\right)$$

or $$\cos\left(\frac{\theta + 2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi}{n}\right)$$

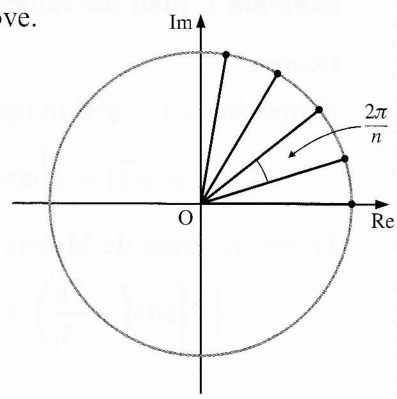
or $$\cos\left(\frac{\theta + 4\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 4\pi}{n}\right)$$

and so on, adding $\frac{2\pi}{n}$ each time until we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^{\frac{1}{n}} \equiv \cos\left[\frac{\theta + (n-1)2\pi}{n}\right] + \mathrm{i}\sin\left[\frac{\theta + (n-1)2\pi}{n}\right]$$

All subsequent values are repeats of the n different values given above. Therefore, $(\cos\theta + \mathrm{i}\sin\theta)^{\frac{1}{n}}$ has n different values.

We note that these n solutions are symmetrically placed on a circle drawn on an Argand diagram.



Example 6 Find the value of $(-64)^{\frac{1}{6}}$.

SOLUTION

Expressing -64 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, we have

$$-64 = 64(\cos\pi + \mathrm{i}\sin\pi)$$

which gives

$$(-64)^{\frac{1}{6}} = 64^{\frac{1}{6}}(\cos\pi + \mathrm{i}\sin\pi)^{\frac{1}{6}}$$
$$= 2\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right] \quad \text{(from de Moivre's theorem)}$$

Using symmetry, we find that the other roots are as shown in the diagram below right. That is,

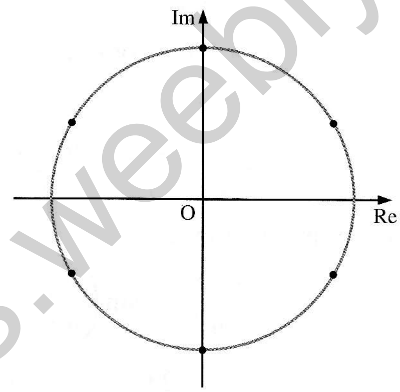
$$2\left[\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right]$$

$$2\left[\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right]$$

$$2\left[\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right]$$

$$2\left[\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]$$

$$2\left[\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]$$



Since all of these values can be expressed simply in the form $a + \mathrm{i}b$, it is common to give these answers in the form

$$\pm\left(\frac{\sqrt{3}}{2} \pm \frac{\mathrm{i}}{2}\right), \ \pm\mathrm{i}$$

Example 7 Find the values of $(-1 - \sqrt{3}\,\mathrm{i})^{\frac{1}{2}}$.

SOLUTION

Expressing $-1 - \sqrt{3}\mathrm{i}$ in the form $\cos\theta + \mathrm{i}\sin\theta$, we have

$$-1 - \sqrt{3}\mathrm{i} = 2\left[\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right]$$

Therefore, from de Moivre's theorem, one value of $(-1 - \sqrt{3}\mathrm{i})^{\frac{1}{2}}$ is

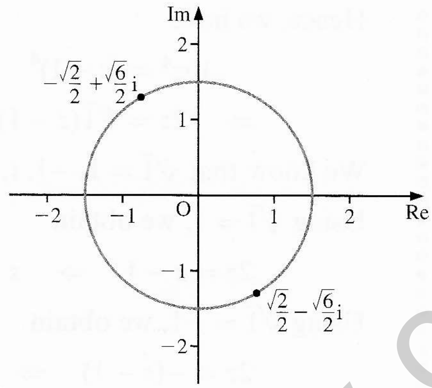
$$\left\{2\left[\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right]\right\}^{\frac{1}{2}} = 2^{\frac{1}{2}}\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(+\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)$$
$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\mathrm{i}$$

By symmetry, the other root is as shown in the diagram on the right. That is,

$$-\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}$$

Therefore, we have

$$\left(-1-\sqrt{3}\mathrm{i}\right)^{\frac{1}{2}}=\pm\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\mathrm{i}\right)$$



Example 8 Find the solutions of $27z^3 = 8$.

SOLUTION

We take the cube root of both sides, remembering to multiply one side of the resulting equation by each of the three cube roots of unity, taken one at a time. In this case, it is simpler to multiply $\sqrt[3]{8}$ by the three cube roots.

Hence, we have

$$27z^3 = 8$$

$$\Rightarrow \quad 3z = \sqrt[3]{1}\times 2$$

From page 18, we know that $\sqrt[3]{1}$ has the following values:

$$1 \qquad -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i} \qquad -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$$

Using $\sqrt[3]{1}=1$, we obtain

$$3z = 2 \quad\Rightarrow\quad z=\frac{2}{3}$$

Using $\sqrt[3]{1}=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$, we obtain

$$3z=-1+\sqrt{3}\mathrm{i} \quad\Rightarrow\quad z=-\frac{1}{3}+\frac{\sqrt{3}}{3}\mathrm{i}$$

Using $\sqrt[3]{1}=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$, we obtain

$$3z=-1-\sqrt{3}\mathrm{i} \quad\Rightarrow\quad z=-\frac{1}{3}-\frac{\sqrt{3}}{3}\mathrm{i}$$

Example 9 Find the solutions of $16z^4 = (z-1)^4$.

