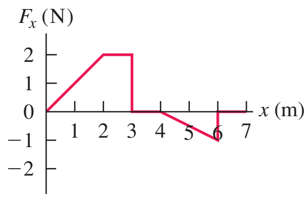
instantaneously? With the air track turned off, the coefficient of kinetic friction is $\mu_k = 0.47$.

6.40 • A 4.00-kg block of ice is placed against a horizontal spring that has force constant $k = 200$ N/m and is compressed 0.025 m. The spring is released and accelerates the block along a horizontal surface. You can ignore friction and the mass of the spring. (a) Calculate the work done on the block by the spring during the motion of the block from its initial position to where the spring has returned to its uncompressed length. (b) What is the speed of the block after it leaves the spring?

6.41 • A force $\vec{F}$ is applied to a 2.0-kg radio-controlled model car parallel to the x-axis as it moves along a straight track. The x-component of the force varies with the x-coordinate of the car as shown in Fig. E6.41. Calculate the work done by the force $\vec{F}$ when the car moves from (a) $x = 0$ to $x = 3.0$ m; (b) $x = 3.0$ m to $x = 4.0$ m; (c) $x = 4.0$ m to $x = 7.0$ m; (d) $x = 0$ to $x = 7.0$ m; (e) $x = 7.0$ m to $x = 2.0$ m.

Figure **E6.41**



6.42 • Suppose the 2.0-kg model car in Exercise 6.41 is initially at rest at $x = 0$ and $\vec{F}$ is the net force acting on it. Use the work–energy theorem to find the speed of the car at (a) $x = 3.0$ m; (b) $x = 4.0$ m; (c) $x = 7.0$ m.

6.43 •• At a waterpark, sleds with riders are sent along a slippery, horizontal surface by the release of a large compressed spring. The spring with force constant $k = 40.0$ N/cm and negligible mass rests on the frictionless horizontal surface. One end is in contact with a stationary wall. A sled and rider with total mass 70.0 kg are pushed against the other end, compressing the spring 0.375 m. The sled is then released with zero initial velocity. What is the sled's speed when the spring (a) returns to its uncompressed length and (b) is still compressed 0.200 m?

6.44 • **Half of a Spring.** (a) Suppose you cut a massless ideal spring in half. If the full spring had a force constant k, what is the force constant of each half, in terms of k? (*Hint:* Think of the original spring as two equal halves, each producing the same force as the entire spring. Do you see why the forces must be equal?) (b) If you cut the spring into three equal segments instead, what is the force constant of each one, in terms of k?

6.45 •• A small glider is placed against a compressed spring at the bottom of an air track that slopes upward at an angle of 40.0° above the horizontal. The glider has mass 0.0900 kg. The spring has $k = 640$ N/m and negligible mass. When the spring is released, the glider travels a maximum distance of 1.80 m along the air track before sliding back down. Before reaching this maximum distance, the glider loses contact with the spring. (a) What distance was the spring originally compressed? (b) When the glider has traveled along the air track 0.80 m from its initial position against the compressed spring, is it still in contact with the spring? What is the kinetic energy of the glider at this point?

6.46 •• An ingenious bricklayer builds a device for shooting bricks up to the top of the wall where he is working. He places a brick on a vertical compressed spring with force constant $k = 450$ N/m and negligible mass. When the spring is released, the brick is propelled upward. If the brick has mass 1.80 kg and is to reach a maximum height of 3.6 m above its initial position on the compressed spring, what distance must the bricklayer compress the spring initially? (The brick loses contact with the spring when the spring returns to its uncompressed length. Why?)

6.47 •• **CALC** A force in the $+x$-direction with magnitude $F(x) = 18.0\text{ N} - (0.530\text{ N/m})x$ is applied to a 6.00-kg box that is sitting on the horizontal, frictionless surface of a frozen lake. $F(x)$ is the only horizontal force on the box. If the box is initially at rest at $x = 0$, what is its speed after it has traveled 14.0 m?

Section 6.4 Power

6.48 •• A crate on a motorized cart starts from rest and moves with a constant eastward acceleration of $a = 2.80\text{ m/s}^2$. A worker assists the cart by pushing on the crate with a force that is eastward and has magnitude that depends on time according to $F(t) = (5.40\text{ N/s})t$. What is the instantaneous power supplied by this force at $t = 5.00$ s?

6.49 • How many joules of energy does a 100-watt light bulb use per hour? How fast would a 70-kg person have to run to have that amount of kinetic energy?

6.50 •• **BIO** **Should You Walk or Run?** It is 5.0 km from your home to the physics lab. As part of your physical fitness program, you could run that distance at 10 km/h (which uses up energy at the rate of 700 W), or you could walk it leisurely at 3.0 km/h (which uses energy at 290 W). Which choice would burn up more energy, and how much energy (in joules) would it burn? Why is it that the more intense exercise actually burns up less energy than the less intense exercise?

6.51 •• **Magnetar.** On December 27, 2004, astronomers observed the greatest flash of light ever recorded from outside the solar system. It came from the highly magnetic neutron star SGR 1806-20 (a *magnetar*). During 0.20 s, this star released as much energy as our sun does in 250,000 years. If P is the average power output of our sun, what was the average power output (in terms of P) of this magnetar?

6.52 •• A 20.0-kg rock is sliding on a rough, horizontal surface at 8.00 m/s and eventually stops due to friction. The coefficient of kinetic friction between the rock and the surface is 0.200. What average power is produced by friction as the rock stops?

6.53 • A tandem (two-person) bicycle team must overcome a force of 165 N to maintain a speed of 9.00 m/s. Find the power required per rider, assuming that each contributes equally. Express your answer in watts and in horsepower.

6.54 •• When its 75-kW (100-hp) engine is generating full power, a small single-engine airplane with mass 700 kg gains altitude at a rate of 2.5 m/s (150 m/min, or 500 ft/min). What fraction of the engine power is being used to make the airplane climb? (The remainder is used to overcome the effects of air resistance and of inefficiencies in the propeller and engine.)

6.55 •• **Working Like a Horse.** Your job is to lift 30-kg crates a vertical distance of 0.90 m from the ground onto the bed of a truck. (a) How many crates would you have to load onto the truck in 1 minute for the average power output you use to lift the crates to equal 0.50 hp? (b) How many crates for an average power output of 100 W?

6.56 •• An elevator has mass 600 kg, not including passengers. The elevator is designed to ascend, at constant speed, a vertical distance of 20.0 m (five floors) in 16.0 s, and it is driven by a motor that can provide up to 40 hp to the elevator. What is the maximum number of passengers that can ride in the elevator? Assume that an average passenger has mass 65.0 kg.

6.57 •• A ski tow operates on a 15.0° slope of length 300 m. The rope moves at 12.0 km/h and provides power for 50 riders at one time, with an average mass per rider of 70.0 kg. Estimate the power required to operate the tow.

6.58 •• The aircraft carrier *John F. Kennedy* has mass 7.4×10^7 kg. When its engines are developing their full power of 280,000 hp, the *John F. Kennedy* travels at its top speed of 35 knots (65 km/h). If 70% of the power output of the engines is applied to pushing the ship through the water, what is the magnitude of the force of water resistance that opposes the carrier's motion at this speed?

6.59 • **BIO** A typical flying insect applies an average force equal to twice its weight during each downward stroke while hovering. Take the mass of the insect to be 10 g, and assume the wings move an average downward distance of 1.0 cm during each stroke. Assuming 100 downward strokes per second, estimate the average power output of the insect.

PROBLEMS

6.60 ••• **CALC** A balky cow is leaving the barn as you try harder and harder to push her back in. In coordinates with the origin at the barn door, the cow walks from $x = 0$ to $x = 6.9$ m as you apply a force with x-component $F_x = -[20.0\text{ N} + (3.0\text{ N/m})x]$. How much work does the force you apply do on the cow during this displacement?

6.61 •• **CALC** **Rotating Bar.** A thin, uniform 12.0-kg bar that is 2.00 m long rotates uniformly about a pivot at one end, making 5.00 complete revolutions every 3.00 seconds. What is the kinetic energy of this bar? (*Hint*: Different points in the bar have different speeds. Break the bar up into infinitesimal segments of mass dm and integrate to add up the kinetic energies of all these segments.)

6.62 •• **A Near-Earth Asteroid.** On April 13, 2029 (Friday the 13th!), the asteroid 99942 Apophis will pass within 18,600 mi of the earth—about $\frac{1}{13}$ the distance to the moon! It has a density of 2600 kg/m^3, can be modeled as a sphere 320 m in diameter, and will be traveling at 12.6 km/s. (a) If, due to a small disturbance in its orbit, the asteroid were to hit the earth, how much kinetic energy would it deliver? (b) The largest nuclear bomb ever tested by the United States was the "Castle/Bravo" bomb, having a yield of 15 megatons of TNT. (A megaton of TNT releases 4.184×10^{15} J of energy.) How many Castle/Bravo bombs would be equivalent to the energy of Apophis?

6.63 • A luggage handler pulls a 20.0-kg suitcase up a ramp inclined at 25.0° above the horizontal by a force $\vec{F}$ of magnitude 140 N that acts parallel to the ramp. The coefficient of kinetic friction between the ramp and the incline is $\mu_k = 0.300$. If the suitcase travels 3.80 m along the ramp, calculate (a) the work done on the suitcase by the force $\vec{F}$; (b) the work done on the suitcase by the gravitational force; (c) the work done on the suitcase by the normal force; (d) the work done on the suitcase by the friction force; (e) the total work done on the suitcase. (f) If the speed of the suitcase is zero at the bottom of the ramp, what is its speed after it has traveled 3.80 m along the ramp?

6.64 • **BIO** **Chin-Ups.** While doing a chin-up, a man lifts his body 0.40 m. (a) How much work must the man do per kilogram of body mass? (b) The muscles involved in doing a chin-up can generate about 70 J of work per kilogram of muscle mass. If the man can just barely do a 0.40-m chin-up, what percentage of his body's mass do these muscles constitute? (For comparison, the *total* percentage of muscle in a typical 70-kg man with 14% body fat is about 43%.) (c) Repeat part (b) for the man's young son, who has arms half as long as his father's but whose muscles can also generate 70 J of work per kilogram of muscle mass. (d) Adults and children have about the same percentage of muscle in their bodies. Explain why children can commonly do chin-ups more easily than their fathers.

6.65 ••• **CP** A 20.0-kg crate sits at rest at the bottom of a 15.0-m-long ramp that is inclined at 34.0° above the horizontal. A constant horizontal force of 290 N is applied to the crate to push it up the ramp. While the crate is moving, the ramp exerts a constant frictional force on it that has magnitude 65.0 N. (a) What is the total work done on the crate during its motion from the bottom to the top of the ramp? (b) How much time does it take the crate to travel to the top of the ramp?

6.66 ••• Consider the blocks in Exercise 6.7 as they move 75.0 cm. Find the total work done on each one (a) if there is no friction between the table and the 20.0-N block, and (b) if $\mu_s = 0.500$ and $\mu_k = 0.325$ between the table and the 20.0-N block.

6.67 • The space shuttle, with mass 86,400 kg, is in a circular orbit of radius 6.66×10^6 m around the earth. It takes 90.1 min for the shuttle to complete each orbit. On a repair mission, the shuttle is cautiously moving 1.00 m closer to a disabled satellite every 3.00 s. Calculate the shuttle's kinetic energy (a) relative to the earth and (b) relative to the satellite.

6.68 •• A 5.00-kg package slides 1.50 m down a long ramp that is inclined at 24.0° below the horizontal. The coefficient of kinetic friction between the package and the ramp is $\mu_k = 0.310$. Calculate (a) the work done on the package by friction; (b) the work done on the package by gravity; (c) the work done on the package by the normal force; (d) the total work done on the package. (e) If the package has a speed of 2.20 m/s at the top of the ramp, what is its speed after sliding 1.50 m down the ramp?

6.69 •• **CP** **BIO** **Whiplash Injuries.** When a car is hit from behind, its passengers undergo sudden forward acceleration, which can cause a severe neck injury known as *whiplash.* During normal acceleration, the neck muscles play a large role in accelerating the head so that the bones are not injured. But during a very sudden acceleration, the muscles do not react immediately because they are flexible, so most of the accelerating force is provided by the neck bones. Experimental tests have shown that these bones will fracture if they absorb more than 8.0 J of energy. (a) If a car waiting at a stoplight is rear-ended in a collision that lasts for 10.0 ms, what is the greatest speed this car and its driver can reach without breaking neck bones if the driver's head has a mass of 5.0 kg (which is about right for a 70-kg person)? Express your answer in m/s and in mph. (b) What is the acceleration of the passengers during the collision in part (a), and how large a force is acting to accelerate their heads? Express the acceleration in m/s^2 and in g's.

6.70 •• **CALC** A net force along the x-axis that has x-component $F_x = -12.0\text{ N} + (0.300\text{ N/m}^2)x^2$ is applied to a 5.00-kg object that is initially at the origin and moving in the $-x$-direction with a speed of 6.00 m/s. What is the speed of the object when it reaches the point $x = 5.00$ m?

6.71 • **CALC** An object is attracted toward the origin with a force given by $F_x = -k/x^2$. (Gravitational and electrical forces have this distance dependence.) (a) Calculate the work done by the force F_x when the object moves in the x-direction from x_1 to x_2. If $x_2 > x_1$, is the work done by F_x positive or negative? (b) The only other force acting on the object is a force that you exert with your hand to move the object slowly from x_1 to x_2. How much work do you do? If $x_2 > x_1$, is the work you do positive or negative? (c) Explain the similarities and differences between your answers to parts (a) and (b).

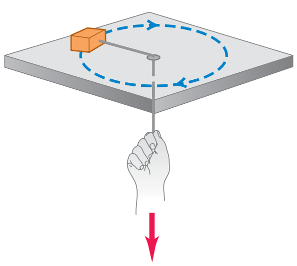
6.72 ••• **CALC** The gravitational pull of the earth on an object is inversely proportional to the square of the distance of the object from the center of the earth. At the earth's surface this force is equal to the object's normal weight mg, where $g = 9.8\ \text{m/s}^2$, and at large distances, the force is zero. If a 20,000-kg asteroid falls to earth from a very great distance away, what will be its minimum speed as it strikes the earth's surface, and how much kinetic energy will it impart to our planet? You can ignore the effects of the earth's atmosphere.

6.73 • **CALC** **Varying Coefficient of Friction.** A box is sliding with a speed of 4.50 m/s on a horizontal surface when, at point P, it encounters a rough section. On the rough section, the coefficient of friction is not constant, but starts at 0.100 at P and increases linearly with distance past P, reaching a value of 0.600 at 12.5 m past point P. (a) Use the work–energy theorem to find how far this box slides before stopping. (b) What is the coefficient of friction at the stopping point? (c) How far would the box have slid if the friction coefficient didn't increase but instead had the constant value of 0.100?

6.74 •• **CALC** Consider a spring that does not obey Hooke's law very faithfully. One end of the spring is fixed. To keep the spring stretched or compressed an amount x, a force along the x-axis with x-component $F_x = kx - bx^2 + cx^3$ must be applied to the free end. Here $k = 100\ \text{N/m}$, $b = 700\ \text{N/m}^2$, and $c = 12{,}000\ \text{N/m}^3$. Note that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. (a) How much work must be done to stretch this spring by 0.050 m from its unstretched length? (b) How much work must be done to *compress* this spring by 0.050 m from its unstretched length? (c) Is it easier to stretch or compress this spring? Explain why in terms of the dependence of F_x on x. (Many real springs behave qualitatively in the same way.)

6.75 •• **CP** A small block with a mass of 0.0900 kg is attached to a cord passing through a hole in a frictionless, horizontal surface (Fig. P6.75). The block is originally revolving at a distance of 0.40 m from the hole with a speed of 0.70 m/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.10 m. At this new distance, the speed of the block is observed to be 2.80 m/s. (a) What is the tension in the cord in the original situation when the block has speed $v = 0.70\ \text{m/s}$? (b) What is the tension in the cord in the final situation when the block has speed $v = 2.80\ \text{m/s}$? (c) How much work was done by the person who pulled on the cord?

Figure **P6.75**

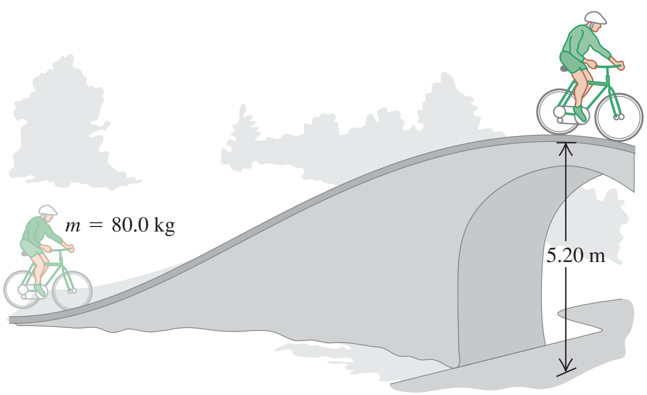


6.76 •• **CALC** **Proton Bombardment.** A proton with mass 1.67×10^{-27} kg is propelled at an initial speed of 3.00×10^5 m/s directly toward a uranium nucleus 5.00 m away. The proton is repelled by the uranium nucleus with a force of magnitude $F = \alpha/x^2$, where x is the separation between the two objects and $\alpha = 2.12 \times 10^{-26}\ \text{N} \cdot \text{m}^2$. Assume that the uranium nucleus remains at rest. (a) What is the speed of the proton when it is 8.00×10^{-10} m from the uranium nucleus? (b) As the proton approaches the uranium nucleus, the repulsive force slows down the proton until it comes momentarily to rest, after which the proton moves away from the uranium nucleus. How close to the uranium nucleus does the proton get? (c) What is the speed of the proton when it is again 5.00 m away from the uranium nucleus?

6.77 •• **CP** **CALC** A block of ice with mass 4.00 kg is initially at rest on a frictionless, horizontal surface. A worker then applies a horizontal force $\vec{F}$ to it. As a result, the block moves along the x-axis such that its position as a function of time is given by $x(t) = \alpha t^2 + \beta t^3$, where $\alpha = 0.200\ \text{m/s}^2$ and $\beta = 0.0200\ \text{m/s}^3$. (a) Calculate the velocity of the object when $t = 4.00$ s. (b) Calculate the magnitude of $\vec{F}$ when $t = 4.00$ s. (c) Calculate the work done by the force $\vec{F}$ during the first 4.00 s of the motion.

6.78 •• You and your bicycle have combined mass 80.0 kg. When you reach the base of a bridge, you are traveling along the road at 5.00 m/s (Fig. P6.78). At the top of the bridge, you have climbed a vertical distance of 5.20 m and have slowed to 1.50 m/s. You can ignore work done by friction and any inefficiency in the bike or your legs. (a) What is the total work done on you and your bicycle when you go from the base to the top of the bridge? (b) How much work have you done with the force you apply to the pedals?

Figure **P6.78**



6.79 •• You are asked to design spring bumpers for the walls of a parking garage. A freely rolling 1200-kg car moving at 0.65 m/s is to compress the spring no more than 0.090 m before stopping. What should be the force constant of the spring? Assume that the spring has negligible mass.

6.80 •• The spring of a spring gun has force constant $k = 400\ \text{N/m}$ and negligible mass. The spring is compressed 6.00 cm, and a ball with mass 0.0300 kg is placed in the horizontal barrel against the compressed spring. The spring is then released, and the ball is propelled out the barrel of the gun. The barrel is 6.00 cm long, so the ball leaves the barrel at the same point that it loses contact with the spring. The gun is held so the barrel is horizontal. (a) Calculate the speed with which the ball leaves the barrel if you can ignore friction. (b) Calculate the speed of the ball as it leaves the barrel if a constant resisting force of 6.00 N acts on the ball as it moves along the barrel. (c) For the situation in part (b), at what position along the barrel does the ball have the greatest speed, and what is that speed? (In this case, the maximum speed does not occur at the end of the barrel.)

6.81 ••• A 2.50-kg textbook is forced against a horizontal spring of negligible mass and force constant 250 N/m, compressing the spring a distance of 0.250 m. When released, the textbook slides on a horizontal tabletop with coefficient of kinetic friction

$\mu_k = 0.30$. Use the work–energy theorem to find how far the textbook moves from its initial position before coming to rest.

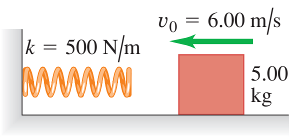
6.82 ••• **Pushing a Cat.** Your cat "Ms." (mass 7.00 kg) is trying to make it to the top of a frictionless ramp 2.00 m long and inclined upward at 30.0° above the horizontal. Since the poor cat can't get any traction on the ramp, you push her up the entire length of the ramp by exerting a constant 100-N force parallel to the ramp. If Ms. takes a running start so that she is moving at 2.40 m/s at the bottom of the ramp, what is her speed when she reaches the top of the incline? Use the work–energy theorem.

6.83 •• **Crash Barrier.** A student proposes a design for an automobile crash barrier in which a 1700-kg sport utility vehicle moving at 20.0 m/s crashes into a spring of negligible mass that slows it to a stop. So that the passengers are not injured, the acceleration of the vehicle as it slows can be no greater than $5.00g$. (a) Find the required spring constant k, and find the distance the spring will compress in slowing the vehicle to a stop. In your calculation, disregard any deformation or crumpling of the vehicle and the friction between the vehicle and the ground. (b) What disadvantages are there to this design?

6.84 ••• A physics professor is pushed up a ramp inclined upward at 30.0° above the horizontal as he sits in his desk chair that slides on frictionless rollers. The combined mass of the professor and chair is 85.0 kg. He is pushed 2.50 m along the incline by a group of students who together exert a constant horizontal force of 600 N. The professor's speed at the bottom of the ramp is 2.00 m/s. Use the work–energy theorem to find his speed at the top of the ramp.

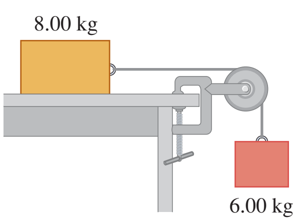
6.85 • A 5.00-kg block is moving at $v_0 = 6.00$ m/s along a frictionless, horizontal surface toward a spring with force constant $k = 500$ N/m that is attached to a wall (Fig. P6.85). The spring has negligible mass. (a) Find the maximum distance the spring will be compressed. (b) If the spring is to compress by no more than 0.150 m, what should be the maximum value of v_0?

Figure **P6.85**



6.86 •• Consider the system shown in Fig. P6.86. The rope and pulley have negligible mass, and the pulley is frictionless. The coefficient of kinetic friction between the 8.00-kg block and the tabletop is $\mu_k = 0.250$. The blocks are released from rest. Use energy methods to calculate the speed of the 6.00-kg block after it has descended 1.50 m.

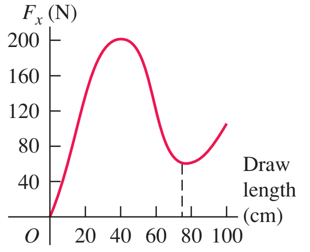
Figure **P6.86**



6.87 •• Consider the system shown in Fig. P6.86. The rope and pulley have negligible mass, and the pulley is frictionless. Initially the 6.00-kg block is moving downward and the 8.00-kg block is moving to the right, both with a speed of 0.900 m/s. The blocks come to rest after moving 2.00 m. Use the work–energy theorem to calculate the coefficient of kinetic friction between the 8.00-kg block and the tabletop.

6.88 ••• **CALC** **Bow and Arrow.** Figure P6.88 shows how the force exerted by the string of a compound bow on an arrow varies as a function of how far back the arrow is pulled (the draw length). Assume that the same force is exerted on the arrow as it moves forward after being released. Full draw for this bow is at a draw length of 75.0 cm. If the bow shoots a 0.0250-kg arrow from full draw, what is the speed of the arrow as it leaves the bow?

Figure **P6.88**



6.89 •• On an essentially frictionless, horizontal ice rink, a skater moving at 3.0 m/s encounters a rough patch that reduces her speed to 1.65 m/s due to a friction force that is 25% of her weight. Use the work–energy theorem to find the length of this rough patch.

6.90 • **Rescue.** Your friend (mass 65.0 kg) is standing on the ice in the middle of a frozen pond. There is very little friction between her feet and the ice, so she is unable to walk. Fortunately, a light rope is tied around her waist and you stand on the bank holding the other end. You pull on the rope for 3.00 s and accelerate your friend from rest to a speed of 6.00 m/s while you remain at rest. What is the average power supplied by the force you applied?

6.91 •• A pump is required to lift 800 kg of water (about 210 gallons) per minute from a well 14.0 m deep and eject it with a speed of 18.0 m/s. (a) How much work is done per minute in lifting the water? (b) How much work is done in giving the water the kinetic energy it has when ejected? (c) What must be the power output of the pump?

