

# Further Pure Mathematics

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# Contents

<b>Preface</b>	<b>v</b>	Cartesian equation of a line	95
<b>1 Complex Numbers</b>	<b>1</b>	Resolved part of a vector	98
What is a complex number?	1	Direction ratios	98
Calculating with complex numbers	2	Direction cosines	99
Argand diagram	6	Vector product	102
Loci in the complex plane	12	Area of a triangle	105
Cube roots of unity	18	Equation of a plane	106
<b>2 Further Trigonometry with Calculus</b>	<b>22</b>	Distance of a plane from the origin	112
General solutions of trigonometric equations	22	Distance of a plane from a point	113
Harmonic form	26	Scalar triple product and its applications	121
Inverse trigonometric functions	30	<b>7 Curve Sketching and Inequalities</b>	<b>130</b>
<b>3 Polar Coordinates</b>	<b>43</b>	Curve sketching	130
Position of a point	43	Sketching rational functions with a quadratic denominator	133
Connection between polar and cartesian coordinates	44	Inequalities	140
Sketching curves given in polar coordinates	45	<b>8 Roots of Polynomial Equations</b>	<b>147</b>
Area of a sector of a curve	48	Roots of a quadratic equation	147
Equations of the tangents to a curve	53	Roots of a cubic equation	149
<b>4 Differential Equations</b>	<b>57</b>	Roots of a polynomial equation of degree $n$	150
First-order equations requiring an integrating factor	57	Equations with related roots	152
Second-order differential equations	61	Complex roots of a polynomial equation	154
Solution of differential equations by substitution	75	<b>9 Proof, Sequences and Series</b>	<b>159</b>
<b>5 Determinants</b>	<b>80</b>	Proof by induction	159
Definition of $2 \times 2$ and $3 \times 3$ determinants	80	Proof by contradiction	164
Rules for the manipulation of determinants	81	Summation of series	168
Factorisation of determinants	84	Convergence	175
Solution of three equations in three unknowns	87	Maclaurin's series	177
<b>6 Vector Geometry</b>	<b>94</b>	Using power series	182
Vector equation of a line	94	Power series for more complicated functions	185
		<b>10 Hyperbolic Functions</b>	<b>189</b>
		Definitions	189
		Graphs of $\cosh x$ , $\sinh x$ and $\tanh x$	190
		Standard hyperbolic identities	191
		Differentiation of hyperbolic functions	192
		Integration of hyperbolic functions	193
		Inverse hyperbolic functions	194

Logarithmic form	198	Diagonalisation	319
Differentiation of inverse hyperbolic functions	201	The characteristic equation	323
'Double-angle' formulae	210	<b>15 Further Complex Numbers</b>	<b>330</b>
Power series	212	De Moivre's theorem	330
Osborn's rule	213	$n$ th roots of unity	334
<b>11 Conics</b>	<b>218</b>	Exponential form of a complex number	338
Generating conics	218	Trigonometric identities	344
Parabola	219	Transformations in a complex plane	355
Ellipse	222	<b>16 Intrinsic Coordinates</b>	<b>361</b>
Hyperbola	227	Trigonometric functions of $\psi$	361
Polar equation of a conic	230	Radius of curvature	361
<b>12 Further Integration</b>	<b>235</b>	Finding intrinsic equations	365
Inverse function of a function rule	235	<b>17 Groups</b>	<b>369</b>
Integration by parts	236	Binary and unary operations	369
Integration of fractions	237	Modular arithmetic	370
Reduction formulae	241	Definition of a group	370
Arc length	250	Group table	373
Area of a surface of revolution	254	Symmetries of a regular $n$ -sided polygon	376
Improper integrals	259	Non-finite groups	378
Summation of series	262	$a^n$ notation	381
<b>13 Numerical Methods</b>	<b>268</b>	Permutation groups	381
Solution of polynomial equations	268	Generator of a group	382
Evaluation of areas under curves	280	Cyclic groups	382
Step-by-step solution of differential equations	286	Abelian groups	383
Taylor's series	292	Order of a group	386
<b>14 Matrices</b>	<b>299</b>	Order of an element	386
Notation	299	Subgroups	387
The order of a matrix	299	Isomorphic groups	388
Addition and subtraction of matrices	300	Lagrange's theorem	389
Multiplication of matrices	300	Groups of order 3	390
Determinant of a matrix	303	Groups of order 4	390
Identity matrices and zero matrices	303	Groups of order 5	391
Inverse matrices	304	Groups of order 6	392
Transformations	309	Real vector spaces	399
Eigenvectors and eigenvalues	314	<b>Answers</b>	<b>405</b>
		<b>Index</b>	<b>419</b>



# 1 Complex numbers

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*That wonder of analysis, that portent of the ideal world, that amphibian between being and not being, which we call the imaginary root of unity*

GOTTFRIED WILHELM LEIBNIZ

In all our previous mathematics work, we have assumed that it is not possible to have a square root of a negative number. For example, on page 26 of *Introducing Pure Mathematics* where we considered the solution of quadratic equations,  $ax^2 + bx + c = 0$ , we noted that when  $b^2 - 4ac$  is less than zero, the equation is said to have no real roots.

In fact, such an equation has **two complex roots**.

Take, for example, the solution of  $x^2 + 2x + 3 = 0$ . Using the quadratic formula, we obtain

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2} \\&= \frac{-2 \pm \sqrt{-8}}{2} \\&= \frac{-2 \pm \sqrt{8}\sqrt{-1}}{2} \\&= \frac{-2 \pm 2\sqrt{2}\sqrt{-1}}{2} \\&= -1 \pm \sqrt{2}\sqrt{-1}\end{aligned}$$

There is no real number which is  $\sqrt{-1}$ , as the square of any real number is always positive.

Therefore, we say that  $\sqrt{-1}$  is an **imaginary number**. We denote  $\sqrt{-1}$  by  $i$ .

So, using  $i$ , we can express the roots of the equation above in the form

$$-1 \pm \sqrt{2}i$$

or  $-1 - \sqrt{2}i$  and  $-1 + \sqrt{2}i$

Note  $j$  is also used to represent  $\sqrt{-1}$ .

## What is a complex number?

A **complex number** is a number of the form

$$a + ib$$

where  $a$  and  $b$  are real numbers and  $i^2 = -1$ .

For example,  $3 + 5i$  is a complex number.

If  $a = 0$ , the number is said to be **wholly imaginary**. If  $b = 0$ , the number is **real**. If a complex number is 0, both  $a$  and  $b$  are 0.

We usually use  $x + iy$  to represent an unknown complex number, and  $z$  to represent  $x + iy$ . So, when the unknown in an equation is a complex number, we denote it by  $z$ : for example,  $z^2 - 40z + 40 = 0$ , whose roots are  $2 \pm 6i$ .

In a similar way, we use  $w$  to represent a second unknown complex number, where  $w = u + iv$ .

### The complex conjugate

The complex number  $x - iy$  is called the **complex conjugate** (or often just the **conjugate**) of  $x + iy$ , and is denoted by  $z^*$  or  $\bar{z}$ .