SOLUTION

We take the fourth root of both sides, remembering to multiply one side of the resulting equation by each of the four fourth roots of unity, taken one at a time.

Hence, we have

$$16z^4 = (z-1)^4$$

$$\Rightarrow \quad 2z = \sqrt[4]{1}(z-1)$$

We know that $\sqrt[4]{1} = 1, -1, \mathrm{i}, -\mathrm{i}$.

Using $\sqrt[4]{1} = 1$, we obtain

$$2z = z - 1 \quad \Rightarrow \quad z = -1$$

Using $\sqrt[4]{1} = -1$, we obtain

$$2z = -(z-1) \quad \Rightarrow \quad 3z = 1 \quad \Rightarrow \quad z = \tfrac{1}{3}$$

Using $\sqrt[4]{1} = \mathrm{i}$, we obtain

$$2z = \mathrm{i}(z-1)$$

$$\Rightarrow \quad z = -\frac{\mathrm{i}}{2-\mathrm{i}}$$

$$\Rightarrow \quad z = -\frac{\mathrm{i}(2+\mathrm{i})}{(2-\mathrm{i})(2+i)}$$

which gives

$$z = \tfrac{1}{5}(1-2\mathrm{i})$$

Using $\sqrt[4]{1} = -\mathrm{i}$, we obtain

$$2z = -\mathrm{i}(z-1)$$

$$\Rightarrow \quad z = \frac{\mathrm{i}}{2+\mathrm{i}}$$

$$\Rightarrow \quad z = \frac{\mathrm{i}(2-\mathrm{i})}{5}$$

which gives

$$z = \tfrac{1}{5}(1+2i)$$

Therefore, the four solutions of $16z^4 = (z-1)^4$ are $-1, \frac{1}{3}, \frac{1}{5}(1 \pm 2\mathrm{i})$.

Exponential form of a complex number

Using the power series expansions studied on pages 177–9, we have

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &= 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \ldots \\
&= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\theta^5}{5!} - \ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)
\end{aligned}$$

$$\Rightarrow \quad \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

This is the **exponential form** of a complex number.

Expressed generally, we have

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \quad \Rightarrow \quad z = r\mathrm{e}^{\mathrm{i}\theta}$$

We can use the exponential form to simplify many types of problem.

Note Using the exponential form of $(\cos\theta + \mathrm{i}\sin\theta)^n$, we have

$$(\cos\theta + \mathrm{i}\sin\theta)^n = (\mathrm{e}^{\mathrm{i}\theta})^n = \mathrm{e}^{\mathrm{i}(n\theta)} = \cos n\theta + \mathrm{i}\sin n\theta$$

which proves de Moivre's theorem.

Example 10 Express $2 + 2\mathrm{i}$ in $r\mathrm{e}^{\mathrm{i}\theta}$ form.

SOLUTION

The modulus of $2 + 2\mathrm{i}$ is $2\sqrt{2}$ and its argument is $\frac{\pi}{4}$. Hence, we have

$$2 + 2\mathrm{i} = 2\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}$$

Example 11 Express $1 - \mathrm{i}\sqrt{3}$ in $r\mathrm{e}^{\mathrm{i}\theta}$ form.

SOLUTION

The modulus of $1 - \mathrm{i}\sqrt{3}$ is 2 and its argument is $-\frac{\pi}{3}$. Hence, we have

$$1 - \mathrm{i}\sqrt{3} = 2\mathrm{e}^{-\mathrm{i}\pi/3}$$

Example 12 Find the values of $(-2 + 2\mathrm{i})^{\frac{1}{3}}$ and show their positions on an Argand diagram.

SOLUTION

We proceed as follows:

- First, express $(-2 + 2\mathrm{i})$ in its (r, θ) form.
- Then find one value of $(-2 + 2\mathrm{i})^{\frac{1}{3}}$
- Finally, use symmetry to find the other roots.

Hence, we have

$$(-2 + 2\mathrm{i})^{\frac{1}{3}} = \left\{2\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right]\right\}^{\frac{1}{3}}$$

$$= \left\{2^{\frac{3}{2}}\left[\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right]\right\}^{\frac{1}{3}}$$

Therefore, from de Moivre's theorem, one value of $(-2 + 2\mathrm{i})^{\frac{1}{3}}$ is

$$2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right]$$

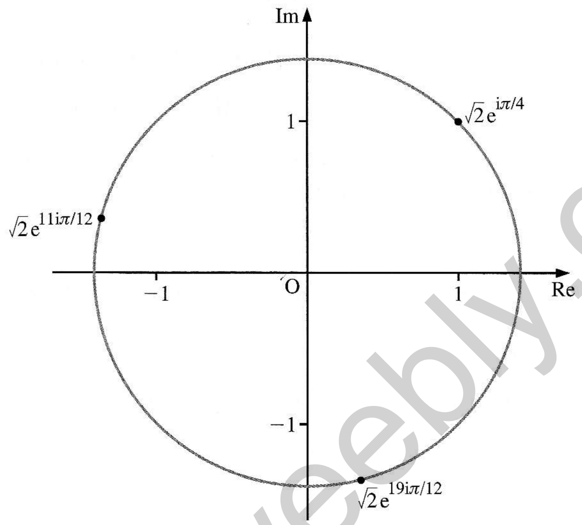
By symmetry, the other roots are

$$2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right]$$

and $$2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)\right]$$

These three roots (see Argand diagram on the right) may be expressed as

$$2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi/4} \qquad 2^{\frac{1}{2}}\mathrm{e}^{11\mathrm{i}\pi/12} \qquad 2^{\frac{1}{2}}\mathrm{e}^{19\mathrm{i}\pi/12}$$



Multiplying one complex number by another

Expressing the two numbers, z_1 and z_2, in their exponential form, we have

$$z_1 z_2 = r_1\mathrm{e}^{\mathrm{i}\theta_1} \times r_2\mathrm{e}^{\mathrm{i}\theta_2}$$

which gives

$$z_1 z_2 = r_1 r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$$

This is a very simple way of showing that to find the product of two complex numbers, we multiply the moduli and add the arguments. (See page 8.)