6.92 •• **BIO** All birds, independent of their size, must maintain a power output of 10–25 watts per kilogram of body mass in order to fly by flapping their wings. (a) The Andean giant hummingbird *(Patagona gigas)* has mass 70 g and flaps its wings 10 times per second while hovering. Estimate the amount of work done by such a hummingbird in each wingbeat. (b) A 70-kg athlete can maintain a power output of 1.4 kW for no more than a few seconds; the *steady* power output of a typical athlete is only 500 W or so. Is it possible for a human-powered aircraft to fly for extended periods by flapping its wings? Explain.

6.93 ••• A physics student spends part of her day walking between classes or for recreation, during which time she expends energy at an average rate of 280 W. The remainder of the day she is sitting in class, studying, or resting; during these activities, she expends energy at an average rate of 100 W. If she expends a total of 1.1×10^7 J of energy in a 24-hour day, how much of the day did she spend walking?

6.94 ••• The Grand Coulee Dam is 1270 m long and 170 m high. The electrical power output from generators at its base is approximately 2000 MW. How many cubic meters of water must flow from the top of the dam per second to produce this amount of power if 92% of the work done on the water by gravity is converted to electrical energy? (Each cubic meter of water has a mass of 1000 kg.)

6.95 • BIO **Power of the Human Heart.** The human heart is a powerful and extremely reliable pump. Each day it takes in and discharges about 7500 L of blood. Assume that the work done by the heart is equal to the work required to lift this amount of blood a height equal to that of the average American woman (1.63 m). The density (mass per unit volume) of blood is 1.05×10^3 kg/m^3. (a) How much work does the heart do in a day? (b) What is the heart's power output in watts?

6.96 ••• Six diesel units in series can provide 13.4 MW of power to the lead car of a freight train. The diesel units have total mass 1.10×10^6 kg. The average car in the train has mass 8.2×10^4 kg and requires a horizontal pull of 2.8 kN to move at a constant 27 m/s on level tracks. (a) How many cars can be in the train under these conditions? (b) This would leave no power for accelerating or climbing hills. Show that the extra force needed to accelerate the train is about the same for a 0.10-m/s^2 acceleration or a 1.0% slope (slope angle $\alpha = \arctan 0.010$). (c) With the 1.0% slope, show that an extra 2.9 MW of power is needed to maintain the 27-m/s speed of the diesel units. (d) With 2.9 MW less power available, how many cars can the six diesel units pull up a 1.0% slope at a constant 27 m/s?

6.97 • It takes a force of 53 kN on the lead car of a 16-car passenger train with mass 9.1×10^5 kg to pull it at a constant 45 m/s (101 mi/h) on level tracks. (a) What power must the locomotive provide to the lead car? (b) How much more power to the lead car than calculated in part (a) would be needed to give the train an acceleration of 1.5 m/s^2, at the instant that the train has a speed of 45 m/s on level tracks? (c) How much more power to the lead car than that calculated in part (a) would be needed to move the train up a 1.5% grade (slope angle $\alpha = \arctan 0.015$) at a constant 45 m/s?

6.98 • CALC An object has several forces acting on it. One of these forces is $\vec{F} = \alpha xy\hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object, with $\alpha = 2.50$ N/m^2. Calculate the work done on the object by this force for the following displacements of the object: (a) The object starts at the point $x = 0$, $y = 3.00$ m and moves parallel to the x-axis to the point $x = 2.00$ m, $y = 3.00$ m. (b) The object starts at the point $x = 2.00$ m, $y = 0$ and moves in the y-direction to the point $x = 2.00$ m, $y = 3.00$ m. (c) The object starts at the origin and moves on the line $y = 1.5x$ to the point $x = 2.00$ m, $y = 3.00$ m.

6.99 •• **Cycling.** For a touring bicyclist the drag coefficient $C(f_{\text{air}} = \frac{1}{2}CA\rho v^2)$ is 1.00, the frontal area A is 0.463 m^2, and the coefficient of rolling friction is 0.0045. The rider has mass 50.0 kg, and her bike has mass 12.0 kg. (a) To maintain a speed of 12.0 m/s (about 27 mi/h) on a level road, what must the rider's power output to the rear wheel be? (b) For racing, the same rider uses a different bike with coefficient of rolling friction 0.0030 and mass 9.00 kg. She also crouches down, reducing her drag coefficient to 0.88 and reducing her frontal area to 0.366 m^2. What must her power output to the rear wheel be then to maintain a speed of 12.0 m/s? (c) For the situation in part (b), what power output is required to maintain a speed of 6.0 m/s? Note the great drop in power requirement when the speed is only halved. (For more on aerodynamic speed limitations for a wide variety of human-powered vehicles, see "The Aerodynamics of Human-Powered Land Vehicles," *Scientific American,* December 1983.)

6.100 •• **Automotive Power I.** A truck engine transmits 28.0 kW (37.5 hp) to the driving wheels when the truck is traveling at a constant velocity of magnitude 60.0 km/h (37.3 mi/h) on a level road. (a) What is the resisting force acting on the truck? (b) Assume that 65% of the resisting force is due to rolling friction and the remainder is due to air resistance. If the force of rolling friction is independent of speed, and the force of air resistance is proportional to the square of the speed, what power will drive the truck at 30.0 km/h? At 120.0 km/h? Give your answers in kilowatts and in horsepower.

6.101 •• **Automotive Power II.** (a) If 8.00 hp are required to drive a 1800-kg automobile at 60.0 km/h on a level road, what is the total retarding force due to friction, air resistance, and so on? (b) What power is necessary to drive the car at 60.0 km/h up a 10.0% grade (a hill rising 10.0 m vertically in 100.0 m horizontally)? (c) What power is necessary to drive the car at 60.0 km/h *down* a 1.00% grade? (d) Down what percent grade would the car coast at 60.0 km/h?

CHALLENGE PROBLEMS

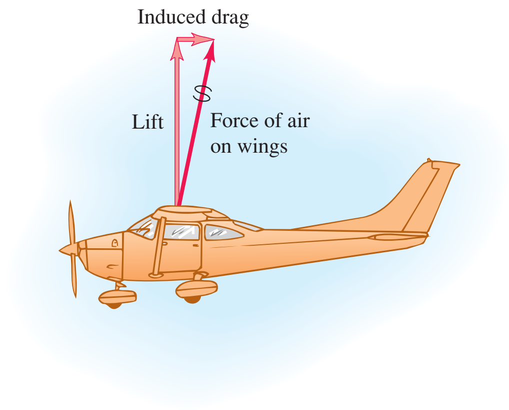
6.102 ••• CALC On a winter day in Maine, a warehouse worker is shoving boxes up a rough plank inclined at an angle α above the horizontal. The plank is partially covered with ice, with more ice near the bottom of the plank than near the top, so that the coefficient of friction increases with the distance x along the plank: $\mu = Ax$, where A is a positive constant and the bottom of the plank is at $x = 0$. (For this plank the coefficients of kinetic and static friction are equal: $\mu_k = \mu_s = \mu$.) The worker shoves a box up the plank so that it leaves the bottom of the plank moving at speed v_0. Show that when the box first comes to rest, it will remain at rest if

$$v_0^2 \geq \frac{3g \sin^2\alpha}{A \cos\alpha}$$

6.103 ••• CALC **A Spring with Mass.** We usually ignore the kinetic energy of the moving coils of a spring, but let's try to get a reasonable approximation to this. Consider a spring of mass M, equilibrium length L_0, and spring constant k. The work done to stretch or compress the spring by a distance L is $\frac{1}{2}kX^2$, where $X = L - L_0$. Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. (a) Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) In a spring gun, a spring of mass 0.243 kg and force constant 3200 N/m is compressed 2.50 cm from its unstretched length. When the trigger is pulled, the spring pushes horizontally on a 0.053-kg ball. The work done by friction is negligible. Calculate the ball's speed when the spring reaches its uncompressed length (b) ignoring the mass of the spring and (c) including, using the results of part (a), the mass of the spring. (d) In part (c), what is the final kinetic energy of the ball and of the spring?

6.104 ••• CALC An airplane in flight is subject to an air resistance force proportional to the square of its speed v. But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward (Fig. P6.104). The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag*. At flying speeds, induced drag is inversely proportional to v^2, so that the total air resistance force can be expressed by $F_{air} = \alpha v^2 + \beta/v^2$, where α and β are positive constants that depend on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, $\alpha = 0.30\ \mathrm{N \cdot s^2/m^2}$ and $\beta = 3.5 \times 10^5\ \mathrm{N \cdot m^2/s^2}$. In steady flight, the engine must provide a forward force that exactly balances the air resistance force. (a) Calculate the speed (in km/h) at which this airplane will have the maximum *range* (that is, travel the greatest distance) for a given quantity of fuel. (b) Calculate the speed (in km/h) for which the airplane will have the maximum *endurance* (that is, remain in the air the longest time).

Figure **P6.104**



Answers

Chapter Opening Question ?

The answer is yes. As the ant was exerting an upward force on the piece of cereal, the cereal was exerting a downward force of the same magnitude on the ant (due to Newton's third law). However, because the ant's body had an upward displacement, the work that the cereal did on the ant was *negative* (see Section 6.1).

Test Your Understanding Questions

6.1 Answer: (iii) The electron has constant velocity, so its acceleration is zero and (by Newton's second law) the net force on the electron is also zero. Therefore the total work done by all the forces (equal to the work done by the net force) must be zero as well. The individual forces may do nonzero work, but that's not what the question asks.

6.2 Answer: (iv), (i), (iii), (ii) Body (i) has kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\ \mathrm{kg})(5.0\ \mathrm{m/s})^2 = 25\ \mathrm{J}$. Body (ii) had zero kinetic energy initially and then had 30 J of work done it, so its final kinetic energy is $K_2 = K_1 + W = 0 + 30\ \mathrm{J} = 30\ \mathrm{J}$. Body (iii) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1.0\ \mathrm{kg})(4.0\ \mathrm{m/s})^2 = 8.0\ \mathrm{J}$ and then had 20 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 8.0\ \mathrm{J} + 20\ \mathrm{J} = 28\ \mathrm{J}$. Body (iv) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2.0\ \mathrm{kg})(10\ \mathrm{m/s})^2 = 100\ \mathrm{J}$; when it did 80 J of work on another body, the other body did -80 J of work on body (iv), so the final kinetic energy of body (iv) is $K_2 = K_1 + W = 100\ \mathrm{J} + (-80\ \mathrm{J}) = 20\ \mathrm{J}$.

6.3 Answers: (a) (iii), (b) (iii) At any point during the pendulum bob's motion, the tension force and the weight both act perpendicular to the motion—that is, perpendicular to an infinitesimal displacement $d\vec{l}$ of the bob. (In Fig. 5.32b, the displacement $d\vec{l}$ would be directed outward from the plane of the free-body diagram.) Hence for either force the scalar product inside the integral in Eq. (6.14) is $\vec{F} \cdot d\vec{l} = 0$, and the work done along any part of the circular path (including a complete circle) is $W = \int \vec{F} \cdot d\vec{l} = 0$.

6.4 Answer: (v) The airliner has a constant horizontal velocity, so the net horizontal force on it must be zero. Hence the backward drag force must have the same magnitude as the forward force due to the combined thrust of the four engines. This means that the drag force must do *negative* work on the airplane at the same rate that the combined thrust force does *positive* work. The combined thrust does work at a rate of $4(108{,}000\ \mathrm{hp}) = 432{,}000\ \mathrm{hp}$, so the drag force must do work at a rate of $-432{,}000$ hp.

Bridging Problem

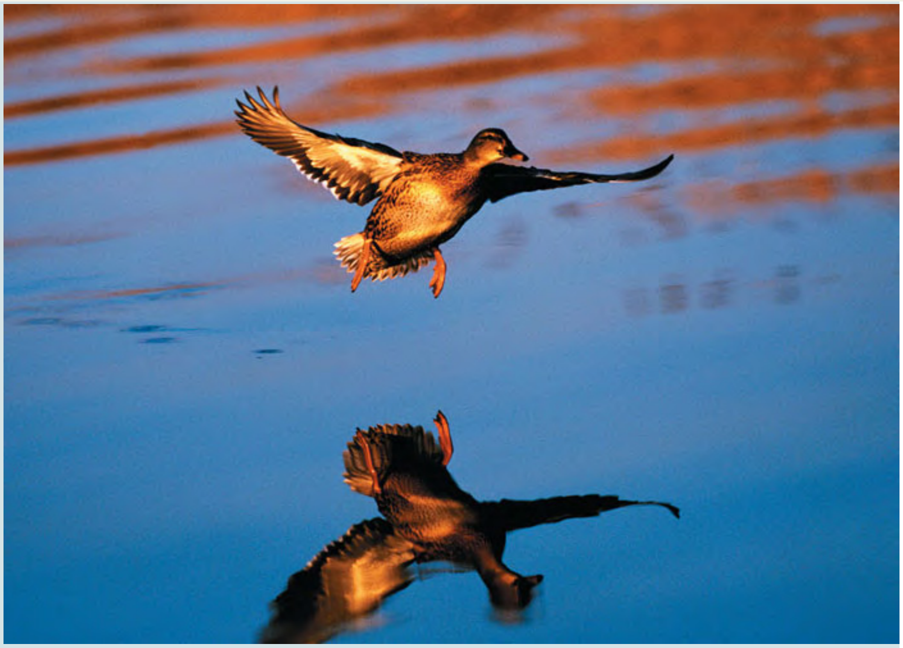
Answers: **(a)** $v_1 = \sqrt{\dfrac{2}{m}\left(mgx_1 - \frac{1}{3}\alpha x_1^3\right)} = \sqrt{2gx_1 - \dfrac{2\alpha x_1^3}{3m}}$

(b) $P = -F_{\text{spring}-1}v_1 = -\alpha x_1^2\sqrt{2gx_1 - \dfrac{2\alpha x_1^3}{3m}}$

(c) $x_2 = \sqrt{\dfrac{3mg}{\alpha}}$ **(d)** No

POTENTIAL ENERGY AND ENERGY CONSERVATION

7



? As this mallard glides in to a landing, it descends along a straight-line path at a constant speed. Does the mallard's mechanical energy increase, decrease, or stay the same during the glide? If it increases, where does the added energy come from? If it decreases, where does the lost energy go?

LEARNING GOALS

By studying this chapter, you will learn:

- How to use the concept of gravitational potential energy in problems that involve vertical motion.
- How to use the concept of elastic potential energy in problems that involve a moving body attached to a stretched or compressed spring.
- The distinction between conservative and nonconservative forces, and how to solve problems in which both kinds of forces act on a moving body.
- How to calculate the properties of a conservative force if you know the corresponding potential-energy function.
- How to use energy diagrams to understand the motion of an object moving in a straight line under the influence of a conservative force.

When a diver jumps off a high board into a swimming pool, he hits the water moving pretty fast, with a lot of kinetic energy. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force (his weight) does work on the diver as he falls. The diver's kinetic energy—energy associated with his *motion*—increases by an amount equal to the work done.

However, there is a very useful alternative way to think about work and kinetic energy. This new approach is based on the concept of *potential energy,* which is energy associated with the *position* of a system rather than its motion. In this approach, there is *gravitational potential energy* even while the diver is standing on the high board. Energy is not added to the earth–diver system as the diver falls, but rather a storehouse of energy is *transformed* from one form (potential energy) to another (kinetic energy) as he falls. In this chapter we'll see how the work–energy theorem explains this transformation.

If the diver bounces on the end of the board before he jumps, the bent board stores a second kind of potential energy called *elastic potential energy*. We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the positions of electrically charged particles relative to each other. We'll encounter this potential energy in Chapter 23.)

We will prove that in some cases the sum of a system's kinetic and potential energy, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy,* one of the most fundamental and far-reaching principles in all of science.

7.1 Gravitational Potential Energy

7.1 As a basketball descends, gravitational potential energy is converted to kinetic energy and the basketball's speed increases.

We learned in Chapter 6 that a particle gains or loses kinetic energy because it interacts with other objects that exert forces on it. During any interaction, the change in a particle's kinetic energy is equal to the total work done on the particle by the forces that act on it.

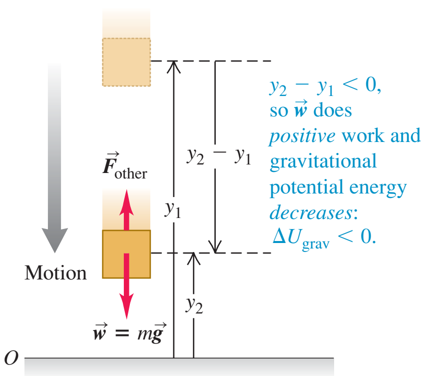
In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

This example points to the idea of an energy associated with the *position* of bodies in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; when a stone is raised into the air, there is a potential for work to be done on it by the gravitational force, but only if the stone is allowed to fall to the ground. For this reason, energy associated with position is called **potential energy.** Our discussion suggests that there is potential energy associated with a body's weight and its height above the ground. We call this *gravitational potential energy* (Fig. 7.1).

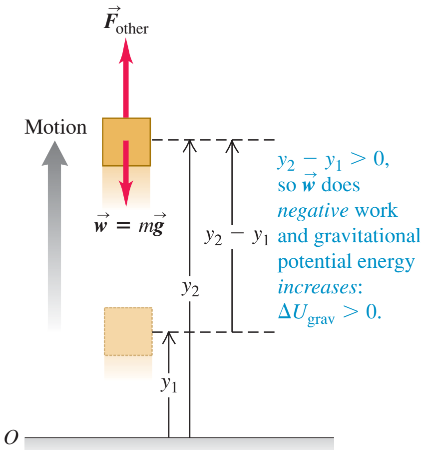
We now have *two* ways to describe what happens when a body falls without air resistance. One way is to say that gravitational potential energy decreases and the falling body's kinetic energy increases. The other way, which we learned in Chapter 6, is that a falling body's kinetic energy increases because the force of the earth's gravity (the body's weight) does work on the body. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

7.2 When a body moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes.

(a) A body moves downward



(b) A body moves upward



To begin with, however, let's derive the expression for gravitational potential energy. Suppose a body with mass m moves along the (vertical) y-axis, as in Fig. 7.2. The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{F}_{other}$. We'll assume that the body stays close enough to the earth's surface that the weight is constant. (We'll find in Chapter 13 that weight decreases with altitude.) We want to find the work done by the weight when the body moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.2a). The weight and displacement are in the same direction, so the work W_{grav} done on the body by its weight is positive;

$$W_{grav} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2 \tag{7.1}$$

This expression also gives the correct work when the body moves *upward* and y_2 is greater than y_1 (Fig. 7.2b). In that case the quantity $(y_1 - y_2)$ is negative, and W_{grav} is negative because the weight and displacement are opposite in direction.

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight mg and the height y above the origin of coordinates, is called the **gravitational potential energy,** U_{grav}:

$$U_{grav} = mgy \qquad \text{(gravitational potential energy)} \tag{7.2}$$

Its initial value is $U_{grav,1} = mgy_1$ and its final value is $U_{grav,2} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{grav} = U_{grav,2} - U_{grav,1}$. We can express the work W_{grav} done by the gravitational force during the displacement from y_1 to y_2 as

$$W_{grav} = U_{grav,1} - U_{grav,2} = -(U_{grav,2} - U_{grav,1}) = -\Delta U_{grav} \tag{7.3}$$

The negative sign in front of ΔU_{grav} is *essential.* When the body moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases ($\Delta U_{grav} > 0$). When the body moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases ($\Delta U_{grav} < 0$). It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). The unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION **To what body does gravitational potential energy "belong"?** It is *not* correct to call $U_{grav} = mgy$ the "gravitational potential energy of the body." The reason is that gravitational potential energy U_{grav} is a *shared* property of the body and the earth. The value of U_{grav} increases if the earth stays fixed and the body moves upward, away from the earth; it also increases if the body stays fixed and the earth is moved away from it. Notice that the formula $U_{grav} = mgy$ involves characteristics of both the body (its mass m) and the earth (the value of g).

Conservation of Mechanical Energy (Gravitational Forces Only)

To see what gravitational potential energy is good for, suppose the body's weight is the *only* force acting on it, so $\vec{F}_{other} = \mathbf{0}$. The body is then falling freely with no air resistance and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the body equals the change in the body's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \qquad \text{or} \qquad K_2 - K_1 = U_{grav,1} - U_{grav,2}$$

which we can rewrite as

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \qquad \text{(if only gravity does work)} \qquad (7.4)$$

or

$$\tfrac{1}{2}mv_1^2 + mgy_1 = \tfrac{1}{2}mv_2^2 + mgy_2 \qquad \text{(if only gravity does work)} \qquad (7.5)$$

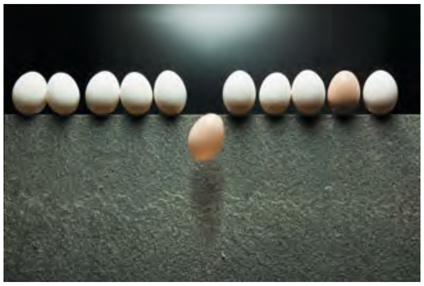
The sum $K + U_{grav}$ of kinetic and potential energy is called E, the **total mechanical energy of the system.** By "system" we mean the body of mass m and the earth considered together, because gravitational potential energy U is a shared property of both bodies. Then $E_1 = K_1 + U_{grav,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{grav,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the body's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since the positions y_1 and y_2 are arbitrary points in the motion of the body, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{grav} = \text{constant} \qquad \text{(if only gravity does work)}$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, it is conserved* (Fig. 7.3). This is our first example of the **conservation of mechanical energy.**

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy; $\Delta K < 0$ and $\Delta U_{grav} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases; $\Delta K > 0$ and $\Delta U_{grav} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be negligible). It's still true that the gravitational force does work on the body as it

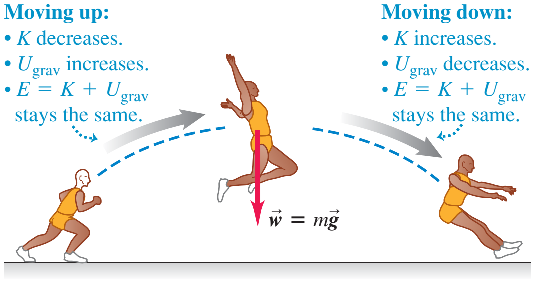
Application Which Egg Has More Mechanical Energy?
The mechanical energy of each of these identical eggs has the *same* value. The mechanical energy for an egg at rest atop the stone is purely gravitational potential energy. For the falling egg, the gravitational potential energy decreases as the egg descends and the egg's kinetic energy increases. If there is negligible air resistance, the mechanical energy of the falling egg remains constant.



MasteringPHYSICS

ActivPhysics 5.2: Upward-Moving Elevator Stops
ActivPhysics 5.3: Stopping a Downward-Moving Elevator
ActivPhysics 5.6: Skier Speed

7.3 While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Mechanical energy E—the sum of kinetic and gravitational potential energy—is conserved.



moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely.

CAUTION **Choose "zero height" to be wherever you like** When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{grav,1}$ and $U_{grav,2}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or on the *difference* in gravitational potential energy $U_{grav,2} - U_{grav,1} = mg(y_2 - y_1)$. As the following example shows, the physically significant quantity is not the value of U_{grav} at a particular point, but only the *difference* in U_{grav} between two points. So we can define U_{grav} to be zero at whatever point we choose without affecting the physics.

Example 7.1 Height of a baseball from energy conservation

You throw a 0.145-kg baseball straight up, giving it an initial velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

SOLUTION

IDENTIFY and SET UP: After the ball leaves your hand, only gravity does work on it. Hence mechanical energy is conserved, and we can use Eqs. (7.4) and (7.5). We take point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0\text{ m/s}$; at its maximum height it is instantaneously at rest, so $v_2 = 0$. We take the origin at point 1, so $y_1 = 0$ (Fig. 7.4). Our target variable, the distance the ball moves vertically between the two points, is the displacement $y_2 - y_1 = y_2 - 0 = y_2$.

EXECUTE: We have $y_1 = 0$, $U_{grav,1} = mgy_1 = 0$, and $K_2 = \frac{1}{2}mv_2^2 = 0$. Then Eq. (7.4), $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

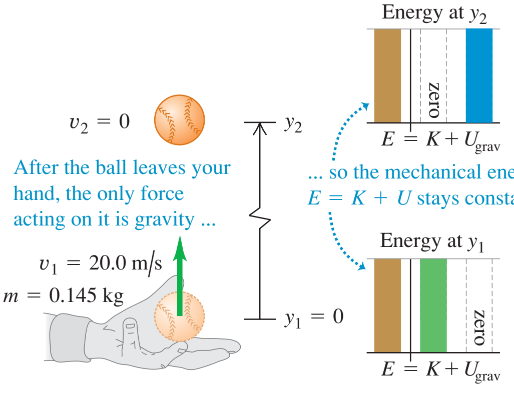
$$K_1 = U_{grav,2}$$

As the energy bar graphs in Fig. 7.4 show, this equation says that the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. We substitute $K_1 = \frac{1}{2}mv_1^2$ and $U_{grav,2} = mgy_2$ and solve for y_2:

$$\frac{1}{2}mv_1^2 = mgy_2$$

$$y_2 = \frac{v_1^2}{2g} = \frac{(20.0\text{ m/s})^2}{2(9.80\text{ m/s}^2)} = 20.4\text{ m}$$

7.4 After a baseball leaves your hand, mechanical energy $E = K + U$ is conserved.



EVALUATE: As a check on our work, use the given value of v_1 and our result for y_2 to calculate the kinetic energy at point 1 and the gravitational potential energy at point 2. You should find that these are equal: $K_1 = \frac{1}{2}mv_1^2 = 29.0\text{ J}$ and $U_{grav,2} = mgy_2 = 29.0\text{ J}$. Note also that we could have found the result $y_2 = v_1^2/2g$ using Eq. (2.13).

