possibility-space diagrammatically. Find the probability that the total is
(a) 4, (b) greater than 4.

17. A coin is tossed and a die is thrown. Draw the possibility-space diagram and find the probability that the outcome of the trial is
(a) a tail and an odd number, (b) a tail and a multiple of 3.

18. Two cards are drawn from a pack of cards, the first being replaced before the second is drawn. Find the probability that the cards will be
(a) a diamond followed by a spade, (b) a king followed by a black card,
(c) a red card followed by a club, (d) a red card and a club in any order,
(e) two aces.

19. A number is selected at random from the set {2, 4, 6, 8} and another number is selected at random from the set {1, 3, 5, 7}. The two numbers so obtained are multiplied together. Draw a possibility-space diagram and find the probability that the product formed will be
(a) even, (b) more than ten.

20. Two numbers are selected at random, one from each of the sets {1, 3, 5, 7} and {2, 4, 6, 8}. The numbers obtained are added together. Draw a possibility-space diagram and find the probability that the sum obtained is (a) less than eight, (b) greater than ten.

21. Three cards are drawn in succession from a pack of cards with replacement taking place. Find the probability that the cards will be
(a) all red, (b) a heart followed by two spades,
(c) a heart and two spades in any order.

22. Two cards are drawn from a pack of cards without replacement. Find the probability that the two cards will be
(a) a club followed by a spade, (b) a red ace followed by a spade,
(c) a black 7 followed by a red card, (d) two hearts.

23. A bag contains 9 discs, 2 of which are green and 7 yellow. Two discs are removed at random in succession, without replacement. Find the probability that the discs will
(a) both be green, (b) be of the same colour, (c) be of different colours.

24. Two different pupils are chosen at random from a group of 3 boys and 5 girls. Find the probability that the two pupils chosen will be
(a) the two youngest pupils, (b) two boys.

25. A coin is tossed and a die is thrown. Find the probability that the result will be a head on the coin or a six on the die (or both).

26. A card is drawn from a pack of 52 cards and a die is thrown. Find the probability of obtaining an ace card or a three on the die (or both).

27. A coin is tossed and a card selected from a pack of 52 cards. Find the probability of obtaining a head on the coin or a court card (or both).

28. A die is biased so that the probability of throwing a six is $\frac{1}{10}$. Two players each throw the die once. Find the probability that one or both players throw a six.

29. Four different numbers are chosen at random from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. Find the probability that the numbers chosen will be
(a) all odd, (b) two odd and two even numbers.

30. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be
(a) all red, (b) all blue, (c) one of each colour.

9.3 Further conditional probability

In the last section we saw that, for conditional events A and B,

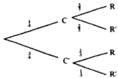
$$P(A \cap B) = P(A) \times P(B|A)$$

Some questions require $P(B|A)$ to be found and the rearranged form $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ is useful.

Example 12

A bag contains five discs, three of which are red. A box contains six discs, four of which are red. A card is selected at random from a normal pack of 52 cards, if the card is a club a disc is removed from the bag and if the card is not a club a disc is removed from the box. Find (a) the probability that the disc removed will be red, (b) the probability that, if the removed disc is red it came from the bag.

Such questions are best solved by means of a tree diagram. Let C be the event of the card being a club and R the event of the disc being red. Draw the tree diagram:
(R′ is the event of the disc *not* being red.)



$P(C \cap R) = \frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$

$P(C \cap R') = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$

$P(C' \cap R) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

$P(C' \cap R') = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

(a) $$P(R) = P(C \cap R) + P(C' \cap R) = \frac{3}{20} + \frac{1}{2} = \frac{13}{20}$$

(b) We now require $P(C|R)$

$$P(C|R) = \frac{P(R \cap C)}{P(R)} = \frac{P(C \cap R)}{P(R)} = \frac{\frac{3}{20}}{\frac{13}{20}} = \frac{3}{13}$$

Exercise 9C

1. If A and B are dependent events such that $P(A) = \frac{5}{8}$ and $P(B|A) = \frac{3}{7}$, find $P(A \cap B)$.
2. If A and B are dependent events such that $P(A) = \frac{2}{5}$ and $P(A \cap B) = \frac{3}{10}$, find $P(B|A)$.
3. Suppose that two different cards are removed from a pack of 52 cards. If event A is that of the first card being a Jack and event B is that of the second card being a court card (J, Q, K), find (a) $P(B|A)$, (b) $P(A \cap B)$.
4. A bag contains 3 yellow and 1 blue disc, whereas a box contains 2 yellow and 3 blue discs. A card is drawn from a pack of cards and if this card is a court card, a disc is drawn from the bag, otherwise a disc is drawn from the box.
   (a) Find the probability that the disc drawn is yellow,
   (b) given that the disc drawn is blue, find the probability that it came from the bag.
5. A bag contains 4 golf balls and 2 other balls, whereas a box contains 2 golf balls and 3 other balls. A die is thrown and if a number less than 3

results a ball is drawn from the bag, otherwise a ball is drawn from the box. Find the probability that
(a) the ball drawn is a golf ball,
(b) if the ball drawn is a golf ball, it came from the box.

6. A school is divided into two parts: Upper school, 400 boys and 200 girls, Lower school, 400 girls and 300 boys. A first pupil is chosen at random from the school. If this pupil comes from the Lower school, a second pupil is chosen from the Upper school; if the first pupil comes from the Upper school, the second pupil is chosen from the Lower school. Find the probability that
(a) the second pupil chosen will be a girl,
(b) if the second pupil chosen is a boy, he is a member of the Upper school.
7. A bag contains 20 discs of which one quarter are white; a similar box contains 15 discs of which one third are white. A card is drawn from a pack, and if the card drawn is a 7, 8 or 9 a disc is drawn from the bag, otherwise a disc is drawn from the box. Find the probability that
(a) the disc drawn is white,
(b) if the disc drawn is white, it came from the box.
8. A hospital diagnoses that a patient has contracted a virus X but it is not known which one of the three strains of the virus X_1, X_2 or X_3 the patient has. For a patient having virus X, the probability of it being X_1, X_2 or X_3 is $\frac{1}{2}$, $\frac{3}{8}$ or $\frac{1}{8}$ respectively. The probability of a recovery (event R) is $\frac{1}{2}$ for X_1, $\frac{2}{3}$ for X_2 and $\frac{1}{4}$ for X_3. Find
(a) the probability that the patient will recover,
(b) the probability that, if the patient recovers, he had virus X_3.
9. A box contains 6 discs of which 2 are blue; a similar bag contains 5 discs of which 3 are blue. A coin, biased so that a head is twice as likely as a tail, is tossed. If the outcome is a head, a disc is removed from the box, otherwise a disc is removed from the bag.
(a) Find the probability that the disc will not be blue,
(b) given that the disc is blue, find the probability that it came from the box.
10. Use the box and bag of question 9. When the weighted coin shows a head, two discs are removed from the box without replacement and, if not a head, two discs are removed from the bag, without replacement. Find the probability that
(a) both discs are blue, (b) if both discs are blue, they came from the bag.

9.4 Probability involving permutations and combinations

Example 13

A team of five children is randomly selected from Alec, Bob, Charles, David, Emma, Frances, Gina and Helen. What is the probability that both Gina and Helen are in the team?

There are 8 children from whom to choose the team.
Number of ways of choosing a team of 5 from 8 = ${}^8C_5$
If Gina and Helen are to be included, the other 3 to complete the team have

to be chosen from the other 6 (A, B, C, D, E, F)
Number of ways of choosing a team including Gina and Helen = ${}^6C_3$

$$P(\text{G, H included}) = \frac{\text{number of teams including G and H}}{\text{total number of possible teams}}$$
$$= \frac{{}^6C_3}{{}^8C_5} = \frac{6!}{3!\,3!} \times \frac{5!\,3!}{8!}$$
$$= \frac{5}{14}$$

Example 14

Find the probability that when a hand of 7 cards is dealt from a shuffled pack of 52 cards it contains (a) all 4 aces, (b) exactly 3 aces, (c) at least 3 aces.

(a) Total number of possible hands = ${}^{52}C_7$
number of hands with 4 aces = ${}^{48}C_3$ (since the other 3 cards must be chosen from the other 48 cards)

$$P(\text{hand has 4 aces}) = \frac{{}^{48}C_3}{{}^{52}C_7} = \frac{1}{7735}$$

(b) number of hands with 3 aces and 4 cards not aces = ${}^4C_3 \times {}^{48}C_4$

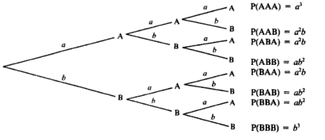
$$P(\text{exactly 3 aces}) = 4 \times \frac{{}^{48}C_4}{{}^{52}C_7} = \frac{9}{1547}$$

(c) P(at least 3 aces) = P(exactly 3 aces) + P(4 aces)

$$= \frac{1}{7735} + \frac{9}{1547}$$
$$= \frac{46}{7735}$$

Binomial Probability

Suppose that a trial is repeated a number of times, say 3, and that in each trial there are two possible outcomes A and B. If P(A) = a and P(B) = b, and the outcome of each trial is independent of the previous trials, the tree diagram would be:



The probability of obtaining outcome B on each occasion = P(BBB) = b^3

or of obtaining outcome A exactly twice = P(AAB) + P(ABA) + P(BAA)

$= 3a^2b$

Clearly for more trials, say 5, the tree diagram would become very large and so an alternative method is needed. If the trial is carried out five times the various outcomes would be:

5 A's   4 A's, 1 B   3 A'S, 2 B's   2 A'S, 3 B's   1 A, 4 B'S   5 B's

The 5 A's occur in five trials in ${}^5C_5$ = 1 way (AAAAA)

The 4 A's occur in five trials in ${}^5C_4$ = 5 ways (BAAAA, ABAAA, AABAA, AAABA, AAAAB)

The 3 A's occur in five trials in ${}^5C_3$ = 10 ways etc.

Thus the probabilities of the various combinations of A's and B's would be

$${}^5C_5a^5 \quad {}^5C_4a^4b^1 \quad {}^5C_3a^3b^2 \quad {}^5C_2a^2b^3 \quad {}^5C_1ab^4 \quad {}^5C_0b^5$$

or $\quad 1a^5 \quad 5a^4b^1 \quad 10a^3b^2 \quad 10a^2b^3 \quad 5ab^4 \quad 1b^5$

Notice that these numbers could also be obtained from the sixth row of Pascal's Triangle:

```
                  1
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
```

Similarly the seventh row would give the coefficients for 6 trials and so on.

Example 15

A coin is biased so that P(head) = $\frac{2}{3}$ and P(tail) = $\frac{1}{3}$. If the coin is tossed 6 times find the probability of obtaining

(a) 6 heads, (b) exactly 5 heads, (c) at least 5 heads, (d) at least 1 tail,
(e) 3 heads and 3 tails with the heads occurring on successive tosses of the coin.

First list the various combinations of probabilities with h = P(head) = $\frac{2}{3}$ and t = P(tail) = $\frac{1}{3}$
These are:

$${}^6C_6h^6 \quad {}^6C_5h^5t^1 \quad {}^6C_4h^4t^2 \quad {}^6C_3h^3t^3 \quad {}^6C_2h^2t^4 \quad {}^6C_1h^1t^5 \quad {}^6C_0t^6$$

or $\quad 1h^6 \quad 6h^5t^1 \quad 15h^4t^2 \quad 20h^3t^3 \quad 15h^2t^4 \quad 6h^1t^5 \quad 1t^6$

(a) P(6 heads) $= 1h^6 = (\frac{2}{3})^6 = \frac{64}{729}$

(b) P(5 heads) $= 6h^5t^1 = 6(\frac{2}{3})^5(\frac{1}{3}) = \frac{64}{243}$

(c) P(at least 5 heads) = P(exactly 5 heads) + P(6 heads)
$= \frac{64}{243} + \frac{64}{729}$
$= \frac{256}{729}$

(d) P(at least 1 tail) = 1 − P(no tails) = 1 − P(6 heads)
$= 1 - \frac{64}{729}$
$= \frac{665}{729}$

(e) $P(HHHTTT) = \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 = \frac{2^3}{3^6}$

However, there are $\frac{4!}{3!}$ ways of arranging the 3 heads and 3 tails with the 3 heads kept together:

HHHTTT, THHHTT, TTHHHT, TTTHHH.

Thus the required probability is $4 \times \frac{2^3}{3^6} = \frac{32}{729}$

Exercise 9D

1. There are only 2 boys in a group of 6 pupils. A group of 5 pupils is to be selected. Find the probability that both boys are in the group selected.
2. A group of 7 pupils are designated T, U, V, W, X, Y and Z. Find the probability that when a group of 4 pupils are selected, at random, it will include X, Y and Z.
3. A bag contains 10 discs of which 3 are red and 7 are blue. If 6 discs are selected at random, find the probability that all the red discs are selected.
4. Four letters are chosen at random from the letters a, b, c, d, e, f. Find the probability that both vowels are in the group chosen.
5. There are 5 green, 4 yellow and 3 blue discs in a bag, from which 4 discs are chosen at random. Find the probability that the 4 discs selected will contain
   (a) exactly 3 blue discs, (b) exactly 3 yellow discs, (c) at least one green disc.
6. From a well shuffled pack of 52 cards a hand of 6 cards is dealt. Find the probability that the hand will contain
   (a) 4 queens, (b) exactly 3 queens, (c) at least 3 queens.
7. A hand of 7 cards is dealt from a shuffled pack of 52 cards. Find the probability that the hand will contain
   (a) exactly 6 black cards, (b) all black cards, (c) at least 6 black cards.
8. An unbiased coin is tossed 7 times. Find the probability of obtaining
   (a) 7 tails, (b) exactly 6 tails, (c) at least one head.
9. A fair coin is tossed ten times. Find the probability of obtaining
   (a) exactly 9 heads, (b) no tails, (c) more than 8 heads.
10. A fair coin is tossed 9 times. Find the probability of obtaining
    (a) 5 heads and 4 tails, (b) 7 tails and 2 heads.
11. A coin is biased so that P(head) $= \frac{1}{4}$ and P(tail) $= \frac{3}{4}$. If the coin is tossed 5 times find the probability of obtaining
    (a) no tails, (b) exactly 1 tail, (c) 3 tails and 2 heads,
    (d) at least 4 heads, (e) 2 heads and 3 tails with the heads occurring in succession.
12. A fair coin is tossed 12 times. Find the probability of obtaining more than 9 tails.
13. A marksman fires at a target and the probability of hitting the bull with any one shot is $\frac{2}{3}$. If he fires 6 shots find the probability that he obtains
    (a) 6 bulls, (b) no bulls, (c) at least 3 bulls.
14. Ten unbiased dice are thrown. Find the probability of obtaining exactly seven sixes.
15. A fair die is thrown 6 times. Find the probability of obtaining
    (a) 6 odd numbers, (b) only 1 even number, (c) at least 4 odd numbers.
16. A firm manufactures radios. The probability that any radio fails the quality check is 0·1. Find the probability that in a case of 15 radios
    (a) all pass the quality check,
    (b) exactly 1 fails the quality check,
    (c) at least 13 pass the quality check.

17. A die is biased so that P(odd number) = $\frac{1}{3}$ and P(even number) = $\frac{2}{3}$. If this die is thrown 8 times, find the probability of obtaining
(a) an even number 7 times, (b) no odd numbers,
(c) an equal number of odd and even numbers,
(d) 5 odd numbers and 3 even numbers with the even numbers occurring in succession,
(e) alternate odd and even numbers.

Exercise 9E Examination questions

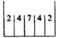
1. From a pack of 52 playing cards two cards are drawn without replacement. Calculate, as a fraction in its lowest terms, the probability that
(a) both are Hearts,
(b) both are red,
(c) neither is a Club. (A.E.B)

2. A card is drawn at random from an ordinary pack of 52 playing cards and it is not returned to the pack. A second card is drawn at random. Calculate, showing details of your working, the probabilities:
(i) that the first card is a spade and the second card is the king of spades;
(ii) that the first card is a king and the second card is the queen of the same suit;
(iii) that the two cards are numerically the same or have the same rank (e.g. are both queens);
(iv) that one card is a heart and the other a spade. (Oxford)

3. Box A contains 5 pieces of paper numbered 1, 3, 5, 7, 9.
Box B contains 3 pieces of paper numbered 1, 4, 9.
One piece of paper is removed at random from each box.
Calculate the probability that the two numbers obtained have
(i) the same value,
(ii) a sum greater than 3,
(iii) a product that is exactly divisible by 3. (Cambridge)

4. Two fair cubical dice are thrown. Calculate the probability that the sum of the scores is
(a) exactly 5,
(b) less than 5,
(c) at least 6. (A.E.B.)

5. In a game, a player rolls two balls down an inclined plane so that each ball finally settles in one of five slots and scores the number of points allotted to that slot as shown in the diagram on the right.



It is possible for both balls to settle in one slot and it may be assumed that each slot is equally likely to accept either ball.
The player's score is the sum of the points scored by each ball.
Draw up a table showing all the possible scores and the probability of each.
If the player pays 10p for each game and receives back a number of pence equal to his score, calculate the player's expected gain or loss per 50 games. (Cambridge)

6. (a) Two dice are thrown together, and the scores added. What is the probability that (i) the total score exceeds 8? (ii) the total score is 9, or the individual scores differ by 1, or both?

(b) A bag contains 3 red balls and 4 black ones. 3 balls are picked out, one at a time and not replaced. What is the probability that there will be 2 red and 1 black in the sample? (S.U.J.B.)

7. A, B and C fire one shot each at a target.
The probability that A will hit the target is $\frac{1}{5}$.
The probability that B will hit the target is $\frac{1}{4}$.
The probability that C will hit the target is $\frac{1}{3}$.
If they fire together, calculate the probability that
(i) all three shots hit the target, (ii) C's shot only hits the target,
(iii) at least one shot hits the target, (iv) exactly two shots hit the target.
(Cambridge)

8. The birthdays of Jack and Jill are in the first seven days in January.
Find the probability that next year
(a) both have their birthday on Monday,
(b) Jack and Jill have their birthdays on the same day,
(c) they have their birthdays on different days,
(d) Monday is the birthday of one or both. (A.E.B.)

9. Calculate
(i) the number of arrangements of 8 different books on a shelf,
(ii) the probability that, in any one of these arrangements, 3 particular books are together. (Cambridge)

10. Three dice are to be rolled. Find the probability of scoring a double but not a triple. (London)

11. The probability that it will rain on a given morning is $\frac{1}{4}$.
If it rains the probability that Mr X misses his train is $\frac{2}{3}$.
If it does not rain the probability that Mr X catches his train is $\frac{5}{6}$.
If he catches his train the probability that he is early for work is $\frac{4}{5}$.
If he misses his train the probability that he is late for work is $\frac{3}{5}$.
Calculate the probability that, on a given morning,
(i) it rains and Mr X is late for work,
(ii) it does not rain and he is early for work. (Cambridge)

12. Two rugby teams, A and B, play a series of three matches. The probability that team A wins any given match is $\frac{1}{2}$ while the probability that team B wins any given match is $\frac{2}{5}$.
Calculate the probability that
(i) all three matches are drawn,
(ii) the teams are level after two matches,
(iii) the series is drawn. (Cambridge)

13. In order to start in a game of chance a player throws a fair cubical die until he obtains a six. He then records whatever scores he obtains on subsequent throws.
For example:
throws of 2, 4, 3, 6, 4, 6, 2, 5 give recorded scores of 4, 6, 2, 5.
Calculate the probability that
(a) the first score recorded is that of the player's fourth throw,
(b) the player does not record a score in his first five throws.
The player has seven throws. Calculate the probability that he will have recorded
(c) exactly three fives and a three,
(d) a total score of three. (A.E.B.)

14. Two events A and B are such that
$P(A) = 0{\cdot}2$, $P(A' \cap B) = 0{\cdot}22$, $P(A \cap B) = 0{\cdot}18$.
Evaluate (i) $P(A \cap B')$, (ii) $P(A|B)$. (J.M.B.)

15. A card is drawn at random from a pack of 52 cards and is then replaced. Let A denote the event 'an ace is drawn', and let R denote the event 'a red card is drawn'. Calculate the values of P(A), P(R), $P(A \cap R)$ and $P(A \cup R)$.
If this experiment is performed four times, find the probability
(a) that an ace will be drawn for the first time at the third attempt,
(b) that a red card will be drawn exactly twice.
(c) that a red card will be drawn at least once.
Find also how many times the experiment must be performed to ensure that the probability of a red card being drawn at least once exceeds 0·99. (A.E.B.)

16. Two players, X and Y, play a game in which X throws 6 coins and Y throws an unbiased die. Player X wins if the number of heads is greater than the number on the die.
(i) If X throws 5 heads find the probability that he wins.
(ii) If Y throws the number 3, show that the probability of X winning is $\frac{11}{32}$. (Cambridge)

17. Previous experience indicates that, of the students entering upon a particular diploma course, 90% will successfully complete it.
One year, 15 students commence the course. Calculate, correct to 3 decimal places, the probability that
(i) all 15 successfully complete the course,
(ii) only 1 student fails,
(iii) no more than 2 students fail,
(iv) at least 2 students fail. (Cambridge)

18. The probability that a marksman will hit a target is $\frac{4}{5}$. He fires 10 shots. Calculate, correct to 3 decimal places, the probability that he will hit the target
(i) at least 8 times,
(ii) no more than 7 times.
If he hits the target exactly 7 times, calculate the probability that the 3 misses are with successive shots. (Cambridge)

19. The probability that a certain football team, playing at home, will win a match is $\frac{4}{5}$.
Calculate the probability that in their next 6 home matches, the team will win
(i) exactly 4 matches,
(ii) at least 4 matches,
(iii) exactly 4 successive matches. (Cambridge)

20. Out of a large population it may be assumed that 5% of unmarried men of forty years of age will marry within five years. Calculate the probabilities that out of a sample of 8 unmarried men who are 40 years old, selected at random,
(i) none will get married within 5 years;
(ii) just three will get married within 5 years;
(iii) at least four will get married within 5 years.
If the size of the sample increases by 4, calculate the probability that just 6 of the 12 men are still unmarried after five years. (Oxford)

21. A player can play on either of two gambling machines, A and B. He chooses one of the two machines at random, and plays two games. The probability of winning a game on A is $\frac{1}{3}$, and the probability of winning a game on B is $\frac{1}{4}$. If he loses both of these two games, he plays a third game on the other machine; otherwise he plays the third game on the same machine. Find the probability that he
(i) wins the first game;
(ii) changes machines after the second game;
(iii) plays the third game on A;
(iv) wins the third game. (Oxford)

22. The events A and B are such that

$$P(A) = \tfrac{1}{2},$$
$$P(A|B') = \tfrac{2}{3},$$
$$P(A|B) = \tfrac{3}{7},$$

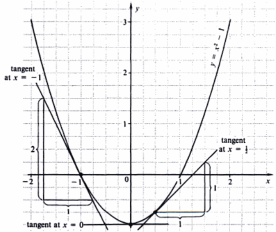
where B′ is the event 'B does not occur'. Find $P(A \cap B)$, $P(A \cup B)$, $P(B)$, $P(B|A)$.
State, with reasons, whether A and B are
(i) independent,
(ii) mutually exclusive. (Cambridge)

23. Three people independently each think of an integer in the set $\{1, 2, 3, 4, 5, 6, 7\}$. Find, in fractional form, the probability that
(a) all three of the integers selected are greater than 4,
(b) all three of the integers selected are greater than 5,
(c) the least integer selected is 5,
(d) the three integers selected are different given that the least integer selected is 5,
(e) the sum of the three integers selected is more than 15. (A.E.B.)

10
Calculus I: Differentiation

10.1 The gradient of a curve

In chapter 3 we saw that the gradient of a straight line is the same at all points on the line. With a curve however the gradient, or steepness, will depend upon where we are on the curve.
The gradient at a point P on a curve is defined as the gradient of the *tangent* drawn to the curve at the point P, i.e. the gradient of the line that just touches the curve at the point P.
If we wished to find the gradient of a curve at some particular point, we could accurately draw the curve, draw the tangent to the curve at the required point and then measure the gradient of this tangent.

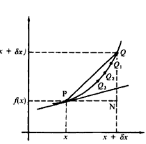


The graph shown is that of $y = x^2 - 1$ for $-2 \leqslant x \leqslant 2$ with the tangents drawn at $x = -1$, $x = 0$ and at $x = \frac{1}{2}$. From the graph we can see that the gradient of $y = x^2 - 1$ is

$-\frac{2}{1} = -2$ at $x = -1$ (the negative sign is because y *decreases* by 2 units as x *increases* by 1 unit)
0 at $x = 0$,
and $\frac{1}{1} = 1$ at $x = \frac{1}{2}$.

Clearly it is not easy to draw these tangents with any high degree of accuracy and so we need an alternative method for finding accurately the gradient at particular points on a curve.

Suppose that we wish to find the gradient of a curve $y = f(x)$ at a point $P(x, y)$ on the curve. Consider a second point, Q, lying on the curve, near to P and with x-coordinate given by $x + \delta x$ where δx is used to denote a small increment of length in the direction of the x-axis.



Thus the gradient of the chord PQ $= \dfrac{QN}{PN}$

$$= \frac{f(x + \delta x) - f(x)}{x + \delta x - x}$$

$$= \frac{f(x + \delta x) - f(x)}{\delta x}$$

Now if we move Q nearer to P, say to points $Q_1, Q_2, \ldots$ then the gradient of the chords $PQ, PQ_1, PQ_2, \ldots$ will give better and better approximations for the gradient of the tangent at P and therefore, the gradient of the curve at P. Thus the gradient of the curve $y = f(x)$ at some point $P(x, y)$ on the curve is given by

$$\lim_{\delta x \to 0} \left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

If we say that P has coordinates (x, y) and Q has coordinates $(x + \delta x, y + \delta y)$

then the gradient $= \displaystyle\lim_{\delta x \to 0} \left[\frac{y + \delta y - y}{x + \delta x - x}\right]$

$$= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right)$$

We write $\dfrac{dy}{dx}$ (pronounced 'dee y by dee x') for $\displaystyle\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right)$ and we call $\dfrac{dy}{dx}$ the gradient function, derived function or differential coefficient of y with respect to x. The gradient function gives a formula by which the gradient at any point on the line can be determined. The process of finding the differential coefficient of a function is called ***differentiation***. A shorthand notation that is sometimes used is that if $y = f(x)$ we write $\dfrac{dy}{dx}$ as $f'(x)$ or simply y'.

$$\text{Thus } \frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} \left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

Note Though we choose to use a fractional form of representation, $\dfrac{dy}{dx}$ is a limit and is not a fraction, i.e. $\dfrac{dy}{dx}$ does not mean $dy \div dx$. $\dfrac{dy}{dx}$ means y differentiated with respect to x. Thus, $\dfrac{dp}{dx}$ means p differentiated with respect to x. The '$\dfrac{d}{dx}$' is the 'operator', operating on some function of x.

Example 1

Find, from first principles, the differential coefficient of y with respect to x if $y = x^2$.

Here $f(x) = x^2$. Using $\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{(x + \delta x)^2 - x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{2x\delta x + (\delta x)^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}(2x + \delta x) = 2x$$

Thus if $y = x^2$, $\frac{dy}{dx} = 2x$.

Alternatively, this could be set out as follows:

$$y = x^2 \qquad \ldots [1]$$

Let x change by a small amount δx and the consequent change in y be δy, then

$$y + \delta y = (x + \delta x)^2 \qquad \ldots [2]$$

Subtracting equation [1] from equation [2]

$$y + \delta y - y = (x + \delta x)^2 - x^2$$

or $\quad \delta y = (x + \delta x)^2 - x^2$

$\therefore \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 - x^2}{\delta x}$

By definition $\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left[\frac{(x + \delta x)^2 - x^2}{\delta x}\right]$

$$= \lim_{\delta x \to 0}\left[\frac{2x\delta x + (\delta x)^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}[2x + \delta x]$$

$$= 2x$$

Example 2

If $f(x) = 4x + 2x^2$, find $f'(x)$ from first principles and hence calculate $f'(2)$ and $f'(-2)$.

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{4(x + \delta x) + 2(x + \delta x)^2 - 4x - 2x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{4\delta x + 4x\delta x + 2(\delta x)^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}[4 + 4x + 2\delta x]$$

$\therefore \quad f'(x) = 4 + 4x$

Since $\quad f'(x) = 4 + 4x$

then $\quad f'(2) = 4 + 4(2) = 12$

and $\quad f'(-2) = 4 + 4(-2) = -4$

Thus if $f(x) = 4x + 2x^2$, then $f'(x) = 4 + 4x$, $f'(2) = 12$ and $f'(-2) = -4$.

Exercise 10A

1. Draw the graph of $y = x^2 + x - 6$ for $-5 \leqslant x \leqslant 6$. Draw the tangents to this curve at $x = 3$, $x = 1$ and $x = -2$, and hence find a value for the gradient of the curve at each of these points.
2. Draw the graph of $y = \dfrac{x^2 - 4x}{4}$ for $0 \leqslant x \leqslant 6$. Draw the tangents to the curve at $x = 4$, $x = 3$ and $x = 2$ and hence find a value for the gradient of the curve at each of these points.
3. Differentiate each of the following from first principles to find $\dfrac{dy}{dx}$.

   (a) $y = 5x$ (b) $y = 9x + 5$ (c) $y = 3x^2$
   (d) $y = x^3$ (e) $y = x^2 + 3x$ (f) $y = 5x - x^2 + 7$
   (g) $y = \dfrac{1}{x}$ (h) $y = \dfrac{1}{x^2}$
4. If $f(x) = 3x - 2x^2$ find $f'(x)$ from first principles and hence evaluate $f'(4)$ and $f'(-1)$.
5. If $f(x) = 2x^2 + 5x - 3$ find $f'(x)$ from first principles and hence evaluate $f'(-1)$ and $f'(-2)$.
6. If $f(x) = x^3 - 2x$ find $f'(x)$ from first principles and hence evaluate $f'(1)$, $f'(0)$ and $f'(-1)$.

10.2 Differentiation of ax^n

In order to avoid differentiating functions from first principles, as we have done in 10.1, we can establish certain rules.

Suppose we wish to find $\dfrac{dy}{dx}$ given that $y = x^n$.

In this case $f(x) = x^n$, and $\dfrac{dy}{dx} = \lim_{\delta x \to 0}\left[\dfrac{f(x + \delta x) - f(x)}{\delta x}\right]$

$$= \lim_{\delta x \to 0}\left[\frac{(x + \delta x)^n - x^n}{\delta x}\right] \qquad \ldots [1]$$

Now for positive integer values of n, $(x + \delta x)^n$ can be expanded by the binomial theorem to give a finite series:

$$(x + \delta x)^n = x^n + {}^nC_1x^{n-1}\delta x + {}^nC_2x^{n-2}(\delta x)^2 + {}^nC_3x^{n-3}(\delta x)^3 + \ldots + (\delta x)^n$$

substituting this in [1], gives

$$\frac{dy}{dx} = \lim_{\delta x \to 0}[nx^{n-1} + {}^nC_2x^{n-2}\delta x + {}^nC_3x^{n-3}(\delta x)^2 + \ldots + (\delta x)^{n-1}]$$

i.e. $\dfrac{dy}{dx} = nx^{n-1}$

Thus the differential coefficient of x^n is nx^{n-1} and although the above proof is for positive integer values of n, the result applies for all rational values of n. This result, together with the three rules stated below, enable us to differentiate many functions.

(i) If $y = f(x) + g(x) - h(x)$ then

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) - \frac{d}{dx}(h(x))$$
$$= f'(x) + g'(x) - h'(x)$$

(ii) If $y = af(x)$ where a is a constant then

$$\frac{dy}{dx} = a\frac{d}{dx}(f(x))$$
$$= af'(x)$$

(iii) If $y = a$ where a is a constant, we can write this as

$$y = ax^0$$

then $\frac{dy}{dx} = (0)ax^{-1} = 0$

Thus the differential coefficient of a constant is zero.

[Rules for differentiating $f(x) \times g(x)$ and $\frac{f(x)}{g(x)}$ will be obtained later.]

Note: The rule for differentiating $y = ax^n$, i.e. $\frac{dy}{dx} = anx^{n-1}$, can be remembered in words as:
'multiply by the power and decrease the power by one'.

Example 3

Differentiate the following with respect to x.

(a) $y = x^8$ (b) $y = 6$ (c) $y = 3x^5$ (d) $y = 8\sqrt{x}$
(e) $y = 3x^2 - 6x + \frac{2}{x^2}$ (f) $y = (2x + 3)(x + 1)$

(a) If $y = x^8$

$$\frac{dy}{dx} = 8x^7$$

(Multiply by the power and decrease the power by one.)

(b) If $y = 6$

$$\frac{dy}{dx} = 0$$

(Differentiation of a constant term gives zero.)

(c) If $y = 3x^5$

$$\frac{dy}{dx} = 15x^4$$

(Multiply by the power and decrease the power by one.)

(d) If $y = 8\sqrt{x}$

$$= 8x^{1/2}$$
$$\frac{dy}{dx} = \tfrac{1}{2} \times 8x^{-1/2}$$
$$= \frac{4}{\sqrt{x}}$$

(e) If $y = 3x^2 - 6x + \frac{2}{x^2}$

$$= 3x^2 - 6x + 2x^{-2}$$
$$\frac{dy}{dx} = 6x - 6 - 4x^{-3}$$
$$= 6x - 6 - \frac{4}{x^3}$$

(f) If $y = (2x + 3)(x + 1)$

$$= 2x^2 + 5x + 3$$
$$\frac{dy}{dx} = 4x + 5$$

Example 4

Find the gradient of the curve $y = x^2 + 7x - 2$ at the point (2, 16).

$$y = x^2 + 7x - 2$$
$$\frac{dy}{dx} = 2x + 7$$

Thus at (2, 16), $\frac{dy}{dx} = 2(2) + 7 = 11$

At the point (2, 16) the curve $y = x^2 + 7x - 2$ has a gradient of 11.

Example 5

Find the points on the curve $y = x^3 + 3x^2 - 6x - 10$ where the gradient is 3.

$$y = x^3 + 3x^2 - 6x - 10$$

$$\frac{dy}{dx} = 3x^2 + 6x - 6$$

If the gradient is 3, then $\frac{dy}{dx} = 3$

i.e. $3x^2 + 6x - 6 = 3$

$x^2 + 2x - 3 = 0$

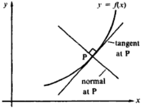
giving $x = -3$ or 1. If $x = -3$ $y = -27 + 27 + 18 - 10 = 8$

and if $x = 1$ $y = 1 + 3 - 6 - 10 = -12$

The curve $y = x^3 + 3x^2 - 6x - 10$ has a gradient of 3 at $(-3, 8)$ and at $(1, -12)$.

Tangents and normals

Suppose that some point P lies on a curve $y = f(x)$. The line passing through P, perpendicular to the tangent to the curve at P, is said to be the *normal* to the curve at P. To find the equation of the tangent or normal to a curve at some point $P(x_1, y_1)$ on the curve we use $y - y_1 = m(x - x_1)$ [see page 87] with the gradient m determined by differentiation.



Example 6

Find the equation of the tangent and normal to the curve $y = x^2 - 4x + 1$ at the point $(-2, 13)$.

$$y = x^2 - 4x + 1$$

$$\therefore \quad \frac{dy}{dx} = 2x - 4$$

at $(-2, 13)$, $\frac{dy}{dx} = 2(-2) - 4 = -8$

Thus the tangent at $(-2, 13)$ has gradient -8 and the normal has gradient $\frac{1}{8}$.

Using $y - y_1 = m(x - x_1)$,

equation of tangent is: $y - 13 = -8(x - -2)$

i.e. $y = -8x - 3$

equation of normal is: $y - 13 = \frac{1}{8}(x - -2)$

i.e. $8y = x + 106$

Higher derivatives

We can repeat the differentiation process to find the differential coefficient of the differential coefficient of y with respect to x. This is called the second differential of y with respect to x and is written $\frac{d^2y}{dx^2}$ or, if $y = f(x)$ it is written as $f''(x)$.

Thus if $y = 3x^3 - 6x + 4$

$$\frac{dy}{dx} = 9x^2 - 6$$

and $$\frac{d^2y}{dx^2} = 18x.$$

Exercise 10B

Differentiate the following functions with respect to x.

1. x^5 **2.** x^3 **3.** $12x^2$ **4.** $5x^4$ **5.** $16x$
6. $3x^2$ **7.** $9x^3$ **8.** $3x^4$ **9.** 7 **10.** $2x$
11. $5x^7$ **12.** 6 **13.** $x^{5/3}$ **14.** $x^{3/4}$ **15.** $x^{2/5}$
16. $x^{3/5}$ **17.** $6x^{2/3}$ **18.** $8x^{1/4}$ **19.** $\sqrt{x}$ **20.** $\sqrt{x^3}$
21. $4\sqrt{x}$ **22.** $\frac{5}{x}$ **23.** $\frac{3}{x^2}$ **24.** $\frac{8}{\sqrt{x}}$ **25.** $\frac{2}{x^3}$

Find the gradient function $\frac{dy}{dx}$ for each of the following:

26. $y = x^2 + 7x - 4$
27. $y = x - 7x^2$
28. $y = 6x^2 - 7x + 8$
29. $y = 3 + x$
30. $y = x^3 + 7x^2 - 2$
31. $y = 3x^6 - 2x^2 + 6x - 8$
32. $y = 3x^2 + 7x - 4 + \frac{1}{x}$
33. $y = 3x - \frac{5}{x} + \frac{6}{x^2}$
34. $y = 5x - \frac{3}{\sqrt{x}}$
35. $y = (x + 3)(x + 1)$
36. $y = (x + 4)(x - 2)$
37. $y = (2x + 3)(x + 2)$

Find the gradients of the following lines at the points indicated:

38. $y = x^2 + 4x$ at $(0, 0)$
39. $y = 5x - x^2$ at $(1, 4)$
40. $y = 3x^3 - 2x$ at $(2, 20)$
41. $y = 5x + x^3$ at $(-1, -6)$
42. $y = (x + 1)(2x + 3)$ at $(2, 21)$
43. $y = 2x^3 - x^2 - 6$ at $(2, 6)$
44. $y = 3x + \frac{1}{x}$ at $(1, 4)$
45. $y = 2x^2 - x + \frac{4}{x}$ at $(2, 8)$

Find the coordinates of any points on the following lines where the gradient is as stated.

46. $y = x^2$, gradient 8
47. $y = x^2$, gradient -8
48. $y = x^2 - 4x + 5$, gradient 2
49. $y = 5x - x^2$, gradient 3
50. $y = x^4 + 2$, gradient -4
51. $y = x^3 + 3x^2 - 5x - 10$, gradient 4
52. $y = x^3 + x^2 - x + 1$, gradient zero.
53. $y = \frac{12}{x}$, gradient -3
54. If $f(x) = x^3 + 4x$ find
(a) $f(1)$, (b) $f'(x)$, (c) $f'(1)$,
(d) $f''(x)$, (e) $f''(1)$.
55. If $f(x) = 3x^2 + \frac{24}{x}$ find
(a) $f(2)$, (b) $f'(x)$, (c) $f'(2)$,
(d) $f''(x)$, (e) $f''(2)$.
56. If $f(x) = 3x^2 - 4x$ find the value of a given that $f'(a) = 5$
57. If $f(x) = x^3 + 1$ find the value of a given that $f''(a) = 24$
58. Find the second differential of y with respect to x for each of the following:
(a) $y = 6x^2 + 7$
(b) $y = 5x^3 + 6x - 5$
(c) $y = 2 + \frac{3}{x}$.
59. If $y = 3x^2 - x$ show that

$$y\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y + 1 = 6x.$$

60. Find the gradient of the curve $y = x^2 + 6x - 4$ at the point where the curve cuts the y-axis.
61. Find the coordinates of the points where the curve $y = x^2 - x - 12$ cuts the x-axis and determine the gradient of $y = x^2 - x - 12$ at these points.

62. The line $y = 4x - 5$ cuts the curve $y = x^2 - 2x$ at two points. Find the gradient of $y = x^2 - 2x$ at these two points.

63. The gradient of $y = x^2 - 4x + 6$ at $(3, 3)$ is the same as the gradient of $y = 8x - 3x^2$ at (a, b). Find the values of a and b.

64. Find the coordinates of the point on the curve $y = x^2 - 6x + 3$ where the tangent is parallel to the line $y = 2x + 3$.

65. Find the coordinates of the point on the curve $y = x - x^2$ where the tangent is parallel to the line $2y + x - 3 = 0$.

66. Find the coordinates of the point on the curve $y = 3x^2 - 9x + 10$ where the normal is parallel to the line $3y - x + 4 = 0$.

67. Find the equations of the tangent and the normal to the following curves at the points indicated:
(a) $y = x^2$ at $(3, 9)$
(b) $y = 5 - 2x^2$ at $(-1, 3)$
(c) $y = x^2 - 5x - 4$ at the point on the curve with an x-coordinate of 5,
(d) $y = 4 + x - 2x^2$ at the point on the curve with an x-coordinate of 1.

68. T is the tangent to the curve $y = x^2 + 6x - 4$ at $(1, 3)$ and N is the normal to the curve $y = x^2 - 6x + 18$ at $(4, 10)$. Find the coordinates of the point of intersection of T and N.

69. The tangent to the curve $y = ax^2 + bx + 2$ at $(1, \frac{1}{2})$ is parallel to the normal to the curve $y = x^2 + 6x + 10$ at $(-2, 2)$. Find the values of a and b.

10.3 Maximum, minimum and points of inflexion

Figure 1 shows the sketch of a function of x, say $y = f(x)$.
At the points A, B and C the tangent to the curve is parallel to the x-axis and therefore the gradient of the curve at A, B and C is zero,

i.e. $\frac{dy}{dx} = 0$ at A, B and C.

The point A is said to be a maximum point on the curve since the value of y at A is clearly greater than the values of y at points on the curve close to A.

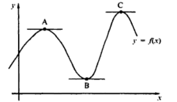


Fig. 1

It is important to realise that at a maximum point such as A, the value of y may not be the greatest value of y on the entire curve. The important fact is that the value of y at A is greater than at points close to A. Thus A is really a *local* maximum but we usually refer to such points simply as maximum points.

In a similar way C is another (local) maximum point on the curve.

The value of y at B is less than the values of y at points on the curve close to B and so B is said to be a (local) minimum point on the curve.

Such maximum and minimum points are said to be *turning* points on the curve. To locate such points without having to draw the graph we have only to find the points at which $\frac{dy}{dx} = 0$. However there may be some point on a curve at which $\frac{dy}{dx} = 0$, but the point is neither a maximum nor a minimum.

In Figure 2, D is such a point because the tangent is parallel to the x-axis but D is neither a maximum point nor a minimum point. D is called a point of *inflexion*.

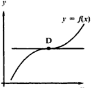
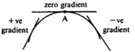


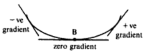
Fig. 2

A point on a curve at which $\frac{dy}{dx} = 0$, i.e. maximum points like A or C in Figure 1, minimum points like B in Figure 1 or points of inflexion such as D in Figure 2 are called *stationary* points.

Having located these stationary points, we can distinguish between them i.e. determine whether they are maximum points, minimum points or points of inflexion, by considering the sign of the gradient at points close to, and on either side of these points.

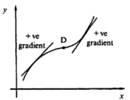
For increasing values of x, as we approach and pass through a maximum point such as A, the gradient of the curve changes from positive ('uphill' for increasing x) to zero, at A, to negative ('downhill' for increasing x).

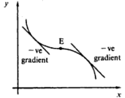




On the other hand, as we approach and pass through a minimum point such as B, the gradient changes from negative through zero to positive.

For points of inflexion, such as D and E, the gradient of the curve does not change sign as we pass through the point.





Thus, to find the stationary points on a curve $y = f(x)$ and to distinguish between them, we proceed as follows:

1. Find the gradient function $\frac{dy}{dx}$ of the curve.
2. Equate to zero the expression for $\frac{dy}{dx}$.
3. Find the values of x (i.e. $x_1, x_2, x_3 \ldots$) which satisfy this equation.
4. Consider the sign of $\frac{dy}{dx}$ on either side of these points.
5. Find the values $y_1, y_2, y_3 \ldots$ which correspond to $x_1, x_2, x_3 \ldots$

Example 7

Find the coordinates of any stationary points on the curve $y = x^3 - 2x^2 - 4x$ and distinguish between these points.

$$y = x^3 - 2x^2 - 4x$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 4x - 4$$

At stationary points, $\frac{dy}{dx} = 0$ i.e. $3x^2 - 4x - 4 = 0$

$$(3x + 2)(x - 2) = 0$$

$$\text{thus} \quad x = -\tfrac{2}{3} \text{ or } 2$$

We must now consider the sign of $\frac{dy}{dx}$ as x increases through these points,

$$\text{now,} \quad \frac{dy}{dx} = (3x + 2)(x - 2)$$

$\therefore$ If x is 'just' less than $-\frac{2}{3}$ (say $-0{\cdot}8$) $\quad \frac{dy}{dx} = (-\text{ve})(-\text{ve}) = +\text{ve}$

and if x is 'just' more than $-\frac{2}{3}$ (say $-0{\cdot}5$) $\quad \frac{dy}{dx} = (+\text{ve})(-\text{ve}) = -\text{ve}$

Thus the gradient changes from $+$ve (/) to $-$ve (\) as we pass through $x = -\frac{2}{3}$, i.e. a maximum point.

If x is 'just' less than 2 (say $1{\cdot}5$) $\quad \frac{dy}{dx} = (+\text{ve})(-\text{ve}) = -\text{ve}$

and if x is 'just' more than 2 (say $2{\cdot}5$) $\quad \frac{dy}{dx} = (+\text{ve})(+\text{ve}) = +\text{ve}$

Thus the gradient changes from $-$ve (\) to $+$ve (/) as we pass through $x = 2$, i.e. a minimum point.

When $x = -\frac{2}{3}$, $\quad y = (-\frac{2}{3})^3 - 2(-\frac{2}{3})^2 - 4(-\frac{2}{3})$

$$= 1\tfrac{13}{27}$$

when $x = 2$, $\quad y = (2)^3 - 2(2)^2 - 4(2)$

$$= -8$$

Thus a maximum point occurs at $(-\frac{2}{3}, 1\frac{13}{27})$ and a minimum point occurs at $(2, -8)$.

Notice that we did not need to evaluate $\frac{dy}{dx}$ at points on either side of the stationary points, we had only to determine the *sign* of the gradient function. For this reason the factorised form of $\frac{dy}{dx}$ was useful.

Example 8

Find the coordinates of any stationary points on the curve $y = x^4 + 2x^3$ and distinguish between them. Hence sketch the curve.

$$y = x^4 + 2x^3$$

$$\frac{dy}{dx} = 4x^3 + 6x^2 = 2x^2(2x + 3)$$

At stationary points $\quad \frac{dy}{dx} = 0 \quad$ i.e. $\quad x = 0 \quad$ or $\quad -\frac{3}{2}$

Using $\frac{dy}{dx} = 2x^2(2x + 3)$

For x just less than 0 (say $-0{\cdot}1$) $\frac{dy}{dx} = (+\text{ve})(+\text{ve}) = +\text{ve}$ (/)

For x just more than 0 (say $0{\cdot}1$) $\frac{dy}{dx} = (+\text{ve})(+\text{ve}) = +\text{ve}$ (/)

Thus at $x = 0$ there is a point of inflexion.

For x just less than $-\frac{3}{2}$ (say $-1{\cdot}6$) $\frac{dy}{dx} = (+\text{ve})(-\text{ve}) = -\text{ve}$ (\)

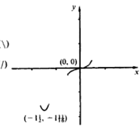
For x just more than $-\frac{3}{2}$ (say -1) $\frac{dy}{dx} = (+\text{ve})(+\text{ve}) = +\text{ve}$ (/)

Thus at $x = -\frac{3}{2}$ there is a minimum point.

If $x = 0$, $y = 0$. If $x = -\frac{3}{2}$, $y = -1\frac{11}{16}$

Thus a minimum point occurs at $(-1\frac{1}{2}, -1\frac{11}{16})$ and a point of inflexion occurs at (0, 0).

These facts can be used to start the sketch as shown.



When sketching, it is also useful to know where the curve cuts the axes.

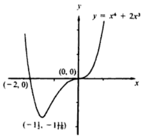
Clearly when $x = 0$, $y = 0$

but also, when $y = 0$, $x^4 + 2x^3 = 0$

i.e. $x^3(x + 2) = 0$

giving $x = 0$ or -2.

Thus we can complete the sketch as shown on the right.



Note

When sketching the graph of $y = f(x)$, it is useful to examine the behaviour of $f(x)$ as $x \to \pm\infty$.

In the above example $y = x^4 + 2x^3$ and so as $x \to \pm\infty$ the x^4 term will dominate.

Thus as $x \to +\infty$, $y \to +\infty$

and as $x \to -\infty$, $y \to +\infty$ which agrees with our sketch.

Curve sketching is considered in greater detail in the next chapter.

Exercise 10C

For questions **1** to **8**, find the coordinates of any stationary points on the given curves and distinguish between them.

1. $y = 2x^2 - 8x$	**2.** $y = 18x - 20 - 3x^2$
3. $y = x^3 - x^2 - x + 7$	**4.** $y = x^3 + 3x^2 - 9x - 5$
5. $y = 1 - 3x + x^3$	**6.** $y = x^3 - 3x^2 + 3x - 1$
7. $y = (x - 1)(x^2 - 6x + 2)$	**8.** $y = x^3 + 6x^2 + 12x + 12$

For questions **9** to **14**, sketch the curves of the given equations, clearly indicating on your sketch the coordinates of any stationary points and of any points where the lines cut the axes.

9. $y = (1 - x)(x - 5)$	**10.** $y = x^2 + 2x - 3$
11. $y = x^2 - 8x - 20$	**12.** $y = x^3 - 4x^2 + 4x$
13. $y = x(x - 3)(x + 5)$	**14.** $y = 3x^4 - 4x^3$

Use of the second differential

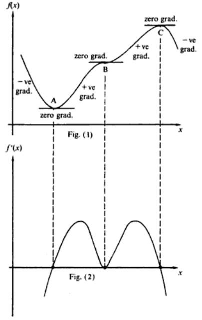


Fig. (1)
Fig. (2)

Figure 1 shows the graph of some function $y = f(x)$. A is a minimum point, B is a point of inflexion and C is a maximum point. Figure 2 shows the corresponding graph of $f'(x)$ plotted against x.

The gradient of the curve shown in Figure 2 will be given by $\frac{d}{dx}[f'(x)]$, i.e. $f''(x)$ or $\frac{d^2y}{dx^2}$.

Notice that the minimum point A, in Figure 1, corresponds to a point at which there is a +ve gradient in Figure 2,

i.e. $\frac{d^2y}{dx^2} > 0$ for a minimum point.

The maximum point C, in Figure 1, corresponds to a point at which there is a −ve gradient in Figure 2,

i.e. $\frac{d^2y}{dx^2} < 0$ for a maximum point.

The point of inflexion B, in Figure 1, corresponds to a point where the gradient is zero in Figure 2,

i.e. $\frac{d^2y}{dx^2} = 0$ for a point of inflexion.

In fact it can also be true that $\frac{d^2y}{dx^2} = 0$ at maximum and minimum points and so, if we wish to use the second differential as an aid to determining the nature of stationary points, we can say: For any point (x_1, y_1) on the curve $y = f(x)$ for which $f'(x_1) = 0$ then,

(a) if $f''(x_1) > 0$ then the point is a minimum point,
(b) if $f''(x_1) < 0$ then the point is a maximum point,
(c) if $f''(x_1) = 0$ then the point is either a maximum point, a minimum point or a point of inflexion and, to determine which it is, we consider the sign of $f'(x)$ on either side of the point.

Example 9

Find the coordinates of any stationary points on the curve $y = 5x^6 - 12x^5$ and distinguish between them. Hence sketch the curve.

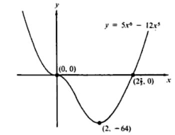
$$y = 5x^6 - 12x^5 = x^5(5x - 12)$$
$$y' = 30x^5 - 60x^4 = 30x^4(x - 2)$$
$$y'' = 150x^4 - 240x^3 = 30x^3(5x - 8)$$

At stationary points $y' = 0$ i.e. $30x^4(x - 2) = 0$ giving $x = 0$ or $x = 2$

when $x = 2$, $y'' = 30(2)^3(10 - 8)$ i.e. positive ⇒ minimum,
when $x = 0$, $y'' = 0$ ⇒ further investigation of y' needed either side of $x = 0$,

when x is just less than zero, say $-0{\cdot}1$, y' is negative (\\)
when x is just more than zero, say $+0{\cdot}1$, y' is negative (\\) } ⇒ point of inflexion

when $x = 0$, $y = 0$ and when $x = 2$, $y = -64$.



Thus (0, 0) is a point of inflexion
and (2, −64) is a minimum point.

From $y = x^5(5x - 12)$ we see that the axes are cut at (0, 0) and $(2\frac{2}{5}, 0)$ and a sketch of the curve can be made as shown.

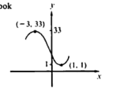
Note also that as $x \to +\infty, y \to +\infty$
and as $x \to -\infty, y \to +\infty$

Example 10

Find (a) the coordinates and nature of any turning points on the curve $y = x^3 + 3x^2 - 9x + 6$
(b) the maximum value of $x^3 + 3x^2 - 9x + 6$ in the range
(i) $-4 \leqslant x \leqslant 2$ (ii) $-4 \leqslant x \leqslant 4$

(a) $y = x^3 + 3x^2 - 9x + 6$
$y' = 3x^2 + 6x - 9 \quad = 3(x - 1)(x + 3)$
$y'' = 6x + 6$
At stationary points $y' = 0$ i.e. $(x - 1)(x + 3) = 0$
giving $x = 1$ or $x = -3$
when $x = 1$ $y'' = 12$ i.e. positive ⇒ minimum
when $x = -3$ $y'' = -12$ i.e. negative ⇒ maximum
when $x = 1$ $y = 1$ and when $x = -3$ $y = 33$
Thus (1, 1) is a minimum point and (−3, 33) is a maximum point.

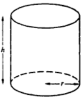
(b) From (a) we know that the graph of $y = x^3 + 3x^2 - 9x + 6$ must look something like that shown on the right.



Thus, remembering that (−3, 33) is a *local* maximum the expression $x^3 + 3x^2 - 9x + 6$ will exceed the value 33 for sufficiently large x.
(i) for $-4 \leqslant x \leqslant 2$:
if $x = 2$, $x^3 + 3x^2 - 9x + 6 = 8 + 12 - 18 + 6$
$= 8$
∴ in the range $-4 \leqslant x \leqslant 2$,
$x^3 + 3x^2 - 9x + 6$ has a maximum value of 33
(ii) for $-4 \leqslant x \leqslant 4$:
if $x = 4$, $x^3 + 3x^2 - 9x + 6 = 64 + 48 - 36 + 6$
$= 82$
∴ in the range $-4 \leqslant x \leqslant 4$,
$x^3 + 3x^2 - 9x + 6$ has a maximum value of 82.

The above examples show how the stationary points of a curve may be located when the equation of the curve is given. However, in some instances the equation may not be given. This is likely to be the case when the problem relates to a real life or practical situation and then the first task is to express the given information as an equation relating two variables. The stationary points can then be determined and their practical significance interpreted. The following example illustrates such a case.

Example 11



A company that manufactures dog food wishes to pack the food in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 128π cm³ and the minimum possible surface area?

Suppose that each tin has base radius r and perpendicular height h. We require the dimensions for the surface area S to be a minimum and so we need an expression for S.

$$S = 2\pi r^2 + 2\pi rh$$

We cannot differentiate this expression, in this form, as S is given as a function of *two* variables r and h.
However the volume $V = \pi r^2 h$ and this volume is to be 128π cm³.

$\therefore \quad 128\pi = \pi r^2 h \quad$ or $\quad h = \dfrac{128}{r^2}$

$\therefore \quad S = 2\pi r^2 + \dfrac{256\pi}{r}$

and $\quad \dfrac{dS}{dr} = 4\pi r - \dfrac{256\pi}{r^2}$

When $\quad \dfrac{dS}{dr} = 0$, then $\quad 4\pi r - \dfrac{256\pi}{r^2} = 0$

giving $\quad r = 4$ cm

and $\quad h = \dfrac{128}{4^2} = 8$ cm

$\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{512\pi}{r^3}$, which is positive for $r = 4 \Rightarrow$ minimum.

Thus the surface area S has a minimum value when $r = 4$ cm and $h = 8$ cm. Each tin should have a base radius of 4 cm and perpendicular height 8 cm.

Exercise 10D

For questions 1 to 8 find the coordinates of any stationary points on the given curves and distinguish between them.

1. $y = x^2 + 10x + 10$ **2.** $y = \frac{1}{4} + 9x - 3x^2$
3. $y = x^3 - 3x^2 - 24x$ **4.** $y = 3x^2 + 45x - 75 - x^3$
5. $y = 2x^3 - 24x$ **6.** $y = x^4 + 3$
7. $y = x^6 - 6x^4$ **8.** $y = 15x^3 - x^5$

For questions 9 to 14, sketch the curves of the given equations clearly indicating on your sketch the coordinates of any stationary points and of any points where the lines cut the axes.

9. $y = 3x - x^2$ **10.** $y = x^2 - 4$
11. $y = (x - 2)^2$ **12.** $y = x^4 - 4x^3$
13. $y = (x - 2)^2(x - 1)$ **14.** $y = x^5 - 5x^4$

15. Find (a) the coordinates and nature of any turning points on the curve $y = x^3 - \frac{3}{2}x^2 - 6x + 12$,
(b) the maximum value of $x^3 - \frac{3}{2}x^2 - 6x + 12$ in the range $-3 \leqslant x \leqslant 3$,
(c) the minimum value of $x^3 - \frac{3}{2}x^2 - 6x + 12$ in the range $-3 \leqslant x \leqslant 3$.

16. Find (a) the coordinates and nature of any turning points on the curve $y = 36x - 3x^2 - 2x^3$,
(b) the minimum and maximum values of $36x - 3x^2 - 2x^3$ in the range $-5 \leqslant x \leqslant 5$.

17. If $S = 4r^2 - 10r + 7$, find the minimum value of S and the value of r which gives this minimum value.

18. If $V = 30t - 6t^2$, find the maximum value of V and the value of t for which it occurs.

19. If $A = xy$ and $2x + 5y = 100$, find the maximum value of A and the values of x and y which give this maximum value.

20. If $V = 4rx + 2r^2$ and $3r + x = 5$, find the maximum value of V and the values of r and x that give this maximum value.

21. A rectangular enclosure is formed by using 1200 m of fencing. Find the greatest possible area that can be enclosed in this way and the corresponding dimensions of the rectangle.

22. An open metal tank with a square base is made from 12 m^2 of sheet metal. Find the length of the side of the base for the volume of the tank to be a maximum and find this maximum volume.

23. A piece of wire of length 60 cm is cut into two pieces. Each piece is then bent to form the perimeter of a rectangle which is twice as long as it is wide. Find the lengths of the two pieces of wire if the sum of the areas of the rectangles is to be a minimum.

24. A cylindrical can is made so that the sum of its height and the circumference of its base is 45π cm. Find the radius of the base of the cylinder if the volume of the can is a maximum.

25. A ship is to make a voyage of 200 km at a constant speed. When the speed of the ship is v km/h the cost is £$(v^2 + 4000/v)$ per hour. Find the speed at which the ship should travel so that the cost of the voyage is a minimum.

10.4 Small changes

In section 10.1 we defined $\frac{dy}{dx}$ as $\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right)$. Thus, if δx is small then

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \quad \text{or} \quad \delta y \approx \frac{dy}{dx}\delta x.$$

Thus if y is given as a function of x we can determine the change in y corresponding to some given small change in x.

Example 12

If $y = 2x^2 - 3x$ find the approximate change in y when x increases from 6 to 6·02.

$$y = 2x^2 - 3x$$
$$\frac{dy}{dx} = 4x - 3$$
using $\delta y \approx \frac{dy}{dx}\delta x$
$$\delta y \approx (4x - 3)\delta x \quad \text{and if } x = 6 \quad \text{and} \quad \delta x = 0{\cdot}02$$
$$\delta y \approx (24 - 3)(0{\cdot}02) = 0{\cdot}42$$

When x changes from 6 to 6·02 the value of y changes by approximately 0·42.

Example 13

In calculating the area of a circle it is known that an error of $\pm 3\%$ could have been made in the measurement of the radius. Find the possible percentage error in the area.

We are told that $\frac{\delta r}{r} = \pm\frac{3}{100}$ and we require $\frac{\delta A}{A}$.

For small δr $\quad \frac{\delta A}{\delta r} \approx \frac{dA}{dr}$

but $\quad A = \pi r^2 \quad$ i.e. $\quad \frac{dA}{dr} = 2\pi r$

$\therefore \quad \frac{\delta A}{\delta r} \approx 2\pi r$

$\quad \delta A \approx 2\pi r\delta r$

$\therefore \quad \frac{\delta A}{A} \approx \frac{2\pi r\delta r}{\pi r^2} = \frac{2\delta r}{r}$

$\quad = \pm\frac{6}{100}$

Thus the possible percentage error in the area will be $\pm 6\%$.

Example 14

Find an approximate value for $\sqrt{(16{\cdot}08)}$

For this we use $y = \sqrt{x}$ with $(x, y) = (16, 4)$ and $(x + \delta x, y + \delta y) = (16 + 0{\cdot}08, 4 + \delta y)$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

For small δx, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ i.e. $\delta y \approx \frac{dy}{dx}\delta x$

$\therefore \quad \delta y \approx \frac{1}{2\sqrt{x}}\delta x$

$\quad = \frac{1}{2\sqrt{16}}(0{\cdot}08)$

$\quad = 0{\cdot}01$

$\therefore \quad y + \delta y \approx 4 + 0{\cdot}01$ and so $\sqrt{(16{\cdot}08)} \approx 4{\cdot}01$

Exercise 10E

1. If $y = x^2 + 2x$ find the approximate increase in y when x increases from 9 to 9·01.

2. If $y = x + \frac{1}{x}$ find the approximate increase in y when x increases from 2 to 2·04.

3. It is noticed that when x increases slightly from an initial value of 7, the value of y increases by 0·7. Find the approximate increase in x that caused this increase in y given that x and y are related according to the rule $y = (2x + 3)(x + 2)$.

4. If $y = 5x^4$ and x is increased by 2% of its original value, find the corresponding percentage increase in y.

5. Find an approximate value for the increase in the volume of a sphere when the radius of the sphere increases from 10 cm to 10·1 cm. (Leave π in your answer).

6. Find the percentage increase in the volume of a cube when all the edges of the cube are increased in length by 2%.

7. Find an approximate value for the increase in the radius of a sphere given that this increase causes the surface area to increase from 100π cm² to $100{\cdot}4\pi$ cm².

8. A rectangle is known to be twice as long as it is wide. If the width is measured as 20 cm $\pm$ 0·2 cm find the area in the form $A \pm b$.

9. The time period, T, of a pendulum of length l is given by $T = 2\pi\sqrt{\frac{l}{g}}$ where π and g are constants. Find the approximate percentage increase in T when the length of the pendulum increases by 4%.

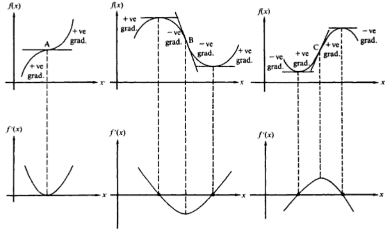
10. A solid circular cylinder has base radius 5 cm and height 12·5 cm. Find the approximate increase in the surface area of the cylinder when the radius of the base increases to 5·04 cm. (Leave π in your answer).

11. Find an approximate value for $(5{\cdot}02)^3$.

12. Find an approximate value for $\sqrt[3]{(64{\cdot}96)}$.

10.5 More about points of inflexion

We saw on page 266 that when examining points on the curve $y = f(x)$ for which $f'(x) = 0$, if $f''(x) < 0$ we have a maximum point, if $f''(x) > 0$ we have a minimum point and if $f''(x) = 0$ we *could* have a point of inflexion. Indeed at all points of inflexion $f''(x) = 0$ but it is possible to have a point of inflexion (a, b) i.e. $f''(a) = 0$, for which $f'(a) \neq 0$. A point of inflexion is said to occur at any point P on a curve at which the tangent to the curve at P crosses the curve at P. Thus in the diagrams shown below A, B and C are all points of inflexion, but only A is a stationary point on the curve as the tangent at A is parallel to the x-axis, (i.e. $f'(x) = 0$ at A). The second set of diagrams show that the gradient of $f'(x)$, i.e. $f''(x)$, is zero at A, B and C.



Thus if $y = f(x)$ stationary points occur where $f'(x) = 0$. These stationary points may be turning points (maximum or minimum) or points of inflexion. Points of inflexion that are not turning points may also occur if $f'(x) \neq 0$ but $f''(x) = 0$.

Example 15

Find the coordinates of any points of inflexion on the curve $y = x^3(x - 1)$ and determine whether they are horizontal points of inflexion.

$$y = x^3(x - 1)$$
$$= x^4 - x^3$$
$$y' = 4x^3 - 3x^2 = x^2(4x - 3)$$
$$y'' = 12x^2 - 6x = 6x(2x - 1)$$

At points of inflexion $y'' = 0$ i.e. $6x(2x - 1) = 0$

giving $x = 0$ or $x = \frac{1}{2}$

9. (i) A rectangular block has a base x centimetres square. Its total surface area is 150 cm². Prove that the volume of the block is $\frac{1}{2}(75x - x^3)$ cm³.
Calculate (a) the dimensions of the block when its volume is maximum; (b) this maximum volume. Show that your answer is a maximum rather than a minimum.
(ii) Write down the gradient of the function $4x^2 + \dfrac{27}{x}$. Hence find the value of x for which the function has a maximum or minimum value, and state which it is, giving reasons. (Oxford)

10. A solid right circular cylinder of height h and radius r has a *total* external surface area of 600 cm².
Show that $h = \dfrac{300}{\pi r} - r$ and hence express the volume, V, in terms of r.
If h and r can vary, find $\dfrac{dV}{dr}$ and $\dfrac{d^2V}{dr^2}$ in terms of r. Show that V has a maximum and find the corresponding value of r in terms of π.
Calculate the ratio $h:r$ in this maximum case. (A.E.B.)

11. A cylinder of volume V is to be cut from a solid sphere of radius R.
Prove that the maximum value of V is $\dfrac{4\pi R^3}{3\sqrt{3}}$. (A.E.B.)

12. The equation of a curve is $y = 2x^3 - 7x^2 + 15$. Write down an expression for $\dfrac{dy}{dx}$ and hence find
(i) the equation of the tangent to the curve at (2, 3),
(ii) the approximate change in y as x increases from 2 to 2·03, stating whether this is an increase or a decrease. (Cambridge)

13. Given that $y = x^{-1/3}$ use the calculus to determine an approximate value for $\dfrac{1}{\sqrt[3]{0{\cdot}9}}$. (Cambridge)

14. (i)

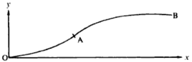


Fig. 1

Fig. 1 represents part of a switch-back ride at a fair. The section O to A is part of the curve whose equation is $y = x^2$ and the section from A to B is part of the curve whose equation is $3y = -20 + 20x - 2x^2$. Find the coordinates of A. Show that the gradients for each section are equal at A, and find this gradient.
(ii) A solid metal cylinder has a height of 10 cm. It is machined until its radius is reduced by 0·5%, its height remaining at 10 cm. Use the method of small increments to find the percentage decrease in volume. No credit will be given for any other method. (Oxford)

11
Sketching functions I

11.1 General methods

For the function $y = f(x)$, we can plot values of x against the corresponding values of y and obtain an accurate graph of the function. A less accurate representation, which we call a sketch, is adequate for many purposes provided that the sketch still shows the salient and noteworthy features of the function.

In earlier chapters we have already encountered the idea of sketching straight lines, quadratic curves, cubic curves and various other functions. Thus if asked to sketch the graph of an equation of the type $y = mx + c$, we know it will be a straight line and need only find two points on the line. If asked to sketch the graph of a function of the type $f(x) = ax^2 + bx + c$, we expect a quadratic curve which will be either a 'hill' (if $a < 0$) or a 'valley' (if $a > 0$). By finding (a) where the curve cuts the axes and (b) the coordinates and nature of any turning points, a good sketch can be made.

If asked to sketch the graph of a function of the type $f(x) = ax^3 + bx^2 + cx + d$, we expect a cubic curve which will have both a 'hill' and a 'valley': (if $a > 0$) or (if $a < 0$). [These 'hills' and 'valleys' may merge into a point of inflexion: or]. Again, by finding intercepts with the axes and the coordinates and nature of any turning points, the curve can be sketched.

However, not all functions are of one of the above types and so we need a general approach to sketching. The following steps will give information from which the graphs of many types of functions, $y = f(x)$, can be sketched.

1. Is there any obvious symmetry?
   If the equation is unchanged when $(-x)$ is substituted for x, the graph will be symmetrical about the y-axis (i.e. an even function, $f(-x) = f(x)$).
   If the equation is unchanged when $(-y)$ is substituted for y, the graph will be symmetrical about the x-axis. [Applies to graphs of the type $y^2 = f(x)$]
   If $f(-x) = -f(x)$ the function is an odd function and the graph is symmetrical for a 180° turn about the origin.
2. Find where the line cuts the x and y axes.
3. Examine the behaviour of the function as $x \to \pm\infty$.
4. Investigate any places where the function is undefined (e.g. $f(x) = 1/x$ is not defined for $x = 0$).
5. If the above steps indicate the presence of a turning point, find its location and nature.

Important notes

(a) For simple functions, e.g. $f(x) = x$, $g(x) = x^2$, $h(x) = 1/x$, point **1**, the symmetry aspect is worthy of consideration but for more complicated functions it is not always easy to consider this aspect. Thus only consider **1** if symmetry is obvious.
(b) For most functions, it is not necessary to follow all five steps; as soon as sufficient information has been gained to enable a sketch of the function to be made, there is no need to search for more. Indeed, point **5** can be difficult unless the function is easy to differentiate, and even when turning points are present a reasonably accurate sketch can often be made without having to locate them precisely.
(c) The behaviour of a function as $x \to \pm\infty$ (point **3**), and values of x for which a function is undefined (point **4**), are illustrated in the following example.

Example 1

For each of the following functions, (i) examine the behaviour of the function as $x \to \pm\infty$, (ii) find any values of x for which the function is undefined and investigate the function on either side of this value of x.

(a) $f(x) = \dfrac{3}{x}$, (b) $f(x) = 3 + \dfrac{2}{x}$, (c) $f(x) = \dfrac{5x-1}{x}$, (d) $f(x) = 3 - \dfrac{2}{x^2}$.

(a) $f(x) = \dfrac{3}{x}$

(i) For x, a large +ve number, $f(x)$ is small and positive we write this: as $x \to +\infty, f(x) \to 0^+$.
(i.e. $f(x)$ is 'just' greater than zero)
For x, a large −ve number, $f(x)$ is small and negative we write this: as $x \to -\infty, f(x) \to 0^-$.
(i.e. $f(x)$ is 'just' less than zero)
(ii) $f(x)$ is undefined for $x = 0$
For x *just* greater than zero, $f(x)$ is a large +ve number.
For x *just* less than zero, $f(x)$ is a large −ve number.
Thus: as $x \to 0^+, f(x) \to +\infty$
as $x \to 0^-, f(x) \to -\infty$.

(b) $f(x) = 3 + \dfrac{2}{x}$

(i) For x, a large +ve number, $\dfrac{2}{x}$ is small and positive.
Thus: as $x \to +\infty, f(x) \to 3^+$. (i.e. $f(x)$ is 'just' greater than 3)
For x, a large −ve number, $\dfrac{2}{x}$ is small and negative.
Thus: as $x \to -\infty, f(x) \to 3^-$. (i.e. $f(x)$ is 'just' less than 3)
(ii) $f(x)$ is undefined for $x = 0$
For x *just* greater than zero, $f(x)$ is a large +ve number.
For x *just* less than zero, $f(x)$ is a large −ve number.
Thus: as $x \to 0^+, f(x) \to +\infty$.
as $x \to 0^-, f(x) \to -\infty$.

(c) $f(x) = \dfrac{5x - 1}{x}$

$= 5 - \dfrac{1}{x}$ (i) As $x \to +\infty, f(x) \to 5^-$.
As $x \to -\infty, f(x) \to 5^+$.
(ii) $f(x)$ is undefined for $x = 0$
As $x \to 0^+, f(x) \to -\infty$.
As $x \to 0^-, f(x) \to +\infty$.

(d) $f(x) = 3 - \dfrac{2}{x^2}$ (i) As $x \to +\infty, f(x) \to 3^-$.
As $x \to -\infty, f(x) \to 3^-$.
(ii) $f(x)$ is undefined for $x = 0$
As $x \to 0^+, f(x) \to -\infty$.
As $x \to 0^-, f(x) \to -\infty$.

Example 2

Sketch the graph of the function
(a) $f(x) = 5 + 4x - x^2$,
(b) $g(x) = \begin{cases} 5 & \text{for } x \leqslant 0 \\ 5 + 4x - x^2 & \text{for } 0 \leqslant x \leqslant 5 \\ 0 & \text{for } x \geqslant 5. \end{cases}$

(a) Let $y = 5 + 4x - x^2$

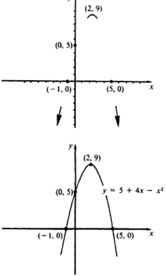
x-axis When $y = 0$, $5 + 4x - x^2 = 0$
i.e. $(5 - x)(1 + x) = 0$
cuts x-axis at (5, 0) and (−1, 0)

y-axis When $x = 0$, $y = 5$
cuts y-axis at (0, 5)

$x \to \pm\infty$ For large x the x^2 term dominates
Thus as $x \to +\infty, y \to -\infty$
as $x \to -\infty, y \to -\infty$

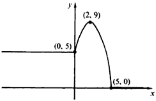
f(x) undefined There is no value of x for which $f(x)$ is undefined.

Max/min $y = 5 + 4x - x^2$
$y' = 4 - 2x$
$y'' = -2$
Thus a maximum exists at (2, 9)



The sketch can then be completed.