Simplifying certain integrals

We can simplify integrals of the type $\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$ using the exponential form, as shown in Examples 13 and 14.

Example 13 Find $\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x$.

SOLUTION

We have

$$\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x = \mathrm{Im}\int \mathrm{e}^{2x}(\cos x + \mathrm{i}\sin x)\,\mathrm{d}x$$

where $\mathrm{Im}\int$ is the imaginary part of the given integral.

Using the exponential form of $\cos x + \mathrm{i}\sin x$, we obtain

$$\int e^{2x}\sin x\,dx = \mathrm{Im}\int e^{2x}\times e^{\mathrm{i}x}\,dx$$

$$= \mathrm{Im}\int e^{(2+\mathrm{i})x}\,dx$$

which gives

$$\int e^{2x}\sin x\,dx = \mathrm{Im}\left(\frac{1}{2+\mathrm{i}}e^{(2+\mathrm{i})x}\right)$$

$$= \mathrm{Im}\left[\frac{2-\mathrm{i}}{(2+\mathrm{i})(2-\mathrm{i})}e^{2x}(\cos x + \mathrm{i}\sin x) + c\right]$$

$$= \mathrm{Im}\left[e^{2x}\,\frac{(2\cos x + \sin x + 2\mathrm{i}\sin x - \mathrm{i}\cos x)}{2^2 - (\mathrm{i})^2}\right] + c$$

Hence, we find that

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + c$$

Example 14 Find $\int e^{4x}\cos 3x\,dx$.

SOLUTION

We have

$$\int e^{4x}\cos 3x = \mathrm{Re}\int e^{4x}(\cos 3x + \mathrm{i}\sin 3x)\,dx$$

where $\mathrm{Re}\int$ is the real part of the given integral.

Using the exponential form of $\cos 3x + \mathrm{i}\sin 3x$, we obtain

$$\int e^{4x}\cos 3x\,dx = \mathrm{Re}\int e^{(4+3\mathrm{i})x}\,dx$$

$$= \mathrm{Re}\left(\frac{1}{4+3\mathrm{i}}e^{(4+3\mathrm{i})x} + c\right)$$

$$= \mathrm{Re}\left[\frac{4-3\mathrm{i}}{(4+3\mathrm{i})(4-3\mathrm{i})}e^{4x}(\cos 3x + \mathrm{i}\sin 3x) + c\right]$$

Hence, we have

$$\int e^{4x}\cos 3x\,dx = \frac{e^{4x}}{25}\left(\frac{4\cos 3x + 3\sin 3x}{25}\right) + c$$

Exercise 15B

1 For each of the following, find the possible values of z, giving your answers in

i) $a + ib$ form **ii)** $re^{i\theta}$ form

a) $z^4 = -16$ **b)** $z^3 = -8 + 8i$ **c)** $z^3 = 27i$

d) $z^2 = 16i$ **e)** $z^2 = -25i$ **f)** $z^5 = -32$

2 Find the six sixth roots of unity.

3 Solve each of these.

a) $(z + 2i)^2 = 4$ **b)** $(z - 1)^3 = 8$ **c)** $z^2 = (z + 1)^2$

d) $(z + 3i)^2 = (2z - 1)^2$ **e)** $(z - i)^4 = 81(z + 2)^4$

4 Find the seven seventh roots of unity in the $e^{i\theta}$ form.

5 Solve $z^5 = 32i$. Give your answers in the $re^{i\theta}$ form, and show them on an Argand diagram.

6 By considering the ninth roots of unity, show that

$$\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{6\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = -\frac{1}{2}$$

7 By considering the seventh roots of unity, show that

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}$$

8 When $\cos 4\theta = \cos 3\theta$, prove that $\theta = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$.

Hence prove that $\cos\left(\frac{2\pi}{7}\right), \cos\left(\frac{4\pi}{7}\right), \cos\left(\frac{6\pi}{7}\right)$ are the roots of $8x^3 + 4x^2 - 4x - 1 = 0$.

9 Evaluate each of these.

a) $\int e^{4x} \cos 5x \, dx$ **b)** $\int e^{3x} \sin 7x \, dx$

c) $\int e^{-2x} \sin 4x \, dx$ **d)** $\int e^{-4x} \cos 3x \, dx$

10 Find, in polar form, each of the fourth roots of $-8 - 8\sqrt{3}i$. (WJEC)

11 Verify that $(3 - 2i)^2 = 5 - 12i$, showing your working clearly. Find the two roots of the equation $(z - i)^2 = 5 - 12i$ (OCR)

12 i) Find the exact modulus and argument of the complex number $-4\sqrt{3} - 4i$.

ii) Hence obtain the roots of the equation

$$z^3 + 4\sqrt{3} + 4i = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. (OCR)

13 Express $(8\sqrt{2})(1+\mathrm{i})$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Hence, or otherwise, solve the equation $z^4 = (8\sqrt{2})(1+\mathrm{i})$, giving your answers in polar form. (OCR)