What if we put the origin somewhere else? For example, what if we put it 5.0 m below point 1, so that $y_1 = 5.0\text{ m}$? Then the total mechanical energy at point 1 is part kinetic and part potential; at point 2 it's still purely potential because $v_2 = 0$. You'll find that this choice of origin yields $y_2 = 25.4\text{ m}$, but again $y_2 - y_1 = 20.4\text{ m}$. In problems like this, you are free to choose the height at which $U_{grav} = 0$. The physics doesn't depend on your choice, so don't agonize over it.

When Forces Other Than Gravity Do Work

If other forces act on the body in addition to its weight, then $\vec{F}_{other}$ in Fig. 7.2 is *not* zero. For the pile driver described in Example 6.4 (Section 6.2), the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be included in $\vec{F}_{other}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{other}$. We will call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1 \tag{7.6}$$

Also, from Eq. (7.3), $W_{grav} = U_{grav,1} - U_{grav,2}$, so

$$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

which we can rearrange in the form

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2} \quad \text{(if forces other than gravity do work)} \tag{7.7}$$

Finally, using the appropriate expressions for the various energy terms, we obtain

$$\tfrac{1}{2}mv_1^2 + mgy_1 + W_{other} = \tfrac{1}{2}mv_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)} \tag{7.8}$$

The meaning of Eqs. (7.7) and (7.8) is this: *The work done by all forces other than the gravitational force equals the change in the total mechanical energy $E = K + U_{grav}$ of the system, where U_{grav} is the gravitational potential energy.* When W_{other} is positive, E increases and $K_2 + U_{grav,2}$ is greater than $K_1 + U_{grav,1}$. When W_{other} is negative, E decreases (Fig. 7.5). In the special case in which no forces other than the body's weight do work, $W_{other} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4) or (7.5).

7.5 As this skydiver moves downward, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases: The skydiver's speed and kinetic energy K stay the same, while the gravitational potential energy U decreases.

Problem-Solving Strategy 7.1 Problems Using Mechanical Energy I

IDENTIFY *the relevant concepts:* Decide whether the problem should be solved by energy methods, by using $\sum\vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces or motion along a curved path (discussed later in this section). If the problem involves elapsed time, the energy approach is usually *not* the best choice because it doesn't involve time directly.

SET UP *the problem* using the following steps:

1. When using the energy approach, first identify the initial and final states (the positions and velocities) of the bodies in question. Use the subscript 1 for the initial state and the subscript 2 for the final state. Draw sketches showing these states.
2. Define a coordinate system, and choose the level at which $y = 0$. Choose the positive y-direction to be upward, as is assumed in Eq. (7.1) and in the equations that follow from it.
3. Identify any forces that do work on each body and that *cannot* be described in terms of potential energy. (So far, this means any forces other than gravity. In Section 7.2 we'll see that the work done by an ideal spring can also be expressed as a change in potential energy.) Sketch a free-body diagram for each body.
4. List the unknown and known quantities, including the coordinates and velocities at each point. Identify the target variables.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, $U_{grav,1}$, and $U_{grav,2}$. If no other forces do work, use Eq. (7.4). If there are other forces that do work, use Eq. (7.7). Draw bar graphs showing the initial and final values of K, $U_{grav,1}$, and $E = K + U_{grav}$. Then solve to find your target variables.

EVALUATE *your answer:* Check whether your answer makes physical sense. Remember that the gravitational work is included in ΔU_{grav}, so do not include it in W_{other}.

Example 7.2 Work and energy in throwing a baseball

In Example 7.1 suppose your hand moves upward by 0.50 m while you are throwing the ball. The ball leaves your hand with an upward velocity of 20.0 m/s. (a) Find the magnitude of the force (assumed constant) that your hand exerts on the ball. (b) Find the speed of the ball at a point 15.0 m above the point where it leaves your hand. Ignore air resistance.

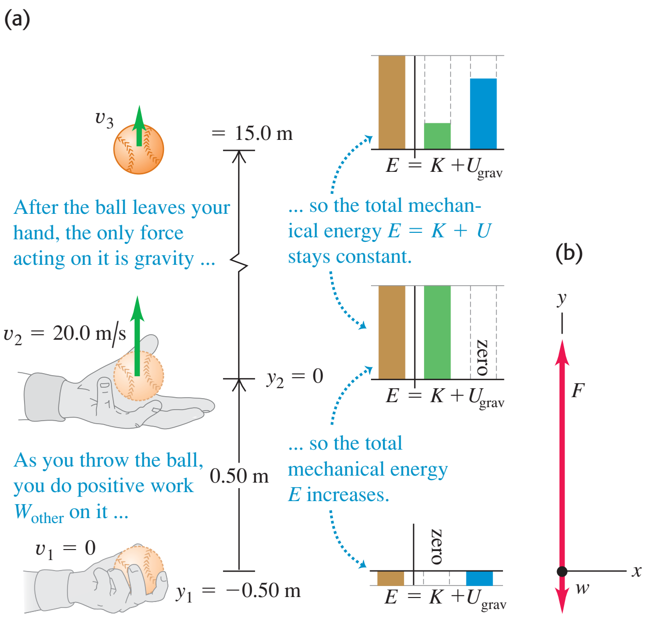
SOLUTION

IDENTIFY and SET UP: In Example 7.1 only gravity did work. Here we must include the nongravitational, "other" work done by your hand. Figure 7.6 shows a diagram of the situation, including a free-body diagram for the ball while it is being thrown. We let point 1 be where your hand begins to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above point 2. The nongravitational force $\vec{F}$ of your hand acts only between points 1 and 2. Using the same coordinate system as in Example 7.1, we have $y_1 = -0.50$ m, $y_2 = 0$, and $y_3 = 15.0$ m. The ball starts at rest at point 1, so $v_1 = 0$, and the ball's speed as it leaves your hand is $v_2 = 20.0$ m/s. Our target variables are (a) the magnitude F of the force of your hand and (b) the ball's velocity v_{3y} at point 3.

EXECUTE: (a) To determine F, we'll first use Eq. (7.7) to calculate the work W_{other} done by this force. We have

$$K_1 = 0$$
$$U_{\text{grav},1} = mgy_1 = (0.145\ \text{kg})(9.80\ \text{m/s}^2)(-0.50\ \text{m}) = -0.71\ \text{J}$$

7.6 (a) Applying energy ideas to a ball thrown vertically upward. (b) Free-body diagram for the ball as you throw it.



$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(0.145\ \text{kg})(20.0\ \text{m/s})^2 = 29.0\ \text{J}$$
$$U_{\text{grav},2} = mgy_2 = (0.145\ \text{kg})(9.80\ \text{m/s}^2)(0) = 0$$

(Don't worry that $U_{\text{grav},1}$ is less than zero; all that matters is the *difference* in potential energy from one point to another.) From Eq. (7.7),

$$K_1 + U_{\text{grav},1} + W_{\text{other}} = K_2 + U_{\text{grav},2}$$
$$W_{\text{other}} = (K_2 - K_1) + (U_{\text{grav},2} - U_{\text{grav},1})$$
$$= (29.0\ \text{J} - 0) + [0 - (-0.71\ \text{J})] = 29.7\ \text{J}$$

But since $\vec{F}$ is constant and upward, the work done by $\vec{F}$ equals the force magnitude times the displacement: $W_{\text{other}} = F(y_2 - y_1)$. So

$$F = \frac{W_{\text{other}}}{y_2 - y_1} = \frac{29.7\ \text{J}}{0.50\ \text{m}} = 59\ \text{N}$$

This is more than 40 times the weight of the ball (1.42 N).

(b) To find v_{3y}, note that between points 2 and 3 only gravity acts on the ball. So between these points mechanical energy is conserved and $W_{\text{other}} = 0$. From Eq. (7.4), we can solve for K_3 and from that solve for v_{3y}:

$$K_2 + U_{\text{grav},2} = K_3 + U_{\text{grav},3}$$
$$U_{\text{grav},3} = mgy_3 = (0.145\ \text{kg})(9.80\ \text{m/s}^2)(15.0\ \text{m}) = 21.3\ \text{J}$$
$$K_3 = (K_2 + U_{\text{grav},2}) - U_{\text{grav},3}$$
$$= (29.0\ \text{J} + 0\ \text{J}) - 21.3\ \text{J} = 7.7\ \text{J}$$

Since $K_3 = \tfrac{1}{2}mv_{3y}^2$, we find

$$v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7\ \text{J})}{0.145\ \text{kg}}} = \pm 10\ \text{m/s}$$

The plus-or-minus sign reminds us that the ball passes point 3 on the way up and again on the way down. The total mechanical energy E is constant and equal to $K_2 + U_{\text{grav},2} = 29.0$ J while the ball is in free fall, and the potential energy at point 3 is $U_{\text{grav},3} = mgy_3 = 21.3$ J whether the ball is moving up or down. So at point 3, the ball's kinetic energy K_3 (and therefore its speed) don't depend on the direction the ball is moving. The velocity v_{3y} is positive $(+10\ \text{m/s})$ when the ball is moving up and negative $(-10\ \text{m/s})$ when it is moving down; the speed v_3 is 10 m/s in either case.

EVALUATE: In Example 7.1 we found that the ball reaches a maximum height $y = 20.4$ m. At that point all of the kinetic energy it had when it left your hand at $y = 0$ has been converted to gravitational potential energy. At $y = 15.0$ m, the ball is about three-fourths of the way to its maximum height, so about three-fourths of its mechanical energy should be in the form of potential energy. (The energy bar graphs in Fig. 7.6a show this.) Can you show that this is true from our results for K_3 and $U_{\text{grav},3}$?

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the body moved along a straight vertical line. What happens when the path is slanted or curved (Fig. 7.7a)? The body is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{other}$. To find the work done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.7b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{j}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{\imath} + \Delta y\hat{j}$, so the work done by the gravitational force is

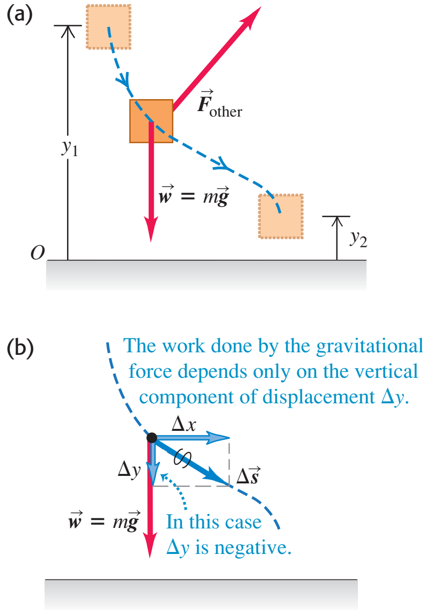
$$\vec{w} \cdot \Delta\vec{s} = -mg\hat{j} \cdot (\Delta x\hat{\imath} + \Delta y\hat{j}) = -mg\Delta y$$

The work done by gravity is the same as though the body had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{grav} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2}$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path a body follows between two points is curved, the total work done by the gravitational force depends only on the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the body's path is curved or straight.*

7.7 Calculating the change in gravitational potential energy for a displacement along a curved path.



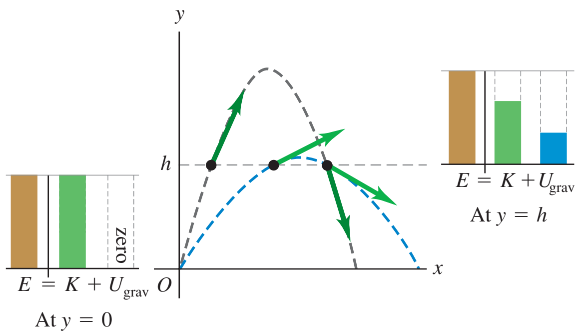
Conceptual Example 7.3 Energy in projectile motion

A batter hits two identical baseballs with the same initial speed and from the same initial height but at different initial angles. Prove that both balls have the same speed at any height h if air resistance can be neglected.

SOLUTION

The only force acting on each ball after it is hit is its weight. Hence the total mechanical energy for each ball is constant. Figure 7.8 shows the trajectories of two balls batted at the same height with the same initial speed, and thus the same total mechanical energy, but with different initial angles. At all points at the same height the potential energy is the same. Thus the kinetic energy at this height must be the same for both balls, and the speeds are the same.

7.8 For the same initial speed and initial height, the speed of a projectile at a given elevation h is always the same, neglecting air resistance.



Example 7.4 Speed at the bottom of a vertical circle

Your cousin Throckmorton skateboards from rest down a curved, frictionless ramp. If we treat Throcky and his skateboard as a particle, he moves through a quarter-circle with radius $R = 3.00$ m (Fig. 7.9). Throcky and his skateboard have a total mass of 25.0 kg. (a) Find his speed at the bottom of the ramp. (b) Find the normal force that acts on him at the bottom of the curve.

SOLUTION

IDENTIFY: We can't use the constant-acceleration equations of Chapter 2 because Throcky's acceleration isn't constant; the slope decreases as he descends. Instead, we'll use the energy approach. Throcky moves along a circular arc, so we'll also use what we learned about circular motion in Section 5.4.

SET UP: The only forces on Throcky are his weight and the normal force $\vec{n}$ exerted by the ramp (Fig. 7.9b). Although $\vec{n}$ acts all along the path, it does zero work because $\vec{n}$ is perpendicular to Throcky's displacement at every point. Hence $W_{other} = 0$ and mechanical energy is conserved. We take point 1 at the starting point and point 2 at the bottom of the ramp, and we let $y = 0$ be at the bottom of the ramp (Fig. 7.9a). We take the positive y-direction upward; then $y_1 = R$ and $y_2 = 0$. Throcky starts at rest at the top, so $v_1 = 0$. In part (a) our target variable is his speed v_2 at the bottom; in part (b) the target variable is the magnitude n of the normal force at point 2. To find n, we'll use Newton's second law and the relation $a = v^2/R$.

Continued

EXECUTE: (a) The various energy quantities are

$$K_1 = 0 \qquad U_{\text{grav},1} = mgR$$
$$K_2 = \tfrac{1}{2}mv_2^2 \qquad U_{\text{grav},2} = 0$$

From conservation of mechanical energy, Eq. (7.4),

$$K_1 + U_{\text{grav},1} = K_2 + U_{\text{grav},2}$$
$$0 + mgR = \tfrac{1}{2}mv_2^2 + 0$$
$$v_2 = \sqrt{2gR}$$
$$= \sqrt{2(9.80\ \text{m/s}^2)(3.00\ \text{m})} = 7.67\ \text{m/s}$$

This answer doesn't depend on the ramp being circular; Throcky will have the same speed $v_2 = \sqrt{2gR}$ at the bottom of any ramp of height R, no matter what its shape.

(b) To find n at point 2 using Newton's second law, we need the free-body diagram at that point (Fig. 7.9b). At point 2, Throcky is moving at speed $v_2 = \sqrt{2gR}$ in a circle of radius R; his acceleration is toward the center of the circle and has magnitude

$$a_{\text{rad}} = \frac{v_2^2}{R} = \frac{2gR}{R} = 2g$$

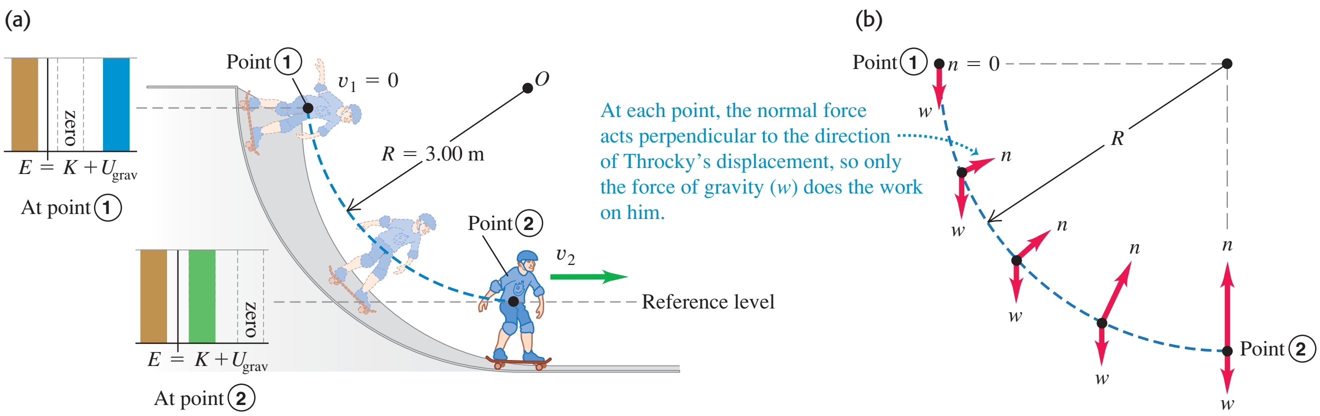
The y-component of Newton's second law is

$$\sum F_y = n + (-w) = ma_{\text{rad}} = 2mg$$
$$n = w + 2mg = 3mg$$
$$= 3(25.0\ \text{kg})(9.80\ \text{m/s}^2) = 735\ \text{N}$$

At point 2 the normal force is three times Throcky's weight. This result doesn't depend on the radius R of the ramp. We saw in Examples 5.9 and 5.23 that the magnitude of n is the *apparent weight,* so at the bottom of the *curved part* of the ramp Throcky feels as though he weighs three times his true weight mg. But when he reaches the *horizontal* part of the ramp, immediately to the right of point 2, the normal force decreases to $w = mg$ and thereafter Throcky feels his true weight again. Can you see why?

EVALUATE: This example shows a general rule about the role of forces in problems in which we use energy techniques: What matters is not simply whether a force *acts,* but whether that force *does work.* If the force does no work, like the normal force $\vec{n}$ here, then it does not appear in Eqs. (7.4) and (7.7).

7.9 (a) Throcky skateboarding down a frictionless circular ramp. The total mechanical energy is constant. (b) Free-body diagrams for Throcky and his skateboard at various points on the ramp.



Example 7.5 A vertical circle with friction

Suppose that the ramp of Example 7.4 is not frictionless, and that Throcky's speed at the bottom is only 6.00 m/s, not the 7.67 m/s we found there. What work was done on him by the friction force?

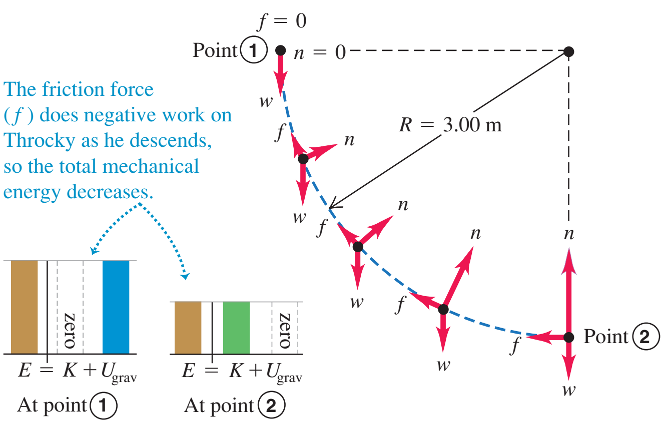
SOLUTION

IDENTIFY and SET UP: Figure 7.10 shows that again the normal force does no work, but now there is a friction force $\vec{f}$ that *does* do work W_f. Hence the nongravitational work W_{other} done on Throcky between points 1 and 2 is equal to W_f and is not zero. We use the same coordinate system and the same initial and final points as in Example 7.4. Our target variable is $W_f = W_{\text{other}}$, which we'll find using Eq. (7.7).

EXECUTE: The energy quantities are

$$K_1 = 0$$
$$U_{\text{grav},1} = mgR = (25.0\ \text{kg})(9.80\ \text{m/s}^2)(3.00\ \text{m}) = 735\ \text{J}$$
$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(25.0\ \text{kg})(6.00\ \text{m/s})^2 = 450\ \text{J}$$
$$U_{\text{grav},2} = 0$$

7.10 Energy bar graphs and free-body diagrams for Throcky skateboarding down a ramp with friction.



From Eq. (7.7),

$$W_f = W_{\text{other}} = K_2 + U_{\text{grav},2} - K_1 - U_{\text{grav},1}$$
$$= 450\text{ J} + 0 - 0 - 735\text{ J} = -285\text{ J}$$

The work done by the friction force is -285 J, and the total mechanical energy *decreases* by 285 J.

EVALUATE: Our result for W_f is negative. Can you see from the free-body diagrams in Fig. 7.10 why this must be so?

It would be very difficult to apply Newton's second law, $\sum\vec{F} = m\vec{a}$, directly to this problem because the normal and friction forces and the acceleration are continuously changing in both magnitude and direction as Throcky descends. The energy approach, by contrast, relates the motions at the top and bottom of the ramp without involving the details of the motion in between.

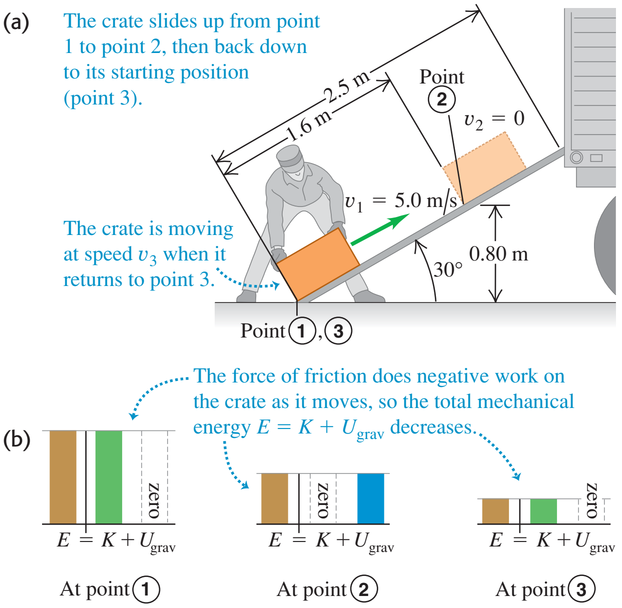
Example 7.6 An inclined plane with friction

We want to slide a 12-kg crate up a 2.5-m-long ramp inclined at 30°. A worker, ignoring friction, calculates that he can do this by giving it an initial speed of 5.0 m/s at the bottom and letting it go. But friction is *not* negligible; the crate slides only 1.6 m up the ramp, stops, and slides back down (Fig. 7.11a). (a) Find the magnitude of the friction force acting on the crate, assuming that it is constant. (b) How fast is the crate moving when it reaches the bottom of the ramp?

SOLUTION

IDENTIFY and SET UP: The friction force does work on the crate as it slides. The first part of the motion is from point 1, at the bottom of the ramp, to point 2, where the crate stops instantaneously ($v_2 = 0$). In the second part of the motion, the crate returns to the bottom of the ramp, which we'll also call point 3 (Fig. 7.11a). We take the positive y-direction upward. We take $y = 0$ (and hence $U_{\text{grav}} = 0$) to be at ground level (point 1), so that $y_1 = 0$, $y_2 = (1.6\text{ m})\sin 30° = 0.80\text{ m}$, and $y_3 = 0$. We are given $v_1 = 5.0\text{ m/s}$. In part (a) our target variable is f, the magnitude of the friction force as the crate slides up; as in Example 7.2, we'll find this using the energy approach. In part (b) our target variable is v_3, the crate's speed at the bottom of the ramp. We'll calculate the work done by friction as the crate slides back down, then use the energy approach to find v_3.

7.11 (a) A crate slides partway up the ramp, stops, and slides back down. (b) Energy bar graphs for points 1, 2, and 3.



EXECUTE: (a) The energy quantities are

$$K_1 = \tfrac{1}{2}(12\text{ kg})(5.0\text{ m/s})^2 = 150\text{ J}$$
$$U_{\text{grav},1} = 0$$
$$K_2 = 0$$
$$U_{\text{grav},2} = (12\text{ kg})(9.8\text{ m/s}^2)(0.80\text{ m}) = 94\text{ J}$$
$$W_{\text{other}} = -fs$$

Here $s = 1.6$ m. Using Eq. (7.7), we find

$$K_1 + U_{\text{grav},1} + W_{\text{other}} = K_2 + U_{\text{grav},2}$$
$$W_{\text{other}} = -fs = (K_2 + U_{\text{grav},2}) - (K_1 + U_{\text{grav},1})$$
$$= (0 + 94\text{ J}) - (150\text{ J} + 0) = -56\text{ J} = -fs$$
$$f = \frac{W_{\text{other}}}{s} = \frac{56\text{ J}}{1.6\text{ m}} = 35\text{ N}$$

The friction force of 35 N, acting over 1.6 m, causes the mechanical energy of the crate to decrease from 150 J to 94 J (Fig. 7.11b).