For example,  $2 - 3i$  is the complex conjugate of  $2 + 3i$ , and the complex conjugate of  $-8 - 9i$  is  $-8 + 9i$ .

## Calculating with complex numbers

When we work with complex numbers, we use ordinary algebraic methods. That means that we **cannot** combine a real number with an  $i$ -term. For example,  $2 + 3i$  cannot be simplified.

For two complex numbers to be equal, **their real parts must be equal and their imaginary parts must be equal**.

This is a **necessary condition** for the equality of two complex numbers.

Hence, if  $a + ib = c + id$ , then  $a = c$  and  $b = d$ .

For example, if  $2 + 3i = x + iy$ , then  $x = 2$  and  $y = 3$ .

### Addition and subtraction

When adding two complex numbers, we add the real terms and **separately** add the  $i$ -terms. For example,

$$\begin{aligned}(3 + 7i) + (4 - 6i) &= (3 + 4) + (7i - 6i) \\ &= 7 + i\end{aligned}$$

Generally, for addition we have

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

and for subtraction

$$(x + iy) - (u + iv) = (x - u) + i(y - v)$$

**Example 1** Subtract  $8 - 4i$  from  $7 + 2i$ .

**SOLUTION**

$$\begin{aligned}7 + 2i - (8 - 4i) &= 7 - 8 + (2i + 4i) \\ &= -1 + 6i\end{aligned}$$



**Example 2** Find  $x$  and  $y$  if  $x + 2i + 2(3 - 5iy) = 8 - 13i$ .

**SOLUTION**

Equating real terms, we get

$$x + 6 = 8$$

$$\Rightarrow x = 2$$

Equating imaginary terms, we get

$$2 - 10y = -13$$

$$\Rightarrow 15 = 10y$$

$$\Rightarrow y = 1\frac{1}{2}$$

## Multiplication

We apply the general algebraic method for multiplication. For example,

$$\begin{aligned}(2 + 3i)(4 - 5i) &= 2(4 - 5i) + 3i(4 - 5i) \\ &= 8 - 10i + 12i - 15i^2\end{aligned}$$

Since  $i^2 = -1$ , this simplifies to

$$\begin{aligned}8 - 10i + 12i - 15 \times -1 &= 8 - 10i + 12i + 15 \\ &= 23 + 2i\end{aligned}$$

Generally, we have

$$(a + ib)(c + id) = ac - bd + i(ad + bc) \quad \text{since } i^2 = -1$$

**Note** It is simpler to multiply out the numbers every time than to memorise this formula.

## Division

To be able to divide by a complex number, we have to change it to a real number. Take, for example, the fraction

$$\frac{2 + 3i}{4 + 5i}$$

In the simplification of surds on page 408 of *Introducing Pure Mathematics*, we noted that  $\frac{1}{1 + \sqrt{3}}$  could be simplified by multiplying the numerator and the denominator of this fraction by  $1 - \sqrt{3}$ .

Similarly, to simplify  $\frac{2 + 3i}{4 + 5i}$  we multiply its numerator and its denominator by

$4 - 5i$ , which is the **complex conjugate** of the denominator. Thus, we have

$$\begin{aligned}\frac{2 + 3i}{4 + 5i} &= \frac{(2 + 3i)(4 - 5i)}{(4 + 5i)(4 - 5i)} \\ &= \frac{8 + 12i - 10i - 15i^2}{4^2 - (5i)^2}\end{aligned}$$

$$= \frac{23 + 2i}{16 + 25} \quad [\text{Note: } -(5i)^2 = -(-25) = +25]$$

$$= \frac{23}{41} + \frac{2}{41}i$$

**Example 3** Simplify  $\frac{3 + i}{7 - 3i}$ .

**SOLUTION**

Multiplying the numerator and the denominator by the complex conjugate of  $7 - 3i$ , which is  $7 + 3i$ , we obtain

$$\begin{aligned} \frac{3 + i}{7 - 3i} &= \frac{(3 + i)(7 + 3i)}{(7 - 3i)(7 + 3i)} \\ &= \frac{21 + 7i + 9i + 3i^2}{7^2 - (3i)^2} \\ &= \frac{21 + 16i - 3}{49 + 9} \quad [\text{Note: } -(3i)^2 = -(-9) = +9] \\ &= \frac{18}{58} + \frac{16i}{58} \\ &= \frac{9}{29} + \frac{8}{29}i \quad \text{or} \quad \frac{1}{29}(9 + 8i) \end{aligned}$$

**Example 4** Simplify  $\frac{(5 - 3i)(7 + i)}{2 - i}$ .

**SOLUTION**

First, we simplify the numerator:

$$\begin{aligned} \frac{(5 - 3i)(7 + i)}{2 - i} &= \frac{35 + 5i - 21i - 3i^2}{2 - i} \\ &= \frac{35 - 16i + 3}{2 - i} \\ &= \frac{38 - 16i}{2 - i} \end{aligned}$$

We then multiply the numerator and the denominator of this fraction by the complex conjugate of  $2 - i$ , which is  $2 + i$ :

$$\begin{aligned} \frac{(38 - 16i)(2 + i)}{(2 - i)(2 + i)} &= \frac{76 + 16 + 38i - 32i}{4 + 1} \\ &= \frac{92 + 6i}{5} \quad \text{or} \quad 18\frac{2}{5} + 1\frac{1}{5}i \end{aligned}$$



## Exercise 1A

1 Simplify each of the following.

a)  $i^3$       b)  $i^4$       c)  $i^6$       d)  $i^9$

2 Express each of the following complex numbers in the form  $a + ib$ .

a)  $3 + 2\sqrt{-1}$       b)  $6 - 3\sqrt{-1}$       c)  $-4 + \sqrt{-9}$   
 d)  $-2 + \sqrt{-8}$       e)  $\sqrt{-100} - \sqrt{-64}$

3 Write down the complex conjugate of  $z$  when  $z$  is:

a)  $3 + 4i$       b)  $2 - 6i$       c)  $-4 - 3i$       d)  $-8 + 5i$

4 Solve each of the following equations.

a)  $z^2 + 2z + 4 = 0$       b)  $z^2 - 3z + 6 = 0$       c)  $2z^2 + z + 1 = 0$       d)  $4z - 3 - 2z^2 = 0$

5 Simplify each of the following.

a)  $(8 + 4i) + (2 - 6i)$       b)  $(-7 + 3i) + (8 - 4i)$       c)  $2 - 4i + 3(-1 + 2i)$   
 d)  $4(-2 + 5i) + 5(2 + 7i)$       e)  $(8 + 3i) - (7 + 2i)$       f)  $(7 + 6i) - (4 - 2i)$   
 g)  $2(9 - 3i) - 4(2 - 6i)$       h)  $3(8 + i) - 2(3 - 5i)$