14 Write each of the complex numbers

$$z_1 = 1 - (\sqrt{3})\mathrm{i} \qquad z_2 = (\sqrt{3}) + \mathrm{i}$$

in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Hence show that if $z_1^7 + z_2^7 = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, then

$$\frac{y}{x} = 2 + \sqrt{3} \qquad \text{(OCR)}$$

15 **a)** State de Moivre's theorem for the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^n$, where n is a positive integer or rational number.
b) Find the modulus and the argument of each of the three cube roots of $1 + \mathrm{i}$.
c) Show that $(1+\mathrm{i})^{51} = 2^{25}(-1+\mathrm{i})$. (WJEC)

16 Write down the modulus and argument of the complex number -64.

Hence solve the equation $z^4 + 64 = 0$, giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Express each of these four roots in the form $a + \mathrm{i}b$ and show, with the aid of a diagram, that the points in the complex plane which represent them form the vertices of a square.

(AEB 96)

17 **a)** Solve the equation $z^5 = 4 + 4\mathrm{i}$, giving your answers in the form $z = r\mathrm{e}^{\mathrm{i}k\pi}$, where r is the modulus of z and k is a rational number such that $0 \leqslant k \leqslant 2$.
b) Show on an Argand diagram the points representing your solutions. (EDEXCEL)

18 **i)** Show that

$$\mathrm{e}^{(3+2\mathrm{i})x} \equiv \mathrm{e}^{3x}(\cos 2x + \mathrm{i}\sin 2x)$$

where x is real.

ii) Find the real and imaginary parts of

$$\frac{\mathrm{e}^{3x}(\cos 2x + \mathrm{i}\sin 2x)}{(3+2\mathrm{i})}$$

iii) If $C = \int \mathrm{e}^{3x}\cos 2x\,\mathrm{d}x$ and $S = \int \mathrm{e}^{3x}\sin 2x\,\mathrm{d}x$, by using parts **i** and **ii** and considering $C + \mathrm{i}S$, or otherwise, find C and S. [You may assume the normal rules of integration apply to $\int \mathrm{e}^{kx}\,\mathrm{d}x$ when k is complex.] (NICCEA)

19 **a)** Verify that $z_1 = 1 + \mathrm{e}^{\pi\mathrm{i}/5}$ is a root of the equation $(z-1)^5 = -1$.
b) Find the other four roots of the equation.
c) Mark on an Argand diagram the points corresponding to the five roots of the equation. Show that these roots lie on a circle, and state the centre and the radius of the circle.
d) By considering the Argand diagram, or otherwise, find
i) $\arg z_1$ in terms of π.
ii) $|z_1|$ in the form $a\cos\dfrac{\pi}{b}$, where a and b are integers to be determined. (NEAB)

20 i) Find the roots of the equation $(z-4)^3 = 8\mathrm{i}$ in the form $a + \mathrm{i}b$, where a and b are real numbers. Indicate, on an Argand diagram, the points A, B and C representing these three roots and find the area of $\triangle$ABC.

ii) The equation $z^3 + pz^2 + 40z + q = 0$, where p and q are real, has a root $3 + \mathrm{i}$. Write down another root of the equation.

Hence, or otherwise, find the values of p and q. (EDEXCEL)

21 Write down the fifth roots of unity in the form $\cos\theta + \mathrm{i}\sin\theta$, where $0 \leqslant \theta < 2\pi$.

i) Hence, or otherwise, find the fifth roots of i in a similar form.

ii) By writing the equation $(z-1)^5 = z^5$ in the form

$$\left(\frac{z-1}{z}\right)^5 = 1$$

show that its roots are

$$\tfrac{1}{2}(1 + \mathrm{i}\cot\tfrac{1}{5}k\pi) \quad k = 1, 2, 3, 4$$ (OCR)

22 i) Find the six complex roots of the equation $z^6 + 8\mathrm{i} = 0$, expressing each in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Give the exact values of θ in radians.

ii) Show that $(1 + \mathrm{i})$ and $(-1 - \mathrm{i})$ are two of the roots.

iii) Sketch the six roots on an Argand diagram, clearly indicating the significant geometrical features. (NICCEA)

Trigonometric identities

Expressions for $\cos^n\theta$ and $\sin^n\theta$ in terms of multiples of θ

Let $z \equiv \cos\theta + \mathrm{i}\sin\theta$. We then have

$$\frac{1}{z} \equiv (\cos\theta + \mathrm{i}\sin\theta)^{-1} \equiv \cos\theta - \mathrm{i}\sin\theta$$

which gives

$$z + \frac{1}{z} \equiv 2\cos\theta$$

$$z - \frac{1}{z} \equiv 2\mathrm{i}\sin\theta$$

We also have

$$z^n \equiv (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

$$\Rightarrow \quad \frac{1}{z^n} \equiv \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta}$$

$$\Rightarrow \quad \frac{1}{z^n} \equiv \cos n\theta - \mathrm{i}\sin n\theta$$

which gives

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$$

$$z^n - \frac{1}{z^n} \equiv 2\mathrm{i}\sin n\theta$$

With the aid of these four identities for $z \pm \frac{1}{z}$ and $z^n \pm \frac{1}{z^n}$, we can write any power of $\cos\theta$ or $\sin\theta$ in terms of multiples of θ.

Example 15 If $z \equiv \cos\theta + \mathrm{i}\sin\theta$, express the following in terms of θ.

a) z^4 **b)** z^{-3}

SOLUTION

We know that, when $z \equiv \cos\theta + \mathrm{i}\sin\theta$,

$$z^n \equiv \cos n\theta + \mathrm{i}\sin n\theta$$

for all integer n.