(b) As the crate moves from point 2 to point 3, the work done by friction has the same negative value as from point 1 to point 2. (The friction force and the displacement both reverse direction but have the same magnitudes.) The total work done by friction between points 1 and 3 is therefore

$$W_{\text{other}} = W_{\text{fric}} = -2fs = -2(56\text{ J}) = -112\text{ J}$$

From part (a), $K_1 = 150$ J and $U_{\text{grav},1} = 0$. Equation (7.7) then gives

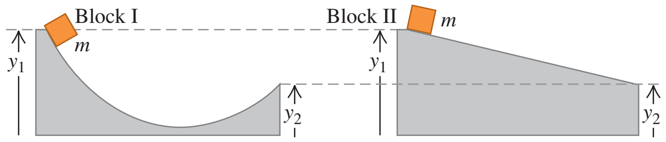
$$K_1 + U_{\text{grav},1} + W_{\text{other}} = K_3 + U_{\text{grav},3}$$
$$K_3 = K_1 + U_{\text{grav},1} - U_{\text{grav},3} + W_{\text{other}}$$
$$= 150\text{ J} + 0 - 0 + (-112\text{ J}) = 38\text{ J}$$

The crate returns to the bottom of the ramp with only 38 J of the original 150 J of mechanical energy (Fig. 7.11b). Since $K_3 = \tfrac{1}{2}mv_3^2$,

$$v_3 = \sqrt{\frac{2K_3}{m}} = \sqrt{\frac{2(38\text{ J})}{12\text{ kg}}} = 2.5\text{ m/s}$$

EVALUATE: Energy was lost due to friction, so the crate's speed $v_3 = 2.5$ m/s when it returns to the bottom of the ramp is less than the speed $v_1 = 5.0$ m/s at which it left that point. In part (b) we applied Eq. (7.7) to points 1 and 3, considering the round trip as a whole. Alternatively, we could have considered the second part of the motion by itself and applied Eq. (7.7) to points 2 and 3. Try it; do you get the same result for v_3?

Test Your Understanding of Section 7.1 The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) block I; (ii) block II; (iii) the speed is the same for both blocks.



7.2 Elastic Potential Energy

There are many situations in which we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

This is the same pattern we saw with the pile driver in Section 7.1: Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable body such as a spring or rubber band in terms of *elastic potential energy* (Fig. 7.12). A body is called *elastic* if it returns to its original shape and size after being deformed.

7.12 The Achilles tendon, which runs along the back of the ankle to the heel bone, acts like a natural spring. When it stretches and then relaxes, this tendon stores and then releases elastic potential energy. This spring action reduces the amount of work your leg muscles must do as you run.

To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring. The ideal spring is a useful idealization because many elastic bodies show this same direct proportionality between force $\vec{F}$ and displacement x, provided that x is sufficiently small.

Let's proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of a body and the earth, but elastic potential energy is stored just in the spring (or other deformable body).

Figure 7.13 shows the ideal spring from Fig. 6.18, with its left end held stationary and its right end attached to a block with mass m that can move along the x-axis. In Fig. 7.13a the body is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, and then let it go. As the block moves from one position x_1 to another position x_2, how much work does the elastic (spring) force do on the block?

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \qquad \text{(work done } on \text{ a spring)}$$

where k is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. We also saw that this expression for work is still correct if the spring is compressed, not stretched, so that x_1 or x_2 or both are negative. Now we need to find the work done *by* the spring. From Newton's third law the two quantities of work are just negatives of each other. Changing the signs in this equation, we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad \text{(work done } by \text{ a spring)}$$

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The subscript "el" stands for *elastic*. When x_1 and x_2 are both positive and $x_2 > x_1$ (Fig. 7.13b), the spring does negative work on the block, which moves in the $+x$-direction while the spring pulls on it in the $-x$-direction. The spring stretches farther, and the block slows down. When x_1 and x_2 are both positive and $x_2 < x_1$ (Fig. 7.13c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1 or x_2 or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.13d, both x_1 and x_2 are negative, but x_2 is less negative than x_1; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express the work done by the spring in terms of a given quantity at the beginning and end of the displacement. This quantity is $\frac{1}{2}kx^2$, and we define it to be the **elastic potential energy:**

$$U_{el} = \tfrac{1}{2}kx^2 \qquad \text{(elastic potential energy)} \tag{7.9}$$

Figure 7.14 is a graph of Eq. (7.9). The unit of U_{el} is the joule (J), the unit used for *all* energy and work quantities; to see this from Eq. (7.9), recall that the units of k are N/m and that $1\ \text{N}\cdot\text{m} = 1\ \text{J}$.

We can use Eq. (7.9) to express the work W_{el} done on the block by the elastic force in terms of the change in elastic potential energy:

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \tag{7.10}$$

When a stretched spring is stretched farther, as in Fig. 7.13b, W_{el} is negative and U_{el} *increases;* a greater amount of elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.13c, x decreases, W_{el} is positive, and U_{el} *decreases;* the spring loses elastic potential energy. Negative values of x refer to a compressed spring. But, as Fig. 7.14 shows, U_{el} is positive for both positive and negative x, and Eqs. (7.9) and (7.10) are valid for both cases. The more a spring is compressed *or* stretched, the greater its elastic potential energy.

CAUTION **Gravitational potential energy vs. elastic potential energy** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we do *not* have the freedom to choose $x = 0$ to be wherever we wish. To be consistent with Eq. (7.9), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, its elastic potential energy and the force that it exerts are both zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on a body. If the elastic force is the *only* force that does work on the body, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

The work–energy theorem, $W_{tot} = K_2 - K_1$, then gives us

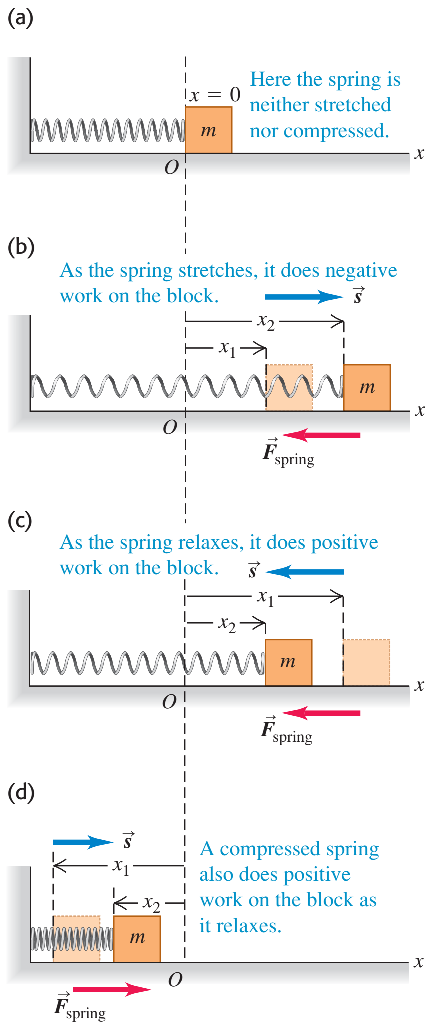
$$K_1 + U_{el,1} = K_2 + U_{el,2} \qquad \text{(if only the elastic force does work)} \tag{7.11}$$

Here U_{el} is given by Eq. (7.9), so

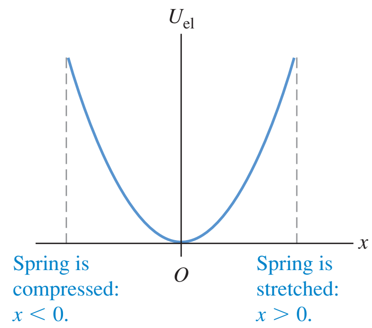
$$\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2 \qquad \text{(if only the elastic force does work)} \tag{7.12}$$

In this case the total mechanical energy $E = K + U_{el}$—the sum of kinetic and *elastic* potential energy—is *conserved*. An example of this is the motion of the

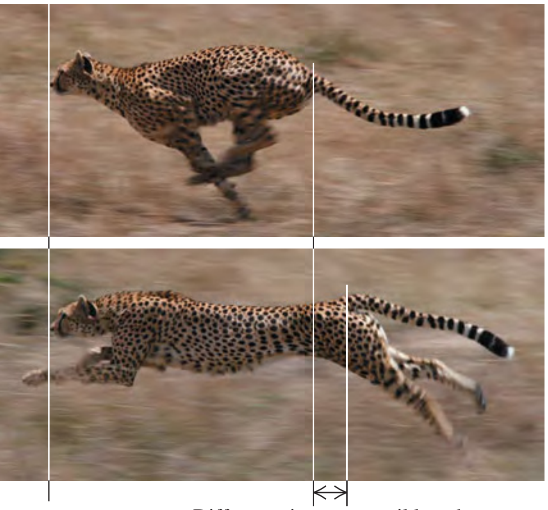
7.13 Calculating the work done by a spring attached to a block on a horizontal surface. The quantity x is the extension or compression of the spring.



7.14 The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative.



Application Elastic Potential Energy of a Cheetah
When a cheetah gallops, its back flexes and extends by an exceptional amount. Flexion of the back stretches elastic tendons and muscles along the top of the spine and also compresses the spine, storing mechanical energy. When the cheetah launches into its next bound, this energy helps to extend the spine, enabling the cheetah to run more efficiently.



Difference in nose-to-tail length

block in Fig. 7.13, provided the horizontal surface is frictionless so that no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless*. If the spring has a mass, it also has kinetic energy as the coils of the spring move back and forth. We can neglect the kinetic energy of the spring if its mass is much less than the mass m of the body attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be neglected if we want to study how a car bounces on its suspension.

Situations with Both Gravitational and Elastic Potential Energy

Equations (7.11) and (7.12) are valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces (W_{other}): $W_{tot} = W_{grav} + W_{el} + W_{other}$. Then the work–energy theorem gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \quad \text{(valid in general)} \tag{7.13}$$

or, equivalently,

$$K_1 + U_1 + W_{other} = K_2 + U_2 \quad \text{(valid in general)} \tag{7.14}$$

where $U = U_{grav} + U_{el} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. For short, we call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where $U = U_{grav} + U_{el}$ is the sum of the gravitational potential energy and the elastic potential energy.

The "system" is made up of the body of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the body, then $W_{other} = 0$ and the total mechanical energy (including both gravitational and elastic potential energy) is conserved. (You should compare Eq. (7.14) to Eqs. (7.7) and (7.8), which describe situations in which there is gravitational potential energy but no elastic potential energy.)

Trampoline jumping (Fig. 7.15) involves transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper descends through the air from the high point of the bounce, gravitational potential energy U_{grav} decreases and kinetic energy K increases. Once the jumper touches the trampoline, some of the mechanical energy goes into elastic potential energy U_{el} stored

7.15 Trampoline jumping involves an interplay among kinetic energy, gravitational potential energy, and elastic potential energy. Due to air resistance and frictional forces within the trampoline, mechanical energy is not conserved. That's why the bouncing eventually stops unless the jumper does work with his or her legs to compensate for the lost energy.

in the trampoline's springs. Beyond a certain point the jumper's speed and kinetic energy K decrease while U_{grav} continues to decrease and U_{el} continues to increase. At the low point the jumper comes to a momentary halt $(K = 0)$ at the lowest point of the trajectory (U_{grav} is minimum) and the springs are maximally stretched (U_{el} is maximum). The springs then convert their energy back into K and U_{grav}, propelling the jumper upward.

Problem-Solving Strategy 7.2 Problems Using Mechanical Energy II

Problem-Solving Strategy 7.1 (Section 7.1) is equally useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{el} = \frac{1}{2}kx^2$, where x is the displacement of the spring *from its unstretched length*. The work done by the gravitational and elastic forces is accounted for by their potential energies; the work done by other forces, W_{other}, must still be included separately.

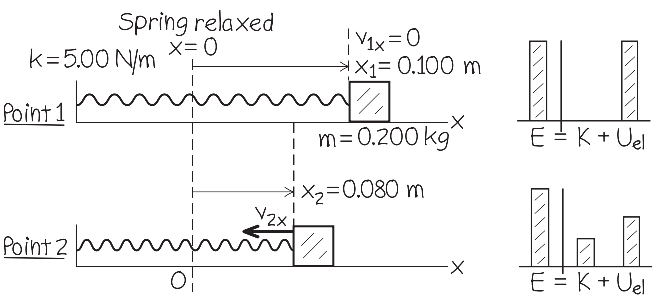
Example 7.7 Motion with elastic potential energy

A glider with mass $m = 0.200$ kg sits on a frictionless horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and release it from rest. The glider moves back toward its equilibrium position $(x = 0)$. What is its x-velocity when $x = 0.080$ m?

SOLUTION

IDENTIFY and SET UP: As the glider starts to move, elastic potential energy is converted to kinetic energy. The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor and $U = U_{el} = \frac{1}{2}kx^2$. Figure 7.16 shows our sketches. Only the spring force does work on the glider, so $W_{other} = 0$ and we may use Eq. (7.11). We designate the point where the glider is released as point 1 (that is, $x_1 = 0.100$ m) and $x_2 = 0.080$ m as point 2. We are given $v_{1x} = 0$; our target variable is v_{2x}.

7.16 Our sketches and energy bar graphs for this problem.



EXECUTE: The energy quantities are

$$K_1 = \tfrac{1}{2}mv_{1x}^2 = \tfrac{1}{2}(0.200\text{ kg})(0)^2 = 0$$

$$U_1 = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$

$$K_2 = \tfrac{1}{2}mv_{2x}^2$$

$$U_2 = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(5.00\text{ N/m})(0.080\text{ m})^2 = 0.0160\text{ J}$$

We use Eq. (7.11) to solve for K_2 and then find v_{2x}:

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m/s}$$

We choose the negative root because the glider is moving in the $-x$-direction. Our answer is $v_{2x} = -0.30$ m/s.

EVALUATE: Eventually the spring will reverse the glider's motion, pushing it back in the $+x$-direction (see Fig. 7.13d). The solution $v_{2x} = +0.30$ m/s tells us that when the glider passes through $x = 0.080$ m on this return trip, its speed will be 0.30 m/s, just as when it passed through this point while moving to the left.

Example 7.8 Motion with elastic potential energy and work done by other forces

Suppose the glider in Example 7.7 is initially at rest at $x = 0$, with the spring unstretched. You then push on the glider with a constant force $\vec{F}$ (magnitude 0.610 N) in the $+x$-direction. What is the glider's velocity when it has moved to $x = 0.100$ m?

SOLUTION

IDENTIFY and SET UP: Although the force $\vec{F}$ you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of

Continued

the work done by the force $\vec{F}$, so we must use the generalized energy relationship given by Eq. (7.13). As in Example 7.7, we ignore gravitational potential energy because the glider's height doesn't change. Hence we again have $U = U_{el} = \frac{1}{2}kx^2$. This time, we let point 1 be at $x_1 = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100$ m. The glider's displacement is then $\Delta x = x_2 - x_1 = 0.100$ m. Our target variable is v_{2x}, the velocity at point 2.

EXECUTE: The force $\vec{F}$ is constant and in the same direction as the displacement, so the work done by this force is $F\Delta x$. Then the energy quantities are

$$\begin{aligned} K_1 &= 0 \\ U_1 &= \tfrac{1}{2}kx_1^2 = 0 \\ K_2 &= \tfrac{1}{2}mv_{2x}^2 \\ U_2 &= \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(5.00\ \text{N/m})(0.100\ \text{m})^2 = 0.0250\ \text{J} \\ W_{other} &= F\Delta x = (0.610\ \text{N})(0.100\ \text{m}) = 0.0610\ \text{J} \end{aligned}$$

The initial total mechanical energy is zero; the work done by $\vec{F}$ increases the total mechanical energy to 0.0610 J, of which $U_2 = 0.0250$ J is elastic potential energy. The remainder is kinetic energy. From Eq. (7.13),

$$K_1 + U_1 + W_{other} = K_2 + U_2$$
$$K_2 = K_1 + U_1 + W_{other} - U_2$$
$$= 0 + 0 + 0.0610\ \text{J} - 0.0250\ \text{J} = 0.0360\ \text{J}$$
$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0360\ \text{J})}{0.200\ \text{kg}}} = 0.60\ \text{m/s}$$

We choose the positive square root because the glider is moving in the $+x$-direction.

EVALUATE: To test our answer, think what would be different if we disconnected the glider from the spring. Then only $\vec{F}$ would do work, there would be zero elastic potential energy at all times, and Eq. (7.13) would give us

$$K_2 = K_1 + W_{other} = 0 + 0.0610\ \text{J}$$
$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0610\ \text{J})}{0.200\ \text{kg}}} = 0.78\ \text{m/s}$$

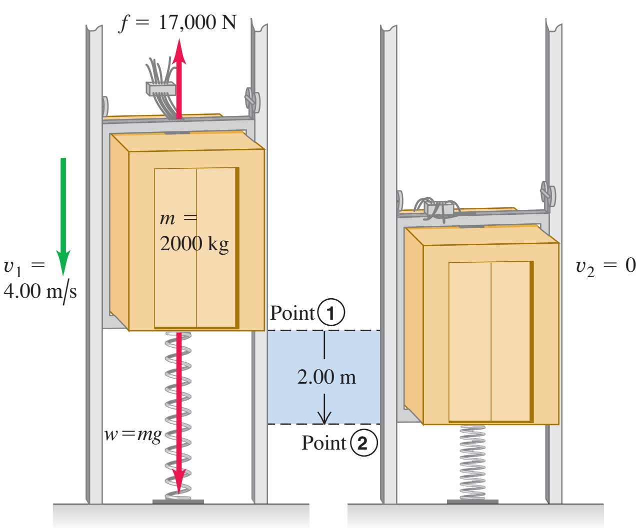
Our answer $v_{2x} = 0.60$ m/s is less than 0.78 m/s because the spring does negative work on the glider as it stretches (see Fig. 7.13b).

If you stop pushing on the glider when it reaches $x = 0.100$ m, only the spring force does work on it thereafter. Hence for $x > 0.100$ m, the total mechanical energy $E = K + U = 0.0610$ J is constant. As the spring continues to stretch, the glider slows down and the kinetic energy K decreases as the potential energy increases. The glider comes to rest at some point $x = x_3$, at which the kinetic energy is zero and the potential energy $U = U_{el} = \frac{1}{2}kx_3^2$ equals the total mechanical energy 0.0610 J. Can you show that $x_3 = 0.156$ m? (It moves an additional 0.056 m after you stop pushing.) If there is no friction, will the glider remain at rest?

Example 7.9 Motion with gravitational, elastic, and friction forces

A 2000-kg (19,600-N) elevator with broken cables in a test rig is falling at 4.00 m/s when it contacts a cushioning spring at the bottom of the shaft. The spring is intended to stop the elevator, compressing 2.00 m as it does so (Fig. 7.17). During the motion a safety clamp applies a constant 17,000-N frictional force to the elevator. What is the necessary force constant k for the spring?

7.17 The fall of an elevator is stopped by a spring and by a constant friction force.



SOLUTION

IDENTIFY and SET UP: We'll use the energy approach to determine k, which appears in the expression for elastic potential energy. This problem involves *both* gravitational and elastic potential energy. Total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator. We'll therefore use the most general form of the energy relationship, Eq. (7.13). We take point 1 as the position of the bottom of the elevator when it contacts the spring, and point 2 as its position when it stops. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -2.00$ m. With this choice the coordinate of the upper end of the spring after contact is the same as the coordinate of the elevator, so the elastic potential energy at any point between points 1 and 2 is $U_{el} = \frac{1}{2}ky^2$. The gravitational potential energy is $U_{grav} = mgy$ as usual. We know the initial and final speeds of the elevator and the magnitude of the friction force, so the only unknown is the force constant k (our target variable).

EXECUTE: The elevator's initial speed is $v_1 = 4.00$ m/s, so its initial kinetic energy is

$$K_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(2000\ \text{kg})(4.00\ \text{m/s})^2 = 16{,}000\ \text{J}$$

The elevator stops at point 2, so $K_2 = 0$. At point 1 the potential energy $U_1 = U_{grav} + U_{el}$ is zero; U_{grav} is zero because $y_1 = 0$, and $U_{el} = 0$ because the spring is uncompressed. At point 2 there is both gravitational and elastic potential energy, so

$$U_2 = mgy_2 + \tfrac{1}{2}ky_2^2$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000\text{ kg})(9.80\text{ m/s}^2)(-2.00\text{ m}) = -39{,}200\text{ J}$$

The "other" force is the constant 17,000-N friction force. It acts opposite to the 2.00-m displacement, so

$$W_{\text{other}} = -(17{,}000\text{ N})(2.00\text{ m}) = -34{,}000\text{ J}$$

We put these terms into Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$:

$$K_1 + 0 + W_{\text{other}} = 0 + \left(mgy_2 + \tfrac{1}{2}ky_2^2\right)$$

$$k = \frac{2(K_1 + W_{\text{other}} - mgy_2)}{y_2^2}$$

$$= \frac{2[16{,}000\text{ J} + (-34{,}000\text{ J}) - (-39{,}200\text{ J})]}{(-2.00\text{ m})^2}$$

$$= 1.06 \times 10^4\text{ N/m}$$

This is about one-tenth the force constant of a spring in an automobile suspension.

EVALUATE: There might seem to be a paradox here. The elastic potential energy at point 2 is

$$\tfrac{1}{2}ky_2^2 = \tfrac{1}{2}(1.06 \times 10^4\text{ N/m})(-2.00\text{ m})^2 = 21{,}200\text{ J}$$

This is *more* than the total mechanical energy at point 1:

$$E_1 = K_1 + U_1 = 16{,}000\text{ J} + 0 = 16{,}000\text{ J}$$

But the friction force *decreased* the mechanical energy of the system by 34,000 J between points 1 and 2. Did energy appear from nowhere? No. At point 2, which is below the origin, there is also *negative* gravitational potential energy $mgy_2 = -39{,}200\text{ J}$. The total mechanical energy at point 2 is therefore not 21,200 J but rather

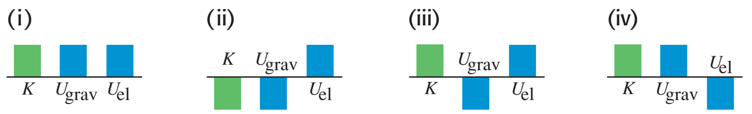
$$E_2 = K_2 + U_2 = 0 + \tfrac{1}{2}ky_2^2 + mgy_2$$

$$= 0 + 21{,}200\text{ J} + (-39{,}200\text{ J}) = -18{,}000\text{ J}$$

This is just the initial mechanical energy of 16,000 J minus 34,000 J lost to friction.

Will the elevator stay at the bottom of the shaft? At point 2 the compressed spring exerts an upward force of magnitude $F_{\text{spring}} = (1.06 \times 10^4\text{ N/m})(2.00\text{ m}) = 21{,}200\text{ N}$, while the downward force of gravity is only $w = mg = (2000\text{ kg})(9.80\text{ m/s}^2) = 19{,}600\text{ N}$. If there were no friction, there would be a net upward force of $21{,}200\text{ N} - 19{,}600\text{ N} = 1600\text{ N}$, and the elevator would rebound. But the safety clamp can exert a kinetic friction force of 17,000 N, and it can presumably exert a maximum static friction force greater than that. Hence the clamp will keep the elevator from rebounding.

Test Your Understanding of Section 7.2 Consider the situation in Example 7.9 at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m. Which of the energy bar graphs in the figure most accurately shows the kinetic energy K, gravitational potential energy U_{grav}, and elastic potential energy U_{el} at this instant?



7.3 Conservative and Nonconservative Forces

In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy. We always have in mind that later we may retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted to gravitational potential energy. But on the way down, the conversion is reversed, and the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

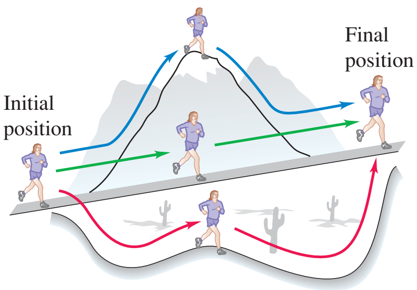
Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper at the end of the track. The glider stops as it compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases we can define a potential-energy function so that the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.** We have seen two examples of

7.18 The work done by a conservative force such as gravity depends only on the end points of a path, not on the specific path taken between those points.

Because the gravitational force is conservative, the work it does is the same for all three paths.