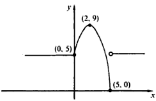
(b) The sketch of $g(x)$ will be the same as that of $f(x)$ for $0 \leqslant x \leqslant 5$, but will equal 5 for $x \leqslant 0$ and 0 for $x \geqslant 5$.
Thus the sketch will be:



Notice that, although the nature of $g(x)$ changes noticeably at $x = 0$ and at $x = 5$, the line itself does not 'break'; $g(x)$ is said to be a *continuous* function (as indeed was $f(x)$).

An example of a *discontinuous* function would be:

$$h(x) = \begin{cases} 5 & \text{for } x \leqslant 0 \\ 5 + 4x - x^2 & \text{for } 0 \leqslant x \leqslant 5 \\ 5 & \text{for } x > 5 \end{cases}$$



Example 3

Show that there is a solution to the equation $x^3 - 3x + 4 = 0$, that lies between $x = -3$ and $x = -2$. Sketch the curve given by $y = x^3 - 3x + 4$.

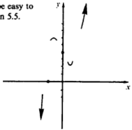
Let $y = x^3 - 3x + 4$, then the solutions of $x^3 - 3x + 4 = 0$ are obtained by putting $y = 0$.
Now y is zero at the points where the curve intersects the x-axis.
When $x = -3$, $y = (-3)^3 - 3(-3) + 4$
or $y = -14$
When $x = -2$, $y = (-2)^3 - 3(-2) + 4$
or $y = +2$
Since for $x = -3$, $y < 0$ and for $x = -2$, $y > 0$ there must exist some value of x between -3 and -2 for which $y = 0$.
Thus there is a solution of the equation $x^3 - 3x + 4 = 0$ lying between $x = -3$ and $x = -2$.
Note that since the solution is not an integer value, it would not be easy to find this solution by means of the factor theorem as used in section 5.5.

$y = x^3 - 3x + 4$

x-axis Curve cuts x-axis between $(-3, 0)$ and $(-2, 0)$. At this stage we do not know if there is any other point of intersection with the x-axis.

y-axis Intersection at (0, 4)

$x \to \pm\infty$ For large x, x^3 dominates. Thus as $x \to +\infty$, $y \to +\infty$ and as $x \to -\infty$, $y \to -\infty$.



y undefined There is no value of x for which y is undefined.

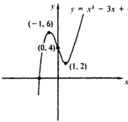
Max/min $y = x^3 - 3x + 4$

$y' = 3x^2 - 3$

$y'' = 6x$

Thus there exists a max at $(-1, 6)$ and a min at $(1, 2)$.

The sketch can then be completed.



Example 4

Make a sketch of the curve given by $y = \dfrac{x + 5}{x}$.

$y = \dfrac{x + 5}{x}$ can be written as $y = 1 + \dfrac{5}{x}$.

x-axis $y = 0$ when $x = -5$

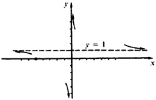
$\therefore$ cuts x-axis at $(-5, 0)$

y-axis No y-axis intercept because y is not defined for $x = 0$

$x \to \pm\infty$ As $x \to +\infty, y \to 1^+$

As $x \to -\infty, y \to 1^-$

The line $y = 1$ is called an *asymptote* to the curve, meaning that it is a straight line that the curve gets nearer and nearer to without actually touching. The line $y = 1$ is a horizontal asymptote and is shown as a broken line on the graph.



y undefined y is undefined for $x = 0$.

As $x \to 0^+, y \to +\infty$.

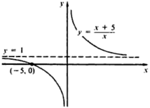
As $x \to 0^-, y \to -\infty$.

The line $x = 0$, i.e. the y-axis, is a vertical asymptote to the curve.

Max/min $y' = \dfrac{-5}{x^2}$. Thus there is no value of x for which $y' = 0$ and therefore no turning points.

Note, also, that the gradient is always negative.

The sketch can then be completed.



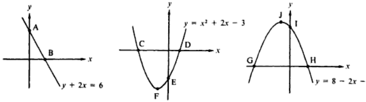
Exercise 11A

1. If each of the following were sketched, state which would give straight lines.

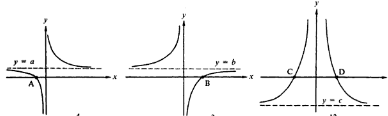
(a) $y = 2x + 3$ (b) $y = x^2 + 4x$ (c) $2y + 3x = 8$

(d) $y = x^3$ (e) $\frac{x}{3} + \frac{y}{4} = 1$ (f) $y = 2 + \frac{3}{x}$

2. Find the coordinates of the points A to J shown in the sketches below.



3. Find the coordinates of the points A to D and the values of the constants a to c shown in the sketches below.



sketch of $y = 2 + \frac{4}{x}$ sketch of $y = \frac{x-2}{x}$ sketch of $y = \frac{12}{x^2} - 3$

4. Make sketch graphs of the following lines giving the coordinates of any maximum points, minimum points and any intersections with the axes.

(a) $5y = 15 - 3x$

(b) $y = |2x + 4|$

(c) $y = 16 - x^2$

(d) $y = |2 - x - x^2|$

5. Show that there is a solution to the equation $x^3 - 6x^2 + 9x + 1 = 0$ between $x = -1$ and $x = 0$. Sketch the curve given by $y = x^3 - 6x^2 + 9x + 1$.

6. Sketch the graphs of the functions:

(a) $f(x) = \begin{cases} 8 & \text{for } x \leqslant -3 \\ x^2 - 1 & \text{for } -3 \leqslant x \leqslant 2 \\ 3 & \text{for } x \geqslant 2 \end{cases}$

(b) $g(x) = \begin{cases} 2 & \text{for } x < -2 \\ x^2 + 2x & \text{for } -2 \leqslant x \leqslant 1 \\ 4 - x & \text{for } x \geqslant 1 \end{cases}$

(c) $h(x) = \begin{cases} x^2 + 2x - 3 & \text{for } x \leqslant 1 \\ 6x - 5 - x^2 & \text{for } x \geqslant 1 \end{cases}$

7. Make a sketch of $y = f(x)$ given that $f(x)$ is an odd function and

$$f(x) = \begin{cases} x^2 & \text{for } 0 \leqslant x \leqslant 2 \\ 4 & \text{for } x > 2 \end{cases}$$

8. Make a sketch of $y = f(x)$ given that $f(x)$ is an even function and

$$f(x) = \begin{cases} x^2 - 1 & \text{for } 0 \leqslant x \leqslant 3 \\ 8 & \text{for } x > 3 \end{cases}$$

9. Make a sketch of $y = f(x)$ for $-6 \leqslant x < 6$ given that

$$f(x) = \begin{cases} x(3 - x) & \text{for } 0 \leqslant x \leqslant 3 \\ x - 3 & \text{for } 3 \leqslant x < 6 \end{cases}$$

and $f(x)$ is periodic with period 6.

Make sketch graphs of the following functions, labelling asymptotes, and giving the coordinates of any turning points and intersections with axes.

10. $f(x) = 1 + \dfrac{3}{x}$ **11.** $f(x) = \dfrac{2}{x} - 2$

12. $f(x) = \left|2 - \dfrac{8}{x}\right|$ **13.** $f(x) = \dfrac{2 - x}{x}$

14. $f(x) = \dfrac{2x - 3}{x}$ **15.** $f(x) = \dfrac{4}{x^2} - 1$

16. $f(x) = \left|1 - \dfrac{4}{x^2}\right|$

11.2 Further considerations

When sketching functions of the type $y = ax \pm \dfrac{b}{cx}$, $y = ax \pm \dfrac{b}{cx^2}$ or $y = ax^2 \pm \dfrac{b}{x}$ the techniques of section 11.1 are still applicable. However, with these functions it should be noted that as $x \to \pm\infty$, $y \to ax$, ax or ax^2 respectively.

Example 6

Make a sketch of the function $f(x) = x - \dfrac{1}{x}$.

Let $y = x - \dfrac{1}{x}$.

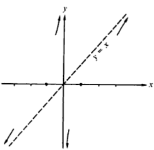
x-axis $y = 0$ when $x - \dfrac{1}{x} = 0$
i.e. $x^2 = 1$ or $x = \pm 1$
Cuts x-axis at $(-1, 0)$ and $(1, 0)$

y-axis No y-axis intersection as y is not defined for $x = 0$.

$x \to \pm\infty$ As $x \to \pm\infty$, $y \to x$ i.e. $y = x$ is an asymptote (called an oblique asymptote).
In fact, for $y = x - \dfrac{1}{x}$
as $x \to +\infty$, $y \to x^-$ and
as $x \to -\infty$, $y \to x^+$

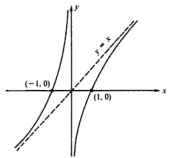
y undefined y is undefined for $x = 0$.
As $x \to 0^+$, $y \to -\infty$.
As $x \to 0^-$, $y \to +\infty$.
$x = 0$ is a vertical asymptote.

Max/min $y' = 1 + \dfrac{1}{x^2}$ which will never equal zero. $\therefore$ no turning points.



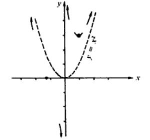
Notice also that the gradient is always positive.

The sketch can then be completed.



Example 7

Make a sketch of the curve given by $y = x^2 + \frac{2}{x}$.



x-axis When $y = 0$, $x^2 + \frac{2}{x} = 0$
i.e. $x = \sqrt[3]{(-2)}$
Cuts x-axis at $(-\sqrt[3]{2}, 0)$.

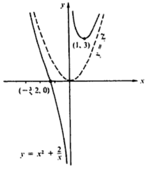
y-axis No y-axis intersection as y is not defined for $x = 0$.

$x \to \pm\infty$ $y \to x^2$, i.e. the curve approximates to that of $y = x^2$.
In fact, for $y = x^2 + \frac{2}{x}$,
as $x \to +\infty$, $y \to (x^2)^+$.
as $x \to -\infty$, $y \to (x^2)^-$.

y undefined y is undefined for $x = 0$.
Thus $x = 0$ is a vertical asymptote.
As $x \to 0^+$, $y \to +\infty$.
As $x \to 0^-$, $y \to -\infty$.

Max/min $y' = 2x - \frac{2}{x^2}$, $y'' = 2 + \frac{4}{x^3}$
Thus min at (1, 3).

The sketch can then be completed.



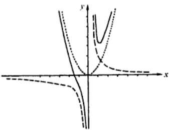
An alternative method for obtaining the sketch of $f(x) = g(x) + h(x)$, is to sketch $y = g(x)$ and $y = h(x)$ on the same axes and then to sum the functions to give $y = f(x)$.

For $f(x) = x^2 + \frac{2}{x}$, i.e. Example 7, this would then appear as follows with:

$y = x^2$ shown as ....

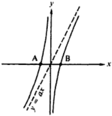
$y = \frac{2}{x}$ shown as _ _ _ _ _ _ _ _ _

and $y = x^2 + \frac{2}{x}$ shown as ____________

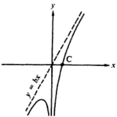


Exercise 11B

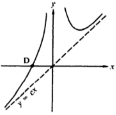
1. Find the coordinates of the points A to D and the values of the constants a to c in the sketches shown below.



sketch of $y = 2x - \frac{8}{x}$



sketch of $y = 2x - \frac{1}{4x^2}$



sketch of $y = x + \frac{1}{x^2}$

Make sketch graphs of each of the following, labelling any asymptotes and showing the coordinates of any turning points and the intersections with the axes.

2. $y = x + \frac{1}{x}$ **3.** $y = x - \frac{9}{x}$ **4.** $y = \frac{9}{x} - x$ **5.** $y = \left|x + \frac{4}{x}\right|$

Make sketch graphs of each of the following. The precise location of any turning points need not be given.

6. $y = x - \frac{1}{x^2}$ **7.** $y = \frac{8}{x^2} - x$ **8.** $y = \left|x^2 - \frac{1}{x}\right|$ **9.** $y = \left|x^2 + \frac{1}{x}\right|$

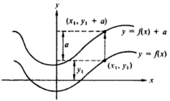
11.3 Simple transformations

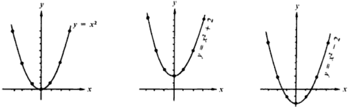
The reader will be familiar with the simple transformations, i.e. translations, reflections, rotations and enlargements. From the graph of $y = f(x)$ it is possible to deduce the graphs of other functions which are transformations of $y = f(x)$.

In this section we shall see how the graph of $y = f(x)$ can help us to draw the graphs of $y = f(x) + a$, $y = f(x - a)$, $y = -f(x)$, $y = f(-x)$, $y = af(x)$ and $y = f(ax)$.

$y = f(x) + a$

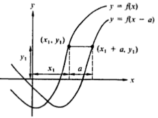
Consider a point (x_1, y_1) on the graph of $y = f(x)$, i.e. $y_1 = f(x_1)$. The point on $y = f(x) + a$ with an x-coordinate of x_1, will have a y-coordinate of $(y_1 + a)$. Thus for every point (x_1, y_1) on $y = f(x)$, there exists a point $(x_1, y_1 + a)$ on $y = f(x) + a$. Therefore, if we translate the graph of $y = f(x)$ by a units parallel to the y-axis, we obtain the graph of $y = f(x) + a$.
e.g.

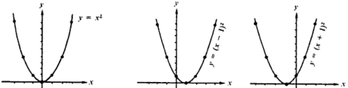




$y = f(x - a)$

Consider a point (x_1, y_1) on the graph of $y = f(x)$, i.e. $y_1 = f(x_1)$. The point on $y = f(x - a)$ with a y-coordinate of y_1 will have an x coordinate of $(x_1 + a)$. Thus for every point (x_1, y_1) on $y = f(x)$, there exists a point $(x_1 + a, y_1)$ on $y = f(x - a)$. Therefore, if we translate the graph of $y = f(x)$ by a units parallel to the x-axis, we obtain the graph of $y = f(x - a)$.
e.g.





Example 8

Find the equation of the curve obtained when the graph of $y = 2x^2 + 3x - 1$ is first reflected in the y-axis and then translated $+2$ units in the direction Ox.

A reflection in the y-axis transforms $y = f(x)$ to $y = f(-x)$

Thus $y = 2x^2 + 3x - 1$ is transformed to $y = 2(-x)^2 + 3(-x) - 1$
$= 2x^2 - 3x - 1$

A transformation of $+2$ units in the direction $0x$ transforms $y = f(x)$ to $y = f(x - 2)$

Thus $y = 2x^2 - 3x - 1$ transforms to $y = 2(x - 2)^2 - 3(x - 2) - 1$
$= 2x^2 - 11x + 13$

Thus under a reflection in the y-axis followed by a translation of $+2$ units in the direction $0x$, the curve $y = 2x^2 + 3x - 1$ transforms to $y = 2x^2 - 11x + 13$.

Alternatively the same result could be obtained using the methods of chapter 6, as follows.

Any point on the line $y = 2x^2 + 3x - 1$ will have coordinates of the form $(k, 2k^2 + 3k - 1)$.

Thus if (x', y') lies on the required image line

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 2k^2 + 3k - 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -k + 2 \\ 2k^2 + 3k - 1 \end{pmatrix}$$

i.e. $x' = -k + 2$ and $y' = 2k^2 + 3k - 1$

Eliminating k from these equations gives $y' = 2(x')^2 - 11x' + 13$.

The required image line has equation $y = 2x^2 - 11x + 13$, as before.

Exercise 11C

1. (a) Draw x and y axes for $-5 \leqslant x \leqslant 5$ and $-6 \leqslant y \leqslant 6$.
   (b) By using integer values of x plot the graph of $y = 3 + 2x - x^2$ for $-2 \leqslant x \leqslant 4$.
   (c) If $f(x) = 3 + 2x - x^2$ draw the graphs of the following functions on the same pair of axes used for (b),
   $y = f(x) + 2,\quad y = f(x - 1),\quad y = f(-x)$
   and give the equation of each function.
2. (a) Draw x- and y-axes for $-6 \leqslant x \leqslant 6$ and $-8 \leqslant y \leqslant 8$.
   (b) By using integer values of x, plot the graph of $y = x^2 - 2x$ for $-2 \leqslant x \leqslant 4$.
   (c) If $f(x) = x^2 - 2x$ draw the graphs of the following functions on the same pair of axes used for (b),
   $y = f(x) - 3,\quad y = -f(x),\quad y = f(x + 4)$
   and give the equation of each function.
3. Draw x- and y-axes for $-4 \leqslant x \leqslant 10$ and $-4 \leqslant y \leqslant 10$.
   By using integer values of x, plot the graph of $y = x^2$ for $-3 \leqslant x \leqslant 3$ and, on the same axes, draw $y = (x - 2)^2$, $y = (\frac{1}{2}x - 2)^2$, $y = (\frac{1}{2}x - 2)^2 - 4$.

4. Find the equation of the curve obtained when the graph of $y = x^2 + x + 1$ is:
(a) Translated $+4$ units in the direction of Oy, (O being the origin).
(b) Translated -3 units in the direction of Oy.
(c) Reflected in the y-axis.
(d) Reflected in the x-axis.
(e) Translated $+2$ units in the direction of Ox.
(f) Translated -2 units in the direction of Ox.
(g) Stretched parallel to the y-axis, scale factor 2, followed by a translation of $+2$ in the direction of Oy.
(h) Translated $+2$ units in the direction of Oy followed by a stretch parallel to Oy, scale factor 2.

5. State the transformation that must be given to the graph of $y = x^n$ to obtain the graph of:
(a) $y = x^n + 2$ (b) $y = (-x)^n$ (c) $y = (x - 3)^n$
(d) $y = 2x^n$ (e) $y = \left(\frac{x}{2}\right)^n$ (f) $y = (x + 2)^n - 3$
(g) $y = (2x)^n - 3$ (h) $y = 2(x - 1)^n + 5$

6. Prove that the curve obtained when $y = \frac{5x}{2} - \frac{x^2}{4} - 6$ is stretched parallel to Ox by a scale factor $\frac{1}{2}$ and then reflected in the x-axis is the same as that obtained when $y = x^2 + x$ is translated by $+3$ units parallel to Ox. Find the equation of this curve.

11.4 $\frac{1}{f(x)}$

A sketch of the graph $y = f(x)$ can be used to obtain a sketch of $y = \frac{1}{f(x)}$.
It is necessary to bear in mind the following important points.

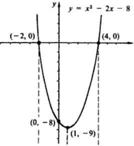
1. For values of x for which $f(x) > 0$, then $\frac{1}{f(x)} > 0$.
2. For values of x for which $f(x) < 0$, then $\frac{1}{f(x)} < 0$.
3. If $f(x) = 0$ when $x = a$, i.e. $f(a) = 0$, then $\frac{1}{f(x)}$ is not defined for this value of x and $x = a$ is a vertical asymptote of $y = \frac{1}{f(x)}$.
4. If $f(x)$ cuts the y-axis at the point $(0, b)$, then $\frac{1}{f(x)}$ will cut the y-axis at the point $\left(0, \frac{1}{b}\right)$.
5. As $f(x) \to \pm\infty$, then $\frac{1}{f(x)} \to 0$.
6. If there is a maximum/minimum at $x = x_1$ on $y = f(x)$ then there is a minimum/maximum at $x = x_1$ on $y = \frac{1}{f(x)}$.

Example 9

Make a sketch of $y = x^2 - 2x - 8$ and hence sketch $y = \dfrac{1}{x^2 - 2x - 8}$.

$y = x^2 - 2x - 8$
$\quad = (x - 4)(x + 2)$

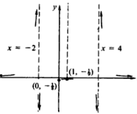
x-axis	Cuts x-axis at $(-2, 0)$ and at $(4, 0)$.
y-axis	Cuts y-axis at $(0, -8)$.
$x \to \pm\infty$	As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to +\infty$.
y undefined	There is no value of x for which y is undefined.
Max/min	$y' = 2x - 2$ and $y'' = 2$ $\therefore$ min at $(1, -9)$



$y = \dfrac{1}{x^2 - 2x - 8}$
$\quad = \dfrac{1}{f(x)}$

Vertical asymptote where $f(x) = 0$, i.e. $x = -2$ and $x = 4$. $\dfrac{1}{f(x)}$ has the same sign as $f(x)$ on either side of these asymptotes.

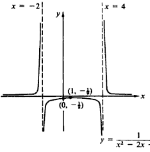
As $f(x) \to \infty$, $\dfrac{1}{f(x)} \to 0$, i.e. when $x \to \pm\infty$.



$f(x)$ cuts y-axis at $(0, -8) \Rightarrow \dfrac{1}{f(x)}$ cuts y-axis at $(0, -\frac{1}{8})$

$f(x)$ has min at $(1, -9) \Rightarrow \dfrac{1}{f(x)}$ has max at $(1, -\frac{1}{9})$

The sketch can then be completed.



Notice from the graph of $y = \frac{1}{(x-4)(x+2)}$ that the ranges of values y can take for real x are $y \leqslant -\frac{1}{9}$, $y > 0$.

We say that the range of the function $f(x) = \frac{1}{(x-4)(x+2)}$ for the domain $\{x \in \mathbb{R}: x \neq 4, x \neq -2\}$ is $\{y \in \mathbb{R}: y \leqslant -\frac{1}{9} \text{ or } y > 0\}$. This range of values of y can be determined algebraically using the method shown in Example 10.

Example 10

Calculate the range of the function $f(x) = \frac{1}{(x-4)(x+2)}$ for real x.

Let $y = \frac{1}{(x-4)(x+2)}$ $\quad \therefore y = \frac{1}{x^2 - 2x - 8}$ $\quad \ldots [1]$

From [1] $\quad yx^2 - 2xy - 8y = 1$

i.e. $\quad yx^2 - 2xy - 8y - 1 = 0$

For $y \neq 0$ this is a quadratic in x.

Thus, for real x, $4y^2 - 4(y)(-8y - 1) \geqslant 0$

i.e. $\quad y(9y + 1) \geqslant 0$

Solving this inequality using the methods of chapter 5, page 143:

For $y = 0$ the result $-1 = 0$ is obtained which is not possible. $\therefore y \neq 0$

	$y < -\frac{1}{9}$	$-\frac{1}{9} < y < 0$	$y > 0$
y	$-$ve	$-$ve	+ve
$9y + 1$	$-$ve	+ve	+ve
$y(9y + 1)$	+ve	$-$ve	+ve

Thus the range of the function is $\{y \in \mathbb{R}: y \leqslant -\frac{1}{9} \text{ or } y > 0\}$.
(Note: this technique is practised further in Chapter 14.)

Exercise 11D

For questions 1 to 9, make sketch graphs of $f(x)$ and $1/f(x)$ showing clearly the coordinates of any turning points and of any intersections with the axes.

1. $f(x) = x^2 - 6x + 5$

2. $f(x) = 1 - x^2$

3. $f(x) = (x - 2)(x - 5)$

4. $f(x) = (3 - x)(x + 1)$

5. $f(x) = 4x - x^2 - 3$

6. $f(x) = \frac{x^2 - 4x - 12}{8}$

7. $f(x) = x(x - 4)$

8. $f(x) = (x - 2)^2$

9. $f(x) = (x + 2)(x - 1)^2$

10. Sketch the graph of $y = \sin\theta$ for $-360° \leqslant \theta \leqslant 360°$. Hence sketch $y = \operatorname{cosec}\theta$ for $-360° \leqslant \theta \leqslant 360°$.

11. Sketch the graph of $y = \cos\theta$ for $-360° \leqslant \theta \leqslant 360°$. Hence sketch $y = \sec\theta$ for $-360° \leqslant \theta \leqslant 360°$.

12. Sketch the graph of $y = \tan\theta$ for $-360° \leqslant \theta \leqslant 360°$. Hence sketch $y = \cot\theta$ for $-360° \leqslant \theta \leqslant 360°$.

13. Calculate the range of the following functions for real x.

(a) $\frac{1}{x^2 - x - 6}$ (b) $\frac{1}{2x^2 - 7x + 3}$ (c) $\frac{1}{4x - 3 - x^2}$ (d) $\frac{6}{x^2 + 3x - 4}$

Exercise 11E Examination questions

1. On two separate diagrams sketch the graphs of
(i) $y = x^2 + 2$ for $-6 \leqslant x \leqslant 4$,
(ii) $y = |8 - 3x|$ for $1 \leqslant x \leqslant 5$.
In each case state the range of values of y. (Cambridge)

2. Using a *separate* diagram for each, *sketch* the curves
(a) $y = x(x - 1)$,
(b) $y = x^2(x - 1)$,
(c) $y = x^2(x - 1)^2$. (A.E.B.)

3. Calculate the range of values of m for which the line $y = mx$ intersects the curve $y = x^2 - 4x + 9$ in two distinct points. Sketch the curve and indicate on it the significance of the values of m you have found. (S.U.J.B.)

4. The curve with equation

$$y = ax^3 + bx^2 + cx + d,$$

where a, b, c and d are constants, passes through the points (0, 3) and (1, 0). At these points the curve has gradients -7 and 0, respectively.
(i) Find the values of a, b, c and d.
(ii) Show that the curve crosses the x-axis at the point (3, 0).
(iii) Find the x-coordinate of the maximum point on the curve.
(iv) Sketch the curve. (Cambridge)

5. On separate diagrams, **sketch** the graphs of
(i) $y = (x - 3)^2 + 2$,
(ii) $y = 2\sin\left(x - \frac{\pi}{2}\right)$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$.
In each case state the range of values of y. (Cambridge)

6. Sketch the curve $y = \cos x + 2$ for the interval $0 \leqslant x \leqslant 2\pi$.
On the same diagram sketch the curve $y = \frac{1}{\cos x + 2}$ for the same interval. (Cambridge)

7. Find $\frac{dy}{dx}$ when $y = x + \frac{9}{x}$. Find also the maxima and minima of y, distinguishing carefully between them.
Show that $\frac{dy}{dx} < 1$ for all values of x.
Draw a sketch of the curve $y = x + \frac{9}{x}$. (Oxford)

8. On separate diagrams, sketch the graphs of
(a) $y = x(x - 5)^2$ (b) $y^2 = x(x - 5)^2$

9. (a) Sketch the graph of $y = (x + 1)(x - 2)$ and hence sketch the graph of $z = 1/\{(x + 1)(x - 2)\}$. By calculation find the range of values which z can take. (S.U.J.B.)

10. Prove that, for all real x,

$$0 < \frac{1}{x^2 + 6x + 10} \leqslant 1.$$

Sketch the curve

$$y = \frac{1}{x^2 + 6x + 10}.$$

(Oxford)

11. Sketch separately the graphs of
(i) $y = 2(x + 2)(x - 1)(x - 2)$; (ii) $y = (x - 1)(x - 3)^2$
showing the coordinates of the points where they cross or touch the axes. Find a cubic function $f(x)$ such that the graph of $y = f(x)$ crosses the x-axis at $x = 1$, touches the x-axis at $x = 3$ and crosses the y-axis at $y = 18$. From consideration of the sketch graph of $y = f(x)$ show that if the equation $f(x) = k$ has three real roots then k must be negative. Give the sign of each root in this case. Show, also, that if $k > 0$ then the equation $f(x) = k$ has just one real root and give the range of values of k for which this root is negative. (S.U.J.B.)

12. Write down the condition for the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ to have no real roots. Sketch the graph of $y = x^2 + 3x + 7$ and hence sketch the graph of $y = 1/(x^2 + 3x + 7)$.
Sketch the graph of $y = (x - 1)(x + 5)$ and hence sketch the graph of $z = 1/((x - 1)(x + 5))$.
By calculation find the range of values which z can take. (S.U.J.B.)

12
Calculus II: Integration

12.1 The reverse of differentiation

In chapter 10 we saw that if we know the equation of a curve, say $y = f(x)$ then, by differentiation, we can find the gradient function $\frac{dy}{dx}$.

If, instead, we are given the gradient function $\frac{dy}{dx}$, can we obtain the equation of the curve?
This reverse process is called **integration**.

Suppose $\frac{dy}{dx} = 2x$. To find y, we require a function that differentiates to give $2x$. Clearly, $y = x^2$ is such a function, but there are many others, e.g. $y = x^2 - 3$, $y = x^2 + 1$, $y = x^2 + 7$, etc.
We say that $y = x^2 + c$, where c is a constant, gives $2x$ when differentiated. Thus if we **integrate** $2x$, we obtain $x^2 + c$. Given the gradient function of a curve, integration gives the 'family' of curves to which that gradient function applies. Given more information about the curve, we may be able to determine the value of c, the constant of integration, and hence the equation of a particular curve.
In simple cases this process of integration can be carried out by inspection.

Thus, given $\frac{dy}{dx} = 2x$ then $y = x^2 + c$

given $\frac{dy}{dx} = 4x^3$ then $y = x^4 + c$

given $\frac{dy}{dx} = x^4$ then $y = \frac{1}{5}x^5 + c$

In the general case, given $\frac{dy}{dx} = ax^n$ then $y = \frac{ax^{n+1}}{n+1} + c$.

This general statement can be verified by differentiation:

If $y = \frac{ax^{n+1}}{n+1} + c$

$$\frac{dy}{dx} = \frac{a(n+1)x^{n+1-1}}{n+1}$$

$$= ax^n \quad \text{as required.}$$

Note that the statement 'If $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1} + c$' does not apply for $n = -1$, because in such cases we have $y = \frac{ax^0}{0} + c$, which is meaningless.

We can now state the rule for integrating ax^n:

If $\frac{dy}{dx} = ax^n$ then $y = \frac{ax^{n+1}}{n+1} + c$ provided $n \neq -1$

The integral sign

We have seen that, if $\frac{dy}{dx} = ax^n$ then, by integration $y = \frac{ax^{n+1}}{n+1} + c$ (provided $n \neq -1$).

Using the integral sign: $\int$ we write this as $\boxed{\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1}$

with the 'dx' signifying that the integration is carried out with respect to the variable x.

$\int g(x)dx$ is read as 'the integral of $g(x)$ with respect to x' and we use the extended S symbol because, as we shall see, integration can be considered as a summation.

From the rules given in chapter 10 for differentiation, it follows that

$$\int af(x)dx = a\int f(x)dx$$

and $\int [af(x) + bg(x) - ch(x)]dx = a\int f(x)dx + b\int g(x)dx - c\int h(x)dx$

Example 3

Find (a) $\int 4x^5 dx$ (b) $\int \frac{6}{x^3}dx$ (c) $\int \left(5 - x^2 + \frac{18}{x^4}\right)dx$

(a) $\int 4x^5 dx$

$= \frac{4x^6}{6} + c$

$= \frac{2x^6}{3} + c$

(b) $\int \frac{6}{x^3}dx$

$= \int 6x^{-3}dx$

$= \frac{6x^{-2}}{-2} + c$

$= -\frac{3}{x^2} + c$

(c) $\int \left(5 - x^2 + \frac{18}{x^4}\right)dx$

$= \int (5 - x^2 + 18x^{-4})dx$

$= 5x - \frac{x^3}{3} + \frac{18x^{-3}}{-3} + c$

$= 5x - \frac{x^3}{3} - \frac{6}{x^3} + c$

Since there is an arbitrary constant, c, in each of these solutions, we say that these are *indefinite* integrals.

Second Order Differentials

Remembering that the differential of $\frac{dy}{dx}$ with respect to x is $\frac{d^2y}{dx^2}$, it follows that

$$\int \frac{d^2y}{dx^2}dx = \frac{dy}{dx} + c$$

Example 4

Find y as a function of x, given that $\frac{d^2y}{dx^2} = 15x - 2$ and that when $x = 2$, $\frac{dy}{dx} = 25$ and $y = 20$.

Given $\frac{d^2y}{dx^2} = 15x - 2$

Integrating both sides of this equation with respect to x,

$$\frac{dy}{dx} = \int (15x - 2)dx$$

$$= \frac{15x^2}{2} - 2x + c$$

But $\frac{dy}{dx} = 25$ when $x = 2$,

$\therefore \quad 25 = \frac{15(4)}{2} - 2(2) + c$ i.e. $c = -1$

$\therefore \quad \frac{dy}{dx} = \frac{15x^2}{2} - 2x - 1$

Integrating both sides of this equation with respect to x,

$$y = \frac{5x^3}{2} - x^2 - x + d$$

But $y = 20$ when $x = 2$, giving $d = 6$.

The required equation is $y = \frac{5x^3}{2} - x^2 - x + 6$.

Exercise 12A

1. Find an expression for y if $\frac{dy}{dx}$ is given by
   (a) $3x^2$ (b) $2x$ (c) x^3 (d) $2x^4$ (e) 5
   (f) $3x^5$ (g) $\sqrt{x}$ (h) $\frac{6}{x^2}$ (i) $-\frac{4}{x^3}$ (j) $\frac{1}{\sqrt{x}}$
   (k) $2x^3 + 3x^2$ (l) $5x + 1$ (m) $2x + 9x^2$
   (n) $5x^4 - 6x$ (o) $8x^3 - 12x^2$ (p) $x(4 - 3x)$
   (q) $3x(x - 2)$ (r) $2x(x^3 - 4)$ (s) $(3x - 1)(x + 1)$
   (t) $(x - 6)(x - 2)$
2. Integrate the following functions with respect to x.
   (a) $8x^3$ (b) $12x$ (c) $5x^2$ (d) 7 (e) $7 - 2x$
   (f) $\frac{6}{x^3}$ (g) $-\frac{12}{x^5}$ (h) $\frac{3x}{\sqrt{x}}$ (i) $\frac{5x^2}{\sqrt{x}}$ (j) $\frac{3x^4 + 6}{x^2}$
   (k) $4x^3 + 3x^2 + 2x + 1$ (l) $2x^2(3 - 4x)$
   (m) $x^4 + x^2 + \frac{1}{x^2} + \frac{1}{x^4}$.

3. Find

(a) $\int 12x\,dx$ (b) $\int (x^3 + x)dx$ (c) $\int x(x + 1)dx$

(d) $\int (x + 6)(x - 4)dx$ (e) $\int \frac{5}{x^4}dx$ (f) $\int \left(10x^4 + 8x^3 - \frac{6}{x^2}\right)dx$

(g) $\int \frac{x^4 + 1}{x^2}dx$ (h) $\int \frac{(1 - 3x)}{\sqrt{x}}dx$ (i) $\int \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)dx$

4. The gradient of a curve at the point (x, y) on the curve is given by $6x$. If the curve passes through the point $(1, 4)$, find the equation of the curve.
5. Find the equation of the curve passing through the point $(-2, 6)$ and having gradient function $(3x^2 - 2)$.
6. The gradient of a curve at the point (x, y) on the curve is given by $2(1 - x)$ and the curve passes through the point $(-1, 5)$. Find the equation of the curve.
7. Find S as a function of t given that $\frac{dS}{dt} = 6t^2 + 12t + 1$ and when $t = -2, S = 5$.
8. Find V as a function of h given that $\frac{dV}{dh} = 2(7h - 2)$ and when $h = 2$, $V = 21$.
9. Find A as a function of p given that $\frac{dA}{dp} = 5 - 4p$ and when $p = 3$, $A = -2$.
10. The gradient of a curve at the point (x, y) on the curve is given by $(3x^2 + 8)$. If the curve and the line $2x - y - 1 = 0$ cut the y-axis at the same point, find the equation of the curve.
11. The gradient function of a curve is given by $(2x - 3)$ and the curve cuts the x-axis at two points: A(5, 0) and B. Find the equation of the curve and the coordinates of B.
12. The gradient of a curve at the point (x, y) on the curve is given by $(2x - 4)$. If the minimum value of y is 3, find the equation of the curve.
13. Find y as a function of x given that $\frac{d^2y}{dx^2} = 4 - 6x$ and that when $x = 2, \frac{dy}{dx} = -4$ and $y = 7$.
14. Find y as a function of x given that $\frac{d^2y}{dx^2} = 6x - 4$, $y = 4$ when $x = 1$ and $y = 2$ when $x = -1$.
15. Find y as a function of x given that $\frac{d^2y}{dx^2} = 30x$, $y = 32$ when $x = 2$ and $y = 5$ when $x = -1$.

12.2 The area under a curve

Suppose A_1 is the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ (see Figure 1). We say that A_1 is the area 'under' the curve from $x = a$ to $x = b$. One way to estimate this area would be to divide it into strips. Since each of these strips approximates to a rectangle (Figure 2), we can then sum the areas of these rectangles. This would give an approximate value for A_1; the more rectangles we use, the greater is the accuracy. Consider one such rectangle, width δx (Figure 3).

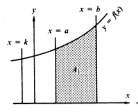


Fig. 1

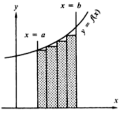


Fig. 2

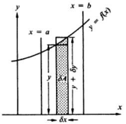


Fig. 3

In Figure 3, if δA is the shaded area, $y\,\delta x < \delta A < (y + \delta y)\delta x$

Thus $y < \dfrac{\delta A}{\delta x} < y + \delta y$

Now as $\delta x \to 0$ (i.e. we increase the number of rectangles)

$\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$ and $\delta y \to 0$

Thus $\dfrac{dA}{dx} = y$ or $A = \displaystyle\int y\,dx$

This integration will give an area function $A(x)$ and will involve a constant of integration c. As we substitute a value for x into the function $A(x)$, say $x = b$, we will obtain an answer for the area under the curve from a right-hand boundary of $x = b$ to some left-hand boundary. The position of the left-hand boundary will determine the value of c, the constant of integration. Suppose we take $x = k$ as the left-hand boundary, then

$A(a) =$ Area from $x = k$ to $x = a$.

$A(b) =$ Area from $x = k$ to $x = b$.

$\therefore\ A(b) - A(a) =$ Area from $x = a$ to $x = b$.

As $A = \displaystyle\int y\,dx$, we write $A(b) - A(a)$ as $\displaystyle\int_a^b y\,dx$.

$\therefore\ A_1 = \displaystyle\int_a^b y\,dx.$

$y\,\delta x < \delta A < y\,\delta x + \delta y\delta x$

As $\delta x \to 0$ $\delta y \to 0$ and so $\delta y\delta x$ becomes negligible compared with $y\,\delta x$.

Thus as $\delta x \to 0$, $\delta A \to y\,\delta x$.

But $A_1 = \displaystyle\sum_{x=a}^{x=b} \delta A$

$\therefore\ A_1 = \displaystyle\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x$

The area under the curve can therefore be found as the limit of a sum or by integration. Thus integration is a process of summation and

$$A_1 = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx, \quad \text{where} \quad y = f(x).$$

Definite integrals

Note that $\int_a^b f(x)dx$ is known as a *definite* integral because the limits of integration, i.e. $x = a$ and $x = b$, are known.

Suppose $\int f(x)dx = F(x) + c$

then
$$\int_{x=a}^{x=b} f(x)dx = (F(b) + c) - (F(a) + c)$$
$$= F(b) - F(a)$$

We usually write this:
$$\int_{x=a}^{x=b} f(x)dx = \Big[F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

We see that the constants of integration cancel out so that in the case of a definite integral there is no need to give an arbitrary constant in the result.

Example 5

Evaluate the following definite integrals: (a) $\int_{-1}^{1} (2x - 3)dx$ (b) $\int_{1/4}^{1/2} \frac{1}{x^3}dx$

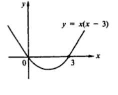
(a)
$$\int_{-1}^{1} (2x - 3)dx$$
$$= \Big[x^2 - 3x\Big]_{-1}^{1}$$
$$= [1^2 - 3(1)] - [(-1)^2 - 3(-1)]$$
$$= -2 - 4$$
$$= -6$$

(b)
$$\int_{1/4}^{1/2} \frac{1}{x^3}dx = \int_{1/4}^{1/2} x^{-3}dx$$
$$= \left[\frac{x^{-2}}{-2}\right]_{1/4}^{1/2}$$
$$= \left[-\frac{1}{2x^2}\right]_{1/4}^{1/2}$$
$$= \left(-\frac{1}{2(\frac{1}{2})^2}\right) - \left(-\frac{1}{2(\frac{1}{4})^2}\right)$$
$$= -2 + 8$$
$$= +6$$

Calculation of the area under a curve

When we calculate the area under a curve, the important first step is to make a sketch of the curve. We must then remember that an area lying 'above' the x-axis will have a positive value, whereas areas lying 'below' the x-axis will be negative. In some cases the required area may lie both 'above' and 'below' the x-axis and particular care is needed in these situations.

Example 6



Find the area between the curve $y = x(x - 3)$ and the x-axis.

First, make a sketch of the curve $y = x(x - 3)$

In this case the required area lies 'below' the x-axis.

Using $A = \int y\,dx$ and substituting for y from the equation of the curve, as we cannot integrate y with respect to x.

$$\therefore \quad A = \int_0^3 x(x - 3)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$$

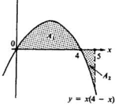
$$= \left(9 - \frac{27}{2}\right) - (0) = -4\tfrac{1}{2}$$

The area has a negative sign, as was anticipated, and the numerical value is $4\frac{1}{2}$ sq. units.

Example 7

Find the area between the curve $y = x(4 - x)$ and the x-axis from $x = 0$ to $x = 5$.

First, make a sketch of the curve $y = x(4 - x)$



The sketch shows that the required area is in two parts; one part lies above the x-axis and therefore has a positive area, the other part lies below the x-axis and has a negative area.

Using $A = \int y\,dx$ and calculating the two areas separately,

$$A_1 = \int_0^4 x(4 - x)dx \qquad A_2 = \int_4^5 x(4 - x)dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 \qquad = \left[2x^2 - \frac{x^3}{3}\right]_4^5$$

$$= \left(32 - \frac{64}{3}\right) - (0) \qquad = \left(50 - \frac{125}{3}\right) - \left(32 - \frac{64}{3}\right)$$

$$= +\frac{32}{3} = +10\tfrac{2}{3} \qquad = -2\tfrac{1}{3}$$

The total area under the curve between $x = 0$ and $x = 5$ is given by the sum of the *numerical* values of these two areas:

required area $= 10\frac{2}{3} + 2\frac{1}{3} = 13$ sq. units.

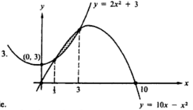
Note In the last example, it is possible to calculate $\int_0^5 x(4 - x)dx$, but this would not give the correct answer for the required area. Instead we would obtain an answer of $10\frac{2}{3} - 2\frac{1}{3}$ i.e. $8\frac{1}{3}$, as the following working shows:

$$\int_0^5 x(4 - x)dx = \left[2x^2 - \frac{x^3}{3}\right]_0^5$$

$$= \left(50 - \frac{125}{3}\right) - (0) = 8\tfrac{1}{3}$$

Example 8

Find the area enclosed between the curves $y = 2x^2 + 3$ and $y = 10x - x^2$.



First, make a sketch of the two curves, noting that the curves will intersect at the points where

$$2x^2 + 3 = 10x - x^2$$
$$3x^2 - 10x + 3 = 0$$
$$(3x - 1)(x - 3) = 0 \quad \text{i.e. at} \quad x = \tfrac{1}{3} \quad \text{and} \quad x = 3.$$

The curve $y = 2x^2 + 3$ intersects the y-axis at $(0, 3)$ and does not cut the x-axis.

The curve $y = 10x - x^2$ intersects the axes at $(0, 0)$ and at $(10, 0)$.

The information is sufficient for a sketch to be made.

The area enclosed by $y = 10x - x^2$, the ordinates $x = \frac{1}{3}$ and $x = 3$ and the x-axis is

$$\int_{1/3}^{3} (10x - x^2)dx.$$

The area enclosed by $y = 2x^2 + 3$, the ordinates $x = \frac{1}{3}$ and $x = 3$ and the x-axis is

$$\int_{1/3}^{3} (2x^2 + 3)dx.$$

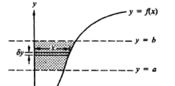
The shaded area is

$$\int_{1/3}^{3} (10x - x^2)dx - \int_{1/3}^{3} (2x^2 + 3)dx$$
$$= \int_{1/3}^{3} (10x - x^2 - 2x^2 - 3)dx$$
$$= \int_{1/3}^{3} (10x - 3x^2 - 3)dx$$
$$= \left[5x^2 - x^3 - 3x\right]_{1/3}^{3}$$
$$= (45 - 27 - 9) - (\tfrac{5}{9} - \tfrac{1}{27} - 1)$$
$$= 9\tfrac{13}{27} \text{ sq. units.}$$

Area between a curve and the y-axis

Suppose we wish to find the area between some curve $y = f(x)$ and the y-axis, from $y = a$ to $y = b$.

Considering a strip of length x and width δy, drawn parallel to the x-axis, we see that



$$A = \lim_{\delta y \to 0} \sum_{y=a}^{y=b} x\,\delta y$$

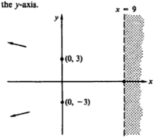
$$\int^{b}$$

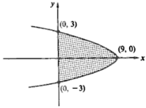
Example 9

Find the area enclosed between the curve $y^2 = 9 - x$ and the y-axis.

First we make a sketch of the curve $y^2 = 9 - x$.

Symmetry	The equation is unchanged if y is replaced by $(-y)$. Hence the curve is symmetrical about the x-axis.
y-axis	Cuts y-axis at (0, 3) and at (0, −3).
x-axis	Cuts x-axis at (9, 0).
$x \to \pm\infty$	$y = \pm\sqrt{(9 - x)}$ ∴ as $x \to +\infty$, y is undefined. As $x \to -\infty$, $y = \pm\sqrt{(9 + \infty)}$ i.e. as $x \to -\infty$, $y \to \pm\infty$ slowly by comparison with x.
y undefined	y is undefined for $x > 9$ because $(9 - x)$ will be negative. Thus the sketch can be completed and the required area shown shaded:





$$\text{Required area} = \int_{y=-3}^{y=3} x\,dy$$

Now we cannot integrate x with respect to y, so we substitute for x,

$$\therefore\ A = \int_{y=-3}^{y=3} (9 - y^2)dy$$

which gives $A = 36$ sq. units

Exercise 12B

1. Evaluate the following definite integrals.

(a) $\int_1^5 2x\,dx$ (b) $\int_0^2 3x^2\,dx$

(c) $\int_{-1}^4 (6 - 2x)dx$ (d) $\int_{-1}^1 (1 + x)dx$

(e) $\int_{-1}^3 (3x - 2)dx$ (f) $\int_{-4}^0 (x^2 + x + 1)dx$

(g) $\int_1^2 \frac{1}{x^2}dx$ (h) $\int_4^9 \frac{1}{\sqrt{x}}dx$

(i) $\int_0^4 (x^3 - 2x - 3\sqrt{x})dx$ (j) $\int_1^4 \left(\frac{x^4 - x^3 + \sqrt{x} - 1}{x^2}\right)dx$

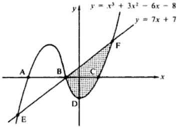
16. Find the area enclosed between the curve $y^2 = 4 - x$ and the y-axis.

17. Find the area enclosed by the curve $(y - 1)^2 = x$, the y-axis and the line $y = 3$.

18. Find the area enclosed between the curve $y = x^2$ and the straight line $y = x$.

19. The line $y = x + 8$ cuts the curve $y = 12 + x - x^2$ at two points A and B (B being in the first quadrant). Find the coordinates of A and B and find the area enclosed between the curve $y = 12 + x - x^2$ and the straight line AB.

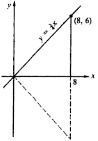
20. The sketch graph shows the lines $y = 7x + 7$ and $y = x^3 + 3x^2 - 6x - 8$. Find the coordinates of the points A, B, C, D, E and F and find the shaded area.



21. Find the area enclosed between the curves $y = x^2 + 6$ and $y = 12 + 4x - x^2$.

22. Find the area enclosed between the curves $y = x^2 - 4x$ and $y = 6x - x^2$.

12.3 Volume of revolution



Suppose that the area enclosed by the line $y = \frac{3}{4}x$, the x-axis and the line $x = 8$ is rotated about the x-axis through one revolution. The solid formed is referred to as a solid of revolution and, in this case, will be a cone. In this particular case the volume of the cone can be readily calculated since its base radius is 6 units and the perpendicular height of the cone is 8 units:

volume of solid of revolution (i.e. the cone)

$$= \tfrac{1}{3} \times \pi r^2 \times h$$
$$= \tfrac{1}{3} \times \pi 6^2 \times 8$$
$$= 96\pi \text{ cubic units.}$$

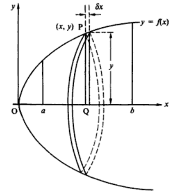
If instead of a straight line, a curve with equation $y = f(x)$ is rotated in the same way, the volume of the solid of revolution can be calculated by the use of calculus.

Suppose the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is rotated about the x-axis through one revolution.

Consider the elementary strip PQ, thickness δx, length y, drawn parallel to the y-axis. The result of rotating this elementary strip through one revolution about the x-axis is to produce an elementary disc of radius y and thickness δx. Let δV be the volume of this disc, then

$$\delta V = \pi y^2 \delta x.$$

Thus, as we allow $\delta x \to 0$, the sum of all these volumes δV will tend towards V, the volume of the solid of revolution.



$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \, \delta x$$

$$= \int_a^b \pi y^2 \, dx$$

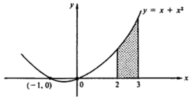
Thus $\boxed{V = \int_a^b \pi y^2 dx \quad \text{where} \quad y = f(x)}$

This definite integral can then be evaluated to find V, the volume of revolution.

Example 10

Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = x + x^2$, the x-axis and the ordinates $x = 2$ and $x = 3$ through one revolution about the x-axis.

First sketch the situation.



The volume of revolution V is given by

$$V = \int_a^b \pi y^2 \, dx$$

$$= \int_2^3 \pi(x + x^2)^2 dx$$

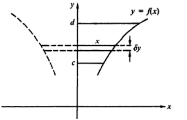
$$= \pi \int_2^3 (x^2 + 2x^3 + x^4)dx$$

$$= \pi\left[\frac{x^3}{3} + \frac{x^4}{2} + \frac{x^5}{5}\right]_2^3$$

$$= \pi\left[\left(9 + \frac{81}{2} + \frac{243}{5}\right) - \left(\frac{8}{3} + 8 + \frac{32}{5}\right)\right] = 81\tfrac{1}{30}\pi \text{ cubic units}$$

The volume of revolution is $81\frac{1}{30}\pi$ cubic units.

Rotation about the y-axis



The volume of the solid of revolution formed by rotating an area through one revolution about the y-axis can be found in a similar way.
The elementary strip will be drawn parallel to the x-axis. In this case, the elementary discs will have radius x and thickness δy.

Thus $V = \lim_{\delta y \to 0} \sum \pi x^2 \, \delta y$

$$V = \int_{y=c}^{y=d} \pi x^2 \, dy$$

Note that the integration will be with respect to y and hence the limits of the integration will be values of y.

Exercise 12C

Find the volumes of the solids formed when each of the areas of questions **1** to **10** perform one revolution about the x-axis.

1. The area enclosed by the curve $y = x^2$, the x-axis and the line $x = 2$.
2. The area between the line $y = x + 1$ and the x-axis from $x = 1$ to $x = 3$.
3. The area between the line $y = 3x + 2$ and the x-axis from $x = 0$ to $x = 1$.
4. The area enclosed by the curve $y = x^3$, the x-axis and the line $x = 2$.
5. The area enclosed by the curve $y = x^2 + 1$, the x-axis, $x = -1$ and $x = 1$.
6. The area enclosed by the curve $y = x^3 + 1$, the x-axis and the line $x = 1$.
7. The area enclosed by the curve $y = 4x - x^2$ and the x-axis.
8. The area enclosed by the curve $y = x^2 - x^3$ and the x-axis.
9. The area enclosed by the curve $y = x - \frac{1}{x}$ the x-axis and the line $x = 2$.
10. The area enclosed between the curve $y = 4x - x^2$ and the line $y = 2x$.
11. Find the equation of the straight line joining the origin to the point with coordinates (h, r). Hence find a formula for the volume of a right circular cone of base radius r and height h.

Find the volumes of the solids formed when each of the areas of questions **12** to **14** perform one revolution about the y-axis.

12. The area lying in the first quadrant and bounded by the curve $y = x^2$, the y-axis and the line $y = 4$.
13. The area lying in the first quadrant and bounded by the curve $y = 2x^2 + 1$, the y-axis and the lines $y = 2$ and $y = 5$.
14. The area lying in the first quadrant and bounded by the y-axis, the curve $y = x^3$ and the line $y = 3x + 2$.

when $t = 1\frac{1}{2}$, $v = 4(1\frac{1}{2}) - 3$
i.e. $v = 3$ m/s

(c) Using $a = \frac{dv}{dt}$

$a = 4$ m/s^2

The displacement is 6 m, the velocity is 3 m/s and the acceleration is constant and is 4 m/s^2.

Example 12

A body moves in a straight line. At time t seconds its acceleration is given by $a = 6t + 1$. When $t = 0$, the velocity v of the body is 2 m/s and its displacement s from the origin O is 1 metre. Find expressions for v and s in terms of t.

Given that $a = \frac{dv}{dt} = 6t + 1$

$$v = \int (6t + 1)dt$$

$$v = 3t^2 + t + c$$

but when $t = 0$, $v = 2$ thus $2 = 0 + c$ $\therefore\ c = 2$

Substituting for c, $v = 3t^2 + t + 2$

Using $v = \frac{ds}{dt} = 3t^2 + t + 2$

$$s = \int (3t^2 + t + 2)dt$$

i.e. $s = t^3 + \frac{t^2}{2} + 2t + d$

but when $t = 0$, $s = 1$ thus $1 = 0 + d$ $\therefore\ d = 1$

Substituting for d, $s = t^3 + \frac{t^2}{2} + 2t + 1$

Thus at time t, $v = 3t^2 + t + 2$ and $s = t^3 + \frac{t^2}{2} + 2t + 1$.

B. Determination of centroids

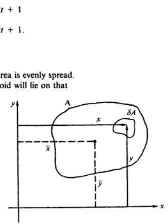
The centroid of an area is that point about which the area is evenly spread. Thus if the area possesses a line of symmetry, the centroid will lie on that line.

If $(\bar{x}, \bar{y})$ is the centroid of an area A, then $A\bar{x}$ is called the first moment of area about the y-axis and $A\bar{y}$ is called the first moment of area about the x-axis.

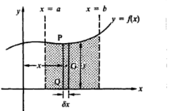
Considering A as the sum of a number of smaller areas of which δA, centroid (x, y), is typical,

then $A\bar{x} = \lim_{\delta A \to 0} \sum (\delta A \times x)$

and $A\bar{y} = \lim_{\delta A \to 0} \sum (\delta A \times y)$



Suppose we wish to find the coordinates of the centroid of an area under a curve $y = f(x)$ from $x = a$ to $x = b$.
Consider the total area A, centroid $(\bar{x}, \bar{y})$, divided into elementary strips like PQ, parallel to the y-axis and at a distance x from it. For small values of δx, the centroid of PQ can be considered to be at G $\left(x, \frac{y}{2}\right)$ and if the area of the strip is δA, it follows that $\delta A \approx y\,\delta x$.



Considering the moment of area about the y-axis

$$A\bar{x} = \lim_{\delta A \to 0} \sum_{x=a}^{x=b} x\,\delta A$$

$$= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\,\delta x \quad \text{because as} \quad \delta A \to 0,\ \delta x \to 0$$

$$= \int_{x=a}^{x=b} xy\,dx$$

Similarly, considering the moment of area about the x-axis

$$A\bar{y} = \lim_{\delta A \to 0} \sum_{x=a}^{x=b} \frac{y}{2}\,\delta A$$

$$= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{y^2}{2}\,\delta x$$

$$= \int_{x=a}^{x=b} \frac{y^2}{2}\,dx$$

Thus $\boxed{A\bar{x} = \int_{x=a}^{x=b} xy\,dx \quad \text{and} \quad A\bar{y} = \int_{x=a}^{x=b} \frac{y^2}{2}\,dx}$

Example 13

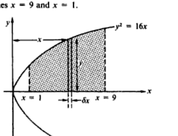
Find the coordinates of the centroid of the area lying in the first quadrant and enclosed by the curve $y^2 = 16x$, the x-axis and the lines $x = 9$ and $x = 1$.

Thus $\quad A\bar{x} = \int_1^9 xy\,dx \quad \ldots [1]$

and $\quad A = \int_1^9 y\,dx \quad \ldots [2]$

From [2] $\quad A = \int_1^9 4x^{1/2}\,dx$

$$= \frac{8}{3}(27 - 1) = \frac{8 \times 26}{3}$$



Substituting this into [1] gives

$$\frac{8 \times 26}{3}\bar{x} = \int_1^9 4x^{3/2}\,dx$$

$$= \frac{242 \times 8}{5}$$

$\therefore \qquad \bar{x} = \frac{363}{65}$ or 5·58 correct to 2 decimal places

Considering moment of area about the x-axis

$$A\bar{y} = \int_{x=1}^{x=9} \frac{y^2}{2}\,dx$$

$$\frac{8 \times 26}{3}\bar{y} = \int_1^9 8x\,dx \quad \text{(substituting for } y \text{ from } y^2 = 16x\text{)}$$

$$= 320$$

$\therefore \qquad \bar{y} = \frac{60}{13}$ or 4·62 correct to 2 decimal places

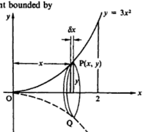
Notes: (i) In the last example the area considered had no lines of symmetry and so both coordinates $\bar{x}$ and $\bar{y}$ had to be calculated using integration.

(ii) For a body of uniform density the centroid of the body will coincide with its centre of gravity.

A similar technique may be used to determine the centroid of a solid of revolution.

Example 14

Find the centroid of the volume of revolution formed by rotating, through one revolution about the x-axis, the area in the first quadrant bounded by the curve $y = 3x^2$, the x-axis and the line $x = 2$.



The centroid of the volume of revolution will lie on the axis of symmetry, i.e. the x-axis. Consider an elementary disc PQ formed by rotating an elementary strip, thickness δx and radius y, through one revolution about the x-axis.

Volume of elementary disc PQ $= \pi y^2\,\delta x$

moment of PQ about y-axis $= x\pi y^2\,\delta x$

Taking moments about the y-axis:

$$V\bar{x} = \int_0^2 \pi x y^2\,dx \quad \text{where} \quad V = \int_0^2 \pi y^2\,dx$$

$$\bar{x}\int_0^2 \pi 9x^4\,dx = \int_0^2 9\pi x^5\,dx \quad \text{substituting for } y \text{ from } y = 3x^2$$

giving $\qquad \bar{x} = \frac{5}{3}$

The centroid of the volume of revolution is at the point $(1\frac{2}{3}, 0)$.

Exercise 12D

Displacement, velocity and acceleration

In questions 1 to 6, s metres, v m/s and a m/s^2 represent respectively the displacement, velocity and acceleration of a body, relative to an origin O, at time t seconds.

1. If $s = 5t^3 - t$ find expressions for v and a in terms of t.

2. If $s = 4t^2 - t^3$ find the displacement, velocity and acceleration when (a) $t = 0$, (b) $t = 2$.

3. If $v = 3t^2 - 8t$ and $s = 3$ when $t = 0$, find expressions for a and s in terms of t.

4. If $v = t^2 - 4t + 3$ and $s = 4$ when $t = 3$, find
(a) the values of t when the body is at rest,
(b) the acceleration when $t = 5$,
(c) the displacement when $t = 1$.

5. If $a = 1 - t$ and, when $t = 2$, $v = 1$ and $s = 4\frac{2}{3}$ find expressions for v and s in terms of t.

6. If $a = 6t - 12$ and, when $t = 0$, $v = 9$ and $s = 6$ find the values of t when the body is at rest and the displacement of the body from O at these times.

7. A body starts from rest at an origin O and its acceleration at any time t seconds later is given by $a = (3 - 2t)$ m/s^2. Find the displacement of the body from O when it is next at rest.

8. A body starts from rest at an origin O and its acceleration at time t seconds later is given by
$a = t + 3$ for $0 \leqslant t \leqslant 6$ and $a = 3t/2$ for $t \geqslant 6$.
Find an expression for s, the displacement from O at time t for $t \geqslant 6$ and hence find s when $t = 8$.

Centroids

Find the coordinates of the centroids of the following areas.

9. The area enclosed by the curve $y = 4 - x^2$ and the x-axis.

10. The area enclosed by the curve $y = 3x - x^2$ and the x-axis.

11. The area bounded by the curve $y = x^2 + 2$, $x = -2$, $x = 2$ and the x-axis.

12. The area bounded by the curve $y = x^2$, $x = 3$ and the x-axis.

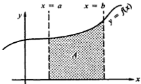
13. The area bounded by the curve $y = x^3$, $x = 3$ and the x-axis.

Find the centroid of the solid of revolution formed by rotating the following areas one revolution about the x-axis.

14. The area bounded by the curve $y = x^2$, the x-axis and the line $x = 4$.

15. The area between $y = x^3 + 1$ and the x-axis from $x = 0$ to $x = 2$.

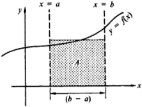
12.5 Mean value of a function



Consider the area under a curve $y = f(x)$ from $x = a$ to $x = b$.

If this area is A, then $A = \int_{x=a}^{x=b} y\,dx$.

Now consider a rectangle of the same area A and of the same base $(b - a)$. The height of this rectangle would equal the mean value of the function $f(x)$ in the range $a \leqslant x \leqslant b$
i.e. (mean value of $f(x)$ in $a \leqslant x \leqslant b$) $\times (b - a) = A$



Thus we define the mean value of a function $y = f(x)$ in the range $a \leqslant x \leqslant b$ as $\frac{1}{(b-a)}\int_a^b f(x)\,dx$.

Example 15

Find the mean value with respect to x of the function $(5x^2 - 4x)$ for $1 \leqslant x \leqslant 3$.

Given: $f(x) = 5x^2 - 4x$

By definition, the mean value is
$$\frac{1}{(b-a)}\int_a^b f(x)\,dx$$
$$= \frac{1}{(3-1)}\int_1^3 (5x^2 - 4x)\,dx$$
$$= \tfrac{1}{2}\left[\tfrac{5}{3}x^3 - 2x^2\right]_1^3$$
$$= \tfrac{1}{2}(\tfrac{5}{3} \times 27 - 2 \times 9 - \tfrac{5}{3} \times 1 + 2 \times 1)$$
$$= 27\tfrac{1}{3}$$

The mean value of $(5x^2 - 4x)$ over the range $1 \leqslant x \leqslant 3$ is $27\tfrac{1}{3}$.

Example 16

A thin rod of length $2l$ is of variable density. Calculate the mean value of the density of the rod, if at a distance x from one end, the density of the rod is cx^2, where c is a constant.

Given that the density is a function of x such that
$$f(x) = cx^2 \quad \text{for} \quad 0 \leqslant x \leqslant 2l$$
$$\text{mean value of density} = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$
$$= \frac{1}{(2l-0)}\int_0^{2l} cx^2\,dx$$
$$= \frac{1}{2l}\left[\frac{cx^3}{3}\right]_0^{2l}$$
$$= \frac{1}{2l}\left(\frac{8l^3c}{3}\right)$$
$$= \frac{4l^2c}{3}$$

The mean value of the density of the rod is $\frac{4l^2c}{3}$.

Exercise 12E

1. Find the mean value of x^2 for $0 \leqslant x \leqslant 2$.
2. Find the mean value of $2x + 3$ for $1 \leqslant x \leqslant 7$.
3. Find the mean value of x^3 for $0 \leqslant x \leqslant 4$.
4. Find the mean value of $3x^2 + 2x$ for $0 \leqslant x \leqslant 2$.
5. Find the mean value of $\frac{1}{x^2}$ for $1 \leqslant x \leqslant 5$.
6. Find the mean value of $x^2 + 4$ for $-2 \leqslant x \leqslant 3$.
7. Find the mean value of $4 - x^2$ for $-2 \leqslant x \leqslant 2$.
8. Find the mean value of $4x^3 - 6x^2 + 2x - 1$ for $-1 \leqslant x \leqslant 3$.
9. The velocity v in m/s of a body t seconds after timing commences is given by $v = 3t^2 - 4$.
   (a) Find the mean velocity during the interval $t = 1$ to $t = 3$.
   (b) Find the mean acceleration during the interval $t = 1$ to $t = 3$.
10. The tension T newtons in a particular spring depends on the extension x metres of the spring from its natural length in accordance with the rule $T = 30x$. Find the mean tension in the spring as x increases from 0·1 m to 0·2 m.
11. The kinetic energy k joules of a 10 kg body depends on the velocity v m/s in accordance with the rule $k = 5v^2$. Find the mean kinetic energy possessed by the body as v increases from 1 m/s to 7 m/s.

Exercise 12F Examination questions

1. Integrate with respect to x:
   (i) $\frac{1}{x^3}$ (ii) $\sqrt{x} + \frac{1}{\sqrt{x}}$. (S.U.J.B.)
2. Evaluate $\int_1^2 \frac{x^3 + 1}{x^2}\,dx$. (S.U.J.B.)
3. (a) Integrate with respect to x,
   $(\sqrt{x} - x)^2$
   (b) Evaluate
   $$\int_1^2 \left(3x + \frac{1}{x^2} - \frac{1}{x^4}\right)dx$$ (A.E.B.)
4. At any point (x, y) on a certain curve,
   $$\frac{dy}{dx} = (3x - 2)(x + 2).$$
   Given that it passes through $(-1, 1)$ find the equation of the curve. Find, and distinguish between, the turning points of the curve. (A.E.B.)
5. A particle P moves in a straight line and passes a fixed point O with a velocity of V m/s. Its acceleration, a m/s², is given by $a = 16 - 4t$ for $0 \leqslant t \leqslant 3$ and $a = t + 1$ for $t \geqslant 3$, where t is the time in seconds after passing O. Given that the velocity of P when $t = 3$ is 38 m/s, find
   (i) the value of V,
   (ii) the velocity of P when $t = 4$. (Cambridge)

11. Find the equation of the chord which joins the points $A(-2, 3)$ and $B(0, 15)$ on the curve $y = 15 - 3x^2$.
(a) Show that the finite area enclosed by the curve and the chord AB is 4 square units.
(b) Find the volume generated when this area is rotated through 360° about the x-axis, leaving your answer in terms of π. (A.E.B.)

12. Given the curve whose equation is

$$y = x^{-1/2},$$

find (i) the mean value of $\frac{1}{y}$, with respect to x, in the interval $1 \leqslant x \leqslant 4$;
(ii) the area of the region R bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$; (J.M.B.)

13. Sketch the curve with equation

$$y = (x^2 - 1)(x + 2).$$

(a) Calculate the area of the finite region above the x-axis bounded by the curve and the x-axis.
(b) Find the coordinates of the point of inflexion on the curve and the equation of the tangent at this point. (London)

14. The points P(3, 2) and Q(0, 1) lie on the curve $y^2 = x + 1$. Calculate the volume of the solid generated when the region bounded by the lines $y = 0$, $x = 0$, $x = 3$ and the arc PQ of the curve is rotated completely about the x-axis. Give your answer as a multiple of π.
S is the region bounded by the lines $y = 1$, $x = 3$ and the arc PQ of the curve. Show that when S is rotated completely about the x-axis the volume of the solid generated is $\frac{9\pi}{2}$.
When S is rotated completely about the line $x = 3$ show that the volume of the solid generated is

$$\pi \int_1^2 (4 - y^2)^2 dy.$$

Calculate this volume in terms of π. (J.M.B.)

13
Calculus III: Further techniques

13.1 Function of a function

When $y = x^2 + 4$, we say that y is a function of x.
If $y = (x^2 + 4)^5$, we say that y is a function (the fifth power) of a function $(x^2 + 4)$ of x.
Suppose that $y = u^5$ and that $u = x^2 + 4$, then this again leads to $y = (x^2 + 4)^5$. In this case we have used another variable u which links together the variables y and x.
Suppose $y = f(u)$ and $u = g(x)$ and let x change by a small amount δx, and the consequent change in u be δu and the change in y be δy.

Then $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$

As $\delta x \to 0$, so also $\delta u \to 0$ and $\delta y \to 0$.
So taking limits as $\delta x \to 0$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is also referred to as the **chain rule**.
The introduction of a third variable may be used to enable us to differentiate a function of a function.

Example 1

Find $\dfrac{dy}{dx}$ if (a) $y = (3x^2 - 2)^4$, (b) $y = \dfrac{1}{\sqrt{(x^2 - 2)}}$

(a)
$$y = (3x^2 - 2)^4$$
Let $u = 3x^2 - 2$ then $y = u^4$

$\therefore$ $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = 4u^3$

Using
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 4u^3 \times 6x$$
$$\frac{dy}{dx} = 24x(3x^2 - 2)^3$$

(b)
$$y = \frac{1}{\sqrt{(x^2 - 2)}} = (x^2 - 2)^{-1/2}$$
Let $u = x^2 - 2$ then $y = u^{-1/2}$

$\therefore$ $\dfrac{du}{dx} = 2x$ and $\dfrac{dy}{du} = -\dfrac{1}{2}u^{-3/2} = -\dfrac{1}{2u^{3/2}}$

Using
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{2u^{3/2}} \times 2x$$
$$\frac{dy}{dx} = -\frac{x}{(x^2 - 2)^{3/2}}$$

Note: The final answer should always be given in terms of the variable used in the question and not the third variable, u, which was of our invention.

In general if $y = [f(x)]^n$, letting $u = f(x)$ then $y = u^n$

then $\frac{du}{dx} = f'(x)$ and $\frac{dy}{du} = nu^{n-1}$

Thus if $y = [f(x)]^n$, $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

With some practice it is quite possible, and permissible, to write down the answer to differentiations of this type without showing the substitution and the introduction of the third variable.

Example 2

Find $\frac{dy}{dx}$ if (a) $y = (4x^3 - 7x)^6$, (b) $y = \sqrt{(5x - 2x^2)}$, (c) $y = \frac{1}{3x^3 - 4x}$.

(a) $y = (4x^3 - 7x)^6$

$\frac{dy}{dx} = 6(4x^3 - 7x)^5(12x^2 - 7)$

(b) $y = \sqrt{(5x - 2x^2)}$

$= (5x - 2x^2)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(5x - 2x^2)^{-1/2} \times (5 - 4x)$

i.e. $\frac{dy}{dx} = \frac{5 - 4x}{2\sqrt{(5x - 2x^2)}}$

(c) $y = \frac{1}{3x^3 - 4x} = (3x^3 - 4x)^{-1}$

$\frac{dy}{dx} = (-1)(3x^3 - 4x)^{-2} \times (9x^2 - 4)$

i.e. $\frac{dy}{dx} = \frac{4 - 9x^2}{(3x^3 - 4x)^2}$

Example 3

Find the equation of (a) the tangent and (b) the normal to the curve $y = \frac{5}{x^2 - 3}$ at the point (2, 5).

Note Substituting $x = 2$ in the equation of the curve gives $y = \frac{5}{2^2 - 3} = 5$.

So the point (2, 5) does lie on the curve.

(a) $y = \frac{5}{x^2 - 3} = 5(x^2 - 3)^{-1}$

$\therefore \frac{dy}{dx} = 5(-1)(x^2 - 3)^{-2} \times (2x)$

$= \frac{-10x}{(x^2 - 3)^2}$

Gradient of tangent at the point (2, 5) is $\frac{-10(2)}{(2^2 - 3)^2} = -20$.

Using $y - y_1 = m(x - x_1)$, equation of tangent at (2, 5) is $y - 5 = -20(x - 2)$

or $y + 20x = 45$

25. Find the coordinates of any stationary points on the following curves and state the nature of each.

(a) $y = \dfrac{1}{1 + x^2}$ (b) $y = (2x - 5)^4$

(c) $y = (2x - 5)^3$ (d) $y = \dfrac{1}{x^2 + 4x}$

26. The displacement, s metres, of a body from an origin O at time t seconds is given by $s = \sqrt{(1 + 2t)}$. When $t = 12$ find
(a) the displacement from O (b) the velocity and
(c) the acceleration of the body.

13.2 Integration of $f'(x)[f(x)]^n$

Although we have established a rule for integrating a power of x, i.e. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$, there are few such rules and we are dependent upon our ability to recognize the type of expression to be integrated.

Consider $\int 24x^3(x^4 + 7)^5 dx$.

If we go back to our idea of integration being the reverse of differentiation, we can see that this integrand, i.e. the expression which is to be integrated, may have come from differentiating $(x^4 + 7)^6$.
The expression $(x^4 + 7)^6$ is a function of a function and as we saw in section 13.1, the differential of this expression is $6(x^4 + 7)^5 \times (4x^3)$ or $24x^3(x^4 + 7)^5$.
Thus the integrand is exactly the differential of $(x^4 + 7)^6$ with respect to x, and hence

$$\int 24x^3(x^4 + 7)^5 dx = (x^4 + 7)^6 + c.$$

This is rather an artificial example in that the numbers have been carefully chosen to produce the desired result. The method would still work had a number other than 24 appeared in the integrand.
In this type of integration we are expecting to see an integrand which has come from differentiating a function of a function of x. The key to the method lies in recognizing that the integrand is a function to some power multiplied by the differential of that function (or some scalar multiple of it). We can state this in general terms:

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c.$$

It is better *not* to remember such a statement, but rather to understand how the integrand has been built up by differentiating a function of a function.

Example 5

Find (a) $\int 20(7 + 5x)^3\,dx$ (b) $\int (2 - 3x)^6\,dx$.

(a) $\int 20(7 + 5x)^3\,dx$

We suspect that this may have come from differentiating $(7 + 5x)^4$ as a function of a function. This would give $4(7 + 5x)^3 \times (5)$ which is exactly the integrand,

$$\therefore \quad \int 20(7 + 5x)^3\,dx = (7 + 5x)^4 + c$$

(b) $\int (2 - 3x)^6\,dx$

We suspect that this may have come from differentiating $(2 - 3x)^7$ as a function of a function. This would give $7(2 - 3x)^6 \times (-3)$ which is $-21(2 - 3x)^6$ and this is a scalar multiple of the integrand,

$$\therefore \quad \int (2 - 3x)^6\,dx = -\frac{1}{21}(2 - 3x)^7 + c$$

Example 6

Find (a) $\int x^4(1 + 2x^5)^3\,dx$ (b) $\int (4x - 2)(x^2 - x + 4)^5\,dx$.

(a) $\int x^4(1 + 2x^5)^3\,dx$

If this has come from differentiating $(1 + 2x^5)^4$, then we expect to see the differential of $(1 + 2x^5)^4$, or a multiple of it, in the integrand.

