Further Pure Mathematics

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OXFORD

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

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First published 2001

British Library Cataloguing in Publication Data

Data available

ISBN 978-0-19-914735-9

20 19 18 17 16 15 14

Typeset and illustrated by Tech-Set Ltd, Gateshead, Tyne and Wear
Printed in Great Britain by Bell and Bain Ltd., Glasgow

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1 Complex numbers

That wonder of analysis, that portent of the ideal world, that amphibian between being and not being, which we call the imaginary root of unity
GOTTFRIED WILHELM LEIBNIZ

In all our previous mathematics work, we have assumed that it is not possible to have a square root of a negative number. For example, on page 26 of *Introducing Pure Mathematics* where we considered the solution of quadratic equations, $ax^2 + bx + c = 0$, we noted that when $b^2 - 4ac$ is less than zero, the equation is said to have no real roots.

In fact, such an equation has **two complex roots**.

Take, for example, the solution of $x^2 + 2x + 3 = 0$. Using the quadratic formula, we obtain

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 12}}{2} \\ &= \frac{-2 \pm \sqrt{-8}}{2} \\ &= \frac{-2 \pm \sqrt{8}\sqrt{-1}}{2} \\ &= \frac{-2 \pm 2\sqrt{2}\sqrt{-1}}{2} \\ &= -1 \pm \sqrt{2}\sqrt{-1} \end{aligned}$$

There is no real number which is $\sqrt{-1}$, as the square of any real number is always positive.

Therefore, we say that $\sqrt{-1}$ is an **imaginary number**. We denote $\sqrt{-1}$ by i.

So, using i, we can express the roots of the equation above in the form

$$-1 \pm \sqrt{2}\mathrm{i}$$

or $\quad -1 - \sqrt{2}\mathrm{i} \quad$ and $\quad -1 - \sqrt{2}\mathrm{i}$

Note j is also used to represent $\sqrt{-1}$.

What is a complex number?

A **complex number** is a number of the form

$$a + \mathrm{i}b$$

where a and b are real numbers and $\mathrm{i}^2 = -1$.

For example, $3 + 5\mathrm{i}$ is a complex number.

If $a = 0$, the number is said to be **wholly imaginary**. If $b = 0$, the number is **real**. If a complex number is 0, both a and b are 0.

We usually use $x + \mathrm{i}y$ to represent an unknown complex number, and z to represent $x + \mathrm{i}y$. So, when the unknown in an equation is a complex number, we denote it by z: for example, $z^2 - 40z + 40 = 0$, whose roots are $2 \pm 6\mathrm{i}$.

In a similar way, we use w to represent a second unknown complex number, where $w = u + \mathrm{i}v$.

The complex conjugate

The complex number $x - \mathrm{i}y$ is called the **complex conjugate** (or often just the **conjugate**) of $x + \mathrm{i}y$, and is denoted by z^* or $\bar{z}$.

For example, $2 - 3\mathrm{i}$ is the complex conjugate of $2 + 3\mathrm{i}$, and the complex conjugate of $-8 - 9\mathrm{i}$ is $-8 + 9\mathrm{i}$.

Calculating with complex numbers

When we work with complex numbers, we use ordinary algebraic methods. That means that we **cannot** combine a real number with an i-term. For example, $2 + 3\mathrm{i}$ cannot be simplified.

For two complex numbers to be equal, **their real parts must be equal and their imaginary parts must be equal.**

This is a **necessary condition** for the equality of two complex numbers.

Hence, if $a + \mathrm{i}b = c + \mathrm{i}d$, then $a = c$ and $b = d$.

For example, if $2 + 3\mathrm{i} = x + \mathrm{i}y$, then $x = 2$ and $y = 3$.

Addition and subtraction

When adding two complex numbers, we add the real terms and **separately** add the i-terms. For example,

$$\begin{aligned}(3 + 7\mathrm{i}) + (4 - 6\mathrm{i}) &= (3 + 4) + (7\mathrm{i} - 6\mathrm{i}) \\ &= 7 + \mathrm{i}\end{aligned}$$

Generally, for addition we have

$$(x + \mathrm{i}y) + (u + \mathrm{i}v) = (x + u) + \mathrm{i}(y + v)$$

and for subtraction

$$(x + \mathrm{i}y) - (u + \mathrm{i}v) = (x - u) + \mathrm{i}(y - v)$$

Example 1 Subtract $8 - 4\mathrm{i}$ from $7 + 2\mathrm{i}$.

SOLUTION

$$\begin{aligned}7 + 2\mathrm{i} - (8 - 4\mathrm{i}) &= 7 - 8 + (2\mathrm{i} + 4\mathrm{i}) \\ &= -1 + 6\mathrm{i}\end{aligned}$$

Example 2 Find x and y if $x + 2i + 2(3 - 5iy) = 8 - 13i$.

SOLUTION

Equating real terms, we get

$$x + 6 = 8$$
$$\Rightarrow \quad x = 2$$

Equating imaginary terms, we get

$$2 - 10y = -13$$
$$\Rightarrow \quad 15 = 10y$$
$$\Rightarrow \quad y = 1\tfrac{1}{2}$$

Multiplication

We apply the general algebraic method for multiplication. For example,

$$(2 + 3i)(4 - 5i) = 2(4 - 5i) + 3i(4 - 5i)$$
$$= 8 - 10i + 12i - 15i^2$$

Since $i^2 = -1$, this simplifies to

$$8 - 10i + 12i - 15 \times -1 = 8 - 10i + 12i + 15$$
$$= 23 + 2i$$

Generally, we have

$$(a + ib)(c + id) = ac - bd + i(ad + bc) \quad \text{since } i^2 = -1$$

Note It is simpler to multiply out the numbers every time than to memorise this formula.

Division

To be able to divide by a complex number, we have to change it to a real number. Take, for example, the fraction

$$\frac{2 + 3i}{4 + 5i}$$

In the simplification of surds on page 408 of *Introducing Pure Mathematics*, we noted that $\frac{1}{1 + \sqrt{3}}$ could be simplified by multiplying the numerator and the denominator of this fraction by $1 - \sqrt{3}$.