Mastering PHYSICS
PhET: The Ramp

conservative forces: the gravitational force and the spring force. (Later in this book we will study another conservative force, the electric force between charged objects.) An essential feature of conservative forces is that their work is always *reversible*. Anything that we deposit in the energy "bank" can later be withdrawn without loss. Another important aspect of conservative forces is that a body may move from point 1 to point 2 by various paths, but the work done by a conservative force is the same for all of these paths (Fig. 7.18). Thus, if a body stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent of height, and the work done by this force depends only on the change in height. If the body moves around a closed path, ending at the same point where it started, the *total* work done by the gravitational force is always zero.

The work done by a conservative force *always* has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the body slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. When a car with its brakes locked skids across the pavement with decreasing speed (and decreasing kinetic energy), the lost kinetic energy cannot be recovered by reversing the motion or in any other way, and mechanical energy is *not* conserved. There is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

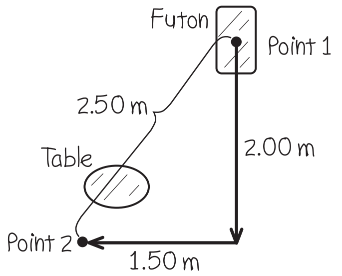
Example 7.10 Frictional work depends on the path

You are rearranging your furniture and wish to move a 40.0-kg futon 2.50 m across the room. A heavy coffee table, which you don't want to move, blocks this straight-line path. Instead, you slide the futon along a dogleg path; the doglegs are 2.00 m and 1.50 m long. How much more work must you do to push the futon along the dogleg path than along the straight-line path? The coefficient of kinetic friction is $\mu_k = 0.200$.

SOLUTION

IDENTIFY and SET UP: Here both you and friction do work on the futon, so we must use the energy relationship that includes "other" forces. We'll use this relationship to find a connection between the work that *you* do and the work that *friction* does. Figure 7.19 shows our sketch. The futon is at rest at both point 1 and point 2, so

7.19 Our sketch for this problem.



$K_1 = K_2 = 0$. There is no elastic potential energy (there are no springs), and the gravitational potential energy does not change because the futon moves only horizontally, so $U_1 = U_2$. From Eq. (7.14) it follows that $W_{\text{other}} = 0$. That "other" work done on the futon is the sum of the positive work you do, W_{you}, and the negative work done by friction, W_{fric}. Since the sum of these is zero, we have

$$W_{\text{you}} = -W_{\text{fric}}$$

Thus we'll calculate the work done by friction to determine W_{you}.

EXECUTE: The floor is horizontal, so the normal force on the futon equals its weight mg and the magnitude of the friction force is $f_k = \mu_k n = \mu_k mg$. The work you do over each path is then

$$\begin{aligned} W_{\text{you}} &= -W_{\text{fric}} = -(-f_k s) = +\mu_k mgs \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.50\ \text{m}) \\ &= 196\ \text{J} \quad \text{(straight-line path)} \\ W_{\text{you}} &= -W_{\text{fric}} = +\mu_k mgs \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.00\ \text{m} + 1.50\ \text{m}) \\ &= 274\ \text{J} \quad \text{(dogleg path)} \end{aligned}$$

The extra work you must do is 274 J − 196 J = 78 J.

EVALUATE: Friction does different amounts of work on the futon, −196 J and −274 J, on these different paths between points 1 and 2. Hence friction is a *nonconservative* force.

Example 7.11 Conservative or nonconservative?

In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (Fig. 7.20). Calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?

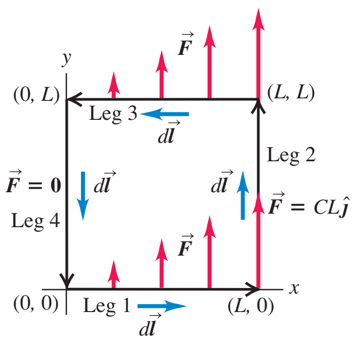
SOLUTION

IDENTIFY and SET UP: The force $\vec{F}$ is not constant, and in general it is not in the same direction as the displacement. To calculate the work done by $\vec{F}$, we'll use the general expression for work, Eq. (6.14):

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is an infinitesimal displacement. We'll calculate the work done on each leg of the square separately, and add the results to find the work done on the round trip. If this round-trip work is zero, force $\vec{F}$ is conservative and can be represented by a potential-energy function.

7.20 An electron moving around a square loop while being acted on by the force $\vec{F} = Cx\hat{j}$.



EXECUTE: On the first leg, from $(0, 0)$ to $(L, 0)$, the force is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value $\vec{F} = CL\hat{j}$ everywhere on the second leg, from $(L, 0)$ to (L, L). The displacement on this leg is in the $+y$-direction, so $d\vec{l} = dy\hat{j}$ and

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\,dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\,dy = CL \int_0^L dy = CL^2$$

On the third leg, from (L, L) to $(0, L)$, $\vec{F}$ is again perpendicular to the displacement and so $W_3 = 0$. The force is zero on the final leg, from $(0, L)$ to $(0, 0)$, so $W_4 = 0$. The work done by $\vec{F}$ on the round trip is therefore

$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a *nonconservative* force; it *cannot* be represented by a potential-energy function.

EVALUATE: Because W is positive, the mechanical energy *increases* as the electron goes around the loop. This is not a mathematical curiosity; it's a much-simplified description of what happens in an electrical generating plant. There, a loop of wire is moved through a magnetic field, which gives rise to a nonconservative force similar to the one here. Electrons in the wire gain energy as they move around the loop, and this energy is carried via transmission lines to the consumer. (We'll discuss how this works in Chapter 29.)

If the electron went *clockwise* around the loop, $\vec{F}$ would be unaffected but the direction of each infinitesimal displacement $d\vec{l}$ would be reversed. Thus the sign of work would also reverse, and the work for a clockwise round trip would be $W = -CL^2$. This is a different behavior than the nonconservative friction force. The work done by friction on a body that slides in any direction over a stationary surface is always negative (see Example 7.6 in Section 7.1).

The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic and potential energy. When a car with locked brakes skids to a stop, the tires and the road surface both become hotter. The energy associated with this change in the state of the materials is called **internal energy.** Raising the temperature of a body increases its internal energy; lowering the body's temperature decreases its internal energy.

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive.* Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{\text{int}} = -W_{\text{other}}$$

where ΔU_{int} is the change in internal energy. If we substitute this into Eq. (7.7) or (7.14), we find

$$K_1 + U_1 - \Delta U_{\text{int}} = K_2 + U_2$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0 \qquad \text{(law of conservation of energy)} \qquad (7.15)$$

7.21 When 1 liter of gasoline is burned in an automotive engine, it releases 3.3×10^7 J of internal energy. Hence $\Delta U_{\text{int}} = -3.3 \times 10^7$ J, where the minus sign means that the amount of energy stored in the gasoline has decreased. This energy can be converted to kinetic energy (making the car go faster) or to potential energy (enabling the car to climb uphill).

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (Fig. 7.21). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules to kinetic energy of the baseball. This is converted to gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back to the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

In Chapters 19 and 20, we will study the relationship of internal energy to temperature changes, heat, and work. This is the heart of the area of physics called *thermodynamics.*

Conceptual Example 7.12 Work done by friction

Let's return to Example 7.5 (Section 7.1), in which Throcky skateboards down a curved ramp. He starts with zero kinetic energy and 735 J of potential energy, and at the bottom he has 450 J of kinetic energy and zero potential energy; hence $\Delta K = +450$ J and $\Delta U = -735$ J. The work $W_{\text{other}} = W_{\text{fric}}$ done by the friction forces is -285 J, so the change in internal energy is $\Delta U_{\text{int}} = -W_{\text{other}} = +285$ J. The skateboard wheels and bearings and the ramp all get a little warmer. In accordance with Eq. (7.15), the sum of the energy changes equals zero:

$$\Delta K + \Delta U + \Delta U_{\text{int}} = +450\text{ J} + (-735\text{ J}) + 285\text{ J} = 0$$

The total energy of the system (including internal, nonmechanical forms of energy) is conserved.

Test Your Understanding of Section 7.3 In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced? (i) the same; (ii) more; (iii) less.

7.4 Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for a body with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. To stretch an ideal spring by a distance x, we exert a force equal to $+kx$. By Newton's third law the force that an ideal spring exerts on a body is opposite this, or $F_x = -kx$. The corresponding potential energy function is $U(x) = \frac{1}{2}kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: It's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$, and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\,\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx. But it is at least approximately true that

$$F_x(x)\,\Delta x = -\Delta U \qquad \text{and} \qquad F_x(x) = -\frac{\Delta U}{\Delta x}$$

You can probably see what's coming. We take the limit as $\Delta x \rightarrow 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

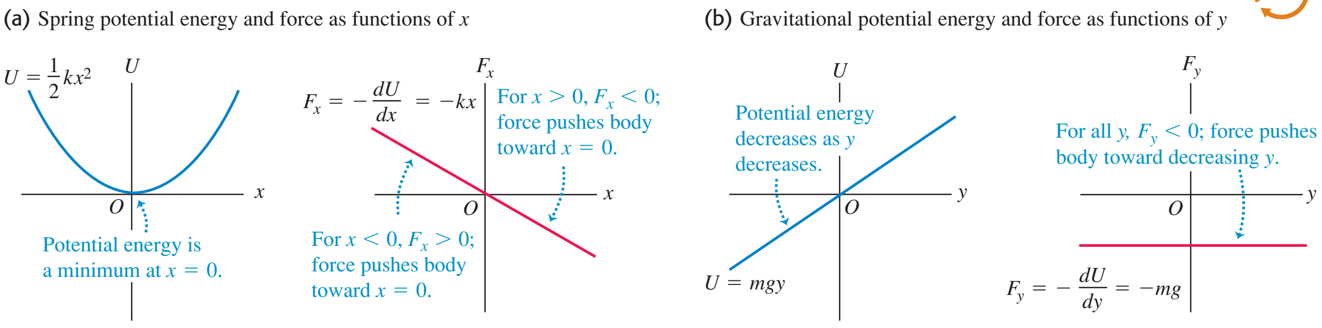
$$F_x(x) = -\frac{dU(x)}{dx} \qquad \text{(force from potential energy, one dimension)} \qquad (7.16)$$

This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

As a check, let's consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$

which is the correct expression for the force exerted by an ideal spring (Fig. 7.22a). Similarly, for gravitational potential energy we have $U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.22b).



7.22 A conservative force is the negative derivative of the corresponding potential energy.

Example 7.13 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$; a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is $U(x) = C/x$, where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable particle as a function of its position.

SOLUTION

IDENTIFY and SET UP: We are given the potential-energy function $U(x)$. We'll find the corresponding force function using Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE: The derivative of $1/x$ with respect to x is $-1/x^2$. So for $x > 0$ the force on the movable charged particle $x > 0$ is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

EVALUATE: The x-component of force is positive, corresponding to a repulsion between like electric charges. Both the potential energy and the force are very large when the particles are close together (small x), and both get smaller as the particles move farther apart (large x); the force pushes the movable particle toward large positive values of x, where the potential energy is lower. (We'll study electric forces in detail in Chapter 21.)

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions, where the particle may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. We can now use Eq. (7.16) to find each component of force. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x \, \Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

The y- and z-components of force are determined in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relationships exact, we take the limits $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, and $\Delta z \rightarrow 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a *partial derivative*. The usual notation for a partial derivative is $\partial U/\partial x$ and so on; the symbol ∂ is a modified d. So we write

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \qquad \text{(force from potential energy)} \tag{7.17}$$

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \qquad \text{(force from potential energy)} \tag{7.18}$$

The expression inside the parentheses represents a particular operation on the function U, in which we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla}U$. Thus the force is the negative of the gradient of the potential-energy function:

$$\vec{F} = -\vec{\nabla}U \tag{7.19}$$

As a check, let's substitute into Eq. (7.19) the function $U = mgy$ for gravitational potential energy:

$$\vec{F} = -\vec{\nabla}(mgy) = -\left(\frac{\partial(mgy)}{\partial x}\hat{\imath} + \frac{\partial(mgy)}{\partial y}\hat{\jmath} + \frac{\partial(mgy)}{\partial z}\hat{k}\right) = (-mg)\hat{\jmath}$$

This is just the familiar expression for the gravitational force.

Application Topography and Potential Energy Gradient

The greater the elevation of a hiker in Canada's Banff National Park, the greater is the gravitational potential energy U_{grav}. Think of an x-axis that runs horizontally from west to east and a y-axis that runs horizontally from south to north. Then the function $U_{\text{grav}}(x, y)$ tells us the elevation as a function of position in the park. Where the mountains have steep slopes, $\vec{F} = -\vec{\nabla}U_{\text{grav}}$ has a large magnitude and there's a strong force pushing you along the mountain's surface toward a region of lower elevation (and hence lower U_{grav}). There's zero force along the surface of the lake, which is all at the same elevation. Hence U_{grav} is constant at all points on the lake surface, and $\vec{F} = -\vec{\nabla}U_{\text{grav}} = \mathbf{0}$.

Example 7.14 Force and potential energy in two dimensions

A puck with coordinates x and y slides on a level, frictionless air-hockey table. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \tfrac{1}{2}k(x^2 + y^2)$$

Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

SOLUTION

IDENTIFY and SET UP: Starting with the function $U(x, y)$, we need to find the vector components and magnitude of the corresponding force $\vec{F}$. We'll find the components using Eq. (7.18). The function U doesn't depend on z, so the partial derivative of U with respect to z is $\partial U/\partial z = 0$ and the force has no z-component. We'll determine the magnitude F of the force using $F = \sqrt{F_x^2 + F_y^2}$.

EXECUTE: The x- and y-components of $\vec{F}$ are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18), the vector expression for the force is

$$\vec{F} = (-kx)\hat{\imath} + (-ky)\hat{\jmath} = -k(x\hat{\imath} + y\hat{\jmath})$$

The magnitude of the force is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

EVALUATE: Because $x\hat{\imath} + y\hat{\jmath}$ is just the position vector $\vec{r}$ of the particle, we can rewrite our result as $\vec{F} = -k\vec{r}$. This represents a force that is opposite in direction to the particle's position vector—that is, a force directed toward the origin, $r = 0$. This is the force that would be exerted on the puck if it were attached to one end of a spring that obeys Hooke's law and has a negligibly small unstretched length compared to the other distances in the problem. (The other end is attached to the air-hockey table at $r = 0$.)

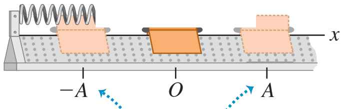
To check our result, note that $U = \tfrac{1}{2}kr^2$, where $r^2 = x^2 + y^2$. We can find the force from this expression using Eq. (7.16) with x replaced by r:

$$F_r = -\frac{dU}{dr} = -\frac{d}{dr}\left(\tfrac{1}{2}kr^2\right) = -kr$$

As we found above, the force has magnitude kr; the minus sign indicates that the force is toward the origin (at $r = 0$).

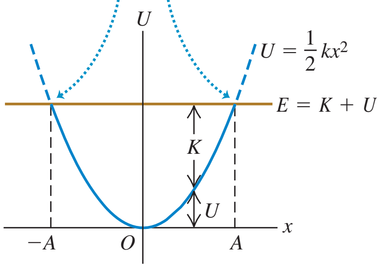
7.23 (a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function.

(a)



(b)

On the graph, the limits of motion are the points where the U curve intersects the horizontal line representing total mechanical energy E.



Test Your Understanding of Section 7.4 A particle moving along the x-axis is acted on by a conservative force F_x. At a certain point, the force is zero. (a) Which of the following statements about the value of the potential-energy function $U(x)$ at that point is correct? (i) $U(x) = 0$; (ii) $U(x) > 0$; (iii) $U(x) < 0$; (iv) not enough information is given to decide. (b) Which of the following statements about the value of the derivative of $U(x)$ at that point is correct? (i) $dU(x)/dx = 0$; (ii) $dU(x)/dx > 0$; (iii) $dU(x)/dx < 0$; (iv) not enough information is given to decide. ❙

7.5 Energy Diagrams

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. Figure 7.23a shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.23b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

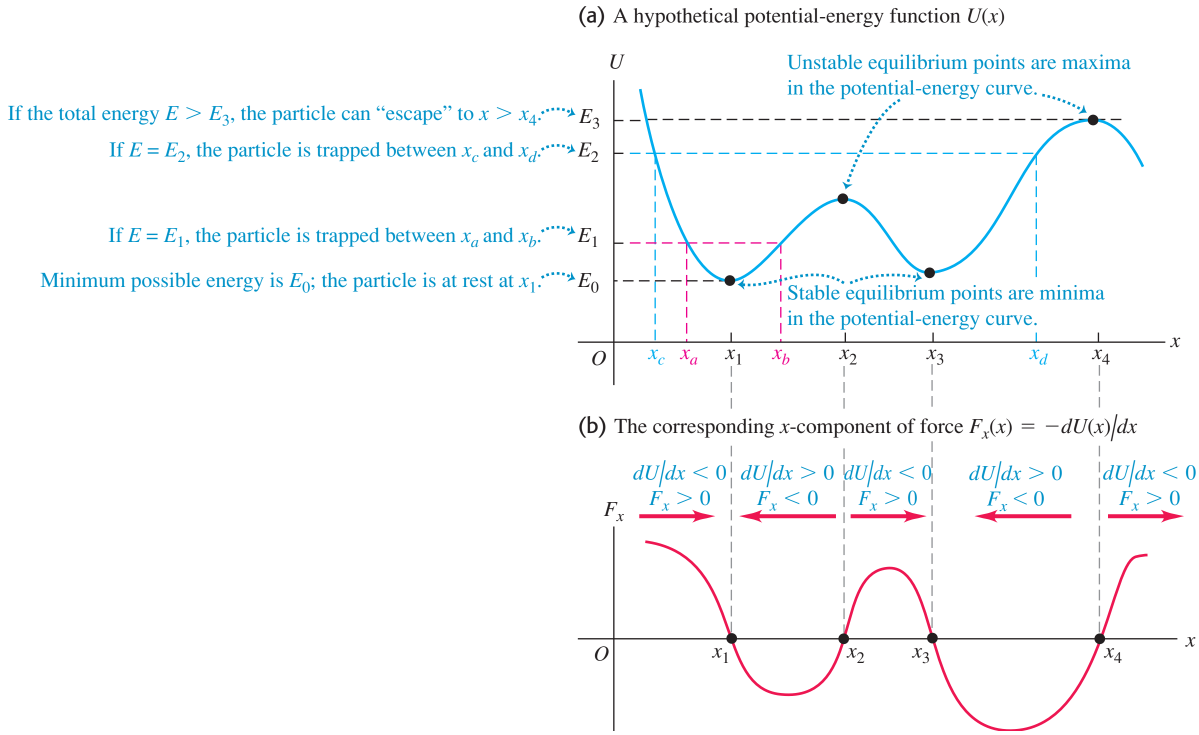
The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in the diagram. Thus the speed v is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

At each point, the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.22a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again directed toward the origin. Such a force is called a *restoring force;* when the glider is displaced to either side of $x = 0$, the force tends to "restore" it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium.** More generally, *any minimum in a potential-energy curve is a stable equilibrium position.*

Figure 7.24a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.24b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At each of these points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position.*

Application Acrobats in Equilibrium
Each of these acrobats is in *unstable* equilibrium. The gravitational potential energy is lower no matter which way an acrobat tips, so if she begins to fall she will keep on falling. Staying balanced requires the acrobats' constant attention.

7.24 The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$.



CAUTION **Potential energy and the direction of a conservative force** The direction of the force on a body is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. As we discussed in Section 7.1, the physically significant quantity is the *difference* in the values of U between two points, which is just what the derivative $F_x = -dU/dx$ measures. This means that you can always add a constant to the potential-energy function without changing the physics of the situation.

PhET: Energy Skate Park

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.24a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well,* and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the least possible total energy the system can have.

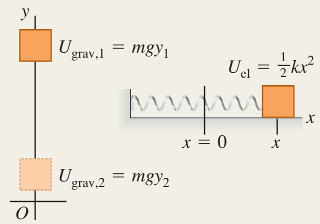
Test Your Understanding of Section 7.5 The curve in Fig. 7.24b has a maximum at a point between x_2 and x_3. Which statement correctly describes what happens to the particle when it is at this point? (i) The particle's acceleration is zero. (ii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (iii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3. (iv) The particle accelerates in the negative x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (v) The particle accelerates in the negative x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3.

CHAPTER 7 SUMMARY

Gravitational potential energy and elastic potential energy: The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy $U_{grav} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{el} = \frac{1}{2}kx^2$.

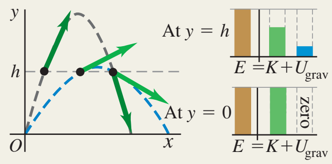
$$\begin{aligned} W_{grav} &= mgy_1 - mgy_2 \\ &= U_{grav,1} - U_{grav,2} \\ &= -\Delta U_{grav} \end{aligned} \qquad (7.1), (7.3)$$

$$\begin{aligned} W_{el} &= \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \\ &= U_{el,1} - U_{el,2} = -\Delta U_{el} \end{aligned} \qquad (7.10)$$



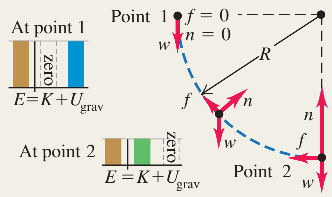
When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energy: $U = U_{grav} + U_{el}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energy is conserved. This sum $E = K + U$ is called the total mechanical energy. (See Examples 7.1, 7.3, 7.4, and 7.7.)

$$K_1 + U_1 = K_2 + U_2 \qquad (7.4), (7.11)$$



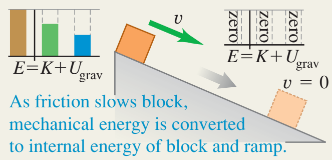
When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy). (See Examples 7.2, 7.5, 7.6, 7.8, and 7.9.)

$$K_1 + U_1 + W_{other} = K_2 + U_2 \qquad (7.14)$$



Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of bodies. The sum of kinetic, potential, and internal energy is always conserved. (See Examples 7.10–7.12.)

$$\Delta K + \Delta U + \Delta U_{int} = 0 \qquad (7.15)$$



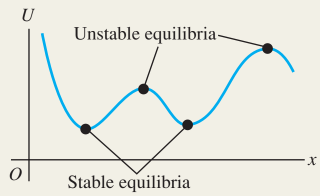
Determining force from potential energy: For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U. (See Examples 7.13 and 7.14.)

$$F_x(x) = -\frac{dU(x)}{dx} \qquad (7.16)$$

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad (7.17)$$

$$F_z = -\frac{\partial U}{\partial z}$$

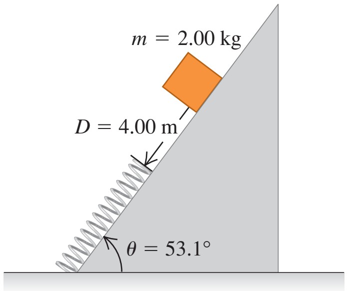
$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \qquad (7.18)$$



BRIDGING PROBLEM A Spring and Friction on an Incline

A 2.00-kg package is released on a 53.1° incline, 4.00 m from a long spring with force constant 1.20×10^2 N/m that is attached at the bottom of the incline (Fig. 7.25). The coefficients of friction between the package and incline are $\mu_s = 0.400$ and $\mu_k = 0.200$. The mass of the spring is negligible. (a) What is the maximum compression of the spring? (b) The package rebounds up the incline. How close does it get to its original position? (c) What is the change in the internal energy of the package and incline from when the package is released to when it rebounds to its maximum height?

7.25 The initial situation.



SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. This problem involves the gravitational force, a spring force, and the friction force, as well as the normal force that acts on the package. Since the spring force isn't constant, you'll have to use energy methods. Is mechanical energy conserved during any part of the motion? Why or why not?
2. Draw free-body diagrams for the package as it is sliding down the incline and sliding back up the incline. Include your choice of coordinate axis. (*Hint:* If you choose $x = 0$ to be at the end of the uncompressed spring, you'll be able to use $U_{el} = \frac{1}{2}kx^2$ for the elastic potential energy of the spring.)
3. Label the three critical points in the package's motion: its starting position, its position when it comes to rest with the spring maximally compressed, and its position when it's rebounded as far as possible up the incline. (*Hint:* You can assume that the package is no longer in contact with the spring at the last of these positions. If this turns out to be incorrect, you'll calculate a value of x that tells you the spring is still partially compressed at this point.)
4. Make a list of the unknown quantities and decide which of these are the target variables.

EXECUTE

5. Find the magnitude of the friction force that acts on the package. Does the magnitude of this force depend on whether the package is moving up or down the incline, or on whether or not the package is in contact with the spring? Does the *direction* of the normal force depend on any of these?
6. Write the general energy equation for the motion of the package between the first two points you labeled in step 3. Use this equation to solve for the distance that the spring is compressed when the package is at its lowest point. (*Hint:* You'll have to solve a quadratic equation. To decide which of the two solutions of this equation is the correct one, remember that the distance the spring is compressed is positive.)
7. Write the general energy equation for the motion of the package between the second and third points you labeled in step 3. Use this equation to solve for how far the package rebounds.
8. Calculate the change in internal energy for the package's trip down and back up the incline. Remember that the amount the internal energy *increases* is equal to the amount the total mechanical energy *decreases*.