6 Evaluate each of these expressions.

a)  $(3 + i)(2 + 3i)$       b)  $(4 - 2i)(5 + 3i)$       c)  $(8 - i)(9 + 2i)$   
 d)  $(9 - 3i)(5 - i)$       e)  $i(2 - 3i)(i + 4)$       f)  $(3 - 2i)(7 - 5i)$

7 Express each of these fractions in the form  $a + ib$ , where  $a, b \in \mathbb{R}$ .

a)  $\frac{2 + 3i}{4 - i}$       b)  $\frac{4 + 3i}{5 + i}$       c)  $\frac{8 - i}{2 + 3i}$       d)  $\frac{2 + 5i}{-3 + 2i}$

8 Solve each of the following equations in  $x$  and  $y$ .

a)  $x + iy = 4 - 2i$       b)  $x + iy + 3 - 2i = 4(-2 + 5i)$   
 c)  $x + iy = (2 + i)(3 - 2i)$       d)  $x + iy = (3 - 5i)(4 + i)$   
 e)  $x + iy = \frac{7 + i}{2 - i}$       f)  $x + iy = (2 - 3i)^2$

9 If  $z = 3 + i$ , find the value of  $z + \frac{1}{z}$ .

10 Find the solution of each of the following equations.

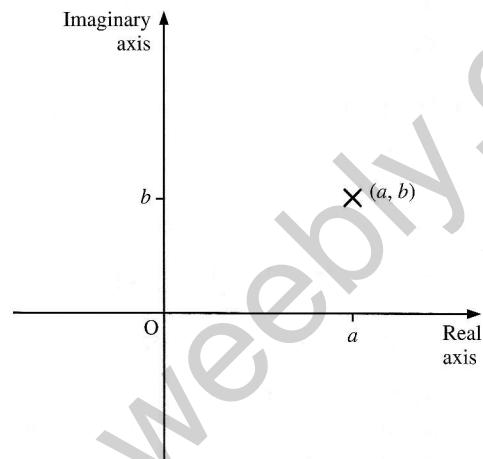
a)  $x^2 + 4x + 7 = 0$       b)  $x^2 + 2x + 6 = 0$       c)  $2x^2 + 6x + 9 = 0$       d)  $x^2 - 5x + 25 = 0$

## Argand diagram

The French mathematician Jean Robert Argand (1768–1822) is credited with the invention and development of the graphical representation of complex numbers and the operations upon them, although others had anticipated his work. So, this graphical representation has become known as the **Argand diagram**.

In the Argand diagram, the complex number  $a + ib$  is represented by the point  $(a, b)$ , as shown on the right.

Real numbers are represented on the  $x$ -axis and imaginary numbers on the  $y$ -axis. Thus, the general complex number  $(x + iy)$  is represented by the point  $(x, y)$ .

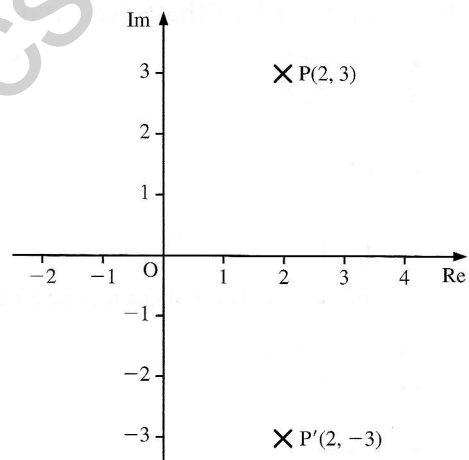


**Example 5** Represent the complex number  $2 + 3i$  on an Argand diagram. Show its complex conjugate.

**SOLUTION**

The number  $2 + 3i$  is represented by the point  $P(2, 3)$ .

The complex conjugate is  $2 - 3i$ , which is represented by the point  $P'(2, -3)$ .



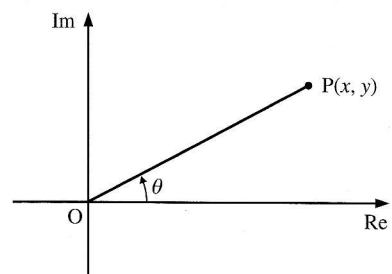
**Note** The position of the complex conjugate  $z^*$  can always be obtained by reflecting the position of  $z$  in the real axis.

## Modulus–argument or polar form of complex numbers

The position of point  $P(x, y)$  on the Argand diagram can be given in terms of  $OP$ , the distance of  $P$  from the origin, and  $\theta$ , the angle in the **anticlockwise** sense which  $OP$  makes with the positive real axis.

The length  $OP$  is the **modulus** of  $z$ , denoted by  $|z|$ , and this length  $|z|$  is **always** taken to be **positive**.

The angle  $\theta$  (normally in radians) is the **argument** of  $z$ , denoted by  $\arg z$ . The **principal value** of  $\theta$  is taken to be between  $-\pi$  and  $\pi$ .





# Connection between the $x + iy$ form and the modulus-argument form

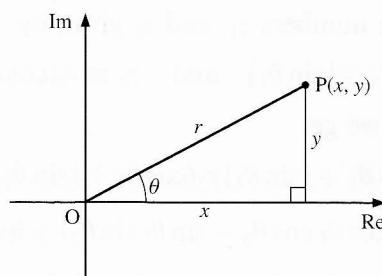
From the diagram on the right, we have

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

which give

$$\begin{aligned} z &\equiv x + iy = r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$



To find  $\theta$ , we use

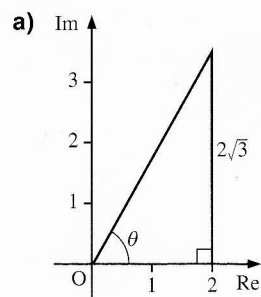
$$\tan \theta = \frac{y}{x}$$

but we need to take care when either  $x$  or  $y$  is **negative**. (See part **b** in Example 6.)

**Example 6** Find the modulus and argument of each of these complex numbers.

a)  $2 + 2\sqrt{3}i$       b)  $-1 - i$

**SOLUTION**

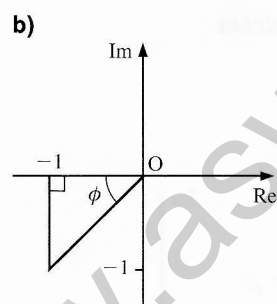


The modulus of  $2 + 2\sqrt{3}i$  is given by

$$\sqrt{2^2 + (2\sqrt{3})^2} = 4$$

Its argument,  $\theta$ , is given by

$$\tan^{-1} \sqrt{3} = \frac{\pi}{3}$$



The modulus of  $-1 - i$  is given by

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

Angle  $\phi$  is  $\frac{\pi}{4}$ . Therefore, the argument

(the angle from the positive real axis) is

$$-\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

**Note** If the angle in Example 6 is measured **anticlockwise** from the positive real axis, its value is  $\frac{5\pi}{4}$ , but this is not between  $\pi$  and  $-\pi$ . Thus, we take the clockwise angle, which is  $-\frac{3\pi}{4}$ . The minus sign denotes that the angle is measured in the clockwise sense.