Similarly, to simplify $\frac{2 + 3i}{4 + 5i}$ we multiply its numerator and its denominator by $4 - 5i$, which is the **complex conjugate** of the denominator. Thus, we have

$$\frac{2 + 3i}{4 + 5i} = \frac{(2 + 3i)(4 - 5i)}{(4 + 5i)(4 - 5i)}$$
$$= \frac{8 + 12i - 10i - 15i^2}{4^2 - (5i)^2}$$

$$= \frac{23 + 2i}{16 + 25} \qquad [\text{Note: } -(5i)^2 = -(-25) = +25]$$

$$= \frac{23}{41} + \frac{2}{41}i$$

Example 3 Simplify $\dfrac{3 + i}{7 - 3i}$.

SOLUTION

Multiplying the numerator and the denominator by the complex conjugate of $7 - 3i$, which is $7 + 3i$, we obtain

$$\frac{3 + i}{7 - 3i} = \frac{(3 + i)(7 + 3i)}{(7 - 3i)(7 + 3i)}$$

$$= \frac{21 + 7i + 9i + 3i^2}{7^2 - (3i)^2}$$

$$= \frac{21 + 16i - 3}{49 + 9} \qquad [\text{Note: } -(3i)^2 = -(-9) = +9]$$

$$= \frac{18}{58} + \frac{16i}{58}$$

$$= \frac{9}{29} + \frac{8}{29}i \quad \text{or} \quad \frac{1}{29}(9 + 8i)$$

Example 4 Simplify $\dfrac{(5 - 3i)(7 + i)}{2 - i}$.

SOLUTION

First, we simplify the numerator:

$$\frac{(5 - 3i)(7 + i)}{2 - i} = \frac{35 + 5i - 21i - 3i^2}{2 - i}$$

$$= \frac{35 - 16i + 3}{2 - i}$$

$$= \frac{38 - 16i}{2 - i}$$

We then multiply the numerator and the denominator of this fraction by the complex conjugate of $2 - i$, which is $2 + i$:

$$\frac{(38 - 16i)(2 + i)}{(2 - i)(2 + i)} = \frac{76 + 16 + 38i - 32i}{4 + 1}$$

$$= \frac{92 + 6i}{5} \quad \text{or} \quad 18\tfrac{2}{5} + 1\tfrac{1}{5}i$$

Exercise 1A

1 Simplify each of the following.

a) i^3 **b)** i^4 **c)** i^6 **d)** i^9

2 Express each of the following complex numbers in the form $a + ib$.

a) $3 + 2\sqrt{-1}$ **b)** $6 - 3\sqrt{-1}$ **c)** $-4 + \sqrt{-9}$
d) $-2 + \sqrt{-8}$ **e)** $\sqrt{-100} - \sqrt{-64}$

3 Write down the complex conjugate of z when z is:

a) $3 + 4i$ **b)** $2 - 6i$ **c)** $-4 - 3i$ **d)** $-8 + 5i$

4 Solve each of the following equations.

a) $z^2 + 2z + 4 = 0$ **b)** $z^2 - 3z + 6 = 0$ **c)** $2z^2 + z + 1 = 0$ **d)** $4z - 3 - 2z^2 = 0$

5 Simplify each of the following.

a) $(8 + 4i) + (2 - 6i)$ **b)** $(-7 + 3i) + (8 - 4i)$ **c)** $2 - 4i + 3(-1 + 2i)$
d) $4(-2 + 5i) + 5(2 + 7i)$ **e)** $(8 + 3i) - (7 + 2i)$ **f)** $(7 + 6i) - (4 - 2i)$
g) $2(9 - 3i) - 4(2 - 6i)$ **h)** $3(8 + i) - 2(3 - 5i)$

6 Evaluate each of these expressions.

a) $(3 + i)(2 + 3i)$ **b)** $(4 - 2i)(5 + 3i)$ **c)** $(8 - i)(9 + 2i)$
d) $(9 - 3i)(5 - i)$ **e)** $i(2 - 3i)(i + 4)$ **f)** $(3 - 2i)(7 - 5i)$

7 Express each of these fractions in the form $a + ib$, where $a, b \in \mathbb{R}$.

a) $\dfrac{2 + 3i}{4 - i}$ **b)** $\dfrac{4 + 3i}{5 + i}$ **c)** $\dfrac{8 - i}{2 + 3i}$ **d)** $\dfrac{2 + 5i}{-3 + 2i}$

8 Solve each of the following equations in x and y.

a) $x + iy = 4 - 2i$ **b)** $x + iy + 3 - 2i = 4(-2 + 5i)$
c) $x + iy = (2 + i)(3 - 2i)$ **d)** $x + iy = (3 - 5i)(4 + i)$
e) $x + iy = \dfrac{7 + i}{2 - i}$ **f)** $x + iy = (2 - 3i)^2$

9 If $z = 3 + i$, find the value of $z + \dfrac{1}{z}$.

10 Find the solution of each of the following equations.

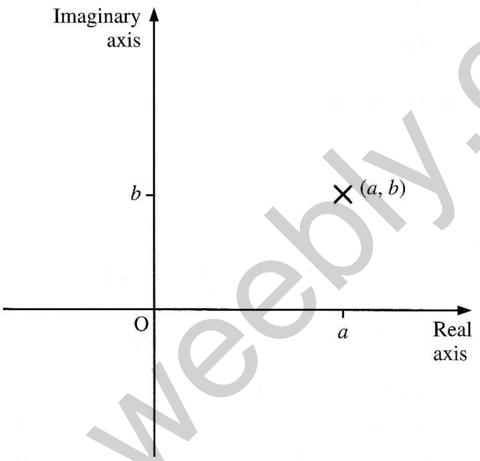
a) $x^2 + 4x + 7 = 0$ **b)** $x^2 + 2x + 6 = 0$ **c)** $2x^2 + 6x + 9 = 0$ **d)** $x^2 - 5x + 25 = 0$

Argand diagram

The French mathematician Jean Robert Argand (1768–1822) is credited with the invention and development of the graphical representation of complex numbers and the operations upon them, although others had anticipated his work. So, this graphical representation has become known as the **Argand diagram**.

In the Argand diagram, the complex number $a + ib$ is represented by the point (a, b), as shown on the right.

Real numbers are represented on the x-axis and imaginary numbers on the y-axis. Thus, the general complex number $(x + iy)$ is represented by the point (x, y).