Hence, we have

a) $$z^4 \equiv \cos 4\theta + \mathrm{i}\sin 4\theta$$

b) $$z^{-3} \equiv \cos(-3\theta) + \mathrm{i}\sin(-3\theta)$$

$$\Rightarrow \quad z^{-3} \equiv \cos 3\theta - \mathrm{i}\sin 3\theta$$

Example 16 If $z \equiv \cos\theta + \mathrm{i}\sin\theta$, express the following in terms of z.

a) $\cos 6\theta$ **b)** $\sin 3\theta$

SOLUTION

When $z \equiv \cos\theta + \mathrm{i}\sin\theta$, we know that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$$

and $$z^n - \frac{1}{z^n} \equiv 2\mathrm{i}\sin n\theta$$

Hence, we have

a) $$2\cos 6\theta \equiv z^6 + \frac{1}{z^6}$$

$$\Rightarrow \quad \cos 6\theta \equiv \frac{1}{2}\left(z^6 + \frac{1}{z^6}\right)$$

b) $$2\mathrm{i}\sin 3\theta \equiv z^3 - \frac{1}{z^3}$$

$$\Rightarrow \quad \sin 3\theta \equiv \frac{1}{2\mathrm{i}}\left(z^3 - \frac{1}{z^3}\right)$$

Example 17 Express $\cos^3\theta$ as the cosines of multiples of θ.

SOLUTION

We proceed as follows:

- Express $\cos\theta$ in terms of z, and hence find $\cos^3\theta$.
- Collect terms of the type $z^n + \dfrac{1}{z^n}$, according to the values of n (as we are required to give the answer as the cosines of multiples of θ).
- Finally, convert these terms into cosines of multiples of θ.

Hence, we have

$$\cos\theta \equiv \frac{1}{2}\left(z + \frac{1}{z}\right)$$

which gives

$$\begin{aligned}\cos^3\theta &\equiv \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^3 \\ &\equiv \frac{1}{2^3}\left(z + \frac{1}{z}\right)^3 \\ &\equiv \frac{1}{8}\left(z^3 + 3z^2 \times \frac{1}{z} + 3z \times \frac{1}{z^2} + \frac{1}{z^3}\right) \\ &\equiv \frac{1}{8}\left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right)\end{aligned}$$

Rearranging the terms on the RHS, we obtain

$$\cos^3\theta \equiv \frac{1}{8}\left[\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)\right]$$

Converting the RHS, we have

$$\cos^3\theta \equiv \frac{1}{8}(2\cos 3\theta + 3 \times 2\cos\theta)$$

which gives

$$\cos^3\theta \equiv \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

Example 18 Express $\cos^6\theta$ as the cosines of multiples of θ.

SOLUTION

We have

$$\cos^6\theta \equiv \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^6$$

where $z \equiv \cos\theta + \mathrm{i}\sin\theta$.

Using the binomial theorem, we obtain

$$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5 \times \frac{1}{z} + 15z^4 \times \frac{1}{z^2} + \ldots + \frac{1}{z^6}$$

which gives

$$\cos^6\theta \equiv \frac{1}{64}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right)$$

$$\equiv \frac{1}{64}\left[\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right]$$

Converting the RHS, we have

$$\cos^6\theta \equiv \frac{1}{64}(2\cos 6\theta + 6 \times 2\cos 4\theta + 15 \times 2\cos 2\theta + 20)$$

$$\Rightarrow \quad \cos^6\theta \equiv \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$$

Example 19 Express $\sin^5\theta$ as the sines of multiples of θ.

SOLUTION

We have

$$\sin^5\theta \equiv \left[\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\right]^5$$

where $z = \cos\theta + \mathrm{i}\sin\theta$.

Using the binomial theorem, we obtain

$$\sin^5\theta \equiv \frac{1}{32\mathrm{i}^5}\left(z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}\right)$$

$$\equiv \frac{1}{32\mathrm{i}}\left[\left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)\right]$$

Converting the RHS, we have

$$\sin^5\theta \equiv \frac{1}{32\mathrm{i}}[2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta]$$

$$\Rightarrow \quad \sin^5\theta \equiv \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

Expansions of $\cos n\theta$ and $\sin n\theta$ as powers of $\cos\theta$ and $\sin\theta$

To change a function such as $\cos 6\theta$ into powers of $\cos\theta$, we express $\cos 6\theta$ as the **real part** of $\cos 6\theta + \mathrm{i}\sin 6\theta$.

By de Moivre's theorem, we have

$$\cos 6\theta + \mathrm{i}\sin 6\theta = (\cos\theta + \mathrm{i}\sin\theta)^6$$

the RHS of which we expand by the binomial theorem. We then extract the real terms from this expansion.

Similarly, we express, for example, $\sin 7\theta$ as the **imaginary part** of $\cos 7\theta + \mathrm{i}\sin 7\theta$.

Example 20 Express $\sin 3\theta$ in terms of $\sin\theta$.

SOLUTION

We put

$$\sin 3\theta = \operatorname{Im}(\cos 3\theta + \mathrm{i}\sin 3\theta)$$

where $\operatorname{Im}(z)$ is the imaginary part of z.