### Multiplication of two complex numbers in modulus–argument form

Consider the complex numbers  $z_1$ , and  $z_2$  given by

$$z_1 \equiv r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 \equiv r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplying  $z_1$  by  $z_2$ , we get

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

We can state this result as follows:

To find the product of two complex numbers, **multiply their moduli** and **add their arguments**.

### Division of two complex numbers in modulus–argument form

Dividing  $z_1$  by  $z_2$ , we get

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2}$$

Multiplying the numerator and the denominator by the complex conjugate of  $\cos \theta_2 + i \sin \theta_2$ , we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2} \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \\ &= \frac{r_1}{r_2} \frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{(\cos^2 \theta_2 + \sin^2 \theta_2)} \\ &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad \text{since } \cos^2 \theta_2 + \sin^2 \theta_2 \equiv 1 \end{aligned}$$

We can state this result as follows:

To find the quotient of two complex numbers, **divide their moduli** and **subtract their arguments**.

**Example 7** Find the modulus and argument of each of the following.

- a)  $z = 1 + i$     b)  $w = -1 + \sqrt{3}i$     c)  $zw$     d)  $z^2$     e)  $\frac{w}{z}$

**SOLUTION**

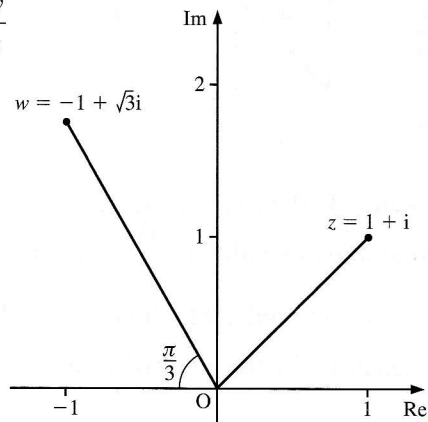
a) From the diagram, we have

$$\text{Modulus of } z = \sqrt{2}$$

$$\text{Argument of } z = \frac{\pi}{4}$$

b) Modulus of  $w = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$$\text{Argument of } w = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$





c) Modulus of  $zw = |z| \times |w| = 2\sqrt{2}$

Argument of  $zw$  is

$$\arg z + \arg w = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$$

d) Using  $z^2 = z \times z$ , we have

$$\text{Modulus of } z^2 = |z| \times |z| = \sqrt{2} \times \sqrt{2} = 2$$

Argument of  $z^2$  is

$$\arg z + \arg z = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

e) Modulus of  $\frac{w}{z} = \frac{|w|}{|z|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Argument of  $\frac{w}{z}$  is

$$\arg w - \arg z = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$$

## Exercise 1B

1 Represent each of the following on an Argand diagram.

a)  $2 + 2i$

b)  $-3 + 3i$

c)  $-2 + 2\sqrt{3}i$

d)  $-1 - i$

e)  $4i$

f)  $5 + 12i$

g)  $-4$

h)  $6 + \sqrt{13}i$

2 Find the modulus and argument of each of the complex numbers in Question 1.

3 Given that  $z = 3 + 4i$ ,

a) calculate i)  $z^2$

ii)  $z^3$

b) find i)  $|z|$

ii)  $|z^2|$

iii)  $|z^3|$

c) evaluate i)  $\arg z$

ii)  $\arg z^2$

iii)  $\arg z^3$

4 Express the complex number  $z$  in its  $a + ib$  form when:

a)  $|z| = 2$  and  $\arg z = \frac{\pi}{3}$

b)  $|z| = 4$  and  $\arg z = \frac{\pi}{4}$

c)  $|z| = 1$  and  $\arg z = -\frac{\pi}{2}$

d)  $|z| = 4$  and  $\arg z = \frac{3\pi}{4}$

e)  $|z| = 2$  and  $\arg z = \frac{5\pi}{6}$

f)  $|z| = 6$  and  $\arg z = \frac{7\pi}{6}$

5 a) Simplify  $\frac{1-i}{-3-i}$ .

b) Find the modulus and argument of the complex number  $-5 + 12i$  (WJEC)

6 Given that  $z = \frac{3+4i}{5-12i}$ , find the modulus and argument of  $z$ . (WJEC)

- 7 Given that  $z = \frac{1+i}{1-2i}$ , find
- $z$  in the form  $a+ib$
  - the modulus and argument of  $z$ . (WJEC)
- 8 i) Given that  $z_1 = 5+i$  and  $z_2 = -2+3i$ ,
- show that  $|z_1|^2 = 2|z_2|^2$
  - find  $\arg(z_1 z_2)$ .
- ii) Calculate, in the form  $a+ib$ , where  $a, b \in \mathbb{R}$ , the square roots of  $16-30i$ . (EDEXCEL)
- 9 Given that
- $$z = \tan \alpha + i, \text{ where } 0 < \alpha < \frac{1}{2}\pi$$
- $$w = 4\left[\cos\left(\frac{1}{10}\pi\right) + i \sin\left(\frac{1}{10}\pi\right)\right]$$
- find in their simplest forms
- $|z|$
  - $|zw|$
  - $\arg z$
  - $\arg\left(\frac{z}{w}\right)$  (OCR)
- 10 The complex number  $z$  is given by  $z = \sin^2 \alpha + i \sin \alpha \cos \alpha$ , where  $0 < \alpha < \frac{1}{2}\pi$ . Simplifying your answers as far as possible, find
- $|z|$
  - $\arg z$  (OCR)
- 11 The complex numbers  $z$  and  $w$  are such that
- $$z = -2 + 5i \quad zw = 14 + 23i$$
- Find  $w$  in the form  $p+qi$ , where  $p$  and  $q$  are real.
  - Display  $z$  and  $w$  on the same Argand diagram.
  - Find  $\arg z$ , in radians, giving your answer to two decimal places.
  - Write down the complex number that represents the mid-point  $M$  of the line joining the points  $z$  and  $zw$ . (EDEXCEL)
- 12 a) Find the roots of the equation  $z^2 + 4z + 7 = 0$ , giving your answers in the form  $p \pm i\sqrt{q}$ , where  $p$  and  $q$  are integers.
- Show these roots on an Argand diagram.
  - Find for each root
    - the modulus
    - the argument, in radians
 giving your answers to three significant figures. (EDEXCEL)
- 13 By putting  $z = x+iy$ , find the complex number  $z$  which satisfies the equation
- $$z + 2z^* = \frac{15}{2-i}$$
- where  $z^*$  denotes the complex conjugate of  $z$ . (NEAB)
- 14 Given that  $z_1 = 1+2i$  and  $z_2 = \frac{3}{5} + \frac{4}{5}i$ , write  $z_1 z_2$  and  $\frac{z_1}{z_2}$  in the form  $p+iq$ , where  $p$  and  $q \in \mathbb{R}$ .
- In an Argand diagram, the origin  $O$  and the points representing  $z_1 z_2$ ,  $\frac{z_1}{z_2}$ ,  $z_3$  are the vertices of a rhombus. Find  $z_3$  and sketch the rhombus on this Argand diagram.
- Show that  $|z_3| = \frac{6\sqrt{5}}{5}$ . (EDEXCEL)