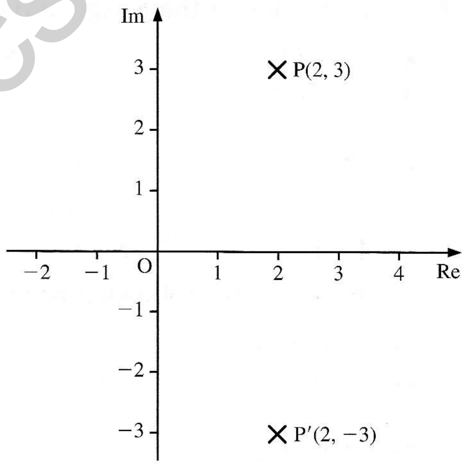


Example 5 Represent the complex number $2 + 3i$ on an Argand diagram. Show its complex conjugate.

SOLUTION

The number $2 + 3i$ is represented by the point $P(2, 3)$.

The complex conjugate is $2 - 3i$, which is represented by the point $P'(2, -3)$.



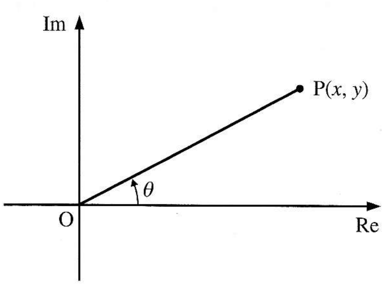
Note The position of the complex conjugate z^* can always be obtained by reflecting the position of z in the real axis.

Modulus–argument or polar form of complex numbers

The position of point $P(x, y)$ on the Argand diagram can be given in terms of OP, the distance of P from the origin, and θ, the angle in the **anticlockwise** sense which OP makes with the positive real axis.

The length OP is the **modulus** of z, denoted by $|z|$, and this length $|z|$ is **always** taken to be **positive**.

The angle θ (normally in radians) is the **argument** of z, denoted by $\arg z$. The **principal value** of θ is taken to be between $-\pi$ and π.

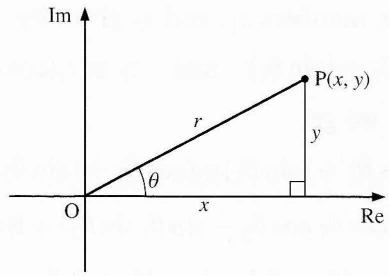


Connection between the $x + iy$ form and the modulus–argument form

From the diagram on the right, we have

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$



which give

$$z \equiv x + iy = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

To find θ, we use

$$\tan\theta = \frac{y}{x}$$

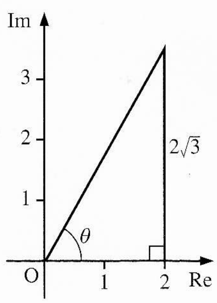
but we need to take care when either x or y is **negative**. (See part **b** in Example 6.)

Example 6 Find the modulus and argument of each of these complex numbers.

a) $2 + 2\sqrt{3}i$ **b)** $-1 - i$

SOLUTION

a)



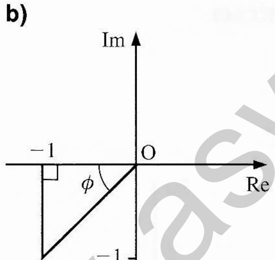
The modulus of $2 + 2\sqrt{3}i$ is given by

$$\sqrt{2^2 + (2\sqrt{3})^2} = 4$$

Its argument, θ, is given by

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

b)



The modulus of $-1 - i$ is given by

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

Angle ϕ is $\frac{\pi}{4}$. Therefore, the argument (the angle from the positive real axis) is

$$-\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

Note If the angle in Example 6 is measured **anticlockwise** from the positive real axis, its value is $\frac{5\pi}{4}$, but this is not between π and $-\pi$. Thus, we take the clockwise angle, which is $-\frac{3\pi}{4}$. The minus sign denotes that the angle is measured in the clockwise sense.

Multiplication of two complex numbers in modulus–argument form

Consider the complex numbers z_1, and z_2 given by

$$z_1 \equiv r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \quad \text{and} \quad z_2 \equiv r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

Multiplying z_1 by z_2, we get

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)\,r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)] \end{aligned}$$

We can state this result as follows:

To find the product of two complex numbers, **multiply their moduli** and **add their arguments**.

Division of two complex numbers in modulus–argument form

Dividing z_1 by z_2, we get

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)} = \frac{r_1}{r_2}\,\frac{\cos\theta_1 + \mathrm{i}\sin\theta_1}{\cos\theta_2 + \mathrm{i}\sin\theta_2}$$

Multiplying the numerator and the denominator by the complex conjugate of $\cos\theta_2 + \mathrm{i}\sin\theta_2$, we have

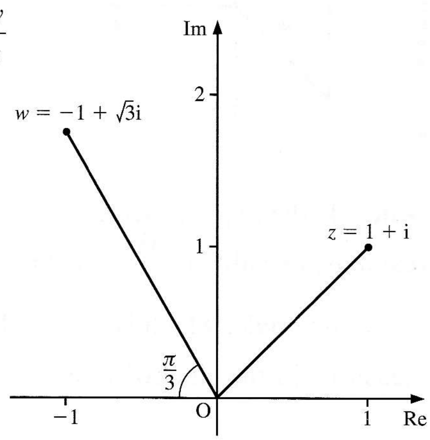
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 + \mathrm{i}\sin\theta_2)(\cos\theta_2 - \mathrm{i}\sin\theta_2)} \\ &= \frac{r_1}{r_2}\,\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2)] \quad \text{since } \cos^2\theta_2 + \sin^2\theta_2 \equiv 1 \end{aligned}$$

We can state this result as follows:

To find the quotient of two complex numbers, **divide their moduli** and **subtract their arguments**.