Hence, we have

$$\sin 3\theta = \operatorname{Im}(\cos\theta + \mathrm{i}\sin\theta)^3$$

Expanding the RHS by the binomial theorem, we obtain

$$\begin{aligned}\sin 3\theta &= \operatorname{Im}[\cos^3\theta + 3\cos^2\theta(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3]\\ &= \operatorname{Im}(\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta)\\ &= 3\cos^2\theta\sin\theta - \sin^3\theta\end{aligned}$$

Using $\cos^2\theta = 1 - \sin^2\theta$ (as the answer has to be in terms of $\sin\theta$), we have

$$\begin{aligned}\sin 3\theta &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\end{aligned}$$

which gives

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Example 21

a) Express $\cos 6\theta$ in terms of powers of $\cos\theta$.

b) Express $\dfrac{\sin 6\theta}{\sin\theta}$ in terms of powers of $\cos\theta$.

SOLUTION

a) We put

$$\cos 6\theta = \operatorname{Re}(\cos 6\theta + \mathrm{i}\sin 6\theta)$$

where $\operatorname{Re}(z)$ means the real part of z.

Hence, we have

$$\cos 6\theta = (\cos\theta + \mathrm{i}\sin\theta)^6$$

Expanding the RHS by the binomial theorem, we obtain

$$\begin{aligned}\cos 6\theta = \operatorname{Re}\Bigg[&\cos^6\theta + 6\cos^5\theta(\mathrm{i}\sin\theta) + \frac{6.5}{2.1}\cos^4\theta(\mathrm{i}\sin\theta)^2 +\\ &+ \frac{6.5.4}{3.2.1}\cos^3\theta(\mathrm{i}\sin\theta)^3 + \frac{6.5.4.3}{4.3.2.1}\cos^2\theta(\mathrm{i}\sin\theta)^4 +\\ &+ \frac{6.5.4.3.2}{5.4.3.2.1}\cos\theta(\mathrm{i}\sin\theta)^5 + (\mathrm{i}\sin\theta)^6\Bigg]\end{aligned}$$

$$\Rightarrow \quad \cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we have

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta -$$
$$- 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

which gives

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

b) We put

$$\sin 6\theta = \mathrm{Im}(\cos 6\theta + \mathrm{i}\sin 6\theta)$$

where $\mathrm{Im}(z)$ means the imaginary part of z.

Hence, we have

$$\sin 6\theta = \mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^6$$

Expanding the RHS by the binomial theorem, we obtain

$$\sin 6\theta = \mathrm{Im}\Big[\cos^6\theta + 6\cos^5\theta\,(\mathrm{i}\sin\theta) + \frac{6.5}{2.1}\cos^4\theta\,(\mathrm{i}\sin\theta)^2 +$$
$$+ \frac{6.5.4}{3.2.1}\cos^3\theta\,(\mathrm{i}\sin\theta)^3 + \frac{6.5.4.3}{4.3.2.1}\cos^2\theta\,(\mathrm{i}\sin\theta)^4 +$$
$$+ \frac{6.5.4.3.2}{5.4.3.2.1}\cos\theta\,(\mathrm{i}\sin\theta)^5 + (\mathrm{i}\sin\theta)^6\Big]$$

$$\Rightarrow \quad \sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

Therefore, we have

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$
$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta - 12\cos^3\theta + 6\cos^5\theta$$

which gives

$$\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

Example 22

a) Express $\sin 5\theta$ in terms of $\sin\theta$.

b) Hence, prove that $\sin\left(\frac{\pi}{5}\right)$, $\sin\left(\frac{2\pi}{5}\right)$, $\sin\left(\frac{6\pi}{5}\right)$ and $\sin\left(\frac{7\pi}{5}\right)$ are the roots of the equation $16x^4 - 20x^2 + 5 = 0$.

c) Deduce that $\sin^2\left(\frac{\pi}{5}\right)$ and $\sin^2\left(\frac{2\pi}{5}\right)$ are roots of the equation $16y^2 - 20y + 5 = 0$, and hence find the exact value of

i) $\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)$ **ii)** $\cos\left(\frac{2\pi}{5}\right)$

SOLUTION

a) We put

$$\sin 5\theta = \mathrm{Im}\,(\cos 5\theta + \mathrm{i}\sin 5\theta)$$

where $\mathrm{Im}\,(z)$ is the imaginary part of z.

Hence, we have

$$\sin 5\theta = \mathrm{Im}\,(\cos\theta + \mathrm{i}\sin\theta)^5$$
$$= \mathrm{Im}\,(\cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta + 10\mathrm{i}^3\cos^2\theta\sin^3\theta + $$
$$+ 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta)$$

which gives

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we obtain

$$\sin 5\theta = 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$\Rightarrow \quad \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

b) From part **a**, we have

$$\frac{\sin 5\theta}{\sin\theta} = 16\sin^4\theta - 20\sin^2\theta + 5$$

When $\sin 5\theta = 0$, $16\sin^4\theta - 20\sin^2\theta + 5 = 0$, which gives

$$16x^4 - 20x^2 + 5 = 0$$

on substituting $x = \sin\theta$.