Now $\frac{d}{dx}(1 + 2x^5)^4$ is $40x^4(1 + 2x^5)^3$,

$$\therefore \quad \int x^4(1 + 2x^5)^3\,dx = \frac{1}{40}(1 + 2x^5)^4 + c$$

(b) $\int (4x - 2)(x^2 - x + 4)^5\,dx$

We suspect this may have come from differentiating $(x^2 - x + 4)^6$.

Now $\frac{d}{dx}(x^2 - x + 4)^6$

$= 6(x^2 - x + 4)^5 \times (2x - 1)$

$$\therefore \quad \int (4x - 2)(x^2 - x + 4)^5\,dx = \tfrac{1}{3}(x^2 - x + 4)^6 + c$$

Note Integrations of the type shown in Examples 5 and 6 can also be performed by the method of substitution which is explained in more detail in chapter 20.

Exercise 13B

Find

1. $\int 24(4 + 3x)^7\,dx$
2. $\int 42(2 + 7x)^5\,dx$
3. $-\int 14(3 - 2x)^6\,dx$
4. $\int 10x(x^2 + 4)^4\,dx$
5. $\int (2 + 3x)^5\,dx$
6. $\int (1 + 2x)^4\,dx$
7. $\int (1 - 6x)^3\,dx$
8. $\int 2(3 + 4x)^4\,dx$
9. $\int 6(1 - 2x)^4\,dx$
10. $\int 2x(1 + x^2)^3\,dx$
11. $\int x^2(1 - x^3)^4\,dx$
12. $\int x^3(x^4 + 6)^3\,dx$
13. $\int (2x + 1)(x^2 + x + 3)^4\,dx$
14. $\int (3x^2 - 1)(x^3 - x + 4)^4\,dx$

15. $\int (1 + 2x)(4 + x + x^2)^5\,dx$ **16.** $\int (x - 1)(x^2 - 2x + 4)^7\,dx$

Evaluate the following definite integrals

17. $\int_0^1 16x(x^2 + 5)^3\,dx$ **18.** $\int_1^7 (x - 7)^5\,dx$

19. $\int_0^1 x(x^2 + 4)^4\,dx$ **20.** $\int_{-1}^1 (3 + 2x)^5\,dx$

21. $\int_1^2 (2x - 1)(x^2 - x + 1)^5\,dx$ **22.** $\int_0^2 (1 - 2x)(x^2 - x - 3)^3\,dx$

23. $\int_1^4 \frac{(1 + \sqrt{x})^5}{\sqrt{x}}\,dx$

13.3 Related rates of change

Suppose that y is expressed as a function of x, say $y = f(x)$. If we know the rate at which x changes with respect to some other variable, say time t $\left(\text{i.e. we know } \frac{dx}{dt}\right)$, then we can use the chain rule to find the rate of change of y with respect to this third variable t, $\left(\text{i.e. we can find } \frac{dy}{dt}\right)$.

i.e. Given $y = f(x)$ and $\frac{dx}{dt}$, we can find $\frac{dy}{dt}$ by using
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = f'(x)\frac{dx}{dt}.$$

Example 7

If the radius r of a sphere is increasing at 2 cm/s, find the rate at which the volume of the sphere is increasing when the radius is 3 cm (leave your answer in terms of π).

We know that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ and we are given that $\frac{dr}{dt} = 2$ cm/s

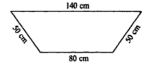
We require $\frac{dV}{dt}$; using
$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \times \frac{dr}{dt}$$
$$= \frac{4}{3}\pi \times 3r^2 \times 2$$
$$= 8\pi r^2 \text{ cm}^3/\text{s}$$

Thus when $r = 3$ cm, $\frac{dV}{dt} = 72\pi$ cm^3/s

When the radius of the sphere is 3 cm, the volume of the sphere is increasing at 72π cm^3/s.

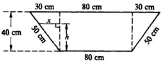
Example 8

Water is pumped into an empty trough which is 200 cm long, at the rate of 33 000 cm³/s. The uniform cross-section of the trough is an isosceles trapezium with the dimensions shown. Find the rate at which the depth of the water is increasing at the instant when this depth is 20 cm.



We require $\frac{dh}{dt}$; thus, given $\frac{dV}{dt}$, we want a formula linking V and h so that we can use the chain rule.

The total depth of the trough is 40 cm.



When the depth of water is h, $\frac{x}{30} = \frac{h}{40}$

or $x = \frac{3}{4}h$

The volume V of water in the trough when the depth of water is h, is given by

$$V = \tfrac{1}{2}(80 + 80 + 2x) \times h \times 200 \text{ cm}^3$$
$$= (160 + \tfrac{3}{2}h) \times 100h$$

Using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{d}{dh}(16\,000h + 150h^2) \times \frac{dh}{dt}$$
$$= (16\,000 + 300h) \times \frac{dh}{dt}$$

But $\frac{dV}{dt} = 33\,000$ cm³/s, so when $h = 20$ cm,

$$33\,000 = (16\,000 + 300(20)) \times \frac{dh}{dt}$$

i.e. $\frac{dh}{dt} = 1\tfrac{1}{2}$ cm/s

When the depth of water is 20 cm, the depth is increasing at the rate of $1\frac{1}{2}$ cm/s.

Note that in the last example we could have used $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$, where

$$\frac{dV}{dt} = 33\,000 \quad\text{and}\quad \frac{dh}{dV} = 1\Big/\frac{dV}{dh} = \frac{1}{16\,000 + 300h}$$
$$\frac{dh}{dt} = \frac{33\,000}{16\,000 + 300h}$$
$$= 1\tfrac{1}{2} \text{ cm/s} \quad \text{when} \quad h = 20 \text{ cm.}$$

The justification that $\frac{dh}{dV} = 1\Big/\frac{dV}{dh}$, or that $\frac{dy}{dx} = 1\Big/\frac{dx}{dy}$, is as follows.

By definition, $\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{1}{\left(\frac{\delta x}{\delta y}\right)}\right] \qquad \ldots [1]$$

but as $\delta x \to 0$, then so also $\delta y \to 0$. Thus [1] can be written

$$\frac{dy}{dx} = \lim_{\delta y \to 0}\left[1 \bigg/ \frac{\delta x}{\delta y}\right]$$

Thus $$\boxed{\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \quad \text{as required.}}$$

Exercise 13C

1. If $T = 5p^2 + \frac{3}{p}$, find $\frac{dT}{dq}$ when $p = 2$ given that for that value of p, $\frac{dp}{dq} = 4$.
2. If $A = (2t + 3)^5$ and $x = 2t^2 + 6t$, find an expression for $\frac{dA}{dx}$ in terms of t.
3. If r, the radius of a circle, increases at the rate of 2 cm/s, find an expression in terms of r for the rate at which the area of the circle is increasing.
4. If the radius of a sphere is increasing at 1 cm/s, find the rate at which the surface area is increasing when the radius is 5 cm. (Leave π in your answer.)
5. Air is being pumped into a spherical balloon at a rate of 54 cm^3/s. Find the rate at which the radius is increasing when the volume of the balloon is 36π cm^3.
6. If the volume of a sphere increases at the rate of 6 cm^3/s, find the rate of increase in the surface area of the sphere at the instant when its radius is 4 cm.
7. Oil is dripping onto a surface at the rate of $\frac{1}{10}\pi$ cm^3/s and forms a circular film which may be considered to have a uniform depth of 0·1 cm. Find the rate at which the radius of the circular film is increasing when this radius is 5 cm.
8. A closed right-circular cylinder has base radius r cm and height $3r$ cm. If r is increased at a rate of 1 millimetre per second, find expressions in terms of r for the rate of increase of
   (a) the total external surface area and (b) the volume of the cylinder.
9. A hollow cone of base radius a and height $3a$ is held vertex downwards. The cone is initially empty and liquid is poured into it at a rate of 4π cm^3/s. Find the rate at which the depth of the liquid in the vessel is increasing 16 seconds after the pouring commenced.
10. A container is in the shape of a cone of semi-vertical angle 30°, with its vertex downwards. Liquid flows into the container at the rate of $\frac{\sqrt{3}\pi}{4}$ cm^3/s. At the instant when the radius of the circular surface of

Example 10

Find $\frac{dy}{dx}$ in terms of the parameter t if: (a) $x = 2t^3, y = 4t^2 + 1$ (b) $x = \frac{3}{t}, y = \sqrt{(1 + t^2)}$

(a) $x = 2t^3, \quad y = 4t^2 + 1$

$\therefore \quad \frac{dx}{dt} = 6t^2, \frac{dy}{dt} = 8t$

Using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 8t \times \frac{1}{6t^2}$

i.e. $\frac{dy}{dx} = \frac{4}{3t}$

(b) $x = \frac{3}{t} = 3t^{-1}, \quad y = \sqrt{(1 + t^2)} = (1 + t^2)^{1/2}$

$\therefore \quad \frac{dx}{dt} = \frac{-3}{t^2}, \qquad \frac{dy}{dt} = \frac{1}{2}(1 + t^2)^{-1/2} \times 2t$

Using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{1}{2}(1 + t^2)^{-1/2} \times 2t \times \left(\frac{t^2}{-3}\right)$

i.e. $\frac{dy}{dx} = \frac{-t^3}{3\sqrt{(1 + t^2)}}$

The last example made use of the fact that $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$, which was proved on page 324.

Second differential

Particular care is needed when finding the second differential from the parametric equations.
One method is to find the cartesian equation by eliminating the parameter and then to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the usual way.
Alternatively, it is often better to work in terms of the parameter throughout as the following example illustrates.

Example 11

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t given that $x = \frac{1}{t}, y = 3t^2 + 2$.

$x = \frac{1}{t} = t^{-1} \qquad y = 3t^2 + 2$

$\frac{dx}{dt} = \frac{-1}{t^2} \qquad \frac{dy}{dt} = 6t$

Using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 6t \times (-t^2) \qquad \left(\frac{dx}{dt} = \frac{-1}{t^2}, \text{ hence } \frac{dt}{dx} = -t^2\right)$

$= -6t^3$

In order to find $\frac{d^2y}{dx^2}$, we need to differentiate $\frac{dy}{dx}$ with respect to x, so we must therefore differentiate $(-6t^3)$ with respect to x.

$\frac{d^2y}{dx^2} = \frac{d}{dx}(-6t^3)$

$= \frac{d}{dt}(-6t^3) \times \frac{dt}{dx} \qquad$ (Chain rule used to enable $-6t^3$ to be differentiated with respect to t)

27. Copy and complete the following table for $x = 4 \sin \theta$, $y = \cos \theta$ (where θ is in radians).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x			3·46		3·46				−3·46		−3·46		
y		0·87				−0·87		−0·87				0·87	

Draw x- and y-axes and hence plot the graph of the curve $x = 4 \sin \theta$, $y = \cos \theta$ for $0 \leqslant \theta \leqslant 2\pi$.

28. Copy and complete the following table for $x = 2t(t^2 - 1)$, $y = 4t^2$

t	−2	$-1\frac{1}{2}$	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
x									
y									

Draw x- and y- axes and hence plot the graph of the curve $x = 2t(t^2 - 1)$, $y = 4t^2$ for $-2 \leqslant t \leqslant 2$.

13.5 Product rule

Suppose u and v are functions of x and

$$y = uv \qquad \ldots [1]$$

Let x change by a small amount δx and let the consequent change in u be δu, in v be δv and in y be δy.

Then
$$y + \delta y = (u + \delta u)(v + \delta v) \qquad \ldots [2]$$

Subtracting equation [1] from equation [2],

$$y + \delta y - y = (u + \delta u)(v + \delta v) - uv$$

i.e.
$$\delta y = u\delta v + v\delta u + \delta u \delta v$$

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

By definition, as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta u}{\delta x} \to \frac{du}{dx}$;

also as $\delta x \to 0$, both δu and δv approach 0.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + 0 \times \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This important result should be memorised.

Example 12

Differentiate the following with respect to x (a) $y = x(x + 3)^4$ (b) $y = (2x + 1)^3(x - 1)^4$

(a) $y = x(x + 3)^4$

This is of the form $y = uv$ where $u = x$ and $v = (x + 3)^4$

Using the product rule

$$\frac{dy}{dx} = x \times 4(x + 3)^3 + (x + 3)^4 \times 1$$
$$= (x + 3)^3[4x + x + 3]$$
$$= (x + 3)^3(5x + 3)$$

(b) $y = (2x + 1)^3(x - 1)^4$

This is of the form $y = uv$ where $u = (2x + 1)^3$ and $v = (x - 1)^4$

Using the product rule

$$\frac{dy}{dx} = (2x + 1)^3 \times 4(x - 1)^3 + (x - 1)^4 \times 3(2x + 1)^2 \times 2$$
$$= 2(2x + 1)^2(x - 1)^3[2(2x + 1) + 3(x - 1)]$$
$$= 2(2x + 1)^2(x - 1)^3(7x - 1)$$

13.6 Quotient rule

Suppose u and v are functions of x and that $y = \dfrac{u}{v}$.

Writing this as $y = uv^{-1}$, we can use the product rule to give

$$\frac{dy}{dx} = u\frac{d}{dx}(v^{-1}) + v^{-1}\frac{du}{dx}$$
$$= u \times -1(v^{-2})\frac{dv}{dx} + \frac{1}{v} \times \frac{du}{dx} \qquad \left[\text{by chain rule } \frac{d}{dx}(v^{-1}) = \frac{d}{dv}(v^{-1}) \times \frac{dv}{dx}\right]$$
$$= \frac{-u\dfrac{dv}{dx} + v\dfrac{du}{dx}}{v^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This important result should be memorised.

Example 13

Differentiate the following with respect to x:

(a) $y = \dfrac{2x + 3}{1 - 5x}$ (b) $y = \dfrac{(3x + 1)^4}{(5x - 2)^3}$

(a) $y = \dfrac{2x + 3}{1 - 5x}$

This is of the form $y = \dfrac{u}{v}$ where

$u = 2x + 3$ and $v = 1 - 5x$.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{(1 - 5x)2 - (2x + 3)(-5)}{(1 - 5x)^2}$$
$$= \frac{2 - 10x + 10x + 15}{(1 - 5x)^2}$$
$$\therefore \quad \frac{dy}{dx} = \frac{17}{(1 - 5x)^2}$$

(b) $y = \dfrac{(3x + 1)^4}{(5x - 2)^3}$

This is of the form $y = \dfrac{u}{v}$ where

$u = (3x + 1)^4$ and $v = (5x - 2)^3$.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{(5x - 2)^3 4(3x + 1)^3(3) - (3x + 1)^4 3(5x - 2)^2(5)}{(5x - 2)^6}$$
$$= \frac{3(5x - 2)^2(3x + 1)^3}{(5x - 2)^6}[4(5x - 2) - 5(3x + 1)]$$
$$\therefore \quad \frac{dy}{dx} = \frac{3(3x + 1)^3}{(5x - 2)^4}(5x - 13).$$

Note In the differentiation of products and quotients, particular care over the algebraic simplification is necessary.

13.7 Implicit functions

An implicit function is one in which a relationship between, say, two variables x and y is given without having y as an explicit or clearly defined function of x.
Thus in $y = 3x^2 - 7x + 1$, y is given as an explicit function of x, whereas in $y^2 - 3yx = x^2$, y is not given explicitly as a function of x.
An implicit function involving y and x can be differentiated with respect to x as it stands.
Suppose $\quad y^2 + 4x = 6x^2,$
Differentiating each term with respect to x,

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(4x) = \frac{d}{dx}(6x^2)$$

i.e. $$\frac{d}{dy}(y^2)\frac{dy}{dx} + \frac{d}{dx}(4x) = \frac{d}{dx}(6x^2)$$

Thus $$2y\frac{dy}{dx} + 4 = 12x$$

It should be noted that we had to rearrange $\frac{d}{dx}(y^2)$ using the chain rule, $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$ to enable y^2 to be differentiated with respect to y.

Example 14

Find $\frac{dy}{dx}$ in terms of x and y if: (a) $y^3 + 6x = x^2$ (b) $3y^2 + 2y + xy = x^3$.

(a) $y^3 + 6x = x^2$
Differentiating each term with respect to x,

$$3y^2\frac{dy}{dx} + 6 = 2x$$

i.e. $$\frac{dy}{dx} = \frac{2x - 6}{3y^2}$$

(b) $3y^2 + 2y + xy = x^3$
Differentiating each term with respect to x,

$$6y\frac{dy}{dx} + 2\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right) = 3x^2$$

$\therefore$ $$\frac{dy}{dx}(6y + 2 + x) = 3x^2 - y$$

i.e. $$\frac{dy}{dx} = \frac{3x^2 - y}{x + 6y + 2}.$$

Example 15

Find the equation of (a) the tangent and (b) the normal to the curve $3x^2 - xy - 2y^2 + 12 = 0$ at the point (2, 3).

$3x^2 - xy - 2y^2 + 12 = 0$

[Note that since $3(2)^2 - 2(3) - 2(3)^2 + 12$
$= 12 - 6 - 18 + 12$
$= 0$, the point (2, 3) does lie on the curve.]

(a) Differentiating the equation of the curve with respect to x,

$$6x - x\frac{dy}{dx} - y - 4y\frac{dy}{dx} = 0$$

thus $$\frac{dy}{dx} = \frac{6x - y}{x + 4y}$$

Gradient at the point (2, 3) $= \dfrac{6(2) - 3}{2 + 4(3)}$

$$= \frac{9}{14}$$

Equation of tangent at (2, 3) is $y - 3 = \frac{9}{14}(x - 2)$

or $14y = 9x + 24$.

(b) Gradient of normal is $\dfrac{-1}{(\text{gradient of tangent})} = -\dfrac{14}{9}$ at the point (2, 3).

Equation of normal at (2, 3) is $y - 3 = -\frac{14}{9}(x - 2)$

or $9y + 14x = 55$

The equation of the tangent is $14y = 9x + 24$ and the equation of the normal is $9y + 14x = 55$.

Exercise 13F

In questions **1** to **9**, find $\frac{dy}{dx}$ in terms of x and y.

1. $x^2 + y^2 = 10$
2. $2x^2 + y^2 = 4x$
3. $6x^2 + 2y^3 = 8x + 4y$
4. $2x^3 - 5y^2 + 6x - 10y = 6$
5. $2x^2 + 2y^2 + 3x = 10 + 7y$
6. $x^2 + xy + y^2 = 0$
7. $x^3 + 3xy - y^2 = 6$
8. $2x^3 + 3xy^2 - y^3 = 0$
9. $3x^2 + 2y^2 - 5x + xy + 6y = 8$
10. Find the gradient of the curve $x^2 + 6y^2 = 10$ at the point $(2, -1)$.
11. Find the gradient of the curve $x^3 + 4xy = 15 + y^2$ at the point (2, 1).

For questions **12** to **14**, find the equation of (a) the tangent and (b) the normal to the curve at the given point on the curve.

12. $5y^2 - 3x^2 - x + y = 0$ at the point $(1, -1)$
13. $3x^2 - 2xy + y^2 = 9$ at the point $(-2, -3)$
14. $x^3 + 3x^2y = 2y^2$ at the point $(-1, 1)$

9. A curve is defined parametrically by the equations

$$x = t^3 - 6t + 4, \; y = t - 3 + \frac{2}{t}.$$

Find

(i) the equations of the normals to the curve at the points where the curve meets the x-axis,

(ii) the coordinates of their point of intersection. (Cambridge)

10. A curve is given by the parametric equations:

$$x = \frac{(1 - t)}{(1 + t)}, \; y = (1 - t)(1 + t)^2.$$

Find dy/dx and d^2y/dx^2 in terms of t. Find also the equation of the tangent to the curve at the point where $t = 2$.

(S.U.J.B.)

11. Find the equations of the tangents to the curve $y^2 + 3xy + 4x^2 = 14$ at the points where $x = 1$. (S.U.J.B.)

12. Given that $y^2 - 5xy + 8x^2 = 2$, prove that $\dfrac{dy}{dx} = \dfrac{5y - 16x}{2y - 5x}$.

The distinct points P and Q on the curve $y^2 - 5xy + 8x^2 = 2$ each have x-coordinate 1. The normals to the curve at P and Q meet at the point N. Calculate the coordinates of N. (A.E.B.)

13. Sketch the curve given parametrically by

$$x = t^2, \; y = t^3.$$

Show that an equation of the normal to the curve at the point A(4, 8) is $x + 3y - 28 = 0$.

This normal meets the x-axis at the point N. Find the area of the region enclosed by the arc OA of the curve, the line segment AN and the x-axis. (London)

14
Sketching functions II

14.1 Rational functions

In this section we consider functions $f(x)$ of the type $f(x) = \frac{g(x)}{h(x)}$. The five basic investigations for curve sketching used in chapter 11 are still applicable for functions of this type, i.e.
(i) symmetry if obvious,
(ii) intersection with axes,
(iii) behaviour as $x \to \pm\infty$
(iv) $f(x)$ undefined,
(v) maximum/minimum.
Examples 1 to 7 show the ways in which these investigations give information from which a sketch of $\frac{g(x)}{h(x)}$ can be made. We restrict our attention to functions $\frac{g(x)}{h(x)}$ for which $h(x)$ is a polynomial of order 2 or less.
The reader should note that in the examples of this chapter, the behaviour of the function on either side of the vertical asymptotes is not investigated. These investigations are not always easy and the information gained from the other investigations usually allows the behaviour of the function near these asymptotes to be determined.

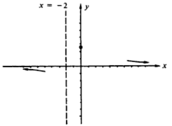
Type I $f(x) = \dfrac{a}{bx + c}$

Example 1

Make a sketch of the curve given by $y = \dfrac{5}{x + 2}$

x-axis No value of x for which $y = 0$
$\therefore$ no intercept with x-axis.
y-axis Cuts y-axis at $(0, 2\frac{1}{2})$
$x \to \pm\infty$ As $x \to \pm\infty$, $y \to \dfrac{5}{x}$

Thus for x a large positive number, y is small and positive, thus as $x \to +\infty$, $y \to 0^+$. For x a large negative number, y is small and negative, thus as $x \to -\infty$, $y \to 0^-$.

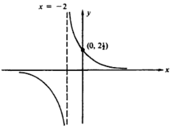


y undefined y is undefined for $x = -2$.
Thus $x = -2$ is a vertical asymptote.

Max/min $y' = \dfrac{-5}{(x+2)^2}$

Thus no turning points and the gradient is always negative.

The sketch can then be completed.



Type II $f(x) = \dfrac{ax+b}{cx+d}$

Example 2

Make a sketch of the curve given by $y = \dfrac{x+3}{x-1}$

x-axis Cuts x-axis at $(-3, 0)$

y-axis Cuts y-axis at $(0, -3)$

$x \to \pm\infty$ By writing $y = \dfrac{1 + 3/x}{1 - 1/x}$

As $x \to \pm\infty$, $y \to 1$

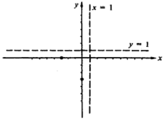
$\therefore$ $y = 1$ is a horizontal asymptote. (It is not necessary to consider $x \to +\infty$ and $x \to -\infty$ separately for functions of this type.)

y undefined y is undefined for $x = 1$.
Thus $x = 1$ is a vertical asymptote.

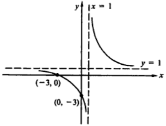
Max/min $y' = \dfrac{1(x-1) - 1(x+3)}{(x-1)^2}$

$= \dfrac{-4}{(x-1)^2}$

Thus no turning points and the gradient is always negative.



The sketch can then be completed.



Type III $f(x) = \dfrac{g(x)}{h(x)}$ for $h(x)$ a quadratic function.

Some functions of this type have a range that is restricted in some way for $x \in \mathbb{R}$. It is useful to examine the range of $f(x)$ first to determine whether such restrictions exist. In addition, such examinations will indicate whether any maximum or minimum points exist.
Note In the following examples, when $f(x)$ is an improper fraction, i.e. order of $g(x) \geqslant$ order of $h(x)$, then $f(x)$ is rearranged to eliminate these improper fractions when considering $x \to \pm\infty$.

Example 3

Sketch the curve given by $y = \dfrac{3x + 3}{x(3 - x)}$

If $y = \dfrac{3x + 3}{x(3 - x)}$ ... [1] then $y = \dfrac{3x + 3}{3x - x^2}$ ... [2]

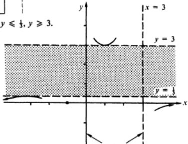
Range of values of y From [2] $3xy - x^2y = 3x + 3$
i.e. $x^2y + 3x(1 - y) + 3 = 0$

For $y \neq 0$, this is a quadratic in x.
Thus, for real x, $[3(1 - y)]^2 - 4y(3) \geqslant 0$
i.e. $(3y - 1)(y - 3) \geqslant 0$
We solve this inequality using the methods of chapter 5, page 143.

For $y = 0$, $3x + 3 = 0$,
$x = -1$
$\therefore$ y can equal zero.

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$3y - 1$	−ve	+ve	+ve
$y - 3$	−ve	−ve	+ve
$(3y - 1)(y - 3)$	+ve	−ve	+ve

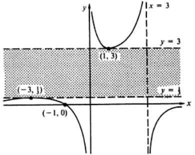
Thus the ranges of values y can take for real x are $y \leqslant \frac{1}{3}$, $y \geqslant 3$.
We can therefore shade a region on our sketch where the curve cannot exist. Note also that for each value of y in the allowed ranges, there will exist two distinct values of x, except at $y = 3$ and $y = \frac{1}{3}$ where a repeated root will occur, and at $y = 0$ where a single root occurs. From $y \leqslant \frac{1}{3}$ we expect a (local) maximum at $y = \frac{1}{3}$, and from $y \geqslant 3$ we expect a (local) minimum at $y = 3$.



x-axis Cuts x-axis at $(-1, 0)$.
y-axis No y-axis intercept as y not defined for $x = 0$.
$x \to \pm\infty$ As $x \to \pm\infty$, $y \to \dfrac{3x}{-x^2}$.
Thus as $x \to +\infty$, $y \to 0^-$; as $x \to -\infty$, $y \to 0^+$
y undefined $x = 0$ and $x = 3$ are vertical asymptotes.

Max/min When $y = \frac{1}{3}$, $x = -3$
max at $(-3, \frac{1}{3})$;
when $y = 3$, $x = 1$
min at $(1, 3)$

The sketch can then be completed.



Note For some sketches it can also be useful to determine the sign of the function throughout its domain by constructing a table. For the last function, remembering that $f(x)$ can only change sign where the curve cuts the x-axis (i.e. $x = -1$) and at vertical asymptotes (i.e. $x = 0$ and $x = 3$), the table would be as shown on the right.

	$x < -1$	$-1 < x < 0$	$0 < x < 3$	$x > 3$
$3x + 3$	−ve	+ve	+ve	+ve
x	−ve	−ve	+ve	+ve
$3 - x$	+ve	+ve	+ve	−ve
$\frac{3x + 3}{x(3 - x)}$	+ve	−ve	+ve	−ve

The reader should verify that these results agree with the sketch.

Example 4

Sketch the curve given by $y = \dfrac{(x - 5)(x - 1)}{(x + 1)(x - 3)}$

Note that the R.H.S. is an improper fraction.

If $y = \dfrac{(x - 5)(x - 1)}{(x + 1)(x - 3)}$ ... [1] then $y = \dfrac{x^2 - 6x + 5}{x^2 - 2x - 3}$ ... [2]

and $y = 1 + \dfrac{8 - 4x}{x^2 - 2x - 3}$ ... [3]

Range of values of y From [2], $(y - 1)x^2 + 2x(3 - y) - 3y - 5 = 0$

For $y \neq 1$, this is a quadratic in x.

Thus, for real x,

$$[2(3 - y)]^2 - 4(y - 1)(-3y - 5) \geqslant 0$$

i.e. $y^2 - y + 1 \geqslant 0$

or $(y - \frac{1}{2})^2 + \frac{3}{4} \geqslant 0$

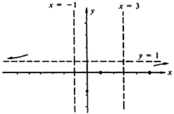
which is true for all real y

Thus there is no restriction on y.

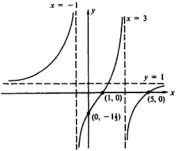
For $y = 1$, $4x - 8 = 0$,
$x = 2$
$\therefore$ y can equal 1.

Note also that, for each value of y, there will exist two real distinct values of x, except where $y = 1$ for which there is one value for x, i.e. $x = 2$.

x-axis	Cuts x-axis at $(1, 0)$ and $(5, 0)$ (from [1])
y-axis	Cuts y-axis at $(0, -1\frac{2}{3})$ (from [2])
$x \to \pm\infty$	As $x \to \pm\infty$, $y \to 1 - \dfrac{4x}{x^2}$ (from [3]) $\therefore$ as $x \to +\infty$ $y \to 1^-$; as $x \to -\infty$ $y \to 1^+$ (see note below)
y undefined	$x = -1$ and $x = 3$ are vertical asymptotes (from [1])
Max/min	Range of values suggest no max/min.



The sketch can then be completed:



Note Alternatively, the horizontal asymptote could be determined by writing
$y = \dfrac{1 - 6/x + 5/x^2}{1 - 2/x - 3/x^2}$, then as $x \to \pm\infty$, $y \to 1$

Example 5

Sketch the curve given by $y = \dfrac{12}{x^2 + 2x - 3}$

If $y = \dfrac{12}{x^2 + 2x - 3}$ ... [1] then $y = \dfrac{12}{(x + 3)(x - 1)}$ ... [2]

Range of values of y From [1], $x^2y + 2xy - 3y - 12 = 0$

For $y \neq 0$, this is a quadratic in x.
Thus, for real x, $4y^2 - 4y(-3y - 12) \geqslant 0$
i.e. $y(y + 3) \geqslant 0$

For $y = 0$, we obtain $-12 = 0$ which is impossible.
$\therefore$ $y \neq 0$.

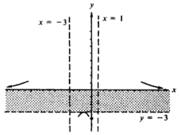
	$y < -3$	$-3 < y < 0$	$y > 0$
y	−ve	−ve	+ve
$y + 3$	−ve	+ve	+ve
$y(y + 3)$	+ve	−ve	+ve

i.e. $y \leqslant -3, y \geqslant 0$

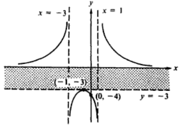
$\therefore$ the ranges of values y can take for real x are: $y \leqslant -3$, $y > 0$.

$y = -3$ will be a local maximum and for the remainder of the range there will be two distinct real values of x for each value of y.

x-axis	No *x*-axis intercept
y-axis	Cuts *y*-axis at $(0, -4)$
$x \to \pm\infty$	As $x \to \pm\infty$, $y \to \frac{12}{x^2}$ (from [1])
	$\therefore$ as $x \to +\infty$, $y \to 0^+$
	as $x \to -\infty$, $y \to 0^+$
y undefined	$x = -3$ and $x = 1$ are vertical asymptotes (from [2])
Max/min	When $y = -3$, $x = -1$. Max at $(-1, -3)$



The sketch can then be completed.



Note Alternatively the method of section 11.4 page 288 could have been used for this

Example 6

Sketch the curve given by $y = \dfrac{2 - 3x}{x^2 + 3x + 3}$

In this case the denominator does not factorise.

Range of values of y From $y = \dfrac{2 - 3x}{x^2 + 3x + 3}$, we have $x^2y + 3x(y + 1) + 3y - 2 = 0$

For $y \neq 0$, this is a quadratic in x.
Thus, for real x, $[3(y + 1)]^2 - 4y(3y - 2) \geqslant 0$
i.e. $(3y + 1)(y - 9) \leqslant 0$

For $y = 0$, $3x = 2$
$x = \frac{2}{3}$
$\therefore$ y can equal zero.

	$y < -\frac{1}{3}$	$-\frac{1}{3} < y < 9$	$y > 9$
$3y + 1$	−ve	+ve	+ve
$y - 9$	−ve	−ve	+ve
$(3y + 1)(y - 9)$	+ve	−ve	+ve

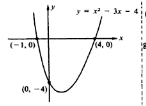
i.e. $-\frac{1}{3} \leqslant y \leqslant 9$

$\therefore$ the range of values y can take for real x is $-\frac{1}{3} \leqslant y \leqslant 9$.
$y = -\frac{1}{3}$ will be a minimum and $y = 9$ a maximum. For every other value of y in the permitted range there will correspond two distinct values for x except $y = 0$ for which there is one value of x, i.e. $x = \frac{2}{3}$.

14.2 Inequalities

We saw in chapter 5 that, for a quadratic inequality, say $x^2 - 3x - 4 < 0$, the range of values that x can take can be found by sketching, by completing the square, or by considering the signs of the factors:

Sketching



Thus for $x^2 - 3x - 4 < 0$,
$-1 < x < 4$

Completing the square

$$x^2 - 3x - 4 < 0$$
$$(x - \tfrac{3}{2})^2 - \tfrac{9}{4} - 4 < 0$$
$$(x - \tfrac{3}{2})^2 < \tfrac{25}{4}$$
$$-\tfrac{5}{2} < (x - \tfrac{3}{2}) < \tfrac{5}{2}$$
giving $-1 < x < 4$

Signs of the factors

$$x^2 - 3x - 4 < 0$$
$\therefore \quad (x - 4)(x + 1) < 0$

	$x < -1$	$-1 < x < 4$	$x > 4$
$x - 4$	−ve	−ve	+ve
$x + 1$	−ve	+ve	+ve
$(x - 4)(x + 1)$	+ve	−ve	+ve

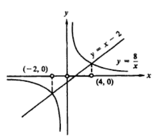
$\therefore \quad -1 < x < 4$

Similar techniques can be used to find the range of possible values x can take in more complicated inequalities, as the following examples show.

Example 8

Find the range (or ranges) of values that x can take if $x - 2 < \dfrac{8}{x}$.

By sketching
In this method we sketch $y = x - 2$ and $y = \dfrac{8}{x}$ on the same graph.
Important points will be where these lines meet.
At points of intersection $x - 2 = \dfrac{8}{x}$ which, provided $x \neq 0$, gives $x^2 - 2x - 8 = 0$ giving $x = -2$ or $x = 4$.
For $x - 2 < \dfrac{8}{x}$, we look for x values for which the line $y = x - 2$ is 'lower' than the curve $y = \dfrac{8}{x}$. (The relevant parts of the x-axis are shown in heavy type on the sketch.)
Thus for $x - 2 < \dfrac{8}{x}$ we must have $x < -2$ or $0 < x < 4$.



By calculation $x - 2 < \frac{8}{x}$

i.e. $x - 2 - \frac{8}{x} < 0$ or $\frac{(x-4)(x+2)}{x} < 0.$

Now the function $y = f(x)$ may change sign as the graph of $y = f(x)$ cuts the x-axis and at vertical asymptotes. Thus for $y = \frac{(x-4)(x+2)}{x}$, the critical values of x are 4, -2 and 0. We can then construct a table from which we can see that $\frac{(x-4)(x+2)}{x} < 0$ for $x < -2$ and for $0 < x < 4$.

	$x < -2$	$-2 < x < 0$	$0 < x < 4$	$x > 4$
$x - 4$	$-$ve	$-$ve	$-$ve	+ve
$x + 2$	$-$ve	+ve	+ve	+ve
x	$-$ve	$-$ve	+ve	+ve
y	$-$ve	+ve	$-$ve	+ve

Example 9

Find the solution set of the inequality $\frac{x+4}{x+1} < \frac{x-2}{x-4}$.

By sketching:

The graphical solution is obtained by sketching $y = \frac{x+4}{x+1}$ and $y = \frac{x-2}{x-4}$ on the same axes.

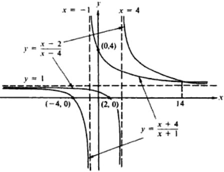
The sketch can be made using the methods developed earlier in this chapter. Notice that the two curves will intersect where

$x^2 - 16 = (x-2)(x+1),$

i.e. $x = 14.$

For $\frac{x+4}{x+1} < \frac{x-2}{x-4}$, we look for values of x for which the curve $y = \frac{x+4}{x+1}$ is 'lower' than the curve $y = \frac{x-2}{x-4}$.

Thus for $\frac{x+4}{x+1} < \frac{x-2}{x-4}$ we must have $x < -1$ or $4 < x < 14$,



i.e. the solution set is $\{x \in \mathbb{R}: x < -1 \text{ or } 4 < x < 14\}$.

By calculation: $\dfrac{x+4}{x+1} < \dfrac{x-2}{x-4}$ i.e. $\dfrac{x+4}{x+1} - \dfrac{x-2}{x-4} < 0$

or $\dfrac{x-14}{(x+1)(x-4)} < 0$

	$x < -1$	$-1 < x < 4$	$4 < x < 14$	$x > 14$
$x - 14$	−ve	−ve	−ve	+ve
$x + 1$	−ve	+ve	+ve	+ve
$x - 4$	−ve	−ve	+ve	+ve
$\dfrac{x-14}{(x+1)(x-4)}$	−ve	+ve	−ve	+ve

Thus for $\dfrac{x+4}{x+1} < \dfrac{x-2}{x-4}$, we must have $x < -1$ or $4 < x < 14$, or in set notation $\{x \in \mathbb{R}: x < -1 \text{ or } 4 < x < 14\}$.

Example 10

Find the range (or ranges) of values x can take if $\dfrac{x-2}{x^2 - x + 1} > 0$.

Since $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$, the denominator of $\dfrac{x-2}{x^2 - x + 1}$ is always positive. Thus the sign of $\dfrac{x-2}{x^2 - x + 1}$ depends on the sign of the numerator, $x - 2$.

Now $x - 2 > 0$ if $x > 2$, hence

$$\frac{x-2}{x^2 - x + 1} > 0 \text{ if } x > 2.$$

This result, as the reader can verify, can also be obtained by sketching.

Modulus inequalities

Simple modulus inequalities have been solved on page 4.

For example, we know that $|3x + 1| > 8$

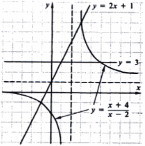
means that $3x + 1 < -8$ or $3x + 1 > 8$

i.e. $x < -3$ or $x > 2\frac{1}{3}$

In order to solve more complicated modulus inequalities, we need other techniques as shown in the following examples.

Exercise 14B

1. The graph shows the lines $y = 2x + 1$, $y = 3$ and $y = \frac{x+4}{x-2}$.
Use the graph to find the solution sets of the following inequalities.
(a) $\frac{x+4}{x-2} < 3$,
(b) $2x + 1 > \frac{x+4}{x-2}$.



For each of the inequalities in questions **2** to **27**, find the range (or ranges) of values x can take for the inequality to be true.

2. $(2x - 3)^2 > 1$ **3.** $(x - 1)^2 < 7 - x$ **4.** $x + 2 > \frac{15}{x}$

5. $13 > 2x + \frac{15}{x}$ **6.** $\frac{7}{4 - x} > 2$ **7.** $x < \frac{8}{x - 2}$

8. $x^4 - 10x^2 + 9 > 0$ **9.** $4x^4 - 17x^2 + 4 < 0$ **10.** $\frac{3x + 2}{x - 2} > 1$

11. $\frac{3x - 10}{x - 4} < 2$ **12.** $\frac{3x + 1}{2x - 5} < 1$ **13.** $x > \frac{4 - x}{x - 1}$

14. $x < \frac{4x}{x + 1}$ **15.** $\frac{x^2 + 3}{x - 1} > 6$ **16.** $\frac{x + 1}{x^2 + x + 1} > 0$

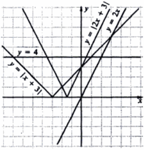
17. $x - 2 < \frac{6 - x}{x}$ **18.** $\frac{(x - 6)(x + 1)}{x} > -4$ **19.** $x - 2 < \frac{6x - 9}{x + 2}$

20. $\frac{3 - x}{x - 8} < \frac{1}{x}$ **21.** $\frac{x - 2}{x + 1} > \frac{x - 6}{x - 2}$ **22.** $\frac{x + 1}{x - 1} < \frac{x + 3}{x + 2}$

23. $\frac{2x + 3}{x + 4} > \frac{x}{x - 2}$ **24.** $\frac{3x - 8}{2x - 1} > \frac{x - 4}{x + 1}$ **25.** $\frac{3x}{x - 8} < \frac{2x - 1}{x - 5}$

26. $\frac{x}{4x - 8} < \frac{1}{x}$ **27.** $\frac{x - 1}{x} > \frac{2}{3 - x}$

28. The graph shows the lines $y = |2x + 3|$, $y = 2x$, $y = |x + 3|$ and $y = 4$. Use the graph to find the solution sets of the following inequalities.
(a) $|x + 3| < 4$,
(b) $2x > |x + 3|$,
(c) $|2x + 3| > |x + 3|$.



For each of the inequalities in questions **29** to **43**, find the range (or ranges) of values x can take for the inequality to be true.

29. $|2x - 3| > 5$ **30.** $|4x + 2| < x + 8$ **31.** $|2x + 1| > x + 5$

32. $|x| > |2x + 3|$ **33.** $|2x + 5| > |x + 1|$ **34.** $|2x - 3| > 4|x|$

35. $|x + 1| > |x - 3|$ **36.** $6 - x > |3x - 2|$ **37.** $3|x + 5| > |x + 3|$

38. $3|x - 1| < |x - 3|$ **39.** $2|x - 3| > |x|$ **40.** $|x| > 2|x + 1|$

41. $\left|\dfrac{x}{x-3}\right| < 2$ **42.** $\left|\dfrac{x}{x-3}\right| < x$ **43.** $\left|\dfrac{2x-4}{x+1}\right| < 4$

Exercise 14C Examination questions

1. Sketch the curve $y = \dfrac{1}{1 + x}$.
Determine (i) the equation of the tangent to the curve at the point where $x = 0$;
(ii) the equation of the other tangent to the curve which is parallel to the tangent in part (i). (S.U.J.B.)

2. Sketch the curve $y = \dfrac{x}{x - 1}$. Find the ranges of values of x for which $\dfrac{x}{x-1} > -1$. (Oxford)

3. Sketch the curve

$$y = \frac{x}{x - 2}$$

and write down the equation of its mirror image in the y-axis. (London)

4. State the equations of the asymptotes of the graph of the function

$$f\colon\ x \to \frac{3x}{x - 2},$$

where $x \in \mathbb{R}$ and $x \neq 2$.
Sketch the graph of the function showing clearly the asymptotes. (London)

5. Prove that the function $y = \dfrac{9}{x + 2} - \dfrac{1}{x}$ has a maximum value at $x = 1$.
Find the value of x for which y has a minimum value.
SKETCH the graph of the function. (Graph paper is not required—a sketch showing the main relevant features will suffice.)
Prove that the x-coordinate of any point of inflexion on the graph will be a solution of the equation $(x + 2)^3 = 9x^3$. (You are NOT asked to solve this equation.) (S.U.J.B.)

6. Show that, for real x,

$$-\frac{1}{4} \leqslant \frac{x}{x^2 + 4} \leqslant \frac{1}{4}.$$

Sketch the curve $y = \dfrac{x}{x^2 + 4}$, showing the coordinates of the turning points. (Cambridge)

7. Given that $y(x - 1) = x^2 + 3$, where x is real, show that y cannot take any value between -2 and 6.
Find the equations of the asymptotes of the curve
$$y = \frac{x^2 + 3}{x - 1}$$
and sketch the curve, showing the coordinates of the turning points. (Cambridge)

8. In each of the following cases, determine the range of values of x for which
(i) $1 - 2x > 0$,
(ii) $(x + 1)(x - 2) > 0$,
(iii) $\frac{(x + 1)(x - 2)}{1 - 2x} > 0$. (Cambridge)

9. Find the set of real values of x for which
$|x - 2| > 2|x + 1|$. (London)

10. Find the set of values of x for which
$|x - 1| - |2x + 1| > 0$. (London)

11. Given that $y = \frac{3x + k}{x^2 - 1}$, where x is real and k is a constant, show that y can take all real values if $|k| < 3$. (London)

12. Find the ranges of values of x which satisfy the inequalities
(a) $x^2 - 3x < 0$,
(b) $\frac{2}{x - 1} < x$. (A.E.B.)

13. Obtain the set of values of x for which
$$\frac{1}{x - 2} > \frac{1}{x + 2},$$ (London)

14. Find, in each case, the set of values of x for which
(i) $x(x - 2) > x + 4$,
(ii) $x - 2 > \frac{x + 4}{x}$, (Cambridge)

15. Find the equation of the tangent to the curve $y = x(x - 3)$ at the point $P(2, -2)$. Using the same axes sketch the curve and the tangent at P.
Using sketches, or otherwise, find the values of x for which
(a) $x(x - 3) \leqslant x - 4$,
(b) $x - 3 < \frac{x - 4}{x}$,
(c) $\frac{1}{x - 3} < \frac{x}{x - 4}$. (London)

16. Given that $y(x^2 - 4) = 2x + 5$ where x is real, prove that y cannot take any value in the range $-1 < y < -\frac{1}{4}$.
Sketch the graph of the curve with equation $y = \frac{2x + 5}{x^2 - 4}$, indicating clearly the asymptotes.
Calculate the ranges of values of x for which $\left|\frac{2x + 5}{x^2 - 4}\right| \geqslant \frac{1}{4}$. (Cambridge)

15
Trigonometry II

15.1 Sum and product formulae

We saw in section 4.7, that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad \ldots [1]$$

and
$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad \ldots [2]$$

adding these we get
$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \qquad \ldots [3]$$

and subtracting we get
$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B. \qquad \ldots [4]$$

If we write $C = A + B$ and $D = A - B$, then $A = \dfrac{C + D}{2}$ and $B = \dfrac{C - D}{2}$

and we then have
$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$$

and
$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

We can, in a similar way, use the expansions of $\cos(A \pm B)$,

i.e.
$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad \ldots [5]$$

and
$$\cos(A - B) = \cos A \cos B + \sin A \sin B, \qquad \ldots [6]$$

to give
$$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$$

and
$$\cos C - \cos D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$$

These four results should be memorised:

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$$

$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$$

$$\cos C - \cos D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$$

Note carefully the pattern which runs through these identities and also the negative sign which appears on the right-hand side of the last one.

Example 1

Prove that $\dfrac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$.

Applying the above results:
$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \frac{2 \cos \dfrac{5A + 3A}{2} \sin \dfrac{5A - 3A}{2}}{2 \cos \dfrac{3A + 5A}{2} \cos \dfrac{3A - 5A}{2}}$$

$$= \frac{2\cos 4A \sin A}{2\cos 4A \cos(-A)}$$

$$= \frac{\sin A}{\cos A} \quad \text{since} \quad \cos(-A) = \cos A$$

$$\therefore \quad \frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A \quad \text{as required.}$$

Note It is also useful to be able to use these four standard results the other way round, i.e. to express products as sums or differences.

From equations [3], [4], [5] and [6], we can write:

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$
$$2\cos A\sin B = \sin(A + B) - \sin(A - B)$$
$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$
$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$

Note very carefully the similarities, and also the differences, between these four relations and the previous ones.

Example 2

Prove $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta.$

$$\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \frac{\frac{1}{2}(\sin 9\theta + \sin 7\theta) - \frac{1}{2}(\sin 9\theta + \sin 3\theta)}{\frac{1}{2}(\cos 3\theta + \cos\theta) - \frac{1}{2}(\cos\theta - \cos 7\theta)}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$$

$$= \frac{2\cos 5\theta\sin 2\theta}{2\cos 5\theta\cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$\therefore \quad \frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$$

Example 3

Solve the equation $\sin x + \sin 5x = \sin 3x$ for $0° \leqslant x \leqslant 180°$.

$\sin x + \sin 5x = \sin 3x$

$\therefore$ $2\sin 3x\cos 2x = \sin 3x$

hence $\sin 3x(2\cos 2x - 1) = 0$

$\therefore$ $\sin 3x = 0$, which gives $3x = 0°, 180°, 360°, 540°, \dots$

i.e. $x = 0°, 60°, 120°, 180°, \dots$

or $\cos 2x = \frac{1}{2}$, which gives $2x = 60°, 300°, 420°, \dots$

i.e. $x = 30°, 150°, 210°, \dots$

Thus, solutions in the range $0° \leqslant x \leqslant 180°$ are $x = 0°, 30°, 60°, 120°, 150°, 180°$.

Also $a\cos x - b\sin x = \sqrt{(a^2 + b^2)}\left[\dfrac{a}{\sqrt{(a^2+b^2)}}\cos x - \dfrac{b}{\sqrt{(a^2+b^2)}}\sin x\right]$

$$= \sqrt{(a^2 + b^2)}[\cos x\cos\alpha - \sin x\sin\alpha]$$

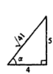
$$= \sqrt{(a^2 + b^2)}\cos(x + \alpha) \quad \text{where} \quad \tan\alpha = \frac{b}{a}.$$

Example 4

Express (a) $4\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$,
(b) $2\sin x + 5\cos x$ in the form $R\sin(x + \alpha)$.

(a) $4\cos x - 5\sin x$

Now $\sqrt{(4^2 + 5^2)} = \sqrt{41}$, and so we write



$$4\cos x - 5\sin x = \sqrt{41}\left[\frac{4}{\sqrt{41}}\cos x - \frac{5}{\sqrt{41}}\sin x\right]$$

$$= \sqrt{41}(\cos x\cos\alpha - \sin x\sin\alpha)$$

where α is given by:

$$= \sqrt{41}\cos(x + \alpha) \quad \text{where} \quad \tan\alpha = \tfrac{5}{4}$$

thus $4\cos x - 5\sin x = \sqrt{41}\cos(x + 51{\cdot}34°)$

(b) $2\sin x + 5\cos x$

Now $\sqrt{(2^2 + 5^2)} = \sqrt{29}$, and so we write

$$2\sin x + 5\cos x = \sqrt{29}\left[\frac{2}{\sqrt{29}}\sin x + \frac{5}{\sqrt{29}}\cos x\right]$$

$$= \sqrt{29}(\sin x\cos\alpha + \cos x\sin\alpha)$$

where α is given by:

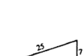
$$= \sqrt{29}\sin(x + \alpha) \quad \text{where} \quad \tan\alpha = \tfrac{5}{2}$$

thus $2\sin x + 5\cos x = \sqrt{29}\sin(x + 68{\cdot}20°)$

Example 5

Find the maximum value of $24\sin\theta - 7\cos\theta$ and the smallest positive value of θ that gives this maximum value.

Now $\sqrt{(24^2 + 7^2)} = 25$ and so we write



$$24\sin\theta - 7\cos\theta = 25\left[\frac{24}{25}\sin\theta - \frac{7}{25}\cos\theta\right]$$

$$= 25(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

where α is given by:

$$= 25\sin(\theta - 16{\cdot}26°)$$

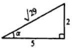
Hence the maximum value of $24\sin\theta - 7\cos\theta$ is 25 and this occurs when $(\theta - 16{\cdot}26°) = 90°$, i.e. when $\theta = 106{\cdot}26°$

Example 6

Solve $5\cos x - 2\sin x = 2$ for $-180° \leqslant x \leqslant 180°$.

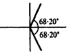
Now $\sqrt{(5^2 + 2^2)} = \sqrt{29}$ and so we rearrange the equation to:

$$\frac{5}{\sqrt{29}}\cos x - \frac{2}{\sqrt{29}}\sin x = \frac{2}{\sqrt{29}}$$



$\therefore$ $\cos x \cos \alpha - \sin x \sin \alpha = \dfrac{2}{\sqrt{29}}$ where α is given by:

i.e. $\cos(x + 21{\cdot}80°) = \dfrac{2}{\sqrt{29}}$



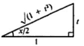
Now $\cos^{-1}\left(\dfrac{2}{\sqrt{29}}\right) = 68{\cdot}20°$ and a positive cosine gives:

Thus $x + 21{\cdot}80° = 68{\cdot}20°$ or $x + 21{\cdot}80° = -68{\cdot}20°$

i.e. $x = 46{\cdot}40°$ or $x = -90°$.

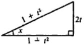
The *t*-substitution

Equations of the type encountered in Example 6 (i.e. $a\cos x + b\sin x = c$) can also be solved by using the substitution $t = \tan\dfrac{x}{2}$:



Then since $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$,

it follows that $\tan x = \dfrac{2t}{1 - t^2}$ i.e.



> Thus, when $t = \tan\dfrac{x}{2}$, $\sin x = \dfrac{2t}{1 + t^2}$ and $\cos x = \dfrac{1 - t^2}{1 + t^2}$.

Example 7

Solve the equation $5\cos x - 2\sin x = 2$ for $-180° \leqslant x \leqslant 180°$ using the substitution $t = \tan\dfrac{x}{2}$.

Substituting $t = \tan\dfrac{x}{2}$, i.e. $\cos x = \dfrac{1 - t^2}{1 + t^2}$ and $\sin x = \dfrac{2t}{1 + t^2}$

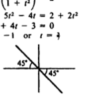
$$5\left(\frac{1 - t^2}{1 + t^2}\right) - 2\left(\frac{2t}{1 + t^2}\right) = 2$$

$$5 - 5t^2 - 4t = 2 + 2t^2$$

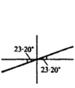
i.e. $7t^2 + 4t - 3 = 0$

giving $t = -1$ or $t = \frac{3}{7}$

hence $\tan\dfrac{x}{2} = -1$



or $\tan\dfrac{x}{2} = \frac{3}{7}$



i.e. $\dfrac{x}{2} = -45°, 135°, \ldots$

$x = -90°, 270°, \ldots$

i.e. $\dfrac{x}{2} = -156{\cdot}80°, 23{\cdot}20°, \ldots$

$x = -313{\cdot}60°, 46{\cdot}40°, \ldots$

The solutions in the range $-180° \leqslant x \leqslant 180°$ are $x = -90°, 46{\cdot}40°$

Note also that α and β must each be between 0 and $\pi/4$ because they are inverse tangents of positive numbers less than one. Thus we expect $(\alpha + \beta)$ to be between 0 and $\pi/2$.

Now
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\sqrt{3}/2 + \sqrt{3}/5}{1 - (\sqrt{3}/2)(\sqrt{3}/5)} = \sqrt{3}$$

$$\therefore \quad \alpha + \beta = \frac{\pi}{3}$$

$$\therefore \quad \tan^{-1}\frac{\sqrt{3}}{2} + \tan^{-1}\frac{\sqrt{3}}{5} = \frac{\pi}{3}.$$

Example 11

Find a positive value of x that satisfies the equation

$\tan^{-1} 3x + \tan^{-1} x = \frac{\pi}{4}$, giving your answer correct to 3 decimal places.

Let $\alpha = \tan^{-1} 3x$ i.e. $\tan\alpha = 3x$

and $\beta = \tan^{-1} x$ i.e. $\tan\beta = x$

Thus $\alpha + \beta = \frac{\pi}{4}$

$$\therefore \quad \tan(\alpha + \beta) = \tan\frac{\pi}{4} = 1 \qquad \ldots [1]$$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$$

$$\frac{3x + x}{1 - (3x)(x)} = 1$$

i.e. $3x^2 + 4x - 1 = 0$

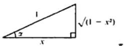
Solving this quadratic by the formula and taking the positive root gives $x = 0{\cdot}215$.

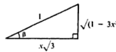
Note: The negative solution of the equation $3x^2 + 4x - 1 = 0$, i.e. $x = -1{\cdot}55$, does not satisfy the equation $\tan^{-1} 3x + \tan^{-1} x = \pi/4$, as can be verified using a calculator. This 'other solution' arises because, for $\alpha + \beta = \pi/4$, we have said $\tan(\alpha + \beta) = \tan\pi/4 = 1$, see [1] above. However $\tan(\alpha + \beta)$ equals 1 for values of $(\alpha + \beta)$ other than the $\pi/4$ value we require it to be. Hence the 'solution' $-1{\cdot}55$ arises but must be discarded as it does not satisfy the orginal equation we are asked to solve. When solving equations of this type it is advisable to check that the solutions obtained do satisfy the equation.

Example 12

Solve the equation $\cos^{-1} x + \cos^{-1} (x\sqrt{3}) = \frac{\pi}{2}$.

Let $\alpha = \cos^{-1} x$ i.e. $\cos \alpha = x$ and $\beta = \cos^{-1} (x\sqrt{3})$ i.e. $\cos \beta = x\sqrt{3}$





Thus $\alpha + \beta = \frac{\pi}{2}$

Taking the cosine of both sides of this equation, (we choose the cosine as $\cos \pi/2 = 0$)

gives $\cos(\alpha + \beta) = \cos \frac{\pi}{2} = 0$

i.e. $\cos \alpha \cos \beta - \sin \alpha \sin \beta = 0$

$$x \times x\sqrt{3} - \sqrt{[(1 - x^2)(1 - 3x^2)]} = 0$$

$$x^2\sqrt{3} = \sqrt{(1 - 4x^2 + 3x^4)}$$

Squaring both sides of this equation:

$$3x^4 = 1 - 4x^2 + 3x^4$$

$$\therefore\ 4x^2 = 1 \text{ giving } x = \pm\tfrac{1}{2}.$$

By substitution in the original equation it can be shown that $x = -\frac{1}{2}$ is not a possible solution for $\cos^{-1} x$ having a range $0 \leqslant x \leqslant \pi$.
Thus the only solution of the equation is $x = \frac{1}{2}$.

Exercise 15C

1. Evaluate the following giving your answers in degrees (do not use a calculator).
   (a) $\sin^{-1} 0$ (b) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\sin^{-1}\left(-\frac{1}{2}\right)$
   (e) $\cos^{-1}\left(-\frac{1}{2}\right)$ (f) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (g) $\tan^{-1}(\sqrt{3})$ (h) $\tan^{-1}(-\sqrt{3})$
2. Evaluate the following giving your answers in radians (leave π in your answers and do not use a calculator).
   (a) $\sin^{-1} 1$ (b) $\sin^{-1}(-1)$ (c) $\cos^{-1}(\frac{1}{2})$ (d) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   (e) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (f) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (g) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (h) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
3. (a) Draw x- and y-axes using a scale of 2 cm to 1 unit and such that $-1 \leqslant x \leqslant 4$ and $-1 \leqslant y \leqslant 4$.
   (b) Copy and complete the following table of values for $y = \cos x$ with $0 \leqslant x \leqslant \pi$. (Use a calculator to obtain the values of y.)

x rad	0	0·25	0·5	0·75	1	1·25	1·5	1·75	2	2·25	2·5	2·75	3	3·14 (i.e. π)
y (correct to 2 d.p's)	1	0·97	0·88									−0·92	−0·99	−1

   Hence draw the graph of $y = \cos x$ for $0 \leqslant x \leqslant \pi$.
   (c) By plotting (1, 0), (0·97, 0·25), (0·88, 0·5) etc. draw the graph of $y = \cos^{-1} x$ on the same pair of axes used for (b).

4. Without using a calculator find
   (a) $\sin\theta$ if $\theta = \sin^{-1}(\frac{3}{5}) + \sin^{-1}(\frac{5}{13})$
   (b) $\cos\theta$ if $\theta = \cos^{-1}(\frac{5}{13}) - \cos^{-1}(\frac{8}{17})$
   (c) $\tan\theta$ if $\theta = \tan^{-1}(\frac{3}{4}) + \tan^{-1}(\frac{7}{24})$
   (d) $\sin\theta$ if $\theta = \sin^{-1}(\frac{4}{5}) - \cos^{-1}(\frac{8}{17})$
   (e) $\cos\theta$ if $\theta = \cos^{-1}(\frac{24}{25}) - \sin^{-1}(\frac{15}{17})$
5. Evaluate the following. (Give your answers in terms of π and do not use a calculator).
   (a) $\sin^{-1}(\frac{1}{3}) + \cos^{-1}(\frac{1}{3})$
   (b) $2\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{7})$
6. Prove that
   $\sin(2\sin^{-1}x + \cos^{-1}x) = \sqrt{(1 - x^2)}$
7. Without the use of a calculator show that
   $2\sin^{-1}(\frac{3}{5}) - \cos^{-1}(\frac{35}{37}) = \sin^{-1}(\frac{756}{925})$
8. Find a positive value of x that satisfies the equation
   $\tan^{-1}(3x) + \tan^{-1}(2x) = \frac{\pi}{4}$
9. Find, correct to three decimal places, a positive value of x that satisfies the equation
   $\tan^{-1}x + \tan^{-1}(2x) = \tan^{-1}2$
10. Solve the equation
    $\cos^{-1}x + \cos^{-1}(x\sqrt{8}) = \frac{\pi}{2}$
11. Solve the equation
    $2\sin^{-1}(x\sqrt{6}) + \sin^{-1}(4x) = \frac{\pi}{2}$
12. Solve the equation
    $$2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}$$

15.4 General solutions

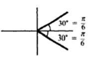
In all the trigonometric equations encountered so far in this book, the required solutions have been restricted to a given interval, say $-\pi$ to π or 0 to 2π, etc. Without such a restriction, there would be an infinite number of solutions. Nevertheless, it is useful to be able to give an expression for the general solution in terms of some letter, usually n. Then, by allowing n to take the integer values (i.e. $n \in \mathbb{Z}$), any particular solution of the equation can be obtained. We now consider three basic types of equation.

Type 1 $\cos\theta = \cos\alpha$

Example 13

Find the general solution of the equation $\cos\theta = \cos\pi/6$.

Now $\cos\pi/6 = \sqrt{3}/2$, and so we require all values of θ for which $\cos\theta = \sqrt{3}/2$.
We can show two solutions in the range $-\pi \leqslant \theta \leqslant \pi$ in a diagram:



Other solutions will occur as we add or subtract multiples of 2π to these two solutions.
Thus the general solution of $\cos\theta = \cos\pi/6$ is $\theta = 2n\pi \pm \pi/6$ for $n \in \mathbb{Z}$.

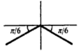
Extending this idea, we can say that the general solution of the equation $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$ for $n \in \mathbb{Z}$ where α is in radians (or $360n° \pm \alpha$ for α in degrees).

To determine the general solution to a trigonometrical equation, it is best to use the first principles method shown in the previous examples, i.e. determine the general solution from a diagram showing the solutions in the range $-\pi$ to π. However, for some types of equation the ability to quote and use the above standard results can lead to a very concise method of solution. The 'alternative solutions' given for examples 18 to 20 demonstrate this technique.

Example 16

Find the general solution of the equation $\sin 2\theta = -\frac{1}{2}$.

For $\sin\theta = -\frac{1}{2}$, we have two solutions in the range $-\pi \leqslant \theta \leqslant \pi$ as shown:



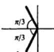
Other solutions will occur as we add or subtract multiples of 2π to these.
Thus for $\sin 2\theta = -\frac{1}{2}$

$$2\theta = \begin{cases} 2n\pi - \pi/6 \\ 2n\pi + \pi + \pi/6 \end{cases} \quad \text{i.e.} \quad \theta = \begin{cases} n\pi - \pi/12 \\ n\pi + 7\pi/12 \end{cases}$$

Example 17

Find the general solution to the equation $2\cos^2\theta + 3\cos\theta + 1 = 0$

$$2\cos^2\theta + 3\cos\theta + 1 = 0$$
$$\therefore \quad (2\cos\theta + 1)(\cos\theta + 1) = 0$$
i.e. $\cos\theta = -\frac{1}{2}$ or $\cos\theta = -1$
For $-\pi \leqslant \theta \leqslant \pi$, solutions are as shown in the diagrams:



Other solutions will occur as we add or subtract multiples of 2π to these.
Thus, for $\cos\theta = -\frac{1}{2}$ and for $\cos\theta = -1$

$$\theta = 2n\pi \pm 2\pi/3 \qquad \theta = 2n\pi + \pi$$

Thus $\theta = 2n\pi \pm 2\pi/3,\ (2n + 1)\pi$

Example 18

Find the general solution of the equation $\cos 5\theta = \cos 3\theta$.

If $\cos 5\theta = \cos 3\theta$
then $\cos 5\theta - \cos 3\theta = 0$

$$\therefore \quad -2\sin\frac{5\theta + 3\theta}{2}\sin\frac{5\theta - 3\theta}{2} = 0$$
$$-2\sin 4\theta \sin\theta = 0$$

Thus, either $\sin 4\theta = 0$ or $\sin\theta = 0$
For $-\pi \leqslant \theta \leqslant \pi$, solutions for $\sin\theta = 0$ are as shown in the diagram:

Thus for $\sin 4\theta = 0$ and for $\sin\theta = 0$

$$4\theta = n\pi$$
$$\theta = n\pi/4 \qquad \theta = n\pi$$

Now, as n takes the integer values, the solutions given by $n\pi/4$ will contain all those given by $n\pi$.
Thus the general solution of $\cos 5\theta = \cos 3\theta$ is $\theta = n\pi/4$.

Alternative solution for $\cos 5\theta = \cos 3\theta$

From the summary on page 362

$\cos\theta = \cos\alpha$ has the general solution $\theta = 2n\pi \pm \alpha$

Thus $\cos 5\theta = \cos 3\theta$ has the general solution $5\theta = 2n\pi \pm 3\theta$

and either $5\theta = 2n\pi + 3\theta$ or $5\theta = 2n\pi - 3\theta$

$\theta = n\pi$ or $\theta = n\pi/4$

Thus, as before, the general solution is $\theta = n\pi/4$

Example 19

Find the general solution of the equation $\sin 5\theta = \cos 3\theta$.

If $\sin 5\theta = \cos 3\theta$

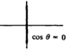
then $\sin 5\theta = \sin(\pi/2 - 3\theta)$

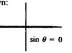
$\sin 5\theta - \sin(\pi/2 - 3\theta) = 0$

$\therefore\quad 2\cos(\theta + \pi/4)\sin(4\theta - \pi/4) = 0$

Thus either $\cos(\theta + \pi/4) = 0$ or $\sin(4\theta - \pi/4) = 0$

Solution for $\cos\theta = 0$ and $\sin\theta = 0$ are shown:





Thus for $\cos(\theta + \pi/4) = 0$ and for $\sin(4\theta - \pi/4) = 0$

$\theta + \pi/4 = n\pi + \pi/2$ $\quad 4\theta - \pi/4 = n\pi$

$\theta = n\pi + \pi/4$ or $\theta = n\pi/4 + \pi/16$

Alternative solution for $\sin 5\theta = \cos 3\theta$

$\sin 5\theta = \cos 3\theta$ can be written $\sin 5\theta = \sin(\pi/2 - 3\theta)$

From the summary on page 362

$\sin\theta = \sin\alpha$ has the general solution $\theta = \begin{cases} 2n\pi + \alpha \\ (2n+1)\pi - \alpha \end{cases}$

Thus $\sin 5\theta = \sin(\pi/2 - 3\theta)$ has the general solution

$$5\theta = \begin{cases} 2n\pi + \pi/2 - 3\theta \\ (2n+1)\pi - \pi/2 + 3\theta \end{cases}$$

and either $5\theta = 2n\pi + \pi/2 - 3\theta$ giving $\theta = n\pi/4 + \pi/16$

or $5\theta = 2n\pi + \pi/2 + 3\theta$ giving $\theta = n\pi + \pi/4$

Example 20

Find the general solution of the equation $\tan 5\theta = \tan(2\theta + \pi/6)$

$\tan 5\theta = \tan(2\theta + \pi/6)$ gives $\dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{\sin(2\theta + \pi/6)}{\cos(2\theta + \pi/6)}$

$\therefore\quad \sin 5\theta\cos(2\theta + \pi/6) - \cos 5\theta\sin(2\theta + \pi/6) = 0$

giving $\sin[5\theta - (2\theta + \pi/6)] = 0$

i.e. $\sin(3\theta - \pi/6) = 0$

Solutions for $\sin\theta = 0$ are as shown in the diagram:

Thus for $\sin(3\theta - \pi/6) = 0$

$3\theta - \pi/6 = n\pi$ giving $\theta = n\pi/3 + \pi/18$

Alternative solution for $\tan 5\theta = \tan(2\theta + \pi/6)$

From the summary on page 362

$\tan\theta = \tan\alpha$ has the general solution $\theta = n\pi + \alpha$

Thus $\tan 5\theta = \tan(2\theta + \pi/6)$ has the general solution $5\theta = n\pi + 2\theta + \pi/6$

i.e. $\theta = n\pi/3 + \pi/18$ as before.

Since $\cos\theta = \sqrt{(1 - \sin^2\theta)}$
$\approx \sqrt{(1 - \theta^2)}$ for small values of θ
$\approx 1 - \frac{1}{2}\theta^2$ expanding by the binomial theorem.

Thus we have, | for small θ: $\sin\theta \approx \theta$; $\cos\theta \approx 1 - \frac{1}{2}\theta^2$; $\tan\theta \approx \theta$ |

Note that these approximations depend upon θ being measured in radians.

Example 21

Find (a) $\lim_{\theta\to 0}\left(\frac{\cos 2\theta - 1}{\theta\sin 5\theta}\right)$ (b) $\lim_{\theta\to 0}\left(\frac{\sin 3\theta + \tan 5\theta}{2\theta}\right)$

(a) $$\lim_{\theta\to 0}\left(\frac{\cos 2\theta - 1}{\theta\sin 5\theta}\right) = \frac{1 - \frac{1}{2}(2\theta)^2 - 1}{\theta(5\theta)} = \frac{-2\theta^2}{5\theta^2} = -\frac{2}{5}$$

(b) $$\lim_{\theta\to 0}\left(\frac{\sin 3\theta + \tan 5\theta}{2\theta}\right) = \frac{3\theta + 5\theta}{2\theta} = 4.$$

Example 22

Find an expression, involving θ, that $\frac{21 + 7\tan\theta - 20\cos\theta}{1 + \sin 2\theta}$ approximates to for small values of θ.

Using the approximations for $\sin\theta$, $\cos\theta$ and $\tan\theta$,

$$\frac{21 + 7\tan\theta - 20\cos\theta}{1 + \sin 2\theta} \approx \frac{21 + 7\theta - 20(1 - \frac{1}{2}\theta^2)}{1 + 2\theta} \approx \frac{1 + 7\theta + 10\theta^2}{1 + 2\theta} \approx \frac{(1 + 2\theta)(1 + 5\theta)}{(1 + 2\theta)}$$

i.e. $$\frac{21 + 7\tan\theta - 20\cos\theta}{1 + \sin 2\theta} \approx 1 + 5\theta.$$

Exercise 15E

1. If θ is a small angle measured in radians approximate values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be obtained using $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. More accurate values can be obtained using a calculator. Using these facts, copy and complete the following tables:

$\theta = 0{\cdot}1$ rad	$\sin\theta$	$\cos\theta$	$\tan\theta$
Approx. value			
Calculator value (correct to 4 decimal places)			

$\theta = 0{\cdot}02$ rad	$\sin\theta$	$\cos\theta$	$\tan\theta$
Approx. value			
Calculator value (correct to 4 decimal places)			

2. Find (a) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$ (b) $\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}$ (c) $\lim_{\theta\to 0}\frac{4\theta}{\sin\theta}$ (d) $\lim_{\theta\to 0}\frac{\tan 2\theta}{\sin\theta}$

3. Find the expressions involving θ that the following approximate to for small values of θ. (Do *not* ignore terms in θ^2).
(a) $\frac{\cos\theta - 1}{\sin\theta}$ (b) $\frac{\sin\theta}{1-\cos 2\theta}$ (c) $\frac{\sin 3\theta + \tan\theta}{\cos 2\theta}$
(d) $\sin(\theta + 45°)$ (e) $\cos(\theta + 30°)$ (f) $\frac{1+\sin\theta}{5 + 3\tan\theta - 4\cos\theta}$

4. Given that $1° \approx 0{\cdot}0175$ radians and $(0{\cdot}0175)^2 \approx 0{\cdot}000\,306$ find, without the use of a calculator, approximate values for (a) $\sin 2°$ (b) $\tan 12'$ (c) $\cos 1°$

15.6 Differentiation of trigonometric functions

Suppose $y = \sin x$ ... [1] where x is measured in radians.
Let x change by a small amount δx and the consequent change in y be δy, then

$$y + \delta y = \sin(x + \delta x) \quad \ldots [2]$$

Subtracting equation [1] from equation [2]

$$\delta y = \sin(x + \delta x) - \sin x$$
$$= 2\cos(x + \tfrac{1}{2}\delta x)\sin(\tfrac{1}{2}\delta x)$$

i.e.

$$\frac{\delta y}{\delta x} = \cos(x + \tfrac{1}{2}\delta x)\frac{\sin(\tfrac{1}{2}\delta x)}{(\tfrac{1}{2}\delta x)}$$

By definition,

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$$
$$= \lim_{\delta x\to 0}\left\{\cos(x + \tfrac{1}{2}\delta x)\frac{\sin(\tfrac{1}{2}\delta x)}{(\tfrac{1}{2}\delta x)}\right\}$$

Now from section 15.5 $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ $\quad\therefore\quad \frac{dy}{dx} = \cos x.$

In a similar way if $y = \cos x$ ... [1]
Let x change by a small amount δx and the consequent change in y be δy,
then $y + \delta y = \cos(x + \delta x)$ ... [2]

Subtracting equation [1] from equation [2]

$$\delta y = \cos(x + \delta x) - \cos x$$
$$= -2\sin(x + \tfrac{1}{2}\delta x)\sin(\tfrac{1}{2}\delta x)$$
$$\frac{\delta y}{\delta x} = -\sin(x + \tfrac{1}{2}\delta x)\frac{\sin(\tfrac{1}{2}\delta x)}{(\tfrac{1}{2}\delta x)}$$

By definition,

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$$
$$= \lim_{\delta x\to 0}\left\{-\sin(x + \tfrac{1}{2}\delta x)\frac{\sin(\tfrac{1}{2}\delta x)}{(\tfrac{1}{2}\delta x)}\right\} \quad\therefore\quad \frac{dy}{dx} = -\sin x$$

From these two results: $\begin{cases} y = \sin x, \\ \frac{dy}{dx} = \cos x \end{cases}$ and $\begin{cases} y = \cos x, \\ \frac{dy}{dx} = -\sin x \end{cases}$

we can obtain the differential coefficients of the other trigonometric ratios.