Example 7 Find the modulus and argument of each of the following.

a) $z = 1 + \mathrm{i}$ **b)** $w = -1 + \sqrt{3}\mathrm{i}$ **c)** zw **d)** z^2 **e)** $\dfrac{w}{z}$



SOLUTION

a) From the diagram, we have

Modulus of $z = \sqrt{2}$

Argument of $z = \dfrac{\pi}{4}$

b) Modulus of $w = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Argument of $w = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

c) Modulus of $zw = |z| \times |w| = 2\sqrt{2}$
Argument of zw is

$$\arg z + \arg w = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$$

d) Using $z^2 = z \times z$, we have

$$\text{Modulus of } z^2 = |z| \times |z| = \sqrt{2} \times \sqrt{2} = 2$$

Argument of z^2 is

$$\arg z + \arg z = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

e) Modulus of $\dfrac{w}{z} = \dfrac{|w|}{|z|} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

Argument of $\dfrac{w}{z}$ is

$$\arg w - \arg z = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$$

Exercise 1B

1 Represent each of the following on an Argand diagram.

a) $2 + 2i$ **b)** $-3 + 3i$ **c)** $-2 + 2\sqrt{3}i$
d) $-1 - i$ **e)** $4i$ **f)** $5 + 12i$
g) -4 **h)** $6 + \sqrt{13}i$

2 Find the modulus and argument of each of the complex numbers in Question **1**.

3 Given that $z = 3 + 4i$,

a) calculate **i)** z^2 **ii)** z^3
b) find **i)** $|z|$ **ii)** $|z^2|$ **iii)** $|z^3|$
c) evaluate **i)** $\arg z$ **ii)** $\arg z^2$ **iii)** $\arg z^3$

4 Express the complex number z in its $a + ib$ form when:

a) $|z| = 2$ and $\arg z = \dfrac{\pi}{3}$ **b)** $|z| = 4$ and $\arg z = \dfrac{\pi}{4}$ **c)** $|z| = 1$ and $\arg z = -\dfrac{\pi}{2}$

d) $|z| = 4$ and $\arg z = \dfrac{3\pi}{4}$ **e)** $|z| = 2$ and $\arg z = \dfrac{5\pi}{6}$ **f)** $|z| = 6$ and $\arg z = \dfrac{7\pi}{6}$

5 a) Simplify $\dfrac{1 - i}{-3 - i}$.

b) Find the modulus and argument of the complex number $-5 + 12i$ (WJEC)

6 Given that $z = \dfrac{3 + 4i}{5 - 12i}$, find the modulus and argument of z. (WJEC)

7 Given that $z = \dfrac{1+\mathrm{i}}{1-2\mathrm{i}}$, find

a) z in the form $a + \mathrm{i}b$

b) the modulus and argument of z. (WJEC)

8 **i)** Given that $z_1 = 5 + \mathrm{i}$ and $z_2 = -2 + 3\mathrm{i}$,

a) show that $|z_1|^2 = 2|z_2|^2$

b) find $\arg(z_1 z_2)$.

ii) Calculate, in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$, the square roots of $16 - 30\mathrm{i}$. (EDEXCEL)

9 Given that

$$z = \tan\alpha + \mathrm{i}, \text{ where } 0 < \alpha < \tfrac{1}{2}\pi$$
$$w = 4[\cos(\tfrac{1}{10}\pi) + \mathrm{i}\sin(\tfrac{1}{10}\pi)]$$

find in their simplest forms

i) $|z|$ **ii)** $|zw|$ **iii)** $\arg z$ **iv)** $\arg\left(\dfrac{z}{w}\right)$ (OCR)

10 The complex number z is given by $z = \sin^2\alpha + \mathrm{i}\sin\alpha\cos\alpha$, where $0 < \alpha < \frac{1}{2}\pi$. Simplifying your answers as far as possible, find

i) $|z|$ **ii)** $\arg z$ (OCR)

11 The complex numbers z and w are such that

$$z = -2 + 5\mathrm{i} \qquad zw = 14 + 23\mathrm{i}$$

a) Find w in the form $p + q\mathrm{i}$, where p and q are real.

b) Display z and w on the same Argand diagram.

c) Find $\arg z$, in radians, giving your answer to two decimal places.

d) Write down the complex number that represents the mid-point M of the line joining the points z and zw. (EDEXCEL)

12 **a)** Find the roots of the equation $z^2 + 4z + 7 = 0$, giving your answers in the form $p \pm \mathrm{i}\sqrt{q}$, where p and q are integers.

b) Show these roots on an Argand diagram.

c) Find for each root

i) the modulus

ii) the argument, in radians

giving your answers to three significant figures. (EDEXCEL)

13 By putting $z = z + \mathrm{i}y$, find the complex number z which satisfies the equation

$$z + 2z^* = \frac{15}{2-\mathrm{i}}$$

where z^* denotes the complex conjugate of z. (NEAB)

14 Given that $z_1 = 1 + 2\mathrm{i}$ and $z_2 = \frac{3}{5} + \frac{4}{5}\mathrm{i}$, write $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in the form $p + \mathrm{i}q$, where p and $q \in \mathbb{R}$.

In an Argand diagram, the origin O and the points representing $z_1 z_2$, $\dfrac{z_1}{z_2}$, z_3 are the vertices of a rhombus. Find z_3 and sketch the rhombus on this Argand diagram.

Show that $|z_3| = \dfrac{6\sqrt{5}}{5}$. (EDEXCEL)

15 The complex numbers z_1 and z_2 are given by

$$z_1 = 5 + \mathrm{i} \qquad z_2 = 2 - 3\mathrm{i}$$

a) Show the points representing z_1 and z_2 on an Argand diagram.
b) Find the modulus of $z_1 - z_2$.
c) Find the complex number $\frac{z_1}{z_2}$ in the form $a + \mathrm{i}b$, where a and b are rational numbers.
d) Hence find the argument of $\frac{z_1}{z_2}$, giving your answer in radians to three significant figures.
e) Determine the values of the real constants p and q such that

$$\frac{p + \mathrm{i}q + 3z_1}{p - \mathrm{i}q + 3z_2} = 2\mathrm{i} \qquad \text{(EDEXCEL)}$$

16 $$z_1 = -3 + 4\mathrm{i} \qquad z_2 = 1 + 2\mathrm{i}$$

a) Express $z_1 z_2$ and $\frac{z_1}{z_2}$ each in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$.
b) Display z_1 and z_2 on the same Argand diagram.
c) Find $\arg z_1$, giving your answer in radians to one decimal place.