The solutions of $16x^4 - 20x^2 + 5 = 0$ are $x = \sin\theta$, where θ satisfies $\dfrac{\sin 5\theta}{\sin\theta} = 0$. All the x are different, and since $\sin 5\theta$ is divided by $\sin\theta$, we exclude the possible root $\sin\theta = 0$. Hence, we have

$$\sin 5\theta = 0 \quad \Rightarrow \quad \theta = 0 \text{ (excluded)}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \ldots$$

which give the following values for $\sin\theta$:

$$\sin\left(\frac{\pi}{5}\right), \quad \sin\left(\frac{2\pi}{5}\right), \quad \sin\left(\frac{3\pi}{5}\right) \text{ which is the same as } \sin\left(\frac{2\pi}{5}\right),$$

$$\sin\left(\frac{4\pi}{5}\right) \text{ which is the same as } \sin\left(\frac{\pi}{5}\right),$$

$\sin\pi$ which is zero and hence excluded,

$$\sin\left(\frac{6\pi}{5}\right) \quad \text{and} \quad \sin\left(\frac{7\pi}{5}\right)$$

Therefore, the four **different** non-zero values of x for $16x^4 - 20x^2 + 5 = 0$ are

$$\sin\left(\frac{\pi}{5}\right) \quad \sin\left(\frac{2\pi}{5}\right) \quad \sin\left(\frac{6\pi}{5}\right) \quad \sin\left(\frac{7\pi}{5}\right)$$

c) We substitute $y = x^2$ to obtain the equation $16y^2 - 20y + 5 = 0$, whose roots are the two different values for y given by the substitution.

There are just two values of x^2, $\sin^2\left(\frac{\pi}{5}\right)$ and $\sin^2\left(\frac{2\pi}{5}\right)$, since

$\sin\left(\frac{6\pi}{5}\right) = -\sin\left(\frac{\pi}{5}\right)$ and $\sin\left(\frac{7\pi}{5}\right) = -\sin\left(\frac{2\pi}{5}\right)$, which give

$$\sin^2\left(\frac{6\pi}{5}\right) = \sin^2\left(\frac{\pi}{5}\right) \quad \text{and} \quad \sin^2\left(\frac{7\pi}{5}\right) = \sin^2\left(\frac{2\pi}{5}\right)$$

Therefore, the two different roots of the equation $16y^2 - 20y + 5 = 0$ are $y = \sin^2\left(\frac{\pi}{5}\right)$ and $y = \sin^2\left(\frac{2\pi}{5}\right)$.

i) Using the product of the roots of a polynomial (see page 147), we have for $16y^2 - 20y + 5 = 0$,

$$\alpha\beta = \frac{5}{16}$$

$$\Rightarrow \quad \sin^2\left(\frac{\pi}{5}\right)\sin^2\left(\frac{2\pi}{5}\right) = \frac{5}{16}$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right) = \pm\sqrt{\frac{5}{16}}$$

Since both $\sin\left(\frac{\pi}{5}\right)$ and $\sin\left(\frac{2\pi}{5}\right)$ are positive, we obtain

$$\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}}{4}$$

ii) Since $16y^2 - 20y + 5 = 0$ is a quadratic equation, its roots are

$$y = \frac{20 \pm \sqrt{400 - 320}}{32}$$

$$\Rightarrow \quad y = \frac{20 \pm \sqrt{80}}{32} = \frac{5 \pm \sqrt{5}}{8}$$

Since these two roots are $\sin^2\left(\frac{\pi}{5}\right)$ and $\sin^2\left(\frac{2\pi}{5}\right)$, and $\sin\left(\frac{2\pi}{5}\right) > \sin\left(\frac{\pi}{5}\right) > 0$, we have

$$\sin^2\left(\frac{\pi}{5}\right) = \frac{5 - \sqrt{5}}{8}$$

Using the identity $\cos\theta \equiv 1 - \sin^2\left(\frac{\theta}{2}\right)$, we obtain

$$\cos\left(\frac{2\pi}{5}\right) = 1 - 2\sin^2\left(\frac{\pi}{5}\right)$$

$$= 1 - 2 \times \frac{5 - \sqrt{5}}{8}$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$$

Exercise 15C

1 If $z = \cos\theta + \mathrm{i}\sin\theta$, find the values of each of the following.

a) $z^2 - \dfrac{1}{z^2}$ **b)** $z^4 + \dfrac{1}{z^4}$ **c)** $z^5 + \dfrac{1}{z^5}$ **d)** $z^2 - \dfrac{2}{z} + \dfrac{2}{z} - \dfrac{1}{z^2}$

2 Express each of the following in terms of z, where $z = \cos\theta + \mathrm{i}\sin\theta$.

a) $\cos 6\theta$ **b)** $\sin 5\theta$ **c)** $\cos^4\theta$ **d)** $\sin^3\theta$
e) $\sin^2 5\theta$ **f)** $\cos^4 3\theta$

3 Express each of the following in terms of $\cos\theta$.

a) $\cos 6\theta$ **b)** $\cos 4\theta$ **c)** $\dfrac{\sin 4\theta}{\sin\theta}$ **d)** $\dfrac{\sin 6\theta}{\sin\theta}$

4 Express each of the following in terms of $\sin\theta$.

a) $\sin 3\theta$ **b)** $\sin 5\theta$ **c)** $\dfrac{\cos 7\theta}{\cos\theta}$ **d)** $\dfrac{\cos 5\theta}{\cos\theta}$

5 Express each of the following in terms of sines or cosines of multiple angles.

a) $\sin^3\theta$ **b)** $\cos^3\theta$ **c)** $\cos^5\theta$ **d)** $\sin^5\theta$
e) $\cos^6\theta$

6 Prove that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

7 Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. Hence solve $t^3 - 3t^2 - 3t + 1 = 0$.

8 By considering $(\cos\theta + \mathrm{i}\sin\theta)^3$, use de Moivre's theorem to establish the identity

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$$

Write down the coefficient of θ^4 in the series expansion of $\cos 3\theta$.

Hence, using the identity above, obtain the coefficient of θ^4 in the series expansion of $\cos^3\theta$.