EVALUATE

9. Was it correct to assume in part (b) that the package is no longer in contact with the spring when it reaches it maximum rebound height?
10. Check your result for part (c) by finding the total work done by the force of friction over the entire trip. Is this in accordance with your result from step 8?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q7.1 A baseball is thrown straight up with initial speed v_0. If air resistance cannot be ignored, when the ball returns to its initial height its speed is less than v_0. Explain why, using energy concepts.

Q7.2 A projectile has the same initial kinetic energy no matter what the angle of projection. Why doesn't it rise to the same maximum height in each case?

Q7.3 An object is released from rest at the top of a ramp. If the ramp is frictionless, does the object's speed at the bottom of the ramp depend on the shape of the ramp or just on its height? Explain. What if the ramp is *not* frictionless?

Q7.4 An egg is released from rest from the roof of a building and falls to the ground. Its fall is observed by a student on the roof of the building, who uses coordinates with origin at the roof, and by a student on the ground, who uses coordinates with origin at the ground. Do the two students assign the same or different values to the initial gravitational potential energy, the final gravitational potential energy, the change in gravitational potential energy, and the kinetic energy of the egg just before it strikes the ground? Explain.

Q7.5 A physics teacher had a bowling ball suspended from a very long rope attached to the high ceiling of a large lecture hall. To illustrate his faith in conservation of energy, he would back up to one side of the stage, pull the ball far to one side until the taut rope brought it just to the end of his nose, and then release it. The massive ball would swing in a mighty arc across the stage and then return to stop momentarily just in front of the nose of the stationary, unflinching teacher. However, one day after the demonstration he looked up just in time to see a student at the other side of the stage *push* the ball away from his nose as he tried to duplicate the demonstration. Tell the rest of the story and explain the reason for the potentially tragic outcome.

Q7.6 Lost Energy? The principle of the conservation of energy tells us that energy is never lost, but only changes from one form to another. Yet in many ordinary situations, energy may appear to be lost. In each case, explain what happens to the "lost" energy. (a) A box sliding on the floor comes to a halt due to friction. How did friction take away its kinetic energy, and what happened to that energy? (b) A car stops when you apply the brakes. What happened to its kinetic energy? (c) Air resistance uses up some of the original gravitational potential energy of a falling object. What type of energy did the "lost" potential energy become? (d) When a returning space shuttle touches down on the runway, it has lost almost all its kinetic energy and gravitational potential energy. Where did all that energy go?

Q7.7 Is it possible for a frictional force to *increase* the mechanical energy of a system? If so, give examples.

Q7.8 A woman bounces on a trampoline, going a little higher with each bounce. Explain how she increases the total mechanical energy.

Q7.9 Fractured Physics. People often call their electric bill a *power* bill, yet the quantity on which the bill is based is expressed in *kilowatt-hours.* What are people really being billed for?

Q7.10 A rock of mass m and a rock of mass $2m$ are both released from rest at the same height and feel no air resistance as they fall. Which statements about these rocks are true? (There may be more than one correct choice.) (a) Both have the same initial gravitational potential energy. (b) Both have the same kinetic energy when they reach the ground. (c) Both reach the ground with the same speed. (d) When it reaches the ground, the heavier rock has twice the kinetic energy of the lighter one. (e) When it reaches the ground, the heavier rock has four times the kinetic energy of the lighter one.

Q7.11 On a friction-free ice pond, a hockey puck is pressed against (but not attached to) a fixed ideal spring, compressing the spring by a distance x_0. The maximum energy stored in the spring is U_0, the maximum speed the puck gains after being released is v_0, and its maximum kinetic energy is K_0. Now the puck is pressed so it compresses the spring twice as far as before. In this case, (a) what is the maximum potential energy stored in the spring (in terms of U_0), and (b) what are the puck's maximum kinetic energy and speed (in terms of K_0 and x_0)?

Q7.12 When people are cold, they often rub their hands together to warm them up. How does doing this produce heat? Where did the heat come from?

Q7.13 You often hear it said that most of our energy ultimately comes from the sun. Trace each of the following energies back to the sun: (a) the kinetic energy of a jet plane; (b) the potential energy gained by a mountain climber; (c) the electrical energy used to run a computer; (d) the electrical energy from a hydroelectric plant.

Q7.14 A box slides down a ramp and work is done on the box by the forces of gravity and friction. Can the work of each of these forces be expressed in terms of the change in a potential-energy function? For each force explain why or why not.

Q7.15 In physical terms, explain why friction is a nonconservative force. Does it store energy for future use?

Q7.16 A compressed spring is clamped in its compressed position and then is dissolved in acid. What becomes of its potential energy?

Q7.17 Since only changes in potential energy are important in any problem, a student decides to let the elastic potential energy of a spring be zero when the spring is stretched a distance x_1. The student decides, therefore, to let $U = \frac{1}{2}k(x - x_1)^2$. Is this correct? Explain.

Q7.18 Figure 7.22a shows the potential-energy function for the force $F_x = -kx$. Sketch the potential-energy function for the force $F_x = +kx$. For this force, is $x = 0$ a point of equilibrium? Is this equilibrium stable or unstable? Explain.

Q7.19 Figure 7.22b shows the potential-energy function associated with the gravitational force between an object and the earth. Use this graph to explain why objects always fall toward the earth when they are released.

Q7.20 For a system of two particles we often let the potential energy for the force between the particles approach zero as the separation of the particles approaches infinity. If this choice is made, explain why the potential energy at noninfinite separation is positive if the particles repel one another and negative if they attract.

Q7.21 Explain why the points $x = A$ and $x = -A$ in Fig. 7.23b are called *turning points.* How are the values of E and U related at a turning point?

Q7.22 A particle is in *neutral equilibrium* if the net force on it is zero and remains zero if the particle is displaced slightly in any direction. Sketch the potential-energy function near a point of neutral equilibrium for the case of one-dimensional motion. Give an example of an object in neutral equilibrium.

Q7.23 The net force on a particle of mass m has the potential-energy function graphed in Fig. 7.24a. If the total energy is E_1, graph the speed v of the particle versus its position x. At what value of x is the speed greatest? Sketch v versus x if the total energy is E_2.

Q7.24 The potential-energy function for a force $\vec{F}$ is $U = \alpha x^3$, where α is a positive constant. What is the direction of $\vec{F}$?

EXERCISES

Section 7.1 Gravitational Potential Energy

7.1 • In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

7.2 • BIO **How High Can We Jump?** The maximum height a typical human can jump from a crouched start is about 60 cm. By how much does the gravitational potential energy increase for a 72-kg person in such a jump? Where does this energy come from?

7.3 •• CP A 120-kg mail bag hangs by a vertical rope 3.5 m long. A postal worker then displaces the bag to a position 2.0 m sideways from its original position, always keeping the rope taut. (a) What horizontal force is necessary to hold the bag in the new position? (b) As the bag is moved to this position, how much work is done (i) by the rope and (ii) by the worker?

7.4 •• BIO **Food Calories.** The *food calorie,* equal to 4186 J, is a measure of how much energy is released when food is metabolized by the body. A certain brand of fruit-and-cereal bar contains

140 food calories per bar. (a) If a 65-kg hiker eats one of these bars, how high a mountain must he climb to "work off" the calories, assuming that all the food energy goes only into increasing gravitational potential energy? (b) If, as is typical, only 20% of the food calories go into mechanical energy, what would be the answer to part (a)? (*Note:* In this and all other problems, we are assuming that 100% of the food calories that are eaten are absorbed and used by the body. This is actually not true. A person's "metabolic efficiency" is the percentage of calories eaten that are actually used; the rest are eliminated by the body. Metabolic efficiency varies considerably from person to person.)

7.5 • A baseball is thrown from the roof of a 22.0-m-tall building with an initial velocity of magnitude 12.0 m/s and directed at an angle of 53.1° above the horizontal. (a) What is the speed of the ball just before it strikes the ground? Use energy methods and ignore air resistance. (b) What is the answer for part (a) if the initial velocity is at an angle of 53.1° *below* the horizontal? (c) If the effects of air resistance are included, will part (a) or (b) give the higher speed?

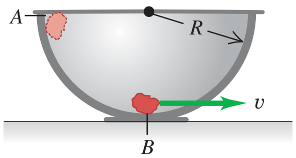
7.6 •• A crate of mass M starts from rest at the top of a frictionless ramp inclined at an angle α above the horizontal. Find its speed at the bottom of the ramp, a distance d from where it started. Do this in two ways: (a) Take the level at which the potential energy is zero to be at the bottom of the ramp with y positive upward. (b) Take the zero level for potential energy to be at the top of the ramp with y positive upward. (c) Why did the normal force not enter into your solution?

7.7 •• BIO **Human Energy vs. Insect Energy.** For its size, the common flea is one of the most accomplished jumpers in the animal world. A 2.0-mm-long, 0.50-mg critter can reach a height of 20 cm in a single leap. (a) Neglecting air drag, what is the takeoff speed of such a flea? (b) Calculate the kinetic energy of this flea at takeoff and its kinetic energy per kilogram of mass. (c) If a 65-kg, 2.0-m-tall human could jump to the same height compared with his length as the flea jumps compared with its length, how high could the human jump, and what takeoff speed would he need? (d) In fact, most humans can jump no more than 60 cm from a crouched start. What is the kinetic energy per kilogram of mass at takeoff for such a 65-kg person? (e) Where does the flea store the energy that allows it to make such a sudden leap?

7.8 •• An empty crate is given an initial push down a ramp, starting with speed v_0, and reaches the bottom with speed v and kinetic energy K. Some books are now placed in the crate, so that the total mass is quadrupled. The coefficient of kinetic friction is constant and air resistance is negligible. Starting again with v_0 at the top of the ramp, what are the speed and kinetic energy at the bottom? Explain the reasoning behind your answers.

7.9 •• CP A small rock with mass 0.20 kg is released from rest at point A, which is at the top edge of a large, hemispherical bowl with radius $R = 0.50$ m (Fig. E7.9). Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolls. The work done by friction on the rock when it moves from point A to point B at the bottom of the bowl has magnitude 0.22 J. (a) Between points A and B, how much work is done on the rock by (i) the normal force and (ii) gravity? (b) What is the speed of the rock as it reaches point B? (c) Of the three forces acting on the rock as it slides down the bowl, which (if any) are constant and which are not? Explain. (d) Just as the rock reaches point B, what is the normal force on it due to the bottom of the bowl?

Figure **E7.9**



7.10 •• BIO **Bone Fractures.** The maximum energy that a bone can absorb without breaking depends on its characteristics, such as its cross-sectional area and its elasticity. For healthy human leg bones of approximately 6.0 cm^2 cross-sectional area, this energy has been experimentally measured to be about 200 J. (a) From approximately what maximum height could a 60-kg person jump and land rigidly upright on both feet without breaking his legs? (b) You are probably surprised at how small the answer to part (a) is. People obviously jump from much greater heights without breaking their legs. How can that be? What else absorbs the energy when they jump from greater heights? (*Hint:* How did the person in part (a) land? How do people normally land when they jump from greater heights?) (c) In light of your answers to parts (a) and (b), what might be some of the reasons that older people are much more prone than younger ones to bone fractures from simple falls (such as a fall in the shower)?

7.11 •• You are testing a new amusement park roller coaster with an empty car of mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m/s, and at the top of the loop (point B) it has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

7.12 • **Tarzan and Jane.** Tarzan, in one tree, sights Jane in another tree. He grabs the end of a vine with length 20 m that makes an angle of 45° with the vertical, steps off his tree limb, and swings down and then up to Jane's open arms. When he arrives, his vine makes an angle of 30° with the vertical. Determine whether he gives her a tender embrace or knocks her off her limb by calculating Tarzan's speed just before he reaches Jane. You can ignore air resistance and the mass of the vine.

7.13 •• CP A 10.0-kg microwave oven is pushed 8.00 m up the sloping surface of a loading ramp inclined at an angle of 36.9° above the horizontal, by a constant force $\vec{F}$ with a magnitude 110 N and acting parallel to the ramp. The coefficient of kinetic friction between the oven and the ramp is 0.250. (a) What is the work done on the oven by the force $\vec{F}$? (b) What is the work done on the oven by the friction force? (c) Compute the increase in potential energy for the oven. (d) Use your answers to parts (a), (b), and (c) to calculate the increase in the oven's kinetic energy. (e) Use $\Sigma\vec{F} = m\vec{a}$ to calculate the acceleration of the oven. Assuming that the oven is initially at rest, use the acceleration to calculate the oven's speed after traveling 8.00 m. From this, compute the increase in the oven's kinetic energy, and compare it to the answer you got in part (d).

Section 7.2 Elastic Potential Energy

7.14 •• An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15-kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.

7.15 •• A force of 800 N stretches a certain spring a distance of 0.200 m. (a) What is the potential energy of the spring when it is stretched 0.200 m? (b) What is its potential energy when it is compressed 5.00 cm?

7.16 • BIO **Tendons.** Tendons are strong elastic fibers that attach muscles to bones. To a reasonable approximation, they obey Hooke's law. In laboratory tests on a particular tendon, it was found that, when a 250-g object was hung from it, the tendon stretched 1.23 cm. (a) Find the force constant of this tendon in N/m. (b) Because of its thickness, the maximum tension this tendon can support without rupturing is 138 N. By how much can the tendon stretch without rupturing, and how much energy is stored in it at that point?

7.17 • A spring stores potential energy U_0 when it is compressed a distance x_0 from its uncompressed length. (a) In terms of U_0, how much energy does it store when it is compressed (i) twice as much and (ii) half as much? (b) In terms of x_0, how much must it be compressed from its uncompressed length to store (i) twice as much energy and (ii) half as much energy?

7.18 • A slingshot will shoot a 10-g pebble 22.0 m straight up. (a) How much potential energy is stored in the slingshot's rubber band? (b) With the same potential energy stored in the rubber band, how high can the slingshot shoot a 25-g pebble? (c) What physical effects did you ignore in solving this problem?

7.19 •• A spring of negligible mass has force constant $k = 1600\ \text{N/m}$. (a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then drop a 1.20-kg book onto it from a height of 0.80 m above the top of the spring. Find the maximum distance the spring will be compressed.

7.20 • A 1.20-kg piece of cheese is placed on a vertical spring of negligible mass and force constant $k = 1800\ \text{N/m}$ that is compressed 15.0 cm. When the spring is released, how high does the cheese rise from this initial position? (The cheese and the spring are *not* attached.)

7.21 •• Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the displacement x of the glider from its equilibrium position when its speed is 0.20 m/s? (You should get more than one answer. Explain why.)

7.22 •• Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. (a) As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the speed of the glider when it returns to $x = 0$? (b) What must the initial displacement of the glider be if its maximum speed in the subsequent motion is to be 2.50 m/s?

7.23 •• A 2.50-kg mass is pushed against a horizontal spring of force constant 25.0 N/cm on a frictionless air table. The spring is attached to the tabletop, and the mass is not attached to the spring in any way. When the spring has been compressed enough to store 11.5 J of potential energy in it, the mass is suddenly released from rest. (a) Find the greatest speed the mass reaches. When does this occur? (b) What is the greatest acceleration of the mass, and when does it occur?

7.24 •• (a) For the elevator of Example 7.9 (Section 7.2), what is the speed of the elevator after it has moved downward 1.00 m from point 1 in Fig. 7.17? (b) When the elevator is 1.00 m below point 1 in Fig. 7.17, what is its acceleration?

7.25 •• You are asked to design a spring that will give a 1160-kg satellite a speed of 2.50 m/s relative to an orbiting space shuttle. Your spring is to give the satellite a maximum acceleration of $5.00g$. The spring's mass, the recoil kinetic energy of the shuttle, and changes in gravitational potential energy will all be negligible. (a) What must the force constant of the spring be? (b) What distance must the spring be compressed?

7.26 •• A 2.50-kg block on a horizontal floor is attached to a horizontal spring that is initially compressed 0.0300 m. The spring has force constant 840 N/m. The coefficient of kinetic friction between the floor and the block is $\mu_k = 0.40$. The block and spring are released from rest and the block slides along the floor. What is the speed of the block when it has moved a distance of 0.0200 m from its initial position? (At this point the spring is compressed 0.0100 m.)

Section 7.3 Conservative and Nonconservative Forces

7.27 • A 10.0-kg box is pulled by a horizontal wire in a circle on a rough horizontal surface for which the coefficient of kinetic friction is 0.250. Calculate the work done by friction during one complete circular trip if the radius is (a) 2.00 m and (b) 4.00 m. (c) On the basis of the results you just obtained, would you say that friction is a conservative or nonconservative force? Explain.

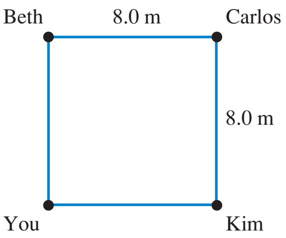
7.28 • A 75-kg roofer climbs a vertical 7.0-m ladder to the flat roof of a house. He then walks 12 m on the roof, climbs down another vertical 7.0-m ladder, and finally walks on the ground back to his starting point. How much work is done on him by gravity (a) as he climbs up; (b) as he climbs down; (c) as he walks on the roof and on the ground? (d) What is the total work done on him by gravity during this round trip? (e) On the basis of your answer to part (d), would you say that gravity is a conservative or nonconservative force? Explain.

7.29 • A 0.60-kg book slides on a horizontal table. The kinetic friction force on the book has magnitude 1.2 N. (a) How much work is done on the book by friction during a displacement of 3.0 m to the left? (b) The book now slides 3.0 m to the right, returning to its starting point. During this second 3.0-m displacement, how much work is done on the book by friction? (c) What is the total work done on the book by friction during the complete round trip? (d) On the basis of your answer to part (c), would you say that the friction force is conservative or nonconservative? Explain.

7.30 •• **CALC** In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2 \hat{\imath}$, where $\alpha = 12\ \text{N/m}^2$. (a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point $(0.10\ \text{m}, 0)$ to the point $(0.10\ \text{m}, 0.40\ \text{m})$? (b) Along the straight-line path from the point $(0.10\ \text{m}, 0)$ to the point $(0.30\ \text{m}, 0)$? (c) Along the straight-line path from the point $(0.30\ \text{m}, 0)$ to the point $(0.10\ \text{m}, 0)$? (d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential-energy function for it? Let $U = 0$ when $x = 0$.

7.31 • You and three friends stand at the corners of a square whose sides are 8.0 m long in the middle of the gym floor, as shown in Fig. E7.31. You take your physics book and push it from one person to the other. The book has a mass of 1.5 kg, and the coefficient of kinetic friction between the book and the floor is $\mu_k = 0.25$. (a) The book slides from you to Beth and then from Beth to Carlos, along the lines connecting these people. What is the work done by friction during this displacement? (b) You slide the book from you to Carlos along the diagonal of the square. What is the work done by friction during this displacement? (c) You slide the book to Kim, who then slides it back to you. What is the total work done by friction during this motion of the book? (d) Is the friction force on the book conservative or nonconservative? Explain.

Figure **E7.31**



7.32 • While a roofer is working on a roof that slants at 36° above the horizontal, he accidentally nudges his 85.0-N toolbox, causing it to start sliding downward, starting from rest. If it starts 4.25 m from the lower edge of the roof, how fast will the toolbox be moving just as it reaches the edge of the roof if the kinetic friction force on it is 22.0 N?

7.33 •• A 62.0-kg skier is moving at 6.50 m/s on a frictionless, horizontal, snow-covered plateau when she encounters a rough patch 3.50 m long. The coefficient of kinetic friction between this patch and her skis is 0.300. After crossing the rough patch and returning to friction-free snow, she skis down an icy, frictionless hill 2.50 m high. (a) How fast is the skier moving when she gets to the bottom of the hill? (b) How much internal energy was generated in crossing the rough patch?

Section 7.4 Force and Potential Energy

7.34 •• **CALC** The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

7.35 •• **CALC** A force parallel to the x-axis acts on a particle moving along the x-axis. This force produces potential energy $U(x)$ given by $U(x) = \alpha x^4$, where $\alpha = 1.20\ \text{J/m}^4$. What is the force (magnitude and direction) when the particle is at $x = -0.800$ m?

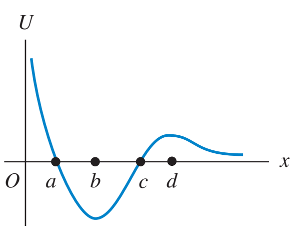
7.36 •• **CALC** An object moving in the xy-plane is acted on by a conservative force described by the potential-energy function $U(x, y) = \alpha(1/x^2 + 1/y^2)$, where α is a positive constant. Derive an expression for the force expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

7.37 •• **CALC** A small block with mass 0.0400 kg is moving in the xy-plane. The net force on the block is described by the potential-energy function $U(x, y) = (5.80\ \text{J/m}^2)x^2 - (3.60\ \text{J/m}^3)y^3$. What are the magnitude and direction of the acceleration of the block when it is at the point $x = 0.300$ m, $y = 0.600$ m?

Section 7.5 Energy Diagrams

7.38 • A marble moves along the x-axis. The potential-energy function is shown in Fig. E7.38. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?

Figure **E7.38**



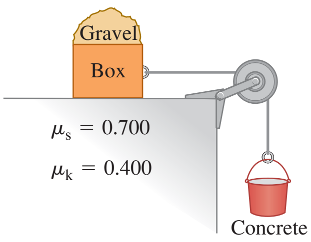
7.39 • **CALC** The potential energy of two atoms in a diatomic molecule is approximated by $U(r) = a/r^{12} - b/r^6$, where r is the spacing between atoms and a and b are positive constants. (a) Find the force $F(r)$ on one atom as a function of r. Draw two graphs: one of $U(r)$ versus r and one of $F(r)$ versus r. (b) Find the equilibrium distance between the two atoms. Is this equilibrium stable? (c) Suppose the distance between the two atoms is equal to the equilibrium distance found in part (b). What minimum energy must be added to the molecule to *dissociate* it—that is, to separate the two atoms to an infinite distance apart? This is called the *dissociation energy* of the molecule. (d) For the molecule CO, the equilibrium distance between the carbon and oxygen atoms is 1.13×10^{-10} m and the dissociation energy is 1.54×10^{-18} J per molecule. Find the values of the constants a and b.

PROBLEMS

7.40 •• Two blocks with different masses are attached to either end of a light rope that passes over a light, frictionless pulley suspended from the ceiling. The masses are released from rest, and the more massive one starts to descend. After this block has descended 1.20 m, its speed is 3.00 m/s. If the total mass of the two blocks is 15.0 kg, what is the mass of each block?

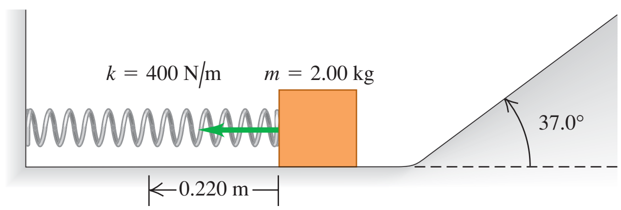
7.41 ••• At a construction site, a 65.0-kg bucket of concrete hangs from a light (but strong) cable that passes over a light, friction-free pulley and is connected to an 80.0-kg box on a horizontal roof (Fig. P7.41). The cable pulls horizontally on the box, and a 50.0-kg bag of gravel rests on top of the box. The coefficients of friction between the box and roof are shown. (a) Find the friction force on the bag of gravel and on the box. (b) Suddenly a worker picks up the bag of gravel. Use energy conservation to find the speed of the bucket after it has descended 2.00 m from rest. (You can check your answer by solving this problem using Newton's laws.)

Figure **P7.41**



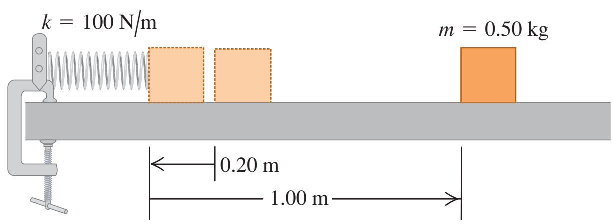
7.42 • A 2.00-kg block is pushed against a spring with negligible mass and force constant $k = 400$ N/m, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0° (Fig. P7.42). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the incline before starting to slide back down?

Figure **P7.42**



7.43 • A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m (Fig. P7.43). When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The spring constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

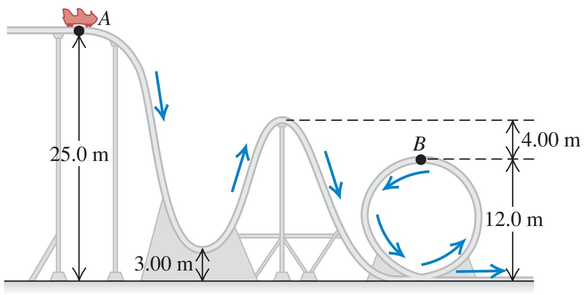
Figure **P7.43**



7.44 • On a horizontal surface, a crate with mass 50.0 kg is placed against a spring that stores 360 J of energy. The spring is released, and the crate slides 5.60 m before coming to rest. What is the speed of the crate when it is 2.00 m from its initial position?