Given that $z_1 + (p + \mathrm{i}q)z_2 = 0$, where $p, q \in \mathbb{R}$,

d) obtain the value of p and the value of q. (EDEXCEL)

17 The complex number z is given by $z = -2 + 2\mathrm{i}$.

a) Find the modulus and argument of z.
b) Write down the modulus and argument of $\frac{1}{z}$.
c) Show on an Argand diagram the points A, B and C representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ respectively.
d) State the value of $\angle$ACB. (EDEXCEL)

18 $$z_1 = -30 + 15\mathrm{i}$$

a) Find $\arg z_1$, giving your answer in radians to two decimal places.

The complex numbers z_2 and z_3 are given by $z_2 = -3 + p\mathrm{i}$ and $z_3 = q + 3\mathrm{i}$, where p and q are real constants and $p > q$.

b) Given that $z_2 z_3 = z_1$, find the value of p and the value of q.
c) Using your values of p and q, plot the points corresponding to z_1, z_2 and z_3 on an Argand diagram.
d) Verify that $2z_2 + z_3 - z_1$ is real and find its value. (EDEXCEL)

19 i) Evaluate the square roots of the complex number $5 + 12i$ in the form $a + bi$, where a and b are real.

ii) If θ is the argument of either of these square roots, obtain the value of $\cos 4\theta$ as an **exact** fraction. (NICCEA)

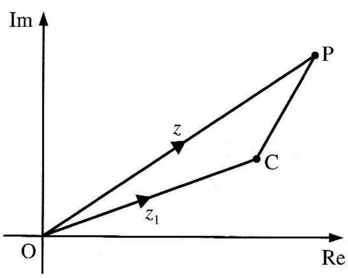
20 a) The complex numbers z and w are such that $z = (4 + 2i)(3 - i)$ and $w = \dfrac{4 + 2i}{3 - i}$. Express each of z and w in the form $a + ib$, where a and b are real.

b) i) Write down the modulus and argument of each of the complex numbers $4 + 2i$ and $3 - i$. Give each modulus in an exact surd form and each argument in radians between $-\pi$ and π.

ii) The points O, P and Q in the complex plane represent the complex numbers $0 + 0i$, $4 + 2i$ and $3 - i$ respectively. Find the exact length of PQ and hence, or otherwise, show that triangle OPQ is right-angled. (AEB 97)

Loci in the complex plane

We know from our previous work on vector geometry that the vector $\mathbf{a} - \mathbf{b}$ connects the point with position vector $\mathbf{b}$ to the point with position vector $\mathbf{a}$. (See *Introducing Pure Mathematics*, page 498.) Similarly, in the complex plane, $z - z_1$ joins the point z_1 to the point z.



From the diagram, we have

$$\overrightarrow{OC} = z_1 \quad \text{and} \quad \overrightarrow{OP} = z$$

Therefore, we obtain

$$\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$$
$$= -z_1 + z$$
$$= z - z_1$$

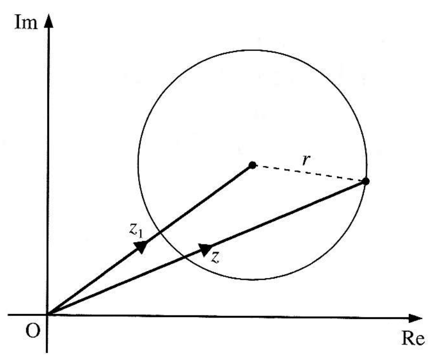
Using this fact, we can identify a number of loci.

Loci which should be recognised

- $|z - z_1| = r$

$|z - z_1|$ is the modulus or length of $z - z_1$. That is, the length of the line joining z_1 to a variable point z.

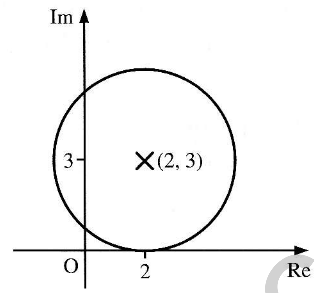
Thus, $|z - z_1| = r$ is the locus of a point, z, moving so that the length of the line joining a fixed point z_1 to z is always r. Hence, the locus of z is a circle, centre z_1 and radius r.



Example 8 State and sketch the locus of $|z - 2 - 3\mathrm{i}| = 3$.

SOLUTION

This locus is $|z - (2 + 3\mathrm{i})| = 3$, which is a circle, centre $(2, 3)$ and radius 3.

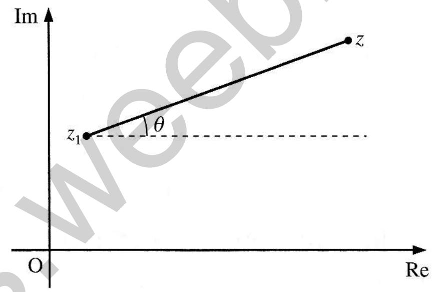


Note When sketching this locus, show clearly that the circle **touches** the x-axis and **cuts** the y-axis twice.

- $\arg(z - z_1) = \theta$

The point z satisfies this locus when the line joining z_1 to z has argument θ.

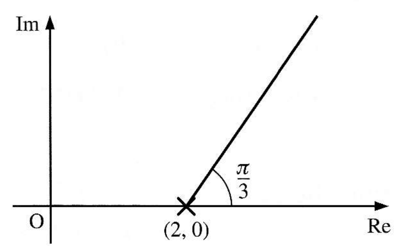
This is the **half-line**, starting at z_1, inclined at θ to the real axis. (It is called a half-line because we want only that part of the line which starts at z_1.)



Example 9 State and sketch the locus of $\arg(z - 2) = \dfrac{\pi}{3}$.

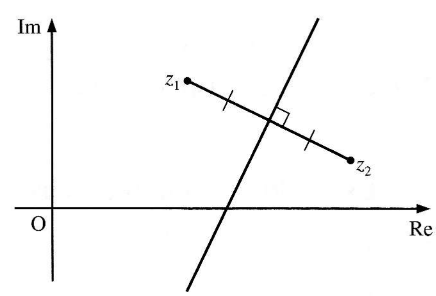
SOLUTION

This locus is the half-line starting at $(2, 0)$, inclined at an angle of $\dfrac{\pi}{3}$ to the real axis.



- $|z - z_1| = |z - z_2|$

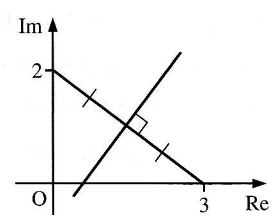
The line joining z to z_1 is equal in length to the line joining z to z_2. Therefore, the locus of z is the perpendicular bisector of the line joining z_1 to z_2.