(AEB 96)

9 i) Show that $(2+\mathrm{i})^4 = -7 + 24\mathrm{i}$.

ii) Use de Moivre's theorem to show that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and $$\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$$

iii) If $t = \tan\theta$, show that

$$\tan 4\theta = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

iv) By considering the argument of $(2+\mathrm{i})$, explain why $t = \frac{1}{2}$ is a root of the following equation

$$\frac{4t - 4t^3}{1 - 6t^2 + t^4} = -\frac{24}{7}$$

v) Using the symmetry properties of the four roots of $z^4 = a^4$, draw an Argand diagram showing the four roots of $z^4 = -7 + 24\text{i}$.
vi) Find one other root of the equation in part **iv**. (NICCEA)

10 Use de Moivre's theorem to prove that

$$\sin 6\theta = 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$$

By putting $x = \sin\theta$, deduce that, for $|x| \leqslant 1$,

$$-\tfrac{1}{2} \leqslant x(16x^4 - 16x^2 + 3)\sqrt{(1 - x^2)} \leqslant \tfrac{1}{2} \qquad \text{(OCR)}$$

11 Use de Moivre's theorem to prove that

$$\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$$

By considering the equation $\cos 5\theta = 0$, show that the exact value of $\cos^2(\frac{1}{10}\pi)$ is $\dfrac{5 + \sqrt{5}}{8}$.

(OCR)

12 Use de Moivre's theorem to show that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

where $t = \tan\theta$. (OCR)

13 Let $z = \cos\theta + \text{i}\sin\theta$.

a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ.
b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ.
c) Use your answers to parts **a** and **b** to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.
d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$, where k, m, n, p are integers. State the values of k, m, n, p. (SQA/CSYS)

14 Use de Moivre's theorem to show that

$$\sin 5\theta = a\cos^4\theta \sin\theta + b\cos^2\theta \sin^3\theta + c\sin^5\theta$$

where a, b and c are integers to be determined.

Hence show that

$$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1 \quad (\theta \neq k\pi, \text{ where } k \in \mathbb{Z})$$

By means of the substitution $x = 2\cos\theta$, find, in trigonometric form, the roots of the equation

$$x^4 - 3x^2 + 1 = 0$$

Hence, or otherwise, show that

$$\cos^2(\tfrac{1}{5}\pi) + \cos^2(\tfrac{2}{5}\pi) = \tfrac{3}{4} \qquad \text{(NEAB)}$$

15 Find each of the roots of the equation $z^5 - 1 = 0$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

a) Given that α is the complex root of this equation with the smallest positive argument, show that the roots of $z^5 - 1 = 0$ can be written as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

b) Show that $\alpha^4 = \alpha^*$ and hence, or otherwise, obtain $z^5 - 1$ as a product of real linear and quadratic factors, giving the coefficients in terms of integers and cosines.

c) Show also that

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$$

and hence, or otherwise, find $\cos(\frac{2}{5}\pi)$, giving your answer in terms of surds. (EDEXCEL)

16 **a)** Use mathematical induction to prove that when n is a positive integer

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

b) Hence show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$ (EDEXCEL)

17 In the polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 = 0$$

all the coefficients $a_n, a_{n-1}, \ldots, a_0$ are real. Given that $x + iy$ is a root of the equation, show that the complex conjugate $x - iy$ is also a root.

Show that $e^{\pi i/6}$ is one root of the equation $z^3 = i$. Find the other two roots and mark on an Argand diagram the points representing the three roots. Show that these three roots are also roots of the equation

$$z^6 + 1 = 0$$

and write down the remaining three roots of this equation. Hence, or otherwise, express $z^6 + 1$ as the product of three quadratic factors each with coefficients in integer or surd form.

(NEAB)

Transformations in a complex plane

We need to be able to transform simple loci in a complex plane, such as straight lines and circles, into new loci, which are again usually straight lines and circles.

The method we usually use is to identify the general point on the original locus and find its image.

Example 23 Under the transformation $w = z^2$, find the image of

a) circle, centre O, radius 3, and

b) line $\arg z = \frac{\pi}{2}$.

SOLUTION

The original locus is $z \equiv x + \mathrm{i}y$, and the new locus is $w \equiv u + \mathrm{i}v$.

a) The general point on the original circle is $z = 3\mathrm{e}^{\mathrm{i}\theta}$, or $z = 3\cos\theta + 3\mathrm{i}\sin\theta$. Its image point is $w = z^2 = 9\mathrm{e}^{2\mathrm{i}\theta}$, or $z = 9(\cos 2\theta + \mathrm{i}\sin 2\theta)$. Therefore, the locus of the image is a circle, centre O, radius 9.

b) The general point on the line $\arg z = \frac{\pi}{2}$ is

$$z = r\mathrm{e}^{\mathrm{i}\pi/2} = r\left[\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad z = \mathrm{i}r$$

Hence, we have for the image point

$$w = z^2 = r^2\mathrm{e}^{\mathrm{i}\pi} \quad \text{or} \quad -r^2$$

Therefore, the locus of the image is a line along the real axis in the negative direction from O.

Example 24 Find the image under the transformation $w = \dfrac{2 + z}{\mathrm{i} - z}$, where z is the circle $|z| = 1$.

SOLUTION

To find the image of $|z| = k$, we usually express z in terms of w and then apply this expression to $|z| = k$.

Hence, we have

$$w(\mathrm{i} - z) = 2 + z$$

$$\Rightarrow \quad w\mathrm{i} - 2 = (1 + w)z$$

$$\Rightarrow \quad z = \frac{w\mathrm{i} - 2}{1 + w}$$