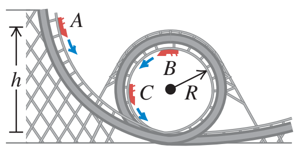
7.45 •• A 350-kg roller coaster starts from rest at point A and slides down the frictionless loop-the-loop shown in Fig. P7.45. (a) How fast is this roller coaster moving at point B? (b) How hard does it press against the track at point B?

Figure **P7.45**



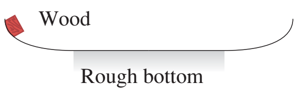
7.46 •• **CP** **Riding a Loop-the-Loop.** A car in an amusement park ride rolls without friction around the track shown in Fig. P7.46. It starts from rest at point A at a height h above the bottom of the loop. Treat the car as a particle. (a) What is the minimum value of h (in terms of R) such that the car moves around the loop without falling off at the top (point B)? (b) If $h = 3.50R$ and $R = 20.0$ m, compute the speed, radial acceleration, and tangential acceleration of the passengers when the car is at point C, which is at the end of a horizontal diameter. Show these acceleration components in a diagram, approximately to scale.

Figure **P7.46**



7.47 •• A 2.0-kg piece of wood slides on the surface shown in Fig. P7.47. The curved sides are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom. (a) Where will this wood eventually come to rest? (b) For the motion from the initial release until the piece of wood comes to rest, what is the total amount of work done by friction?

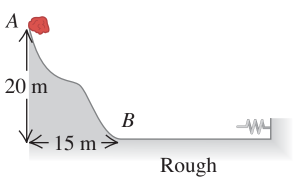
Figure **P7.47**



7.48 •• **Up and Down the Hill.** A 28-kg rock approaches the foot of a hill with a speed of 15 m/s. This hill slopes upward at a constant angle of 40.0° above the horizontal. The coefficients of static and kinetic friction between the hill and the rock are 0.75 and 0.20, respectively. (a) Use energy conservation to find the maximum height above the foot of the hill reached by the rock. (b) Will the rock remain at rest at its highest point, or will it slide back down the hill? (c) If the rock does slide back down, find its speed when it returns to the bottom of the hill.

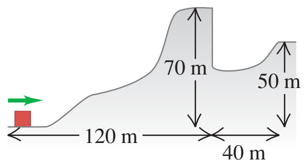
7.49 •• A 15.0-kg stone slides down a snow-covered hill (Fig. P7.49), leaving point A with a speed of 10.0 m/s. There is no friction on the hill between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. After entering the rough horizontal region, the stone travels 100 m and then runs into a very long, light spring with force constant 2.00 N/m. The coefficients of kinetic and static friction between the stone and the horizontal ground are 0.20 and 0.80, respectively. (a) What is the speed of the stone when it reaches point B? (b) How far will the stone compress the spring? (c) Will the stone move again after it has been stopped by the spring?

Figure **P7.49**



7.50 •• **CP** A 2.8-kg block slides over the smooth, icy hill shown in Fig. P7.50. The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the hill in order for it to pass over the pit at the far side of the hill?

Figure **P7.50**



7.51 ••• **Bungee Jump.** A bungee cord is 30.0 m long and, when stretched a distance x, it exerts a restoring force of magnitude kx. Your father-in-law (mass 95.0 kg) stands on a platform 45.0 m above the ground, and one end of the cord is tied securely to his ankle and the other end to the platform. You have promised him that when he steps off the platform he will fall a maximum distance of only 41.0 m before the cord stops him. You had several bungee cords to select from, and you tested them by stretching them out, tying one end to a tree, and pulling on the other end with a force of 380.0 N. When you do this, what distance will the bungee cord that you should select have stretched?

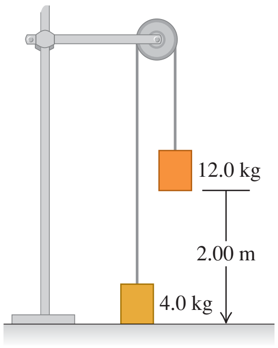
7.52 •• **Ski Jump Ramp.** You are designing a ski jump ramp for the next Winter Olympics. You need to calculate the vertical height h from the starting gate to the bottom of the ramp. The skiers push off hard with their ski poles at the start, just above the starting gate, so they typically have a speed of 2.0 m/s as they reach the gate. For safety, the skiers should have a speed no higher than 30.0 m/s when they reach the bottom of the ramp. You determine that for a 85.0-kg skier with good form, friction and air resistance will do total work of magnitude 4000 J on him during his run down the ramp. What is the maximum height h for which the maximum safe speed will not be exceeded?

7.53 ••• The Great Sandini is a 60-kg circus performer who is shot from a cannon (actually a spring gun). You don't find many men of his caliber, so you help him design a new gun. This new gun has a very large spring with a very small mass and a force constant of 1100 N/m that he will compress with a force of 4400 N. The inside of the gun barrel is coated with Teflon, so the average friction force will be only 40 N during the 4.0 m he moves in the barrel. At what speed will he emerge from the end of the barrel, 2.5 m above his initial rest position?

7.54 ••• You are designing a delivery ramp for crates containing exercise equipment. The 1470-N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22.0°. The ramp exerts a 550-N kinetic friction force on each crate, and the maximum static friction force also has this value. Each crate will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 8.0 m along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the force constant of the spring that will be needed in order to meet the design criteria.

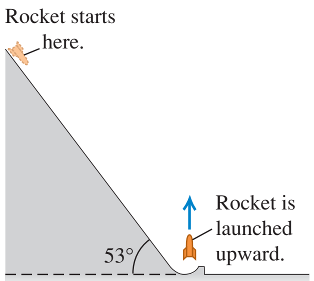
7.55 •• A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0-kg bucket 2.00 m above the floor (Fig. P7.55). Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. You can ignore friction and the mass of the pulley.

Figure **P7.55**



7.56 •• A 1500-kg rocket is to be launched with an initial upward speed of 50.0 m/s. In order to assist its engines, the engineers will start it from rest on a ramp that rises 53° above the horizontal (Fig. P7.56). At the bottom, the ramp turns upward and launches the rocket vertically. The engines provide a constant forward thrust of 2000 N, and friction with the ramp surface is a constant 500 N. How far from the base of the ramp should the rocket start, as measured along the surface of the ramp?

Figure **P7.56**



7.57 • **Legal Physics.** In an auto accident, a car hit a pedestrian and the driver then slammed on the brakes to stop the car. During the subsequent trial, the driver's lawyer claimed that he was obeying the posted 35-mph speed limit, but that the legal speed was too high to allow him to see and react to the pedestrian in time. You have been called in as the state's expert witness. Your investigation of the accident found that the skid marks made while the brakes were applied were 280 ft long, and the tread on the tires produced a coefficient of kinetic friction of 0.30 with the road. (a) In your testimony in court, will you say that the driver was obeying the posted speed? You must be able to back up your conclusion with clear reasoning because one of the lawyers will surely cross-examine you. (b) If the driver's speeding ticket were \$10 for each mile per hour he was driving above the posted speed limit, would he have to pay a fine? If so, how much would it be?

7.58 ••• A wooden rod of negligible mass and length 80.0 cm is pivoted about a horizontal axis through its center. A white rat with mass 0.500 kg clings to one end of the stick, and a mouse with mass 0.200 kg clings to the other end. The system is released from rest with the rod horizontal. If the animals can manage to hold on, what are their speeds as the rod swings through a vertical position?

7.59 •• **CP** A 0.300-kg potato is tied to a string with length 2.50 m, and the other end of the string is tied to a rigid support. The potato is held straight out horizontally from the point of support, with the string pulled taut, and is then released. (a) What is the speed of the potato at the lowest point of its motion? (b) What is the tension in the string at this point?

7.60 •• These data are from a computer simulation for a batted baseball with mass 0.145 kg, including air resistance:

t	x	y	v_x	v_y
0	0	0	30.0 m/s	40.0 m/s
3.05 s	70.2 m	53.6 m	18.6 m/s	0
6.59 s	124.4 m	0	11.9 m/s	−28.7 m/s

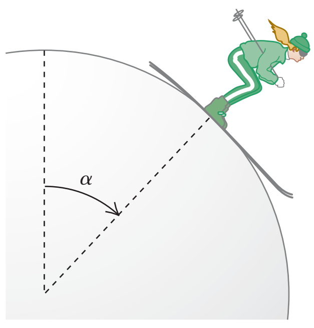
(a) How much work was done by the air on the baseball as it moved from its initial position to its maximum height? (b) How much work was done by the air on the baseball as it moved from its maximum height back to the starting elevation? (c) Explain why the magnitude of the answer in part (b) is smaller than the magnitude of the answer in part (a).

7.61 •• **Down the Pole.** A fireman of mass m slides a distance d down a pole. He starts from rest. He moves as fast at the bottom as if he had stepped off a platform a distance $h \le d$ above the ground and descended with negligible air resistance. (a) What average friction force did the fireman exert on the pole? Does your answer make sense in the special cases of $h = d$ and $h = 0$? (b) Find a numerical value for the average friction force a 75-kg fireman exerts, for $d = 2.5$ m and $h = 1.0$ m. (c) In terms of g, h, and d, what is the speed of the fireman when he is a distance y above the bottom of the pole?

7.62 •• A 60.0-kg skier starts from rest at the top of a ski slope 65.0 m high. (a) If frictional forces do −10.5 kJ of work on her as she descends, how fast is she going at the bottom of the slope? (b) Now moving horizontally, the skier crosses a patch of soft snow, where $\mu_k = 0.20$. If the patch is 82.0 m wide and the average force of air resistance on the skier is 160 N, how fast is she going after crossing the patch? (c) The skier hits a snowdrift and penetrates 2.5 m into it before coming to a stop. What is the average force exerted on her by the snowdrift as it stops her?

7.63 • **CP** A skier starts at the top of a very large, frictionless snowball, with a very small initial speed, and skis straight down the side (Fig. P7.63). At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?

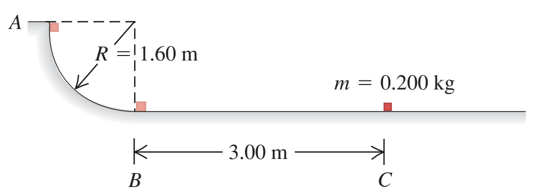
Figure **P7.63**



7.64 •• A ball is thrown upward with an initial velocity of 15 m/s at an angle of 60.0° above the horizontal. Use energy conservation to find the ball's greatest height above the ground.

7.65 •• In a truck-loading station at a post office, a small 0.200-kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m (Fig. P7.65). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 4.80 m/s. From point B, it slides on a level surface a distance of

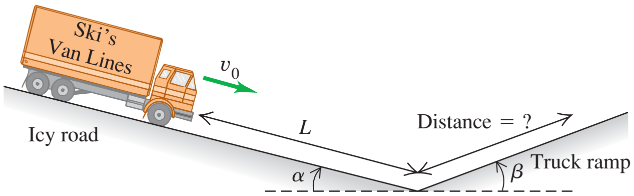
Figure **P7.65**



3.00 m to point C, where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from A to B?

7.66 ••• A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α (Fig. P7.66). Initially the truck is moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_r. What is the distance that the truck moves up the ramp before coming to a halt? Solve using energy methods.

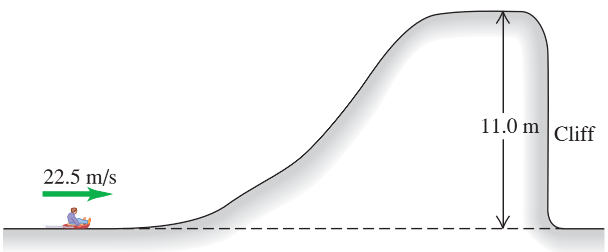
Figure **P7.66**



7.67 •• **CALC** A certain spring is found *not* to obey Hooke's law; it exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha = 60.0\ \text{N/m}$ and $\beta = 18.0\ \text{N/m}^2$. The mass of the spring is negligible. (a) Calculate the potential-energy function $U(x)$ for this spring. Let $U = 0$ when $x = 0$. (b) An object with mass 0.900 kg on a frictionless, horizontal surface is attached to this spring, pulled a distance 1.00 m to the right (the $+x$-direction) to stretch the spring, and released. What is the speed of the object when it is 0.50 m to the right of the $x = 0$ equilibrium position?

7.68 •• **CP** A sled with rider having a combined mass of 125 kg travels over the perfectly smooth icy hill shown in Fig. 7.68. How far does the sled land from the foot of the cliff?

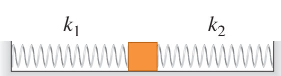
Figure **P7.68**



7.69 •• A 0.150-kg block of ice is placed against a horizontal, compressed spring mounted on a horizontal tabletop that is 1.20 m above the floor. The spring has force constant 1900 N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the block of ice and the tabletop, what is the speed of the block of ice when it reaches the floor?

7.70 •• A 3.00-kg block is connected to two ideal horizontal springs having force constants $k_1 = 25.0\ \text{N/cm}$ and $k_2 = 20.0\ \text{N/cm}$ (Fig. P7.70). The system is initially in equilibrium on a horizontal, frictionless surface. The block is now pushed 15.0 cm to the right and released from rest. (a) What is the maximum speed of the block? Where in the motion does the maximum speed occur? (b) What is the maximum compression of spring 1?

Figure **P7.70**



7.71 •• An experimental apparatus with mass m is placed on a vertical spring of negligible mass and pushed down until the spring is compressed a distance x. The apparatus is then released and reaches its maximum height at a distance h above the point where it is released. The apparatus is not attached to the spring, and at its maximum height it is no longer in contact with the spring. The maximum magnitude of acceleration the apparatus can have without being damaged is a, where $a > g$. (a) What should the force constant of the spring be? (b) What distance x must the spring be compressed initially?

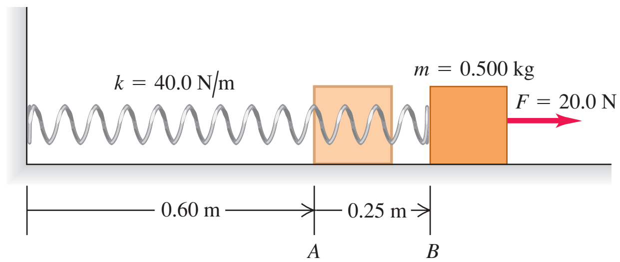
7.72 •• If a fish is attached to a vertical spring and slowly lowered to its equilibrium position, it is found to stretch the spring by an amount d. If the same fish is attached to the end of the unstretched spring and then allowed to fall from rest, through what maximum distance does it stretch the spring? (*Hint:* Calculate the force constant of the spring in terms of the distance d and the mass m of the fish.)

7.73 ••• **CALC** A 3.00-kg fish is attached to the lower end of a vertical spring that has negligible mass and force constant 900 N/m. The spring initially is neither stretched nor compressed. The fish is released from rest. (a) What is its speed after it has descended 0.0500 m from its initial position? (b) What is the maximum speed of the fish as it descends?

7.74 •• A basket of negligible weight hangs from a vertical spring scale of force constant 1500 N/m. (a) If you suddenly put a 3.0-kg adobe brick in the basket, find the maximum distance that the spring will stretch. (b) If, instead, you release the brick from 1.0 m above the basket, by how much will the spring stretch at its maximum elongation?

7.75 • A 0.500-kg block, attached to a spring with length 0.60 m and force constant 40.0 N/m, is at rest with the back of the block at point A on a frictionless, horizontal air table (Fig. P7.75). The mass of the spring is negligible. You move the block to the right along the surface by pulling with a constant 20.0-N horizontal force. (a) What is the block's speed when the back of the block reaches point B, which is 0.25 m to the right of point A? (b) When the back of the block reaches point B, you let go of the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

Figure **P7.75**



7.76 •• **Fraternity Physics.** The brothers of Iota Eta Pi fraternity build a platform, supported at all four corners by vertical springs, in the basement of their frat house. A brave fraternity brother wearing a football helmet stands in the middle of the platform; his weight compresses the springs by 0.18 m. Then four of his fraternity brothers, pushing down at the corners of the platform, compress the springs another 0.53 m until the top of the brave brother's helmet is 0.90 m below the basement ceiling. They then simultaneously release the platform. You can ignore the masses of the springs and platform. (a) When the dust clears, the fraternity asks you to calculate their fraternity brother's speed just before his helmet hit the flimsy ceiling. (b) Without the ceiling, how high would he have gone? (c) In discussing their probation, the dean of students suggests that the next time they try this, they do it outdoors on another planet. Would the answer to part (b) be the same if this stunt were performed on a planet with a different value of g? Assume that the fraternity brothers push the platform down 0.53 m as before. Explain your reasoning.

7.77 ••• **CP** A small block with mass 0.0500 kg slides in a vertical circle of radius $R = 0.800$ m on the inside of a circular track. There is no friction between the track and the block. At the bottom of the block's path, the normal force the track exerts on the block has magnitude 3.40 N. What is the magnitude of the normal force that the track exerts on the block when it is at the top of its path?

7.78 ••• **CP** A small block with mass 0.0400 kg slides in a vertical circle of radius $R = 0.500$ m on the inside of a circular track. During one of the revolutions of the block, when the block is at the bottom of its path, point A, the magnitude of the normal force exerted on the block by the track has magnitude 3.95 N. In this same revolution, when the block reaches the top of its path, point B, the magnitude of the normal force exerted on the block has magnitude 0.680 N. How much work was done on the block by friction during the motion of the block from point A to point B?

7.79 •• A hydroelectric dam holds back a lake of surface area $3.0 \times 10^6\ \text{m}^2$ that has vertical sides below the water level. The water level in the lake is 150 m above the base of the dam. When the water passes through turbines at the base of the dam, its mechanical energy is converted to electrical energy with 90% efficiency. (a) If gravitational potential energy is taken to be zero at the base of the dam, how much energy is stored in the top meter of the water in the lake? The density of water is $1000\ \text{kg/m}^3$. (b) What volume of water must pass through the dam to produce 1000 kilowatt-hours of electrical energy? What distance does the level of water in the lake fall when this much water passes through the dam?

7.80 •• **CALC** How much total energy is stored in the lake in Problem 7.79? As in that problem, take the gravitational potential energy to be zero at the base of the dam. Express your answer in joules and in kilowatt-hours. (*Hint:* Break the lake up into infinitesimal horizontal layers of thickness dy, and integrate to find the total potential energy.)

7.81 ••• A wooden block with mass 1.50 kg is placed against a compressed spring at the bottom of an incline of slope 30.0° (point A). When the spring is released, it projects the block up the incline. At point B, a distance of 6.00 m up the incline from A, the block is moving up the incline at 7.00 m/s and is no longer in contact with the spring. The coefficient of kinetic friction between the block and the incline is $\mu_k = 0.50$. The mass of the spring is negligible. Calculate the amount of potential energy that was initially stored in the spring.

7.82 •• **CP** **Pendulum.** A small rock with mass 0.12 kg is fastened to a massless string with length 0.80 m to form a pendulum. The pendulum is swinging so as to make a maximum angle of 45° with the vertical. Air resistance is negligible. (a) What is the speed of the rock when the string passes through the vertical position? (b) What is the tension in the string when it makes an angle of 45° with the vertical? (c) What is the tension in the string as it passes through the vertical?

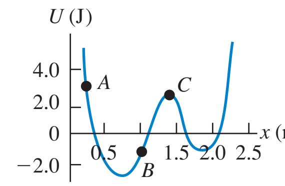
7.83 ••• **CALC** A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{F} = -\alpha x y^2 \hat{j}$, a force in the negative y-direction whose magnitude depends on the position of the tool. The constant is $\alpha = 2.50\ \text{N/m}^3$. Consider the displacement of the tool from the origin to the point $x = 3.00$ m, $y = 3.00$ m. (a) Calculate the work done on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points. (b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point $x = 3.00$ m, $y = 0$ and then moved parallel to the y-axis to the point $x = 3.00$ m, $y = 3.00$ m. (c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or nonconservative? Explain.

7.84 • **CALC** (a) Is the force $\vec{F} = Cy^2\hat{j}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer. (b) Is the force $\vec{F} = Cy^2\hat{\imath}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer.

7.85 •• **CALC** An object has several forces acting on it. One force is $\vec{F} = \alpha x y \hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object. (See Problem 6.98.) The constant is $\alpha = 2.00\ \text{N/m}^2$. The object moves along the following path: (1) It starts at the origin and moves along the y-axis to the point $x = 0$, $y = 1.50$ m; (2) it moves parallel to the x-axis to the point $x = 1.50$ m, $y = 1.50$ m; (3) it moves parallel to the y-axis to the point $x = 1.50$ m, $y = 0$; (4) it moves parallel to the x-axis back to the origin. (a) Sketch this path in the xy-plane. (b) Calculate the work done on the object by $\vec{F}$ for each leg of the path and for the complete round trip. (c) Is $\vec{F}$ conservative or nonconservative? Explain.

7.86 • A particle moves along the x-axis while acted on by a single conservative force parallel to the x-axis. The force corresponds to the potential-energy function graphed in Fig. P7.86. The particle is released from rest at point A. (a) What is the direction of the force on the particle when it is at point A? (b) At point B? (c) At what value of x is the kinetic energy of the particle a maximum? (d) What is the force on the particle when it is at point C? (e) What is the largest value of x reached by the particle during its motion? (f) What value or values of x correspond to points of stable equilibrium? (g) Of unstable equilibrium?

Figure **P7.86**



CHALLENGE PROBLEM

7.87 ••• **CALC** A proton with mass m moves in one dimension. The potential-energy function is $U(x) = \alpha/x^2 - \beta/x$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that $U(x)$ can be written as

$$U(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$, the speed of the proton as a function of position. Graph $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton a maximum? What is the value of that maximum speed? (d) What is the force on the proton at the point in part (c)? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate the point x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each release point ($x = x_0$ and $x = x_1$), what are the maximum and minimum values of x reached during the motion?

Answers

Chapter Opening Question ?

The mallard's kinetic energy K remains constant because the speed remains the same, but the gravitational potential energy U_{grav} decreases as the mallard descends. Hence the total mechanical energy $E = K + U_{grav}$ decreases. The lost mechanical energy goes into warming the mallard's skin (that is, an increase in the mallard's internal energy) and stirring up the air through which the mallard passes (an increase in the internal energy of the air). See the discussion in Section 7.3.

Test Your Understanding Questions

7.1 Answer: (iii) The initial kinetic energy $K_1 = 0$, the initial potential energy $U_1 = mgy_1$, and the final potential energy $U_2 = mgy_2$ are the same for both blocks. Mechanical energy is conserved in both cases, so the final kinetic energy $K_2 = \frac{1}{2}mv_2^2$ is also the same for both blocks. Hence the speed at the right-hand end is the *same* in both cases!

7.2 Answer: (iii) The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$ and $U_{grav} < 0$; and the spring is compressed, so $U_{el} > 0$.

7.3 Answer: (iii) Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism.

7.4 Answers: (a) (iv), (b) (i) If $F_x = 0$ at a point, then the derivative of $U(x)$ must be zero at that point because $F_x = -dU(x)/dx$. However, this tells us absolutely nothing about the *value* of $U(x)$ at that point.

7.5 Answers: (iii) Figure 7.24b shows the x-component of force, F_x. Where this is maximum (most positive), the x-component of force and the x-acceleration have more positive values than at adjacent values of x.

Bridging Problem

Answers: **(a)** 1.06 m
(b) 1.32 m
(c) 20.7 J

MOMENTUM, IMPULSE, AND COLLISIONS

8

? Which could potentially do greater damage to this carrot: a .22-caliber bullet moving at 220 m/s as shown here, or a lightweight bullet of the same length and diameter but half the mass moving at twice the speed?

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of the momentum of a particle, and how the impulse of the net force acting on a particle causes its momentum to change.
- The conditions under which the total momentum of a system of particles is constant (conserved).
- How to solve problems in which two bodies collide with each other.
- The important distinction among elastic, inelastic, and completely inelastic collisions.
- The definition of the center of mass of a system, and what determines how the center of mass moves.
- How to analyze situations such as rocket propulsion in which the mass of a body changes as it moves.

There are many questions involving forces that cannot be answered by directly applying Newton's second law, $\sum\vec{F} = m\vec{a}$. For example, when a moving van collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite's kinetic energy is released in the impact?

A common theme of all these questions is that they involve forces about which we know very little: the forces between the car and the moving van, between the two pool balls, or between the meteorite and the earth. Remarkably, we will find in this chapter that we don't have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, *momentum* and *impulse,* and a new conservation law, *conservation of momentum.* This conservation law is every bit as important as the law of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton's laws are inadequate, such as bodies moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton's laws directly. Among these are *collision* problems, in which two bodies collide and can exert very large forces on each other for a short time.