Example 10 State the locus of $|z - 3| = |z - 2\mathrm{i}|$.

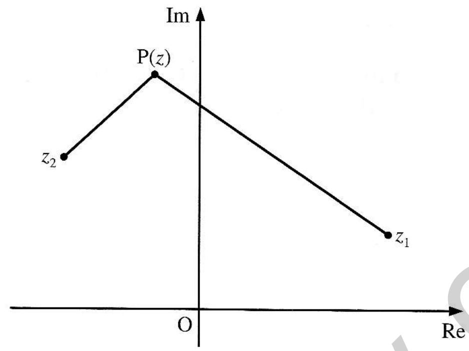
SOLUTION

This locus is the perpendicular bisector of the line joining $+3$ to $+2\mathrm{i}$.



- **$|z - z_1| = k|z - z_2|$, where $k \neq 1$**

The locus of $\mathrm{P}(z)$ is drawn so that the length of the line joining P to z_1 is k times the length of the line joining P to z_2.



Assuming $z \equiv x + \mathrm{i}y$, $z_1 \equiv x_1 + \mathrm{i}y_1$ and $z_2 \equiv x_2 + \mathrm{i}y_2$, Pythagoras' theorem gives

$$|z - z_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

and $$|z - z_2| = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

Therefore, $|z - z_1| = k|z - z_2|$ can be expressed as

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = k\sqrt{(x - x_2)^2 + (y - y_2)^2}$$

Squaring both sides, we get

$$(x - x_1)^2 + (y - y_1)^2 = k^2[(x - x_2)^2 + (y - y_2)^2]$$
$$\Rightarrow \quad x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = k^2x^2 - 2k^2xx_2 + k^2x_2^2 + k^2y^2 - 2k^2yy_2 + k^2y_2^2$$
$$\Rightarrow \quad (1 - k^2)x^2 + (1 - k^2)y^2 - x(2x_1 - 2k^2x_2) - y(2y_1 - 2k^2y_2) + x_1^2 + y_1^2 - k^2x_2^2 - k^2y_2^2 = 0$$

In this equation, the coefficients of x and y are the same, and there is no term in xy. Therefore, the locus of z is a circle.

By symmetry, a diameter of this circle lies on the line joining z_1 to z_2.

Note We recall from earlier work (*Introducing Pure Mathematics*, page 220) that the equation of a circle, centre (a, b) and radius r, is

$$(x - a)^2 + (y - b)^2 = r^2$$

This equation may also be written as

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

To find the centre and the radius of a circle when its equation is written in this form, we use the method of completing the square:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

Therefore, the centre of the circle is $(-g, -f)$, and its radius is $\sqrt{g^2 + f^2 - c}$.

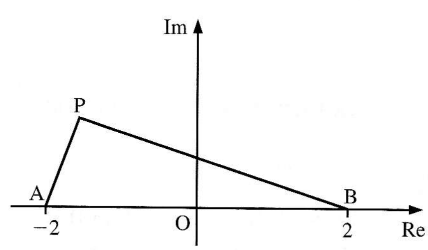
Example 11 Find the locus of $|z - 2| = 3|z + 2|$.

SOLUTION

Let A be $(-2, 0)$ and B be $(2, 0)$.

The locus required is the locus of P when $\mathrm{BP} = 3\mathrm{AP}$.

To find this circle, we determine the two points at which it intersects the line joining A to B.



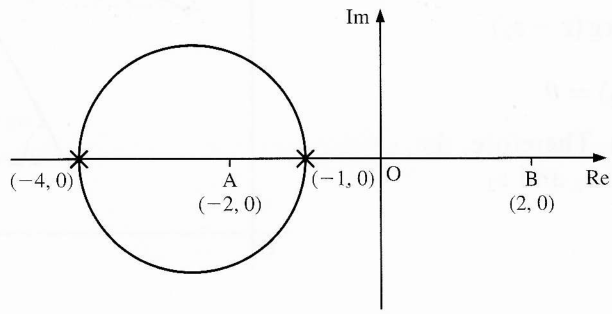
The point $(-1, 0)$ satisfies this condition.

The other point on the line AB which satisfies this condition is never between A and B, but on the line AB produced.

The point $(-4, 0)$ is the other point which satisfies the locus.

The points $(-1, 0)$ and $(-4, 0)$ identify the diameter of the locus's circle. Therefore, the circle has centre $(-2\frac{1}{2}, 0)$ and radius $1\frac{1}{2}$.

Its equation is $|z + 2\frac{1}{2}| = \frac{3}{2}$.



Example 12 Find the locus of $|z - 18| = 2|z + 18\mathrm{i}|$.

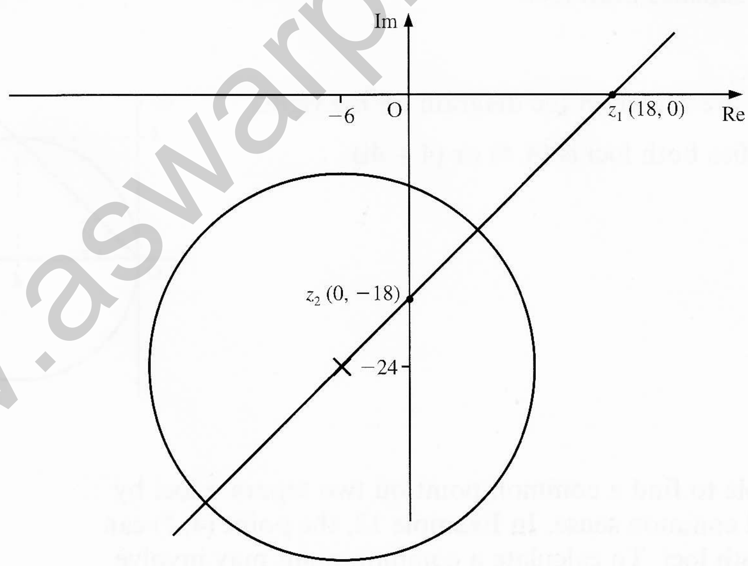
SOLUTION

To find the circle, we determine the two points at which it intersects the line joining z_1 to z_2, where $z_1 = 18$ and $z_2 = -18\mathrm{i}$.