- 15 The complex numbers  $z_1$  and  $z_2$  are given by

$$z_1 = 5 + i \quad z_2 = 2 - 3i$$

- Show the points representing  $z_1$  and  $z_2$  on an Argand diagram.
- Find the modulus of  $z_1 - z_2$ .
- Find the complex number  $\frac{z_1}{z_2}$  in the form  $a + ib$ , where  $a$  and  $b$  are rational numbers.
- Hence find the argument of  $\frac{z_1}{z_2}$ , giving your answer in radians to three significant figures.
- Determine the values of the real constants  $p$  and  $q$  such that

$$\frac{p + iq + 3z_1}{p - iq + 3z_2} = 2i \quad (\text{EDEXCEL})$$

- 16  $z_1 = -3 + 4i \quad z_2 = 1 + 2i$

- Express  $z_1 z_2$  and  $\frac{z_1}{z_2}$  each in the form  $a + ib$  where  $a, b \in \mathbb{R}$ .
- Display  $z_1$  and  $z_2$  on the same Argand diagram.
- Find  $\arg z_1$ , giving your answer in radians to one decimal place.

Given that  $z_1 + (p + iq)z_2 = 0$ , where  $p, q \in \mathbb{R}$ ,

- obtain the value of  $p$  and the value of  $q$ . (EDEXCEL)

- 17 The complex number  $z$  is given by  $z = -2 + 2i$ .

- Find the modulus and argument of  $z$ .
- Write down the modulus and argument of  $\frac{1}{z}$ .
- Show on an Argand diagram the points A, B and C representing the complex numbers  $z$ ,  $\frac{1}{z}$  and  $z + \frac{1}{z}$  respectively.
- State the value of  $\angle ACB$ . (EDEXCEL)

- 18  $z_1 = -30 + 15i$

- Find  $\arg z_1$ , giving your answer in radians to two decimal places.

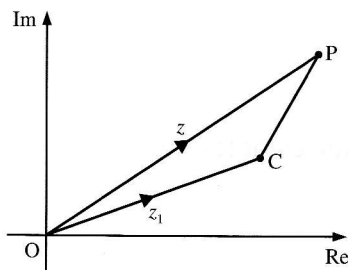
The complex numbers  $z_2$  and  $z_3$  are given by  $z_2 = -3 + pi$  and  $z_3 = q + 3i$ , where  $p$  and  $q$  are real constants and  $p > q$ .

- Given that  $z_2 z_3 = z_1$ , find the value of  $p$  and the value of  $q$ .
- Using your values of  $p$  and  $q$ , plot the points corresponding to  $z_1$ ,  $z_2$  and  $z_3$  on an Argand diagram.
- Verify that  $2z_2 + z_3 - z_1$  is real and find its value. (EDEXCEL)

- 19 i)** Evaluate the square roots of the complex number  $5 + 12i$  in the form  $a + bi$ , where  $a$  and  $b$  are real.
- ii)** If  $\theta$  is the argument of either of these square roots, obtain the value of  $\cos 4\theta$  as an **exact** fraction. (NICCEA)
- 20 a)** The complex numbers  $z$  and  $w$  are such that  $z = (4 + 2i)(3 - i)$  and  $w = \frac{4 + 2i}{3 - i}$ . Express each of  $z$  and  $w$  in the form  $a + ib$ , where  $a$  and  $b$  are real.
- b) i)** Write down the modulus and argument of each of the complex numbers  $4 + 2i$  and  $3 - i$ . Give each modulus in an exact surd form and each argument in radians between  $-\pi$  and  $\pi$ .
- ii)** The points  $O$ ,  $P$  and  $Q$  in the complex plane represent the complex numbers  $0 + 0i$ ,  $4 + 2i$  and  $3 - i$  respectively. Find the exact length of  $PQ$  and hence, or otherwise, show that triangle  $OPQ$  is right-angled. (AEB 97)

## Loci in the complex plane

We know from our previous work on vector geometry that the vector  $\mathbf{a} - \mathbf{b}$  connects the point with position vector  $\mathbf{b}$  to the point with position vector  $\mathbf{a}$ . (See *Introducing Pure Mathematics*, page 498.) Similarly, in the complex plane,  $z - z_1$  joins the point  $z_1$  to the point  $z$ .



From the diagram, we have

$$\overrightarrow{OC} = z_1 \quad \text{and} \quad \overrightarrow{OP} = z$$

Therefore, we obtain

$$\begin{aligned} \overrightarrow{CP} &= \overrightarrow{CO} + \overrightarrow{OP} \\ &= -z_1 + z \\ &= z - z_1 \end{aligned}$$

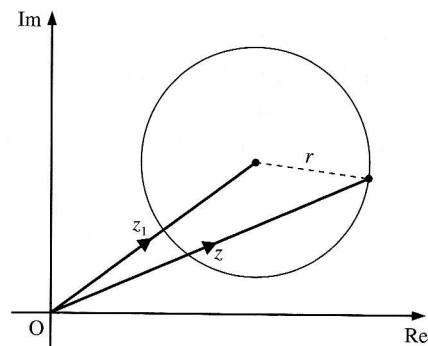
Using this fact, we can identify a number of loci.

### Loci which should be recognised

•  $|z - z_1| = r$

$|z - z_1|$  is the modulus or length of  $z - z_1$ . That is, the length of the line joining  $z_1$  to a variable point  $z$ .

Thus,  $|z - z_1| = r$  is the locus of a point,  $z$ , moving so that the length of the line joining a fixed point  $z_1$  to  $z$  is always  $r$ . Hence, the locus of  $z$  is a circle, centre  $z_1$  and radius  $r$ .

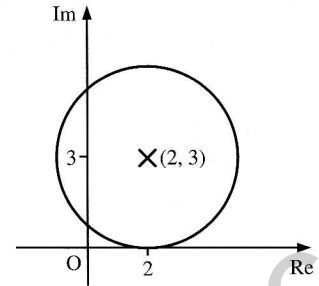




**Example 8** State and sketch the locus of  $|z - 2 - 3i| = 3$ .

**SOLUTION**

This locus is  $|z - (2 + 3i)| = 3$ , which is a circle, centre  $(2, 3)$  and radius 3.

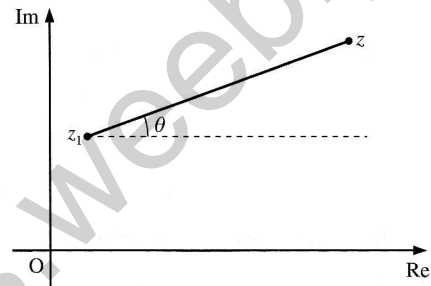


**Note** When sketching this locus, show clearly that the circle **touches** the  $x$ -axis and **cuts** the  $y$ -axis twice.