Differential of tan x and cot x

Since $y = \tan x$

$= \dfrac{\sin x}{\cos x}$

$\dfrac{dy}{dx} = \dfrac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$

$= \sec^2 x$

and $y = \cot x$

$= \dfrac{\cos x}{\sin x}$

$\dfrac{dy}{dx} = \dfrac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x}$

$= -\text{cosec}^2\, x$

Differential of sec x and cosec x

Since $y = \sec x$

$= \dfrac{1}{\cos x} = (\cos x)^{-1}$

$\dfrac{dy}{dx} = (-1)(\cos x)^{-2}(-\sin x)$

$= \dfrac{\sin x}{\cos^2 x}$

$\therefore \quad \dfrac{dy}{dx} = \sec x \tan x$

and $y = \text{cosec}\, x$

$= \dfrac{1}{\sin x} = (\sin x)^{-1}$

$\dfrac{dy}{dx} = (-1)(\sin x)^{-2}(\cos x)$

$= \dfrac{-\cos x}{\sin^2 x}$

$\therefore \quad \dfrac{dy}{dx} = -\text{cosec}\, x \cot x$

To summarise these results, for the angle x in radians,

$y = \sin x$ $y' = \cos x$	$y = \cos x$ $y' = -\sin x$	$y = \tan x$ $y' = \sec^2 x$
$y = \text{cosec}\, x$ $y' = -\text{cosec}\, x \cot x$	$y = \sec x$ $y' = \sec x \tan x$	$y = \cot x$ $y' = -\text{cosec}^2\, x$

(Note that all of the 'co-' functions give a minus sign on differentiation)

If $y = \sin f(x)$ this can be written as $y = \sin u$, where $u = f(x)$ and then $\dfrac{dy}{dx}$ can be found from $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$. Thus expressions such as $\sin f(x)$ can be treated as a function of a function as in section 13.1. Similarly, if $y = \tan^n x$, this may be written as $y = u^n$ where $u = \tan x$ and then $\dfrac{dy}{dx}$ is again $\dfrac{dy}{du} \times \dfrac{du}{dx}$.

The following examples illustrate this method.

15.7 Integration of trigonometric functions

It follows from the differentiation of sin x and cos x that

$$\int \sin x\,dx = -\cos x + c \quad \text{and} \quad \int \cos x\,dx = \sin x + c,$$

and since if $y = \sin ax$, $\dfrac{dy}{dx} = a\cos ax$ and if $y = \cos ax$, $\dfrac{dy}{dx} = -a\sin ax$

it follows that

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c \quad \text{and} \quad \int \cos ax\,dx = \frac{1}{a}\sin ax + c$$

In some cases, trigonometric functions can be integrated directly as the reverse of differentiation, as we can see what function would differentiate to give the integrand.

Example 27

Find the following indefinite integrals (a) $\int \cos 7x\,dx$, (b) $\int 4\operatorname{cosec} x \cot x\,dx$, (c) $\int \sec^2 6x\,dx$.

(a) $\int \cos 7x\,dx$

Now $\dfrac{d}{dx}(\sin 7x) = 7\cos 7x$

$\therefore \int \cos 7x\,dx = \frac{1}{7}\sin 7x + c$

(b) $\int 4\operatorname{cosec} x \cot x\,dx$

Now $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\therefore \int 4\operatorname{cosec} x \cot x\,dx = -4\operatorname{cosec} x + c$

(c) $\int \sec^2 6x\,dx$

Now $\dfrac{d}{dx}(\tan 6x) = 6\sec^2 6x$

$\therefore \int \sec^2 6x\,dx = \frac{1}{6}\tan 6x + c$

Example 28

Find the following indefinite integrals (a) $\int 6\cos x \sin^2 x\,dx$, (b) $\int \sin^4 2x \cos 2x\,dx$.

(a) $\int 6\cos x \sin^2 x\,dx$

Now $\dfrac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$

$\therefore \int 6\cos x \sin^2 x\,dx = 2\sin^3 x + c$

(b) $\int \sin^4 2x \cos 2x\,dx$

Now $\dfrac{d}{dx}(\sin^5 2x) = 5\sin^4 2x\, 2\cos 2x$

$= 10\sin^4 2x \cos 2x$

$\therefore \int \sin^4 2x \cos 2x\,dx = \frac{1}{10}\sin^5 2x + c$

Example 29

Find the following indefinite integrals (a) $\int \cot^5 x \operatorname{cosec}^2 x\,dx$, (b) $\int \sec^4 x \tan x\,dx$.

(a) $\int \cot^5 x \operatorname{cosec}^2 x\,dx$

Now $\frac{d}{dx}(\cot^6 x) = -6\cot^5 x \operatorname{cosec}^2 x$

$\therefore \quad \int \cot^5 x \operatorname{cosec}^2 x\,dx = -\frac{1}{6}\cot^6 x + c$

(b) $\int \sec^4 x \tan x\,dx = \int \sec^3 x(\sec x \tan x)\,dx$

Now $\frac{d}{dx}(\sec^4 x) = 4\sec^3 x \sec x \tan x$

$\therefore \quad \int \sec^4 x \tan x\,dx = \frac{1}{4}\sec^4 x + c.$

Example 30

Find $\int 8 \sin 4x \sin x\,dx$.

Now $8 \sin 4x \sin x = 4(\cos 3x - \cos 5x)$ (using the result of section 15.1)

$\therefore \quad \int 8 \sin 4x \sin x\,dx = \frac{4}{3}\sin 3x - \frac{4}{5}\sin 5x + c.$

$\int \sin^n x\,dx$, $\int \cos^n x\,dx$

Example 31 shows the method to use for these integrations when n is odd and Example 32 shows the method to use when n is even.

Example 31

Find $\int \cos^7 x\,dx$

$$\begin{aligned}\int \cos^7 x\,dx &= \int \cos x(\cos^6 x)\,dx \\ &= \int \cos x(1 - \sin^2 x)^3\,dx \\ &= \int \cos x(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\,dx \\ &= \int (\cos x - 3\sin^2 x \cos x + 3\sin^4 x \cos x - \sin^6 x \cos x)\,dx\end{aligned}$$

$\therefore \quad \int \cos^7 x\,dx = \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

Example 32

Find $\int \sin^4 x\,dx$.

35. Find the following indefinite integrals (hint: simplify each expression first).

(a) $\int \sin x \cot x\,dx$ (b) $\int (\sin^2 x + \cos^2 x)\,dx$ (c) $\int (\cos^2 x - \sin^2 x)\,dx$

(d) $\int \frac{1}{1 - \sin^2 x}\,dx$ (e) $\int \cos x(\sec x + \tan x)\,dx$ (f) $\int \frac{2\tan x}{\sin 2x}\,dx$

Evaluate the following definite integrals

36. $\int_0^{\pi/2} (1 - \sin x)\,dx$ **37.** $\int_0^{\pi/4} \sin 2x\,dx$ **38.** $\int_0^{\pi} \cos\left(3x + \frac{\pi}{2}\right)dx$

39. $\int_0^{\pi/4} \sin^3 x \cos x\,dx$ **40.** $\int_0^{\pi/4} \sec^2 x \tan^3 x\,dx$ **41.** $\int_{\pi/8}^{\pi/4} 2\cos^2 2x\,dx$

42 $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$ **43.** $\int_0^{\pi/2} \sin 3x \cos 5x\,dx$

44. Find the area between the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

45. Find the area between the curve $y = 3\cos x$ and the x-axis from $x = 0$ to $x = \pi/2$.

46. Find the area between the curve $y = \sin x + 3\cos x$ and the x-axis from $x = 0$ to $x = \pi/2$.

47. Find the volume of the solid of revolution formed by rotating about the x-axis the area between the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \pi$.

48. Find the volume of the solid of revolution formed by rotating about the x-axis the area between the curve $y = \sin x + \cos x$ and the x-axis, from $x = 0$ to $x = \pi/2$.

15.8 Differentiation of inverse trigonometric functions

Suppose $y = \sin^{-1} x$

then $x = \sin y$

and $\frac{dx}{dy} = \cos y \quad \ldots [1]$

Using $\frac{dy}{dx} = 1/(dx/dy)$, (see page 324) and $\sin^2 y + \cos^2 y = 1$, we can

rearrange [1] to give $\frac{dy}{dx} = \frac{1}{\sqrt{(1 - \sin^2 y)}}$

$= \frac{1}{\sqrt{(1 - x^2)}}$

Thus, if $y = \sin^{-1} x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{(1 - x^2)}}$

In a similar way, it can be shown that

if $y = \cos^{-1} x$, $\frac{dy}{dx} = \frac{-1}{\sqrt{(1 - x^2)}}$

and if $y = \tan^{-1} x$, $\frac{dy}{dx} = \frac{1}{1 + x^2}$.

Example 33

Find $\frac{dy}{dx}$ if $y = \sin^{-1}\frac{x}{a}$ where a is a constant.

Let $u = \frac{x}{a}$, then $y = \sin^{-1} u$

thus $\frac{du}{dx} = \frac{1}{a}$ and $\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}}$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-u^2)}} \times \frac{1}{a}$$

$$= \frac{1}{\sqrt{(1-x^2/a^2)}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}.$$

Example 34

Find $\frac{dy}{dx}$ if (a) $y = \cos^{-1}\frac{x}{5}$, (b) $y = \cos^{-1}(x^2 - 1)$.

(a)

$$y = \cos^{-1}\frac{x}{5}$$

Let $u = \frac{x}{5}$, then $y = \cos^{-1} u$

$$\frac{du}{dx} = \frac{1}{5} \text{ and } \frac{dy}{du} = \frac{-1}{\sqrt{(1-u^2)}}$$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{-1}{\sqrt{(1-u^2)}} \times \frac{1}{5}$$

$$= \frac{-1}{\sqrt{\left(1-\frac{x^2}{25}\right)}} \times \frac{1}{5}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{(25-x^2)}}$$

(b)

$$y = \cos^{-1}(x^2 - 1)$$

Let $u = x^2 - 1$, then $y = \cos^{-1} u$

$$\frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = \frac{-1}{\sqrt{(1-u^2)}}$$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{-1}{\sqrt{(1-u^2)}} \times 2x$$

$$= \frac{-2x}{\sqrt{[1-(x^2-1)^2]}}$$

$$= \frac{-2x}{\sqrt{(2x^2-x^4)}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2}{\sqrt{(2-x^2)}}$$

$\int \frac{1}{\sqrt{(a^2-x^2)}}dx$ and $\int \frac{1}{(a^2+x^2)}dx$

From $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$, it can be shown that $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(a^2-x^2)}}$

and from $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{(1+x^2)}$, it can be shown that $\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{(a^2+x^2)}$.

Thus to integrate expressions involving $\frac{1}{\sqrt{(a^2-x^2)}}$ we look for a solution of the form $\sin^{-1}\frac{x}{a} + c$ and to integrate an expression of the form $\frac{1}{(a^2+x^2)}$ we look for a solution of the form $\tan^{-1}\frac{x}{a}$.

Example 35

Find (a) $\int \frac{4}{x^2 + 16} dx$ (b) $\int \frac{2}{36 + x^2} dx$.

(a) $\int \frac{4}{x^2 + 16} dx$

Since this involves $\frac{1}{x^2 + 16}$ i.e. of the form $\frac{1}{x^2 + a^2}$, we suspect it has come from the differentiation of $\tan^{-1} \frac{x}{4}$ which is $\frac{4}{x^2 + 16}$.

Thus $\int \frac{4}{x^2 + 16} dx = \tan^{-1} \frac{x}{4} + c.$

(b) $\int \frac{2}{36 + x^2} dx$

The differential of $\tan^{-1} \frac{x}{6}$ is $\frac{6}{x^2 + 36}$.

Thus $\int \frac{2}{36 + x^2} dx = \frac{1}{3} \int \frac{6}{36 + x^2} dx$

$= \frac{1}{3} \tan^{-1} \frac{x}{6} + c.$

Example 36

Evaluate $\int_{-1 \cdot 5}^{0} \frac{1}{\sqrt{(9 - x^2)}} dx$.

This is of the form $\int \frac{1}{\sqrt{(a^2 - x^2)}} dx$ and we can therefore quote the solution:

$$\int_{-1 \cdot 5}^{0} \frac{1}{\sqrt{(9 - x^2)}} dx = \left[\sin^{-1} \frac{x}{3}\right]_{-1 \cdot 5}^{0}$$

$$= \sin^{-1}(0) - \sin^{-1}\left(-\frac{1 \cdot 5}{3}\right)$$

$$= 0 - \left(-\frac{\pi}{6}\right) \qquad \therefore \quad \int_{-1 \cdot 5}^{0} \frac{1}{\sqrt{(9 - x^2)}} dx = \frac{\pi}{6}$$

The reader should note that in order to integrate more complicated functions of this type, e.g. $\int \frac{1}{\sqrt{(a^2 - Ax^2)}} dx$, for $A \neq 1$ or $\int \frac{1}{a^2 + Ax^2} dx$, for $A \neq 1$, we use the method of 'changing the variable' which is explained in chapter 20.

Exercise 15H

Differentiate the following with respect to x

1. $\sin^{-1} x$ **2.** $\tan^{-1} \frac{x}{a}$ **3.** $\sin^{-1} \frac{x}{4}$

4. $\cos^{-1} 3x$ **5.** $\tan^{-1} 4x$ **6.** $\sin^{-1} 6x$

7. $\sin^{-1}(2x - 1)$ **8.** $\tan^{-1}(1 - 3x)$ **9.** $\sin^{-1}(x^2 - 1)$

10. $x \sin^{-1} x$ **11.** $x \tan^{-1} x$ **12.** $(x^2 + 1) \tan^{-1} x$

Find the following indefinite integrals

13. $\int \frac{1}{\sqrt{(4 - x^2)}} dx$ **14.** $\int \frac{1}{\sqrt{(16 - x^2)}} dx$ **15.** $\int \frac{3}{9 + x^2} dx$

6. Given that
$$3 \sin x - \cos x \equiv R \sin (x - \alpha),$$
where $R > 0$ and $0^\circ < \alpha < 90^\circ$, find the values of R and α correct to one decimal place.
Hence find one value of x between 0° and 360° for which the curve $y = 3 \sin x - \cos x$ has a turning point. (London)

7. Find the general solution of the equation
$$\tan \theta - \sin \theta = 1 - \cos \theta.$$ (Cambridge)

8. (i) Find the general solution, in radians, of the equation
$$\sin 3x = -\tfrac{1}{2}.$$
(ii) By putting $\tan (\theta/2) = t$, or otherwise, find the general solution of the equation
$$2 \cos \theta - \sin \theta = 1,$$
giving your answers to the nearest tenth of a degree. (London)

9. (a) Find the general solution of the equation
$$\tan x + \sec x = 3 \cos x.$$
(b) Express $5 \sin^2 x - 3 \sin x \cos x + \cos^2 x$ in the form $a + b \cos (2x - \alpha)$ where a, b, α are independent of x.
Hence, or otherwise, find the maximum and minimum values of
$$5 \sin^2 x - 3 \sin x \cos x + \cos^2 x$$
as x varies. (Cambridge)

10. If $x = t^2 \sin 3t$ and $y = t^2 \cos 3t$, find $\dfrac{dy}{dx}$ in terms of t, and show that the curve defined by these parametric equations is parallel to the x-axis at points where $\tan 3t = \dfrac{2}{3t}$. (S.U.J.B.)

11. Find
$$\int \sin x(1 + \cos^2 x)\, dx.$$ (Cambridge)

12. Prove from first principles that the derivative of $\sin x$ is $\cos x$.
$\left[\text{You may use } \lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1 \text{ without proof.}\right]$
When x increases from π to $\pi + \varepsilon$, where ε is small, the increment in
$$\frac{\sin x}{x}$$
is approximately equal to $p\varepsilon$. Find p in terms of π. (J.M.B.)

13. (a) Differentiate with respect to x
(i) $\tan 2x$, (ii) $\sin^3 3x$, (iii) $(1 + \sin x)^5$.
(b) If $y = \dfrac{\sin x}{x}$, prove that
$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0.$$ (S.U.J.B.)

14. Sketch the curve $y = 1 + \sin x$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$.
Show that $(1 + \sin x)^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$.
Hence show that the volume of the solid of revolution formed when the region $(0 \leqslant x \leqslant \frac{3}{2}\pi)$ bounded by the curve $y = 1 + \sin x$, the y-axis and the x-axis is rotated through one revolution about the x-axis is

$$\tfrac{1}{4}\pi(9\pi + 8).$$

(Cambridge)

15. (a) Differentiate with respect to x:

(i) $x \sin 2x$; (ii) $\tan\left(3x + \frac{\pi}{4}\right)$; (iii) $(1 + \cos x)^2$.

(b) Evaluate the following integrals:

(i) $\int_0^{\pi/3} \sin\left(2x + \frac{\pi}{3}\right)dx$; (ii) $\int_{\pi/6}^{\pi/3} \cos 3x\,dx$. (S.U.J.B.)

16. Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1 - x)$ are acute angles,
(i) prove that $\sin[\sin^{-1} x - \cos^{-1} x] = 2x^2 - 1$,
(ii) solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1 - x)$. (A.E.B.)

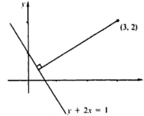
17. Differentiate $x \sin^{-1} mx$, where m is a constant.
Hence, or otherwise, integrate $\sin^{-1} mx$. (J.M.B.)

18. (i) For the curve $y = \sin x \cos^3 x$, where $0 \leqslant x \leqslant \pi$, find the x- and y-coordinates of the points at which $\frac{dy}{dx} = 0$.
Sketch the curve.
(ii) For the curve $y^2 = \sin x \cos^3 x$, where $0 \leqslant x \leqslant \frac{1}{2}\pi$, show that

$$\left(\frac{dy}{dx}\right)^2 = \tfrac{1}{4}\cot x(\cos^2 x - 3\sin^2 x)^2,$$

provided that $x \neq 0$.
Sketch the curve. (Cambridge)

16.2 Distance of a point from a line



Suppose we want to find the shortest distance from the point (3, 2) to the line $y + 2x = 1$.
We could proceed as follows:

gradient of $y + 2x = 1$ is -2

$\therefore$ gradient of line perpendicular to $y + 2x = 1$ is $\frac{1}{2}$.

Line through the point (3, 2) perpendicular to $y + 2x = 1$ has equation $y - 2 = \frac{1}{2}(x - 3)$

i.e. $2y = x + 1$

Solving $2y = x + 1$ and $y + 2x = 1$ simultaneously gives $x = \frac{1}{5}$ and $y = \frac{3}{5}$.
Thus the lines $2y = x + 1$ and $y + 2x = 1$ intersect at $(\frac{1}{5}, \frac{3}{5})$.
Length of the line joining the points (3, 2) and $(\frac{1}{5}, \frac{3}{5})$ is given by

$$\sqrt{[(3 - \tfrac{1}{5})^2 + (2 - \tfrac{3}{5})^2]} = \tfrac{7}{5}\sqrt{5} \text{ units.}$$

Alternatively we could express the equation $y + 2x = 1$ in vector form and proceed as in chapter 2 (see page 67 example 22). However, at times it is more convenient to quote and use the formula stated and proved below.

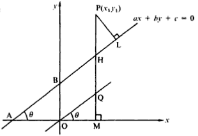
> The perpendicular distance from the point (x_1, y_1) to the line $ax + by + c = 0$ is given by $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$,
> i.e. we substitute the coordinates of the point into the equation of the line and divide by $\sqrt{(a^2 + b^2)}$.

For the point (3, 2) and the line $y + 2x = 1$ (i.e. $2x + y - 1 = 0$) this formula gives the perpendicular distance as

$$\frac{2(3) + 2 - 1}{\sqrt{(2^2 + 1^2)}} = \frac{7}{\sqrt{5}} = \frac{7}{5}\sqrt{5} \quad \text{as required.}$$

The formula quoted above may be proved as follows:

Let P be the point (x_1, y_1) and AB the line $ax + by + c = 0$. PL is the perpendicular from P to the line AB.
The line through P, parallel to the y-axis, cuts the x-axis at M, AB at H and the line through the origin parallel to AB at Q.



Gradient of $ax + by + c = 0$ is $\dfrac{-a}{b} = \tan\theta$

where θ is the angle made with the x-axis, and

the y-intercept, i.e. OB, is $-\dfrac{c}{b}$.

Then
$$\begin{aligned} \text{PL}\sec\theta &= \text{PH} \\ &= \text{PM} - \text{MQ} - \text{QH} \\ &= y_1 - \text{OM}\tan\theta - \text{OB} \\ &= y_1 - x_1\left(-\frac{a}{b}\right) - \left(-\frac{c}{b}\right) \end{aligned}$$

$$PL\sqrt{(1+\tan^2\theta)} = \frac{ax_1 + by_1 + c}{b}$$

$$PL = \frac{ax_1 + by_1 + c}{b\sqrt{(1 + a^2/b^2)}} = \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \quad \text{as required.}$$

Example 3

Find the perpendicular distance from the line $3y = 4x - 1$ to the points (a) (1, 3), (b) (1, − 2).

$$3y = 4x - 1$$

or $$0 = 4x - 3y - 1$$

(a) Using $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$

Required distance $= \dfrac{4(1) - 3(3) - 1}{\sqrt{(4^2 + 3^2)}}$

$= -\dfrac{6}{5}$

(b) Using $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$

Required distance $= \dfrac{4(1) - 3(-2) - 1}{\sqrt{(4^2 + 3^2)}}$

$= \dfrac{9}{5}$

Notice that one result is negative and the other positive. This is due to the fact that the two points lie on opposite sides of the line.
Thus the points (1, 3) and (1, −2) are respectively $1\frac{1}{5}$ and $1\frac{4}{5}$ units from the line $3y = 4x - 1$, and they lie on different sides of the line.

Example 4

Find the equations of the lines that bisect the angles between the lines $3x - 4y + 13 = 0$ and $12x + 5y - 32 = 0$.

Any point on a line that bisects one of the angles between the line $3x - 4y + 13 = 0$ and the line $12x + 5y - 32 = 0$ will be equidistant from these two lines. Thus we require the locus of all the points which are equidistant from $3x - 4y + 13 = 0$ and $12x + 5y - 32 = 0$. Suppose that (x, y) is the general point equidistant from the two lines, then using $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$ it follows that

$$\frac{3x - 4y + 13}{\sqrt{(3^2 + 4^2)}} = \pm\frac{12x + 5y - 32}{\sqrt{(12^2 + 5^2)}}$$

i.e. $13(3x - 4y + 13) = \pm 5(12x + 5y - 32)$
which gives $3x + 11y - 47 = 0$ and $11x - 3y + 1 = 0$.

The equations of the bisectors of the angles between the lines $3x - 4y + 13 = 0$ and $12x + 5y - 32 = 0$ are $3x + 11y - 47 = 0$ and $11x - 3y + 1 = 0$.

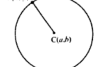
Note: The two bisectors will be at right angles to each other (i.e. in the example above the gradients of the bisectors are $-\frac{3}{11}$ and $\frac{11}{3}$).

Exercise 16B

1. In each of the following find the perpendicular distance from the given point to the given line.
   (a) $4y + 3x + 6 = 0$, (2, − 1) (b) $12y = 5x - 1$, (−3, −2)

We now consider the four types of curve in more detail and although we shall define each curve using the idea of a locus, it can be shown that such definitions are consistent with the conic section idea explained above.

16.3 The circle



The circle is defined as the locus of all points, P(x, y), which are equidistant from some given point C, (a, b). Suppose that the distance of the points P, from the given point C(a, b) is r, then

$$CP^2 = (x - a)^2 + (y - b)^2$$
$$= r^2$$

Thus the required locus is $(x - a)^2 + (y - b)^2 = r^2$, where r is the radius of the circle and the point (a, b) is its centre.

If the coordinates of the point C are $(0, 0)$, i.e. C is at the origin, then the equation becomes $x^2 + y^2 = r^2$, thus

Circle centre $(0, 0)$ and radius r has equation	$x^2 + y^2 = r^2$
Circle centre (a, b) and radius r has equation	$(x - a)^2 + (y - b)^2 = r^2$

Example 5

Find the centre and the radius of the circles with equations:
(a) $(x - 2)^2 + y^2 = 25$ (b) $x^2 + y^2 - 4x - 2y = 4$

(a) $(x - 2)^2 + y^2 = 25$

Comparing this equation with

$(x - a)^2 + (y - b)^2 = r^2$

the centre is $(2, 0)$ and the radius is 5 units.

(b) $x^2 + y^2 - 4x - 2y = 4$

which can be rearranged as

$(x^2 - 4x + 4) + (y^2 - 2y + 1) = 4 + 4 + 1$

i.e. $(x - 2)^2 + (y - 1)^2 = 3^2$

Comparing this equation with

$(x - a)^2 + (y - b)^2 = r^2$

the centre is $(2, 1)$ and the radius is 3 units

Note: the general equation of the circle, $(x - a)^2 + (y - b)^2 = r^2$, may be written as $x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$. Notice that the coefficients of x^2 and y^2 are equal in this second order equation and that there is no xy term. This will be true for all equations of circles.

Intersection of a line and a circle

Consider a straight line $y = mx + c$ and a circle $(x - a)^2 + (y - b)^2 = r^2$.

There are three possible situations:

1. the line cuts the circle in two distinct places, i.e. part of the line is a chord of the circle,
2. the line *touches* the circle, i.e. the line is a tangent to the circle,
3. the line neither cuts nor touches the circle.

Gradient at a given point on a curve

The gradient at a given point is the gradient of the tangent to that curve at the given point, and we can find this directly from the equation of the curve by implicit differentiation.

Example 7

Find the equation of (i) the tangent and (ii) the normal to the circle with equation $x^2 + y^2 - 6x - 2y - 3 = 0$ at the point (5, 4) on the circle.

$$x^2 + y^2 - 6x - 2y - 3 = 0$$

differentiating with respect to x,

$$2x + 2y\frac{dy}{dx} - 6 - 2\frac{dy}{dx} = 0$$

i.e.

$$\frac{dy}{dx} = \frac{3 - x}{y - 1}$$

at the point (5, 4)

$$\frac{dy}{dx} = \frac{3 - 5}{4 - 1} = -\frac{2}{3}$$

(i) gradient of tangent is $-\frac{2}{3}$
Using $y - y_1 = m(x - x_1)$,
the equation of the tangent is
$y - 4 = -\frac{2}{3}(x - 5)$
i.e. $3y + 2x = 22$

(ii) gradient of normal is $\frac{3}{2}$
Using $y - y_1 = m(x - x_1)$,
the equation of the normal is
$y - 4 = \frac{3}{2}(x - 5)$
i.e. $2y - 3x + 7 = 0$

At the point (5, 4) on the circle $x^2 + y^2 - 6x - 2y - 3 = 0$, the equations of the tangent and of the normal are respectively $3y + 2x = 22$ and $2y - 3x + 7 = 0$.

Circle through three given points

Three non-collinear points define a circle, i.e. there is one, and only one circle which can be drawn through three non-collinear points. The equation of any circle may be written as $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are constants.
Thus, if we are given the coordinates of three points on the circumference of a circle, we can substitute these values of x and y into the equation of the circle and obtain three equations which can be solved simultaneously to find the constants g, f and c.

Example 8

Find the equation of the circle passing through the points (0, 1), (4, 3) and (1, −1).

Suppose the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Substituting the coordinates of each of the three points into this equation gives:

$$1 + 2f + c = 0 \quad \ldots [1]$$
$$25 + 8g + 6f + c = 0 \quad \ldots [2]$$
$$2 + 2g - 2f + c = 0 \quad \ldots [3]$$

Multiplying equation [3] by 4 and then subtracting from equation [2], gives

$$17 + 14f - 3c = 0 \qquad \ldots [4]$$

Multiplying equation [1] by 3 and adding to equation [4], gives

$$20 + 20f = 0 \text{ or } f = -1.$$

Then from equation [1] $\qquad c = 1$

and from equation [3] $\qquad g = -2\frac{1}{2}$.

The equation of the circle which passes through (0, 1), (4, 3) and (1, − 1) is $x^2 + y^2 - 5x - 2y + 1 = 0$.

Note: In the above example, the coordinates of three points on the circumference were given. The data may sometimes be given in a different form and it is often necessary to use some other geometrical fact concerning the circle, e.g. the perpendicular bisector of a chord passes through the centre of the circle, in order to find the equation of the circle.

Intersecting circles

Consider two circles, of radius r_1 and r_2 ($r_1 > r_2$) with their centres distance d apart. There are a number of possible situations:

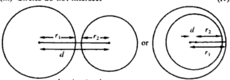
(i) Circles *touch* externally

$d = r_1 + r_2$

(ii) Circles *touch* internally

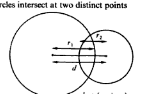
$d = r_1 - r_2$

(iii) Circles do not intersect



$d > (r_1 + r_2)$ $\qquad$ $d < (r_1 - r_2)$

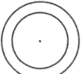
(iv) Circles intersect at two distinct points



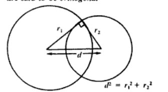
$d < (r_1 + r_2)$

Note also that:

(a) Circles having the same centre are said to be *concentric*



(b) Two circles that cut at right-angles are said to be *orthogonal*



$d^2 = r_1^2 + r_2^2$

Example 9

Prove that the circles $x^2 + y^2 + 2x - 8y + 5 = 0$ and $x^2 + y^2 - 4x - 4y + 7 = 0$ are orthogonal.

$x^2 + y^2 + 2x - 8y + 5 = 0$
i.e. $(x + 1)^2 + (y - 4)^2 = 12$
centre $(-1, 4)$ and radius is $\sqrt{12}$ units

$x^2 + y^2 - 4x - 4y + 7 = 0$
i.e. $(x - 2)^2 + (y - 2)^2 = 1$
centre $(2, 2)$ and radius is 1 unit

Now $r_1^2 + r_2^2 = (\sqrt{12})^2 + 1^2$
$= 13$

and the (distance between their centres)$^2 = (2 - (-1))^2 + (2 - 4)^2$
$= 9 + 4 = 13.$

Thus, since $d^2 = r_1^2 + r_2^2$, the circles $x^2 + y^2 + 2x - 8y + 5 = 0$ and $x^2 + y^2 - 4x - 4y + 7 = 0$ are orthogonal.

Example 10

Show that the circles $x^2 + y^2 - 2y - 4 = 0$ and $x^2 + y^2 - x + y - 12 = 0$ intersect in two distinct points and find the equation of the common chord.

The circles have equations $x^2 + y^2 - 2y - 4 = 0$ ... [1]
and $x^2 + y^2 - x + y - 12 = 0$ ... [2]
Consider the equation
$x^2 + y^2 - 2y - 4 - (x^2 + y^2 - x + y - 12) = 0$ ... [3]
This simplifies to $x - 3y + 8 = 0$, i.e. the equation of a straight line.
If the point, P(a, b), satisfies both equations [1] and [2], then it will also satisfy $x - 3y + 8 = 0$.
If some other point, Q(c, d), satisfies equations [1] and [2], then it also will satisfy $x - 3y + 8 = 0$.
But if this is so, then PQ is the common chord of the two circles.
Thus $x - 3y + 8 = 0$ is the equation of the common chord of the two circles.

Solving the equation of the common chord, $x - 3y + 8 = 0$
with one of the circles $x^2 + y^2 - 2y - 4 = 0,$
i.e. $(3y - 8)^2 + y^2 - 2y - 4 = 0$
giving $y^2 - 5y + 6 = 0$
i.e. $y = 2$, or $y = 3$, and then
substituting in the equation of the common chord, the distinct points of intersection are $(-2, 2)$ and $(1, 3)$.

Note: In this example we were able to find the common chord without having first to find the points of intersection.
If two circles are such that they do not intersect, then the equation analagous to equation [3] above, would, when solved with one of the circle equations, not yield any *real* points.

Notice that the graph of $(y - 2)^2 = 12(x - 1)$ is that of $y^2 = 12x$ translated by 1 unit parallel to the x-axis and 2 units parallel to the y-axis which is consistent with the ideas of section 11.3, page 284.

Example 13

Show that the line $y = 3x + 1$ *touches* the parabola $y^2 = 12x$.

If the given line *touches* the parabola, i.e. is a tangent to the parabola, solving the two equations $y = 3x + 1$ and $y^2 = 12x$ simultaneously should give a quadratic equation with a repeated root.

Substituting $y = 3x + 1$ into $y^2 = 12x$ gives $(3x + 1)^2 = 12x$

$\therefore \quad 9x^2 + 6x + 1 = 12x$

$9x^2 - 6x + 1 = 0$

$(3x - 1)(3x - 1) = 0$

$\therefore \quad x = \frac{1}{3}$ or $\frac{1}{3}$, i.e. a repeated root.

Thus the line $y = 3x + 1$ touches the parabola $y^2 = 12x$ at the point $(\frac{1}{3}, 2)$.

Gradient at a particular point

As with the circle, the gradient at a particular point on a parabola can be determined by implicit differentiation.

Example 14

Show that the point A(2, −4) lies on the parabola $y^2 = 8x$ and find the equation of the normal to the parabola at the point A.

Substituting $x = 2$, $y = -4$ in the equation $y^2 = 8x$, gives $(-4)^2 = 8(2)$, so the point A does lie on the parabola $y^2 = 8x$.

Differentiating implicitly, with respect to x,

$$2y\frac{dy}{dx} = 8$$

i.e. $$\frac{dy}{dx} = \frac{4}{y}.$$

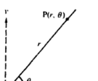
$\therefore$ the gradient at the point A(2, −4) is $\frac{4}{-4} = -1$

Thus the gradient of the normal at A is $+1$ and, using $y - y_1 = m(x - x_1)$, the equation of the normal at A is $(y - -4) = 1(x - 2)$.

Thus the equation of the normal at the point (2, −4) on the parabola $y^2 = 8x$ is $y = x - 6$.

3. Find the equation of the tangent to each of the following curves at the given point on the curve.
(a) $x^2 + 4y^2 = 4$ at $(\sqrt{3}, \frac{1}{2})$, (b) $x^2 + 4y^2 = 100$ at $(-8, 3)$,
(c) $9x^2 - y^2 = 9$ at $(-\frac{5}{3}, 4)$, (d) $4x^2 - y^2 = 4$ at $(\sqrt{2}, -2)$,
(e) $xy = 9$ at $(-3, -3)$, (f) $xy = 16$ at $(2, 8)$.
4. Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$ and find the coordinates of the point where this tangent cuts the y-axis.
5. (a) Find the equation of the tangent to the curve $xy = c^2$ at the point $(ct_1, c/t_1)$.
(b) Find the equation of the normal to the curve $xy = c^2$ at the point $(ct_2, c/t_2)$.
(c) If the tangent of (a) meets the normal of (b) on the y-axis show that $2t_2 = t_1(1 - t_2^4)$.
6. (a) Find the equation of the chord joining the point $(ct_1, c/t_1)$ to the point $(ct_2, c/t_2)$ on the hyperbola $xy = c^2$.
(b) By letting $t_1 = t_2 = t$ use your answer to part (a) to obtain the tangent to the curve $xy = c^2$ at the point $(ct, c/t)$.
7. Given that $y = mx + c$ is a tangent to $xy = d^2$ prove that $m = -\frac{c^2}{4d^2}$.
8. Given that $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove that $c^2 = a^2m^2 + b^2$.

16.6 Polar coordinates



We have, so far, used cartesian coordinates (x, y) in order to give the position of a point in a plane relative to two axes.
We will now investigate an alternative system of coordinates in which the position of any point P can be described in terms of its distance and direction from a fixed point.
Suppose that OA is a straight line with the point O fixed.
If the distance OP $= r$ and the angle AOP $= \theta$, then the position of the point P is defined by (r, θ) and we call these the polar coordinates of the point P. The angle θ is positive when measured in an anticlockwise direction from the line OA, and is called the vectorial angle.
r is known as the radius vector.
It should be noted that different polar coordinates may be used to describe the same point, e.g. $\left(2, \frac{\pi}{2}\right)$ could also be written as $\left(2, \frac{5\pi}{2}\right)$, $\left(2, \frac{9\pi}{2}\right)$, $\left(2, -\frac{3\pi}{2}\right)$ etc.
For this reason it is usual to state the polar coordinates of a point (r, θ) with $-\pi \leqslant \theta \leqslant \pi$ or $0 \leqslant \theta \leqslant 2\pi$.
Also, it is possible to give polar coordinates with a negative value of r so that $\left(-1, \frac{\pi}{3}\right)$ and $\left(1, -\frac{2\pi}{3}\right)$ in fact represent the same point. In practice it is usual to state (r, θ) with $r \geqslant 0$.
If OA in the diagram is taken as the x-axis and the y-axis be taken through the point O as indicated, then clearly the cartesian coordinates of P(x, y) can be written as $x = r \cos \theta$, and $y = r \sin \theta$.

In the same way that all the points lying on a line can be described in terms of an equation involving the cartesian coordinates x and y, so also can we describe the line in terms of a polar equation involving r and θ.
Using this polar form of the equation of a curve we can plot values of r against θ and hence represent the curve graphically. Graph paper designed for this specific purpose is obtainable, but the reader will find it possible to obtain sufficiently accurate graphs using plane paper.

Example 21

Copy and complete the following table for $r = 3 \sin \theta$.

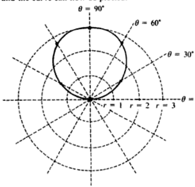
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	0		2·6		2·6			−1·5	−2·6		−2·6		

Hence plot the graph of $r = 3 \sin \theta$ for $0° \leqslant \theta \leqslant 360°$.

Completing the table for $r = 3 \sin \theta$, gives

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	0	1·5	2·6	3·0	2·6	1·5	0	−1·5	−2·6	−3·0	−2·6	−1·5	0

and the curve can now be plotted.



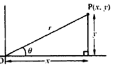
Converting between polar equations and cartesian equations

From the diagram on the right, we have

$x = r \cos \theta, \qquad y = r \sin \theta$

and $r^2 = x^2 + y^2, \qquad \tan \theta = \dfrac{y}{x}$.

Using these four relationships, we can convert polar equations to the equivalent cartesian equations and vice-versa.



Example 22

Express (a) $x^2 + xy = 3$ in polar form, (b) $r = 3 - \sin\theta$ in cartesian form.

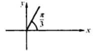
(a)
$$x^2 + xy = 3$$
Substituting $x = r\cos\theta$ and $y = r\sin\theta$
gives $r^2\cos^2\theta + r^2\cos\theta\sin\theta = 3$.
i.e. $r^2\cos\theta(\cos\theta + \sin\theta) = 3$.

(b)
$$r = 3 - \sin\theta$$
In order to be able to use $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we first multiply the polar equation by r:
$$r^2 = 3r - r\sin\theta$$
$\therefore$ $x^2 + y^2 = 3\sqrt{(x^2 + y^2)} - y$
i.e. $(x^2 + y^2 + y)^2 = 9(x^2 + y^2)$
which is the required cartesian form.
Note that in this case the polar form of the equation is simpler than the cartesian form.

The following are some particular types of polar equations:

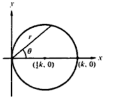
(a) $r = k$, a constant — From $r^2 = x^2 + y^2$, we see that $r = k$ gives $x^2 + y^2 = k^2$, i.e. $r = k$ is the equation of a circle, centre (0, 0), radius k.

(b) $\theta = k$, a constant — Now $\tan\theta = \dfrac{y}{x}$ and so $\dfrac{y}{x} = m$ where $m = \tan k$
i.e. $y = mx$
Thus $\theta = k$ is the equation of a straight line.
In fact, if we restrict r to positive values only, then $\theta = k$ describes a 'half line', e.g. $\theta = \dfrac{\pi}{3}$ as shown.

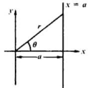


(c) $r = k\theta$ — This form of polar equation will give a spiral (see Exercise 16F Question 6).

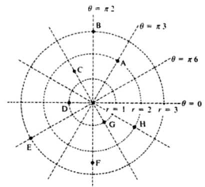
(d) $r = k\cos\theta$ — Multiplying by r gives $r^2 = kr\cos\theta$
$\therefore$ $x^2 + y^2 = kx$
i.e. $(x - \tfrac{1}{2}k)^2 + y^2 = \tfrac{1}{4}k^2$
Thus $r = k\cos\theta$ is the equation of a circle centre $(\tfrac{1}{2}k, 0)$ and radius $\tfrac{1}{2}k$ as shown.



(e) $r = a\sec\theta$ — If $r = a\sec\theta$ then $r = \dfrac{a}{\cos\theta}$
$\therefore$ $1 = \dfrac{a}{r\cos\theta}$ or $1 = \dfrac{a}{x}$
thus $x = a$.
i.e. $r = a\sec\theta$ is the polar equation of the straight line $x = a$ as shown.



Exercise 16F



1. Write down the polar coordinates of the points A to H shown in the diagram, giving your answers in the form (r, θ) for $0 \leqslant \theta \leqslant 2\pi$ and $r \geqslant 0$

2. Find the cartesian coordinates of the points P to U having polar coordinates as follows: P(4, 30°) Q(5, 90°) R(2, −90°) S($5\sqrt{2}$, 135°) T(6, −120°) U(−3, 90°).

3. Express each of the following cartesian equations in polar form, giving your answers in the form $r = f(\theta)$ or $r^2 = f(\theta)$ as appropriate.

(a) $x = 4$ (b) $x^2 + y^2 = 4y$ (c) $2xy = 1$ (d) $y^2 = 8x$

(e) $(x - 1)^2 + y^2 = 1$ (f) $x^2 + y^2 = 4x$ (g) $x^2 - y^2 = 8$ (h) $(x - y)^2 = 4$

4 Express each of the following polar equations in cartesian form.

(a) $r \sin \theta = 1$ (b) $r = 3$ (c) $\theta = \frac{\pi}{3}$ (d) $r = \sin \theta$

(e) $r = \sin \theta + \cos \theta$ (f) $r^2 = \sec 2\theta$ (g) $r = 3 \tan \theta$ (h) $3 = r(1 - \sin \theta)$

5. Copy and complete the following table for $r = 3 \cos \theta$.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r		2·6	1·5		−1·5	−2·6		−2·6				2·6	

Hence plot the graph of $r = 3 \cos \theta$ for $0° \leqslant \theta \leqslant 360°$.

6. Copy and complete the following table for $r = \frac{3\theta}{2\pi}$, (θ in radians).

θ	0	$\frac{1}{3}\pi$	$\frac{2}{3}\pi$	π	$\frac{4}{3}\pi$	$\frac{5}{3}\pi$	2π	$\frac{7}{3}\pi$	$\frac{8}{3}\pi$	3π	$\frac{10}{3}\pi$	$\frac{11}{3}\pi$	4π
r	0	$\frac{1}{2}$	1	$1\frac{1}{2}$									

Hence plot the graph of $r = \frac{3\theta}{2\pi}$ for $0 \leqslant \theta \leqslant 4\pi$.

7. Copy and complete the following table for $r = 2(1 + \cos \theta)$

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	4	3·73				0·27		0·27				3·73	

Hence plot the graph of $r = 2(1 + \cos \theta)$ for $0° \leqslant \theta \leqslant 360°$.

Exercise 16G Examination questions

1. Find the coordinates of the maximum point T and the minimum point B of the curve

$$y = \frac{x^3}{3} - 2x^2 + 3x.$$

Find also the point of inflexion I and show that T, I, B are collinear. Calculate to the nearest 0·1° the acute angle between TIB and the normal to the curve at I. (London)

2. Obtain the equation of the normal to the curve

$$y^2 = x^3$$

at the point (t^2, t^3). Show that the equation of the normal at the point where $t = \frac{1}{2}$ is

$$32x + 24y - 11 = 0.$$

Find the perpendicular distance from the point $(-1, 2)$ to this normal. (J.M.B.)

3. The equation of a circle is $x^2 + y^2 - 3x - 4 = 0$.
Find (i) the coordinates of its centre,
(ii) its radius,
(iii) the coordinates of the points at which it cuts the axes.
Show that the line whose equation is $3x + 4y = 17$ touches the circle and find the coordinates of its point of contact.
Show also that this line and the tangent to the circle at the point $(3, -2)$ intersect at a point on the x-axis and find its coordinates. (J.M.B.)

4. Find, *by calculation*, the coordinates of the centre and the radius of the circle which passes through the points $(1, 1)$, $(3, 5)$ and $(-3, 1)$. (A.E.B)

5. Verify that the circle with equation

$$x^2 + y^2 - 2rx - 2ry + r^2 = 0$$

touches both the coordinate axes.
Find the radii of the two circles which pass through the point $(16, 2)$ and touch both the coordinate axes. (Cambridge)

6. Find the values of m such that $y = mx$ is a tangent from the origin O $(0, 0)$ to the circle whose equation is $(x - 3)^2 + (y - 4)^2 = 1$.
Find the cosine of the acute angle between these tangents. (A.E.B)

7. Find the equations of the two circles which pass through the point $(2, 0)$ and have both the y-axis and the line $y - 1 = 0$ as tangents.
Calculate the coordinates of the second point at which the circles intersect. (A.E.B)

8. The fixed points A and B have coordinates $(-3a, 0)$ and $(a, 0)$ respectively. Find the equation of the locus of a point P which moves in the coordinate plane so that AP = 3PB. Show that the locus is a circle, S, which touches the axis of y and has its centre at the point $(\frac{3}{2}a, 0)$.
A point Q moves in such a way that the perpendicular distance of Q from the axis of y is equal to the length of a tangent from Q to the circle S. Find the equation of the locus of Q.
Show that this locus is also the locus of points which are equidistant from the line $4x + 3a = 0$ and the point $(\frac{3}{4}a, 0)$. (Cambridge)

9. A point P moves in the x–y plane so that its distance from the origin, O, is twice its distance from the point with coordinates $(3a, 0)$. Show that the locus of P is a circle and obtain the coordinates of its centre and its radius. If the circle meets the x-axis in A and B, where $OA < OB$, find the coordinates of A and B. If the tangents from O to the circle are OL and OM, find the angle LOM and the equations of OL and OM. Calculate the area of the triangle enclosed by the lines OL, OM and the tangent to the circle at A. (S.U.J.B)

10. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. The straight line $4x - 9y + 8a = 0$ meets the parabola at the points P and Q; the normals to the parabola at the points P and Q meet at R. Find the coordinates of R, and verify that it lies on the parabola. (Oxford)

11. Prove that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ on the curve is

$$py = x + ap^2.$$

Find the coordinates of the point of intersection, T, of the tangents at P and $Q(ap^2, 2aq)$, simplifying your answers where possible.
Given that S is the point $(a, 0)$, verify that $SP \,.\, SQ = ST^2$. (Cambridge)

12. The tangent to the curve $4ay = x^2$ at the point $P(2at, at^2)$ meets the x-axis at the point Q. The point S is $(0, a)$.
(a) Prove that PQ is perpendicular to SQ.
(b) Find a cartesian equation for the locus of the point, M, the mid-point of PS. (A.E.B.)

13. Prove that the chord joining the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ has the equation

$$(p + q)y = 2r + 2apq.$$

A variable chord PQ of the parabola is such that the lines OP and OQ are perpendicular, where O is the origin.
(i) Prove that the chord PQ cuts the axis of x at a fixed point, and give the x-coordinate of this point.
(ii) Find the equation of the locus of the mid-point of PQ. (Cambridge)

14. The point P lies on the ellipse $x^2 + 4y^2 = 1$ and N is the foot of the perpendicular from P to the line $x = 2$. Find the equation of the locus of the mid-point of PN as P moves on the ellipse. (Cambridge)

15. Show that the x coordinates of any points of intersection of the line $y = mx + c$ and the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are given by the solutions of the quadratic equation $(4 + 9m^2)x^2 + 18mcx + (9c^2 - 36) = 0$.
If the line $y = mx + c$ is a tangent to the ellipse, prove that $c^2 = 4 + 9m^2$. The line $y = mx + c$ passes through the point $(2, 3)$. Write down a second equation connecting m and c, and hence prove that m must satisfy the equation $5m^2 + 12m - 5 = 0$.
Prove that the two tangents drawn from the point $(2, 3)$ to the ellipse are perpendicular to each other. (S.U.J.B.)

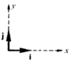
20. The tangent at the point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, on the rectangular hyperbola $xy = c^2$ meets the x-axis at A and the y-axis at B. The normal at P to the rectangular hyperbola meets the line $y = x$ at C and the line $y = -x$ at D.
(a) Show that P is the mid-point of both AB and CD.
(b) Prove that the points A, B, C and D form the vertices of a square. The normal at P meets the hyperbola again at the point Q and the mid-point of PQ is M.
(c) Prove that, as t varies, the point M lies on the curve
$$c^2(x^2 - y^2)^2 + 4x^3y^3 = 0.$$
(A.E.B.)

21. Find a polar equation of the curve $x^2 + y^2 = 4x$ and calculate the polar coordinates of the two points P and Q where the curve intersects the line $r = 2\sqrt{2} \sec\left(\frac{\pi}{4} - \theta\right)$.
Find the polar equations of the two half-lines from the origin which are tangents to the circle which has PQ as diameter. (London)

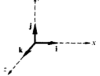
17
Three-dimensional work: vectors and matrices

17.1 Three-dimensional work

From the work of earlier chapters the reader is familar with the use of coordinates to define a point in a plane in terms of its distance from two mutually perpendicular axes x and y. The position vector of such a point has been expressed in the form $a\mathbf{i} + b\mathbf{j}$, where $\mathbf{i}$ is a unit vector in the direction of the x-axis and $\mathbf{j}$ is a unit vector in the direction of the y-axis.



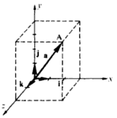
To consider points that are not coplanar we must introduce a third axis, the z-axis, and a corresponding third unit vector $\mathbf{k}$ in the direction of this axis.



Suppose the point A shown in the diagram has coordinates (3, 4, 2).
If $\mathbf{a}$ is the position vector of point A then

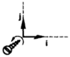
$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

or this may be written as $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

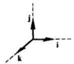


Note The convention for determining the direction of $\mathbf{k}$ is to use the 'right-handed screw rule'. For this we consider the motion of a screw at O, lying perpendicular to the $\mathbf{i} - \mathbf{j}$ plane, turning through the right angle from $\mathbf{i}$ to $\mathbf{j}$.

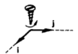
Thus if we draw:



the rotation will move the screw 'out of the page':



if we draw:



the rotation will move the screw 'upwards':

The reader will remember from chapter 2 that if $\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{s} = x_2\mathbf{i} + y_2\mathbf{j}$ then $|\mathbf{r}| = \sqrt{(x_1^2 + y_1^2)}$ and $\mathbf{r} \cdot \mathbf{s} = x_1x_2 + y_1y_2$
$= |\mathbf{r}||\mathbf{s}| \cos\theta$ where θ is the angle between **r** and **s**.

These results can be extended to three-dimensional vectors so that if $\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{s} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ then
$|\mathbf{r}| = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$ and $\mathbf{r} \cdot \mathbf{s} = x_1x_2 + y_1y_2 + z_1z_2$
$= |\mathbf{r}||\mathbf{s}| \cos\theta$
where θ is the angle between **r** and **s**.

Example 1

Find, to the nearest degree, the angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

If θ is the angle between the vectors then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
thus $2 + 2 - 3 = \sqrt{14}\sqrt{6} \cos\theta$
giving $\theta = 84°$ to the nearest degree.

As we saw in chapter 2, if three vectors are coplanar any one of the vectors can be expressed as a combination of scalar multiples of the other two. The converse statement is also true, i.e. if a vector can be expressed as a combination of scalar multiples of two other vectors then the three vectors are coplanar.

Example 2

Show that the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ -10 \\ 7 \end{pmatrix}$ are coplanar.

Suppose that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$:

$$\begin{pmatrix} 1 \\ -10 \\ 7 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

Then $1 = \lambda + 6\mu$, $-10 = -3\lambda - 4\mu$ and $7 = 2\lambda + 2\mu$.
Solving the first two of these simultaneously gives $\lambda = 4$ and $\mu = -\frac{1}{2}$ and these values are compatible with $7 = 2\lambda + 2\mu$.
Thus we can write $\mathbf{c} = 4\mathbf{a} - \frac{1}{2}\mathbf{b}$, hence **a**, **b** and **c** are coplanar.

Example 3

Find a vector that is perpendicular to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

There are many vectors of the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ that are perpendicular to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ but for all of them
$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 0$
i.e. $2a + 3b - c = 0$
Suppose we take $a = b = 1$ then $c = 5$
Thus the vector $\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ is perpendicular to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Example 6

Show that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ form a set of base vectors for three dimensional space and express the vector $\mathbf{d} = 5\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$.

The vectors **a**, **b** and **c** will form a set of base vectors provided
(a) no two of the vectors are parallel to each other and
(b) the vectors are not coplanar.

Condition (a) is true as no one of the vectors **a**, **b** or **c** is a scalar multiple of either of the other two.
(b) If **a**, **b** and **c** were coplanar we could write that
$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ where α and β are scalars
i.e. $3\mathbf{i} + 8\mathbf{j} - \mathbf{k} = \alpha(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \beta(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
Equating coefficients of **i**, **j** and **k**: $3 = 2\alpha + \beta$, $8 = 3\alpha - 2\beta$ and $-1 = \alpha + \beta$.
Solving the first two of these simultaneously gives $\alpha = 2$ and $\beta = -1$.
However, these values are not consistent with $-1 = \alpha + \beta$ and so we cannot write $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$. Thus **a**, **b** and **c** are not coplanar.
Therefore **a**, **b** and **c** form a set of base vectors for three dimensional space.

If $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$

$5\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} = \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \eta(3\mathbf{i} + 8\mathbf{j} - \mathbf{k})$

$\therefore \quad 5 = 2\lambda + \mu + 3\eta \quad \ldots [1]$

$-3 = 3\lambda - 2\mu + 8\eta \quad \ldots [2]$

$6 = \lambda + \mu - \eta \quad \ldots [3]$

equation [1] minus equation [3] gives: $-1 = \lambda + 4\eta$
equation [2] added to twice equation [3] gives: $9 = 5\lambda + 6\eta$
Thus $\lambda = 3$, $\mu = 2$ and $\eta = -1$
$\therefore \quad \mathbf{d} = 3\mathbf{a} + 2\mathbf{b} - \mathbf{c}$

Exercise 17A

1. If $\mathbf{a} = 9\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find
(a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) $|\mathbf{c}|$ (d) $\mathbf{a}\cdot\mathbf{b}$ (e) $\mathbf{b}\cdot\mathbf{c}$
(f) the angle between **a** and **b** (to the nearest degree)
(g) the angle between **b** and **c** (to the nearest degree).
2. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ find
(a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) the angle between **a** and **b** (to the nearest degree).
3. If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ find $\hat{\mathbf{a}}$, a unit vector in the direction of **a**.
4. If $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ find $\hat{\mathbf{b}}$, a unit vector in the direction of **b**.
5. Find a vector that is perpendicular to $5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
6. Find a unit vector that is perpendicular to $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
7. State which of the vectors **a**, **b**, **c** or **d** listed below are perpendicular to the vector $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.
$\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ $\quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\mathbf{c} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ $\quad \mathbf{d} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

8. Find the direction ratios and direction cosines of the vector $14\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.
9. Vector $\mathbf{r}$ is at right angles to both $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. Find the direction ratios of $\mathbf{r}$.
10. (a) Find a vector that is perpendicular to both $3\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
    (b) Find the direction cosines of the vector obtained for part (a)
    (c) Find a unit vector that is perpendicular to both $3\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
11. Points A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ and $5\mathbf{i} + 12\mathbf{j} - 7\mathbf{k}$ respectively. Prove that A, B and C are collinear.
12. Points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} - \mathbf{k}$ respectively. Prove that angle $B\hat{A}C$ is a right angle.
13. Prove that the vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = 4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ are coplanar.
14. Prove that the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ are coplanar.
15. If the points A, B and C have position vectors $\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} + 9\mathbf{k}$ and $8\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ respectively find the angles of triangle ABC giving your answers to the nearest degree.
16. Prove that the sum of the squares of the direction cosines of any vector is one.
17. Prove that the vectors $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ form a set of base vectors for three dimensional space and express the vector $\mathbf{d} = -3\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$.
18. Prove that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ form a set of base vectors for three dimensional space and express the vector $\mathbf{d} = \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$.

17.2 Differentiation and integration of vectors

Suppose that $\mathbf{a} = f(x)\mathbf{i} + g(x)\mathbf{j} + h(x)\mathbf{k}$,

then $\frac{d\mathbf{a}}{dx} = f'(x)\mathbf{i} + g'(x)\mathbf{j} + h'(x)\mathbf{k}$.

i.e. to differentiate a vector with respect to some variable x we differentiate each component with respect to that variable.

Similarly, to integrate a vector with respect to some variable x we integrate each component with respect to that variable.
In particular, for a particle which has a displacement vector $\mathbf{s}$, velocity

vector **v** and acceleration vector **a** given in terms of the time, t, we can link these vectors by the method shown on page 308 i.e.

Differentiation →

displacement **s** velocity **v** acceleration **a**

← Integration

Example 7

If $\mathbf{y} = 8p^2\mathbf{i} + (8p - 4)\mathbf{j} + (p^3 + 3)\mathbf{k}$, find $\frac{d\mathbf{y}}{dp}$ and $\frac{d^2\mathbf{y}}{dp^2}$

$$\mathbf{y} = 8p^2\mathbf{i} + (8p - 4)\mathbf{j} + (p^3 + 3)\mathbf{k}$$

$$\therefore \quad \frac{d\mathbf{y}}{dp} = 16p\mathbf{i} + 8\mathbf{j} + 3p^2\mathbf{k}$$

$$\text{and} \quad \frac{d^2\mathbf{y}}{dp^2} = 16\mathbf{i} + 6p\mathbf{k}$$

Example 8

If the velocity of a body at time t is given by $\mathbf{v} = 3t^2\mathbf{i} - 2t\mathbf{j} + 4\mathbf{k}$, find expressions for the acceleration, **a**, and the displacement, **s**, of the body at time t, given that when $t = 1$, $\mathbf{s} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$$\mathbf{v} = 3t^2\mathbf{i} - 2t\mathbf{j} + 4\mathbf{k}$$

$$\therefore \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = 6t\mathbf{i} - 2\mathbf{j}$$

$$\text{also} \quad \mathbf{s} = \int \mathbf{v}\,dt = t^3\mathbf{i} - t^2\mathbf{j} + 4t\mathbf{k} + \mathbf{c}$$

but when $t = 1$, $\quad \mathbf{s} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\quad \therefore \quad \mathbf{c} = 2\mathbf{i} - 2\mathbf{k}$

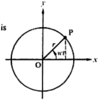
$\therefore \quad \mathbf{s} = (t^3 + 2)\mathbf{i} - t^2\mathbf{j} + 2(2t - 1)\mathbf{k}$

Thus at time t the displacement of the body is given by $\mathbf{s} = (t^3 + 2)\mathbf{i} - t^2\mathbf{j} + 2(2t - 1)\mathbf{k}$ and the acceleration is given by $\mathbf{a} = 6t\mathbf{i} - 2\mathbf{j}$.

Exercise 17B

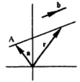
1. Differentiate each of the following vectors with respect to t.
   (a) $3t\mathbf{i} + 4t^2\mathbf{j}$ (b) $(3t + 1)\mathbf{i} + 2t\mathbf{j} - 5t^2\mathbf{k}$
   (c) $3\mathbf{i} + 2t\mathbf{j} - 5\mathbf{k}$ (d) $(6t - 1)\mathbf{i} + 3t^3\mathbf{j} - 6t\mathbf{k}$
2. Differentiate each of the following vectors with respect to θ.
   (a) $\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ (b) $2\cos\theta\mathbf{i} + 2\theta\mathbf{j}$
   (c) $\sin\theta\mathbf{i} - \cos\theta\mathbf{j} + \theta^2\mathbf{k}$ (d) $\sin 3\theta\mathbf{j} - \cos^3\theta\mathbf{k}$
3. The position vector of a particle at time **t** seconds is given by $\mathbf{s} = (2t^3\mathbf{i} - 2t\mathbf{j} + \mathbf{k})$ metres.
   Find the displacement, velocity and acceleration vectors when $\mathbf{t} = 2$.

4. The velocity vector of a particle at time t is given by $\mathbf{v} = 2t\mathbf{i} + 3t^2\mathbf{j} + 2\mathbf{k}$. Given that when $t = 0$ the particle has position vector $\mathbf{i} + \mathbf{j}$ with respect to an origin O, find the position vector of the particle with respect to O when $t = 3$.
5. At time t the displacement of a particle from an origin O is given by $\mathbf{s} = (2 \sin(\pi t)\mathbf{i} + 2 \cos(\pi t)\mathbf{j})$ metres. Prove that the particle is always 2 metres from O and find the velocity and speed of the particle when $t = 2$ seconds.
6. The acceleration of a body at time t is given by $\mathbf{a} = 4t\mathbf{j}$. Find expressions for the velocity, $\mathbf{v}$, and the displacement, $\mathbf{s}$, from an origin O at time t given that when $t = 0$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{s} = 3\mathbf{k}$.
7. A body moves such that its position vector when at point P is given by $\overrightarrow{OP} = 3 \sin 5t\mathbf{i} + 3 \cos 5t\mathbf{j}$ where O is the origin and t is the time. Prove that the velocity of the particle when at P is perpendicular to $\overrightarrow{OP}$.
8. For a body moving in a circle of radius r, at constant angular speed ω, the position vector $\mathbf{s}$, of the body at time t is given by $\mathbf{s} = \overrightarrow{OP} = r \cos \omega t\mathbf{i} + r \sin \omega t\mathbf{j}$.
Prove that the acceleration of the body has constant magnitude $r\omega^2$ and is directed along $\overrightarrow{PO}$ (see diagram).



17.3 Equations of a straight line in 3-D

As we saw in section 2.5 a line that passes through the point with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$ has vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. Thus a line passing through the point A, position vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and which is parallel to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ has vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. This may also be written in column vector form, $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$



Example 9

Find the vector equation of the line which is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and passes through a point with position vector $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

The vector equation is $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$ where $\mathbf{r}$ is the position vector of a general point on the line and λ is a scalar.

Example 10

Show that the point with position vector $\mathbf{i} - 9\mathbf{j} + \mathbf{k}$ lies on the line L, vector equation $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 6\mathbf{j} - \mathbf{k})$.

The position vector of any point on a line must satisfy the equation of the line. Thus if $\mathbf{i} - 9\mathbf{j} + \mathbf{k}$ lies on L there must exist some valve of λ such that

$$\mathbf{i} - 9\mathbf{j} + \mathbf{k} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 6\mathbf{j} - \mathbf{k})$$

i.e. $1 = 3 + \lambda$, $-9 = 3 + 6\lambda$ and $1 = -1 - \lambda$.
The first of these gives $\lambda = -2$ which is consistent with $-9 = 3 + 6\lambda$ and $1 = -1 - \lambda$. Thus the point with position vector $\mathbf{i} - 9\mathbf{j} + \mathbf{k}$ does lie on L.

Example 11

The vector equations of three lines are stated below:
line 1: $\mathbf{r} = 17\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} + \lambda(-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k})$
line 2: $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(6\mathbf{i} + 7\mathbf{j} - \mathbf{k})$
line 3: $\mathbf{r} = 2\mathbf{i} - 12\mathbf{j} - \mathbf{k} + \eta(-3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
State which pair of lines (a) are parallel to each other,
(b) intersect with each other,
(c) are not parallel and do not intersect (i.e. are skew).

(a) line 1 is parallel to $-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$
line 2 is parallel to $6\mathbf{i} + 7\mathbf{j} - \mathbf{k}$
line 3 is parallel to $-3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$(-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k})$ is parallel to $(-3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ as one is a scalar multiple of the other. Thus lines 1 and 3 are parallel.

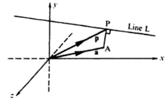
(b) If two lines intersect there will exist a point whose position vector satisfies the vector equations of both lines.
If lines 1 and 2 intersect there will exist values of λ and μ such that
$17\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} + \lambda(-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(6\mathbf{i} + 7\mathbf{j} - \mathbf{k})$
i.e. $17 - 9\lambda = 2 + 6\mu$, $2 + 3\lambda = -3 + 7\mu$ and $-6 + 9\lambda = 4 - \mu$
Solving $15 = 9\lambda + 6\mu$ and $10 = 9\lambda + \mu$ simultaneously gives $\mu = 1$ and $\lambda = 1$
However this is inconsistent with $5 = 7\mu - 3\lambda$
Thus lines 1 and 2 do not intersect.

If lines 2 and 3 intersect there will exist values of μ and η such that
$2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(6\mathbf{i} + 7\mathbf{j} - \mathbf{k}) = 2\mathbf{i} - 12\mathbf{j} - \mathbf{k} + \eta(-3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
i.e. $2 + 6\mu = 2 - 3\eta$, $-3 + 7\mu = -12 + \eta$ and $4 - \mu = -1 + 3\eta$
or $2\mu = -\eta$, $9 = \eta - 7\mu$ and $5 = \mu + 3\eta$
Solving $2\mu = -\eta$ and $9 = \eta - 7\mu$ simultaneously gives $\eta = 2$ and $\mu = -1$ which is consistent with $5 = \mu + 3\eta$
Thus lines 2 and 3 intersect.

(c) From the working of parts (a) and (b) it is clear that lines 1 and 2 are not parallel and do not intersect.

Example 12

Find the perpendicular distance from the point A, position vector $\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ to the line L, vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$



Let P be the point where the perpendicular from the point A meets the line L, and let P have position vector **p**.
Suppose that P is the point on the line L for which $\lambda = \lambda_1$, then

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 3\lambda_1 \\ 2 - \lambda_1 \\ 3 + 2\lambda_1 \end{pmatrix}$$

$$\therefore \quad \vec{PA} = \mathbf{a} - \mathbf{p} = \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 + 3\lambda_1 \\ 2 - \lambda_1 \\ 3 + 2\lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 3\lambda_1 \\ -5 + \lambda_1 \\ 7 - 2\lambda_1 \end{pmatrix}$$

But $\vec{PA}$ is perpendicular to L and L is parallel to $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Thus $\begin{pmatrix} 3 - 3\lambda_1 \\ -5 + \lambda_1 \\ 7 - 2\lambda_1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$ i.e. $9 - 9\lambda_1 + 5 - \lambda_1 + 14 - 4\lambda_1 = 0$
giving $\lambda_1 = 2$

Thus $\vec{PA} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$ and $|\vec{PA}| = \sqrt{[(-3)^2 + (-3)^2 + (3)^2]}$
$= 3\sqrt{3}$ units

The perpendicular distance from the point A to the line L is $3\sqrt{3}$ units.

Cartesian equation of a line in three dimensions

Consider a line that is parallel to the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and which passes through the point A, position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If the general point on this line has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then the vector equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Thus $\left.\begin{aligned} x &= a + \lambda p \\ y &= b + \lambda q \\ \text{and} \quad z &= c + \lambda r \end{aligned}\right\}$ These are the parametric equations of the line, using the parameter λ.

Isolating λ in each equation gives $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} (= \lambda)$. These are the cartesian equations of the line

Thus the line with vector equation $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ has cartesian

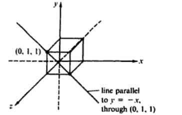
equations: $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$

Notes

1. Given the cartesian equation of a line in the above form it is easy to obtain the vector equation by remembering that the numerator gives the position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ of a point on the line and the denominator gives the direction vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$.
2. It is acceptable to give the cartesian equation of a line in the above form even when one or more of p, q and r are zero.

For example, the line through (0, 1, 1) and parallel to $y = -x$, i.e. parallel to the vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, has vector equation

$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, (see diagram).



This gives the parametric equations $\begin{cases} x = -\lambda \\ y = 1 + \lambda \\ z = 1 \end{cases}$

The cartesian equations of the line can then be written $\frac{x}{-1} = \frac{y-1}{1} = \frac{z-1}{0}$ or simply as $\frac{x}{-1} = \frac{y-1}{1}$, $z = 1$.

Example 13

Find the cartesian equations of the line that is parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and which passes through the point A, position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

The vector equation of the line is $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

thus
$$x = 3 + 2\lambda$$
$$y = -1 + 3\lambda$$
$$z = 2 + 4\lambda$$

The cartesian equations are therefore $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{4} (= \lambda)$

solving simultaneously gives $x_1 = 4$ and $y_1 = 6$
Using these values of x_1 and y_1 equations [1] and equations [2] both give $z_1 = 1$.
Thus the lines L_1 and L_2 do intersect and the point of intersection has coordinates (4, 6, 1).

Exercise 17C

1. State the vector equation of the line which is parallel to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and which passes through the point A, position vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
2. State the vector equation of the line which passes through the point B, position vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and which is parallel to the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
3. State a vector that is parallel to the line with vector equation $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$.
4. Show that the point with position vector $4\mathbf{i} - \mathbf{j} + 12\mathbf{k}$ lies on the line with vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
5. Points A, B and C have position vectors $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.
6. Points D, E and F have position vectors $\mathbf{i} - 2\mathbf{j}$, $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$.
7. If the point A, position vector $a\mathbf{i} + b\mathbf{j} + 3\mathbf{k}$, lies on the line L, vector equation $\mathbf{r} = (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, find the values of a and b.
8. Points A and B have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ repectively. Find
   (a) $\overrightarrow{AB}$ in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$,
   (b) the vector equation of the line that passes through A and B.
9. Find the cartesian equations of the lines with vector equations
   (a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$,
   (b) $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$,
   (c) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \eta(2\mathbf{i} - \mathbf{j} - \mathbf{k})$.
10. Find the vector equation of the line with parametric equations $x = 2 + 3\lambda$, $y = 5 - 2\lambda$, $z = 4 - \lambda$
11. Find the vector equations of the lines with the following cartesian equations
    (a) $\frac{x-2}{3} = \frac{y-2}{2} = \frac{z+1}{4}$ (b) $x - 3 = \frac{y+2}{4} = \frac{z-3}{-1}$
12. Lines L_1 and L_2 have vector equations $\mathbf{r} = 8\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = -2\mathbf{i} + 8\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ respectively. Show that L_1 and L_2 intersect and find the position vector of the point of intersection.

13. Lines L_1 and L_2 have vector equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \text{ respectively. Show}$$

that L_1 and L_2 intersect and find the position vector of the point of intersection.

14. For each of the following pairs of lines state whether the two lines intersect and, for those that do, give the coordinates of the point of intersection.

(a) $\dfrac{x-1}{2} = \dfrac{y+2}{1} = \dfrac{z-3}{-1}$; $\dfrac{x+1}{-2} = y - 3 = \dfrac{z-7}{2}$.

(b) $\dfrac{x-1}{1} = \dfrac{y+1}{3} = \dfrac{z-2}{-1}$; $\dfrac{x+3}{-2} = \dfrac{y-8}{1} = \dfrac{z+2}{-1}$.

(c) $x - 2 = \dfrac{y+3}{4} = \dfrac{z-5}{2}$; $\dfrac{x-1}{-1} = \dfrac{y-8}{1} = \dfrac{z-3}{-2}$.

(d) $\dfrac{x-2}{5} = \dfrac{y-3}{-3} = \dfrac{z+1}{2}$; $\dfrac{x-9}{-3} = \dfrac{y-2}{5} = \dfrac{z-2}{-1}$.

15. For each of the pairs of lines given by the following vector equations state whether the lines are parallel lines, non-parallel coplanar lines or skew lines.

(a) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$.

(b) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j})$.

(c) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$.

(d) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = -8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

16. Find the acute angle between the lines with vector equations $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, giving your answer to the nearest degree.

17. Find the acute angle between the lines whose equations are $\dfrac{x-2}{-4} = \dfrac{y-3}{3} = \dfrac{z+1}{-1}$ and $\dfrac{x-3}{2} = \dfrac{y-1}{6} = \dfrac{z+1}{-5}$, giving your answer to the nearest degree.

18. The vector equations of three lines are:

line 1 $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$
line 2 $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \mu(-\mathbf{i} - 2\mathbf{k})$
line 3 $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \eta(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

(a) Show that lines 1 and 2 intersect and find the position vector of the point of intersection.
(b) Show that lines 2 and 3 intersect and find the position vector of the point of intersection.
(c) Find the distance between these two points of intersection.

19. Two lines L_1 and L_2 lie in the x–y plane and have cartesian equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Show that the vector equations of L_1 and L_2 can be written

$$\mathbf{r}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m_1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ m_2 \end{pmatrix}.$$

Use vector methods to show that if θ is the angle between L_1 and L_2

then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

(i.e. obtain the result of page 380 by vector methods).

20. For each of the following parts find the perpendicular distance from the given point to the given line,

(a) the point with position vector $\begin{pmatrix}4\\2\\2\end{pmatrix}$ and the line $\mathbf{r} = \begin{pmatrix}3\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$

(b) the point with position vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and the line $\mathbf{r} = \mathbf{i} - 6\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

(c) the point (1, 1, 3) and the line $\dfrac{x+4}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{3}$

(d) the point (−6, −4, −5) and the line $x - 5 = \dfrac{y-6}{2} = \dfrac{z-3}{4}$

21. Find the distance between the pairs of parallel lines listed below

(a) $\mathbf{r} = \begin{pmatrix}2\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$

(b) $\dfrac{x-2}{1} = \dfrac{y-1}{-1} = \dfrac{z-3}{2}$ and $\dfrac{x+1}{1} = \dfrac{y-3}{-1} = \dfrac{z-1}{2}$

22. Find the shortest distance between the two skew lines L_1 and L_2 given that L_1 has vector equation $\mathbf{r} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ and L_2 has vector equation $\mathbf{r} = \begin{pmatrix}0\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\1\\3\end{pmatrix}$

23. Find the shortest distance between the two skew lines given in each of the following parts.

(a) $\mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$

(b) $\dfrac{x-2}{0} = \dfrac{y+1}{1} = \dfrac{z}{2}$ and $\dfrac{x+1}{1} = \dfrac{y-1}{-3} = \dfrac{z-1}{-2}$

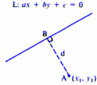
24. The diagram shows the line L and the point A both lying in the x-y plane. L has cartesian equation $ax + by + c = 0$ and A is the point (x_1, y_1). A is a perpendicular distance d from L.

Show that the vector equation $\mathbf{r} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\-c/b\end{pmatrix} + \lambda\begin{pmatrix}b\\-a\end{pmatrix}$ also represents the line L.

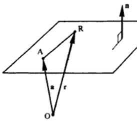
If λ takes the value λ_1 at point B show that $\lambda_1 = \dfrac{b^2x_1 - aby_1 - ac}{b(a^2 + b^2)}$

Hence show that $d = \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$

(Note: this has proved the result on page 382 by vector methods).