The two points satisfying the locus are $6 - 12\mathrm{i}$ and $-18 - 36\mathrm{i}$.

These two points identify the diameter of the locus's circle. Therefore, the circle has its centre at $-6 - 24\mathrm{i}$ and has a radius of $12\sqrt{2}$.

Hence, its equation is $|z + 6 + 24\mathrm{i}| = 12\sqrt{2}$.



- $\arg \dfrac{(z - z_1)}{(z - z_2)} = \theta$

To find this locus, we use the relationship

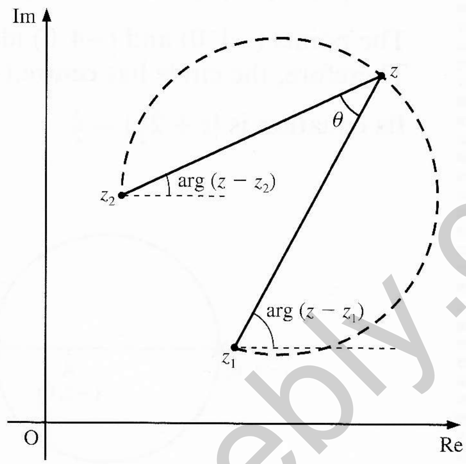
$$\arg \frac{u}{v} = \arg u - \arg v$$

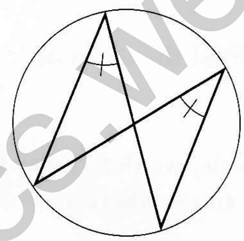
Putting $u = z - z_1$ and $v = z - z_2$, we get

$$\arg \frac{z - z_1}{z - z_2} = \arg (z - z_1) - \arg (z - z_2)$$

$$\Rightarrow \quad \arg (z - z_1) - \arg (z - z_2) = \theta$$

Angles in the same segment are equal. Therefore, the locus of z is part of the circle through z_1 and z_2 (shown dashed).





Example 13 Show the locus of z when

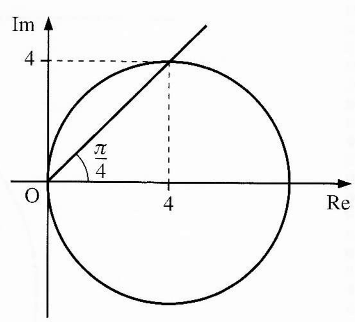
a) $|z - 4| = 4$ **b)** $\arg z = \dfrac{\pi}{4}$

Find the point which satisfies both loci.

SOLUTION

The two loci required are shown in the diagram on the right.

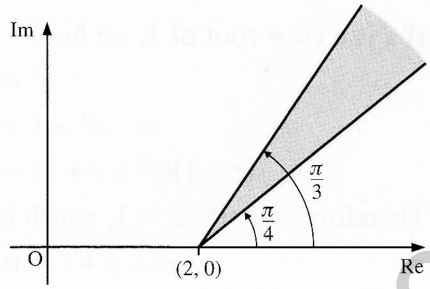
The point which satisfies both loci is $(4, 4)$ or $(4 + 4\text{i})$.



Note Usually, it is possible to find a common point on two separate loci by using simple geometry and common sense. In Example 12, the point $(4, 4)$ can readily be seen to be on both loci. To calculate a common point may involve complicated algebra.

Example 14 Find the locus of $\frac{\pi}{4} < \arg(z-2) < \frac{\pi}{3}$.

SOLUTION



We draw the two separate loci

$$\frac{\pi}{4} = \arg(z-2) \quad \text{and} \quad \arg(z-2) = \frac{\pi}{3}$$

ensuring that we select the correct sector.

- $|z - z_1| + |z - z_2| = c$

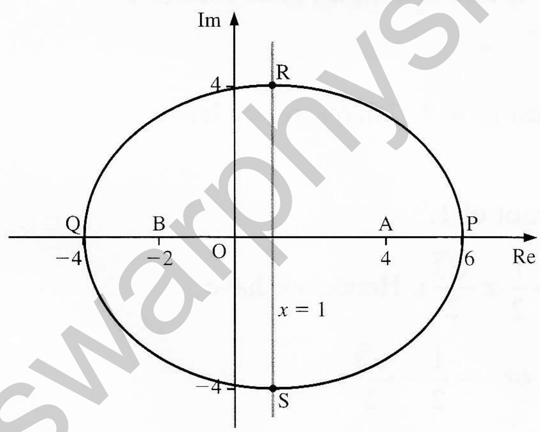
This locus is an ellipse, with z_1 and z_2 as foci (see section on ellipses, pages 222–6). To find the position of the ellipse, we have to find four points which satisfy the locus:

- two points on the line joining z_1 to z_2 produced, and
- two points on the perpendicular bisector of the line joining z_1 to z_2.

Example 15 Find the locus of z when $|z-4| + |z+2| = 10$.

SOLUTION

First, we identify on the diagram the points A and B representing z_1 and z_2. These are $(4, 0)$ and $(-2, 0)$.



We then extend AB in both directions, where AB is of length 6.

Therefore, the points satisfying the locus are $P(6, 0)$ and $Q(-4, 0)$, so that $PA = 2$ and $PB = 8$, which gives $PA + PB = 10$.

Also, we have $QA = 8$ and $QB = 2$, which gives $QA + QB = 10$.

The perpendicular bisector of PQ is the line $x = 1$.

The points satisfying the locus on this line are $R(1, 4)$ and $S(1, -4)$, so that $RA = 5$, $RB = 5$ and hence $RA + RB = 10$.

These four points, P, Q, R and S, identify the major and minor axes of the ellipse.

Cube roots of unity

If z is a cube root of 1, we have

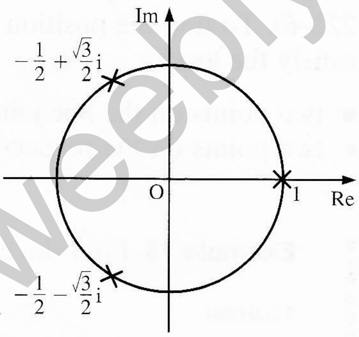
$$z^3 = 1$$
$$\Rightarrow \quad z^3 - 1 = 0$$
$$\Rightarrow \quad (z-1)(z^2+z+1) = 0$$

Therefore, either: $z = 1$, which is the real root, or

$$z^2 + z + 1 = 0$$

If w is a **complex** cube root of 1, $w \neq 1$ and satisfies the equation $z^2 + z + 1 = 0$. Hence, we have

$$w^2 + w + 1 = 0$$
$$\Rightarrow \quad w = \frac{-1 \pm \sqrt{1-4}}{2}$$
$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$



If we plot these three roots of 1 on an Argand diagram, we find them to be symmetrically positioned on the circumference of a circle of radius 1, as shown in the diagram on the right.