•  $\arg(z - z_1) = \theta$

The point  $z$  satisfies this locus when the line joining  $z_1$  to  $z$  has argument  $\theta$ .

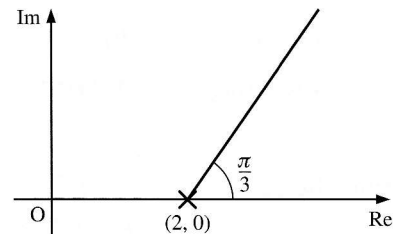
This is the **half-line**, starting at  $z_1$ , inclined at  $\theta$  to the real axis. (It is called a half-line because we want only that part of the line which starts at  $z_1$ .)



**Example 9** State and sketch the locus of  $\arg(z - 2) = \frac{\pi}{3}$ .

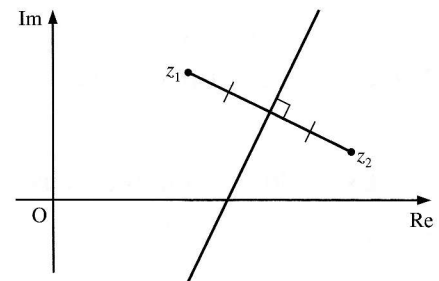
**SOLUTION**

This locus is the half-line starting at  $(2, 0)$ , inclined at an angle of  $\frac{\pi}{3}$  to the real axis.



•  $|z - z_1| = |z - z_2|$

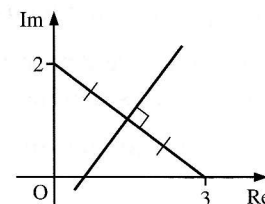
The line joining  $z$  to  $z_1$  is equal in length to the line joining  $z$  to  $z_2$ . Therefore, the locus of  $z$  is the perpendicular bisector of the line joining  $z_1$  to  $z_2$ .



**Example 10** State the locus of  $|z - 3| = |z - 2i|$ .

**SOLUTION**

This locus is the perpendicular bisector of the line joining  $+3$  to  $+2i$ .



- $|z - z_1| = k|z - z_2|$ , where  $k \neq 1$

The locus of  $P(z)$  is drawn so that the length of the line joining  $P$  to  $z_1$  is  $k$  times the length of the line joining  $P$  to  $z_2$ .

Assuming  $z \equiv x + iy$ ,  $z_1 \equiv x_1 + iy_1$  and  $z_2 \equiv x_2 + iy_2$ , Pythagoras' theorem gives

$$|z - z_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

$$\text{and } |z - z_2| = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

Therefore,  $|z - z_1| = k|z - z_2|$  can be expressed as

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = k\sqrt{(x - x_2)^2 + (y - y_2)^2}$$

Squaring both sides, we get

$$\begin{aligned} (x - x_1)^2 + (y - y_1)^2 &= k^2[(x - x_2)^2 + (y - y_2)^2] \\ \Rightarrow x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 &= k^2x^2 - 2k^2xx_2 + k^2x_2^2 + k^2y^2 - 2k^2yy_2 + k^2y_2^2 \\ \Rightarrow (1 - k^2)x^2 + (1 - k^2)y^2 - x(2x_1 - 2k^2x_2) - y(2y_1 - 2k^2y_2) + x_1^2 + y_1^2 - k^2x_2^2 - k^2y_2^2 &= 0 \end{aligned}$$

In this equation, the coefficients of  $x$  and  $y$  are the same, and there is no term in  $xy$ . Therefore, the locus of  $z$  is a circle.

By symmetry, a diameter of this circle lies on the line joining  $z_1$  to  $z_2$ .

**Note** We recall from earlier work (*Introducing Pure Mathematics*, page 220) that the equation of a circle, centre  $(a, b)$  and radius  $r$ , is

$$(x - a)^2 + (y - b)^2 = r^2$$

This equation may also be written as

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

To find the centre and the radius of a circle when its equation is written in this form, we use the method of completing the square:

$$\begin{aligned} x^2 + y^2 + 2gx + 2fy + c &= 0 \\ (x + g)^2 + (y + f)^2 &= g^2 + f^2 - c \end{aligned}$$

Therefore, the centre of the circle is  $(-g, -f)$ , and its radius is  $\sqrt{g^2 + f^2 - c}$ .

**Example 11** Find the locus of  $|z - 2| = 3|z + 2|$ .

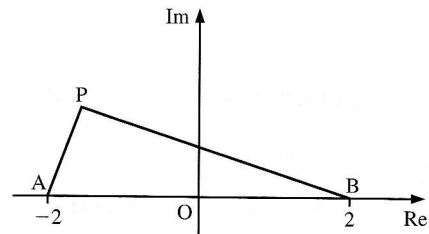
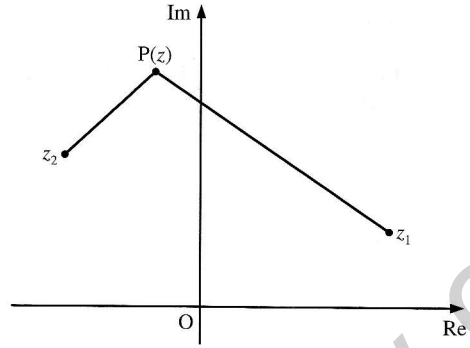
**SOLUTION**

Let  $A$  be  $(-2, 0)$  and  $B$  be  $(2, 0)$ .

The locus required is the locus of  $P$  when  $BP = 3AP$ .

To find this circle, we determine the two points at which it intersects the line joining  $A$  to  $B$ .

The point  $(-1, 0)$  satisfies this condition.

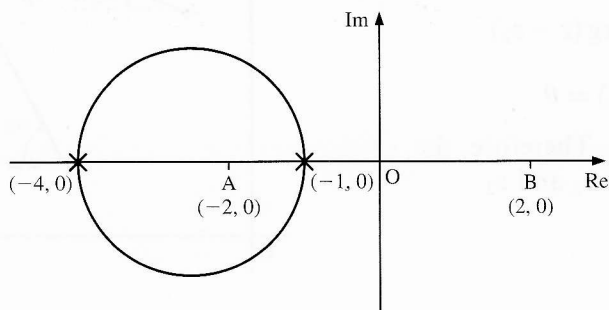


The other point on the line AB which satisfies this condition is never between A and B, but on the line AB produced.

The point  $(-4, 0)$  is the other point which satisfies the locus.

The points  $(-1, 0)$  and  $(-4, 0)$  identify the diameter of the locus's circle. Therefore, the circle has centre  $(-2\frac{1}{2}, 0)$  and radius  $1\frac{1}{2}$ .

Its equation is  $|z + 2\frac{1}{2}| = \frac{3}{2}$ .