A plane can also be uniquely defined by stating a vector that is perpendicular to the plane and the position vector of a point on the plane.
Suppose that **n** is a vector perpendicular to the plane and A is a point in the plane having position vector **a**.
Consider some general point R in the plane having position vector **r**.
Thus $\overrightarrow{AR}$ will lie in the plane and will therefore be perpendicular to **n**.



i.e. $\overrightarrow{AR} \cdot \mathbf{n} = 0$

thus $(-\mathbf{a} + \mathbf{r}) \cdot \mathbf{n} = 0$

or $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

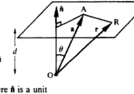
The equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is the vector equation of the plane that is perpendicular to the vector **n** and that contains the point with position vector **a**.
With **a** and **n** known the scalar product $\mathbf{a} \cdot \mathbf{n}$ can be evaluated. If $\mathbf{a} \cdot \mathbf{n} = p$, a scalar, then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = p$$

This form of the equation is referred to as the scalar product form of the vector equation of the plane. The position vector **r**, of *any* point in the plane will satisfy this equation.

Distance of a plane from the origin

Suppose that $\hat{\mathbf{n}}$ is the *unit* vector perpendicular to the plane and d is the perpendicular distance from the plane to the origin.
The scalar product vector equation of the plane will be



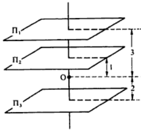
$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}}$

$= |\mathbf{a}| \times 1 \times \cos\theta$ where θ is the angle between **a** and $\hat{\mathbf{n}}$

$= a\cos\theta$

$= d$

Thus if the equation of the plane is in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ where $\hat{\mathbf{n}}$ is a unit vector normal to the plane then d is the perpendicular distance of the plane from the origin.
Consider the three parallel planes Π_1, Π_2 and Π_3 with vector equations $\mathbf{r} \cdot \hat{\mathbf{n}} = 3$, $\mathbf{r} \cdot \hat{\mathbf{n}} = 1$ and $\mathbf{r} \cdot \hat{\mathbf{n}} = -2$ respectively. The perpendicular distance of these planes from the origin will be 3 units, 1 unit and 2 units respectively and the significance of the negative sign is that Π_3 is on the other side of the origin from Π_1 and Π_2. Thus Π_1 is 2 units from Π_2 and 5 units from Π_3.



Example 18

A plane contains a point A, position vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Find

(a) a vector equation of the plane,
(b) the perpendicular distance of the plane from the origin,
(c) the perpendicular distance from this plane to the parallel plane $\mathbf{r} . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -3$

(a) The scalar product form of the vector equation will be

$$\begin{aligned}\mathbf{r} . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) &= (3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= 3 + 8 - 4 \\ &= 7\end{aligned}$$

The scalar product vector equation of the plane is $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 7$

(b) If we can write the vector equation in the form $\mathbf{r} . \hat{\mathbf{n}} = c$ where $\hat{\mathbf{n}}$ is a unit vector normal to the plane, then c is the required distance.

Now $|\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}| = 3$

From $\mathbf{r} . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 7$ we can write $\mathbf{r} . \dfrac{(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})}{3} = \dfrac{7}{3}$

giving the perpendicular distance of the plane from the origin as $2\frac{1}{3}$ units.

(c) The plane $\mathbf{r} . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -3$ is on the other side of the origin from $\mathbf{r} . (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 7$ and is a perpendicular distance of $\dfrac{3}{\sqrt{(1^2 + 2^2 + 2^2)}} = 1$ unit from the origin. Thus the perpendicular distance between the planes is $3\frac{1}{3}$ units.

Example 19

Find the position vector of the point where the line

$$\mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \text{ meets the plane } \mathbf{r} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12.$$

The position vector of the point of intersection will satisfy both the equation of the line and that of the plane.

If $\mathbf{r}_1$ is the position vector of the point of intersection then

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_1 . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$$

thus

$$\begin{pmatrix} 5 + \lambda \\ 3 - 4\lambda \\ -1 + 2\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$$

$$\therefore \quad 10 + 2\lambda + 3 - 4\lambda - 3 + 6\lambda = 12$$

$$\lambda = \tfrac{1}{2}$$

Thus the required position vector is $\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

Example 20

Show that the line $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$ lies in the plane $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$.

If the line and the plane (i) are parallel and (ii) contain a common point, then the line must lie in the plane.

(i) The line is parallel to $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and the plane is perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
However $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 3 - 2 - 1$
$= 0$
Hence the vectors $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ are perpendicular to each other.
Thus the line and plane are both perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and must therefore be parallel to each other

(ii) The line passes through the point with position vector $3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.
Also $(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 9 - 6 - 2$
$= 1$
i.e. $(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ satisfies $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$
Thus the point with position vector $3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ is common to both the line and the plane.
With conditions (i) and (ii) proved the line must lie in the plane.

Alternatively example 20 can be solved by showing that the line and plane contain two common points. Taking any two values for λ, for example $\lambda = 0$ and $\lambda = 1$:
If $\lambda = 0$ $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ Thus the point with position vector $3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ lies on the given line.
If $\lambda = 1$ $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ Thus the point with position vector $4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ lies on the given line.
But $(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$
and $(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$
i.e. both points also lie in the plane $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$.
Therefore the line and plane contain two common points and so the line must lie in the plane.

Cartesian equation of a plane

Consider a plane with equation $\mathbf{r} \cdot \mathbf{n} = p$ where $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Writing $\mathbf{r}$, the position vector of a general point in the plane, as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$$

i.e. $ax + by + cz = p$

Thus a plane perpendicular to $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has cartesian equation $ax + by + cz = p$.

2. The plane Π has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

Show that the point with position vector $\begin{pmatrix} 2 \\ 2 \\ -7 \end{pmatrix}$ lies in the plane Π.

3. The plane Π has vector equation $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 5$. Show that the point with position vector $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ lies in the plane Π.

4. Find the scalar product vector equation of the plane that is perpendicular to $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and contains the point with position vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

5. Find the scalar product vector equation of the plane that is perpendicular to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and contains the point with position vector $3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

6. Find the vector equation of the line that passes through the point A, position vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, and is perpendicular to the plane Π, vector equation $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2$

7. State which of the lines given by the vector equations below are perpendicular to the plane $\mathbf{r} \cdot (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 6$
(a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$
(b) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$
(c) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \eta(4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$.

8. State which of the lines given by the vector equations below are parallel to the plane $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 5$
(a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(6\mathbf{i} + 3\mathbf{j})$
(b) $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mu(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
(c) $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \eta(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$.

9. Write down the vector equations of the following planes in the form $\mathbf{r} \cdot \mathbf{n} = p$
(a) $4x + 2y + 3z = 4$ (b) $2x - 3y + 4z = 5$

10. Find the cartesian equation of the plane with parametric vector equation
$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

11. Find the cartesian equation of the plane containing the point with position vector $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and parallel to the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

12. Find the cartesian equation of the plane containing the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$.

13. Find the perpendicular distance from the plane $\mathbf{r} \cdot (2\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}) = 10$ to the origin.

14. Find the perpendicular distance from the plane $2x + 3y - 6z = 21$ to the origin.

15. The plane Π contains the point A, position vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and is perpendicular to the vector $4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$. Find the perpendicular distance from the plane Π to the origin.

26. A plane Π is perpendicular to the vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and contains the point A, position vector $\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Find the position vector of the point where the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ meets the plane Π.

27. Points D, E and F have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ respectively. Find the equation of the plane containing D, E and F giving your answer both in parametric vector form and cartesian form.

28. Find the acute angle between the line $\dfrac{x-6}{5} = \dfrac{y-1}{-1} = z + 1$ and the plane $7x - y + 5z = -5$ giving your answer to the nearest degree.

29. The point A has position vector $3\mathbf{i} - \mathbf{j} - \mathbf{k}$ and the plane Π_1 has vector equation $\mathbf{r} \cdot (3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) = 3\sqrt{2}$.

(a) Find in scalar product form, the vector equation of a plane Π_2 that is parallel to Π_1 and contains the point A.

(b) Find the perpendicular distance from A to the plane Π_1.

30. For each of the following find the perpendicular distance from the given point to the given plane.

(a) The point with position vector $3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ and the plane $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 6$,

(b) the point $(4, -1, 2)$ and the plane $2x - 2y + z = 21$.

31. The line L has vector equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ and the plane Π has vector equation $\mathbf{r} \cdot (6\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$

(a) Show that the line L is parallel to plane Π.

(b) Obtain the scalar product vector equation of the plane that is parallel to Π and that contains the line L.

(c) Find the perpendicular distance from the line L to the plane Π.

32. Show that the line $\dfrac{x-2}{2} = \dfrac{y-2}{-1} = \dfrac{z-3}{3}$ is parallel to the plane $4x - y - 3z = 4$ and find the perpendicular distance from the line to the plane.

33. Find the cartesian equation of the line of intersection of the two planes $2x - 3y - z = 1$ and $3x + 4y + 2z = 3$

34. Two lines have vector equations $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Find the position vector of the point of intersection of the two lines and the cartesian equation of the plane containing the two lines.

35. The four points A, B, C and D have position vectors

$\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$ respectively.

The perpendicular from D to the plane containing A, B and C meets the plane at E. Find

(a) the scalar product vector equation of the plane containing A, B and C,

(b) the vector equation of the straight line through D and E,

(c) the position vector of the point E.

17.5 3 × 3 Matrices

In chapter 5 we saw that a linear transformation on 2 dimensional space has an associated 2 × 2 matrix. Similarly any linear transformation on 3 dimensional space has an associated 3 × 3 matrix. If under this transformation the points with position vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are transformed to the points with position vectors $\begin{pmatrix}a\\b\\c\end{pmatrix}$, $\begin{pmatrix}d\\e\\f\end{pmatrix}$ and $\begin{pmatrix}g\\h\\i\end{pmatrix}$ respectively then the associated 3 × 3 matrix is $\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}$.

Note that under a linear transformation on 3-D space the origin (0, 0, 0) is mapped onto itself: $\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

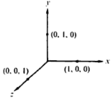
Example 28

Find the 3 × 3 matrix representing the transformation of 3-D space that reflects all points in the x–y plane (i.e. the plane $z = 0$)

Under this transformation

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\-1\end{pmatrix}.$$

Thus the required matrix is $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$



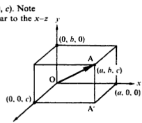
Example 29

Find the 3 × 3 matrix representing an orthogonal projection onto the x–z plane.

This projection maps any point A(a, b, c) onto the point A′(a, 0, c). Note that A′ lies in the x–z plane and is such that AA′ is perpendicular to the x–z plane—hence the term 'orthogonal' projection.

Thus $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$

and the required matrix is $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$



Example 30

Show that the matrix $\begin{pmatrix} -1 & 1 & 3 \\ 2 & -2 & -6 \\ -4 & 4 & 12 \end{pmatrix}$ maps all points in 3-D space onto a line and find the cartesian equations of this line.

Under this transformation the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) have images A′, B′, C′, position vectors $\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix}$ respectively, i.e. all the images are points whose position vectors are multiples of $\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$. Thus this transformation maps all points onto the line passing through the origin and the point with position vector $\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$.

The vector equation of this line will be $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$ giving the cartesian equations $-x = \frac{y}{2} = -\frac{z}{4} (= \lambda)$.

Example 31

Show that the matrix $\begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & 7 \\ 2 & -1 & 1 \end{pmatrix}$ maps all points in 3-D space onto a plane and find the cartesian equation of this plane.

Under this transformation the points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) are mapped onto A′(3, −1, 2), B′(−2, 3, −1) and C′(0, 7, 1). For these points A′, B′ and C′ to be coplanar there must exist values for λ and μ such that

$$\lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$$

Solving $3\lambda - 2\mu = 0$ and $-\lambda + 3\mu = 7$ simultaneously gives $\lambda = 2$ and $\mu = 3$ which is consistent with $2\lambda - \mu = 1$. Hence C′ lies in the plane containing A′ and B′ [and the origin O (0, 0, 0)]. Thus the transformation maps all points in 3-D space on to the plane containing the origin and the points (3, −1, 2) and (−2, 3, −1).

As the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ lie in this plane the parametric vector equation of the plane is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$.

With $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ this gives the cartesian equation of the plane as $5x + y - 7z = 0$.

The determinant of a 3 × 3 matrix

Just as for linear transformations in 2-dimensional space the determinant of the associated matrix gives the area scale factor then similarly the determinant of a 3 × 3 matrix gives the volume scale factor for the 3-dimensional transformation defined by that matrix.

Remembering that for the 2 × 2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can write the determinant of **A** as det **A**, |**A**| or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, the determinant of the 3 × 3

matrix $\mathbf{A} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is defined as

$$\det \mathbf{A} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example 32

If a solid of volume 8 cubic units is transformed using the matrix $\mathbf{T} = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ find the volume of the new solid.

Volume of new body = |det **T**| × 8

Now $\det \mathbf{T} = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$

$$= 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -9 + 4 - 4$$

$$= -9$$

Thus the volume of the new body is 9 × 8 = 72 cubic units.

Note: Just as a 2 × 2 matrix with zero determinant has no inverse then similarly a 3 × 3 matrix with zero determinant has no inverse and is said to be *singular*.

The inverse of a 3 × 3 matrix

If the 3 × 3 matrix **A** transforms a body B to its image B′, then $\mathbf{A}^{-1}$ (i.e. the inverse of **A**) will transform B′ back to B. By definition

$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where, for 3 × 3 matrices, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ but can $\mathbf{A}^{-1}$ be determined from **A**?

One method, called the *adjoint* method, first requires an understanding of certain terms, namely the *minor* of an element of a matrix, the *cofactor* of an element of a matrix and the *adjoint* of a matrix.

Consider the matrix $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

If we delete the row and column that contains the element a_1 and find the determinant of the 2×2 matrix that we are left with we obtain the *minor* of a_1.

Thus the minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of $b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ etc.

The *cofactor* of each element is then obtained by multiplying the minor by ± 1 according to the pattern $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$, i.e. the cofactor is the minor with an appropriate sign attached to it.

Thus the cofactors of a_1, b_1 and c_1 are $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ respectively.

If we form a new 3×3 matrix by replacing each element of **A** by its cofactor and then transpose this new matrix, we obtain the *adjoint of* **A**, written adj **A**. By forming the product (adj **A**)**A**, or **A**(adj **A**), we can then determine $\mathbf{A}^{-1}$ as the following example shows.

Example 33

Find the inverse of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Let the given matrix be **A** and first check that det $\mathbf{A} \neq 0$. In this case det $\mathbf{A} = -4$.

Then find the minors of each element: $\begin{matrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & -6 & 5 \end{matrix}$

and write the matrix of cofactors: $\begin{pmatrix} 1 & -2 & -3 \\ -1 & -2 & -1 \\ -3 & 6 & 5 \end{pmatrix}$

Hence obtain the adjoint of **A**: $\begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix}$

and determine (adj **A**) × **A**: $\begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$

Hence $(\text{adj } \mathbf{A})\mathbf{A} = -4\mathbf{I}$

or $-\frac{1}{4}(\text{adj } \mathbf{A})\mathbf{A} = \mathbf{I}$

Thus $\mathbf{A}^{-1}$ is given by $-\frac{1}{4}\begin{pmatrix} 1 & -1 & -3 \\ -2 & -2 & 6 \\ -3 & -1 & 5 \end{pmatrix}$ or $\frac{1}{4}\begin{pmatrix} -1 & 1 & 3 \\ 2 & 2 & -6 \\ 3 & 1 & -5 \end{pmatrix}$.

7. Show that each of the following matrices map any points in 3-D space onto a plane and find the cartesian equation of the plane in each case.

(a) $\begin{pmatrix} 0 & -2 & -2 \\ 1 & 1 & 5 \\ 1 & 3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 10 \\ 2 & -3 & -8 \\ -3 & 1 & -2 \end{pmatrix}$

8. Find the determinants of the following matrices.

(a) $\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \\ 1 & -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}$

9. Find, where possible, the inverses of the following matrices.

(a) $\begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -1 & -4 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 3 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -2 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 5 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

10. The transformations $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined by the matrices

$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ respectively. If $\mathbf{T}_1$ transforms a body S, volume 10 cubic units, to its image S′ and $\mathbf{T}_2$ transforms S′ to S″ find

(a) the single matrix that will transform S to S″,
(b) the single matrix that will transform S′ back to S,
(c) the volume of S′,
(d) the volume of S″.

11. Find the inverse of the matrix $\begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$ and *hence* solve the equations $\begin{cases} -x - y + z = 1 \\ -y + 3z = 8 \\ 2x + y + 2z = 8 \end{cases}$

12. Find the inverse of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 3 & 1 & 2 \end{pmatrix}$ and *hence* solve the equations $\begin{cases} 2x - y + 3z = 0 \\ x + y - 2z = -1 \\ 3x + y + 2z = 5 \end{cases}$

13. (a) Find the inverse of the matrix $\begin{pmatrix} -1 & -3 & -4 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$

(b) A point (x, y, z) in three-dimensional space is transformed to its image (x', y', z') according to the rule

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & -3 & -4 \\ 2 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -7 \\ -3 \\ 5 \end{pmatrix}$$

Find the coordinates of the point in three dimensional space that is invariant under the transformation.

17.6 Sets of linear equations

In chapter 6 we saw that a pair of linear equations in two unknowns could be solved simultaneously by matrix methods to find the one set of values that satisfied both equations. In questions 11 and 12 of exercise 17E we used our ability to determine the inverse of a 3 × 3 matrix to solve a set of 3 linear equations in 3 unknowns. Matrix methods are not the only method of solution since the equations could be solved by progressively eliminating the variables.

Example 34

Solve the following simultaneous equations by eliminating the variables.

$$2x + y - 2z = -4 \qquad \ldots [1]$$
$$2x + 3y + z = -2 \qquad \ldots [2]$$
$$-x + 2z = 3 \qquad \ldots [3]$$

3 × [1] − [2]: $4x - 7z = -10$ ... [4]

4 × [3]: $-4x + 8z = 12$ ... [5]

[4] + [5]: $z = 2$ ... [5]

By substituting into [3] and then into [1], we obtain $x = 1$ and $y = -2$.
Thus the solution of the given set of equations is $x = 1$, $y = -2$ and $z = 2$.

The geometrical interpretation of this solution is that the point $(1, -2, 2)$ is the point that is common to the three planes
$2x + y - 2z = -4$, $2x + 3y + z = -2$ and $-x + 2z = 3$.

This method of eliminating the variables gives another method for determining the inverse of a matrix.

Suppose we wish to find the inverse of $\begin{pmatrix} -1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

First let $$\begin{pmatrix} -1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

i.e.
$$-x + y + 3z = a \qquad \ldots [1]$$
$$2x + y + z = b \qquad \ldots [2]$$
$$x + y + 2z = c \qquad \ldots [3]$$

Example 36

Solve the equations $\begin{cases} 2x - y + z = 5 \\ x - 3y + 2z = 2 \\ 2x + y + 4z = -3 \end{cases}$

Writing the equations in matrix form $\begin{pmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

$$\begin{matrix} r_1: \\ \text{new } r_2 = 2r_2 - r_1: \\ \text{new } r_3 = r_3 - r_1: \end{matrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & -5 & 3 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -8 \end{pmatrix}$$

$$\begin{matrix} r_1: \\ r_2: \\ \text{new } r_3 = 5r_3 + 2r_2: \end{matrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & 21 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -42 \end{pmatrix}$$

Row 3 gives $z = -2$.
Using this value in row 2 gives $-5y - 6 = -1$ i.e. $y = -1$
and these values in row 1 gives $2x + 1 - 2 = 5$ i.e. $x = 3$
Thus $x = 3$, $y = -1$ and $z = -2$ is the required solution.

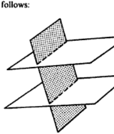
Given three equations $a_1x + b_1y + c_1z = p$
$a_2x + b_2y + c_2z = q$
$a_3x + b_3y + c_3z = r$
the geometrical interpretation of the solution $x = \alpha$, $y = \beta$ and $z = \gamma$ is that the three planes defined by the three equations have a common point

(α, β, γ). However if the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is singular then the three

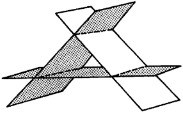
equations do not have a unique solution. In such a situation the equations could either
have (a) no solutions, and the equations are said to be *inconsistent*,
or (b) an infinite number of solutions.

The geometrical interpretation of these two possibilities is as follows:

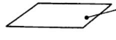
(a) *No solutions* (i.e. inconsistent equations).
This situation will arise when
1. two or more of the planes are parallel (but not coincident). In such cases there is no point common to all three planes.



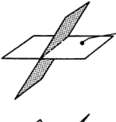
or 2. each plane is parallel to the line of intersection of the other two. Again there will be no point that is common to all three planes.



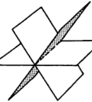
(b) *An infinite number of solutions.*
This situation will arise when
1. all three equations represent the same plane. (i.e. all 3 planes are coincident). Any point in the plane will provide a solution to the equation.



or 2. two of the three planes are coincident and the third plane is not parallel to these two. The planes will intersect in a line and any point on the line will provide a solution.



or 3. the three planes have a common line and any point on this line provides a solution to the equation.



Example 37
For what value of a does the equation $\begin{pmatrix} 2 & -4 & 2 \\ 4 & -1 & 2 \\ 1 & a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$ not have a unique solution?

The equation will not have a unique solution if $\begin{vmatrix} 2 & -4 & 2 \\ 4 & -1 & 2 \\ 1 & a & 0 \end{vmatrix}$ is zero.

i.e. if $\quad -4a - 8 + 8a + 2 = 0$
giving $\quad a = 1\frac{1}{2}$
The equation will not have a unique solution if $a = 1\frac{1}{2}$.

Example 38

Discuss the solution of the following equations when (a) $a = 6$ and $b = 6$,
(b) $a = 5$ and $b = 0$,
giving a full geometric interpretation for each situation.

$$\begin{aligned} 2x - y + 3z &= 2 \\ -4x + 2y - bz &= -4 \\ 6x - 3y + 9z &= a \end{aligned}$$

(a) With $a = 6$ and $b = 6$ the equations become

$$\begin{aligned} 2x - y + 3z &= 2 \\ -4x + 2y - 6z &= -4 \quad \text{i.e.} \quad -2[2x - y + 3z = 2] \\ 6x - 3y + 9z &= 6 \quad \text{i.e.} \quad 3[2x - y + 3z = 2] \end{aligned}$$

These three equations represent the same plane, $2x - y + 3z = 2$ and so any point in the plane provides a solution to the equations. Thus there are an infinite number of solutions. Letting $x = \lambda$ and $y = \mu$ we can write these solutions in parametric form as $x = \lambda$, $y = \mu$, $z = \frac{1}{3}(2 - 2\lambda + \mu)$. This gives the parametric vector equation of the plane as

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3} \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$

(b) With $a = 5$ and $b = 0$ the equations become

$$\begin{aligned} 2x - y + 3z &= 2 \quad &\ldots [1] \\ -4x + 2y &= -4 \quad &\ldots [2] \\ 6x - 3y + 9z &= 5 \quad &\ldots [3] \end{aligned}$$

Equations [1] and [3] represent parallel planes and so there will be no point common to all three planes. The equations have no solutions i.e. they are inconsistent.

Example 39

Discuss the solution of the following equations when (a) $a = 5$, (b) $a = -2$, giving a geometrical interpretation for each case.

$$\begin{aligned} -2x + y - 5z &= 4 \\ 3x - y + 2z &= -1 \\ -4x + y + z &= a \end{aligned}$$

These equations do not represent coincident or parallel planes whatever the value of a and so we attempt a solution by the diagonal matrix method:

$$\begin{matrix} \mathbf{r}_1: \\ \mathbf{r}_2: \\ \mathbf{r}_3: \end{matrix} \begin{pmatrix} -2 & 1 & -5 \\ 3 & -1 & 2 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ a \end{pmatrix}$$

$$\begin{matrix} \mathbf{r}_1: \\ \text{new } \mathbf{r}_2 = 2\mathbf{r}_2 + 3\mathbf{r}_1: \\ \text{new } \mathbf{r}_3 = \mathbf{r}_3 - 2\mathbf{r}_1: \end{matrix} \begin{pmatrix} -2 & 1 & -5 \\ 0 & 1 & -11 \\ 0 & -1 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ a-8 \end{pmatrix}$$

$$\begin{matrix} r_1: \\ r_2: \\ \text{new } r_3 = r_3 + r_2: \end{matrix} \begin{pmatrix} -2 & 1 & -5 \\ 0 & 1 & -11 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ a+2 \end{pmatrix}$$

(a) for $a = 5$ row 3, $0 = a + 2$, gives a contradiction and so the equations are inconsistent. The equations have no solutions for $a = 5$ and this corresponds to the situation of each plane being parallel to the line of intersection of the other two.

(b) for $a = -2$ row 3 gives no contradiction and is true for any z. Using the parameter λ we let $z = \lambda$.
Then from row_1 $\quad -2x + y - 5z = 4$
and from row_2 $\quad y - 11z = 10$
we obtain $y = 10 + 11\lambda$ and $x = 3(1 + \lambda)$.
i.e. the solutions to the equations are all of the form $x = 3 + 3\lambda$, $y = 10 + 11\lambda$, $z = \lambda$. Thus the 3 given equations represent planes having a common line, cartesian equation $\frac{x-3}{3} = \frac{y-10}{11} = z$ or, in parametric vector form $\mathbf{r} = \begin{pmatrix} 3 \\ 10 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$.

Exercise 17F

1. Find, where possible, the inverses of the following matrices by the method of row reduction.

(a) $\begin{pmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & -1 \\ 3 & 2 & 1 \\ -5 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & -4 & 2 \\ 3 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & 2 \\ 2 & -2 & 2 \end{pmatrix}$

2. Solve the following sets of equations by eliminating the variables.

(a) $x + y = 2$, $2y - z = -7$, $3x + z = 16$
(b) $2x + y - 3z = -5$, $x - 2y + 3z = -1$, $3x + y + z = 2$
(c) $x + y - z = -4$, $2x - 3y + 2z = 15$, $5x + 2y + z = 1$

3. Solve the following matrix equations by reducing each 3×3 matrix to a triangular matrix.

(a) $\begin{pmatrix} -1 & 2 & -1 \\ 2 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -1 & 1 \\ -2 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ -1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 4 \end{pmatrix}$

4. If matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ find the 3×3 matrices **B**, **C** and **D** such that

(a) $\mathbf{AB} = \begin{pmatrix} 10 & 8 & 2 \\ 8 & 5 & 2 \\ 7 & 6 & 1 \end{pmatrix}$ (b) $\mathbf{CA} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & -1 \\ 2 & 2 & -2 \end{pmatrix}$ (c) $\mathbf{AD} + \mathbf{C} = \mathbf{B}$

5. Find the value of a for which the equation

$\begin{pmatrix} -3 & -1 & 1 \\ 2 & -3 & 0 \\ a & 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ does not have a unique solution.

6. For the set of equations
$$ax + by + z = 2$$
$$-x + 2y + 3z = -2$$
$$3x - y + z = 1$$
give a geometric interpretation for
(a) $a = 3, b = -1$ (b) $a = 5, b = -2$ (c) $a = 4, b = -1$.
(Give any solutions in parametric form where appropriate).

7. For the set of equations
$$-3x + ay - 6z = b$$
$$x - y + 2z = 1$$
$$2x - 2y + 4z = 2$$
give a geometric interpretation for (a) $a = 3, b = 7$ (b) $a = 4, b - 2$.
(Give any solutions in parametric form).

8. For the set of equations
$$x + y + z = -1$$
$$2x - 2y + z = 5$$
$$3x - y + 2z = a$$
give a geometric interpretation for (a) $a = -1$ (b) $a = 4$.
(Give any solutions in parametric form).

9. For each of the following sets of equations find the value a must take for the equations to have solutions and find these solutions in parametric form.

(a)
$$x + 2y + 3z = 2$$
$$-x - 2y - 3z = -2$$
$$2x + 4y + 6z = a$$

(b)
$$x + 3y - z = -3$$
$$x - 2y + z = 4$$
$$-2x + 4y - 2z = a$$

(c)
$$x + y + 3z = -1$$
$$2x - y + 4z = 1$$
$$-x + 5y + z = a$$

(d)
$$2x - 2y + z = 3$$
$$-3x + 3y + 2z = -1$$
$$2x - 2y + 3z = a$$

10. For the set of equations
$$2x + y - 3z = 5$$
$$x + 2y + 3z = 1$$
$$2x - y + az = b$$
(a) Find the value of a for which the equations have no unique solution.
(b) With a taking this value find the value b must take for there to be an infinite number of solutions to the equations. Interpret this situation geometrically giving any relevant equations in cartesian form.

11. Show that the equations
$$x + y - z = -1$$
$$5x + 3y + z = 3$$
$$2x + y + z = a$$
are inconsistent for $a = 4$.
Find the value a must take for the equations to represent three planes that have a common line and find the vector equation of this line.

12. Find the inverse of the 4 × 4 matrix $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 2 & -1 \end{pmatrix}$

Hence solve the equations $\begin{cases} x + 2y + t + 1 = 0 \\ x + y + 2z + t + 2 = 0 \\ 2x + y + 3z + t + 3 = 0 \\ x + 2z - t + 7 = 0 \end{cases}$

Exercise 17G Examination questions

1. The position vectors of the points A and B relative to an origin O are $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ respectively. Write down the vector $\overrightarrow{AB}$. Calculate the size of angle OAB. (A.E.B.)

2. The triangle ABC is such that $\mathbf{AB} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{AC} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Find $B\hat{A}C$ and the area of the triangle. (Cambridge)

3. Relative to an origin O, points A and B have position vectors $2\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$ and $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ respectively, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ being orthogonal unit vectors. C is the point such that $\mathbf{OC} = 2\mathbf{OA}$ and D is the mid-point of AB. Find
   (i) the position vectors of C and D;
   (ii) a vector equation of the line CD;
   (iii) the position vector of the point of intersection of CD and OB;
   (iv) the angle AOB correct to the nearest degree. (S.U.J.B)

4. Find the angle between the lines with vector equations $\mathbf{r} = \mathbf{a} + t\mathbf{c}$ and $\mathbf{r} = \mathbf{b} + s\mathbf{d}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ -4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

   Show that these lines intersect, and find the position vector of P, the point of intersection. Express in the vector form $\mathbf{r} \cdot \mathbf{n} = k$ the equation of the plane which passes through the point P, and which is perpendicular to the line joining the two points with position vectors $\mathbf{a}$ and $\mathbf{b}$. (Oxford)

5. The points P and Q have coordinates (1, 2, 3) and (4, 6, −2) respectively and the plane π has equation $x + y - z = 24$.
   (i) Write down, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, the equation of the line PQ, and find the coordinates of the point where PQ meets π.
   (ii) Calculate the cosine of the angle that PQ makes with the normal to the plane π.
   (iii) Find the equation of the plane perpendicular to π containing the line PQ. (Oxford)

6. The point A has position vector $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ referred to the origin O. The line L has vector equation $\mathbf{r} = t\mathbf{i}$. The plane Π contains the line L and the point A. Find
(a) a vector which is normal to the plane Π,
(b) a vector equation for the plane Π,
(c) the cosine of the acute angle between OA and the line L. (London)

7. The coordinates of the points A and B are (0, 2, 5) and (−1, 3, 1), respectively, and the equations of the line L are

$$\frac{x-3}{2} = \frac{y-2}{-2} = \frac{z-2}{-1}.$$

(i) Find the equation of the plane π which contains A and is perpendicular to L, and verify that B lies in π.
(ii) Show that the point C in which L meets π is (1, 4, 3), and find the angle between CA and CB.
(iii) Find the coordinates of the two points P and Q on L which are such that the volume of each of the tetrahedra PABC and QABC is 9.
(J.M.B.)

8. The lines l_1 and l_2 have the vector equations

$$l_1\colon \mathbf{r} = (1 + s)\mathbf{i} + (1 - s)\mathbf{j} - 2\mathbf{k},$$
$$l_2\colon \mathbf{r} = \mathbf{i} + (1 + t)\mathbf{j} - (2 - t)\mathbf{k},$$

where **i**, **j** and **k** are perpendicular unit vectors. Show that l_1 and l_2 meet in the point A with position vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Show also that the lines l_3 and l_4,

$$l_3\colon \mathbf{r} = (3 - u)\mathbf{i} - (1 - u)\mathbf{j} + (-2 + u)\mathbf{k},$$
$$l_4\colon \mathbf{r} = (1 + v)\mathbf{i} - (1 + 3v)\mathbf{j} - v\mathbf{k},$$

form the other two sides of a quadrilateral ABCD, and find the position vectors of B, C and D, which are the intersections of l_1 and l_3, l_3 and l_4, l_2 and l_4 respectively. Find the angle between the diagonals AC and BD. Determine whether or not the four points A, B, C and D lie in a plane, explaining your method carefully. (Oxford)

9. The planes p and q are given by the equations $3x + 2y + z = 4$ and $2x + 3y + z = 5$ respectively. The plane π containing the point A(2, 2, 1) is perpendicular to each of the planes p and q. Find
(a) the distance from the point A to the plane p,
(b) the cosine of the angle between the planes p and q,
(c) a cartesian equation for the plane π,
(d) cartesian equations for the line of intersection, l, of the planes p and q,
(e) the point P on line l, which is nearest to the point A. (A.E.B.)

10. Let **A** be matrix

$$\begin{pmatrix} 1 & 2 & 7 \\ 1 & 3 & 0 \\ 0 & -1 & 8 \end{pmatrix}.$$

By performing elementary row operations on the matrix (**A**|**I**) find $\mathbf{A}^{-1}$. Solve the equation $\mathbf{AX} = \mathbf{K}$, when

(i) $\mathbf{K} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (ii) $\mathbf{K} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (iii) $\mathbf{K} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. (Cambridge)

11. Find values of a, b, c, d, e, f, so that the linear transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

maps the points with position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$

to the points with position vectors $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},$

respectively.
Find the inverse matrix $\mathbf{M}^{-1}$.
Show that the transformation with matrix $\mathbf{M}^{-1}$ maps points of the line $2x + y = 0, z = 0$ to points of the line $x = 0, 5y + 2z = 0$. (London)

12. Show that $x + a + y$ is a factor of the determinant

$$\begin{vmatrix} a & x & y \\ x & a & y \\ x & y & a \end{vmatrix}.$$

Express the determinant as a product of three factors.
Hence find all the values of θ in the range $0 \leqslant \theta \leqslant \pi$ which satisfy the equation

$$\begin{vmatrix} 1 & \cos\theta & \cos 2\theta \\ \cos\theta & 1 & \cos 2\theta \\ \cos\theta & \cos 2\theta & 1 \end{vmatrix} = 0. \qquad \text{(J.M.B.)}$$

13. Find the complete set of solutions of the system of equations

$$2x - 3y + 4z = 11,$$
$$3x + 2y - z = 9,$$
$$\lambda x + 12y - 11z = \lambda$$

(i) when $\lambda = 4$ (ii) when $\lambda = 5$. (Oxford)

14. Find all the solutions of the system of equations

$$x + 2y + 2z = -1,$$
$$3x + 2y + 10z = 5,$$
$$2x - y + 9z = 8,$$

and the particular solution for which $x + y + z = 0$.
Find also the solutions (if any) of the systems obtained
(a) by replacing the first equation by

$$5x + y + 19z = 0;$$

(b) by replacing the first two equations by

$$4x - 2y + 18z = 16.$$
$$-2x + y - 9z = -8.$$

(Oxford).

15. Show that the only values of λ for which the simultaneous equations
$$x + (\lambda - 4)y + 2z = 0$$
$$2x - 6y + (\lambda + 3)z = 0$$
$$(\lambda - 5)x + 12y - 8z = 0$$
have a solution in which x, y and z are not all zero are $\lambda = 1$ and $\lambda = 4$.
Show that, when $\lambda = 4$, the three planes represented by the above equations meet in a line L. Show also that L intersects the line whose equations are
$$\frac{x-8}{4} = y + 1 = \frac{z-2}{10}.$$
Interpret the three equations geometrically in the case when $\lambda = 1$.
(J.M.B.)

16. Show that the simultaneous equations
$$kx + y + z = 1,$$
$$2x + ky - 2z = -1,$$
$$x - 2y + kz = -2$$
have a unique solution except for three values of k which are to be found.
Show that, when $k = 1$, the planes represented by the equations meet at a point P which lies in the plane $y = 1$, and that, when $k = -1$, the planes meet in a line L which also lies in the plane $y = 1$. Find the perpendicular distance of P from L. (J.M.B.)

17. Find all the values of (x, y, z) which satisfy
$$\begin{bmatrix} 7 & 2 & 4 \\ 4 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$
if (p, q, r) satisfies
$$\begin{bmatrix} 2 & -7 & 5 \\ 6 & -9 & -1 \\ -4 & 5 & 2 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -4 \end{bmatrix}.$$
(Oxford)

18
Algebra II

18.1 Partial fractions

We know that $\frac{1}{x+3} + \frac{2}{x-1}$ can be expressed as the single fraction $\frac{3x+5}{(x+3)(x-1)}$ but, if we were given $\frac{3x+5}{(x+3)(x-1)}$ how would we get back to $\frac{1}{x+3} + \frac{2}{x-1}$?

This reverse process is called expressing $\frac{3x+5}{(x+3)(x-1)}$ in partial fractions.

The method used is to assume that $\frac{3x+5}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$,

where A and B are constants which have to be determined. Remember that the sign $\equiv$ means that the equality holds for *all* values of x.

The right hand side (RHS) of this identity is then expressed as a single fraction

i.e. $$\frac{A(x-1) + B(x+3)}{(x+3)(x-1)}$$

This fraction and the left hand side (LHS) of the identity now have the same denominators so their numerators must be identical

i.e. $$3x + 5 \equiv A(x-1) + B(x+3)$$

One, or both, of the following techniques is used to find A and B from this identity

(i) substituting suitable values of x in both sides of the identity,

(ii) equating the coefficients of particular powers of x.

It should be noted that the given fraction, $\frac{3x+5}{(x+3)(x-1)}$, is a proper fraction and so the partial fractions assumed, $\frac{A}{x+3} + \frac{B}{x-1}$, are also proper fractions. (Remember that an improper algebraic fraction is one for which the degree of the numerator is equal to or greater than the degree of the denominator, see page 7).

Thus, given $\frac{5x^2+13x+1}{(x+5)(x^2-2)}$, we would assume the partial fractions to be $\frac{A}{x+5} + \frac{Bx+C}{x^2-2}$

The denominators of the algebraic fractions encountered will be of three basic types:

Type	Denominator of fraction	Example	Expression used
1.	has linear factors	$\frac{5}{(x-2)(x+3)}$	$\frac{A}{x-2}+\frac{B}{x+3}$
2.	has a quadratic factor which does not factorise	$\frac{2x+3}{(x-1)(x^2+4)}$	$\frac{A}{x-1}+\frac{Bx+C}{x^2+4}$
3.	has a repeated factor	$\frac{5x+3}{(x-2)(x+3)^2}$	$\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$

Notes 1. When expressing an algebraic fraction in partial fractions, always factorise the denominator as far as possible first.

2. Given an improper algebraic fraction, we must first use the methods of page 7 to obtain an expression which does not contain any improper fractions (see examples 5 and 6).

3. For a fraction with a denominator having a repeated factor,
e.g. $\frac{5x+3}{(x-2)(x+3)^2}$, we may initially think of using $\frac{A}{x-2}+\frac{Bx+C}{(x+3)^2}$
but this is not the simplest form because

$$\frac{Bx+C}{(x+3)^2}=\frac{B(x+3)}{(x+3)^2}+\frac{C-3B}{(x+3)^2}$$
$$=\frac{B}{x+3}+\frac{C'}{(x+3)^2} \quad \text{where } C'=C-3B.$$

Example 1 (denominator has linear factors)

Express $\frac{1}{x^3-9x}$ in partial fractions.

The denominator factorises to $x(x-3)(x+3)$

Assume $\frac{1}{x(x-3)(x+3)} \equiv \frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+3}$

$$\text{R.H.S.} = \frac{A(x-3)(x+3)+B(x)(x+3)+C(x)(x-3)}{x(x-3)(x+3)}$$

$\therefore$ $\quad 1 \equiv A(x-3)(x+3)+B(x)(x+3)+C(x)(x-3)$
as L.H.S. and R.H.S. have the same denominators.

Put $x=0$, $\quad 1 = A(-3)(+3)+B(0)+C(0)$ $\quad$ hence $A=-\frac{1}{9}$
Put $x=3$, $\quad 1 = A(0)+B(3)(3+3)+C(0)$ $\quad$ hence $B=\frac{1}{18}$
Put $x=-3$, $\quad 1 = A(0)+B(0)+C(-3)(-6)$ $\quad$ hence $C=\frac{1}{18}$

The partial fractions are $-\frac{1}{9x}+\frac{1}{18(x-3)}+\frac{1}{18(x+3)}$.

Example 2 (denominator has quadratic factor)

Express $\frac{5x^2-2x-1}{(x+1)(x^2+1)}$ in partial fractions.

Assume $\dfrac{5x^2 - 2x - 1}{(x + 1)(x^2 + 1)} \equiv \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 + 1}$

$\equiv \dfrac{A(x^2 + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 + 1)}$

$\therefore \quad 5x^2 - 2x - 1 \equiv A(x^2 + 1) + (Bx + C)(x + 1)$

Put $x = -1$, $\quad 5(-1)^2 - 2(-1) - 1 = A(1 + 1) + 0$

$\therefore \quad 5 + 2 - 1 = 2A \qquad$ hence $A = 3$

Put $x = 0$, $\quad 0 - 0 - 1 = A(1) + C$

$\therefore \quad -1 = 3 + C \qquad$ hence $C = -4$

Equating the coefficients of x^2 on the two sides of the identity:

$5 = A + B \qquad$ hence $B = 2$

The partial fractions are $\dfrac{3}{x + 1} + \dfrac{2x - 4}{x^2 + 1}$.

Example 3 (denominator has a repeated factor)

Express $\dfrac{x + 4}{(x + 1)(x - 2)^2}$ in partial fractions.

Assume $\dfrac{x + 4}{(x + 1)(x - 2)^2} \equiv \dfrac{A}{x + 1} + \dfrac{B}{x - 2} + \dfrac{C}{(x - 2)^2}$

(Note carefully the form which the partial fractions will take in this case.)

$\therefore \quad x + 4 \equiv A(x - 2)^2 + B(x + 1)(x - 2) + C(x + 1)$

Put $x = 2$, $\quad 2 + 4 = A(0) + B(0) + C(2 + 1) \qquad$ hence $C = 2$

Put $x = -1$, $\quad -1 + 4 = A(-1-2)^2 + B(0) + C(0) \qquad$ hence $A = \frac{1}{3}$

Equating coefficients of x^2 on both sides of the identity:

$0 = A + B \qquad$ hence $B = -\frac{1}{3}$

The partial fractions are $\dfrac{1}{3(x + 1)} - \dfrac{1}{3(x - 2)} + \dfrac{2}{(x - 2)^2}$.

Example 4 (denominator has a repeated factor)

Express $\dfrac{4x + 3}{(x - 1)^2}$ in partial fractions.

METHOD 1 as for Example 3

Assume $\dfrac{4x + 3}{(x - 1)^2} \equiv \dfrac{A}{x - 1} + \dfrac{B}{(x - 1)^2}$

$\therefore \quad 4x + 3 \equiv A(x - 1) + B$

Put $x = 1$, $\quad 4 + 3 = A(0) + B$

$B = 7$

Equating coefficients of x:

$4 = A$

METHOD 2

$\dfrac{4x + 3}{(x - 1)^2} \equiv \dfrac{4(x - 1) + 7}{(x - 1)^2}$

$\equiv \dfrac{4(x - 1)}{(x - 1)^2} + \dfrac{7}{(x - 1)^2}$

$\equiv \dfrac{4}{x - 1} + \dfrac{7}{(x - 1)^2}$

The partial fractions are $\dfrac{4}{(x - 1)} + \dfrac{7}{(x - 1)^2}$.

Example 5

Express $\dfrac{x(x + 3)}{x^2 + x - 12}$ in partial fractions.

22. $\dfrac{x^3 + 3x^2 + 10}{(x + 1)(x + 4)}$ 23. $\dfrac{2x^2 + 2x + 3}{x^2 - 1}$ 24. $\dfrac{x^4 - 3x^3 - 3}{x^2(x - 1)}$

Miscellaneous.

25. $\dfrac{7}{(2x - 3)(x + 2)}$ 26. $\dfrac{10 + 6x - 3x^2}{(2x - 1)(x + 3)^2}$ 27. $\dfrac{2x - 3}{x^2 - 1}$

28. $\dfrac{3x^2 - 1}{(x - 1)(2x - 1)^2}$ 29. $\dfrac{2x^2 - 7}{(x - 3)(2x + 5)}$ 30. $\dfrac{2x^3 + 11}{(x^2 + 4)(x - 3)}$

31. $\dfrac{37x - 81}{(x - 3)(x + 7)(2x - 3)}$ 32. $\dfrac{3x + 7}{(1 + 2x)(x^2 - x + 2)}$ 33. $\dfrac{2x^3 + 46}{(x - 1)^2(x + 3)}$

34. $\dfrac{3x^3 + 6x^2 - 11x + 1}{(1 + x)^3(x - 2)}$ 35. $\dfrac{3x^2 - 10x - 24}{x(x^2 - 4)}$ 36. $\dfrac{x^3 + 2x^2 + 61}{(x + 3)^2(x^2 + 4)}$

18.2 Partial fractions and series

Series expansions

Partial fractions can be used when finding the series expansions of some functions.

Example 7

Express $\dfrac{32x^2 + 17x + 18}{(2 - 3x)(1 + 2x)^2}$ in partial fractions and hence obtain its series expansion in ascending powers of x, stating the terms up to and including the term in x^3, the term in x^r and the values of x for which the expansion is valid.

Using the methods of the previous section, we find that $\dfrac{32x^2 + 17x + 18}{(2 - 3x)(1 + 2x)^2} = \dfrac{8}{2 - 3x} + \dfrac{5}{(1 + 2x)^2}$

Using the binomial expansion

$$\begin{aligned}\frac{8}{2 - 3x} &= \frac{8}{2(1 - \frac{3}{2}x)}\\ &= 4(1 - \tfrac{3}{2}x)^{-1} = 4\left[1 + (-1)(-\tfrac{3}{2}x) + \frac{(-1)(-2)}{1 \times 2}(-\tfrac{3}{2}x)^2 + \frac{(-1)(-2)(-3)}{1 \times 2 \times 3}(-\tfrac{3}{2}x)^3 + \ldots\right]\\ &= 4[1 + \tfrac{3}{2}x + (\tfrac{3}{2}x)^2 + (\tfrac{3}{2}x)^3 + \ldots + (\tfrac{3}{2}x)^r + \ldots]\\ &= 4 + 6x + 9x^2 + \tfrac{27}{2}x^3 + \ldots + \frac{3^r x^r}{2^{r-2}} + \ldots \qquad \ldots[1]\end{aligned}$$

$$\begin{aligned}\frac{5}{(1 + 2x)^2} &= 5(1 + 2x)^{-2} = 5\left[1 + (-2)(2x) + \frac{(-2)(-3)(2x)^2}{1 \times 2} + \frac{(-2)(-3)(-4)(2x)^3}{1 \times 2 \times 3} + \ldots\right]\\ &= 5[1 - 2(2x) + 3(2x)^2 - 4(2x)^3 + \ldots + (-1)^r(r + 1)(2x)^r + \ldots]\\ &= 5 - 20x + 60x^2 - 160x^3 + \ldots + (-2)^r 5(r + 1)x^r + \ldots \qquad \ldots[2]\end{aligned}$$

Adding [1] and [2]:

$$\frac{8}{2 - 3x} + \frac{5}{(1 + 2x)^2} = 9 - 14x + 69x^2 - \tfrac{293}{2}x^3 + \ldots + [3^r 2^{2-r} + (-2)^r 5(r + 1)]x^r \ldots$$

Notice that, substituting $r = 0$ into the term in x^r gives the term in x^0, i.e. 9,
substituting $r = 1$ into the term in x^r gives the term in x^1, i.e. $-14x$ etc.

The expansion of $(1 - \frac{3}{2}x)^{-1}$ is valid for $|\frac{3}{2}x| < 1$ i.e. $|x| < \frac{2}{3}$ and
the expansion of $(1 + 2x)^{-2}$ is valid for $|2x| < 1$ i.e. $|x| < \frac{1}{2}$.
Thus if $|x| < \frac{1}{2}$, both of these conditions will be satisfied.

$$\therefore \quad \frac{32x^2 + 17x + 18}{(2 - 3x)(1 + 2x)^2} = 9 - 14x + 69x^2 - \tfrac{293}{2}x^3 + \ldots + [3^r 2^{2-r} + (-2)^r 5(r + 1)]x^r \ldots$$

provided $|x| < \frac{1}{2}$.

Exercise 18B

Express each of the given functions in partial fractions and hence obtain the series expansion of each function, stating the terms up to and including the term in x^3, the term in x^r and the values of x for which the expansion is valid.

1. $\dfrac{2+x}{(1-x)(1+2x)}$ **2.** $\dfrac{7x+1}{(1+x)(1+3x)}$ **3.** $\dfrac{8x}{(1+5x)(1-3x)}$

4. $\dfrac{2}{(1-x)(3-x)}$ **5.** $\dfrac{4x}{(3+2x)(2x-1)}$ **6.** $\dfrac{11x+6}{(3+x)(3+4x)}$

7. $\dfrac{1+2x-x^2}{(1+x)^2(1+2x)}$ **8.** $\dfrac{6-14x-7x^2}{(1-4x)(1-x)(2+3x)}$ **9.** $\dfrac{x^2-4x-8}{(2+x)^2(1+x)}$

10. Express $\dfrac{11x+3}{(1+x^2)(1-5x)}$ in partial fractions and hence obtain its series expansion in ascending powers of x up to and including the term in x^3. State the range of values of x for which the expansion is valid.

11. Express $\dfrac{18-x-x^2}{(1+x+x^2)(2-3x)}$ in partial fractions and hence obtain the first four terms in the series expansion of the function in ascending powers of x.

Summation of series by the method of differences

Partial fractions enable us to sum certain series using the method of differences as the following examples show.

Example 8

(a) Express $\dfrac{2}{(2x-1)(2x+1)}$ in partial fractions.

(b) Find an expression for $\sum_{r=1}^{n}\dfrac{2}{(2r-1)(2r+1)}$ and state whether or not the series is convergent.

(a) Using the methods of 18.1 we obtain $\dfrac{2}{(2x-1)(2x+1)} = \dfrac{1}{2x-1} - \dfrac{1}{2x+1}$.

(b) $$\sum_{r=1}^{n}\frac{2}{(2r-1)(2r+1)} = \sum_{r=1}^{n}\left[\frac{1}{2r-1} - \frac{1}{2r+1}\right] = \begin{array}{l} \frac{1}{1} - \frac{1}{3} \\ + \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{5} - \frac{1}{7} \\ \quad\vdots \qquad \vdots \\ + \frac{1}{2n-1} - \frac{1}{2n+1} \end{array} = \frac{1}{1} - \frac{1}{2n+1}$$

Then, if $f(r) = g(r) - g(r+1)$

$$\sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} [g(r) - g(r+1)] = \begin{array}{l} g(1) - g(2) \\ + g(2) - g(3) \\ + g(3) - g(4) \\ \quad \vdots \qquad \vdots \\ + g(n-1) - g(n) \\ + g(n) - g(n+1) = g(1) - g(n+1). \end{array}$$

[Clearly Example 9 above is an extension of this basic technique and uses the relationship $f(r) = g(r) + g(r+1) - 2g(r+2)$.]

Example 10

(a) Show that $(x - 3)^2 - (x - 2)^2 \equiv 5 - 2x$.
(b) Use this identity from part (a) and the method of differences to obtain an expansion for $\sum_{r=1}^{n}(5 - 2r)$. Hence sum the series $3 + 1 - 1 - 3 - 5 \ldots - 35$.

(a) L.H.S. $= (x - 3)^2 - (x - 2)^2$
$= (x^2 - 6x + 9) - (x^2 - 4x + 4)$
$= x^2 - 6x + 9 - x^2 + 4x - 4$
$= -2x + 5 =$ R.H.S. as required.

(b) $$\sum_{r=1}^{n} (5 - 2r) = \sum_{r=1}^{n} [(r - 3)^2 - (r - 2)^2] = \begin{array}{l} (-2)^2 - (-1)^2 \\ + (-1)^2 - (0)^2 \\ + (0)^2 - (1)^2 \\ \quad \vdots \qquad \vdots \\ + (n-4)^2 - (n-3)^2 \\ + (n-3)^2 - (n-2)^2 \end{array}$$

$$\therefore \quad \sum_{r=1}^{n} (5 - 2r) = 4 - (n - 2)^2$$

$$3 + 1 - 1 - 3 - 5 \ldots - 35 = \sum_{r=1}^{20}(5 - 2r)$$
$$= 4 - (20 - 2)^2$$
$$= -320$$

Exercise 18C

1. Using the identity $2r + 1 \equiv (r + 1)^2 - r^2$ obtain an expression for $\sum_{r=1}^{n} (2r + 1)$.
2. (a) Show that $r(r + 1)(r + 2) - (r - 1)(r)(r + 1) \equiv 3r(r + 1)$
(b) Using the identity from part (a) and the method of differences obtain an expression for $\sum_{r=1}^{n} r(r + 1)$.

3. (a) Express $\frac{1}{x(x+1)}$ in partial fractions.

(b) Show that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

(c) Find the sum of the series

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots + \frac{1}{15\times 16}$$

4. (a) Express $\frac{3}{(3x-1)(3x+2)}$ in partial fractions.

(b) Show that $\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$

(c) Find the sum of the series

$\frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \frac{1}{11\times 14} + \ldots \frac{1}{29\times 32}$ and find

$\sum_{r=1}^{\infty} \frac{1}{(3r-1)(3r+2)}$

5. (a) Express $\frac{2}{x(x+2)}$ in partial fractions.

(b) Show that $\sum_{x=1}^{n} \frac{1}{x(x+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

and determine $\sum_{x=1}^{\infty} \frac{1}{x(x+2)}$.

(c) Find the sum of the series

$$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \frac{1}{4\times 6} + \ldots + \frac{1}{9\times 11}$$

6. (a) Show that

$(r+1)(2r+1)(2r+3) - r(2r-1)(2r+1) \equiv 3(2r+1)^2$

(b) Using the identity from part (a) and the method of differences show that

$$\sum_{r=1}^{n} (2r+1)^2 = \frac{n}{3}(4n^2 + 12n + 11)$$

(c) Hence find the sum of the series $3^2 + 5^2 + 7^2 + \ldots + 21^2$.

7. Find the sums of the following series. [Express your answers in the form $a - f(n)$]

(a) $\frac{4}{1\times 5} + \frac{4}{5\times 9} + \frac{4}{9\times 13} + \frac{4}{13\times 17} + \ldots + \frac{4}{(4n-3)(4n+1)}$

(b) $\frac{1}{1\times 5} + \frac{1}{3\times 7} + \frac{1}{5\times 9} + \frac{1}{7\times 11} + \ldots + \frac{1}{(2n-1)(2n+3)}$

(c) $\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \frac{1}{4\times 5\times 6} + \ldots + \frac{1}{n(n+1)(n+2)}$

(d) $\frac{6}{1\times 2\times 4} + \frac{6}{2\times 3\times 5} + \frac{6}{3\times 4\times 6} + \frac{6}{4\times 5\times 7} + \ldots + \frac{1}{n(n+1)(n+3)}$

8. (a) Use partial fractions to show that

$$\frac{7}{1\times 2\times 3} + \frac{10}{2\times 3\times 4} + \frac{13}{3\times 4\times 5} + \ldots + \frac{3n+4}{n(n+1)(n+2)} = \frac{n(5n+9)}{2(n+1)(n+2)}$$

(b) State whether the series $\sum_{r=1}^{n} \frac{3r+4}{r(r+1)(r+2)}$ converges as $n \to \infty$ and,

if it does, find its sum to infinity.

9. (a) Use partial fractions to show that

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\ldots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

(b) State whether the series $\sum_{r=1}^{n}\frac{r+1}{(2r-1)(2r+1)(2r+3)}$ converges as $n \to \infty$ and, if it does, find its sum to infinity.

18.3 Complex numbers

Page 1 discussed the extension of the basic ideas of number from positive and negative integers to rational numbers and eventually to irrational numbers. We can therefore solve equations like:

(i) $x^2 = 49$
$x = +7$ or -7

(ii) $x^2 = 6\frac{1}{4} = \frac{25}{4}$
$x = \frac{5}{2}$ or $-\frac{5}{2}$

(iii) $x^2 = 17$
$x = \sqrt{17}$ or $-\sqrt{17}$
i.e. $x = 4{\cdot}123\ldots$ or $-4{\cdot}123\ldots$

We now need to extend our ideas of number still further, since the equation

(iv) $x^2 = -16$

has no solution in terms of rational or irrational numbers.

We introduce, therefore, the concept of an imaginary number.

If we write i for $\sqrt{(-1)}$, then $x^2 = -16 = (16)(-1)$
and $x = \pm\sqrt{(16)}\sqrt{(-1)}$
or $x = \pm 4\mathrm{i}$ i.e. $x = 4\mathrm{i}$ or $x = -4\mathrm{i}$

Thus, by the use of the imaginary number i, to stand for $\sqrt{(-1)}$, we have found two solutions of the equation $x^2 = -16$.

We say that $b\mathrm{i}$, where $b \in \mathbb{R}$, is an *imaginary* number and that $a + \mathrm{i}b$, where a and $b \in \mathbb{R}$, is a *complex* number.

The following points about complex numbers should be noted:

If we write $z = a + \mathrm{i}b$, then
the real part of the complex number z is a, written $\mathrm{Re}(z) = a$,
and the imaginary part of the complex number z is b, written $\mathrm{Im}(z) = b$.

Two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal.
Thus $a + \mathrm{i}b = c + \mathrm{i}d \Rightarrow a = c$ and $b = d$.

Complex numbers are added (or subtracted) by adding (or subtracting) their real parts and also their imaginary parts.

The complex conjugate of $z = a + \mathrm{i}b$ is $a - \mathrm{i}b$ and is denoted by z^*.

It follows that the product zz^* is therefore a wholly real number, since

$$\begin{aligned} zz^* &= (a + \mathrm{i}b)(a - \mathrm{i}b) \\ &= a^2 - \mathrm{i}ab + \mathrm{i}ab - \mathrm{i}^2b^2 \\ &= a^2 + b^2 \end{aligned}$$

The manipulation of complex numbers and their use is illustrated in the following examples.

Example 11

(a) Add $(4 - 2i)$ and $(3 + 7i)$ (b) Simplify $(5 + 4i) - (3 - 2i)$ (c) Find $z + z^*$ if $z = 3 - 7i$.

(a) $$\begin{aligned}&(4 - 2i) + (3 + 7i)\\ &= (4 + 3) + (-2i + 7i)\\ &= 7 + 5i\end{aligned}$$

(b) $$\begin{aligned}&(5 + 4i) - (3 - 2i)\\ &= 5 + 4i - 3 + 2i\\ &= 2 + 6i\end{aligned}$$

(c) $$\begin{aligned}z &= 3 - 7i\\ \therefore\ z^* &= 3 + 7i\\ z + z^* &= 3 - 7i + 3 + 7i\\ &= 6\end{aligned}$$

Example 12

Multiply $(2 - 7i)$ by $(3 + 2i)$

$$\begin{aligned}&(2 - 7i)(3 + 2i)\\ &= 6 + 4i - 21i - 14i^2 && i^2 \text{ being written for } i \times i\\ &= 6 - 17i - 14(-1) && -1 \text{ being written for } i^2, \text{ since } i = \sqrt{(-1)}\\ &= 20 - 17i\end{aligned}$$

Example 13

Divide $(5 + 2i)$ by $(1 - 3i)$

$$\begin{aligned}(5 + 2i) \div (1 - 3i) &= \frac{5 + 2i}{1 - 3i}\\ &= \frac{(5 + 2i)}{(1 - 3i)} \times \frac{(1 + 3i)}{(1 + 3i)}\\ &= \frac{5 + 15i + 2i + 6i^2}{1 - 9i^2}\\ &= \frac{5 + 17i - 6}{1 - 9(-1)} \quad \text{since} \quad i^2 = -1\\ &= \frac{1}{10}(-1 + 17i) \quad \text{or} \quad -\frac{1}{10} + \frac{17}{10}i\end{aligned}$$

The denominator (z) is multiplied by z^* so as to make it a real number.

Example 14

Find the square roots of $(3 + 4i)$

Suppose the square root of $(3 + 4i)$ is $a + ib$ where a and $b \in \mathbb{R}$.

Then $$\begin{aligned}3 + 4i &= (a + ib)^2\\ &= a^2 + 2abi + i^2b^2\\ &= a^2 - b^2 + 2abi\end{aligned}$$

$\therefore \quad 3 = a^2 - b^2$ and $4 = 2ab$

Solving simultaneously gives $a = 2$ and $b = 1$ or $a = -2$ and $b = -1$.

The square roots of $(3 + 4i)$ are $\pm(2 + i)$.

Equations

On page 11 we saw that many quadratic equations could be solved using the formula. With our extended idea of number, solutions can now be obtained for *all* quadratic equations.

Example 15

Solve $x^2 + 2x + 6 = 0$.

Using $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ with $a = 1$, $b = 2$ and $c = 6$,

$$x = \frac{-2 \pm \sqrt{(4 - 24)}}{2}$$
$$= \frac{-2 \pm \sqrt{(-20)}}{2}$$
$$= -1 \pm i\sqrt{5}$$

The solutions of the equation are $x = -1 + i\sqrt{5}$ or $x = -1 - i\sqrt{5}$.
Thus, in the cases where $b^2 - 4ac < 0$, as in the above example, we can improve on our previous comment, in chapter 5, of 'no real roots,' by saying that the equation has 'two complex conjugate roots', i.e. one root is the complex conjugate of the other.

Example 16

Find the equation having roots of $(1 + 5i)$ and $(1 - 5i)$.

Sum of the roots of the equation $= 1 + 5i + 1 - 5i$
$= 2$

Product of the roots $= (1 + 5i)(1 - 5i)$
$= 26$

Thus, for $ax^2 + bx + c = 0$, $\dfrac{c}{a} = 26$ and $-\dfrac{b}{a} = 2$, so the equation can be written as $x^2 - 2x + 26 = 0$.

Example 17

Solve $2x^3 - 12x^2 + 25x - 21 = 0$.

Consider $f(x) = 2x^3 - 12x^2 + 25x - 21$. Since $f(3) = 0$, then it follows by the factor theorem that $f(x)$ is divisible by $(x - 3)$,
hence if $2x^3 - 12x^2 + 25x - 21 = 0$
then $(x - 3)(2x^2 - 6x + 7) = 0$ [$2x^2 - 6x + 7$ obtained by long division]
$\therefore$ $x = 3$, or $2x^2 - 6x + 7 = 0$

Solving the quadratic $x = \dfrac{6 \pm \sqrt{(36 - 56)}}{4}$

$$= \frac{3}{2} \pm \frac{i\sqrt{5}}{2}$$

The roots of the cubic equation are $x = 3$, $x = \dfrac{3}{2} + \dfrac{i\sqrt{5}}{2}$, $x = \dfrac{3}{2} - \dfrac{i\sqrt{5}}{2}$ i.e. $x = 3, \dfrac{3}{2} \pm \dfrac{i\sqrt{5}}{2}$.

Note that the complex roots of polynomial equations occur in conjugate pairs. Thus if $a + ib$ is a root of $f(x) = 0$, then so is $a - ib$. It follows that a polynomial equation cannot have an odd number of complex roots.

Example 18

Given that $(2 - i)$ and $(1 + 3i)$ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, find (i) the other two roots and (ii) the sum, and the product, of the four roots of the equation.

(i) Complex roots occur in conjugate pairs, hence
since $x = 2 - i$ is a root of the equation, so also is $x = 2 + i$,
and since $x = 1 + 3i$ is a root of the equation, so also is $x = 1 - 3i$.

(ii) Sum of roots $= (2 - i) + (2 + i) + (1 + 3i) + (1 - 3i)$
$= 6$
Product of roots $= (2 - i)(2 + i)(1 + 3i)(1 - 3i)$
$= (4 - i^2)(1 - 9i^2)$
$= 50$
The roots are $(2 \pm i)$, $(1 \pm 3i)$; their sum is 6 and their product 50.

Exercise 18D

1. If $z = 3 - 4i$ find (a) $\text{Re}(z)$, (b) $\text{Im}(z)$, (c) z^*, (d) zz^*, (e) $(zz)^*$.

2. Simplify each of the following,
(a) $(3 + 4i) + (2 + 3i)$ (b) $(2 - 4i) - 3(5 - 3i)$ (c) $(2i)^2$ (d) i^4
(e) $\frac{1}{i^2}$ (f) $(2 + 3i)(2 - 3i)$ (g) $\frac{1}{(1 + i)(1 - i)}$ (h) $-\frac{1}{i}$

3. Simplify each of the following,
(a) $(2 + i)(3 - i)$ (b) $(5 - 2i)(6 + i)$ (c) $(4 - 3i)(1 - i)$
(d) $(3 + i)(2 - 5i)$ (e) $(3 + 4i)(1 - 2i)$ (f) $(2 + i)(2 - i)$
(g) $(6 + 9i)(4 - 6i)$ (h) $(2 + i)(1 - 2i)(1 + i)$

4. Express each of the following in the form $a + ib$.
(a) $\frac{20}{3 + i}$ (b) $\frac{4}{1 + i}$ (c) $\frac{2i}{1 - i}$ (d) $\frac{1}{1 - 2i}$
(e) $\frac{5i}{1 + 2i}$ (f) $\frac{5}{4 - 3i}$ (g) $\frac{2 + 3i}{1 - i}$ (h) $\frac{3 - i}{1 + 2i}$
(i) $\frac{3 + 2i}{3 - 2i}$ (j) $\frac{4i - 3}{2 + 3i}$

5. Simplify $\frac{a + ib}{b - ai}$

6. Solve the following equations,
(a) $x^2 + 25 = 0$ (b) $2x^2 + 32 = 0$ (c) $4x^2 + 9 = 0$
(d) $x^2 + 2x + 5 = 0$ (e) $x^2 - 4x + 5 = 0$ (f) $2x^2 + x + 1 = 0$.

7. Find the square roots of the following complex numbers,
(a) $5 + 12i$ (b) $15 + 8i$ (c) $7 - 24i$

8. Find the quadratic equations having roots,
(a) $3i, -3i$ (b) $1 + 2i, 1 - 2i$ (c) $2 + i, 2 - i$
(d) $2 + 3i, 2 - 3i$ (e) $3 + 4i\ 3 - 4i$ (f) $3 + 5i, 3 - 5i$

9. If $a + ib$ is a root of the quadratic equation $x^2 + cx + d = 0$ show that $a^2 + b^2 = d$ and $2a + c = 0$.

10. Solve the following equations (all have at least one real root).
(a) $x^3 - 7x^2 + 19x - 13 = 0$
(b) $2x^3 - 2x^2 - 3x - 2 = 0$
(c) $x^3 + 3x^2 + 5x + 3 = 0$

(d) $4x^4 - 20x^3 + 37x^2 - 31x + 10 = 0$
(e) $5x^4 + 8x^3 - 8x - 5 = 0$

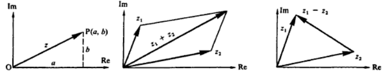
11. If $5x^4 - 14x^3 + 18x^2 + 40x + 16 = (x^2 - 4x + 8)(ax^2 + bc + c)$ find a, b and c and hence find the four solutions of the equation $5x^4 + 18x^2 + 16 = 14x^3 - 40x$.

12. If $3 - 2i$ and $1 + i$ are two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ find the values of a, b, c, d and e.

13. Given that $x^3 - 1 = (x - 1)(ax^2 + bx + c)$ find the values of a, b and c and hence find the three roots of the equation $x^3 = 1$
[These three roots can be referred to as 'the cube roots of unity' and are often written as $1, \omega, \omega^2$].

Geometrical representation

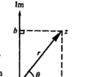
Complex numbers can be represented geometrically using the x and y axes as the Real (Re) and Imaginary (Im) axes. The plane of the axes is then referred to as the complex plane and a diagram showing complex numbers is said to be an Argand diagram.
On the Argand diagram, each complex number is represented by a line of a certain length in a particular direction. Thus each complex number is shown as a vector on the Argand diagram.
Thus if $P(a, b)$ is a point on the Argand diagram, the vector $\overrightarrow{OP}$ represents the complex number $(a + ib)$; a and b are the real and imaginary components of the complex number $(a + ib)$.
The sum and the difference of two complex numbers can be shown on an Argand diagram in the same way as we show vectors which are added or subtracted.



$z = a + ib$

Modulus and argument of complex numbers

The modulus of a complex number, $z = a + ib$, is a measure of the magnitude of z, and it is written as $|z|$.
Thus modulus $z = |z| = \sqrt{(a^2 + b^2)}$.
The argument of a complex number, $z = a + ib$, is the magnitude in radians of the angle between the positive real axis and the line representing the complex number on the Argand diagram. For the complex number z shown in the diagram, the argument is θ where $\tan \theta = b/a$. Clearly we could also refer to $\theta \pm 2\pi$, $\theta \pm 4\pi$, $\theta \pm 6\pi$, etc, as being the argument of z. To avoid this complication, we say that the value of θ lying in the range $-\pi < \theta \leqslant \pi$ is the **principal** value of the argument. We use the abbreviation 'arg' for the principal argument.



Example 20

Prove that for the complex numbers z_1 and z_2, (a) $|z_1z_2| = |z_1||z_2|$
(b) $\arg(z_1z_2) = \arg z_1 + \arg z_2$. (Assume $\pi < \arg z_1 + \arg z_2 \leqslant \pi$).

Also, if $z_1 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ find $|z_1z_2|$ and $\arg(z_1z_2)$.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then
$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2) \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \end{aligned}$$

thus $|z_1z_2| = r_1r_2 = |z_1||z_2|$
and $\arg(z_1z_2) = (\theta_1 + \theta_2) = \arg z_1 + \arg z_2$ as required.

For $z_1 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $|z_1| = 3$ and $\arg z_1 = \frac{\pi}{3}$

For $z_2 = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $|z_2| = 5$ and $\arg z_2 = \frac{\pi}{4}$

$$\begin{aligned} |z_1z_2| &= |z_1||z_2| & \arg(z_1z_2) &= \arg z_1 + \arg z_2 \\ &= 3 \times 5 & &= \frac{\pi}{3} + \frac{\pi}{4} \\ &= 15. & &= \frac{7\pi}{12}. \end{aligned}$$

Note: 1. The results $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$ can also be proved using $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.
2. The assumption that $\pi < \arg z_1 + \arg z_2 \leqslant \pi$ is necessary as we use the abbreviation arg for the principal value of the argument. Thus for $\arg(z_1z_2) = \arg z_1 + \arg z_2$ we must have $-\pi < \arg z_1 + \arg z_2 \leqslant \pi$
3. Similar results to those obtained for $|z_1z_2|$ and $\arg(z_1z_2)$ can be obtained for $\frac{z_1}{z_2}$, namely $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

The proofs of these are left as an exercise for the reader (Exercise 18E, number 9).

De Moivre's theorem

From the result that $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ it follows
that $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$.

This result can be extended to give the statement that
if $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$
i.e. $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$.

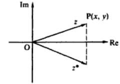
For $r = 1$, we have

> $(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta$
> and this is known as de Moivre's theorem.

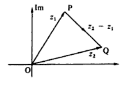
This theorem can be proved by the method of induction and is set as an exercise for the reader (Exercise 18E, number 11).

15. Use de Moivre's theorem to show that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.
Obtain an expression for $\tan 3\theta$ in terms of $\tan\theta$.
16. Use de Moivre's theorem to show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.
17. Use de Moivre's theorem to obtain an expression for $\tan 6\theta$ in terms of $\tan\theta$.
18. Given that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for a positive integral value of n show that it is also true for n a negative integer. [Hint: For n negative let $n = -m$, m positive).

Further geometrical considerations

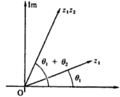


For the complex number $z = x + iy$, there is an associated point P on the Argand diagram with coordinates (x, y). The vector $\overrightarrow{OP}$ then represents the complex number z. Since $z = x + iy$, then z^* will be the reflection of z in the real axis.



Consider the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ on the Argand diagram. $|z_2 - z_1|$ is then the length of the line PQ.

On page 466 we saw that if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then
$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

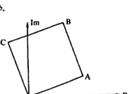


Thus the effect of multiplying some complex number z_1 by another complex number z_2, modulus r_2 and argument θ_2, is to rotate z_1 anticlockwise through an angle θ_2 and multiply its length by r_2.

In particular, multiplying a complex number by i ($|i| = 1$, $\arg i = \pi/2$) has the effect of rotating the original complex number through 90° anticlockwise.

Example 22

The diagram shows the square OABC on an Argand diagram.
If $\overrightarrow{OA} = 5 + 2i$, express the following vectors in the form $a + ib$,
(a) $\overrightarrow{CB}$, (b) $\overrightarrow{BC}$, (c) $\overrightarrow{OC}$.



$\overrightarrow{OA} = 5 + 2i$

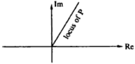
(a) $\overrightarrow{CB}$ is parallel to, and the same length as, $\overrightarrow{OA}$
$\overrightarrow{CB} = \overrightarrow{OA}$
$= 5 + 2i.$

(b) $\overrightarrow{BC} = -\overrightarrow{CB}$
$= -5 - 2i.$

(c) $\overrightarrow{OC} = i(\overrightarrow{OA})$
$= 5i + 2i^2$
$= -2 + 5i.$

(c) By geometry

If $\arg z = \frac{\pi}{3}$, then OP makes an angle of 60° with the Real axis.
Thus the locus is the set of all points such that the line joining them to the origin is at 60° to the x-axis.



By algebra
Suppose $z = x + iy$.
As $\arg z = \frac{\pi}{3}$ then

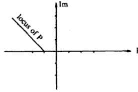
$$\frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3},$$

giving the line $y = x\sqrt{3}$ as the locus of P.
In fact, it is that part of $y = x\sqrt{3}$ for which $y \geq 0$, as for negative y points on the line have argument $-\frac{2\pi}{3}$.