Square of a complex cube root of unity

If w is a complex cube root of 1, w^2 is also a complex cube root of 1.

Proof

If w is a complex cube root of 1, then $w^3 = 1$. Therefore, we have

$$(w^2)^3 = w^6 = (w^3)^2 = 1$$

That is, w^2 is also a complex cube root of 1.

Note We found earlier that $w = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$. Hence, we have

$$w^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)^2 \quad \text{or} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

Or we have

$$w^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)^2 \quad \text{or} \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

Thus, we obtain

$$1 + w + w^2 = \left[1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)\right] = 0$$

which agrees with the equation found above.

Example 16 If w is the complex root of 1, find the value of $w^4 + w^8$.

SOLUTION

$$w^4 + w^8 = w \times w^3 + w^2(w^3)^2$$

Since $w^3 = 1$, we get

$$w^4 + w^8 = w + w^2$$

Since $1 + w + w^2 = 0$, we find

$$w^4 + w^8 = -1$$

Example 17 If p is a cube root of 1, find the possible values of $p^2 + p^4$.

SOLUTION

$$p^2 + p^4 = p^2 + p \times p^3$$
$$= p^2 + p \quad \text{since } p^3 = 1$$

If p is real, $p = 1$, and thus $p^2 + p = 2$.

If p is a **complex** cube root, we have

$$p^2 + p = -1$$

Therefore, the possible values of $p^2 + p^4$ are 2 and -1.

Exercise 1C

1 Sketch the locus of z when:

a) $|z| = 5$ **b** $|z| = 3$ **c)** $|z - 2| = 3$ **d)** $|z - 2\mathrm{i}| = 4$

e) $|z + 2 + 2\mathrm{i}| = 2\sqrt{2}$ **f)** $|z + 3 - \sqrt{3}\mathrm{i}| = 2\sqrt{3}$ **g)** $2|z - \mathrm{i}| = 3$

2 Sketch the locus of z when:

a) $\arg z = \frac{\pi}{3}$ **b)** $\arg z = -\frac{3\pi}{4}$ **c)** $\arg(z + 2) = \frac{\pi}{2}$

d) $\arg(z - 3\mathrm{i}) = \frac{\pi}{3}$ **e)** $\arg(z + 1 + \mathrm{i}) = \frac{\pi}{4}$ **f)** $\arg(z - 2 - \sqrt{3}\mathrm{i}) = -\frac{2\pi}{3}$

3 Sketch the locus of z when:

a) $|z - 2| = |z - 4|$ **b)** $|z - 6| = |z + 3|$ **c)** $|z - \mathrm{i}| = |z - 2\mathrm{i}|$

d) $|z + 2\mathrm{i}| = |z - 2|$ **e)** $\left|\frac{z - 1 - \mathrm{i}}{z + 2 + 2\mathrm{i}}\right| = 1$ **f)** $\left|\frac{z - 4\mathrm{i}}{z + 4}\right| = 1$

4 Sketch the locus of z when:

a) $|z - 1| = 3|z + 2|$ **b)** $|z + \mathrm{i}| = 2|z - 2\mathrm{i}|$ **c)** $|z - \mathrm{i}| = 4|z + 3\mathrm{i}|$

d) $|z - 2 - \mathrm{i}| = 3|z + 6 + 3\mathrm{i}|$ **e)** $\left|\frac{z - 2}{z + 2\mathrm{i}}\right| = 3$

5 Sketch each of the following.

a) $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{4}$ **b)** $\arg\left(\frac{z-1}{z-3}\right) = \frac{\pi}{3}$

c) $\arg\left(\frac{z+2i}{z-2i}\right) = \frac{\pi}{4}$ **d)** $\arg\left(\frac{z}{z+4i}\right) = \frac{\pi}{6}$

6 If w is a complex root of 1, simplify each of these.

a) $w^4 + w^8$ **b)** $w^9 + w^{18}$ **c)** $w^3 + w^7 + w^{11}$

7 If w is a cube root of 1, find the possible values of each of the following.

a) $1 + w^4 + w^8$ **b)** $w^3 + w^6$ **c)** $\frac{w + w^4}{w^2 + w^5}$ **d)** $w^8 + w^{10}$

8 Find the solutions of $(z-2)^3 = 1$.

9 With the aid of a sketch, explain why there is no complex number which satisfies both

$$\arg z = \frac{\pi}{3} \quad \text{and} \quad |z-2-i| = |z-4+i|$$

10 The complex number $z = x + iy$ satisfies the equation

$$|z-9+4i| = 3|z-1-4i|$$

The complex number z is represented by the point P in the Argand diagram.

a) Show that the locus of P is a circle.
b) State the centre and radius of this circle.
c) Sketch the circle on an Argand diagram. (EDEXCEL)

11 A complex number z satisfies the inequality

$$|z+2-(2\sqrt{3})i| \leqslant 2$$

Describe in geometrical terms, with the aid of a sketch, the corresponding region in an Argand diagram. Find

i) the least possible value of $|z|$
ii) the greatest possible value of $\arg z$. (OCR)

12 The region R in an Argand diagram is defined by the inequalities

$$|z| \leqslant 4 \quad \text{and} \quad |z| \geqslant |z-2|$$

Draw a clearly labelled diagram to illustrate R. (OCR)

13 The region R of an Argand diagram is defined by the inequalities

$$0 \leqslant \arg(z+4i) \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad |z| \leqslant 4$$

Draw a clearly labelled diagram to illustrate R. (OCR)

14 Two complex numbers, z and w, satisfy the inequalities

$$|z-3-2i| \leqslant 2 \quad \text{and} \quad |w-7-5i| \leqslant 1$$

By drawing an Argand diagram, find the least possible value of $|z-w|$. (OCR)