**Example 12** Find the locus of  $|z - 18| = 2|z + 18i|$ .

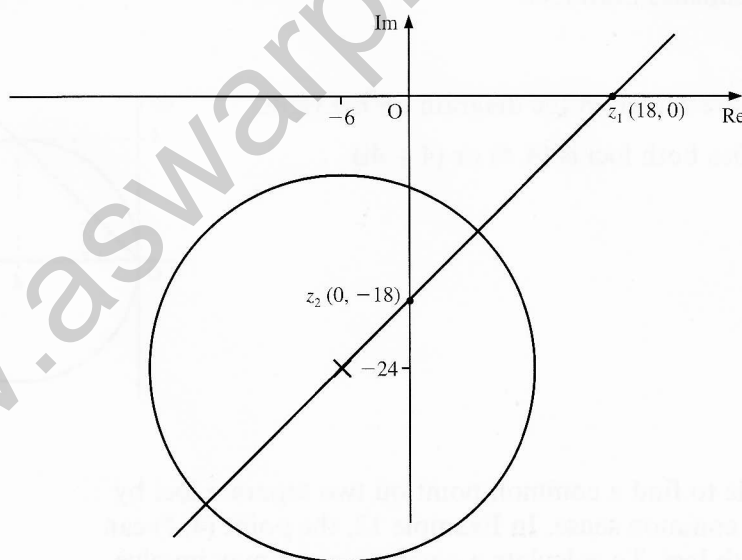
**SOLUTION**

To find the circle, we determine the two points at which it intersects the line joining  $z_1$  to  $z_2$ , where  $z_1 = 18$  and  $z_2 = -18i$ .

The two points satisfying the locus are  $6 - 12i$  and  $-18 - 36i$ .

These two points identify the diameter of the locus's circle. Therefore, the circle has its centre at  $-6 - 24i$  and has a radius of  $12\sqrt{2}$ .

Hence, its equation is  $|z + 6 + 24i| = 12\sqrt{2}$ .



•  $\arg \frac{(z - z_1)}{(z - z_2)} = \theta$

To find this locus, we use the relationship

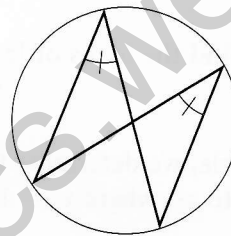
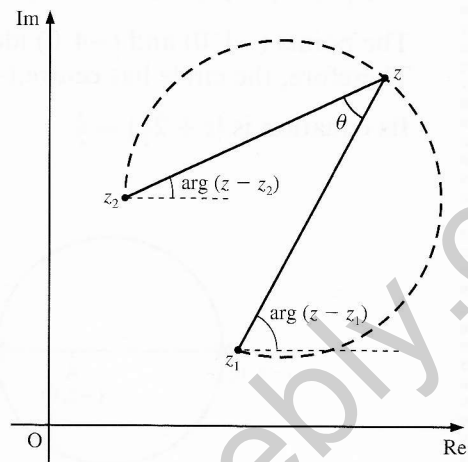
$$\arg \frac{u}{v} = \arg u - \arg v$$

Putting  $u = z - z_1$  and  $v = z - z_2$ , we get

$$\arg \frac{z - z_1}{z - z_2} = \arg(z - z_1) - \arg(z - z_2)$$

$$\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \theta$$

Angles in the same segment are equal. Therefore, the locus of  $z$  is part of the circle through  $z_1$  and  $z_2$  (shown dashed).



**Example 13** Show the locus of  $z$  when

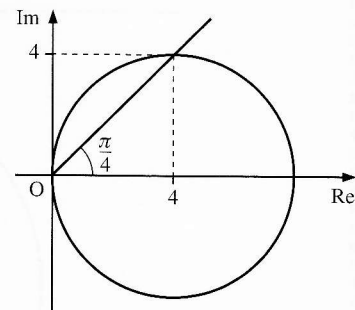
a)  $|z - 4| = 4$       b)  $\arg z = \frac{\pi}{4}$

Find the point which satisfies both loci.

**SOLUTION**

The two loci required are shown in the diagram on the right.

The point which satisfies both loci is  $(4, 4)$  or  $(4 + 4i)$ .



**Note** Usually, it is possible to find a common point on two separate loci by using simple geometry and common sense. In Example 12, the point  $(4, 4)$  can readily be seen to be on both loci. To calculate a common point may involve complicated algebra.



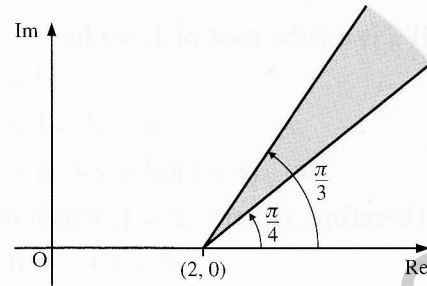
**Example 14** Find the locus of  $\frac{\pi}{4} < \arg(z - 2) < \frac{\pi}{3}$ .

**SOLUTION**

We draw the two separate loci

$$\frac{\pi}{4} = \arg(z - 2) \quad \text{and} \quad \arg(z - 2) = \frac{\pi}{3}$$

ensuring that we select the correct sector.



$$|z - z_1| + |z - z_2| = c$$

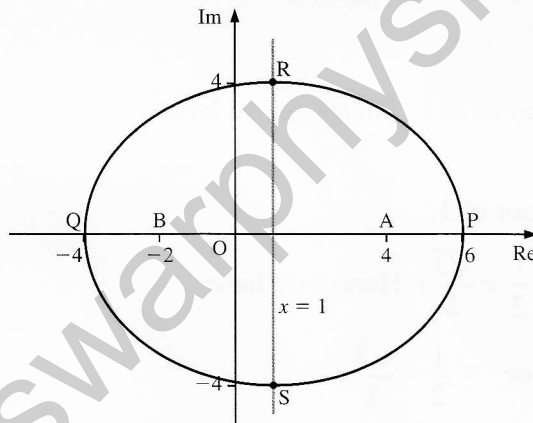
This locus is an ellipse, with  $z_1$  and  $z_2$  as foci (see section on ellipses, pages 222–6). To find the position of the ellipse, we have to find four points which satisfy the locus:

- two points on the line joining  $z_1$  to  $z_2$  produced, and
- two points on the perpendicular bisector of the line joining  $z_1$  to  $z_2$ .

**Example 15** Find the locus of  $z$  when  $|z - 4| + |z + 2| = 10$ .

**SOLUTION**

First, we identify on the diagram the points A and B representing  $z_1$  and  $z_2$ . These are (4, 0) and (−2, 0).



We then extend AB in both directions, where AB is of length 6.

Therefore, the points satisfying the locus are P(6, 0) and Q(−4, 0), so that PA = 2 and PB = 8, which gives PA + PB = 10.

Also, we have QA = 8 and QB = 2, which gives QA + QB = 10.

The perpendicular bisector of PQ is the line  $x = 1$ .

The points satisfying the locus on this line are R(1, 4) and S(1, −4), so that RA = 5, RB = 5 and hence RA + RB = 10.

These four points, P, Q, R and S, identify the major and minor axes of the ellipse.