(d) By geometry

Suppose $Z = z + 1$, then $\arg Z = \frac{3\pi}{4}$.

This is a line making an angle of $\frac{3\pi}{4}$ with the Real axis, through the point where $Z = 0$,
i.e. where $z + 1 = 0$ or $z = -1$.
The locus is the set of all points on the line through the point $(-1, 0)$ making an angle of $\frac{3\pi}{4}$ with the Real axis.



By algebra
Suppose $z = x + iy$.
As $\arg(z + 1) = \frac{3\pi}{4}$ then

$$\arg(x + 1 + iy) = \frac{3\pi}{4},$$

i.e. $$\frac{y}{x + 1} = \tan\frac{3\pi}{4} = -1,$$

giving the line $y = -x - 1$ as the locus of P.
As with (c) we must restrict the line to that part for which $y \geq 0$.

Exercise 18F

For questions **1** to **11**, find the locus of the point P corresponding to the complex number z where z obeys the given law.

For **1** to **6**, give *both* a geometrical interpretation and the cartesian equation for each locus.

1. $|z - 2i| = 4$

2. $|z - 1 - 3i| = 5$

3. $|z + 2 - 3i| = 4$

4. $|z| = |z - 2i|$

5. $|z - i| = |z - 1|$

6. $|z - 4 + i| = |z - 1 - 2i|$

For questions **7** and **8**, give the cartesian equation for each locus.

7. $z = z^*$ **8.** $z + z^* = 0$

For questions **9**, **10** and **11**, show each locus on a separate Argand diagram.

9. $\arg z = \frac{\pi}{4}$ **10.** $\arg(z + 2) = \frac{\pi}{4}$ **11.** $\arg(z - i) = \frac{3\pi}{4}$

12. If the point P in the complex plane corresponds to the complex number $z = x + iy$ show that if $|z - 1| = 2|z + 2 - 3i|$ then the locus of P is a circle centre at $-3 + 4i$, and find the radius of the circle.

13. Find by drawing or calculation the vector $\overrightarrow{OX}$ where X is the point in the complex plane where the locus given by $|z - 3 - i| = |z - 1 - 3i|$ intersects with the locus given by $|z + 3i| = |z + 2 - i|$.

14. Find the cartesian equation for
(a) the locus given by $|z| = \sqrt{17}$,
(b) the locus given by $|z| = |z - 2|$.
Hence find the coordinates of the points in the complex plane where these two loci intersect. Show the two loci and their points of intersection on a sketch.

15. If z is the general complex number on an Argand diagram make a sketch of the Argand diagram and shade the region in which $|z + 1 - 4i| \geqslant |z - 2 - i|$.

16. If z is the general complex number on an Argand diagram sketch the Argand diagram and shade the region in which $2 \leqslant |z - i| \leqslant 3$.

17. Show by shading on an Argand diagram the region in which both $|z| \leqslant 4$ and $|z - 3 - i| \geqslant |z - 3 - 5i|$.

Exercise 18G Examination questions

1. Given that

$$\frac{p}{2x + 3} + \frac{q}{3x + 2} \equiv \frac{1}{(2x + 3)(3x + 2)},$$

find the values of the constants p and q. (London)

2. Express $f(x) = \dfrac{x}{(1 - 2x)^2(1 - 3x)}$ in partial fractions.
Hence, or otherwise, find the expansion of $f(x)$ in ascending powers of x up to and including the term in x^3. For what range of values of x is the expansion valid? (Oxford)

3. Show that $\dfrac{3}{(1 + x - 2x^2)}$ can be expressed as

$$\frac{1}{(1 - x)} + \frac{2}{(1 + 2x)}$$

Hence or otherwise expand $(1 + x - 2x^2)^{-1}$ as far as the term in x^4, the values of x being such as to make the expansion valid.
(a) By putting $x = -0{\cdot}02$, use your expansion to evaluate, correct to seven decimal places, $(0{\cdot}9792)^{-1}$,
(b) by giving x a suitable value, use your expansion to evaluate $(1{\cdot}0098)^{-1}$, to seven decimal places. (A.E.B.)

4. Express the function $f(x) = \dfrac{2x - 1}{(1 - x)(3 + x)}$ as the sum of partial fractions.
Obtain the expansion of $f(x)$ in ascending powers of x, in the form $f(x) = a + bx + cx^2 + \ldots$, and find the values of a, b and c. Show that, when $x = 0{\cdot}1$, the error caused by using only these three terms in evaluating $f(x)$ is slightly greater than $0{\cdot}1\%$. (Oxford)

5. Express the function

$$f(x) = \frac{4 - 3x}{(1 - 2x)(2 + x)}$$

in partial fractions.
Find the first three terms in the expansion of $f(x)$ in ascending powers of x, and show that the coefficient of x^n is

$$\frac{4^n + (-1)^n}{2^n}.$$

(J.M.B.)

6. Express

$$\frac{x^2 + 5x + 2}{(x + 1)^2(x - 1)}$$

in the form

$$\frac{A}{(x + 1)} + \frac{B}{(x + 1)^2} + \frac{C}{(x - 1)},$$

where A, B, C are constants to be determined.
Obtain the expansion of

$$\frac{x^2 + 5x + 2}{(x + 1)^2(x - 1)}$$

in ascending powers of $\left(\dfrac{1}{x}\right)$ up to and including the term in $\left(\dfrac{1}{x}\right)^5$

Prove that if n is even the coefficient of $\left(\dfrac{1}{x}\right)^n$ is $n + 2$. Find the coefficient of $\left(\dfrac{1}{x}\right)^n$ if n is odd. (Cambridge)

7. Express $\dfrac{1}{r^2 - 1}$ in partial fractions and hence evaluate

$$\sum_{r=2}^{n} \frac{1}{r^2 - 1}.$$

(Cambridge)

8. Express

$$\frac{1}{(2r - 1)(2r + 3)}$$

in partial fractions.
By multiplying by $\dfrac{1}{2r + 1}$, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{1}{(2r - 1)(2r + 1)(2r + 3)} = \frac{1}{12} - \frac{1}{4(2n + 1)(2n + 3)}.$$

(J.M.B.)

9. Show that

$$\frac{1}{r!} - \frac{1}{(r + 1)!} = \frac{r}{(r + 1)!}$$

and find the corresponding expression for

$$\frac{1}{r!} - \frac{1}{(r+1)!}.$$

S_n and T_n are the sums of the first n terms of the series whose rth terms are

$$\frac{r}{(r+1)!} \quad\text{and}\quad \frac{(-1)^r(r+2)}{(r+1)!},$$

respectively. Find S_{2n} and T_{2n}, and show that

$$S_{2n} = -T_{2n}.$$ (J.M.B.)

10. Given that $z_1 = 3 + 2i$ and $z_2 = 4 - 3i$,

(i) find $z_1 z_2$ and $\frac{z_1}{z_2}$, each in the form $a + ib$;

(ii) verify that $|z_1 z_2| = |z_1||z_2|$. (Cambridge)

11. (a) Find $(2 - i)^3$, expressing your answer in the form $a + ib$.

(b) Verify that $2 + 3i$ is one of the square roots of $-5 + 12i$. Write down the other square root. (Cambridge)

12. One root of the equation

$$z^2 + az + b = 0,$$

where a and b are real constants, is $2 + 3i$. Find the values of a and b. (London)

13. Given that $2 + i$ is a root of the equation

$$z^3 - 11z + 20 = 0,$$

find the remaining roots. (London)

14. Show that $1 + i$ is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down one quadratic factor of $x^4 + 3x^2 - 6x + 10$, and find all the roots of the equation. (Oxford)

15. The complex number z satisfies $\frac{z}{z+2} = 2 - i$. Find the real and imaginary parts of z, and the modulus and argument of z. (Oxford)

16. Expand $z = (1 + ic)^6$ in powers of c and find the five real finite values of c for which z is real. (J.M.B.)

17. (a) If z_1 and z_2 are complex numbers, solve the simultaneous equations $4z_1 + 3z_2 = 23$, $z_1 + iz_2 = 6 + 8i$, giving both answers in the form $x + yi$.

(b) If $(a + bi)^2 = -5 + 12i$, find a and b given that they are both real. Give the two square roots of $-5 + 12i$.

(c) In each of the following cases define the locus of the point which represents z in the Argand diagram. Illustrate each statement by a sketch.

(i) $|z - 2| = 3$, (ii) $|z - 2| = |z - 3|$. (S.U.J.B.)

18. In the Argand diagram, the point P represents the complex number z. Given that

$$|z - 1 - i| = \sqrt{2},$$

sketch the locus of P.

Deduce the greatest and least values of $|z|$ for points P lying on the locus. (Cambridge)

19. Prove that the non-real cube roots of unity are

$$-\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}.$$

These roots are represented in an Argand diagram by the points A, B and the number $z = -2$ is represented by the point C. Show that the area of the sector of the circle with centre C through A and B which is bounded by CA, CB and the minor arc AB is $\frac{1}{2}\pi$. (J.M.B.)

20. By using de Moivre's theorem, or otherwise, find the roots of the equation $z^4 + 4 = 0$.
Hence, or otherwise, express $z^4 + 4$ as the product of two quadratic polynomials in z with real coefficients (Oxford)

21. State de Moivre's theorem, and hence express $\cos 5\theta$ as a polynomial in $\cos \theta$. By considering the roots of the equation $\cos 5\theta = 0$, prove that

$$\cos 5\theta = 16 \cos \theta\left(\cos \theta - \cos \frac{\pi}{10}\right)\left(\cos \theta - \cos \frac{3\pi}{10}\right)\left(\cos \theta - \cos \frac{7\pi}{10}\right)\left(\cos \theta - \cos \frac{9\pi}{10}\right).$$

(Oxford)

22. Prove that, when n is a positive integer,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Given that

$$z = \cos \theta + i \sin \theta,$$

and assuming that the result above is also true for negative integers, show that

$$z^n - z^{-n} = 2i \sin n\theta.$$

Hence, or otherwise, prove that

$$16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta.$$

Find all the solutions of the equation

$$4 \sin^5 \theta + \sin 5\theta = 0$$

which lie in the interval $0 \leqslant \theta \leqslant 2\pi$. (J.M.B.)

19
Exponential and logarithmic functions

An exponential function is one in which the variable appears as an index. For example, $f(x) = 2^x$ or more generally, $f(x) = a^x$.

19.1 Differentiating the exponential function

If we wish to find $\frac{d}{dx}(a^x)$ we must use the basic definition of chapter 10, i.e.:

$$\frac{d}{dx}f(x) = \lim_{\delta x \to 0}\left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

$$\text{Thus} \quad \frac{d}{dx}(a^x) = \lim_{\delta x \to 0}\left[\frac{a^{x+\delta x} - a^x}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{a^x(a^{\delta x} - 1)}{\delta x}\right]$$

$$= a^x \lim_{\delta x \to 0}\left(\frac{a^{\delta x} - 1}{\delta x}\right)$$

The table below shows the values of $\left(\frac{a^{\delta x} - 1}{\delta x}\right)$, correct to four decimal places, for $a = 1, 2, 3$ and 4 as δx decreases.

	$\delta x = 1$	$\delta x = 0{\cdot}1$	$\delta x = 0{\cdot}01$	$\delta x = 0{\cdot}001$	$\delta x = 0{\cdot}0001$	$\delta x = 0{\cdot}00001$	$\delta x = 0{\cdot}000001$
$a = 1$	0	0	0	0	0	0	0
$a = 2$	1	0·7177	0·6956	0·6934	0·6932	0·6931	0·6931
$a = 3$	2	1·1612	1·1047	1·0992	1·0987	1·0986	1·0986
$a = 4$	3	1·4870	1·3959	1·3873	1·3864	1·3863	1·3862

This table suggests that there exists a value of a between 2 and 3 for which $\lim_{\delta x \to 0}\left(\frac{a^{\delta x} - 1}{\delta x}\right) = 1$, i.e. for this value of a, $\frac{d}{dx}(a^x) = a^x$. If we call this number e, then

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Thus the significance of the number e (lying between 2 and 3) is that the function e^x differentiates to give itself. It is possible to show that e is an irrational number and, correct to 5 decimal places, its value is 2·71828. The function $f(x) = e^x$ is known as *the* exponential function.

We can then use this result when differentiating exponential functions:

$$\text{If} \quad y = e^{f(x)}, \ \frac{dy}{dx} = f'(x) \times e^{f(x)}$$

Example 3

Differentiate with respect to x:
(a) $y = e^{4x}$ (b) $y = 5e^{(x^2+1)}$ (c) $y = 5x^2 + 3/e^{x^2}$ (d) $y = e^x \sin 2x$

Using the above rule:

(a) $y = e^{4x}$

$$\frac{dy}{dx} = 4e^{4x}$$

(b) $y = 5e^{(x^2+1)}$

$$\frac{dy}{dx} = 2x \times 5e^{(x^2+1)} = 10xe^{(x^2+1)}$$

(c) $y = 5x^2 + 3/e^{x^2}$

$$= 5x^2 + 3e^{-x^2}$$

$$\frac{dy}{dx} = 10x + (-2x)3e^{-x^2} = 10x - \frac{6x}{e^{x^2}}$$

(d) $y = e^x \sin 2x$

Using the product rule,

$$\frac{dy}{dx} = e^x(2 \cos 2x) + e^x \sin 2x = e^x(2 \cos 2x + \sin 2x)$$

Integration of exponential functions

From the last section we know that if $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$.

Noticing that $e^{f(x)}$ features in both the expression for y and that for $\frac{dy}{dx}$, we must expect integration of $g(x)e^{f(x)}$ to feature $e^{f(x)}$. Thus when integrating $g(x)e^{f(x)}$ we first consider $\frac{d}{dx}e^{f(x)}$.

Example 4

Find the following indefinite integrals:

(a) $\int 10e^{5x}\,dx$ (b) $\int xe^{(x^2+1)}\,dx$ (c) $\int (\sin x + 2/e^x)\,dx$

(a) $\int 10e^{5x}\,dx$

Now $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$$\therefore \int 10e^{5x}\,dx = 2e^{5x} + c$$

(b) $\int xe^{(x^2+1)}\,dx$

Now $\frac{d}{dx}(e^{(x^2+1)}) = 2xe^{(x^2+1)}$

$$\therefore \int xe^{(x^2+1)}\,dx = \tfrac{1}{2}e^{(x^2+1)} + c$$

(c) $\int (\sin x + 2e^{-x})\,dx$

Now $\frac{d}{dx}(e^{-x}) = -e^{-x}$

$\therefore \quad \int (\sin x + 2e^{-x})\,dx = -\cos x - 2e^{-x} + c$

Example 5

Evaluate $\int_0^1 (4xe^{x^2} + 1)\,dx$, leaving your answer in terms of e.

Now $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}, \quad \therefore \quad \int_0^1 (4xe^{x^2} + 1)\,dx = \left[2e^{x^2} + x\right]_0^1$

$$= (2e + 1) - (2e^0 + 0)$$
$$= 2e - 1$$

Thus $\quad \int_0^1 (4xe^{x^2} + 1)\,dx = 2e - 1.$

Exercise 19A

Differentiate the following with respect to x.

1. e^x **2.** e^{3x} **3.** e^{2x} **4.** e^{x^2} **5.** $e^{(x^2+1)}$

6. $\frac{1}{e^x}$ **7.** $4e^{2x}$ **8.** $5e^{2x^2}$ **9.** $\frac{3}{e^{2x}}$ **10.** $e^{(5x+3)} + \frac{2}{x^2}$

11. $2e^{x^2} + 3e^x + \frac{4}{e^x}$ **12.** x^2e^x **13.** $3x^3e^{2x}$ **14.** $\frac{3x^2}{e^x}$

15. $\frac{e^x + 4}{x^2}$

Find the following indefinite integrals

16. $\int e^x\,dx$ **17.** $\int 5e^x\,dx$ **18.** $\int 2e^{2x}\,dx$ **19.** $\int 6e^{3x}\,dx$

20. $\int 2xe^{x^2}\,dx$ **21.** $\int 4xe^{(x^2+1)}\,dx$ **22.** $\int \left(-\frac{4}{e^{2x}}\right)dx$ **23.** $\int \frac{e^{(x+2)} + 4}{e^x}\,dx$

Evaluate the following giving your answers in terms of e.

24. $\int_1^3 e^x\,dx$ **25.** $\int_0^3 e^{-x}\,dx$ **26.** $\int_1^2 2e^{(2x+1)}\,dx$ **27.** $\int_{-1}^1 2e^{(1-2x)}\,dx$

28. Find the gradient of the curve $y = xe^{(x-3)}$ at the point (3, 3).

29. Find the equation of the curve that has a gradient function given by $e^x + 2x$ and that passes through the point $(0, -3)$.

30. With the help of the sketch of $y = e^x$ on page 478 make sketch graphs of (a) $y = -e^x$ (b) $y = e^{-x}$ (c) $y = e^{x-1}$

31. Find the coordinates and nature of any turning points on the curve $y = xe^x$

32. Find the coordinates and nature of any turning points on the curve $y = x - e^x$

33. Sketch the graph of $y = x^2e^x$ indicating clearly the coordinates of any stationary points on the curve and of any points where the curve cuts the axes.

34. If $y = -e^x \cos 2x$, show that $\dfrac{d^2y}{dx^2} = 5e^x \sin(2x + \alpha)$ where $\alpha = \tan^{-1}(\frac{3}{4})$.

35. If $y = e^x \tan x$, show that $\dfrac{d^2y}{dx^2} = e^x(2\tan^3 x + 2\tan^2 x + 3\tan x + 2)$

19.2 The logarithmic function

In chapter 5, we saw that if $y = \log_e x$, which may also be written as $y = \ln x$, then

$$x = e^y$$

$$\therefore \quad \frac{dx}{dy} = e^y$$

but $\dfrac{dy}{dx} = 1\bigg/\left(\dfrac{dx}{dy}\right)$, (see page 324) Thus $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

> Thus, if $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$

Alternatively, this result could be obtained from $x = e^y$ by differentiating implicitly.

Example 6

Make a sketch of the curve given by $y = \ln x$.

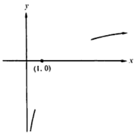
x-axis If $y = 0$, then $\ln x = 0$
i.e. $x = 1$.
Curve cuts x-axis at $(1, 0)$

y-axis If $x = 0$, then $y = \ln 0$,
i.e. no value of y for $x = 0$
$\therefore$ curve does not cut y-axis.

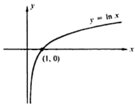
$x \to \pm\infty$ $x \to +\infty$, y increases slowly
$x \to -\infty$ need not be considered as x cannot be negative.

y undefined y is undefined for negative x and for $x = 0$

max/min $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$.
$\therefore$ no turning points.
Also, as x cannot be negative the gradient is always positive and as $x \to 0^+$, the gradient $\to +\infty$.



Thus the sketch can be completed.



Example 7

Differentiate $y = \ln(x^2 + 1)$ with respect to x.

$$y = \ln(x^2 + 1)$$

Let $u = x^2 + 1$ then $y = \ln u$

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = \frac{1}{u}$$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, $\frac{dy}{dx} = \frac{1}{u}2x$

$$\frac{dy}{dx} = \frac{2x}{x^2 + 1}.$$

Note: If we consider the more general case

$$y = \ln[f(x)]$$

and let $u = f(x)$ then $y = \ln u$

$$\frac{du}{dx} = f'(x) \qquad \frac{dy}{du} = \frac{1}{u}$$

thus $\frac{dy}{dx} = \frac{1}{u}f'(x)$

or $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Thus we have a rule for differentiating functions of the type $\ln[f(x)]$:

If $y = \ln[f(x)]$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Example 8

Find $\frac{dy}{dx}$ for each of the following: (a) $y = \ln(5x^2 - 6)$ (b) $y = \ln(\sin 2x)$ (c) $y = \ln\left(\frac{x+2}{x+3}\right)$

Using the above rule:

(a) $y = \ln(5x^2 - 6)$

$$\frac{dy}{dx} = \frac{10x}{5x^2 - 6}$$

(b) $y = \ln(\sin 2x)$

$$\frac{dy}{dx} = \frac{2\cos 2x}{\sin 2x}$$

$$\frac{dy}{dx} = 2\cot 2x$$

(c) $y = \ln\left(\frac{x+2}{x+3}\right)$

$$= \ln(x+2) - \ln(x+3)$$

$$\frac{dy}{dx} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$= \frac{1}{(x+2)(x+3)}$$

Example 9

Find an expression for $\frac{dy}{dx}$, given that $y = 2^x$.

If $y = 2^x$

then $\ln y = x \ln 2$ ... [1]

Example 15

Evaluate $\int_3^5 \frac{3}{x-2}\,dx$ giving your answer correct to 3 decimal places.

METHOD 1

$$\int_3^5 \frac{3}{x-2}\,dx = \Big[3\ln|x-2|\Big]_3^5$$
$$= 3\ln 3 - 3\ln 1$$
$$= 3{\cdot}296$$

METHOD 2

$(x-2)$ is positive for $3 \leqslant x \leqslant 5$,

$$\therefore \quad \int_3^5 \frac{3}{x-2}\,dx = \Big[3\ln(x-2)\Big]_3^5$$
$$= 3\ln 3 - 3\ln 1 = 3{\cdot}296$$

Example 16

Evaluate $\int_3^5 \frac{4}{1-x}\,dx$.

METHOD 1

$$\int_3^5 \frac{4}{1-x}\,dx = \Big[-4\ln|1-x|\Big]_3^5$$
$$= (-4\ln 4) - (-4\ln 2)$$
$$= 4\ln 2 - 4\ln 4$$
$$= 4\ln\tfrac{1}{2}$$
$$= -2{\cdot}773$$

METHOD 2

$(1-x)$ is negative for $3 \leqslant x \leqslant 5$,

$$\therefore \quad \int_3^5 \frac{4}{1-x}\,dx = \int_3^5 \frac{-4}{x-1}\,dx$$

Now, $(x-1)$ is positive for $3 \leqslant x \leqslant 5$,

$$\text{so} \quad \int_3^5 \frac{4}{1-x}\,dx = \Big[-4\ln(x-1)\Big]_3^5$$
$$= (-4\ln 4) - (-4\ln 2)$$
$$= 4\ln 2 - 4\ln 4$$
$$= -2{\cdot}773$$

Note: (i) $\int_a^b \frac{f'(x)}{f(x)}\,dx$ is meaningless if $f(a)$ and $f(b)$ are of opposite sign because, in such cases, there will be a value c between a and b for which $f(c) = 0$ and so $\frac{f'(x)}{f(x)}$ is undefined.

For example: $\int_{-1}^2 \frac{1}{x}\,dx$ is meaningless because $\frac{1}{x}$ is undefined for $x = 0$, which lies between -1 and 2.

$\int_2^4 \frac{2}{2x-5}\,dx$ is meaningless because $\frac{2}{2x-5}$ is undefined for $x = 2\frac{1}{2}$ which lies between 2 and 4.

(ii) It can also be the case that $f(x) = 0$ for a value of x between a and b, even though $f(a)$ and $f(b)$ are of the same sign.

Consider $\int_0^5 \frac{2x-3}{x^2-3x+2}\,dx$. If $f(x) = x^2 - 3x + 2$, then $f(5)$ and $f(0)$ are both positive.

But $f(x) = 0$ for $x = 1$ and $x = 2$ and both of these values lie in the range 0 to 5.

So this integral is meaningless as the integrand is undefined at $x = 1$ and $x = 2$.

19.4 The logarithmic series

From the binomial expansion, we know that for $|x| < 1$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

Integrating both sides with respect to x gives

$$\ln(1+x) + c = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Substituting $x = 0$ in both sides gives $c = 0$

Thus for $|x| < 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots$$

Notes: 1. This expansion depends on the assumption that the sum of the series obtained by integrating each term of a series is the integral of the sum of that series. Again the proof of this assumption is beyond the scope of this book.

2. It can be shown that the expansion is valid for $x = 1$ as well as for $|x| < 1$.

> Thus: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots \quad \text{for} \quad -1 < x \leqslant 1$

Example 18

Find an expansion in x for $\ln(1-x)$ and state the range of values of x for which it is valid.

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots$

for $-1 < x \leqslant 1$

substituting $(-x)$ for x gives

$$\ln(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \dots + (-1)^{n+1}\frac{(-x)^n}{n} + \dots$$

for $-1 < -x \leqslant 1$

i.e. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots - \frac{x^n}{n} - \dots$

for $-1 \leqslant x < 1$.

Example 19

Find an expansion, in ascending powers of x, for $\ln(2-5x)$ stating the terms up to and including the term in x^4 and the range of values of x for which the expansion is valid.

We know that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots$ for $-1 < x \leqslant 1$

Now $\ln(2-5x) = \ln[2(1-\frac{5}{2}x)]$

$= \ln 2 + \ln(1-\frac{5}{2}x)$

thus $\ln(2-5x) = \ln 2 + [-\frac{5}{2}x - \frac{1}{2}(-\frac{5}{2}x)^2 + \frac{1}{3}(-\frac{5}{2}x)^3 - \frac{1}{4}(-\frac{5}{2}x)^4 \dots]$

$= \ln 2 - \frac{5}{2}x - \frac{25}{8}x^2 - \frac{125}{24}x^3 - \frac{625}{64}x^4 \dots$ for $-1 < -\frac{5}{2}x \leqslant 1$

or $-\frac{2}{5} \leqslant x < \frac{2}{5}$

Thus $\ln(2-5x) = \ln 2 - \frac{5}{2}x - \frac{25}{8}x^2 - \frac{125}{24}x^3 - \frac{625}{64}x^4 \dots$ for $-\frac{2}{5} \leqslant x < \frac{2}{5}$

Example 20

Find an expansion in ascending powers of x for $\ln(1 - x - 6x^2)$, giving the terms up to and including that in x^4, and state the values of x for which the expansion is valid.

$$\begin{aligned}\ln(1 - x - 6x^2) &= \ln[(1 - 3x)(1 + 2x)] \\ &= \ln(1 - 3x) + \ln(1 + 2x) \\ &= \left(-3x - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} - \frac{(-3x)^4}{4} \dots\right) \\ &\qquad + \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots\right)\end{aligned}$$

Thus $\ln(1 - x - 6x^2) = -x - \frac{13}{2}x^2 - \frac{19}{3}x^3 - \frac{97}{4}x^4 \dots$

The expansion for $\ln(1 - 3x)$ is valid for $-1 < -3x \leqslant 1$, i.e. $-\frac{1}{3} \leqslant x < \frac{1}{3}$.

The expansion for $\ln(1 + 2x)$ is valid for $-1 < 2x \leqslant 1$, i.e. $-\frac{1}{2} < x \leqslant \frac{1}{2}$.

As both expansions have been used, the expansion is valid for $-\frac{1}{3} \leqslant x < \frac{1}{3}$.

Thus $\ln(1 - x - 6x^2) = -x - \frac{13}{2}x^2 - \frac{19}{3}x^3 - \frac{97}{4}x^4 \dots$ for $-\frac{1}{3} \leqslant x < \frac{1}{3}$

Example 21

Find an expansion in ascending powers of x for $\ln\left(\frac{1 + x}{1 - x}\right)$ and state the values of x for which the expansion is valid.

$$\ln\left(\frac{1 + x}{1 - x}\right) = \ln(1 + x) - \ln(1 - x)$$

Hence $$\ln\left(\frac{1 + x}{1 - x}\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

The two expansions used are valid for $-1 < x \leqslant 1$ and $-1 \leqslant x < 1$ respectively.
The combined expansion is then valid for $-1 < x < 1$.

Thus $$\ln\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) \quad \text{for } |x| < 1.$$

Note: This expansion can be used to evaluate logarithms and since it converges so much more quickly than $\ln(1 + x)$, it is particularly useful for this purpose.

Using $\ln(1 + x)$, and substituting $x = 1$,
$\ln(1 + 1) = \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ i.e. the fifth term is 0·2.

Using $\ln\left(\frac{1 + x}{1 - x}\right)$ and substituting $x = \frac{1}{3}$,

$$\ln\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right) = \ln 2 = 2\left(\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \dots\right)$$

$$= \frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \dots$$ i.e. the third term is 0·0016.

6. Using an appropriate number of terms of the series expansion of $\ln(1 + x)$ find the value of $\ln(1{\cdot}1)$ and $\ln(0{\cdot}9)$ correct to four decimal places.
7. (a) Show that for $|x| < 1$, $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$
   (b) Using the result of part (a) evaluate (i) $\ln \frac{1}{3}$ correct to 4 decimal places,
   (ii) $\ln 3$ correct to 3 decimal places.
8. If $x > 1$ show that $\ln(1 + x) = \ln x + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots$
9. Find the coefficient of x^r in the series expansions of the following
   (a) e^{2x} (b) $\ln\left(1 + \frac{x}{2}\right)$ (c) $(1 + x)e^x$ (d) $\left(2 - x - \frac{1}{x}\right)\ln(1 - x)$
10. Expand $e^{2x} - 3e^x - e^{-x}$ in ascending powers of x up to and including the term in x^4.
11. Obtain the series expansion of $e^x/(1 + 2x)$ up to and including the term in x^3 and state the values of x for which the expansion is valid.
12. Obtain the series expansion of $[\ln(1 + x)]/(2 + x)$ up to and including the term in x^4 and state the values of x for which the expansion is valid.
13. Find the first four non-zero terms in the series expansion of each of the following in ascending powers of x, and state the values of x for which the expansion is valid
   (a) $e^x \ln(1 + x)$ (b) $e^{-x}\ln(1 + 2x)$ (c) $e^{6x}\ln\left(1 + \frac{x}{2}\right)^2$
   (d) $e^x \ln\sqrt{(1 - 2x)}$
14. Find the values of the positive constants a, b and c given that when x is sufficiently small for terms in x^4, and higher powers of x, to be neglected
   then $\frac{e^{ax}}{2 + bx} = \frac{1}{2} + \frac{x^2}{4} - cx^3$ (assume $|bx| < 2$)
15. Find the values of a, b and c given that when x is sufficiently small for terms in x^4 and higher powers of x to be neglected
   then $e^{ax}\ln(1 + bx) = -3x - \frac{21}{2}x^2 + cx^3$ (assume $-1 < bx \leqslant 1$)

Exercise 19D Examination Questions

1. Sketch the curves $y = e^x$ and $y = e^{-2x}$, using the same axes.
   The line $y = 4$ intersects the first curve at A and the second curve at B.
   Calculate the length of AB to two decimal places. (A.E.B.)
2. Differentiate with respect to x
   (a) $\frac{e^{-x}}{x}$ (b) $\ln(1 + \sin x)$ (London)
3. Differentiate $x \log_e x$ with respect to x.
   Use your result to find $\int \log_e x\,dx$.
   Hence evaluate, to three significant figures, $\int_1^2 \log_e x\,dx$. (A.E.B.)

12. Given that $\frac{dy}{dx} = \frac{2x}{1 + x^2} - 2xe^{-x^2}$ and that $y = 0$ when $x = 0$, express y in terms of x.
Show that, for small values of $|x|$, $y \approx px^6 + qx^8$ finding the values of the constants p and q. (A.E.B.)

13. Write down, in ascending powers of y, the first four terms in the series for e^y. By taking $e^y = 2^x$, show that

$$2^x = 1 + x \log_e 2 + \tfrac{1}{2}x^2 (\log_e 2)^2 + \tfrac{1}{6}x^3 (\log_e 2)^3 + \ldots.$$

Given that $2^{3x} + 5(2^x) = 6 + Ax + Bx^2 + Cx^3 + \ldots$, for all real x, find the values of the constants A, B and C in terms of $\log_e 2$. (A.E.B.)

14. Find the first three non-zero terms in the expansion, in ascending powers of x, of

$$e^{2x} + e^{-2x}.$$

Given that x is so small that its fifth and higher powers may be neglected, find the numerical values of the constants a, b and c in the approximate formula

$$\log_e (e^{2x} + e^{-2x}) \approx a + bx^2 + cx^4. \qquad \text{(J.M.B.)}$$

15. (a) Solve the equation $\log_3 x = \log_x 9$, giving your answers correct to three significant figures.
(b) Show that $\ln (1 + x + x^2) = \ln (1 - x^3) - \ln (1 - x)$. Hence obtain the expansion of

$$\ln (1 + x + x^2)$$

in ascending powers of x up to and including the term in x^5.
Show that the coefficient of x^n in this expansion is $-\frac{2}{n}$ or $\frac{1}{n}$ according as n is or is not a multiple of 3.
Write down the expansion of $\ln (1 - x + x^2)$ in ascending powers of x up to and including the term in x^5. (Cambridge)

16. Given that

$$f(x) = e^{-3x} \quad \text{and} \quad g(x) = (1 - x)^{-3},$$

find the first three non-zero terms in the expansions in ascending powers of x of

(i) $f(x) - g(x)$, (ii) $\log_e f(x) - \log_e g(x)$, (iii) $\frac{f(x)}{g(x)}$.

In each of the expansions (i) and (ii) obtain the coefficient of x^n for $n > 1$.
(J.M.B.)

20
Calculus IV: Further integration

20.1 Change of variable

Suppose we want to integrate $x^2(2x^3 + 3)^5$ with respect to x. We could expand the bracket $(2x^3 + 3)^5$, multiply by x^2 and then integrate each term. It is, however, much better to notice that x^2 is a scalar multiple of $\frac{d}{dx}(2x^3 + 3)$ and then it follows that

$$\int x^2(2x^3 + 3)^5\,dx = \frac{1}{36}(2x^3 + 3)^6 + c.$$

If we fail to notice this 'function of a function', an alternative method is to change the variable from x to some other suitably chosen variable. This method depends on the following theory:

Suppose $y = \int f(x)\,dx$ where $x = g(u)$ i.e. $\frac{dy}{dx} = f(x)$ and $f(x) = f[g(u)]$

Now $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$\therefore \quad \frac{dy}{du} = f(x) \times \frac{dx}{du}$

integrating, with respect to u, gives

$$y = \int f(x)\frac{dx}{du}\,du$$

$$= \int f[g(u)]\frac{dx}{du}\,du \quad \text{but } y = \int f(x)\,dx$$

$$\therefore \int f(x)\,dx = \int f[g(u)]\frac{dx}{du}\,du$$

Thus, to integrate $f(x)$ with respect to x, dx is replaced by $\frac{dx}{du}du$ and the integrand is then expressed in terms of the new variable, u.

Example 1

Find $\int x^2(2x^3 + 3)^5\,dx$.

Let $\quad u = 2x^3 + 3$ then $\frac{du}{dx} = 6x^2$

Thus $\int x^2(2x^3 + 3)^5\,dx = \int x^2(2x^3 + 3)^5\frac{dx}{du}\,du$

$$= \int x^2(2x^3 + 3)^5\frac{1}{6x^2}\,du$$

$$= \int \frac{u^5}{6} du$$

$$= \frac{u^6}{36} + c$$

$$\therefore \quad \int x^2(2x^3 + 3)^5 \, dx = \frac{1}{36}(2x^3 + 3)^6 + c$$

As was noted at the begining of the chapter, $x^2(2x^3 + 3)^5$ could be integrated by inspection. However, there are many cases for which the method of changing the variable enables us to integrate an expression which we could not integrate by inspection.

Example 2

Find $\int 2x(x + 2)^5 \, dx$.

Let $\qquad u = x + 2 \quad \text{then} \quad \frac{du}{dx} = 1$

$$\int 2x(x + 2)^5 \, dx = \int 2x(x + 2)^5 \frac{dx}{du} du$$

$$= \int 2(u - 2)u^5 \, du$$

$$= \tfrac{2}{7}u^7 - \tfrac{2}{3}u^6 + c$$

$$= \tfrac{2}{21}u^6(3u - 7) + c$$

$$\therefore \int 2x(x + 2)^5 \, dx = \tfrac{2}{21}(x + 2)^6(3x - 1) + c$$

Example 3

Find $\int x\sqrt{(3x - 2)} \, dx$

Let $\qquad u = 3x - 2 \quad \text{then} \quad \frac{du}{dx} = 3$

$$\int x\sqrt{(3x - 2)} \, dx = \int x\sqrt{(3x - 2)} \frac{dx}{du} du$$

$$= \int \left(\frac{u + 2}{3}\right)\sqrt{u}\,\frac{1}{3} du$$

$$= \int \left(\frac{u^{3/2}}{9} + \frac{2u^{1/2}}{9}\right) du$$

$$= \tfrac{2}{45}u^{5/2} + \tfrac{4}{27}u^{3/2} + c$$

$$= \tfrac{2}{135}u^{3/2}(3u + 10) + c$$

$$\therefore \int x\sqrt{(3x - 2)} \, dx = \tfrac{2}{135}(3x - 2)^{3/2}(9x + 4) + c$$

Example 4

Evaluate $\int_0^5 \frac{x}{\sqrt{(x + 4)}} dx$.

Let $\qquad u = x + 4 \quad \text{then} \quad \frac{du}{dx} = 1$

Notice, also, that when $x = 0$, $u = 4$ and when $x = 5$, $u = 9$.

$$\int_0^5 \frac{x}{\sqrt{(x + 4)}} dx = \int_{u=4}^{u=9} \frac{x}{\sqrt{(x + 4)}} \frac{dx}{du} du$$

$$= \int_4^9 \frac{u - 4}{\sqrt{u}} du$$

$$= \left[\frac{2}{3}u^{3/2} - 8u^{1/2}\right]_4^9$$

$$= (\tfrac{2}{3} \times 27 - 8 \times 3) - (\tfrac{2}{3} \times 8 - 8 \times 2)$$

$$\therefore \quad \int_0^5 \frac{x}{\sqrt{(x + 4)}} dx = 4\tfrac{2}{3}$$

In chapter 15 we saw that $\int \frac{1}{(a^2 + x^2)}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ and

$$\int \frac{1}{\sqrt{(a^2 - x^2)}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c.$$

However, as was indicated in chapter 15, for more difficult functions of this type, integration by change of variable is advisable. These more difficult integrations will be of the form $\int \frac{1}{a^2 + b^2x^2}dx$, $|b| \neq 1$ and

$$\int \frac{1}{\sqrt{(a^2 - c^2x^2)}}dx, \ |c| \neq 1.$$

We can then substitute

either $u = bx$ or $x = \frac{a}{b}\tan u$ into $\int \frac{1}{a^2 + b^2x^2}dx$

and either $u = cx$ or $x = \frac{a}{c}\sin u$ into $\int \frac{1}{\sqrt{(a^2 - c^2x^2)}}dx$.

Example 5

Use a suitable substitution to find

(a) $\int \frac{1}{25 + 16x^2}dx$ (b) $\int \frac{1}{\sqrt{(2 - 9x^2)}}dx$.

METHOD 1

(a) Let $u = 4x$ then $\frac{du}{dx} = 4$

$$\int \frac{1}{25 + 16x^2}dx = \int \frac{1}{25 + u^2} \times \frac{1}{4}du$$
$$= \frac{1}{4}\int \frac{1}{25 + u^2}du$$
$$= \frac{1}{20}\tan^{-1}\left(\frac{u}{5}\right) + c$$
$$\therefore \int \frac{1}{25 + 16x^2}dx = \frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right) + c$$

(b) Let $u = 3x$ then $\frac{du}{dx} = 3$

$$\int \frac{1}{\sqrt{(2 - 9x^2)}}dx = \int \frac{1}{\sqrt{(2 - u^2)}} \times \frac{1}{3}du$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{(2 - u^2)}}du$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$
$$\therefore \int \frac{1}{\sqrt{(2 - 9x^2)}}dx = \frac{1}{3}\sin^{-1}\left(\frac{3x}{\sqrt{2}}\right) + c$$

METHOD 2

Let $x = \frac{5}{4}\tan u$ then $\frac{dx}{du} = \frac{5}{4}\sec^2 u$

$$\int \frac{1}{25 + 16x^2}dx = \int \frac{1}{25(1 + \tan^2 u)} \times \frac{5}{4}\sec^2 u\,du$$
$$= \frac{1}{20}\int du$$
$$= \frac{u}{20} + c$$
$$\int \frac{1}{25 + 16x^2}dx = \frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right) + c$$

Let $x = \frac{\sqrt{2}}{3}\sin u$ then $\frac{dx}{du} = \frac{\sqrt{2}}{3}\cos u$

$$\int \frac{1}{\sqrt{(2 - 9x^2)}}dx = \int \frac{1}{\sqrt{[2(1 - \sin^2 u)]}} \times \frac{\sqrt{2}}{3}\cos u\,du$$
$$= \frac{1}{3}\int du$$
$$= \frac{1}{3}u + c$$
$$\int \frac{1}{\sqrt{(2 - 9x^2)}}dx = \frac{1}{3}\sin^{-1}\left(\frac{3x}{\sqrt{2}}\right) + c$$

Example 6

Evaluate $\int_0^1 \sqrt{(1 - x^2)}\,dx$.

For integrations of the type $\sqrt{(a^2 - b^2x^2)}$, $x = \frac{a}{b}\sin u$ is a useful substitution.

Thus, for $\int_0^1 \sqrt{(1 - x^2)}\,dx$, we let $x = \sin u$ then $\frac{dx}{du} = \cos u$.

Notice also that when $x = 0$, $u = 0$ and when $x = 1$, $u = \frac{\pi}{2}$.

$$\int_0^1 \sqrt{(1 - x^2)}\,dx = \int_{u=0}^{u=\pi/2} \sqrt{(1 - \sin^2 u)}\cos u\,du$$

$$= \int_0^{\pi/2} \cos^2 u\,du$$

$$= \int_0^{\pi/2} \frac{1}{2}(\cos 2u + 1)\,du$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2u + u\right]_0^{\pi/2}$$

$$= \tfrac{1}{2}(\tfrac{1}{2}\sin \pi + \pi/2) - \tfrac{1}{2}(\tfrac{1}{2}\sin 0 + 0)$$

$$\therefore \quad \int_0^1 \sqrt{(1 - x^2)}\,dx = \frac{\pi}{4}$$

This method of substituting in order to change the variable can also be used to integrate expressions of the type $\int \frac{1}{ax^2 + bx + c}\,dx$ when $(ax^2 + bx + c)$ does not factorise.

Example 7

Find $\int \frac{1}{x^2 - 4x + 29}\,dx$.

Now $\quad \int \frac{1}{x^2 - 4x + 29}\,dx = \int \frac{1}{(x - 2)^2 + 25}\,dx$

Let $u = x - 2$ then $\frac{du}{dx} = 1$

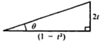
and $\therefore \quad \int \frac{1}{x^2 - 4x + 29}\,dx = \int \frac{1}{u^2 + 25}\,du$

$$= \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + c$$

$$\therefore \quad \int \frac{1}{x^2 - 4x + 29}\,dx = \frac{1}{5}\tan^{-1}\left(\frac{x - 2}{5}\right) + c$$

The substitution $t = \tan \frac{1}{2}\theta$

If we let $t = \tan \frac{1}{2}\theta$ then, from $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$, it follows that $\tan \theta = \dfrac{2t}{(1 - t^2)}$.
It also follows, from the right angled triangle shown, that $\sin \theta = \dfrac{2t}{(1 + t^2)}$ and that $\cos \theta = \dfrac{(1 - t^2)}{(1 + t^2)}$.
This substitution can be useful in finding some integrals.



Example 8

Use the substitution, $t = \tan \dfrac{\theta}{2}$, to evaluate $\displaystyle\int_0^{\pi/2} \frac{4}{3 + 5 \sin \theta} d\theta$.

Let $t = \tan \dfrac{\theta}{2}$ then $\sin \theta = \dfrac{2t}{(1 + t^2)}$ and $\dfrac{dt}{d\theta} = \frac{1}{2} \sec^2 \dfrac{\theta}{2} = \dfrac{1 + t^2}{2}$.

Notice also that when $\theta = 0$, $t = 0$ and when $\theta = \dfrac{\pi}{2}$, $t = 1$.

$$\text{Thus} \quad \int_0^{\pi/2} \frac{4}{3 + 5 \sin \theta} d\theta = \int_{t=0}^{t=1} \frac{4}{[3 + 10t/(1 + t^2)]} \times \frac{2}{(1 + t^2)} dt$$
$$= \int_0^1 \frac{8}{3t^2 + 10t + 3} dt$$
$$= \int_0^1 \left(\frac{3}{3t + 1} - \frac{1}{t + 3}\right) dt$$
$$= \Big[\ln (3t + 1) - \ln (t + 3)\Big]_0^1$$
$$= (\ln 4 - \ln 4) - (\ln 1 - \ln 3)$$
$$= \ln 3$$
$$\therefore \quad \int_0^{\pi/2} \frac{4}{3 + 5 \sin \theta} d\theta = \ln 3.$$

Exercise 20A

Find the following indefinite integrals, in terms of x, using the suggested substitution

1. $\int 6x \sin (x^2 - 4)\, dx$, $u = x^2 - 4$ **2.** $\int 5x \cos (5 - x^2)\, dx$, $u = 5 - x^2$

3. $\int 3x\sqrt{(1 + x^2)}\, dx$, $u = 1 + x^2$ **4.** $\int 3x (x^2 + 6)^5\, dx$, $u = x^2 + 6$

5. $\int x(x + 2)^9\, dx$, $u = x + 2$ **6.** $\int 5x^2(x - 3)^8\, dx$, $u = x - 3$

7. $\int 9x(3x + 2)^3\, dx$, $u = 3x + 2$ **8.** $\int 7x(2x + 3)^5\, dx$, $u = 2x + 3$

9. $\int \frac{3x}{\sqrt{(2x+3)}}dx,\ u = 2x + 3$ **10.** $\int \frac{1}{\sqrt{(1-x^2)}}dx,\ x = \sin u$

11. $\int \frac{1}{4+9x^2}dx,\ x = \frac{2}{3}\tan u$ **12.** $\int \frac{1}{\sqrt{(25-4x^2)}}dx,\ x = \frac{5}{2}\sin u$

13. $\int \frac{1}{\sqrt{(4-9x^2)}}dx,\ x = \frac{2}{3}\sin u$ **14.** $\int \frac{1}{4+5x^2}dx,\ x = \frac{2}{\sqrt{5}}\tan u$

15. $\int \sqrt{(25-x^2)}\,dx,\ x = 5\sin\theta$ **16.** $\int \sqrt{(1-4x^2)}\,dx,\ x = \frac{1}{2}\sin\theta$

17. Given that $\int \frac{1}{\sqrt{(a^2-x^2)}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ and $\int \frac{a}{a^2+x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c$ find the following indefinite integrals using the suggested substitution

(a) $\int \frac{1}{1+16x^2}dx,\ u = 4x$ (b) $\int \frac{1}{\sqrt{(9-16x^2)}}dx,\ u = 4x$

(c) $\int \frac{1}{4+5x^2}dx,\ u = x\sqrt{5}$ (d) $\int \frac{1}{\sqrt{(5-2x^2)}}dx,\ u = x\sqrt{2}$

18. Use the method of example 7 to find the following integrals

(a) $\int \frac{3}{x^2-2x+10}dx$ (b) $\int \frac{1}{x^2+6x+13}dx$ (c) $\int \frac{3}{4x^2-12x+13}dx$

Find the following indefinite integrals by using some suitable substitution.

19. $\int (x+2)(2x-3)^6\,dx$ **20.** $\int x\sqrt{(2x+1)}\,dx$

21. $\int \frac{x}{\sqrt{(2x+1)}}dx$ **22.** $\int \frac{1}{\sqrt{(25-4x^2)}}dx$

23. $\int \sqrt{(9-x^2)}\,dx$ **24.** $\int \frac{4}{x^2+2x+17}dx$

Evaluate the following definite integrals by using some suitable substitution.

25. $\int_0^3 x\sqrt{(x+1)}\,dx$ **26.** $\int_1^2 (x+2)(x-1)^5\,dx$

27. $\int_1^2 x^2(x-1)^5\,dx$ **28.** $\int_1^2 x(2x-3)^4\,dx$

29. $\int_0^1 4x(2x-1)^4\,dx$ **30.** $\int_0^{1/2} \frac{1}{1+4x^2}dx$

31. $\int_0^{3/2} \sqrt{(9-4x^2)}\,dx$

Find the following definite integrals using the substitution $t = \tan\frac{1}{2}\theta$

32. $\int_0^{\pi/2} \frac{3}{1+\sin\theta}d\theta$ **33.** $\int_0^{2\pi/3} \frac{3}{5+4\cos\theta}d\theta$

34. $\int_{-\pi/2}^{\pi/2} \frac{3}{4+5\cos\theta}d\theta$ **35.** $\int_0^{\pi/2} \frac{5}{3\sin\theta+4\cos\theta}d\theta$

20.2 Integration by parts

In section 13.5 we obtained a rule for differentiating the product of u and v, two functions of x.
From this rule we can obtain a result which enables us to integrate certain products.
If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

integrating both sides of this equation, with respect to x

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

i.e. $$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

This is known as the formula for integration by parts.

Example 9

Find $\int x(3x-2)^4\,dx$, using the method of integration by parts.

[Note that this integral could be found by expanding the integrand, or by means of a substitution.]

Let $\quad u = x \quad \text{and} \quad \frac{dv}{dx} = (3x-2)^4$

then $\quad \frac{du}{dx} = 1 \quad \text{and} \quad v = \frac{1}{15}(3x-2)^5$

Using $\quad \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

$$\int x(3x-2)^4\,dx = \frac{x}{15}(3x-2)^5 - \int \frac{1}{15}(3x-2)^5\,dx$$

$$= \frac{x}{15}(3x-2)^5 - \frac{1}{15\times 18}(3x-2)^6 + c$$

$$\therefore \quad \int x(3x-2)^4\,dx = \frac{1}{270}(3x-2)^5(15x+2) + c$$

Note: The choice to substitute u for x is made, since when differentiated this function becomes 1; thus the expression, $\int v\frac{du}{dx}dx$, will no longer be a product of two functions of x.

Example 10

Find $\int x \ln x \, dx$.

We know that we shall have to differentiate one, and integrate the other, function (x or $\ln x$) with respect to x. Although we would like to put $u = x$, because the differential $\frac{du}{dx} = 1$, we cannot do this since it would then mean integrating $\ln x$.

Let $\quad u = \ln x \quad$ and $\quad \frac{dv}{dx} = x$

then $\quad \frac{du}{dx} = \frac{1}{x} \quad$ and $\quad v = \frac{1}{2}x^2$

Using $\quad \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

$$\therefore \quad \int x \ln x \, dx = \ln x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \times \frac{1}{x}dx$$

$$\therefore \quad \int x \ln x \, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Example 11

Find $\int \ln x \, dx$.

This is not seemingly a product, but we write it as one so that we can then choose to differentiate $\ln x$.

Writing $\int \ln x \, dx$ as $\int 1 \times \ln x \, dx$

Let $\quad u = \ln x \quad$ and $\quad \frac{dv}{dx} = 1$

then $\quad \frac{du}{dx} = \frac{1}{x} \quad$ and $\quad v = x$

Using $\quad \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

$$\int \ln x \, dx = (\ln x)x - \int x\frac{1}{x}dx$$

$$= x \ln x - \int dx$$

$$\therefore \quad \int \ln x \, dx = x \ln x - x + c.$$

$$\int e^x \sin x\, dx$$

Let $\quad u = e^x \quad$ and $\quad \dfrac{dv}{dx} = \sin x$

then $\quad \dfrac{du}{dx} = e^x \quad$ and $\quad v = -\cos x$

Using $\quad \displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

$$\int e^x \sin x\, dx = e^x(-\cos x) - \int(-\cos x)e^x\, dx$$

$$= -e^x \cos x + \int e^x \cos x\, dx$$

Using this result in [1] above:

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$\therefore \quad 2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\therefore \quad \int e^x \cos x\, dx = \frac{1}{2}e^x(\sin x + \cos x) + c$$

Exercise 20B

Use the method of integration by parts to find the following indefinite integrals

1. $\int x(x+1)^4\, dx$ **2.** $\int x\sqrt{(x-3)}\, dx$ **3.** $\int \frac{x}{\sqrt{(x+3)}}dx$

4. $\int x \sin x\, dx$ **5.** $\int x^2 \sin x\, dx$ **6.** $\int 2x \sin(3x-1)\, dx$

7. $\int x^2 \ln x\, dx$ **8.** $\int (x+1)\ln x\, dx$ **9.** $\int \frac{1}{x^3}\ln x\, dx$

10. $\int e^x \sin x\, dx$ **11.** $\int xe^x\, dx$ **12.** $\int x^2 e^{3x}\, dx$

13. $\int e^x \cos 2x\, dx$ **14.** $\int \ln(2x+1)\, dx$ **15.** $\int \tan^{-1} x\, dx$

16. $\int \sin^{-1} x\, dx$ **17.** $\int 2x \ln(x+1)\, dx$ **18.** $\int x \ln(x-2)\, dx$

Evaluate the following definite integrals using the method of integration by parts.

19. $\int_0^1 x(x-1)^5\, dx$ **20.** $\int_3^6 \frac{x}{\sqrt{(x-2)}}dx$ **21.** $\int_0^2 x(x-2)^4\, dx$

22. $\int_0^{\pi/4} x \cos 2x\, dx$ **23.** $\int_2^4 (x-1)\ln(2x)\, dx$ **24.** $\int_1^4 \frac{\ln x}{x^2}dx$

25. $\int_0^2 x^3 e^x\, dx$

3. Is the integrand a quotient? If so,

(a) those of the type $\int \frac{f'(x)}{f(x)} dx$ can be integrated to give $\ln |f(x)| + c$,

(b) those of the type $\int \frac{f'(x)}{[f(x)]^n} dx$ can be integrated by inspection or by change of variable:

e.g. $\int \frac{2x}{(x^2 + 4)^7} dx$ is $-\frac{1}{6}(x^2 + 4)^{-6} + c$ by inspection or by the substitution $u = x^2 + 4$,

(c) remember that expressing the integrand in partial fractions can enable some integrations to be carried out:

e.g. $$\int \frac{x^3 - 3x - 5}{(x + 1)(x - 2)} dx = \int \left(x + 1 + \frac{1}{x + 1} - \frac{1}{x - 2} \right) dx$$
$$= \frac{x^2}{2} + x + \ln \left| \frac{x + 1}{x - 2} \right| + c,$$

(d) writing the integrand as two (or more) fractions by 'splitting' the numerator makes some integrations possible:

e.g. $$\int \frac{x + 1}{x^2 + 1} dx = \int \left(\frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} \right) dx$$
$$= \frac{1}{2} \ln (x^2 + 1) + \tan^{-1} x + c$$

4. Remember the standard techniques for integrating odd and even powers of $\sin x$ and $\cos x$, and that there are seven functions for which integration is straightforward, provided that you remember the initial rearrangement.
Note, particularly, parts (vi) and (vii) below, which have not been encountered earlier in this book:

(i) $\int \tan x\,dx$: write as $\int \frac{\sin x}{\cos x} dx$, which integrates to give a log function.

(ii) $\int \cot x\,dx$: write as $\int \frac{\cos x}{\sin x} dx$, which integrates to give a log function.

(iii) $\int \ln x\,dx$: write as $\int 1 \times \ln x\,dx$ and integrate by parts.

(iv) $\int \sin^{-1} x\,dx$: by a similar method to that used for $\int \ln x\,dx$.

(v) $\int \cos^{-1} x\,dx$: by a similar method to that used for $\int \ln x\,dx$.

(vi) $\int \sec x\,dx$: write as $\int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$ to give $\ln |\sec x + \tan x| + c$.

(vii) $\int \operatorname{cosec} x\,dx$: write as $\int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} dx$ to give $\ln |\operatorname{cosec} x - \cot x| + c$.

5. If consideration of points 1 to 4 above do not yield a solution, try rearranging the integrand or making a suitable substitution:

   e.g. $\int 2\sin 8x \cos 4x\,dx$ rearranges to $\int (\sin 12x + \sin 4x)\,dx$

   to give $-\frac{1}{12}\cos 12x - \frac{1}{4}\cos 4x + c$.

Exercise 20C

Find the following indefinite integrals

1. $\int (3x^2 - 6)\,dx$	**2.** $\int \tan x\,dx$	**3.** $\int \sin 2x\,dx$
4. $\int \frac{2}{x^2 - 1}\,dx$	**5.** $\int 3e^x\,dx$	**6.** $\int e^{3x}\,dx$
7. $\int \frac{1}{x^2}\,dx$	**8.** $\int \frac{1}{x}\,dx$	**9.** $\int \cos^3 2x\,dx$
10. $\int \frac{2}{(2x - 3)^3}\,dx$	**11.** $\int \frac{2}{2x + 3}\,dx$	**12.** $\int \frac{1}{\sqrt{(1 - x^2)}}\,dx$
13. $\int x(x - 1)^6\,dx$	**14.** $\int \frac{1}{4 + x^2}\,dx$	**15.** $\int \frac{8x}{(x^2 - 4)^5}\,dx$
16. $\int 2\sin 5x \cos 4x\,dx$	**17.** $\int (x + 3)(x + 7)^5\,dx$	**18.** $\int \frac{1 + x}{\sqrt{(1 - x^2)}}\,dx$
19. $\int x\cos x\,dx$	**20.** $\int 4x(3x - 5)\,dx$	**21.** $\int \frac{10}{25 + x^2}\,dx$
22. $\int 9\cos 3x\,dx$	**23.** $\int x^3 \ln x\,dx$	**24.** $\int 2^x\,dx$
25. $\int x\sqrt{(2x + 3)}\,dx$	**26.** $\int \frac{2x}{x^2 + 4}\,dx$	**27.** $\int \frac{6 - 2x}{(x + 1)(x^2 + 3)}\,dx$
28. $\int \ln x\,dx$	**29.** $\int 30\cos 4x \cos x\,dx$	**30.** $\int 4\sec^2 x\,dx$
31. $\int \frac{1}{\sqrt{(x + 5)}}\,dx$	**32.** $\int \sin^7 x\,dx$	**33.** $\int \cos^4 2x\,dx$
34. $\int \frac{\sin x}{\cos^2 x}\,dx$	**35.** $\int \frac{x}{x + 10}\,dx$	**36.** $\int (2x + 3)(x - 2)\,dx$
37. $\int xe^{-x}\,dx$	**38.** $\int \frac{2x^3 + 7x^2 + 2x - 10}{(x + 3)(2x - 1)}\,dx$	**39.** $\int \frac{1}{9 + 4x^2}\,dx$
40. $\int \frac{\sin 2x}{\cos x}\,dx$	**41.** $\int \frac{\sin 2x}{\cos 2x}\,dx$	**42.** $\int \frac{x + 2}{x^2 + 4x + 7}\,dx$
43. $\int \cos 7x \sin 5x\,dx$	**44.** $\int \sin^{-1}\left(\frac{x}{3}\right)dx$	**45.** $\int \sec 4x\,dx$

Example 13

If the general equation of a family of curves is given by $y = Ax^2$, find the corresponding differential equation for the family, giving your answer independent of the constant A.

$$y = Ax^2$$

$$\therefore \quad \frac{dy}{dx} = 2Ax$$

but $A = \frac{y}{x^2}$ $\quad \therefore \quad \frac{dy}{dx} = \frac{2y}{x}$ is the required differential equation.

Solving first order differential equations

Any first order differential equation that can be expressed in the form

$$f(y)\frac{dy}{dx} = g(x),$$

can be solved by separating the variables:

Suppose $\quad f(y)\frac{dy}{dx} = g(x)$

Integrating both sides with respect to x $\quad \int f(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$

Earlier in this chapter we saw that the operation $\ldots \frac{dy}{dx}dx$ can be replaced by $\ldots dy$.

$$\therefore \quad \int f(y)dy = \int g(x)\,dx$$

Thus, if $f(y)\frac{dy}{dx} = g(x)$, it follows that $\int f(y)dy = \int g(x)dx$.

This result is easily remembered if we think of $\frac{dy}{dx}$ as a 'fraction' and view the rearrangement from $f(y)\frac{dy}{dx} = g(x)$ to $f(y)dy = g(x)dx$ as 'putting the x's with the dx and the y's with the dy'. It is important to realise, though, that this is simply a useful way of remembering what to do; in fact, $\frac{dy}{dx}$ is not a fraction and the method really depends on $\ldots \frac{dy}{dx}dx$ being equivalent to $\ldots dy$.

The simplest case of $f(y)\frac{dy}{dx} = g(x)$ is for $f(y) = 1$, in which case we have $\frac{dy}{dx} = g(x)$ which we can integrate directly to obtain y (see examples 14 and 15).

Remember also that integration gives rise to a constant c, and so we obtain the *general* solution of the differential equation. Only if we are given additional information, can we find a *particular* solution.

Example 14

Find the general solution of the differential equation, $\frac{dy}{dx} = 3x^2 - 4$.

$$\frac{dy}{dx} = 3x^2 - 4$$

By direct integration: or separating the variables and integrating:

$y = x^3 - 4x + c$

$$\int dy = \int (3x^2 - 4)\,dx$$

$$y = x^3 - 4x + c$$

Thus $y = x^3 - 4x + c$ is the required general solution of the differential equation.

Example 15

Solve the differential equation, $\frac{dy}{dx} + 5 = 6x$, given that when $x = 1$, $y = 2$.

$$\frac{dy}{dx} + 5 = 6x$$

$$\therefore \quad \frac{dy}{dx} = 6x - 5$$

$$y = 3x^2 - 5x + c$$

but when $x = 1$, $y = 2$ $\therefore$ $2 = 3(1)^2 - 5(1) + c$ giving $c = 4$

Thus $y = 3x^2 - 5x + 4$ is the required solution.

Example 16

Solve the differential equation, $\frac{dy}{dx} = \frac{2}{\sin y}$, giving y in terms of x.

$$\frac{dy}{dx} = \frac{2}{\sin y}$$

$$\therefore \quad \int \sin y\,dy = \int 2dx$$

$$-\cos y = 2x + c$$

thus $y = \cos^{-1}(-2x - c)$.

Note As c is the arbitrary constant which can take either positive or negative values, we lose no generality by writing this solution as $y = \cos^{-1}(c - 2x)$.

Example 17

Solve the equation, $\frac{dy}{dx} = \frac{1}{x^2 y}$, given that when $x = 1$, $y = 2$.

$$\frac{dy}{dx} = \frac{1}{x^2 y}$$

$$\therefore \quad \int y\,dy = \int \frac{1}{x^2}dx$$

$$\frac{y^2}{2} = -\frac{1}{x} + c$$

but when $x = 1$, $y = 2$

$\therefore$ $\frac{4}{2} = -\frac{1}{1} + c$ giving $c = 3$.

Thus $y^2 = 6 - \frac{2}{x}$ is the required solution.

Exercise 20D

1. From each of the following equations obtain a first order differential equation that does not contain the arbitrary constant A. (For (g), (h) and (i) use implicit differentiation).

(a) $y = 3x^2 + Ax$ (b) $y = \frac{A}{x}$ (c) $y = 4x^2 - A$
(d) $y = Ae^{x^3}$ (e) $y = A \ln x$ (f) $y = A \cos x$
(g) $x^2 + 4y^2 = A$ (h) $y^3 = A(x^2 + 1)$ (i) $y^2 = xy + A$

Find the general solutions to the following differential equations, giving y in terms of x, in each case.

2. $\frac{dy}{dx} = x + 2$ **3.** $2y\frac{dy}{dx} = 3x^2$ **4.** $x^5\frac{dy}{dx} = 4y^2$

5. $\frac{dy}{dx} - e^x y^2 = 0$ **6.** $\frac{dy}{dx} = \frac{\cos x}{y}$ **7.** $\frac{dy}{dx} = \frac{1}{1 - x}$

8. $\frac{dy}{dx} + 2x \operatorname{cosec} y = 0$ **9.** $\frac{dy}{dx} = 2y$ **10.** $\frac{dy}{dx} = 3x^2(3 + y)$

11. $x\frac{dy}{dx} = \sec y$ **12.** $\frac{dy}{dx} = 2x(1 + y^2)$ **13.** $\frac{dy}{dx} = 4xe^{-y}$

14. $\frac{dy}{dx} = \frac{y}{x + 1}$ **15.** $(x + 1)\frac{dy}{dx} = x(y + 3)$ **16.** $2\frac{dy}{dx} = 3x^2(y^2 - 1)$

Solve the following differential equations, giving y in terms of x, in each case.

17. $\frac{dy}{dx} + 4 = 12x$; when $x = -2$, $y = 30$.

18. $\frac{dy}{dx} = y^2$; when $x = 3$, $y = -1$.

19. $3y^2\frac{dy}{dx} = 2x + 1$; when $x = 2$, $y = 2$.

20. $(\cos y)\frac{dy}{dx} = x^2 \operatorname{cosec}^2 y$; when $x = \frac{1}{2}$, $y = \frac{\pi}{2}$.

21. $x\frac{dy}{dx} = 2$; $x > 0$ and when $x = 1$, $y = -3$.

22. $x\frac{dy}{dx} = 2 + \frac{dy}{dx}$; $x > 1$ and when $x = 2$, $y = 1$.

23. $\frac{dy}{dx} = 4xy$; when $x = 0$, $y = 4$.

24. $\frac{dy}{dx} = 6x(y + 1)$; when $x = 0$, $y = 3$.

25. $\frac{dy}{dx} = (y - 3)(4x + 3)$; when $x = -1$, $y = 3(e^{-1} + 1)$.

26. $e^y\frac{dy}{dx} + \sin x = 0$; when $x = \frac{\pi}{2}$, $y = 1$.

27. $x\frac{dy}{dx} + y^2 = 0$; $x > 0$ and when $x = 1$, $y = \frac{1}{2}$.

28. $x\frac{dy}{dx} = y + 2$; when $x = 3$, $y = 7$.

29. $x\frac{dy}{dx} + 3 = y - 4\frac{dy}{dx}$; when $x = 1$, $y = 13$.

30. $2x\frac{dy}{dx} + 1 = y^2$; when $x = \frac{1}{2}$, $y = 2$.

31. $\frac{dy}{dx} = \frac{x^2y + y}{x^2 - 1}$; when $x = 0$, $y = 2$.

32. $(x^2 + 1)\frac{dy}{dx} = 2(y + 1)(x^2 + x + 1)$; when $x = 0$, $y = 3$.

33. Use the substitution $y = vx$, where v is a function of x, to solve $x\frac{dy}{dx} = 2x - y$, stating
(a) the general solution and
(b) the particular solution for which $y = 5$ when $x = 1$.

34. Use the substitution $4x + y = z$ to solve $\frac{dy}{dx} = 4x + y$ given that $y = 2$ when $x = 0$.

35. Use the substitution $z = 2x - 3y$ to solve $(2x - 3y + 3)\frac{dy}{dx} = 2x - 2y + 1$ given that $y = 1$ when $x = 1$.

36. Use the substitution $y = vx$, where v is a function of x, to solve $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$, stating (a) the general solution and (b) the particular solution for which $y = 4$ when $x = 2$.

37. Use the substitution $y = vx$, where v is a function of x, to find the general solution of $(x - y)\frac{dy}{dx} = 2x + y$

38. Use the substitution $y = vx$, where v is a function of x, to find the general solution of $\frac{dy}{dx} = \frac{x^2 + y^2}{x(x + y)}$

39. By substituting $x = X - 1$ and $y = Y + 3$ reduce the differential equation $\frac{dy}{dx} = \frac{4x - y + 7}{2x + y - 1}$ to a homogeneous equation and hence find the general solution in terms of x and y.

40. Find the general equation of the family of curves for which the gradient at any point on the curve is the same as the y-coordinate at that point.

41. The rate at which a body loses speed at any given instant as it travels through a resistive medium is given by kv m/s² where v is the speed of the body at that instant and k is a positive constant. If its initial speed is u m/s show that the time taken for the body to decrease its speed to $\frac{1}{2}u$ m/s is $\frac{1}{k}\ln 2$ seconds.

42. The rate at which a body loses temperature at any instant is proportional to the amount by which the temperature of the body, at that instant, exceeds the temperature of its surroundings. A container of hot liquid is placed in a room of temperature 18°C and in 6 minutes the liquid cools from 82°C to 50°C. How long does it take for the liquid to cool from 26°C to 20°C?

Example 25

Find the general solution to the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

We try $y = e^{px}$, from which $\frac{dy}{dx} = pe^{px}$ and $\frac{d^2y}{dx^2} = p^2e^{px}$.
Substitution in the differential equation gives

$$p^2e^{px} - 4pe^{px} + 3e^{px} = 0$$
$$p^2 - 4p + 3 = 0$$

i.e. $(p - 3)(p - 1) = 0$ giving $p = 3$ or 1.
The general solution to the differential equation is $y = Ae^{3x} + Be^{x}$.

Example 26

Find the general solution to the differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = e^{2x}$

We first find the complementary function by solving the equation
$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0$.
Try $y = e^{px}$ then substitution in the differential equation gives

$$2p^2e^{px} - pe^{px} - 3e^{px} = 0$$
$$2p^2 - p - 3 = 0$$

i.e. $(2p - 3)(p + 1) = 0$ giving $p = \frac{3}{2}$ or -1
Thus the complementary function is $y = Ae^{3x/2} + Be^{-x}$.
To find a particular solution we try a solution of the form of the R.H.S. of the differential equation i.e. $y = Ce^{2x}$
Substitution in the differential equation gives

$$2 \times 2^2Ce^{2x} - 2Ce^{2x} - 3Ce^{2x} = e^{2x}$$

giving $C = \frac{1}{3}$
Thus a particular solution is $y = \frac{1}{3}e^{2x}$ and the general solution to the differential equation is $y = Ae^{3x/2} + Be^{-x} + \frac{1}{3}e^{2x}$.

Example 27

Find the general solution to the differential equation $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 3x + 4$.

To find the complementary function we solve $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$ by trying $y = e^{px}$

$$4p^2e^{px} + 4pe^{px} + e^{px} = 0$$
$$4p^2 + 4p + 1 = 0$$

i.e. $(2p + 1)^2 = 0$ giving $p = -\frac{1}{2}, -\frac{1}{2}$ a repeated root.
Thus the complementary function is $y = (A + Bx)e^{-x/2}$.
To find a particular solution we try a solution of the form of the R.H.S. of the differential equation i.e. $y = Cx + D$.
Substitution in the differential equation gives $4(0) + 4C + (Cx + D) = 3x + 4$
Equating coefficients gives $C = 3$ and $D = -8$.

8. Use the standard results quoted on page 519 to write down the general solutions of the following differential equations

(a) $\frac{d^2y}{dx^2} + 36y = 1$ (b) $\frac{d^2y}{dx^2} - 9y = 18$

(c) $\frac{d^2y}{dx^2} + 9y = 18$ (d) $4\frac{d^2y}{dx^2} - 9y = 36$

9. Use the standard results quoted on page 519 to solve the following differential equations

(a) $\frac{1}{4}\frac{d^2y}{dx^2} + y = 2$ given that when $x = \frac{\pi}{4}$, $y = 1$ and when $x = \frac{\pi}{2}$, $y = -1$.

(b) $\frac{d^2y}{dx^2} - y = 2$ given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 6$.

(c) $\frac{d^2y}{dx^2} + \frac{1}{4}y = 4$ given that when $x = \pi$, $y = 19$ and $\frac{dy}{dx} = -1$.

(d) $4\frac{d^2y}{dx^2} - y = 4$ given that when $x = 0$, $\frac{dy}{dx} = 2$ and $\frac{d^2y}{dx^2} = \frac{3}{2}$.

Exercise 20F Examination Questions

1. Evaluate

(a) $\int_0^{\pi} \sin\frac{3x}{2}dx$, (b) $\int_0^{a} e^{-2x}\,dx$, where $a = \log_e 2$, (c) $\int_1^2 \frac{1}{3x+2}dx$.

(A.E.B.)

2. Evaluate the integrals

(a) $\int_0^{3/4} x\sqrt{(1 + x^2)}\,dx$, (b) $\int_0^{\pi/4} \tan^2 x\,dx$,

(c) $\int_0^{\pi} x \sin x\,dx$, (d) $\int_3^4 \frac{1}{x^2 - 3x + 2}dx$.

Express your answer to (d) as a natural logarithm. (London)

3. Find

(a) $\int x \ln x\,dx$ (b) $\int \frac{x}{\sqrt{(x-2)}}dx$. (London)

4. Using the substitution $t = \sin x$, evaluate to two decimal places the integral

$$\int_{\pi/6}^{\pi/2} \frac{4\cos x}{3 + \cos^2 x}dx.$$

(A.E.B.)

5. Evaluate the integrals

(i) $\int_0^1 \frac{2x - 1}{(x-3)^3}dx$ (by substitution or otherwise),

(ii) $\int_0^{\pi/4} \sin^2 3x\,dx$. (Oxford)

6. (a) Find the indefinite integrals of

(i) $(x - 2/x)^2$ (ii) $\cot 2x$ (iii) $\dfrac{1}{\sqrt{(1 - 2x)}}$ (iv) $x \log_e 2x$

(b) Evaluate $\displaystyle\int_0^{\pi/2} \cos^3 x \, dx$.

(c) Using the substitution $t = \log_e x$ evaluate $\displaystyle\int_e^{e^2} \frac{dx}{x(\log_e x)^2}$. (S.U.J.B.)

7. Write down, or obtain, the derivative of $\tan^{-1} 2x$.

Find (i) $\displaystyle\int \frac{x + 2}{4x^2 + 1} dx$, (ii) $\displaystyle\int \frac{4x}{4x^4 + 1} dx$.

(J.M.B.)

8. Find the volume generated when the region bounded by the curve

$$y = \left(\frac{1 - x^2}{1 + x^2}\right)^{1/2}$$

and the x-axis is rotated through four right-angles about the x-axis. (Oxford)

9. Find y in terms of x given that

$$x\frac{dy}{dx} = y(y + 1)$$

and $y = 4$ when $x = 2$. (London)

10. Given that

$$\frac{dy}{dx} - 2xy = 0$$

and that $y = 1$ when $x = 1$, show that $y \geqslant \dfrac{1}{e}$ for all values of x. (Cambridge)

11. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2 \tan x}$, given that $y = 3$ when $x = \dfrac{\pi}{2}$.

Hence express y in terms of x. (A.E.B.)

12. For all positive values of x the gradient of a curve at the point (x, y) is $\dfrac{y}{x^2 + x}$. The point $A(3, 6)$ lies on this curve.

(i) Calculate the equation of the normal to the curve at A.
(ii) Find the equation of the curve in the form $y = f(x)$. (A.E.B.)

13. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2}{(x^2 - x - 2)}$$

in the region $x > 2$. Find also the particular solution which satisfies $y = 1$ when $x = 5$. (Oxford)

18. (a) Find the solution of the differential equation

$$x\frac{dy}{dx} = 3x - 2y, \quad \text{for} \quad x > 0$$

given that $y = \frac{3}{4}$ when $x = 1$

(b) For the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 6e^{-3x} - 1$

find (i) the general solution

(ii) the limit to which y tends as x tends to infinity.

19. Prove that

$$\int_a^b f(x)\,dx = \int_a^b f(a + b - t)\,dt.$$

Hence prove that, if $0 < \beta < \frac{1}{2}\pi$,

$$\int_\beta^{\pi-\beta} \frac{\theta\,d\theta}{\sin\theta} = \pi \ln \cot\left(\tfrac{1}{2}\beta\right). \qquad \text{(Oxford)}$$

$$\therefore \quad \int_a^b f(x)\,dx \approx \tfrac{1}{3}h\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

and this is known as the approximation from Simpson's rule.

Note that in order to apply this rule, the interval is divided into an *even* number of strips, i.e. there are an *odd* number of ordinates.
Numerical methods of integration can be particularly useful when the function to be integrated is one that does not integrate readily, or one for which we know the coordinates of various points on the graph of the function, but we do not know the equation of the function (as in Example 2 below).

Example 2

Given that $y = f(x)$ and that for the given values of x, the corresponding values of y are as shown in the table, use Simpson's rule with nine ordinates to find an approximate value for $\int_{-1}^{7} f(x)\,dx$.

x	-1	0	1	2	3	4	5	6	7
y	17	10	5	2	1	2	5	10	17

Applying Simpson's rule, with $h = 1$, and $y_0 = 17$, $y_1 = 10$, etc.

$$\int_{-1}^{7} f(x)\,dx \approx \tfrac{1}{3}\left[17 + 4(10) + 2(5) + 4(2) + 2(1) + 4(2) + 2(5) + 4(10) + 17\right]$$

$$\approx \tfrac{152}{3} = 50{\cdot}7 \quad \text{to one decimal place.}$$

Exercise 21A

In this exercise give your answers correct to 4 significant figures unless told otherwise.

1. Given that $y = f(x)$ and for the given values of x the corresponding values of y are as shown in the table below, use the trapezium rule to find an approximate value for $\int_1^5 f(x)\,dx$.

x	1	2	3	4	5
y	0	3	4	7	4

2. Find an approximate value for $\int_0^1 e^x\,dx$ using the trapezium rule with 5 strips.

3. Find approximate values for $\int_1^2 \frac{1}{x}\,dx$ using the trapezium rule with
   (a) 5 strips (b) 10 strips.

4. Given that $y = f(x)$ and for the given values of x the corresponding values of y are as shown in the table below, use Simpson's rule with eleven ordinates to find an approximate value for $\int_0^{10} f(x)\,dx$

x	0	1	2	3	4	5	6	7	8	9	10
y	2	3	5	6	7	7	8	4	3	5	7

5. Use Simpson's rule with 10 strips to find an approximation for $\int_0^1 \frac{1}{1+x}\,dx$.
6. Use Simpson's rule with (a) 5 ordinates, (b) 10 ordinates to find approximations for $\int_0^2 \frac{1}{1+x^2}\,dx$.
(Give your answers to 4 decimal places in each case).
7. Find an approximate value for $\int_0^1 e^{x^2}\,dx$ by (a) the trapezium rule and (b) Simpson's rule, using 8 strips in each case.
8. Find an approximate value for $\int_0^{\pi/2} \sin x\,dx$ by (a) the trapezium rule and (b) Simpson's rule, using 8 strips in each case.
9. If $A = \int_0^{0.6} \frac{1}{\sqrt{(1-x^2)}}\,dx$
(a) find A by integration,
(b) estimate A using the trapezium rule and six strips
(c) estimate A using Simpson's rule and six strips
(d) estimate A by integrating the first four terms of the series expansion of $(1 - x^2)^{-1/2}$.

21.3 Taylor's theorem

If $f(x + h) = (x + h)^n$, we can use the binomial expansion to obtain a series expansion of $f(x + h)$ in powers of x. However, can we obtain a power series expansion when $f(x + h)$ is not of the form $(x + h)^n$? Let us suppose that such a series expansion does exist.

i.e. $f(x + h) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$

then $f'(x + h) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \dots$

$f''(x + h) = 2c_2 + 3 \times 2c_3x + 4 \times 3c_4x^2 + \dots$

$f'''(x + h) = 3 \times 2c_3 + 4 \times 3 \times 2c_4x + \dots$

$f''''(x + h) = 4 \times 3 \times 2c_4 + \dots$

and so on.

Now if $x = 0$, then the above equations give:

$$c_0 = f(h)$$
$$c_1 = f'(h)$$
$$2c_2 = f''(h) \quad \text{i.e.} \quad c_2 = \frac{1}{2!}f''(h)$$

Example 4

Use Maclaurin's theorem to find the first three non-zero terms in the expansion of $\sin x$. Hence find $\sin(0{\cdot}1)^{\text{rad}}$ correct to seven decimal places.

In this case
$$\begin{aligned} f(x) &= \sin x, & f(0) &= \sin 0 = 0\\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1\\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0\\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1\\ f''''(x) &= \sin x & f''''(0) &= \sin 0 = 0\\ f'''''(x) &= \cos x & f'''''(0) &= \cos 0 = 1 \end{aligned}$$

By Maclaurin's theorem

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \frac{x^5}{5!}f'''''(0) + \dots$$

$$= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \dots$$

$$\therefore \quad \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$ which is the required expansion.

Thus
$$\sin(0{\cdot}1\text{ rad}) \approx \frac{(0{\cdot}1)}{1!} - \frac{(0{\cdot}1)^3}{3!} + \frac{(0{\cdot}1)^5}{5!}$$
$$= 0{\cdot}1 - 0{\cdot}00016667 + 0{\cdot}00000008$$
$$= 0{\cdot}09983341$$

i.e. $\sin(0{\cdot}1\text{ rad}) = 0{\cdot}0998334$ correct to seven decimal places.

Exercise 21B

1. Use Maclaurin's theorem to find series expansions for each of the following giving all terms up to and including that in x^5.
(a) $\cos x$, (b) e^x, (c) $(1 + x)^n$, (d) $\ln(1 + x)$, (e) a^x, (f) $\sin^{-1} x$.
Use your answer to (a) to determine $\cos(0{\cdot}1$ rads) correct to six decimal places. Use your answer to (f) to determine $\sin^{-1}(0{\cdot}2)$ correct to four decimal places.
2. Use Maclaurin's theorem to find the series expansion of $\tan x$ giving all terms up to and including that in x^5. Hence show that if x is sufficiently small for terms in x^6 and higher powers to be neglected then
$$\frac{\tan x}{(1 + x)} = x - x^2 + \frac{4}{3}x^3 - \frac{4}{3}x^4 + \frac{22}{15}x^5.$$
3. Using Taylor's theorem obtain a series expansion for $\sin(\theta + x)$ where θ, an acute angle, is measured in radians and is such that $\sin\theta = \frac{3}{5}$ (give the first five non-zero terms).
Hence find $\sin(\theta + 0{\cdot}1$ rad) correct to four decimal places.
4. Using Taylor's theorem obtain a series expansion for $\tan\left(\frac{\pi}{4} + x\right)$, stating the terms up to and including that in x^3.
Hence find tan 46° correct to four decimal places.

21.5 Solution of equations

In some instances it may be almost impossible to use an exact method to solve an equation. In such cases we may be able to use other techniques which give good approximations to the solution.
Two such approaches, graphical and iterative, are explained overleaf.

Iterative methods

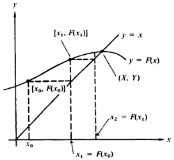
A darts player will frequently use the first of his three darts as a 'marker'. He will then adjust the throwing of the second and third dart dependent upon where the first dart landed on the board. In a similar way a bowls player will use the first wood to gauge the distance of the jack and the speed of the green. Thus we can learn from one trial, or guess, so that our second attempt is an improvement upon the first. We can use similar methods to help us to obtain increasingly accurate approximations to the solutions of equations from an initial educated guess.

If we wish to solve an equation $f(x) = 0$ by an iterative method, we need to find a relationship $x_{r+1} = F(x_r)$ such that x_{r+1} is a better approximation to the solution of the equation than is x_r.

Suppose we can rearrange $f(x) = 0$ into the form $x = F(x)$, e.g. $x^2 - 3x + 1 = 0$ can be written as $x(x - 3) + 1 = 0$ to give $x = \frac{1}{3 - x}$, or alternatively $x = 3 - \frac{1}{x}$, both of which are rearrangements into the form $x = F(x)$.

Suppose that the graphs of $y = x$ and $y = F(x)$ are as shown, $x = X$ is the solution to $F(x) = x$, and that our initial guess is x_0. From the graph we see that $x_1 = F(x_0)$ gives a better approximation than x_0, $x_2 = F(x_1)$ gives a better approximation than x_1, and so on.

In this way we can obtain a value x, which is as close to X as is required.



Example 6

Show that the equation $x^2 - 3x + 1 = 0$ has a root between $x = 2$ and $x = 3$. Using the rearrangement $x = 3 - \frac{1}{x}$, and an initial value of $x = 3$, use an iterative method to find this root correct to two decimal places.

If $f(x) = x^2 - 3x + 1$ then $f(2) = -1$
and $f(3) = 1$.

The change in sign of $f(x)$ between $x = 2$ and $x = 3$ shows that there is some value of x between 2 and 3 for which $f(x) = 0$, i.e. a solution to the equation $x^2 - 3x + 1 = 0$ lies between $x = 2$ and $x = 3$.

From the rearranged form $x = 3 - \frac{1}{x}$, we use the iterative formula

$x_{r+1} = 3 - \frac{1}{x_r}$, with $x_0 = 3$.

$$x_1 = 3 - \frac{1}{3} = 2{\cdot}6667 \quad \left(\text{i.e. } x_1 = 3 - \frac{1}{x_0}\right)$$

$$x_2 = 3 - \frac{1}{2{\cdot}6667} = 2{\cdot}6250 \quad \left(\text{i.e. } x_2 = 3 - \frac{1}{x_1}\right)$$

$$x_3 = 3 - \frac{1}{2\cdot6250} = 2\cdot6190$$
$$x_4 = 3 - \frac{1}{2\cdot6190} = 2\cdot6182$$
$$x_5 = 3 - \frac{1}{2\cdot6182} = 2\cdot6181$$

Thus the required root is $x = 2\cdot62$, correct to two decimal places.

Note: It should not be assumed that this method will always work. In some cases the values $x_0, x_1, x_2, x_3 \ldots$ diverge. For example $x^3 + 2x - 4 = 0$ has a solution near $x = 1$. Using the rearranged form $x_{r+1} = \frac{1}{2}[4 - (x_r)^3]$, we do not obtain converging values for $x_0, x_1, x_2 \ldots$, but using $x_{r+1} = \sqrt[3]{(4 - 2x_r)}$ we obtain $x = 1\cdot18$, correct to two decimal places, as is shown below:

$$x_{r+1} = \sqrt[3]{(4 - 2x_r)}$$

if $x_0 = 1$,

$x_1 = \sqrt[3]{2} = 1\cdot2599$
$x_2 = \sqrt[3]{[4 - 2(1\cdot2599)]} = 1\cdot1397$
$x_3 = \sqrt[3]{[4 - 2(1\cdot1397)]} = 1\cdot1983$
$x_4 = \sqrt[3]{[4 - 2(1\cdot1983)]} = 1\cdot1704$
$x_5 = \sqrt[3]{[4 - 2(1\cdot1704)]} = 1\cdot1839$
$x_6 = \sqrt[3]{[4 - 2(1\cdot1839)]} = 1\cdot1774$

$\therefore\ x = 1\cdot18$ correct to two decimal places.

$$x_{r+1} = \tfrac{1}{2}[4 - (x_r)^3]$$

if $x_0 = 1$,

$x_1 = \frac{1}{2}(4 - 1) = 1\cdot5$
$x_2 = \frac{1}{2}[4 - (1\cdot5)^3] = 0\cdot3125$
$x_3 = \frac{1}{2}[4 - (0\cdot3125)^3] = 1\cdot9847$
$x_4 = \frac{1}{2}[4 - (1\cdot9847)^3] = -1\cdot9089$
$x_5 = \frac{1}{2}[4 - (-1\cdot9089)^3] = 5\cdot4779$

i.e. $x_0, x_1, x_2, x_3 \ldots$ do not converge.

The Newton-Raphson method

Suppose that $x \doteqdot x_r$ is a good approximation for the solution of $f(x) = 0$ and that the exact solution is $x = X$
$= x_r + \varepsilon$ for some small ε.

By Taylor's theorem
$$f(x_r + \varepsilon) = f(x_r) + \varepsilon f'(x_r) + \ldots$$
thus $\quad f(X) \approx f(x_r) + \varepsilon f'(x_r)$

but $\quad f(X) = 0, \quad$ so $\quad \varepsilon \approx -\dfrac{f(x_r)}{f'(x_r)}$

Thus, if x_r is a good approximation for a root of $f(x) = 0$, then $x_r - \dfrac{f(x_r)}{f'(x_r)}$ is a better approximation. The Newton-Raphson iterative method is based on this statement,

i.e. If x_r is a good approximation for a root of $f(x) = 0$, then $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$ is a better approximation.

Note that this is sometimes simply referred to as Newton's method.

Example 7

Use the Newton-Raphson method to determine the root of the equation $x^3 - x = 2$ lying in the range $-2 \leqslant x \leqslant 2$, giving your answer correct to three decimal places.

If $f(x) = x^3 - x - 2$, then

$$\begin{aligned} f(-2) &= -8 + 2 - 2 = -8 \\ f(-1) &= -1 + 1 - 2 = -2 \\ f(0) &= 0 - 0 - 2 = -2 \\ f(1) &= 1 - 1 - 2 = -2 \\ f(2) &= 8 - 2 - 2 = 4 \end{aligned}$$

The change of sign of $f(x)$ between $f(1)$ and $f(2)$ indicates that $x^3 - x = 2$ has a root between $x = 1$ and $x = 2$.

If $f(x) = x^3 - x - 2$, $f'(x) = 3x^2 - 1$, thus using $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$,

$$x_{r+1} = x_r - \left(\frac{x_r^3 - x_r - 2}{3x_r^2 - 1}\right)$$

if $x_0 = 1$,

$$x_1 = 1 - \left(\frac{1 - 1 - 2}{3(1)^2 - 1}\right) = 1 + 1 = 2$$

$$x_2 = 2 - \left(\frac{2^3 - 2 - 2}{3(2)^2 - 1}\right) = 2 - \frac{4}{11} = 1{\cdot}63636$$

$$x_3 = 1{\cdot}63636 - \left[\frac{(1{\cdot}63636)^3 - 1{\cdot}63636 - 2}{3(1{\cdot}63636)^2 - 1}\right] = 1{\cdot}63636 - 0{\cdot}10597 = 1{\cdot}53039$$

$$x_4 = 1{\cdot}53039 - \left[\frac{(1{\cdot}53039)^3 - 1{\cdot}53039 - 2}{3(1{\cdot}53039)^2 - 1}\right] = 1{\cdot}53039 - 0{\cdot}00895 = 1{\cdot}52144$$

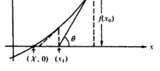
$$x_5 = 1{\cdot}52144 - \left[\frac{(1{\cdot}52144)^3 - 1{\cdot}52144 - 2}{3(1{\cdot}52144)^2 - 1}\right] \approx 1{\cdot}52144 - 0{\cdot}00006 = 1{\cdot}52138$$

Thus the required root, correct to three decimal places, is 1·521.

Note When using the Newton-Raphson method, it can be assumed that the error obtained when finding x_{r+1} is less than the correction $\dfrac{f(x_r)}{f'(x_r)}$ used to obtain x_{r+1} from x_r.
Thus in Example 7, the error in $x_5 = 1{\cdot}52138$ is less than 0·00006. Therefore an answer of 1·521 is indeed correct to three decimal places.

In order to picture the Newton-Raphson method for solving $f(x) = 0$, consider the graph of $y = f(x)$.

Suppose that the exact solution is $x = X$ and that our initial, close estimate is x_0.
Drawing the tangent at $[x_0, f(x_0)]$, we can see that a better solution will be $x_1 = x_0 - \dfrac{f(x_0)}{\tan\theta}$.
However, $\tan\theta$ is the gradient of the curve at $[x_0, f(x_0)]$, i.e. $\tan\theta = f'(x_0)$. Thus if x_0 is our initial estimate, a better approximation is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.
Continuing this, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$ etc.



Thus we obtain the general rule $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$ as before.

Again, it is important to realise that this method will not always work as the values $x_0, x_1, x_2, x_3 \ldots$ will not always converge to X. The following diagrams illustrate situations for which the method would fail.

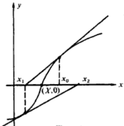


Figure 1

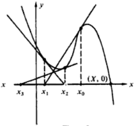


Figure 2

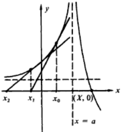


Figure 3

In Fig. 1, the values $x_0, x_1, x_2 \ldots$ diverge.
In Fig. 2, the first value x_0 was not chosen close enough to $(X, 0)$.
In Fig. 3, $y = f(x)$ has a discontinuity at $x = a$ and x_0 needs to be on the same side of $x = a$ as $(X, 0)$.

Note With the use of a calculator and particularly a programmable calculator, it is no longer so tedious to determine the root of an equation by substituting successive values of x into the equation. Again, we use the fact that if $f(a) = 0$ then, as x varies, $f(x)$ will change sign as x passes through the value $x = a$.

Thus to determine the root of $x^3 - x = 2$, lying in the range $-2 \leqslant x \leqslant 2$, (see Example 7) we could proceed as follows:

Let $f(x) = x^3 - x - 2$

x	$f(x)$
−2	−ve
−1	−ve
0	−ve
1	−ve
2	+ve

x	$f(x)$
1·1	−ve
1·2	−ve
1·3	−ve
1·4	−ve
1·5	−ve
1·6	+ve
1·7	
1·8	
1·9	

x	$f(x)$
1·51	−ve
1·52	−ve
1·53	+ve
1·54	
1·55	
1·56	
1·57	
1·58	
1·59	

x	$f(x)$
1·521	−ve
1·522	+ve
1·523	
1·524	
1·525	
1·526	
1·527	
1·528	
1·529	

x	$f(x)$
1·5211	−ve
1·5215	+ve
1·5219	

Thus the solution correct to three decimal places is 1·521.

Exercise 21C

1. Copy and complete the following table for $y = e^x$.

x	-3	$-2\cdot5$	-2	$-1\cdot5$	-1	$-0\cdot5$	0	$0\cdot5$	1	$1\cdot5$	2
y (correct to 2 d.p's)	0·05	0·08	0·14								

On the same pair of axes draw the lines $y = e^x$ and $y = x + 2$ for $-3 \leqslant x \leqslant 2$ and hence find approximate values for all the solutions to the equation $e^x - x = 2$ lying in the range $-3 \leqslant x \leqslant 2$.

2. The equation $x^2 + 4x = 2$ has two roots, one near $x = 0$ and the other near $x = -4$.

(a) Using $x_{r+1} = \dfrac{2}{x_r + 4}$ with $x_0 = 0$ find the root near $x = 0$, correct to 2 decimal places.

(b) Why could we not use the formula $x_{r+1} = \dfrac{2}{x_r + 4}$ with $x_0 = -4$?

(c) Using $x_{r+1} = \dfrac{2}{x_r} - 4$ with $x_0 = -4$ find the root near $x = -4$ correct to 2 decimal places.

(d) Check your answers to (a) and (c) using the quadratic formula.

3. Show that the equation $x^3 - x + 3 = 0$ has a root in the range $-3 \leqslant x \leqslant 3$. Use the rearranged form $x = \sqrt[3]{(x - 3)}$ to obtain this root correct to two decimal places.

For questions **4** to **11**, use the Newton-Raphson iterative method to obtain a root for each equation using the given initial value, x_0, and giving your answer correct to the required degree of accuracy.

4. $x^2 = 5$; $x_0 = 2$, to 3 d.p.

5. $x^2 - 2x - 5 = 0$; $x_0 = 4$, to 4 d.p.

6. $x^3 - 5x - 2 = 0$; $x_0 = 2$, to 3 d.p.

7. $x^3 - 3x + 3 = 0$; $x_0 = -2$, to 3 d.p.

8. $\ln x = 2 - x$; $x_0 = 2$, to 3 d.p.

9. $x \ln x = 1$; $x_0 = 2$, to 4 d.p.

10. $e^x = 3 - x$; $x_0 = 1$, to 5 d.p.

11. $\sin x = x - \frac{1}{2}$, (x is in rads); $x_0 = 1$, to 4 d.p.

21.6 e^z, $\cos z$ and $\sin z$ for complex z.

Having obtained the series expansions for $\sin x$ and $\cos x$ in section 21.4, i.e. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots$ and $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \ldots$, we will now conclude by mentioning how these ideas can be used to assign a meaning to the exponential, sine and cosine functions of a complex number z.

If we are to assign some meaning to the idea of a complex index, it would be useful if our definition is compatible with the existing laws of indices i.e. $e^a \times e^b = e^{a+b}$, $e^a \div e^b = e^{a-b}$, and $(e^a)^b = e^{ab}$

Now we know from earlier work that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots$

If we define e^z, where z is a complex number, by a similar series

i.e. $$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} \ldots$$

it will follow that $$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots\right)$$

$$= \cos\theta + i\sin\theta$$

Thus $$e^{i\theta} = \cos\theta + i\sin\theta \qquad \ldots [1]$$

Similarly $$e^{-i\theta} = \cos\theta - i\sin\theta \qquad \ldots [2]$$

However, as was mentioned earlier, we require that this definition of $e^{i\theta}$ is compatible with the existing laws of indices. The following example and question 1 of Exercise 21D show this to be the case.

Example 8

Using the definition $e^{i\theta} = \cos\theta + i\sin\theta$ prove that $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$

$$e^{i\alpha} \times e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$
$$= e^{i(\alpha+\beta)} \text{ as required.}$$

Let us consider equations [1] and [2]: $$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Adding these two equations gives $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ i.e. $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

and subtracting gives $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ i.e. $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Thus we can now attribute a meaning to the functions $\sin z$ and $\cos z$ for complex values of z:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example 9 below and Exercise 21D question 2 show that, with these definitions, the existing trigonometric identities for $\sin x$ and $\cos x$ also hold for $\sin z$ and $\cos z$.

Example 9

Using the definitions $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, prove that $\cos^2 z + \sin^2 z = 1$

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}}{4} = \frac{1}{4}e^{2iz} + \frac{1}{2} + \frac{1}{4}e^{-2iz}$$

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} - 2e^{iz}e^{-iz} + e^{-2iz}}{-4} = -\frac{1}{4}e^{2iz} + \frac{1}{2} - \frac{1}{4}e^{-2iz}$$

$\therefore \quad \cos^2 z + \sin^2 z = \frac{1}{2} + \frac{1}{2} = 1$ as required.

Answers

page 4 ***Exercise A***

1. (a) F (b) F (c) T (d) T (e) F (f) T (g) T (h) F (i) T (j) F (k) T (l) F (m) T (n) F (o) T (p) F
2. (a) $\{-1, 0, 1, 2, 3\}$ (b) $\{-2, -1, 0, 1, 2, 3, 4\}$ (c) $\{-3, -2, -1, 0, 1\}$ (d) $\{-2, -1, 0, 1, 2\}$ (e) $\{1, 2, 3, 4\}$ (f) $\{1, 2, 3, 4, 5\}$
3. For (a) (b) and (c) there are other possible inequality statements.
(a) $\{x \in \mathbb{Z}: -3 \leqslant x \leqslant 2\}$ (b) $\{x \in \mathbb{Z}: -5 \leqslant x \leqslant -1\}$ (c) $\{x \in \mathbb{Z}: x \leqslant 1\}$
(d) $\{x \in \mathbb{R}: x \geqslant -1\}$ (e) $\{x \in \mathbb{R}: x > -1\}$ (f) $\{x \in \mathbb{R}: x \leqslant 1\}$
(g) $\{x \in \mathbb{R}: x < -2\}$ (h) $\{x \in \mathbb{R}: -1 \leqslant x \leqslant 3\}$ (i) $\{x \in \mathbb{R}: -4 < x < 2\}$
(j) $\{x \in \mathbb{R}: -1 \leqslant x < 2\}$
4. (a) $x \leqslant 3$ (b) $x < 2$ (c) $x \geqslant 4$
(d) $x > -2$ (e) $x < -1$ (f) $x \geqslant 1\frac{1}{2}$
(g) $x \leqslant 2$ (h) $x < 5$ (i) $-1 \leqslant x \leqslant 5$
(j) $-3 \leqslant x \leqslant 1$ (k) $-3 < x < 2$ (l) $-2 < x < 3$
(m) $-5 < x \leqslant -2$ (n) $1 \leqslant x < 6$
5. (a) $\{x \in \mathbb{R}: -3 \leqslant x \leqslant 7\}$ (b) $\{x \in \mathbb{R}: -2 \leqslant x \leqslant 3\}$ (c) $\{x \in \mathbb{R}: x < -1 \text{ or } x > 6\}$
(d) $\{x \in \mathbb{R}: -3\frac{1}{2} < x < 4\}$ (e) $\{x \in \mathbb{R}: x \leqslant -\frac{1}{3} \text{ or } x \geqslant 1\}$ (f) $\{x \in \mathbb{R}: x < -1\frac{1}{3} \text{ or } x > 2\}$

page 7 ***Exercise B***

1. (a) $11x + 6y$ (b) $3x - 8$ (c) $2p^2 + 8p - 3$ (d) $4x^2y + 2xy^2$ (e) $8a^2 - 2ab$
2. (a) $3x^2 - 12x$ (b) $x + 24$ (c) $13x + 3$ (d) $5xy - 6x - 4y$
(e) $x^2 - 2x - 15$ (f) $2x^2 - 5x - 12$ (g) $4x^2 - 12x + 9$ (h) $27x^3 - 27x^2 + 9x - 1$
(i) $5x^3 - 17x^2 - 32x - 18$
3. (a) $8x(2x + 3y)$ (b) $5x(3x + 2y + 4)$ (c) $(x + 5)(x - 2)$ (d) $(x + 7)(x - 2)$
(e) $(2x + 3)(x - 5)$ (f) $(3x - 2)(2x + 1)$ (g) $(2x + 5)(2x - 3)$ (h) $(x + 3)(x - 3)$
(i) $(4x + 7y)(4x - 7y)$ (j) $2x(x + 3)(x - 3)$ (k) $(a + 3)(x + 2)$ (l) $(a - 2b)(x + y)$
4. (a) $x^2 + 3x + 2$ (b) $x^2 + 6x + 7$ (c) $x^2 + 2x - 1$ (d) $3x^2 + 7x - 3$
5. (a) $13x - 13$ (b) $x - 3$ (c) $8x^2 + 11x + 10$ (d) -20
6. (a) $\frac{11}{12}x$ (b) $\frac{8x - 5}{15}$ (c) $\frac{x + 10}{6}$ (d) $\frac{x + 7}{3(x + 1)}$
(e) $\frac{17 - 2x}{4(2x + 3)}$ (f) $\frac{-3x - 5}{(x + 2)^2}$ (g) $\frac{x + 12}{x(x + 4)}$ (h) $\frac{x^2 + 9x + 26}{2(x + 4)^2}$
(i) $\frac{x^2 + 3x + 20}{2(x + 4)^2}$ (j) $\frac{14}{3}$ (k) $\frac{2}{5(x - 4)}$ (l) $\frac{5}{6x}$
(m) $\frac{2(2x - 3)}{x(x - 4)}$ (n) $\frac{3x - 5}{x(x - 1)}$ (o) $\frac{(4 - x)(x + 12)}{x^2}$

7. (a) $1 + \dfrac{2x - 3}{x^2 + 4x + 1}$ (b) $2 + \dfrac{3}{x^2 + 1}$ (c) $5 - \dfrac{3x + 1}{x^2 + x - 2}$

(d) $x - 3 + \dfrac{2}{x^2 - 2x + 3}$

8. (a) $1 + \dfrac{3x - 1}{x^2 + 4}$ (b) $1 - \dfrac{3}{x + 5}$ (c) $2 - \dfrac{5}{x^2 - 2x + 8}$

(d) $2 - \dfrac{5}{2x + 3}$ (e) $x + 1 + \dfrac{2}{x^2 + 3}$ (f) $x - 3 + \dfrac{2}{x + 1}$

(g) $x + 1 - \dfrac{3x}{x^2 + 2x - 4}$ (h) $x^2 + x - 2 + \dfrac{3x}{x^2 + 7}$

page 10 **Exercise C**

1. (a) $2\sqrt{3}$ (b) $7\sqrt{3}$ (c) $11\sqrt{2}$ (d) $8\sqrt{3}$ (e) $\sqrt{5}$ (f) 1 (g) 13 (h) -6

2. (a) $\sqrt{12}$ (b) $\sqrt{12}$ (c) $\sqrt{20}$ (d) $\sqrt{18}$ (e) $2\sqrt{5}$ (f) 36 (g) $\dfrac{1}{\sqrt{3}}$ (h) $\sqrt{12}$

(i) $\dfrac{5}{\sqrt{2}}$

3. (a) $5 + 2\sqrt{6}$ (b) $23 + 4\sqrt{15}$ (c) $7 - 2\sqrt{6}$

4. (a) $\dfrac{\sqrt{3}}{3}$ (b) $3\sqrt{3}$ (c) $\dfrac{8\sqrt{2}}{3}$ (d) $\dfrac{3 + \sqrt{2}}{7}$ (e) $\dfrac{5(3 + \sqrt{5})}{4}$

(f) $4 + \sqrt{15}$ (g) $-7 - 4\sqrt{3}$ (h) $\dfrac{11 + 4\sqrt{6}}{5}$ (i) $\dfrac{2 - 3\sqrt{2}}{7}$

page 12 **Exercise D**

1. (a) $-3, 4$ (b) $-8, 3$ (c) $4, 9$ (d) $-1\frac{1}{2}, 2$ (e) $-\frac{1}{2}, 4$ (f) $-3, 1\frac{1}{3}$ (g) $-6, 1$
(h) $\frac{1}{2}, 1\frac{1}{2}$ (i) $-2, 5$ (j) $-6, 4$ (k) $\frac{1}{3}, 3$ (l) $-1\frac{1}{2}, 8$

2. (a) 0·38, 2·62 (b) −1·56, 2·56 (c) −6·19, −0·81
(d) −1·78, 0·28 (e) 0·42, 1·58 (f) −0·43, 1·18

3. (a) $\frac{1}{2}(3 \pm \sqrt{5})$ (b) $\frac{1}{2}(3 \pm \sqrt{7})$ (c) $\frac{1}{3}(3 \pm \sqrt{6})$

4. (a) $-3 \pm \sqrt{10}$ (b) $-2 \pm \sqrt{7}$ (c) $\frac{1}{2}(-1 \pm \sqrt{5})$ (d) $\frac{1}{2}(1 \pm \sqrt{13})$
(e) $\frac{1}{2}(3 \pm \sqrt{29})$ (f) $\frac{1}{2}(3 \pm \sqrt{7})$ (g) $\frac{1}{3}(2 \pm \sqrt{10})$ (h) $\frac{1}{4}(3 \pm \sqrt{5})$
(i) $\frac{1}{3}(3 \pm 2\sqrt{3})$

5. (a) MIN (b) -7 (c) -2 **6.** (a) MIN (b) -8 (c) 3

7. (a) MAX (b) 4 (c) -1 **8.** (a) MIN (b) $-5\frac{1}{4}$ (c) $2\frac{1}{2}$

9. (a) MAX (b) 6 (c) 1 **10.** (a) MAX (b) 22 (c) 4

11. (a) MIN (b) $-\frac{1}{8}$ (c) $-\frac{3}{4}$ **12.** (a) MIN (b) $1\frac{2}{3}$ (c) $-\frac{1}{3}$

13. (a) MAX (b) $4\frac{1}{8}$ (c) $-1\frac{1}{4}$

page 16 **Exercise E**

1. (a) $\sin 50°$ (b) $-\cos 50°$ (c) $-\tan 50°$ (d) $\sin 40°$
(e) $-\cos 10°$ (f) $-\cos 20°$ (g) $-\tan 80°$ (h) $\sin 85°$

2. (a) $\frac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) 1 (d) $\dfrac{1}{\sqrt{2}}$ (e) $\sqrt{3}$ (f) 1 (g) 0 (h) $\frac{1}{2}$

(i) $\dfrac{1}{\sqrt{2}}$ (j) $-\frac{1}{2}$ (k) 0 (l) -1 (m) -1 (n) $-\dfrac{\sqrt{3}}{2}$ (o) $-\sqrt{3}$ (p) $\dfrac{\sqrt{3}}{2}$

4. (a) 0·342 (b) 0·342 (c) −0·342

5. (a) 0·766 (b) 0·643 (c) 0·643 (d) 0·766 (e) −0·766 (f) −0·643

6. 4·9 **7.** 3·18 **8.** (a) $\frac{2}{3}$ (b) $\frac{3}{4}$

page 57 **Exercise 2B**

1. $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{i}$, $\overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OD} = -3\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OE} = -2\mathbf{i} - \mathbf{j}$, $\overrightarrow{OF} = -2\mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OG} = -2\mathbf{j}$, $\overrightarrow{OH} = 3\mathbf{i} - 3\mathbf{j}$
2. (a) (i) $4\mathbf{i} + 3\mathbf{j}$ (ii) 5 units (iii) 36·8° (b) (i) $-4\mathbf{i} + \mathbf{j}$ (ii) 4·12 units (iii) 14°
(c) (i) $2\mathbf{i} - 3\mathbf{j}$ (ii) 3·61 units (iii) 56·3°
3. 4 **4.** $\frac{\sqrt{5}}{5}(\mathbf{i} + 2\mathbf{j})$ **5.** $\frac{\sqrt{10}}{10}(3\mathbf{i} - \mathbf{j})$ **6.** $15\mathbf{i} + 36\mathbf{j}$ **7.** $6\mathbf{i} - 3\mathbf{j}$
8. $\frac{3}{5}(4\mathbf{i} - 3\mathbf{j})$ **9.** (a) $5\mathbf{i} + \mathbf{j}$ (b) $-5\mathbf{i} - \mathbf{j}$ **10.** (a) $-2\mathbf{i} + 8\mathbf{j}$ (b) $2\mathbf{i} - 8\mathbf{j}$
11. $2\mathbf{i} + 2\mathbf{j}$ **12.** $\mathbf{i} + \mathbf{j}$
13. $\mathbf{a} + 2\mathbf{b}$, $\frac{3}{11}\mathbf{a} + \frac{24}{11}\mathbf{b}$ **14.** (a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $5\mathbf{i}$ (d) 5
15. (a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) 5 (c) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ (d) $\sqrt{58}$ **16.** $6\mathbf{i} + \mathbf{j}$ **17.** $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$
18. (a) $1\frac{6}{7}\mathbf{i} + 2\frac{4}{7}\mathbf{j}$ (b) $-5\mathbf{i} + 6\mathbf{j}$ **21.** C, E and F

page 62 **Exercise 2C**

1. **b** and **d** **2.** **g** and **h** **3.** 14° **4.** 68° **5.** 143° **6.** −4 or 1
7. (a) $x^2 + y^2 = 25$ (b) $2x + y = 0$ (c) $x = 2y$ (d) $4x + y = 9$ **10.** 1
11. $x = 5, y = 2; x = -5, y = 12$ **12.** (a) $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ (b) $\pm\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$ **13.** (a) $\begin{pmatrix} 7\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$ (b) $\pm\begin{pmatrix} 3 \\ 21 \end{pmatrix}$
14. (a) $-3\mathbf{i} + 4\mathbf{j}$ (b) $-6\mathbf{i} - \mathbf{j}$ (c) $9\mathbf{i} - 3\mathbf{j}$ (d) 35°, 117°, 28° **17.** (b) 23 **18.** 131°
19. 4 **20.** $\frac{16}{13}\sqrt{13}$ **21.** $\frac{8}{5}\sqrt{5}$ **22.** (b) $k\sqrt{7}$, $2k\sqrt{21}$

page 64 **Exercise 2D**

1. (a) $\mathbf{a} + \mathbf{c}$ (b) $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ (c) $\frac{1}{2}(\mathbf{c} - \mathbf{a})$ (d) $\frac{1}{2}(\mathbf{a} + \mathbf{c})$
3. (a) $\frac{1}{3}(\mathbf{a} + \mathbf{b})$ (b) $\frac{1}{3}(\mathbf{a} + \mathbf{b})$ (c) $\frac{1}{3}(\mathbf{a} + \mathbf{b})$

page 67 **Exercise 2E**

1. $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + 5\mathbf{j})$ **2.** $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(4\mathbf{i} - \mathbf{j})$ **3.** $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \end{pmatrix}$
4. $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ **5.** $5\mathbf{i} + 10\mathbf{j}$ **6.** $\pm(4\mathbf{i} + 3\mathbf{j})$ **8.** B and C
9. D and F **10.** 4 **11.** (a) $\mathbf{i} + 3\mathbf{j}$ (b) $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} + 3\mathbf{j})$
12. (a) not parallel, $4\mathbf{i} + 5\mathbf{j}$ (b) parallel (c) parallel (d) not parallel, $9\mathbf{i} - \mathbf{j}$
13. $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$ **14.** 77° **15.** $7\mathbf{a} - 12\mathbf{b}$ **16.** $6\mathbf{i} + 6\mathbf{j}$, $6\sqrt{2}$ units
17. (a) $\sqrt{5}$ units (b) 2 units (c) $\sqrt{10}$ units

page 68 **Exercise 2F**

1. (a) (i) $\mathbf{a} + \mathbf{b}$ (ii) $-\mathbf{a}$ (iii) $\mathbf{b} - \mathbf{a}$ (iv) $\frac{1}{2}(\mathbf{a} - \mathbf{b})$ (v) $-\frac{1}{2}(\mathbf{a} + \mathbf{b})$
(b) (i) $4\mathbf{i} + 5\mathbf{j}$, $\sqrt{41}$ (ii) $2\sqrt{58}$
2. 1 : 2, $6\mathbf{q} - 4\mathbf{p}$, $5\mathbf{p} - 3\mathbf{q}$
3. $\mathbf{b} - \mathbf{a}$, $\frac{1}{10}(5\mathbf{b} - 3\mathbf{a})$ (i) $\frac{n}{10}(5\mathbf{b} - 3\mathbf{a})$ (ii) $-\frac{1}{2}\mathbf{a} + (\frac{1}{2} + k)\mathbf{b}$, $\frac{5}{3}$, $\frac{1}{3}$
4. (a) $\frac{1}{2}(3\mathbf{b} - \mathbf{a})$ (b) $\frac{1}{4}(3\mathbf{b} + \mathbf{c})$ (d) 1 : 1

page 92 Exercise 3H

1. 1·8, 32 (a) 176°F (b) 104°F (c) −40°F **2.** 0·4, 10; £62 **3.** 1·5 ohm
4. 0·37, 100 (a) 100 cm^3 (b) 111 cm^3 **5.** 20, 0·4; 110N **6.** 2·4, 3·6
8. 2, 5; 10 **9.** 36, 7; 11 **10.** 1, 1; 2550

page 96 Exercise 3I

1. (a) (4, −1) (b) (2, 1) (c) (2, 3) (d) (3, 5) (e) (1, 1), (−4, 16) (f) (2, 14), (−6, 6)
(g) (1, 1), (5, 9) (h) ($\frac{1}{2}$, $1\frac{1}{4}$), (2, 5) (i) (0, −1), (4, 19) (j) ($\frac{1}{3}$, $\frac{1}{9}$), (−2, 11)
(k) (1, 2), (−4, −3) (l) (−2, −5), (4, 1) **2.** (a) (4, 3) **4.** 3 **5.** Parallel lines, grad. 2
6. (a) No (b) Yes (c) Yes **7.** (1, 6) **8.** (−1, 4), (5, 10), $6\sqrt{2}$
9. (a) $3y + x = 21$ (b) (3, 6) (c) $2\sqrt{10}$ **10.** (a) $3y + 4x = 28$ (b) 5 units
11. $y = 3x$ **12.** (0, −1), (6, 5), (9, 2) **13.** −1, −4, (−1, 0)
14. (a) $x + y = 8$ (b) (4, 4) (c) $3\sqrt{2}$

page 97 Exercise 3J

1. (i) $y = 3x - 6$ (ii) (2, 0) **2.** 9 **3.** $\frac{2}{3}$, $5\frac{1}{2}$ units, $7\frac{1}{3}$ units, $20\frac{1}{6}$ sq. units
4. (i) (6, −3) (ii) $4y + 5x = 18$ **5.** (−3, 5), $3y = 2x + 8$ **6.** 28°
7. (i) $8y = x + 15$, $2y + x + 5 = 0$ (ii) (−7, 1) (iv) 30 sq. units (v) $\frac{8}{13}\sqrt{65}$
8. $2\sqrt{5}$, $4\sqrt{5}$ **9.** $y^2 - 3x^2 + 6x - 6y + 18 = 0$ **10.** (−4, −7), (7, 4) **11.** (6, 2)
12. 140, 0·35 **13.** (i) $\frac{1}{4}$, 9 (ii) 2·7

page 104 Exercise 4A

1. (a) 1st (b) 3rd (c) 4th (d) 2nd (e) 3rd
2. (a) sin 20° (b) −sin 40° (c) cos 50° (d) tan 20° (e) −sin 10° (f) −tan 10°
(g) −cos 40° (h) −sin 40° (i) cos 50° (j) tan 50° (k) −sin 10° (l) −cos 20°
(m) −sin 50° (n) sin 40° (o) −cos 80° (p) tan 20°
3. (a) $\frac{1}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$ (e) $-\sqrt{3}$ (f) 1 (g) −1 (h) 0 (i) $-\frac{1}{2}$
(j) 1 (k) $\frac{\sqrt{3}}{2}$ (l) $-\frac{\sqrt{3}}{2}$
4. (a) (i) 1, 90° (ii) −1, 270° (b) (i) 2, 90° (ii) −2, 270° (c) (i) 3, 0° (ii) −3, 180°
(d) (i) 3, 90° (ii) 1, 270° (e) (i) 4, 180° (ii) 2, 0° (f) (i) 5, 90° (ii) 1, 270°
(g) (i) 1, 45° (ii) −1, 135° (h) (i) 0, 0° (ii) −2, 90°

5. (a)

x	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	1	0·87	0·71	0·5	0	−0·5	−0·71	−0·87	−1	−0·87	−0·71	−0·5	0	0·5	0·71	0·87	1

6. (a)

x	0	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°
y	0	0·5	0·87	1	0·87	0·5	0	−0·5	−0·87	−1	−0·87	−0·5	0

(c) (i) 12°, 48° (ii) 72°, 108° (iii) 15°, 45°

7. (a)

x	0	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
y	2	1·87	1·5	1	0·5	0·13	0	0·13	0·5	1	1·5	1·87	2

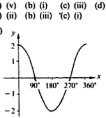
(c) (i) 36°, 144° (ii) 51°, 129° (iii) 67°, 113°

8. (b) 27°, 207° **9.** (b) (i) 104°, 256° (ii) 76°, 284° (iii) 37°, 270°
10. (b) (i) 37°, 143° (ii) 17°, 163° (iii) 24°, 246°

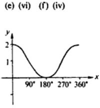
page 108 Exercise 4B

1. (a) (v) (b) (i) (c) (iii) (d) (ii) (e) (vi) (f) (iv)
2. (a) (ii) (b) (iii) (c) (i)

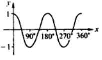
3. (a)



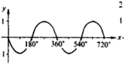
(b)



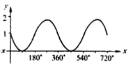
(c)



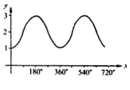
4. (a)



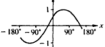
(b)



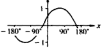
(c)



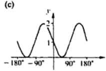
5. (a)



(b)



(c)



6. 2 **7.** 2 **8.** 1

page 114 Exercise 4C

1. (a) 30°, 150° (b) 120°, 240° (c) 45°, 225° (d) 135°, 315° (e) 240°, 300° (f) 45°, 315°
2. (a) −30°, −150° (b) ±60° (c) 60°, 120° (d) −120°, 60° (e) ±135° (f) ±150°

3. 17·5°, 162·5° **4.** 134·4°, 225·6° **5.** 143·1°, 323·1° **6.** 60°, 120°, 240°, 300°
7. 63·4°, 243·4° **8.** 56·3°, 236·3° **9.** 120°, 150°, 300°, 330° **10.** 30°, 150°, 210°, 330°
11. 220°, 280° **12.** 75°, 255° **13.** 48·2°, 311·8° **14.** 0, 60°, 180°, 300°, 360°
15. 90°, 153·4°, 270°, 333·4° **16.** 0, 120°, 180°, 240°, 360° **17.** 48·6°, 90°, 131·4°, 270°
18. 90°, 104·5°, 255·5°, 270° **19.** 0, 75·5°, 180°, 284·5°, 360° **20.** 30°, 150°, 270°
21. 90°, 210°, 330° **22.** 63·4°, 123·7°, 243·4°, 303·7° **23.** 45°, 153·4°, 225°, 333·4°
24. ±30°, ±150° **25.** −148·3°, −58·3°, 31·7°, 121·7° **26.** −115°, 155°
27. ±90° **28.** 0, 33·7°, −146·3°, ±180° **29.** ±60°
30. (a) $(2\sin\theta + 1)(3\cos\theta + 2)$ (b) −30°, ±131·8°, −150°
31. (a) $(\cos\theta - 1)(3\sin\theta + 2)$ (b) 0, 221·8°, 318·2°, 360°

page 117 **Exercise 4D**

1. (a) $\sqrt{2}$ (b) 1 (c) 2 (d) 2 (e) $\sqrt{2}$ (f) −2 (g) −2 (h) −2 (i) 1
(j) 2 (k) −2 (l) $-\sqrt{2}$
2. (a) $\cos A$ (b) $\sec\theta$ (c) 1 (d) $\cos\theta$
5. (a) $x^2 + y^2 = 1$ (b) $xy = 3$ (c) $100x^2 + y^2 = 4$ (d) $x^2 + y^2 - 6x + 8 = 0$
(e) $x^2 + y^2 - 4x + 2y + 4 = 0$
6. (a) 0, 14·5°, 165·5°, ±180° (b) ±180° (c) 0, ±48·2° (d) 19·5°, 160·5° (e) ±45°, ±135°
(f) −63·4°, 0, 116·6°, ±180°
7. (a) 60°, 300° (b) 55·9°, 145·9°, 235·9°, 325·9°
(c) 60°, 120°, 240°, 300° (d) 120°, 240° (e) 26·6°, 135°, 206·6°, 315°
(f) 68·2°, 135°, 248·2°, 315° (g) 194·5°, 345·5°
(h) 26·6°, 45°, 206·6°, 225° (i) 70·5°, 289·5°

page 120 **Exercise 4E**

1. (a) 1 (b) $\frac{1}{2}$ **2.** (a) $\sin 3A$ (b) $\sin\theta$ (c) $\cos 5\theta$ (d) 1 (e) 2
3. (a) $\frac{\sqrt{2}}{4}(\sqrt{3} + 1)$ (b) $\frac{\sqrt{2}}{4}(\sqrt{3} - 1)$ (c) $-2 - \sqrt{3}$
4. (a) (i) 1, 70° (ii) −1, 250° (b) (i) 1, 110° (ii) −1, 290° (c) (i) 1, 320° (ii) −1, 140°
5. (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{4}{7}$ **6.** (a) $\frac{33}{65}$ (b) $\frac{33}{56}$ **7.** (a) $-\frac{56}{65}$ (b) $\frac{63}{65}$
8. (a) 200°, 320° (b) 26·4°, 253·6° (c) 45°, 225°

page 122 **Exercise 4F**

1. (a) 2 sin A (b) sin 2A (c) 1 **2.** $-\frac{1}{6}$ **3.** (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{24}{25}$ (d) $-\frac{7}{25}$
4. (a) $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{3}$ (b) $-\frac{4}{5}$, $\frac{3}{5}$, $-\frac{4}{3}$
5. (a) $x + 2y^2 = 0$ (b) $x = 2y^2 + 4y + 1$ (c) $2x^2 - 8x + y + 4 = 0$
6. (a) $\frac{3}{5}$ (b) $\frac{24}{25}$ (c) $\frac{\sqrt{10}}{10}$ **7.** (a) $\frac{\sqrt{5}}{4}$ (b) $\frac{\sqrt{11}}{4}$
8. (a) $3\sin\theta - 4\sin^3\theta$ (b) $\frac{11}{16}$ **9.** (a) $4\cos^3\theta - 3\cos\theta$ (b) $-\frac{2\sqrt{5}}{25}$
13. (a) 0, ±120°, ±180° (b) −135°, 45°, ±90° (c) ±70·5°, ±120° (d) 0, ±45°, ±135°, ±180°
14. (a) 22·5°, 112·5°, 202·5°, 292·5° (b) 9·6°, 90°, 170·4°, 270° (c) 30°, 90°, 150°
(d) 14·5°, 165·5°
17. (a) 2, 1, 1 (b) −1, 2, −2 (c) −3, 1, 2

page 125 Exercise 4G

1. (a) 180° (b) 30° (c) 270° (d) 120° (e) 330°

2. (a) 2π (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{6}$ (e) $\frac{\pi}{2}$

3. (a) 1 (b) 0 (c) 1 (d) −1 (e) −1 (f) $\frac{1}{2}$ (g) $\frac{1}{2}$ (h) $-\frac{\sqrt{3}}{2}$ (i) −1 (j) $-\frac{1}{\sqrt{2}}$

4. (a) 1·22 (b) 4·36 **5.** (a) 172° (b) 143°

6. (a) 0·84 (b) −0·14 (c) −0·74

7. (a) (i) 5 cm (ii) 12·5 cm^2 (b) (i) 2 cm (ii) 4 cm^2 (c) (i) 12 m (ii) 24 m^2

8. (a) $\frac{\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) π cm (d) 5π cm

9. 1 rad **10.** 8 cm^2 **11.** 1·5 rad **12.** 12 cm

13. (a) $\frac{\pi}{6}, \frac{5\pi}{6}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (c) $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

(d) $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ (e) $\frac{\pi}{2}$ (f) $0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$

14. 6 cm^2 **15.** (a) 9 cm^2 (b) 0·42 cm^2 **16.** (a) 1·70 rad (b) 13·6 cm^2

18. (a) $2\frac{1}{4}$ m^2 (b) 0·47 m^2 **19.** (a) 1·55 (b) 1·19 (c) 7·66 cm^2 **20.** (c) (i) 1·7 (ii) 1·25

21. (c) 2·5 (d) 2·475

page 128 Exercise 4H

2. (a) −146·3, 33·7 (b) 30°, 150° **3.** 48·6°, 131·4°, 210°, 330°

4. (a) (i) $-\frac{24}{7}$ (ii) $-\frac{33}{65}$ (iii) $\frac{33}{65}$ **5.** (a) $-\frac{1}{3}$ (b) $-\frac{4}{9}\sqrt{2}$ (c) $-\frac{7}{9}$ (d) $\frac{1}{\sqrt{3}}$

6. (i) 45°, 135°, 210°, 225°, 315°, 330° (ii) 100·9°, 280·9°

7. (i) 56·3°, 135°, 236·3°, 315° (ii) 26·6°, 63·4°, 206·6°, 243·4°

8. (a) $x = 4y^2(1 - y^2)$ (b) 26·6°, 135°

9. (a) $\frac{1}{15}(8 + 3\sqrt{5})$, $\frac{2}{15}(2\sqrt{5} - 3)$, acute (b) ±35·3°, ±144·7°

11. 1, 4, 3 **12.** (i) 123°, 303° (ii) 32·1°, 147·9° (iii) $\frac{1 + \tan\theta}{1 - \tan\theta}$, 45°, 225°

14. 4 **15.** (a) $\frac{2}{3}\pi, \frac{4}{3}\pi$ (b) $\frac{1}{3}\pi, \frac{4}{3}\pi$ (c) $\frac{3}{4}\pi, \frac{7}{4}\pi$

16. (i) 21·4 cm (ii) 13·6 cm^2 **17.** (a) $\pm\frac{1}{3}\pi$, ±2·3 rads (b) 1·97 rads

page 132 Exercise 5A

1. 27	**2.** 128	**3.** 25	**4.** 49	**5.** 1	**6.** 1	**7.** 2	**8.** 4
9. 12	**10.** 3	**11.** 2	**12.** 64	**13.** $\frac{1}{25}$	**14.** $\frac{1}{7}$	**15.** $\frac{1}{100}$	**16.** $\frac{1}{2}$
17. $\frac{1}{8}$	**18.** $\frac{1}{2}$	**19.** 9	**20.** $\frac{1}{16}$	**21.** $\frac{1}{81}$	**22.** $\frac{1}{64}$	**23.** 4	**24.** $\frac{1}{4}$
25. $\frac{1}{16}$	**26.** $\frac{4}{9}$	**27.** 1	**28.** $1\frac{9}{16}$	**29.** $6\frac{1}{4}$	**30.** $\frac{3}{4}$	**31.** $1\frac{1}{3}$	**32.** $\frac{2}{3}$
33. $15\frac{5}{8}$	**34.** $\frac{8}{27}$	**35.** $\frac{4}{9}$	**36.** 2^6	**37.** 2^4	**38.** 2^6	**39.** 2^{11}	**40.** 2^2
41. 2^3	**42.** 2^7	**43.** 2^6	**44.** x^7	**45.** x^6	**46.** $18x^6$	**47.** $24x^3y$	**48.** $3x^2$
49. $5x^3$	**50.** x^2	**51.** $30x^4y^3$	**52.** $12x^4y^3$	**53.** a^{-1}	**54.** $a^{-1/2}$	**55.** a^{-1}	**56.** a^5
57. $a^{1/2}$	**58.** $a^{-1/2}$	**59.** a^6	**60.** $a^{3/2}$	**61.** $a^{7/2}$	**62.** $a^{5/2}$	**63.** 6	**64.** 0
65. −1	**66.** −2	**67.** −2	**68.** 11	**69.** 3	**70.** 8	**71.** 4	**72.** 3
73. −2	**74.** 6	**75.** 6	**76.** 9				

page 135 **Exercise 5B**

1. (a) $y = kx$ (b) $y = kx^3$ (c) $y = k\sqrt[3]{x}$ (d) $y = \frac{k}{x}$ (e) $y = \frac{k}{\sqrt{x}}$ (f) $y = kxz^2$
(g) $y = \frac{kx^2}{z}$ (h) $y = \frac{kwx}{z}$ (i) $y = k_1x + k_2z$ (j) $y = k_1x^2 + k_2z$ (k) $y = k_1\sqrt{x} + \frac{k_2}{z^2}$

2. (a) $y = \frac{5}{x^2}$ (b) 80 **3.** (a) $y = 3z^2 - \frac{5}{2}x$ (b) $\frac{1}{2}$ **4.** $F = \frac{50}{3}x$, 0·48 m

5. $T = 2\sqrt{l}$, 0·25 m **6.** $F = \frac{3v^2}{2r}$, 12N **7.** $C = 13 + \frac{u}{20}$, £35·50

8. $v = \sqrt{\frac{E}{\rho}}$, 5200 m/s

page 139 **Exercise 5C**

1. (a) $m^d = c$ (b) $b^q = p$ (c) $10^t = s$ (d) $e^r = z$
2. (a) 3 (b) 4 (c) 6 (d) 256 (e) $\frac{1}{100}$ (f) 2048
(g) $1\frac{1}{3}$ (h) $1\frac{1}{2}$ (i) $\frac{1}{8}$
3. (a) 5 (b) 2 (c) 4 (d) -1 (e) -2 (f) 0 (g) 1 (h) $\frac{1}{3}$
(i) $-\frac{1}{3}$ (j) $\frac{1}{3}$ (k) $\frac{2}{3}$ (l) $-\frac{2}{3}$
4. (a) $\log a + \log b + \log c$ (b) $2 \log a + \log b + \log c$ (c) $3 \log a + 2 \log b + \log c$
(d) $\log a + \frac{1}{2} \log b$ (e) $\log a + \log b - \log c$ (f) $2 \log a - \log b - 3 \log c$
(g) $-\log a - \log b$ (h) $\log a + \frac{1}{2} \log b - \frac{3}{2} \log c$ (i) $1 + \log a + 2 \log b$
(j) $\frac{1}{2} + \frac{1}{2} \log a - 2 \log b$
5. (a) $\log_a 10$ (b) $\log_a 3$ (c) $\log_a 2$ (d) $\log_a 20$

6. (a) $\log_a x^2$ (b) $\log_a [x(x + 3)]$ (c) $\log_a \left(\frac{x + 1}{2}\right)$ (d) $\log_a (x - 1)$

(e) $\log_a \left(\frac{x}{x + 1}\right)$ (f) $\log_a [x^3(x + 1)]$ (g) $\log_a \left(\frac{x^2}{1 + x}\right)$ (h) $\log_a [x^2(x + 1)]$

7. (a) 6 (b) 3 (c) 256
8. (a) 1·36 (b) 3·14 (c) 2·09 (d) 25·12 (e) 0·22 (f) 22·20 (g) 1·14 (h) 0·64
9. (a) 0·4 (b) 0·6 (c) 1·4 (d) 1·5 (e) $-0·3$ (f) $2\frac{1}{3}$ (g) $1\frac{3}{7}$
10. (a) 6·4 (b) 4 (c) 0·8 (d) 5·6 (e) $-1·6$ (f) $-3·2$ (g) 1·5

11. (a)

x	-3	-2	-1	0	1	2
y	0·125	0·25	0·5	1	2	4

(b)

x	0·2	0·4	0·6	0·8	1	2	3	4
y	$-2·32$	$-1·32$	$-0·74$	$-0·32$	0	1	1·58	2

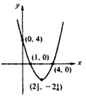
12. (a) 6 (b) 12 (c) 9
13. (a) 1·6 (b) 4·1
14. 2·5, 1·4 **15.** 2·3, 6 **16.** 2, 1·5 **17.** 3, 0·5
18. (a) 2·32 (b) 2·58 (c) 1·43 (d) 1·79 (e) 5·37 (f) $-2·71$ (g) 0·326 (h) $-1·95$ (i) 0·287
19. (a) 2 (b) 1 (c) ± 1 (d) 2·16 (e) 1 or 0·585 (f) $\frac{1}{16}$ or 8
20. (a) 15 (b) 56 (c) 3 (d) 13·5 (e) 64 (f) 5 or 25
21. (a) $x = 16, y = 4$ (b) $x = \sqrt{3}, y = 9$
(c) $x = 12, y = 4$ (d) $x = \frac{1}{2}, y = \frac{1}{2}$; $x = 8, y = 2$
(e) $x = 3, y = 2$ (f) $x = 4, y = 4$; $x = 48, y = \frac{1}{3}$
(g) $x = 9, y = 6$ (h) $x = 12, y = 10$

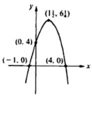
page 144 Exercise 5D

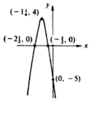
1. (a) (0, 4)
(b) (1, 0)(4, 0)
(c) $(2\frac{1}{2}, -2\frac{1}{4})$ min

2. (a) (0, 4)
(b) (−1, 0)(4, 0)
(c) $(1\frac{1}{2}, 6\frac{1}{4})$ max

3. (a) (0, −5)
(b) $(-\frac{1}{2}, 0)$
$(-2\frac{1}{2}, 0)$
(c) $(-1\frac{1}{2}, 4)$ max







4. $a = 2, b = 2, c = 1$ **5.** $a = 1\frac{1}{2}, b = 3, c = -2$ **6.** $a = 3, b = 2, c = 2$
7. (a) $x \leqslant -3, x \geqslant 5$ (b) $3 \leqslant x \leqslant 5$ (c) $x \leqslant -1, x \geqslant 5$ (d) $x \leqslant -\frac{1}{2}, x \geqslant 3$
(e) $\frac{1}{3} \leqslant x \leqslant 6$ (f) $-2\frac{1}{2} \leqslant x \leqslant 1$
8. (a) $-6 \leqslant x \leqslant 1$ (b) $x \leqslant -4, x \geqslant -3$ (c) $5 < x < 7$ (d) $x \leqslant 2\frac{1}{3}, x \geqslant 5$
(e) $x < -1\frac{1}{2}, x > 1\frac{2}{3}$ (f) $-\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$
9. (a) $x \leqslant -\sqrt{3} - 2, x \geqslant \sqrt{3} - 2$ (b) $\frac{1}{2}(5 - \sqrt{13}) \leqslant x \leqslant \frac{1}{2}(5 + \sqrt{13})$ (c) $\mathbb{R}$
(d) $\frac{1}{4}(3 - \sqrt{17}) \leqslant x \leqslant \frac{1}{4}(3 + \sqrt{17})$ (e) $\mathbb{R}$ (f) $x \leqslant -\frac{1}{3}(\sqrt{7} + 1), x \geqslant \frac{1}{3}(\sqrt{7} - 1)$
12. $k > 5$ **13.** $-1 < k < 3$

page 148 Exercise 5E

1. (a) $g(x) = 0$ (b) $f(x) = 0$ (c) $h(x) = 0$ (d) $f(x), g(x)$ (e) $h(x)$
2. (a) 129, real distinct (b) −3, no real roots (c) 57, real distinct (d) 0, repeated root
(e) 37, real distinct (f) −7, no real roots (g) 0, repeated root (h) −19, no real root
(i) 261, real distinct
3. (a) $\{y \in \mathbb{R}: y \geqslant -6\}$ (b) $\{y \in \mathbb{R}: y \geqslant 2\frac{3}{4}\}$ (c) $\{y \in \mathbb{R}: y \leqslant 6\}$ (d) $\{y \in \mathbb{R}: y \geqslant -5\}$
(e) $\{y \in \mathbb{R}: y \leqslant 2\frac{1}{8}\}$
4. 8 **5.** ±20 **6.** $|b| > 12$ **7.** $k \leqslant -3, k \geqslant 1$ **8.** $k < -1, k > 15$
9. (a) $-4 \leqslant k \leqslant 1$ (b) $-\frac{1}{2} < k < 4\frac{1}{2}$ except $k = 0$ (c) $\{k \in \mathbb{R}: k \neq 0\}$
(d) $k \geqslant 3, k \leqslant \frac{1}{3}$ (e) $k < 2$ except $k = 0$ (f) $k < 0, k \geqslant 1$
10. $0 < \beta < \frac{4}{25}$ **11.** (3, 0) **12.** $(1\frac{1}{2}, -3)$ **13.** (a) (ii), (b) (iii), (c) (i), (d) (iii)

page 151 Exercise 5F

1. (a) −9, 4 (b) 2, −5 (c) 7, 2 (d) 9, −3 (e) $3\frac{1}{2}, \frac{1}{2}$ (f) $-\frac{1}{4}, -\frac{1}{4}$ (g) $-3\frac{1}{3}, -\frac{2}{3}$ (h) $-2, \frac{1}{3}$
2. (a) $x^2 + 3x - 1 = 0$ (b) $x^2 - 6x - 4 = 0$ (c) $x^2 - 7x - 5 = 0$
(d) $3x^2 + 2x - 7 = 0$ (e) $2x^2 + 5x - 4 = 0$ (f) $2x^2 + 3x - 10 = 0$
(g) $12x^2 + 3x - 4 = 0$ (h) $6x^2 + 10x + 3 = 0$
3. (a) 2 (b) −11 (c) 121 (d) 117 (e) 14 (f) −22
4. (a) $\frac{1}{2}$ (b) $2\frac{1}{2}$ (c) $6\frac{3}{4}$ (d) $3\frac{3}{4}$ (e) $2\frac{5}{8}$ (f) 5
5. (a) $-2\frac{5}{8}$ (b) $4\frac{5}{8}$ (c) 3 (d) $-2\frac{2}{3}$ (e) $-\frac{29}{34}$ (f) $4\frac{8}{27}$

page 161 **Exercise 6A**

1. (a) $\begin{pmatrix}5 & 0\\ 3 & 2\end{pmatrix}$ (b) $\begin{pmatrix}-7 & -6\\ -1 & 0\end{pmatrix}$ (c) $\begin{pmatrix}-2 & -6\\ 2 & 2\end{pmatrix}$ (d) Impossible (e) $\begin{pmatrix}13 & 9\\ 3 & 1\end{pmatrix}$

(f) $\begin{pmatrix}-12 & -6\\ 8 & 4\end{pmatrix}$ (g) $\begin{pmatrix}-3 & -15\\ -1 & -5\end{pmatrix}$ (h) $\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$ (i) Impossible (j) $\begin{pmatrix}12 & 6\\ 12 & 6\\ -14 & -7\end{pmatrix}$

(k) Impossible (l) $\begin{pmatrix}2 & 5\\ 5 & -2\\ -4 & -1\end{pmatrix}$ (m) $\frac{1}{2}\begin{pmatrix}1 & 3\\ -1 & -1\end{pmatrix}$ (n) No inverse, **B** is singular

(o) $\begin{pmatrix}-1 & 1\\ -3 & 1\end{pmatrix}$ (p) $\begin{pmatrix}1 & 2 & -3\\ 3 & 0 & 2\end{pmatrix}$

2. $\mathbf{AB} = \begin{pmatrix}4\\ 5\end{pmatrix}$ $\mathbf{AD} = \begin{pmatrix}4 & 4 & -1\\ 5 & -2 & -3\end{pmatrix}$ $\mathbf{BC} = \begin{pmatrix}-2 & 6\\ -1 & 3\end{pmatrix}$ $\mathbf{CD} = (1 \quad 6 \quad 1)$

$\mathbf{CA} = (8 \quad -5)$ $\mathbf{CB} = (1)$

3.

	Won	Drawn	Lost
A	4	1	3
B	3	1	4
C	2	6	0
D	2	2	4
E	3	2	3

A, C, E, B, D; C, A, E, B, D.

4. (a) $-4, 11$ (b) $-2, -3$ (c) $2, 4$ (d) $4, 3$ (e) $2, -1$ (f) $5, 19$ (g) $-1, 4$ (h) $2, -1$
(i) $-3, 2$ (j) $-3, 1\frac{1}{2}$

6. $\mathbf{X} = \begin{pmatrix}1 & -1\\ 3 & 4\end{pmatrix}$ $\mathbf{Y} = \begin{pmatrix}-1 & 1\\ 2 & -3\end{pmatrix}$ $\mathbf{Z} = \begin{pmatrix}2 & -1\\ 4 & 1\end{pmatrix}$ **7.** $4, -11$ **8.** $-\frac{1}{2}$ or 3

9. (a) $x = 3, y = -2$ (b) $x = 1, y = -\frac{2}{3}$ (c) $x = 1\frac{1}{2}, y = \frac{1}{2}$ **10.** $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $-1, 2, 1$.

page 168 **Exercise 6B**

1. $(9, -3), (-3, -13), (6, -13), (3, 4)$ **2.** (a) $(4, 1)$ (b) $(5, 10)$ (c) $(3, -1)$ (d) $(-5, 2)$

3. $\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$, $(-5, 9), (2, -2)$

4. (a) $(5, -3), (15, -9), (13, -7), (3, -1)$ (b) 8 sq. units (c) $\frac{1}{2}\begin{pmatrix}1 & 1\\ 3 & 5\end{pmatrix}$

5. $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}5 & 2\\ 4 & -2\end{pmatrix}$

6. (a) (i) $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ (ii) $\begin{matrix}x' = x + 2y\\ y' = y\end{matrix}$ (b) (i) $\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ (ii) $\begin{matrix}x' = x\\ y' = 3x + y\end{matrix}$

7. $\begin{pmatrix}18 & 14\\ 6 & 5\end{pmatrix}$, 10 sq. units, 30 sq. units

8. (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$

9. (a) reflection in $y = x$ (b) enlargement ($\times$ 5) centre (0, 0)
(c) stretch ($\times$ 3) parallel to x-axis, y-axis fixed (d) mapping onto (0, 0)
(e) mapping onto x-axis (f) mapping onto $y = x$
(g) enlargement ($\times$ 3) centre (0, 0) and reflection in $y = x$
(h) shear with y-axis fixed and $(1, 0) \to (1, 2)$ (i) mapping onto $y = 4x$
(j) shear with $y = x$ fixed and $(1, 0) \to (-2, -3)$

10. $2x + 3y = 0$ **12.** (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (b) $\begin{matrix} x' = -x + 2 \\ y' = y + 3 \end{matrix}$

13. (a) $(-1, -1)$ (b) $(-3, 2)$ (c) $(-1, 2)$

page 176 **Exercise 6C**

1. (a) $(-4, -2)$ (b) $(-3, 2)$ (c) $(4\frac{1}{2}, 1)$

2. $\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -8 \\ -4 \end{pmatrix}$,
$\begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
(a) $(\frac{1}{3}, -\frac{2}{3})$ (b) $(2, -1)$ (c) $(2, 0)$ (d) $(1, -1)$

3. $7y = 5x + 6$, $x = 3y + 2$ **4.** $y = 3x - 4$ **5.** $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 11 \\ 1 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

6. $\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$, $x + 7y = 0$, $4x + y = 27$, $y = 3$ **7.** $y = x + 1$ **8.** $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

9. (a) $3y = 10x + 8$, $y = 5x + 21$ **13.** $\begin{pmatrix} 7 \\ -7 \end{pmatrix}$ **14.** $-2\mathbf{i} + 3\mathbf{j}$ **15.** $(-1, 1)$

16. (a) $y = x$, $y = 2x$ (b) $3y = x$, $y + x = 0$ (c) $y = \pm 2\sqrt{2}x$

17. $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$, $y = x$, $2y = x$; $y = x$ **18.** 8, $y = 2x$

19. (a) $y = x + 6$, $3y = 2x + 12$ (b) $y + 4x = 1$, $y = x + 1$ (c) $y = 2$, $y = 2x - 6$
(d) $y = x - 1$ (e) $y = 3x - 9$

page 182 **Exercise 6D**

1. (a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

2. (a) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

3. (a) rotation of 53·1° anticlockwise about origin (b) reflection in $2y = x$
(c) rotation of 53·1° anticlockwise about origin and enlargement (× 5), centre the origin

4. (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix}$
(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 5 \end{pmatrix}$
(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix}$

5. (a) $x' = -y$, $y' = x - 2$ (b) $x' = 2x + 1$, $y' = 2y - 3$ (c) $x' = y + 5$, $y' = -x + 3$ (d) $x' = -y + 3$, $y' = -x + 3$
(e) $x' = -y + 6$, $y' = -x$

7. (a) 90° rotation anticlockwise about (4, −1) (b) reflection in the line $y = x - 3$
(c) glide reflection in $y = 1$ with (0, 1) → (−6, 1) (d) enlargement (× 5) centre (1, 4)
(e) 180° rotation about $(1, -1\tfrac{1}{2})$ (f) 90° rotation clockwise about $(-\tfrac{1}{2}, -3\tfrac{1}{2})$
(g) 90° rotation anticlockwise about $(\tfrac{1}{2}, 2\tfrac{1}{2})$ and an enlargement (× 3) centre $(\tfrac{1}{2}, 2\tfrac{1}{2})$

8. (a) (3, −2) (b) $y = x - 5$, $y + x = 1$
(c) enlargement (× 2) centre (3, −2) and a reflection in the line $y + x = 1$

page 183 **Exercise 6E**

1. $\tfrac{8}{3}$, −5

2. (a) $\begin{pmatrix} 2 & 0 & 3 & 4 \\ 14 & 0 & 21 & 28 \end{pmatrix}$ (b) $\tfrac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ (i) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 10 & 5\tfrac{1}{2} \\ -10 & -5 \end{pmatrix}$

3. (a) (i) $\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(b) $\text{BA} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 2 & 2 \end{pmatrix}$ $\text{AC} = \begin{pmatrix} 2 & 0 \\ 4 & -2 \\ 0 & 6 \end{pmatrix}$ $\text{EA} = (5 \quad 1)$

4. (a) $-\tfrac{2}{3}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $x = 2, y = -1, z = 3$

5. (a) $x = 3\tfrac{2}{3}, y = 6\tfrac{1}{3}$ (b) $\tfrac{1}{7}\begin{pmatrix} -2 & -3 & 8 \\ 1 & 5 & -4 \\ 3 & 1 & -5 \end{pmatrix}$

6. (c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (iv) (−5, 2) (v) (−4, 6)

7. (1, 2), (−2, 1), $\begin{pmatrix} -\tfrac{3}{5} & \tfrac{4}{5} \\ \tfrac{4}{5} & \tfrac{3}{5} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} \tfrac{4}{5} & \tfrac{3}{5} \\ \tfrac{3}{5} & -\tfrac{4}{5} \end{pmatrix}$

8. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
(i) $(a + 6, -b)$ (ii) $(a, 4 - b)$ (iii) $(6 - a, -b)$ (iv) $(-a, b)$ (v) (a, b)
(i) no values (ii) any points for which $b = 2$ (iii) $a = 3, b = 0$
(iv) any points for which $a = 0$ (v) all values of a and b

9. (i) $\begin{pmatrix} 1 & \frac{1-x}{y} \\ -1 & \frac{x}{y} \end{pmatrix}$ (a) $\frac{1}{2}, -1\frac{1}{2}$ (b) $\frac{3}{4}, -\frac{1}{2}$ (ii) $x = 1, 2y = x + 3$

10. $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}, y = 2x$

11. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} -\frac{9}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{9}{5} \end{pmatrix}, (-15, 45)$

12. (i) enlargement, scale factor $\frac{1}{2}$, centre (0, 0); reflection in the line $y = x$; rotation of 120°, anticlockwise about (0, 0) (ii) 2, 3 (iii) enlargement, scale factor 2, centre (0, 0); reflection in the line $y = x$; rotation of 120° clockwise about the origin.

13. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$, a rotation of 60° anticlockwise about (0, 0), reflection in $x = y\sqrt{3}$

14. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, 13 **15.** $\frac{1}{2}$ **16.** $x = \frac{1}{2}$, glide reflection in $x = \frac{1}{2}$ with $(\frac{1}{2}, 0) \to (\frac{1}{2}, 3)$

17. $(1\frac{3}{5}, -\frac{1}{5})$, 90° rotation anticlockwise about $(1\frac{3}{5}, -\frac{1}{5})$ and an enlargement, scale factor 2, centre $(1\frac{3}{5}, -\frac{1}{5})$. $\begin{matrix} x' = -y + 1 \\ y' = -x + 1 \end{matrix}, \begin{matrix} x' = -2x + 4 \\ y' = 2y \end{matrix}$, centre $(\frac{4}{3}, 0)$, scale factor 2.

18. $2y = x$

page 193 *Exercise 7A*

1. (a) 56 (b) 1008 (c) 600 (d) 4

2. (a) 5! (b) 3 × 5! (c) 5! (d) $\frac{5!}{5}$ (e) 5 × 5! (f) 9 × 5! (g) 2 × 5! (h) 36 × 5! (i) 55 × 5!

3. 18 **4.** 10! **5.** 8! **6.** 24 **7.** 120 **8.** 24, 256 **9.** (a) 81 (b) 256 **10.** 8!

11. 720, 120 **12.** 24 **13.** 48 **14.** 72 **15.** 12 **16.** 10 080 **17.** 80 640

18. (a) 14! (b) 11! (c) 11! × 4! **19.** 1440, 240 **20.** 2520 **21.** 30 240 **22.** 42

23. (a) 12 (b) 12 **24.** 120 (a) 48 (b) 72 (c) 12 **25.** 96 **26.** 1296

page 197 *Exercise 7B*

1. 30 **2.** 151 200 **3.** 6720 **4.** 120

5. (a) 24 (b) 64 **6.** (a) 6 (b) 16 **7.** 36 **8.** 2730

9. 2 522 520 **10.** 3360 **11.** 453 600, 10 080

12. (a) 20 160 (b) 5040 (c) 15 120 **13.** 18

14. 60, 40 **15.** 1260 **16.** 528 **17.** 42

18. 151 (a) 73 (b) 78 (c) 13 (d) 138 **19.** 19 958 400 (a) 3 628 800 (b) 302 400

20. 2 494 800 (a) 453 600 (b) 60 480 **21.** 5472 **22.** 1596, 198

page 201 *Exercise 7C*

1. 120 **2.** 126 **3.** 10 **4.** 330 **5.** 10 **6.** 4845

7. 22 100 **8.** 210, 210 **9.** 1960 **10.** 286 **11.** 150 **12.** 60

13. 2016 **14.** (a) 126 (b) 105 **15.** (a) 15 (b) 1 **16.** (a) 20 (b) 4 **17.** (a) 13 (b) 5

18. (a) 24 (b) 7 (c) 13 **19.** 37 **20.** 6 300 **21.** (a) 5 (b) 85 (c) 365 **22.** 175

23. (a) 105 (b) 21 (c) 231 **24.** 100 **25.** (a) 1 (b) 56 (c) 1231 (d) 1230 **26.** 120

27. 3360, 1200 **28.** 30 240, 12 600 **29.** 7560, 3150 *page 205* **Exercise 7D**

1. 630 630 **2.** 127 **3.** 63 **4.** 330 **5.** 7920 **6.** 60 060 **7.** 63
8. 64 **9.** 255 **10.** 47 **11.** 31 **12.** 255 **13.** 39
14. $(p+1)(q+1)\,2^{n-p-q}-1$ **15.** (a) 126 (b) 280 **16.** 5775 **17.** 126 126
18. 25 740 **19.** 23

page 206 **Exercise 7E**

1. 816 **2.** 286 **3.** 6 **4.** 360 (a) 180 (b) 240 **5.** 100 **6.** (a) 495 (b) 225 (c) 15
7. 25 **8.** 840, 96 **9.** 3528, 8624 **10.** 154 **11.** 10 080, 30 **12.** $(n-2)!(n-3)$
13. 120, 24 **14.** 480, 172 800, 462, 425
15. (a) 38 760 (i) 12 320 (ii) 5320 (b) 86 400, 172 800

page 210 **Exercise 8A**

1. (a) 10, 13 (b) 49, 64 (c) 0·00001, 0·000001, (d) $5\frac{1}{4}$, 6
2. (a) 8, 3 (b) 23, 2 (c) 19, −3 (d) $13\frac{1}{2}$, $1\frac{1}{2}$
(e) −11·5, 2·5 (f) $6\frac{1}{4}$, $\frac{1}{2}$ (g) −8, 1 (h) 6, −3
3. (a) 20 (b) 102 (c) 45 (d) 32 **4.** 56 **5.** 24 **6.** −9 **7.** 13, −15
8. (a) $2+3n$, 302 (b) $8-3n$, −292 (c) $\frac{1}{2}(18+7n)$, 359
9. (a) 200th (b) 411th **10.** (a) 122nd (b) 64th
11. (a) 2, 4, 288 (b) 10, −2, −60 (c) 4·5, 1·5, 342 (d) 15, −2, 0
(e) 7, −4, −620 (f) $-6\frac{1}{2}$, $+1\frac{1}{2}$, 21 (g) −9, 2, 96
12. (a) 5402 (b) 1000 (c) 9464 (d) $13\frac{1}{2}$
13. 2, $1\frac{1}{2}$, $87\frac{1}{2}$ **14.** $6\frac{1}{2}$, $\frac{1}{2}$, 66 **15.** 21, −2, 0 **16.** −10, 3, 18
17. −15, 3, 0 **18.** $13\frac{1}{2}$, $-3\frac{1}{2}$, $-14\frac{1}{2}$ **19.** 0, $\frac{1}{2}$, $9\frac{1}{2}$ **20.** 2160
21. $1575d$ **22.** 20 **23.** 32 **24.** 24 **25.** 28
26. (a) $23n-20-3n^2$ (b) $20-6n$ (c) 14, −6
27. 7, 12, 17 **28.** 2·5 **29.** $\log_e 3$, $\log_e 3^n$, $\frac{1}{2}n(n+1)\log_e 3$
30. $\log_e(ab^n)$, $\frac{1}{2}n\log_e(a^2b^{n+1})$

page 216 **Exercise 8B**

1. (a) 5, 2500, 12 500 (b) $\frac{1}{2}$, 3, $1\frac{1}{2}$ (c) $\frac{1}{3}$, $1\frac{2}{3}$, $\frac{5}{8}$ **2.** 59 048 **3.** $12\frac{3}{8}$, $11\frac{13}{16}$, $-\frac{9}{16}$
4. (a) $2047\frac{7}{8}$ (b) $1228\frac{7}{8}$ **5.** 3 **6.** $-\frac{1}{4}$ **7.** $\pm 1\frac{1}{2}$ **8.** 2, $\frac{3}{4}$; −2, $-\frac{3}{4}$ **9.** 31 **10.** 2186
11. (a) 14 (b) −3 (c) 4 (d) 7·5 **12.** (a) 8 (b) 12 (c) 2·5 (d) 3·5 **13.** 24
14. (a) divergent (b) convergent, 32 (c) convergent, 56 (d) convergent, $156\frac{1}{4}$
(e) divergent (f) convergent, $51\frac{1}{3}$
15. (a) $|x| < 1$ (b) $|x| < 3$ (c) $|x| > 1$ (d) $-\frac{1}{2} < x < \frac{1}{2}$ **16.** $20\frac{1}{4}$
17. $2\frac{1}{2}$, $1\frac{1}{2}$ **18.** 2, 3 **19.** 6 **20.** £11 million **21.** $\frac{2}{3}$
22. 16 **23.** $\frac{5}{8}$ **24.** $\frac{4}{3}$, 5 **25.** 8 **26.** 7
27. (a) $\frac{7}{9}$ (b) $\frac{40}{99}$ (c) $\frac{234}{495}$ (d) $\frac{40}{111}$

page 220 **Exercise 8C**

1. (a) 1 + 2 + 3 + 4 + 5 (b) 1 + 4 + 9 + 16 + 25 (c) 13 + 15 + 17
(d) 504 + 420 + 360 + 315 + 280 + 252
(e) −1 + 2 − 3 + 4 − 5 (f) 1 − 2 + 3 − 4 + 5

2. (a) $\sum_{r=1}^{50} r$ (b) $\sum_{r=1}^{50} r^3$ (c) $\sum_{r=1}^{6} 5r$ (d) $\sum_{r=1}^{6} (2r+1)$
(e) $\sum_{r=1}^{6} (-1)^{r+1}3r$ (f) $\sum_{r=1}^{13} (-1)^r r^2$

4. (a) $\sum_{r=1}^{100} r = 5050$ (b) $\sum_{r=1}^{22} r^2 = 3795$ (c) $\sum_{r=1}^{15} r^3 = 14\,400$
(d) $\sum_{r=1}^{51} (2r-1) = 2601$ (e) $\sum_{r=1}^{20} r(r+2) = 3290$ (f) $\sum_{r=1}^{13} r(2r-1) = 1547$

5. (a) 90 000 (b) 14 400 (c) 75 600
6. (a) $\frac{1}{3}n(n+1)(n+2)$ (b) 8120
7. (a) $\frac{1}{4}n(n+1)(n+2)(n+3)$ (b) 53 130
8. (a) $\frac{1}{6}n(4n^2+15n+17)$ (b) 6390
10. (a) $n(2n+1)$ (b) $\frac{1}{6}(n+1)(n+2)(2n+3)$ (c) $n^2(2n+1)^2$ (d) $4n(n+3)$ (e) $2n^2(n+1)^2$
(f) $8n^2(2n+1)^2$ (g) $\frac{1}{3}n(n+1)(n+5)$ (h) $\frac{1}{12}n(n+1)(n+2)(3n+5)$
(i) $\frac{1}{3}n(n+1)(4n-1)$ (j) $\frac{1}{6}n(n+1)(3n^2+n-1)$

page 226 **Exercise 8D**

1. (a) $27 + 27x + 9x^2 + x^3$ (b) $125 + 150x + 60x^2 + 8x^3$
(c) $16 + 32x + 24x^2 + 8x^3 + x^4$ (d) $16 - 32x + 24x^2 - 8x^3 + x^4$
(e) $32y^5 + 80y^4x + 80y^3x^2 + 40y^2x^3 + 10yx^4 + x^5$
(f) $32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$
(g) $x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$ (h) $x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$
2. (a) $1 + 12x + 54x^2 + 108x^3 + 81x^4$ (b) $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$
(c) $64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 - 2916x^5 + 729x^6$
(d) $32x^5 + 240x^3 + 720x + \frac{1080}{x} + \frac{810}{x^3} + \frac{243}{x^5}$
3. $1 + 12x + 66x^2 + 220x^3$
4. $a^{12} + 12a^{11}x + 66a^{10}x^2 + 220a^9x^3$
5. $a^{10} - 30a^9x + 405a^8x^2 - 3240a^7x^3$
6. $1 + 10x + \frac{95}{2}x^2 + \frac{285}{2}x^3 + \frac{4845}{16}x^4$
7. $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$ (a) 40·84101 (b) 24·76099
8. (a) $1 - 8x + 25x^2 - 30x^3$ (b) $1 + 3x - 3x^2 - 25x^3$ (c) $1 - 16x + 113x^2 - 464x^3$
(d) $1 + 3x + 3x^2 + 2x^3$ (e) $1 + 16x + 111x^2 + 434x^3$ (f) $1 - 14x + 76x^2 - 195x^3$
9. (a) $1 + 6x + 21x^2 + 50x^3$ (b) $1 + 10x + 35x^2 + 40x^3$ (c) $1 - 16x + 144x^2 - 896x^3$
10. $1 - 11x + 55x^2 - 165x^3 + 330x^4$, 0·801
11. $1 + 15x + 105x^2 + 455x^3$, 1·161
12. $1 + 24x + 264x^2$, 1·3
13. $128 + 448x + 672x^2 + 560x^3$, 132·548
14. (a) 1·0937 (b) 0·9860837 (c) 0·9044 (d) 973·9
15. $\frac{1}{2}$, 1, 5, $11\frac{1}{4}$
16. 16, $\frac{1}{8}$
17. $1 - x - 11x^2$

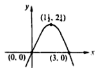
page 228 **Exercise 8E**

1. ${}^{12}C_8 \times 4^4 \times x^8$
2. ${}^{8}C_5 \times 3^3 \times 2^5 \times x^5$
3. $-{}^{9}C_3 \times 2^6 \times x^3$
4. (a) ${}^{8}C_3 \times 5^5 \times 3^3$ (b) $-{}^{7}C_3 \times 7^4 \times 2^3$
5. ${}^{7}C_5 \times 3^2 \times 2^5x^{10}$
6. (a) ${}^{10}C_5 \times 3^5$ (b) ${}^{12}C_8 \times 4^{10}$ (c) ${}^{6}C_4 \times 3^2 \times 2^4$ (d) $2^4(2 \times {}^{10}C_5 + {}^{10}C_6)$

page 268 **Exercise 10D**

1. Min at $(-5, -15)$ **2.** Max at $(1\frac{1}{2}, 7)$ **3.** Max at $(-2, 28)$ Min at $(4, -80)$
4. Min at $(-3, -156)$, Max at $(5,100)$ **5.** Max at $(-2, 32)$, Min $(2, -32)$ **6.** Min at $(0, 3)$
7. Min at $(-2, -32)$, Max at $(0, 0)$, Min at $(2, -32)$
8. Min at $(-3, -162)$, Infl at $(0, 0)$, Max at $(3, 162)$

9.



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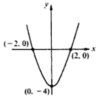

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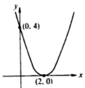

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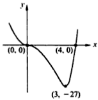
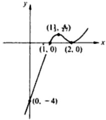
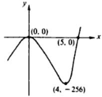


13.



14.



15. (a) Min $(2, 2)$, Max $(-1, 15\frac{1}{2})$ (b) $15\frac{1}{2}$ (c) $-10\frac{1}{2}$
16. (a) Min $(-3, -81)$, Max $(2, 44)$ (b) -145, 44
17. $\frac{3}{4}$, $1\frac{1}{4}$ **18.** $37\frac{1}{2}$, $2\frac{1}{2}$ **19.** 250, 25, 10 **20.** 10, 1, 2 **21.** 90 000 m^2, 300 m square
22. 2 m, 4 m^3 **23.** 30 cm, 30 cm **24.** 15 cm **25.** 20 km/h

page 271 **Exercise 10E**

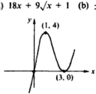
1. 0·2	**2.** 0·03	**3.** 0·02	**4.** 8%	**5.** 40π cm^3	**6.** 6%
7. 0·01 cm	**8.** (800 ± 16) cm^2	**9.** 2%	**10.** $1{\cdot}8\pi$ cm^2	**11.** 126·5	**12.** 4·02

page 273 **Exercise 10F**

1. $(0, 0)$ horizontal, $(1, -1)$ **2.** $(0, 0)$ **3.** $(\frac{1}{2}, \frac{11}{2})$ **4.** $(\frac{1}{3}, 8\frac{23}{27})$
5. $(1, -3)$ horizontal **6.** $(1, 0)$ horizontal **7.** $(4, 2)$ **8.** $(0, -5)$ horizontal
9. $(\frac{1}{3}, \frac{14}{27})$

page 273 **Exercise 10G**

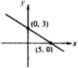
1. 1 **2.** $y = 8x - 11$, $(1\frac{3}{8}, 0)$ **3.** $2y + x + 3 = 0$, $(-3, 0)$, $(0, -1\frac{1}{2})$
4. (a) $18x + 9\sqrt{x} + 1$ (b) $\pm\frac{1}{2}$ **5.** 6, 15 **6.** -3, max **7.** $-2, 8, 10$
8.



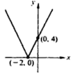
9. (i) (a) 5 cm × 5 cm × 5 cm (b) 125 cm^3
(ii) $8x - \frac{27}{x^2}$, $1\frac{1}{2}$, min
10. $300r - \pi r^3$, $300 - 3\pi r^2$, $-6\pi r$, $\frac{10}{\sqrt{\pi}}$, 2 : 1
12. $6x^2 - 14x$ (i) $y + 4x = 11$ (ii) 0·12 decrease
13. $\frac{11}{36}$ **14.** (i) (2, 4), 4 (ii) 1%

page 281 **Exercise 11A**

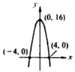
1. (a), (c), (e)
2. A(0, 6), B(3, 0); C(−3, 0), D(1, 0), E(0, −3), F(−1, −4); G(−4, 0), H(2, 0), I(0, 8), J(−1, 9)
3. A(−2, 0), $a = 2$; B(2, 0), $b = 1$; C(−2, 0), D(2, 0), $c = -3$
4. (a)



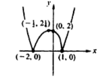
(b)



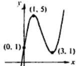
(c)



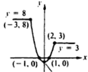
(d)



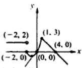
5.



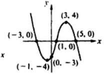
6. (a)



(b)



(c)



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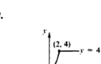

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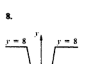

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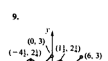
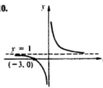
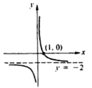


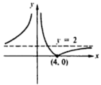
10.



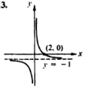
11.



12.



13.



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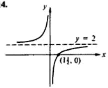

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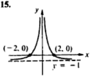
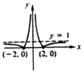


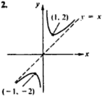
16.



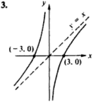
page 284 Exercise 11B

1. A(−2, 0), B(2, 0), $a = 2$; C($\frac{1}{2}$, 0), $b = 2$; D(−1, 0), $c = 1$

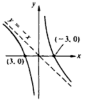
2.



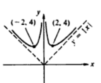
3.



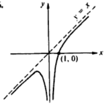
4.



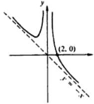
5.



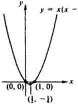
6.



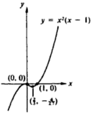
7.



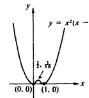
2. (a)



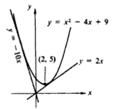
(b)



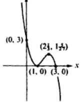
(c)



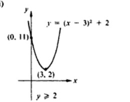
3. $m < -10, m > 2$



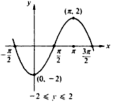
4. (i) $-1, 5, -7, 3$ (iii) $2\frac{1}{3}$
(iv)



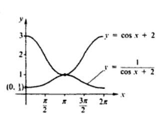
5. (i)



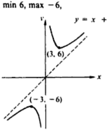
(ii)



6.



7. $1 - \frac{9}{x^2}$, min 6, max -6,



page 296 **Exercise 12A**

1. (a) $x^3 + c$ (b) $x^2 + c$ (c) $\frac{1}{4}x^4 + c$ (d) $\frac{3}{5}x^5 + c$
(e) $5x + c$ (f) $\frac{1}{2}x^6 + c$ (g) $\frac{2}{3}x^{3/2} + c$ (h) $-\frac{6}{x} + c$
(i) $\frac{2}{x^2} + c$ (j) $2\sqrt{x} + c$ (k) $\frac{1}{4}x^4 + x^3 + c$ (l) $\frac{5}{2}x^2 + x + c$
(m) $x^2 + 3x^3 + c$ (n) $x^5 - 3x^2 + c$ (o) $2x^4 - 4x^3 + c$ (p) $2x^2 - x^3 + c$
(q) $x^3 - 3x^2 + c$ (r) $\frac{3}{5}x^5 - 4x^2 + c$ (s) $x^3 + x^2 - x + c$ (t) $\frac{1}{3}x^3 - 4x^2 + 12x + c$

2. (a) $2x^4 + c$ (b) $6x^2 + c$ (c) $\frac{5}{3}x^3 + c$ (d) $7x + c$ (e) $7x - x^2 + c$
(f) $-\frac{3}{x^2} + c$ (g) $\frac{3}{x^4} + c$ (h) $2x^{3/2} + c$ (i) $2x^{5/2} + c$ (j) $x^3 - \frac{6}{x} + c$
(k) $x^4 + x^3 + x^2 + x + c$ (l) $2x^3 - 2x^4 + c$ (m) $\frac{x^5}{5} + \frac{x^3}{3} - \frac{1}{x} - \frac{1}{3x^3} + c$

3. (a) $6x^2 + c$ (b) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + c$ (c) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + c$
(d) $\frac{1}{3}x^3 + x^2 - 24x + c$ (e) $-\frac{5}{3x^3} + c$ (f) $2x^5 + 2x^4 + \frac{6}{x} + c$
(g) $\frac{x^3}{3} - \frac{1}{x} + c$ (h) $2\sqrt{x} - 2x^{3/2} + c$ (i) $\frac{x^3}{3} + \frac{1}{x} + c$

4. $y = 3x^2 + 1$ **5.** $y = x^3 - 2x + 10$ **6.** $y = 8 + 2x - x^2$
7. $s = 2t^3 + 6t^2 + t - 1$ **8.** $V = 7h^2 - 4h + 1$ **9.** $A = 5p - 2p^2 + 1$
10. $y = x^3 + 8x - 1$ **11.** $y = x^2 - 3x - 10, (-2, 0)$ **12.** $y = x^2 - 4x + 7$
13. $y = 7 + 2x^2 - x^3$ **14.** $y = x^3 - 2x^2 + 5$ **15.** $y = 5x^3 - 6x + 4$

page 303 **Exercise 12B**

1. (a) 24 (b) 8 (c) 15 (d) 2 (e) 4 (f) $17\frac{1}{3}$ (g) $\frac{1}{2}$ (h) 2 (i) 32 (j) $13\frac{3}{4}$

2. (a) A(−1, 0), B(0, 4), C(4, 0), $20\frac{5}{6}$ sq. units
(b) A(−1, 0), B(0, −5), C(5, 0), 36 sq. units
(c) A(0, 4), B(1, 0), C(4, 0), $6\frac{1}{3}$ sq. units

3. (a) $\frac{1}{x}$ not defined for $x = 0$, (b) $\frac{1}{x^2}$ not defined for $x = 0$,
(c) $\frac{1}{x-1}$ not defined for $x = 1$, (d) $\frac{1}{x^2-1}$ not defined for $x = -1$
(e) $\sqrt{x}$ not defined for $x < 0$

4. 26 sq. units **5.** $3\frac{3}{4}$ sq. units **6.** $12\frac{2}{3}$ sq. units **7.** $31\frac{1}{2}$ sq. units
8. $10\frac{2}{3}$ sq. units **9.** 36 sq. units **10.** 3 sq. units **11.** 13 sq. units
12. $7\frac{1}{3}$ sq. units **13.** 8 sq. units **14.** $1\frac{1}{3}$ sq. units
15. (a) $5\frac{1}{3}$ sq. units (b) $25\frac{1}{4}$ sq. units (c) 2 sq. units **16.** $10\frac{2}{3}$ sq. units
17. $2\frac{2}{3}$ sq. units **18.** $\frac{1}{6}$ sq. units **19.** A(−2, 6), B(2, 10), $10\frac{2}{3}$ sq. units
20. A(−4, 0), B(−1, 0), C(2, 0), D(0, −8), E(−5, −28), F(3, 28), 64 sq. units
21. $21\frac{1}{3}$ sq. units **22.** $41\frac{2}{3}$ sq. units

page 307 **Exercise 12C**

1. $\frac{32}{5}\pi$ cu. units **2.** $\frac{20}{3}\pi$ cu. units **3.** 13π cu. units **4.** $\frac{128}{7}\pi$ cu. units
5. $\frac{16}{15}\pi$ cu. units **6.** $\frac{16}{7}\pi$ cu. units **7.** $\frac{512}{15}\pi$ cu. units **8.** $\frac{1}{105}\pi$ cu. units
9. $\frac{3}{8}\pi$ cu. units **10.** $\frac{32}{5}\pi$ cu. units **11.** $y = \frac{rx}{h}$, $V = \frac{1}{3}\pi r^2 h$
12. 8π cu. units **13.** $\frac{15}{4}\pi$ cu. units **14.** $\frac{56}{3}\pi$ cu. units

page 319 **Exercise 13A**

1. $12(3x+2)^3$ **2.** $10(2x-3)^4$ **3.** $-28(1-4x)^6$ **4.** $45(2+9x)^4$
5. $10x(3+x^2)^4$ **6.** $-24x^2(1-x^3)^7$ **7.** $48x^3(2x^4+1)^5$ **8.** $20x(x^2+5)^9$
9. $\dfrac{-9}{(1+3x)^4}$ **10.** $\dfrac{8x}{(1-4x^2)^2}$ **11.** $\dfrac{5}{2\sqrt{(5x-3)}}$ **12.** $\dfrac{-3x}{(x^2+1)^{5/2}}$
13. $\dfrac{2}{(1-2x)^2}$ **14.** $-\dfrac{2x}{(x^2+3)^2}$ **15.** $\dfrac{x+2}{\sqrt{(x^2+4x+1)}}$ **16.** $-\dfrac{x}{(x^2-5)^{3/2}}$
17. $-\dfrac{1}{2\sqrt{x}(1+\sqrt{x})^2}$ **18.** $40x(1+(1+x^2)^5)^3(1+x^2)^4$ **19.** $\dfrac{1}{4\sqrt{x}\sqrt{(2+\sqrt{x})}}$
20. $-\dfrac{9x}{(3x^2+1)^{5/2}}$ **21.** (a) $y=12x-23$ (b) $12y=14-x$
22. (a) $y=4x-9$ (b) $4y+x+2=0$ **23.** (a) $8y=x+3$ (b) $4y+32x+31=0$
24. (a) $3y=x+5$ (b) $y+3x=15$
25. (a) Max at (0, 1) (b) Min at $(2\frac{1}{2}, 0)$ (c) Infl at $(2\frac{1}{2}, 0)$ (d) Max at $(-2, -\frac{1}{4})$
26. (a) 5 m (b) 0·2 m/s (c) $-0{\cdot}008$ m/s^2

page 321 **Exercise 13B**

1. $(4+3x)^8+c$ **2.** $(2+7x)^6+c$ **3.** $(3-2x)^7+c$ **4.** $(x^2+4)^5+c$
5. $\frac{1}{18}(2+3x)^6+c$ **6.** $\frac{1}{10}(1+2x)^5+c$ **7.** $-\frac{1}{24}(1-6x)^4+c$ **8.** $\frac{1}{10}(3+4x)^5+c$
9. $-\frac{3}{5}(1-2x)^5+c$ **10.** $\frac{1}{2}(1+x^2)^4+c$ **11.** $\frac{1}{15}(x^3-1)^5+c$ **12.** $\frac{1}{16}(x^4+6)^4+c$
13. $\frac{1}{5}(x^2+x+3)^5+c$ **14.** $\frac{1}{5}(x^3-x+4)^5+c$ **15.** $\frac{1}{6}(4+x+x^2)^6+c$
16. $\frac{1}{16}(x^2-2x+4)^8+c$ **17.** 1342 **18.** -7776 **19.** 210·1
20. 1302 **21.** $121\frac{1}{3}$ **22.** 20 **23.** $221\frac{2}{3}$

page 324 **Exercise 13C**

1. 77 **2.** $5(2t+3)^3$ **3.** $4\pi r$ cm^2/s **4.** 40π cm^2/s **5.** $\dfrac{3}{2\pi}$ cm/s
6. 3 cm^2/s **7.** 0·1 cm/s **8.** (a) $\frac{8}{3}\pi r$ cm^2/s (b) $\frac{9}{16}\pi r^2$ cm^3/s **9.** 0·25 cm/s
10. (a) 0·01 cm/s (b) $0{\cdot}1\pi$ cm^2/s **11.** $2s(s^2-2)$

page 327 **Exercise 13D**

1. $y=x^4+4$ **2.** $y=x^2-8x+6$ **3.** $y=\dfrac{3}{x-2}$ **4.** $x+2y=11$
5. $y=x^2-6x$ **6.** $9y=(x+7)(x-8)$ **7.** $y=\dfrac{x}{2x+1}$ **8.** $x+3y=7xy$
9. $x^2+4y^2=4$ **10.** $x^2=4y^2(1-y^2)$ **11.** $5x^2+5y^2-8xy=9$
12. $\frac{1}{2}t$ **13.** $\frac{1}{t}$ **14.** $\frac{1}{2}t$ **15.** $\dfrac{1}{4t^{3/2}}$
16. $-\dfrac{1}{3t^2}$ **17.** $\dfrac{3}{2t+3}$ **18.** $\left(\dfrac{t-1}{t+1}\right)^2$ **19.** $-2t(t+1)^2$
20. (a) $y=6x-2$ (b) $6y+x=99$ **21.** (a) $y+8x=5$ (b) $16y=2x+15$
22. (a) $y=x+3$ (b) $y+x+5=0$ **23.** (a) $y=6x-5$ (b) $6y+x=44$
24. $(49, -9)$ **25.** (a) 2 (b) $\dfrac{3}{16t^5}$ (c) $\dfrac{3}{4(t+1)}$

26.

t	-4	-3	-2	-1	0	1	2	3	4
x	32	18	8	2	0	2	8	18	32
y	-16	-12	-8	-4	0	4	8	12	16

27.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	0	2	3·46	4	3·46	2	0	-2	$-3·46$	-4	$-3·46$	-2	0
y	1	0·87	0·5	0	$-0·5$	$-0·87$	-1	$-0·87$	$-0·5$	0	0·5	0·87	1

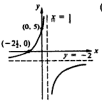
28.

t	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
x	-12	$-3\frac{3}{4}$	0	$\frac{3}{4}$	0	$-\frac{3}{4}$	0	$3\frac{3}{4}$	12
y	16	9	4	1	0	1	4	9	16

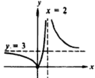
page 330 **Exercise 13E**

1. $(16x + 5)(2x + 5)^6$ **2.** $2x(9x + 5)(2x + 5)^6$ **3.** $3x^3(3x + 4)(x + 3)^4$
4. $2x^3(7x^2 + 6)(x^2 + 3)^4$ **5.** $x^2(9 - 23x^4)(3 - x^4)^4$
6. $2x(x^4 + 1)(5x^4 - 28x^2 + 1)$ **7.** $3(2x^2 + 1)^2(3x - 1)^6(26x^2 - 4x + 7)$
8. $3(30x + 11)(2x - 1)^5(3x + 5)^8$ **9.** $(19 - 16x)(2x + 1)^2(4 - x)^4$
10. $\frac{3(x + 2)}{2\sqrt{(x + 3)}}$ **11.** $\frac{x(5x + 12)}{2\sqrt{(x + 3)}}$ **12.** $\frac{3(3x + 1)(x + 3)^3}{2\sqrt{x}}$
13. $\frac{2(x + 3)(3x^2 + 3x + 1)}{\sqrt{(2x^2 + 1)}}$ **14.** $\frac{6}{(x + 3)^2}$ **15.** $-\frac{15}{(x - 5)^2}$
16. $\frac{11}{(2x + 3)^2}$ **17.** $-\frac{13}{(3x + 2)^2}$ **18.** $\frac{2(2 + x)(2 - x)}{(x^2 + 4)^2}$ **19.** $\frac{(x - 3)(x + 1)}{(x^2 + 3)^2}$
20. $\frac{(x - 8)(x + 2)}{(x^2 + 16)^2}$ **21.** $\frac{x^4(2x + 5)}{(x + 1)^4}$ **22.** $\frac{3(3x + 2)}{(2 - x)^5}$
23. $\frac{-6(2x + 11)(6x + 5)}{(2x - 3)^4}$ **24.** $\frac{4x - 7}{(2x - 1)^{3/2}}$
25. $60(2x + 3)^3(2x + 1)$ **26.** (a) $\frac{1}{(x + 1)^2}$ (b) 1 (c) $-\frac{2}{(x + 1)^3}$ (d) -2
27. $\left(\frac{t + 6}{2 - t}\right)^2$ **28.** $\frac{3(t + 1)^2}{t(t + 2)^3}$ **29.** (a) $y = 24x - 44$ (b) $24y + x = 98$
30. (a) $y = 5x - 8$ (b) $5y + x = 12$ **31.** (a) $2y = 3x - 7$ (b) $3y + 2x + 4 = 0$
32. (a) $3y = 8x - 4$ (b) $8y + 3x = 38$ **33.** Max at (1, 256), Min at (5, 0).
34. Min at (0, 0), Max at (1, 16), Min at (3, 0) **35.** Max at $(-2, \frac{1}{4})$, Min at $(4, -\frac{1}{8})$
36. Infl at (0, 0), Min at $(5, 115\frac{20}{27})$ **37.** $4\left(\frac{t + 1}{t - 1}\right)^3$ **38.** $\frac{10}{9}\left(\frac{t + 3}{t - 2}\right)^3$

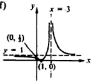
(d)



(e)



(f)



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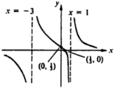
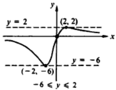


4. $-6 \leqslant y \leqslant 2$



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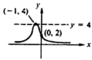

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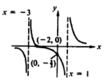

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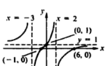

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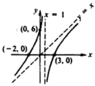
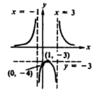
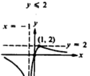


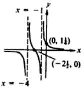
9. $y \leqslant -3, y > 0$



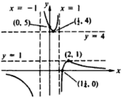
10. $y \leqslant 2$



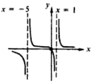
11. $y \in \mathbb{R}$



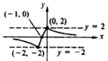
12. $y \leqslant 1, y \geqslant 4$



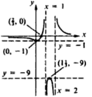
13. $y \in \mathbb{R}$



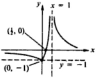
14. $-2 \leqslant y \leqslant 2$



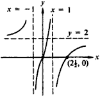
15. $y \leqslant -9, y \geqslant -1$



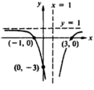
16. $y \geqslant -1$



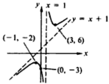
17. $y \in \mathbb{R}$



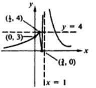
18. $y < 1$



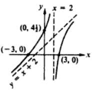
19. $y \leqslant -2, y \geqslant 6$



20. $y \geqslant 0$

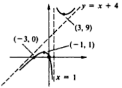


21. $k \geqslant 4$



22. $k \geqslant 2$

23. $0 \leqslant k \leqslant 3$; $f(x) \leqslant 1, f(x) \geqslant 9$

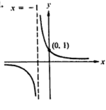


page 348 *Exercise 14B*

1. (a) $\{x \in \mathbb{R}: x < 2 \quad \text{or} \quad x > 5\}$ (b) $\{x \in \mathbb{R}: -1 < x < 2 \quad \text{or} \quad x > 3\}$ **2.** $x < 1, x > 2$
3. $-2 < x < 3$ **4.** $-5 < x < 0, x > 3$ **5.** $x < 0, 1\frac{1}{2} < x < 5$ **6.** $\frac{1}{2} < x < 4$
7. $x < -2, 2 < x < 4$ **8.** $x < -3, -1 < x < 1, x > 3$ **9.** $-2 < x < -\frac{1}{2}, \frac{1}{2} < x < 2$
10. $x < -2, x > 2$ **11.** $2 < x < 4$ **12.** $-6 < x < 2\frac{1}{2}$ **13.** $-2 < x < 1, x > 2$
14. $x < -1, 0 < x < 3$ **15.** $x > 1$ but $\neq 3$ **16.** $x > -1$ **17.** $x < -2, 0 < x < 3$
18. $-2 < x < 0, x > 3$ **19.** $x < -2, 1 < x < 5$ **20.** $x < -2, 0 < x < 4, x > 8$
21. $-10 < x < -1, x > 2$ **22.** $x < -5, -2 < x < 1$ **23.** $x < -4, -1 < x < 2, x > 6$
24. $x < -6, -1 < x < \frac{1}{2}, x > 2$ **25.** $-4 < x < 2, 5 < x < 8$ **26.** $0 < x < 2$
27. $x < 0, x > 3$
28. (a) $\{x \in \mathbb{R}: -7 < x < 1\}$ (b) $\{x \in \mathbb{R}: x > 3\}$ (c) $\{x \in \mathbb{R}: x < -2 \quad \text{or} \quad x > 0\}$
29. $x < -1, x > 4$ **30.** $-2 < x < 2$ **31.** $x < -2, x > 4$ **32.** $-3 < x < -1$
33. $x < -4, x > -2$ **34.** $-1\frac{1}{2} < x < \frac{1}{2}$ **35.** $x > 1$ **36.** $-2 < x < 2$
37. $x < -6, x > -4\frac{1}{2}$ **38.** $0 < x < 1\frac{1}{2}$ **39.** $x < 2, x > 6$ **40.** $-2 < x < -\frac{2}{3}$
41. $x < 2, x > 6$ **42.** $0 < x < 2, x > 4$ **43.** $x < -4, x > 0$

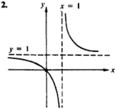
page 349 *Exercise 14C*

1.



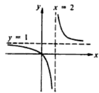
(i) $y + x = 1$,
(ii) $y + x + 3 = 0$.

2.



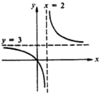
$x < \frac{1}{2}, x > 1$.

3.

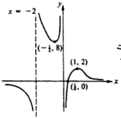


$y = \dfrac{x}{x + 2}$.

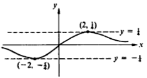
4. $x = 2, y = 3$,



5. $-\frac{1}{2}$,



6.



5. (a) $\sqrt{2}$, 45° (b) 5, 53·1° (c) 17, 28·1° (d) 5, 323·1° (e) $\sqrt{13}$, 123·7° (f) $\sqrt{29}$, 21·8°
6. (a) 118·1°, 323·1° (b) −46·4°, −36·9°, 313·6°, 323·1°
7. (a) 11°, 225·2° (b) 244·1°, 352° (c) 36·9°, 270° (d) 77·9°, 325·7° (e) 15·7°, 282·4° (f) 142·5°, 172·5°, 322·5°, 352·5°
8. (a) −50·5°, 103·6° (b) −17·6°, 130·2° (c) 24·6°, 171·7° (d) −169·1°, 51° (e) −153·9°, 78·1° (f) ±180°, −29·6°, 0, 103·3°

page 360 *Exercise 15C*

1. (a) 0 (b) 60° (c) 30° (d) −30° (e) 120° (f) −60° (g) 60° (h) −60°
2. (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{6}$ (e) $\frac{\pi}{4}$ (f) $\frac{\pi}{6}$ (g) $-\frac{\pi}{4}$ (h) $\frac{3\pi}{4}$

3.

x	0	0·25	0·5	0·75	1	1·25	1·5	1·75	2	2·25	2·5	2·75	3	3·14
y	1	0·97	0·88	0·73	0·54	0·32	0·07	−0·18	−0·42	−0·63	−0·80	−0·92	−0·99	−1

4. (a) $\frac{36}{85}$ (b) $\frac{220}{221}$ (c) $\frac{4}{5}$ (d) $-\frac{13}{85}$ (e) $\frac{297}{425}$ **5.** (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
8. $\frac{1}{6}$ **9.** 0·425 **10.** $\frac{1}{3}$ **11.** $\frac{1}{6}$ **12.** $\sqrt{(6-4\sqrt{2})}$

page 365 *Exercise 15D*

1. $2n\pi \pm \frac{\pi}{3}$ **2.** $n\pi - \frac{\pi}{4}$ **3.** $n\pi + (-1)^{n+1}\frac{\pi}{3}$ **4.** $\frac{2n\pi}{3}$ **5.** $\frac{n\pi}{2} - \frac{\pi}{12}$
6. $\frac{n\pi}{5} + (-1)^n\frac{\pi}{30}$ **7.** $n\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{4}$ **8.** $n\pi + (-1)^{n+1}\frac{\pi}{6}, 2n\pi + \frac{\pi}{2}$
9. $\frac{n\pi}{5}, \frac{n\pi}{2}$ **10.** $\frac{n\pi}{6}$ **11.** $n\pi, (2n+1)\frac{\pi}{8}$ **12.** $\frac{n\pi}{2}$
13. $\frac{1}{3}\left(2n\pi - \frac{\pi}{2}\right), \frac{1}{7}\left(2n\pi - \frac{\pi}{2}\right)$ **14.** $\frac{1}{7}\left(2n\pi + \frac{\pi}{2}\right), 2n\pi + \frac{\pi}{2}$ **15.** $n\pi - \frac{\pi}{3}, \frac{n\pi}{3} - \frac{\pi}{18}$
16. $\frac{n\pi}{4} + \frac{\pi}{16}$ **17.** $180°n + (-1)^n 5·7°$ **18.** $180°n \pm 26·6°$
19. $360°n \pm 120°, 360°n \pm 48·2°$ **20.** $180°n, 360°n \pm 75·5°$
21. $360°n + 115·3°, 360°n - 41·6°$ **22.** $360°n + 46·4°, 360°n - 90°$

page 366 *Exercise 15E*

1.

$\sin\theta$	$\cos\theta$	$\tan\theta$
0·1	0·995	0·1
0·0998	0·995	0·1003

$\sin\theta$	$\cos\theta$	$\tan\theta$
0·02	0·9998	0·02
0·02	0·9998	0·02

2. (a) 2 (b) 1 (c) 4 (d) 2
3. (a) $-\frac{1}{2}\theta$ (b) $\frac{1}{2\theta}$ (c) 4θ (d) $\frac{1}{4}\sqrt{2}(2 + 2\theta - \theta^2)$ (e) $\frac{\sqrt{3}}{2} - \frac{1}{2}\theta - \frac{\sqrt{3}}{4}\theta^2$ (f) $1 - 2\theta + 4\theta^2$
4. (a) 0·035 (b) 0·0035 (c) 0·999 847

page 381 ***Exercise 16A***

1. (a) $\frac{1}{3}$ (b) $\frac{3}{4}$ (c) $5\frac{1}{2}$ **2.** 26·6°, 45°, 108·4° **3.** $\frac{1}{3}$, 3 **4.** 36° **5.** 27°
6. $2y = x + 2,\ y + 2x = 16$

page 383 ***Exercise 16B***

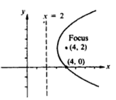
1. (a) $1\frac{2}{3}$ (b) $\frac{8}{13}$ (c) $5\sqrt{2}$ (d) $\sqrt{5}$ (e) $\frac{1}{2}\sqrt{5}$ **2.** A, C, $\frac{4}{13}\sqrt{13}$ 3. $-2,\ -7\frac{1}{4}$
4. (a) $3y = 4x + 10$ (b) 5 units (c) $2\frac{1}{3}$ units (d) $5\frac{1}{2}$ sq. units
5. $x + 4y + 1 = 0,\ y = 4x + 1$
6. (a) $y = 2x + 1,\ x + 2y + 2 = 0$ (b) $x + 3y = 1,\ y = 3x + 2$
(c) $7x + 4y = 18,\ 7y = 4x - 1$

page 390 ***Exercise 16C***

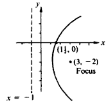
1. (b) (c) (f) **2.** (a) $(0, 0), \sqrt{10}$ (b) $(3, 0), 5$ (c) $(2, -1), 3\sqrt{2}$
(d) $(-1, 1), 2$ (e) $(3, -1), 4$ (f) $(-2, 0), \sqrt{10}$ (g) $(-\frac{1}{2}, 0), 5$
(h) $(\frac{1}{2}, -\frac{1}{2}), 1$ **3.** (2, 3) **8.** $4\sqrt{2}$ **9.** $\sqrt{10}$
10. (a) (i) $3y = x - 20$ (ii) $y + 3x = 0$ (b) (i) $y + 5x = 17$ (ii) $5y = x + 7$
(c) (i) $4y + 2x = 7$ (ii) $y = 2x - 2$
11. (a) $x^2 + y^2 - 4x - 4y + 3 = 0$ (b) $x^2 + y^2 + 9x + y = 22$
(c) $2x^2 + 2y^2 + x - 10y + 2 = 0$
13. (a) external (b) external (c) internal (d) external **14.** $3x + y = 15$ **15.** $y = 7x - 37$
16. $(y - 3)\sin\theta + (x - 2)\cos\theta = 4,\ y\sqrt{3} + x = 10 + 3\sqrt{3}$
17. $x^2 + y^2 - 4x - 22y + 115 = 0$
18. $x^2 + y^2 - 2x + 2y = 8$ **19.** $x^2 + y^2 - 8x - 4y + 15 = 0$

page 395 ***Exercise 16D***

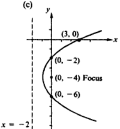
1. (a) (1, 0) (b) (−2, 0) (c) (5, 0) (d) $(2\frac{1}{4}, 0)$
2. (a) $x = -3$ (b) $x = 3$ (c) $x = -5$ (d) $x = \frac{1}{2}$
3. (a)



(b)



(c)



4. (1, 4) **5.** $(-\frac{1}{4}, -1)$ **6.** (2, 8) (18, 24) **7.** $(1, -2), (9, 6),\ y + x + 1 = 0,\ 3y = x + 9$
8. $2y = x + 8,\ y + 2x + 1 = 0$ **9.** $2y = x + 12,\ 2y + 3x + 4 = 0$
10. $yt = x + at^2$ **11.** $y + tx = 2at + at^3$; 0, ±2
12. (a) $[a(2 + p^2), 0]$ (b) $[0, ap(2 + p^2)]$ **13.** $4y^2 + 1 = 4(x + y)$
14. $y(p + q) = 2x + 2apq$ **15.** −1, 3 **19.** $y^2 = 2ax + 4a^3$

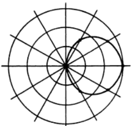
page 400 *Exercise 16E*

1. (a) $(\pm 2\sqrt{2}, 0), \frac{8}{9}$ (b) $(\pm 3, 0), \frac{9}{25}$ (c) $(\pm 2\sqrt{3}, 0), \frac{3}{4}$
2. (a) $(\pm 5, 0), \frac{25}{16}$ (b) $(\pm \sqrt{10}, 0)$, 10 (c) $(\pm 2\sqrt{17}, 0)$, 17
3. (a) $2y + \sqrt{3}\,x = 4$ (b) $3y = 2x + 25$ (c) $4y + 15x + 9 = 0$ (d) $y + 2\sqrt{2}\,x = 2$
(e) $y + x + 6 = 0$ (f) $y + 4x = 16$
4. $bx \cos\theta + ay \sin\theta = ab$, $(0, b \operatorname{cosec}\theta)$
5. (a) $yt_1^2 + x = 2ct_1$ (b) $yt_2 = xt_2^3 - ct_2^4 + c$
6. (a) $yt_1t_2 + x = c(t_1 + t_2)$ (b) $yt^2 + x = 2ct$

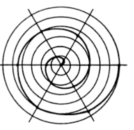
page 404 *Exercise 16F*

1. $A(2, \frac{1}{3}\pi)$, $B(3, \frac{1}{2}\pi)$, $C(1\frac{1}{2}, \frac{2}{3}\pi)$, $D(1, \pi)$, $E(3, \frac{7}{6}\pi)$, $F(2\frac{1}{2}, \frac{3}{2}\pi)$, $G(1, \frac{5}{3}\pi)$, $H(2, \frac{11}{6}\pi)$
2. $P(2\sqrt{3}, 2)$, $Q(0, 5)$, $R(0, -2)$, $S(-5, 5)$, $T(-3, -3\sqrt{3})$, $U(0, -3)$
3. (a) $r = 4 \sec\theta$ (b) $r = 4 \sin\theta$ (c) $r^2 = \operatorname{cosec} 2\theta$
(d) $r = 8 \cot\theta \operatorname{cosec}\theta$ (e) $r = 2\cos\theta$ (f) $r = 4\cos\theta$
(g) $r^2 = 8 \sec 2\theta$ (h) $r^2 = \dfrac{4}{1 - \sin 2\theta}$
4. (a) $y = 1$ (b) $x^2 + y^2 = 9$ (c) $y = \sqrt{3}\,x$ (d) $x^2 + y^2 = y$
(e) $x^2 + y^2 = x + y$ (f) $x^2 - y^2 = 1$ (g) $x^4 + x^2y^2 = 9y^2$ (h) $x^2 = 9 + 6y$

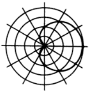
5.



6.



7.



page 405 *Exercise 16G*

1. $(1, 1\frac{1}{3})$, $(3, 0)$, $(2, \frac{2}{3})$, $78{\cdot}7^\circ$
2. $3ty + 2x = t^2(2 + 3t^2)$, $\frac{1}{8}$
3. (i) $(1\frac{1}{2}, 0)$ (ii) $2\frac{1}{2}$ units (iii) $(0,2)$ $(0, -2)$, $(-1, 0)$, $(4, 0)$; $(3, 2)$, $(5\frac{2}{3}, 0)$
4. $(-1, 4\frac{1}{2})$, $\frac{1}{2}\sqrt{65}$
5. 10 units, 26 units
6. $\frac{1}{2}(6 \pm \sqrt{6})$, $\frac{23}{25}$
7. $(x - 1)^2 + y^2 = 1$, $(x - 5)^2 + (y + 4)^2 = 25$, $(1, -1)$
8. $x^2 + y^2 = 3ax$, $y^2 = 3ax$
9. $(4a, 0)$, $2a$, $(2a, 0)$, $(6a, 0)$, 60°, $\sqrt{3}\,y = \pm x$, $\frac{4}{3}a^2\sqrt{3}$
10. $(\frac{81}{4}a, -9a)$
11. $[apq, a(p + q)]$
12. (b) $2ay = x^2 + a^2$
13. (i) $4a$ (ii) $y^2 = 2ax - 8a^2$
14. $4x^2 + 4y^2 - 8x + 3 = 0$
15. $2m + c = 3$
16. $\dfrac{ab}{\sin 2\theta}$
17. $a^2 - b^2 = 2ax - 2by$ (ii) $(0, -b)$, $\left(\dfrac{3}{5}\sqrt{3}\,b, \dfrac{4}{5}b\right)$
18. $x + yt^2 = 6t$, $(-1\frac{1}{2}, -6)$, $(3\frac{3}{4}, 2\frac{2}{3})$, $\frac{21}{20}\sqrt{89}$
21. $r = 4\cos\theta$, $(4, 0)$, $(2\sqrt{2}, \frac{1}{4}\pi)$, $\theta = -0{\cdot}142$ rads, $\theta = \dfrac{\pi}{4}$ rads

page 412 **Exercise 17A**

1. (a) 11 (b) 7 (c) 3 (d) 12 (e) 16 (f) 81° (g) 40°

2. (a) $\sqrt{11}$ (b) $\sqrt{6}$ (c) 76° **3.** $\frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$ **4.** $\begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{7}{9} \end{pmatrix}$

5. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is one example **6.** $\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ is one example

7. $\mathbf{c}$ and $\mathbf{d}$ **8.** $14 : -2 : 5$, $\frac{14}{15}, -\frac{2}{15}, \frac{1}{3}$ **9.** $1 : 2 : -2$
10. (a) $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ or any multiple thereof (b) $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$ (c) $\frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
15. $36^\circ, 68^\circ, 76^\circ$ **17.** $4\mathbf{a} + \mathbf{b} - 2\mathbf{c}$ **18.** $3\mathbf{a} - \mathbf{b} + 2\mathbf{c}$

page 414 **Exercise 17B**

1. (a) $3\mathbf{i} + 8t\mathbf{j}$ (b) $3\mathbf{i} + 2\mathbf{j} - 10t\mathbf{k}$ (c) $2\mathbf{j}$ (d) $6\mathbf{i} + 9t^2\mathbf{j} - 6\mathbf{k}$
2. (a) $\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$ (b) $-2\sin\theta\mathbf{i} + 2\mathbf{j}$ (c) $\cos\theta\mathbf{i} + \sin\theta\mathbf{j} + 2\theta\mathbf{k}$
(d) $3\cos 3\theta\mathbf{j} + 3\cos^2\theta\sin\theta\mathbf{k}$
3. $16\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $24\mathbf{i} - 2\mathbf{j}$, $24\mathbf{i}$ **4.** $10\mathbf{i} + 28\mathbf{j} + 6\mathbf{k}$ **5.** $2\pi\mathbf{i}$ m/s, 2π m/s
6. $2\mathbf{i} + (2t^2 + 1)\mathbf{j} - 4\mathbf{k}$, $2t\mathbf{i} + (\frac{2}{3}t^3 + t)\mathbf{j} + (3 - 4t)\mathbf{k}$

page 420 **Exercise 17C**

1. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ **2.** $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

3. $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ or multiples thereof **5.** B and C **6.** F **7.** 6, 8
8. (a) $3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ (b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$
9. (a) $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z+1}{1}$ (b) $\frac{x-3}{3} = \frac{y+1}{2} = \frac{z-2}{-4}$ (c) $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{-1}$
10. $\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$
11. (a) $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ (b) $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

12. $-4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ **13.** $\begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$

14. (a) (5, 0, 1) (b) lines do not intersect (c) (4, 5, 9) (d) (12, −3, 3)
15. (a) parallel (b) non parallel coplanar (c) skew (d) non parallel coplanar
16. 79° **17.** 69° **18.** (a) $\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ (b) $-4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ (c) $5\sqrt{5}$ units
20. (a) $\sqrt{5}$ units (b) $3\sqrt{2}$ units (c) $\sqrt{11}$ units (d) $4\sqrt{6}$ units
21. (a) $\frac{1}{3}\sqrt{21}$ units (b) $\frac{1}{2}\sqrt{14}$ units
22. $\frac{2}{5}\sqrt{35}$ units **23.** (a) $\sqrt{3}$ units (b) $\frac{4}{7}\sqrt{21}$ units

page 429 **Exercise 17D**

4. $\mathbf{r} . (\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 7$ **5.** $\mathbf{r} . (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 5$ **6.** $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$
7. (a) and (c) **8.** (a) and (b)
9. (a) $\mathbf{r} . (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4$ (b) $\mathbf{r} . (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 5$ **10.** $x + 2y - 3z = 0$
11. $3y + z = 10$ **12.** $3x + 2y + z = 6$ **13.** $\frac{2}{3}$ units **14.** 3 units

15. 2 units **16.** $\begin{pmatrix} 12 \\ 5 \\ 7 \end{pmatrix}$ **17.** $5\mathbf{i} - 5\mathbf{k}$ **20.** (1, −5, 1)

28. $\frac{2}{x-1} - \frac{5}{2(2x-1)} + \frac{1}{2(2x-1)^2}$ **29.** $1 + \frac{1}{x-3} - \frac{1}{2x+5}$

30. $2 + \frac{x-5}{x^2+4} + \frac{5}{x-3}$ **31.** $\frac{1}{x-3} - \frac{2}{x+7} + \frac{2}{2x-3}$

32. $\frac{2}{1+2x} + \frac{3-x}{x^2-x+2}$ **33.** $2 - \frac{1}{2(x+3)} - \frac{3}{2(x-1)} + \frac{12}{(x-1)^2}$

34. $\frac{2}{1+x} + \frac{3}{(1+x)^2} - \frac{5}{(1+x)^3} + \frac{1}{x-2}$ **35.** $\frac{6}{x} - \frac{4}{x-2} + \frac{1}{x+2}$

36. $\frac{3}{x+3} + \frac{4}{(x+3)^2} + \frac{1-2x}{x^2+4}$

page 456 ***Exercise 18B***

1. $\frac{1}{1-x} + \frac{1}{1+2x}$, $2 - x + 5x^2 - 7x^3 + \ldots + [(-2)^r + 1]\,x^r + \ldots, |x| < \frac{1}{2}$

2. $\frac{3}{1+x} - \frac{2}{1+3x}$, $1 + 3x - 15x^2 + 51x^3 - \ldots + (-1)^r 3[1 - 2\times 3^{r-1}]\,x^r + \ldots, |x| < \frac{1}{3}$

3. $\frac{1}{1-3x} - \frac{1}{1+5x}$, $8x - 16x^2 + 152x^3 - \ldots + [3^r - (-5)^r]\,x^r + \ldots, |x| < \frac{1}{5}$

4. $\frac{1}{1-x} - \frac{1}{3-x}$, $\frac{2}{3} + \frac{8x}{9} + \frac{26}{27}x^2 + \frac{80}{81}x^3 + \ldots + \left(1 - \frac{1}{3^{r+1}}\right)x^r + \ldots, |x| < 1$

5. $\frac{3}{2(3+2x)} + \frac{1}{2(2x-1)}$, $-\frac{4}{3}x - \frac{16}{9}x^2 - \frac{112}{27}x^3 - \ldots + 2^{r-1}\left(\left(\frac{-1}{3}\right)^r - 1\right)x^r + \ldots, |x| < \frac{1}{2}$

6. $\frac{3}{3+x} - \frac{1}{3+4x}$, $\frac{2}{3} + \frac{x}{9} - \frac{13}{27}x^2 + \frac{61}{81}x^3 - \ldots + \left(-\frac{1}{3}\right)^r\left(1 - \frac{4^r}{3}\right)x^r + \ldots, |x| < \frac{3}{4}$

7. $\frac{2}{(1+x)^2} - \frac{1}{1+2x}$, $1 - 2x + 2x^2 - 0x^3 + \ldots + (-1)^r\, 2(r + 1 - 2^{r-1})\,x^r + \ldots, |x| < \frac{1}{2}$

8. $\frac{1}{1-4x} + \frac{1}{1-x} + \frac{2}{2+3x}$, $3 + \frac{7}{2}x + \frac{77}{4}x^2 + \frac{493}{8}x^3 + \ldots + \left[4^r + 1 + \left(\frac{-3}{2}\right)^r\right]x^r + \ldots, |x| < \frac{1}{4}$

9. $\frac{4}{2+x} - \frac{4}{(2+x)^2} - \frac{3}{1+x}$, $-2 + 3x - \frac{13}{4}x^2 + \frac{13}{4}x^3 - \ldots + \left(-\frac{1}{2}\right)^r(1 - r - 3\times 2^r)x^r + \ldots, |x| < 1$

10. $\frac{x-2}{1+x^2} + \frac{5}{1-5x}$, $3 + 26x + 127x^2 + 624x^3 + \ldots, |x| < \frac{1}{5}$

11. $\frac{3x+5}{1+x+x^2} + \frac{8}{2-3x}$, $9 + 4x + 6x^2 + \frac{37}{2}x^3 + \ldots$

page 458 ***Exercise 18C***

1. $n(n+2)$ **2.** (b) $\frac{1}{3}n(n+1)(n+2)$ **3.** (a) $\frac{1}{x} - \frac{1}{x+1}$ (c) $\frac{15}{16}$

4. (a) $\frac{1}{3x-1} - \frac{1}{3x+2}$ (c) $\frac{5}{32}, \frac{1}{6}$ **5.** (a) $\frac{1}{x} - \frac{1}{x+2}$ (b) $\frac{3}{4}$ (c) $\frac{36}{55}$

6. (c) 1770 **7.** (a) $1 - \frac{1}{4n+1}$ (b) $\frac{1}{3} - \frac{n+1}{(2n+1)(2n+3)}$

(c) $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ (d) $\frac{7}{6} - \frac{3n+7}{(n+1)(n+2)(n+3)}$

8. (b) Yes, $2\frac{1}{2}$ **9.** (b) Yes, $\frac{5}{24}$

page 463 Exercise 18D

1. (a) 3 (b) -4 (c) $3 + 4i$ (d) 25 (e) $-7 + 24i$

2. (a) $5 + 7i$ (b) $-13 + 5i$ (c) -4 (d) 1 (e) -1 (f) 13 (g) $\frac{1}{2}$ (h) i

3. (a) $7 + i$ (b) $32 - 7i$ (c) $1 - 7i$ (d) $11 - 13i$ (e) $11 - 2i$ (f) 5 (g) 78 (h) $7 + i$

4. (a) $6 - 2i$ (b) $2 - 2i$ (c) $-1 + i$ (d) $\frac{1}{5} + \frac{3}{5}i$ (e) $2 + i$ (f) $\frac{4}{5} + \frac{3}{5}i$ (g) $-\frac{1}{2} + \frac{5}{2}i$ (h) $\frac{1}{5} - \frac{7}{5}i$ (i) $\frac{5}{13} + \frac{12}{13}i$ (j) $\frac{6}{13} + \frac{17}{13}i$

5. i

6. (a) $x = \pm 5i$ (b) $x = \pm 4i$ (c) $x = \pm\frac{3i}{2}$ (d) $x = -1 \pm 2i$ (e) $x = 2 \pm i$ (f) $x = \frac{-1 \pm \sqrt{7}i}{4}$

7. (a) $\pm(3 + 2i)$ (b) $\pm(4 + i)$ (c) $\pm(4 - 3i)$

8. (a) $x^2 + 9 = 0$ (b) $x^2 - 2x + 5 = 0$ (c) $x^2 - 4x + 5 = 0$ (d) $x^2 - 4x + 13 = 0$ (e) $x^2 - 6x + 25 = 0$ (f) $x^2 - 6x + 34 = 0$

10. (a) $x = 1, 3 \pm 2i$ (b) $x = 2, -\frac{1}{2} \pm \frac{1}{2}i$ (c) $x = -1, -1 \pm \sqrt{2}i$ (d) $x = 1, 2, 1 \pm \frac{1}{2}i$ (e) $x = 1, -1, \frac{-4 \pm 3i}{5}$

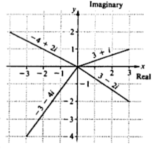
11. $5, 6, 2$; $x = 2 \pm 2i, -\frac{3}{5} \pm \frac{1}{5}i$ **12.** $a = 1, b = -8, c = 27, d = -38, e = 26$

13. $a = b = c = 1, x = 1, \frac{-1 \pm \sqrt{3}i}{2}$

page 467 Exercise 18E

1. $3 + 2i, -1 + 3i, -3 - 4i, 4 - i$

2.



3. $5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), 4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$3\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right], 3\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

4. (a) $\sqrt{2}, \frac{\pi}{4}$ (b) $2\sqrt{2}, \frac{3\pi}{4}$ (c) $2, \pi$ (d) $3, -\frac{\pi}{2}$ (e) $2\sqrt{2}, -\frac{3\pi}{4}$

5. (a) $5, \frac{\pi}{2}$ (b) $7, 0$ (c) $2, -\frac{\pi}{2}$ (d) $3, \pi$ (e) $2, \frac{\pi}{3}$ (f) $10, -\frac{\pi}{6}$ (g) $5, -0{\cdot}93$ (h) $13, 1{\cdot}97$

6. $z_1 = 5i, z_2 = 4 - 4i, z_3 = -4, z_4 = -6 + 6\sqrt{3}i, z_5 = 2 - 2\sqrt{3}i, z_6 = -6 + 6i$

7. (a) $1, -\frac{\pi}{2}$ (b) $\sqrt{2}, -\frac{3\pi}{4}$ (c) $\frac{\sqrt{10}}{5}, 1{\cdot}25$ (d) $5\sqrt{2}, 1{\cdot}43$

8. (b) (i) 6 (ii) $\frac{5\pi}{12}$ (iii) 9 (iv) 4 (v) $\frac{\pi}{3}$ (vi) $\frac{\pi}{2}$

9. (b) (i) $\frac{1}{3}$ (ii) $-\frac{7\pi}{12}$ (iii) 3 (iv) $\frac{7\pi}{12}$ **10.** $1 + \sqrt{3}i, -4\sqrt{2} + 4\sqrt{2}i$

12. 64 **13.** $-64i$ **14.** $1, \frac{1}{2}(-1 \pm i\sqrt{3})$ **15.** $\frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}$

17. $\frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$

page 473 Exercise 18G

1. $-\frac{2}{5}, \frac{3}{5}$ **2.** $\frac{3}{1-3x} - \frac{2}{1-2x} - \frac{1}{(1-2x)^2}, x + 7x^2 + 33x^3, |x| < \frac{1}{3}$

3. $1 - x + 3x^2 - 5x^3 + 11x^4$ (a) 1·0212418 (b) 0·9902951

4. $\frac{1}{4(1-x)} + \frac{-7}{4(3+x)}, -\frac{1}{3}, \frac{4}{9}, \frac{5}{27}$ **5.** $\frac{1}{1-2x} + \frac{2}{2+x}, 2 + \frac{3}{2}x + \frac{17}{4}x^2$

6. $-\frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{x-1}, \frac{1}{x} + \frac{4}{x^2} - \frac{1}{x^3} + \frac{6}{x^4} - \frac{3}{x^5}, (2 - n)$

7. $\frac{1}{2(r-1)} - \frac{1}{2(r+1)}, \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$ **8.** $\frac{1}{4(2r-1)} - \frac{1}{4(2r+3)}$

9. $\frac{r+2}{(r+1)!}, 1 - \frac{1}{(2n+1)!}, \frac{1}{(2n+1)!} - 1$ **10.** (i) $18 - i, \frac{6}{25} + i\frac{17}{25}$

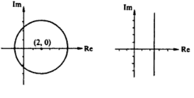
11. (a) $2 - 11i$ (b) $-2 - 3i$ **12.** $-4, 13$

13. $2 - i, -4$ **14.** $x^2 - 2x + 2, 1 \pm i, -1 \pm 2i$

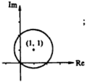
15. $\text{Re}(z) = -3, \text{Im}(z) = -1, \sqrt{10}, -2{\cdot}82$ rads

16. $1 + 6ci - 15c^2 - 20c^3i + 15c^4 + 6c^5i - c^6; 0, \pm \frac{1}{3}\sqrt{3}, \pm\sqrt{3}$

17. (a) $2 + 3i, 5 - 4i$ (b) $a = 2, b = 3; a = -2, b = -3; \pm(2 + 3i)$
(c) (i) circle centre (2, 0) radius 3 (ii) line through $(2\frac{1}{2}, 0)$ perpendicular to real axis



18.



; $2\sqrt{2}, 0$.

20. $\pm(1 + i), \pm(1 - i), (x^2 - 2x + 2)(x^2 + 2x + 2)$

21. $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

22. $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

page 480 Exercise 19A

1. e^x **2.** $3e^{3x}$ **3.** $2e^{2x}$ **4.** $2xe^{x^2}$ **5.** $2xe^{(x^2+1)}$ **6.** $-e^{-x}$ **7.** $8e^{2x}$ **8.** $30x^2e^{2x^3}$

9. $-6e^{-2x}$ **10.** $5e^{5x+3} - 4x^{-3}$ **11.** $4xe^{x^2} + 3e^x - 4e^{-x}$ **12.** $xe^x(2 + x)$

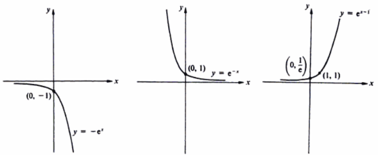
13. $3x^2e^{2x}(3 + 2x)$ **14.** $3xe^{-x}(2 - x)$ **15.** $\frac{xe^x - 2e^x - 8}{x^3}$ **16.** $e^x + c$ **17.** $5e^x + c$

18. $e^{2x} + c$ **19.** $2e^{3x} + c$ **20.** $e^{x^2} + c$ **21.** $2e^{(x^2+1)} + c$ **22.** $2e^{-2x} + c$

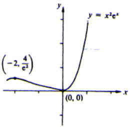
23. $e^2x - 4e^{-x} + c$ **24.** $e(e^2 - 1)$ **25.** $1 - \frac{1}{e^3}$ **26.** $e^3(e^2 - 1)$ **27.** $e^3 - \frac{1}{e}$

28. 4 **29.** $y = e^x + x^2 - 4$

30. (a) (b) (c)



31. min at $\left(-1, -\frac{1}{e}\right)$ **32.** max at $(0, -1)$ **33.**



page 487 Exercise 19B

1. $\frac{1}{x}$ **2.** $\frac{1}{x}$ **3.** $\frac{1}{x}$ **4.** $-\frac{1}{x}$ **5.** $\frac{3}{x}$ **6.** $\frac{2x}{x^2 + 3}$ **7.** $\frac{3}{3x - 4}$ **8.** $\frac{2x + 2}{x^2 + 2x - 1}$

9. $\frac{6}{9 - x^2}$ **10.** $\frac{3}{x(3 - 2x)}$ **11.** $\frac{3x - 4}{x(3x - 2)}$ **12.** $\frac{2}{1 - x^2}$ **13.** $\frac{x + 2}{(x + 3)^2}$

14. $\frac{2x}{x^2 - 1} - \frac{1}{(x + 4)^2}$ **15.** $\frac{1}{x \ln 10}$ **16.** $\frac{1}{x \ln 3}$ **17.** $2e^x + \frac{1}{x}$ **18.** $\frac{e^x}{x}(x \ln x + 1)$

19. $\frac{e^{x^2}}{x}(2x^2 \ln x + 1)$ **20.** $\frac{1 - 2x^2 \ln x}{xe^{x^2}}$

21. (a) $2x^x(1 + \ln x)$ (b) $x2^{x^2+1} \ln 2$ (c) $\frac{(3x + 5)(x - 1)}{2(x + 1)^{3/2}}$ (d) $\frac{2}{(1 - 2x)\sqrt{(1 - 4x^2)}}$

22. $3 \ln 3 - 1$ **23.** 4 **24.** (a) $3^x \ln 3$ (b) $4^x \ln 4$

25. $a^x \ln a$, $\frac{a^x}{\ln a} + c$, 39·09 **26.** $\ln |x| + c$ **27.** $3 \ln |x| + c$

28. $\frac{1}{2} \ln |x| + c$ **29.** $\ln (x^2 + 1) + c$ **30.** $3 \ln (x^2 + 3) + c$

31. $3 \ln |x^2 - 3| + c$ **32.** $\ln |x^2 + 3x - 1| + c$ **33.** $\frac{1}{2} \ln (x^2 - 4x + 7) + c$

page 492 **Exercise 19C**

1. (a) $1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 \ldots\ldots$ (b) $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \ldots\ldots$
(c) $1 + x^2 + \frac{1}{2}x^4 \ldots\ldots$ (d) $1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 \ldots\ldots$
(e) $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 \ldots\ldots$ (f) $1 + 6x + \frac{35}{2}x^2 + \frac{100}{3}x^3 + \frac{375}{8}x^4 \ldots\ldots$
(g) $1 + 2x - \frac{16}{3}x^3 - \frac{32}{3}x^4 \ldots\ldots$ (h) $1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{5}{24}x^4 \ldots\ldots$
(i) $1 + x - \frac{1}{2}x^2 - \frac{5}{6}x^3 + \frac{1}{24}x^4 \ldots\ldots$

2. (a) e^2 (b) e^{-3} (c) $e^{-1/2} - 1$ (d) $e - 2$ (e) $e^x - 1$ (f) e^{x^2}

3. (a) $2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 \ldots\ldots, -\frac{1}{2} < x \leqslant \frac{1}{2}$
(b) $-6x - 18x^2 - 72x^3 - 324x^4 \ldots\ldots, -\frac{1}{6} \leqslant x < \frac{1}{6}$
(c) $\frac{1}{3}x - \frac{1}{18}x^2 + \frac{1}{81}x^3 - \frac{1}{324}x^4 \ldots\ldots, -3 < x \leqslant 3$
(d) $\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 - \frac{1}{64}x^4 \ldots\ldots, -2 < x \leqslant 2$
(e) $\ln 3 + \frac{2}{3}x - \frac{2}{9}x^2 + \frac{8}{81}x^3 - \frac{4}{81}x^4 \ldots\ldots, -\frac{3}{2} < x \leqslant \frac{3}{2}$
(f) $-x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - \frac{17}{4}x^4 \ldots\ldots, -\frac{1}{2} \leqslant x < \frac{1}{2}$
(g) $3x - \frac{5}{2}x^2 + 3x^3 - \frac{17}{4}x^4 \ldots\ldots, -\frac{1}{2} < x \leqslant \frac{1}{2}$
(h) $\ln 3 + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{28}{81}x^3 - \frac{41}{162}x^4 \ldots\ldots, -1 < x \leqslant 1$
(i) $5x - \frac{25}{2}x^2 + \frac{125}{3}x^3 - \frac{625}{4}x^4 \ldots\ldots, -\frac{1}{5} < x \leqslant \frac{1}{5}$
(j) $4 \ln 2 - \frac{3}{2}x + \frac{7}{16}x^2 - \frac{11}{32}x^3 + \frac{127}{512}x^4 \ldots\ldots, -1 < x \leqslant 1$
(k) $x - \frac{5}{2}x^2 + \frac{7}{3}x^3 - \frac{17}{12}x^4 \ldots\ldots, -1 < x \leqslant 1$
(l) $6x + 24x^2 + 66x^3 + 240x^4 \ldots\ldots, -\frac{1}{3} \leqslant x < \frac{1}{3}$

4. (a) $\ln(\frac{3}{2})$ (b) $-\ln 2$ (c) $3 \ln 2$ (d) $\ln 2$

5. 1·1052, 0·9048 **6.** 0·0953, −0·1054 **7.** (i) 0·5108 (ii) 1·099

9. (a) $\dfrac{2^r}{r!}$ (b) $\dfrac{(-1)^{r+1}}{r.2^r}$ (c) $\dfrac{1 + r}{r!}$ (d) $\dfrac{2}{r(r^2 - 1)}$ **10.** $-3 + x^3 + \frac{1}{2}x^4$

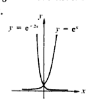
11. $1 - x + \frac{3}{2}x^2 - \frac{29}{6}x^3, |x| < \frac{1}{2}$ **12.** $\frac{1}{2}x - \frac{1}{2}x^2 + \frac{5}{12}x^3 - \frac{1}{3}x^4, -1 < x \leqslant 1$

13. (a) $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{40}x^5, -1 < x \leqslant 1$ (b) $2x - 4x^2 + \frac{17}{3}x^3 - 8x^4, -\frac{1}{2} < x \leqslant \frac{1}{2}$
(c) $x + \frac{23}{4}x^2 + \frac{199}{12}x^3 + \frac{1023}{32}x^4, -2 < x \leqslant 2$ (d) $-x - 2x^2 - \frac{17}{6}x^3 - 4x^4, -\frac{1}{2} \leqslant x < \frac{1}{2}$

14. 1, 2, $\frac{1}{6}$ **15.** 2, −3, −24

page 493 **Exercise 19D**

1.



, 2·08 units

2. (a) $\dfrac{1}{e^x x^2}(x + 1)$ (b) $\dfrac{\cos x}{1 + \sin x}$ **3.** $1 + \ln x, x \ln x - x + c$, 0·386

4. $(\frac{1}{6} \ln 4, 8)$, minimum **5.** $y = e(2x - 1), ey + 1 = 0$, 80°

6. $\dfrac{2}{2x - 1} - \dfrac{x}{x^2 + 1}, \ln\left[\dfrac{|2x - 1|}{\sqrt{(x^2 + 1)}}\right] + c$

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