## Cube roots of unity

If  $z$  is a cube root of 1, we have

$$\begin{aligned} z^3 &= 1 \\ \Rightarrow z^3 - 1 &= 0 \\ \Rightarrow (z - 1)(z^2 + z + 1) &= 0 \end{aligned}$$

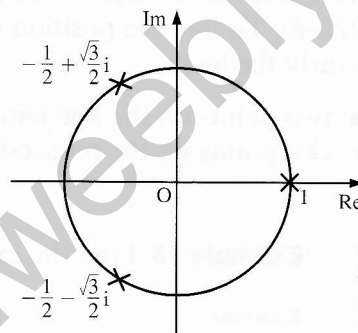
Therefore, either:  $z = 1$ , which is the real root, or

$$z^2 + z + 1 = 0$$

If  $w$  is a **complex** cube root of 1,  $w \neq 1$  and satisfies the equation  $z^2 + z + 1 = 0$ . Hence, we have

$$\begin{aligned} w^2 + w + 1 &= 0 \\ \Rightarrow w &= \frac{-1 \pm \sqrt{1 - 4}}{2} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

If we plot these three roots of 1 on an Argand diagram, we find them to be symmetrically positioned on the circumference of a circle of radius 1, as shown in the diagram on the right.



## Square of a complex cube root of unity

If  $w$  is a complex cube root of 1,  $w^2$  is also a complex cube root of 1.

### Proof

If  $w$  is a complex cube root of 1, then  $w^3 = 1$ . Therefore, we have

$$(w^2)^3 = w^6 = (w^3)^2 = 1$$

That is,  $w^2$  is also a complex cube root of 1.

**Note** We found earlier that  $w = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ . Hence, we have

$$w^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \quad \text{or} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Or we have

$$w^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 \quad \text{or} \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Thus, we obtain

$$1 + w + w^2 = \left[1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right] = 0$$

which agrees with the equation found above.

**Example 16** If  $w$  is the complex root of 1, find the value of  $w^4 + w^8$ .

**SOLUTION**

$$w^4 + w^8 = w \times w^3 + w^2(w^3)^2$$

Since  $w^3 = 1$ , we get

$$w^4 + w^8 = w + w^2$$

Since  $1 + w + w^2 = 0$ , we find

$$w^4 + w^8 = -1$$

**Example 17** If  $p$  is a cube root of 1, find the possible values of  $p^2 + p^4$ .

**SOLUTION**

$$\begin{aligned} p^2 + p^4 &= p^2 + p \times p^3 \\ &= p^2 + p \quad \text{since } p^3 = 1 \end{aligned}$$

If  $p$  is real,  $p = 1$ , and thus  $p^2 + p = 2$ .

If  $p$  is a **complex** cube root, we have

$$p^2 + p = -1$$

Therefore, the possible values of  $p^2 + p^4$  are 2 and  $-1$ .

## Exercise 1C

1 Sketch the locus of  $z$  when:

- a)  $|z| = 5$                       b)  $|z| = 3$                       c)  $|z - 2| = 3$                       d)  $|z - 2i| = 4$   
 e)  $|z + 2 + 2i| = 2\sqrt{2}$                       f)  $|z + 3 - \sqrt{3}i| = 2\sqrt{3}$                       g)  $2|z - i| = 3$

2 Sketch the locus of  $z$  when:

- a)  $\arg z = \frac{\pi}{3}$                       b)  $\arg z = -\frac{3\pi}{4}$                       c)  $\arg(z + 2) = \frac{\pi}{2}$   
 d)  $\arg(z - 3i) = \frac{\pi}{3}$                       e)  $\arg(z + 1 + i) = \frac{\pi}{4}$                       f)  $\arg(z - 2 - \sqrt{3}i) = -\frac{2\pi}{3}$

3 Sketch the locus of  $z$  when:

- a)  $|z - 2| = |z - 4|$                       b)  $|z - 6| = |z + 3|$                       c)  $|z - i| = |z - 2i|$   
 d)  $|z + 2i| = |z - 2|$                       e)  $\left| \frac{z - 1 - i}{z + 2 + 2i} \right| = 1$                       f)  $\left| \frac{z - 4i}{z + 4} \right| = 1$

4 Sketch the locus of  $z$  when:

- a)  $|z - 1| = 3|z + 2|$                       b)  $|z + i| = 2|z - 2i|$                       c)  $|z - i| = 4|z + 3i|$   
 d)  $|z - 2 - i| = 3|z + 6 + 3i|$                       e)  $\left| \frac{z - 2}{z + 2i} \right| = 3$

5 Sketch each of the following.

$$\begin{array}{ll} \text{a) } \arg\left(\frac{z}{z-2}\right) = \frac{\pi}{4} & \text{b) } \arg\left(\frac{z-1}{z-3}\right) = \frac{\pi}{3} \\ \text{c) } \arg\left(\frac{z+2i}{z-2i}\right) = \frac{\pi}{4} & \text{d) } \arg\left(\frac{z}{z+4i}\right) = \frac{\pi}{6} \end{array}$$

6 If  $w$  is a complex root of 1, simplify each of these.

$$\text{a) } w^4 + w^8 \quad \text{b) } w^9 + w^{18} \quad \text{c) } w^3 + w^7 + w^{11}$$

7 If  $w$  is a cube root of 1, find the possible values of each of the following.

$$\text{a) } 1 + w^4 + w^8 \quad \text{b) } w^3 + w^6 \quad \text{c) } \frac{w + w^4}{w^2 + w^5} \quad \text{d) } w^8 + w^{10}$$

8 Find the solutions of  $(z-2)^3 = 1$ .

9 With the aid of a sketch, explain why there is no complex number which satisfies both

$$\arg z = \frac{\pi}{3} \quad \text{and} \quad |z-2-i| = |z-4+i|$$

10 The complex number  $z = x + iy$  satisfies the equation

$$|z-9+4i| = 3|z-1-4i|$$

The complex number  $z$  is represented by the point P in the Argand diagram.

- Show that the locus of P is a circle.
- State the centre and radius of this circle.
- Sketch the circle on an Argand diagram. (EDEXCEL)

11 A complex number  $z$  satisfies the inequality

$$|z+2-(2\sqrt{3})i| \leq 2$$

Describe in geometrical terms, with the aid of a sketch, the corresponding region in an Argand diagram. Find

- the least possible value of  $|z|$
- the greatest possible value of  $\arg z$ . (OCR)

12 The region  $R$  in an Argand diagram is defined by the inequalities

$$|z| \leq 4 \quad \text{and} \quad |z| \geq |z-2|$$

Draw a clearly labelled diagram to illustrate  $R$ . (OCR)

13 The region  $R$  of an Argand diagram is defined by the inequalities

$$0 \leq \arg(z+4i) \leq \frac{1}{4}\pi \quad \text{and} \quad |z| \leq 4$$

Draw a clearly labelled diagram to illustrate  $R$ . (OCR)

14 Two complex numbers,  $z$  and  $w$ , satisfy the inequalities

$$|z-3-2i| \leq 2 \quad \text{and} \quad |w-7-5i| \leq 1$$

By drawing an Argand diagram, find the least possible value of  $|z-w|$ . (OCR)