8.1 Momentum and Impulse

In Chapter 6 we re-expressed Newton's second law for a particle, $\sum\vec{F} = m\vec{a}$, in terms of the work–energy theorem. This theorem helped us tackle a great number of physics problems and led us to the law of conservation of energy. Let's now return to $\sum\vec{F} = m\vec{a}$ and see yet another useful way to restate this fundamental law.

Newton's Second Law in Terms of Momentum

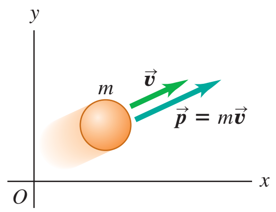
Consider a particle of constant mass m. (Later in this chapter we'll see how to deal with situations in which the mass of a body changes.) Because $\vec{a} = d\vec{v}/dt$, we can write Newton's second law for this particle as

$$\sum\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \tag{8.1}$$

We can move the mass m inside the derivative because it is constant. Thus Newton's second law says that the net force $\sum\vec{F}$ acting on a particle equals the time rate of change of the combination $m\vec{v}$, the product of the particle's mass and velocity. We'll call this combination the **momentum,** or **linear momentum,** of the particle. Using the symbol $\vec{p}$ for momentum, we have

$$\vec{p} = m\vec{v} \qquad \text{(definition of momentum)} \tag{8.2}$$

8.1 The velocity and momentum vectors of a particle.



Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{v}$.

The greater the mass m and speed v of a particle, the greater is its magnitude of momentum mv. Keep in mind, however, that momentum is a *vector* quantity with the same direction as the particle's velocity (Fig. 8.1). Hence a car driving north at 20 m/s and an identical car driving east at 20 m/s have the same *magnitude* of momentum (mv) but different momentum *vectors* $(m\vec{v})$ because their directions are different.

We often express the momentum of a particle in terms of its components. If the particle has velocity components v_x, v_y, and v_z, then its momentum components p_x, p_y, and p_z (which we also call the *x-momentum, y-momentum,* and *z-momentum*) are given by

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{8.3}$$

These three component equations are equivalent to Eq. (8.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are kg·m/s. The plural of momentum is "momenta."

If we now substitute the definition of momentum, Eq. (8.2), into Eq. (8.1), we get

$$\sum\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{(Newton's second law in terms of momentum)} \tag{8.4}$$

The net force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle. This, not $\sum\vec{F} = m\vec{a}$, is the form in which Newton originally stated his second law (although he called momentum the "quantity of motion"). This law is valid only in inertial frames of reference.

According to Eq. (8.4), a rapid change in momentum requires a large net force, while a gradual change in momentum requires less net force. This principle is used in the design of automobile safety devices such as air bags (Fig. 8.2).

8.2 If a fast-moving automobile stops suddenly in a collision, the driver's momentum (mass times velocity) changes from a large value to zero in a short time. An air bag causes the driver to lose momentum more gradually than would an abrupt collision with the steering wheel, reducing the force exerted on the driver as well as the possibility of injury.

The Impulse–Momentum Theorem

A particle's momentum $\vec{p} = m\vec{v}$ and its kinetic energy $K = \frac{1}{2}mv^2$ both depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the *physical* difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called *impulse.*

Let's first consider a particle acted on by a *constant* net force $\sum\vec{F}$ during a time interval Δt from t_1 to t_2. (We'll look at the case of varying forces shortly.) The **impulse** of the net force, denoted by $\vec{J}$, is defined to be the product of the net force and the time interval:

$$\vec{J} = \sum\vec{F}(t_2 - t_1) = \sum\vec{F}\,\Delta t \qquad \text{(assuming constant net force)} \qquad (8.5)$$

Impulse is a vector quantity; its direction is the same as the net force $\sum\vec{F}$. Its magnitude is the product of the magnitude of the net force and the length of time that the net force acts. The SI unit of impulse is the newton-second $(\mathrm{N\cdot s})$. Because $1\ \mathrm{N} = 1\ \mathrm{kg\cdot m/s^2}$, an alternative set of units for impulse is $\mathrm{kg\cdot m/s}$, the same as the units of momentum.

To see what impulse is good for, let's go back to Newton's second law as restated in terms of momentum, Eq. (8.4). If the net force $\sum\vec{F}$ is constant, then $d\vec{p}/dt$ is also constant. In that case, $d\vec{p}/dt$ is equal to the *total* change in momentum $\vec{p}_2 - \vec{p}_1$ during the time interval $t_2 - t_1$, divided by the interval:

$$\sum\vec{F} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1}$$

Multiplying this equation by $(t_2 - t_1)$, we have

$$\sum\vec{F}(t_2 - t_1) = \vec{p}_2 - \vec{p}_1$$

Comparing with Eq. (8.5), we end up with a result called the **impulse–momentum theorem:**

$$\vec{J} = \vec{p}_2 - \vec{p}_1 \qquad \text{(impulse–momentum theorem)} \qquad (8.6)$$

The change in momentum of a particle during a time interval equals the impulse of the net force that acts on the particle during that interval.

The impulse–momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's second law $\sum\vec{F} = d\vec{p}/dt$ over time between the limits t_1 and t_2:

$$\int_{t_1}^{t_2} \sum\vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = \vec{p}_2 - \vec{p}_1$$

The integral on the left is defined to be the impulse $\vec{J}$ of the net force $\sum\vec{F}$ during this interval:

$$\vec{J} = \int_{t_1}^{t_2} \sum\vec{F}\,dt \qquad \text{(general definition of impulse)} \qquad (8.7)$$

With this definition, the impulse–momentum theorem $\vec{J} = \vec{p}_2 - \vec{p}_1$, Eq. (8.6), is valid even when the net force $\sum\vec{F}$ varies with time.

We can define an *average* net force $\vec{F}_{\mathrm{av}}$ such that even when $\sum\vec{F}$ is not constant, the impulse $\vec{J}$ is given by

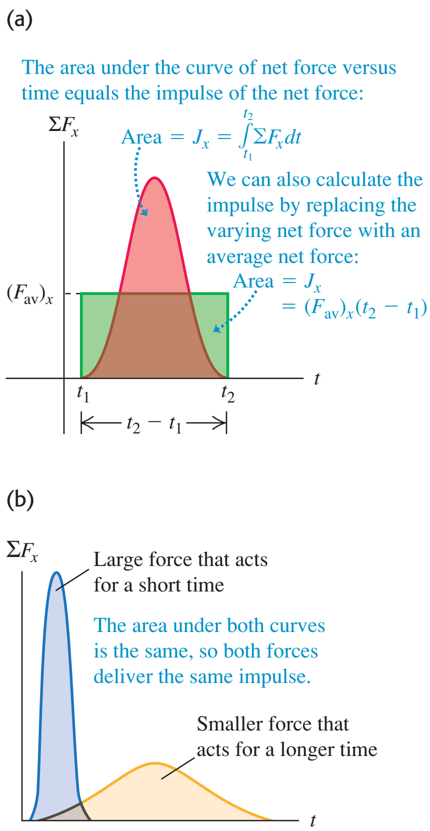
$$\vec{J} = \vec{F}_{\mathrm{av}}(t_2 - t_1) \qquad (8.8)$$

When $\sum\vec{F}$ is constant, $\sum\vec{F} = \vec{F}_{\mathrm{av}}$ and Eq. (8.8) reduces to Eq. (8.5).

Figure 8.3a shows the x-component of net force $\sum F_x$ as a function of time during a collision. This might represent the force on a soccer ball that is in contact with a player's foot from time t_1 to t_2. The x-component of impulse during this interval is represented by the red area under the curve between t_1 and t_2. This

Application Woodpecker Impulse
The pileated woodpecker (*Dryocopus pileatus*) has been known to strike its beak against a tree up to 20 times a second and up to 12,000 times a day. The impact force can be as much as 1200 times the weight of the bird's head. Because the impact lasts such a short time, the impulse—the product of the net force during the impact multiplied by the duration of the impact—is relatively small. (The woodpecker has a thick skull of spongy bone as well as shock-absorbing cartilage at the base of the lower jaw, and so avoids injury.)

8.3 The meaning of the area under a graph of $\sum F_x$ versus t.



area is equal to the green rectangular area bounded by t_1, t_2, and $(F_{av})_x$, so $(F_{av})_x(t_2 - t_1)$ is equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–time curves are the same (Fig. 8.3b). In this language, an automobile airbag (see Fig. 8.2) provides the same impulse to the driver as would the steering wheel or the dashboard by applying a weaker and less injurious force for a longer time.

Impulse and momentum are both vector quantities, and Eqs. (8.5)–(8.8) are all vector equations. In specific problems, it is often easiest to use them in component form:

$$\begin{aligned} J_x &= \int_{t_1}^{t_2} \Sigma F_x\,dt = (F_{av})_x(t_2 - t_1) = p_{2x} - p_{1x} = mv_{2x} - mv_{1x} \\ J_y &= \int_{t_1}^{t_2} \Sigma F_y\,dt = (F_{av})_y(t_2 - t_1) = p_{2y} - p_{1y} = mv_{2y} - mv_{1y} \end{aligned} \tag{8.9}$$

and similarly for the z-component.

Momentum and Kinetic Energy Compared

We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem $\vec{J} = \vec{p}_2 - \vec{p}_1$ says that changes in a particle's momentum are due to impulse, which depends on the *time* over which the net force acts. By contrast, the work–energy theorem $W_{tot} = K_2 - K_1$ tells us that kinetic energy changes when work is done on a particle; the total work depends on the *distance* over which the net force acts. Consider a particle that starts from rest at t_1 so that $\vec{v}_1 = \mathbf{0}$. Its initial momentum is $\vec{p}_1 = m\vec{v}_1 = \mathbf{0}$, and its initial kinetic energy is $K_1 = \frac{1}{2}mv_1^2 = 0$. Now let a constant net force equal to $\vec{F}$ act on that particle from time t_1 until time t_2. During this interval, the particle moves a distance s in the direction of the force. From Eq. (8.6), the particle's momentum at time t_2 is

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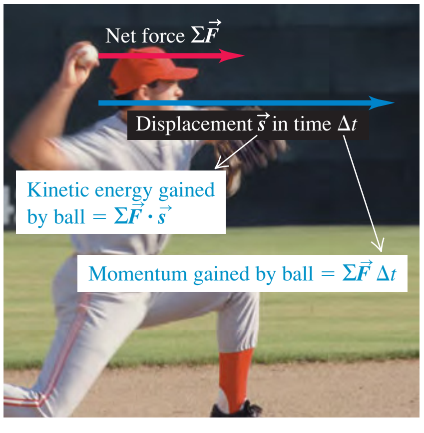
$$\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J}$$

where $\vec{J} = \vec{F}(t_2 - t_1)$ is the impulse that acts on the particle. So *the momentum of a particle equals the impulse that accelerated it from rest to its present speed;* impulse is the product of the net force that accelerated the particle and the *time* required for the acceleration. By comparison, the kinetic energy of the particle at t_2 is $K_2 = W_{tot} = Fs$, the total *work* done on the particle to accelerate it from rest. The total work is the product of the net force and the *distance* required to accelerate the particle (Fig. 8.4).

8.4 The *kinetic energy* of a pitched baseball is equal to the work the pitcher does on it (force multiplied by the distance the ball moves during the throw). The *momentum* of the ball is equal to the impulse the pitcher imparts to it (force multiplied by the time it took to bring the ball up to speed).



Here's an application of the distinction between momentum and kinetic energy. Suppose you have a choice between catching a 0.50-kg ball moving at 4.0 m/s or a 0.10-kg ball moving at 20 m/s. Which will be easier to catch? Both balls have the same magnitude of momentum, $p = mv = (0.50\text{ kg})(4.0\text{ m/s}) = (0.10\text{ kg})(20\text{ m/s}) = 2.0\text{ kg}\cdot\text{m/s}$. However, the two balls have different values of kinetic energy $K = \frac{1}{2}mv^2$; the large, slow-moving ball has $K = 4.0$ J, while the small, fast-moving ball has $K = 20$ J. Since the momentum is the same for both balls, both require the same *impulse* to be brought to rest. But stopping the 0.10-kg ball with your hand requires five times more *work* than stopping the 0.50-kg ball because the smaller ball has five times more kinetic energy. For a given force that you exert with your hand, it takes the same amount of time (the duration of the catch) to stop either ball, but your hand and arm will be pushed back five times farther if you choose to catch the small, fast-moving ball. To minimize arm strain, you should choose to catch the 0.50-kg ball with its lower kinetic energy.

Both the impulse–momentum and work–energy theorems are relationships between force and motion, and both rest on the foundation of Newton's laws. They are *integral* principles, relating the motion at two different times separated by a finite interval. By contrast, Newton's second law itself (in either of the forms $\sum\vec{F} = m\vec{a}$ or $\sum\vec{F} = d\vec{p}/dt$) is a *differential* principle, relating the forces to the rate of change of velocity or momentum at each instant.

Conceptual Example 8.1 Momentum versus kinetic energy

Consider again the race described in Conceptual Example 6.5 (Section 6.2) between two iceboats on a frictionless frozen lake. The boats have masses m and $2m$, and the wind exerts the same constant horizontal force $\vec{F}$ on each boat (see Fig. 6.14). The boats start from rest and cross the finish line a distance s away. Which boat crosses the finish line with greater momentum?

SOLUTION

In Conceptual Example 6.5 we asked how the *kinetic energies* of the boats compare when they cross the finish line. We answered this by remembering that *a body's kinetic energy equals the total work done to accelerate it from rest*. Both boats started from rest, and the total work done was the same for both boats (because the net force and the displacement were the same for both). Hence both boats had the same kinetic energy at the finish line.

Similarly, to compare the *momenta* of the boats we use the idea that *the momentum of each boat equals the impulse that accelerated it from rest*. As in Conceptual Example 6.5, the net force on each boat equals the constant horizontal wind force $\vec{F}$. Let Δt be the time a boat takes to reach the finish line, so that the impulse on the boat during that time is $\vec{J} = \vec{F}\,\Delta t$. Since the boat starts from rest, this equals the boat's momentum $\vec{p}$ at the finish line:

$$\vec{p} = \vec{F}\,\Delta t$$

Both boats are subjected to the same force $\vec{F}$, but they take different times Δt to reach the finish line. The boat of mass $2m$ accelerates more slowly and takes a longer time to travel the distance s; thus there is a greater impulse on this boat between the starting and finish lines. So the boat of mass $2m$ crosses the finish line with a greater magnitude of momentum than the boat of mass m (but with the same kinetic energy). Can you show that the boat of mass $2m$ has $\sqrt{2}$ times as much momentum at the finish line as the boat of mass m?

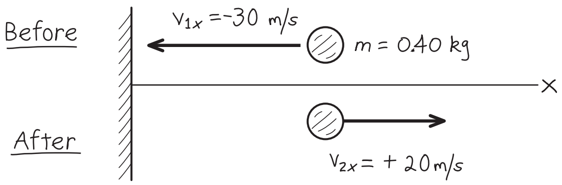
Example 8.2 A ball hits a wall

You throw a ball with a mass of 0.40 kg against a brick wall. It hits the wall moving horizontally to the left at 30 m/s and rebounds horizontally to the right at 20 m/s. (a) Find the impulse of the net force on the ball during its collision with the wall. (b) If the ball is in contact with the wall for 0.010 s, find the average horizontal force that the wall exerts on the ball during the impact.

SOLUTION

IDENTIFY and SET UP: We're given enough information to determine the initial and final values of the ball's momentum, so we can use the impulse–momentum theorem to find the impulse. We'll then use the definition of impulse to determine the average force. Figure 8.5 shows our sketch. We need only a single axis because the motion is purely horizontal. We'll take the positive x-direction to be to the right. In part (a) our target variable is the x-component of impulse, J_x, which we'll find from the x-components of momentum before and after the impact, using Eqs. (8.9). In part (b), our target variable is the average x-component of force $(F_{av})_x$; once we know J_x, we can also find this force by using Eqs. (8.9).

8.5 Our sketch for this problem.



EXECUTE: (a) With our choice of x-axis, the initial and final x-components of momentum of the ball are

$$p_{1x} = mv_{1x} = (0.40\text{ kg})(-30\text{ m/s}) = -12\text{ kg}\cdot\text{m/s}$$
$$p_{2x} = mv_{2x} = (0.40\text{ kg})(+20\text{ m/s}) = +8.0\text{ kg}\cdot\text{m/s}$$

From the x-equation in Eqs. (8.9), the x-component of impulse equals the *change* in the x-momentum:

$$\begin{aligned} J_x &= p_{2x} - p_{1x} \\ &= 8.0\text{ kg}\cdot\text{m/s} - (-12\text{ kg}\cdot\text{m/s}) = 20\text{ kg}\cdot\text{m/s} = 20\text{ N}\cdot\text{s} \end{aligned}$$

(b) The collision time is $t_2 - t_1 = \Delta t = 0.010$ s. From the x-equation in Eqs. (8.9), $J_x = (F_{av})_x(t_2 - t_1) = (F_{av})_x\,\Delta t$, so

$$(F_{av})_x = \frac{J_x}{\Delta t} = \frac{20\text{ N}\cdot\text{s}}{0.010\text{ s}} = 2000\text{ N}$$

EVALUATE: The x-component of impulse J_x is positive—that is, to the right in Fig. 8.5. This is as it should be: The impulse represents the "kick" that the wall imparts to the ball, and this "kick" is certainly to the right.

CAUTION **Momentum is a vector** Because momentum is a vector, we had to include the negative sign in writing $p_{1x} = -12\text{ kg}\cdot\text{m/s}$. Had we carelessly omitted it, we would have calculated the impulse to be $8.0\text{ kg}\cdot\text{m/s} - (12\text{ kg}\cdot\text{m/s}) = -4\text{ kg}\cdot\text{m/s}$. This would say that the wall had somehow given the ball a kick to the *left!* Make sure that you account for the *direction* of momentum in your calculations.

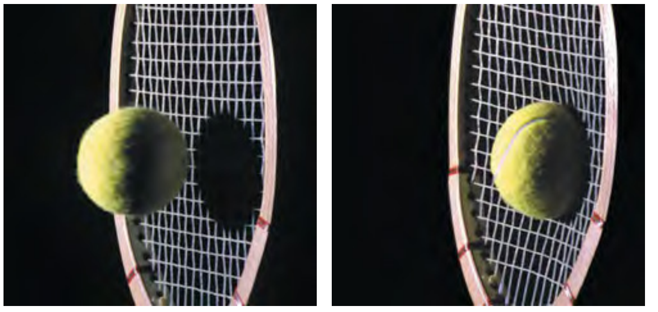
The force that the wall exerts on the ball must have such a large magnitude (2000 N, equal to the weight of a 200-kg object) to

Continued

change the ball's momentum in such a short time. Other forces that act on the ball during the collision are comparatively weak; for instance, the gravitational force is only 3.9 N. Thus, during the short time that the collision lasts, we can ignore all other forces on the ball. Figure 8.6 shows the impact of a tennis ball and racket.

Note that the 2000-N value we calculated is the *average* horizontal force that the wall exerts on the ball during the impact. It corresponds to the horizontal line $(F_{av})_x$ in Fig. 8.3a. The horizontal force is zero before impact, rises to a maximum, and then decreases to zero when the ball loses contact with the wall. If the ball is relatively rigid, like a baseball or golf ball, the collision lasts a short time and the maximum force is large, as in the blue curve in Fig. 8.3b. If the ball is softer, like a tennis ball, the collision time is longer and the maximum force is less, as in the orange curve in Fig. 8.3b.

8.6 Typically, a tennis ball is in contact with the racket for approximately 0.01 s. The ball flattens noticeably due to the tremendous force exerted by the racket.



Example 8.3 Kicking a soccer ball

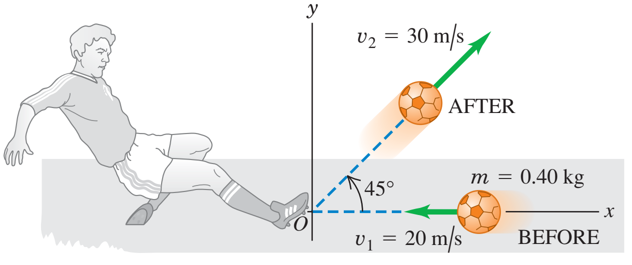
A soccer ball has a mass of 0.40 kg. Initially it is moving to the left at 20 m/s, but then it is kicked. After the kick it is moving at 45° upward and to the right with speed 30 m/s (Fig. 8.7a). Find the impulse of the net force and the average net force, assuming a collision time $\Delta t = 0.010$ s.

SOLUTION

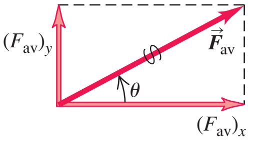
IDENTIFY and SET UP: The ball moves in two dimensions, so we must treat momentum and impulse as vector quantities. We take the x-axis to be horizontally to the right and the y-axis to be vertically upward. Our target variables are the components of the net impulse on the ball, J_x and J_y, and the components of the average net force on the ball, $(F_{av})_x$ and $(F_{av})_y$. We'll find them using the impulse–momentum theorem in its component form, Eqs. (8.9).

8.7 (a) Kicking a soccer ball. (b) Finding the average force on the ball from its components.

(a) Before-and-after diagram



(b) Average force on the ball



EXECUTE: Using $\cos 45° = \sin 45° = 0.707$, we find the ball's velocity components before and after the kick:

$$v_{1x} = -20 \text{ m/s} \qquad v_{1y} = 0$$
$$v_{2x} = v_{2y} = (30 \text{ m/s})(0.707) = 21.2 \text{ m/s}$$

From Eqs. (8.9), the impulse components are

$$\begin{aligned} J_x &= p_{2x} - p_{1x} = m(v_{2x} - v_{1x}) \\ &= (0.40 \text{ kg})[21.2 \text{ m/s} - (-20 \text{ m/s})] = 16.5 \text{ kg}\cdot\text{m/s} \end{aligned}$$

$$\begin{aligned} J_y &= p_{2y} - p_{1y} = m(v_{2y} - v_{1y}) \\ &= (0.40 \text{ kg})(21.2 \text{ m/s} - 0) = 8.5 \text{ kg}\cdot\text{m/s} \end{aligned}$$

From Eq. (8.8), the average net force components are

$$(F_{av})_x = \frac{J_x}{\Delta t} = 1650 \text{ N} \qquad (F_{av})_y = \frac{J_y}{\Delta t} = 850 \text{ N}$$

The magnitude and direction of the average net force $\vec{F}_{av}$ are

$$F_{av} = \sqrt{(1650 \text{ N})^2 + (850 \text{ N})^2} = 1.9 \times 10^3 \text{ N}$$
$$\theta = \arctan\frac{850 \text{ N}}{1650 \text{ N}} = 27°$$

The ball was not initially at rest, so its final velocity does *not* have the same direction as the average force that acted on it.

EVALUATE: $\vec{F}_{av}$ includes the force of gravity, which is very small; the weight of the ball is only 3.9 N. As in Example 8.2, the average force acting during the collision is exerted almost entirely by the object that the ball hit (in this case, the soccer player's foot).

Test Your Understanding of Section 8.1 Rank the following situations according to the magnitude of the impulse of the net force, from largest value to smallest value. In each situation a 1000-kg automobile is moving along a straight east–west road. (i) The automobile is initially moving east at 25 m/s and comes to a stop in 10 s. (ii) The automobile is initially moving east at 25 m/s and comes to a stop in 5 s. (iii) The automobile is initially at rest, and a 2000-N net force toward the east is applied to it for 10 s. (iv) The automobile is initially moving east at 25 m/s, and a 2000-N net force toward the west is applied to it for 10 s. (v) The automobile is initially moving east at 25 m/s. Over a 30-s period, the automobile reverses direction and ends up moving west at 25 m/s.

8.2 Conservation of Momentum

The concept of momentum is particularly important in situations in which we have two or more bodies that *interact*. To see why, let's consider first an idealized system of two bodies that interact with each other but not with anything else—for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (Fig. 8.8). Think of the astronauts as particles. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal and opposite, and the changes in momentum of the two particles are equal and opposite.

Let's go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces.** Forces exerted on any part of the system by some object outside it are called **external forces.** For the system shown in Fig. 8.8, the internal forces are $\vec{F}_{B\text{ on }A}$, exerted by particle B on particle A, and $\vec{F}_{A\text{ on }B}$, exerted by particle A on particle B. There are *no* external forces; when this is the case, we have an **isolated system.**

The net force on particle A is $\vec{F}_{B\text{ on }A}$ and the net force on particle B is $\vec{F}_{A\text{ on }B}$, so from Eq. (8.4) the rates of change of the momenta of the two particles are

$$\vec{F}_{B\text{ on }A} = \frac{d\vec{p}_A}{dt} \qquad \vec{F}_{A\text{ on }B} = \frac{d\vec{p}_B}{dt} \tag{8.10}$$

The momentum of each particle changes, but these changes are related to each other by Newton's third law: The two forces $\vec{F}_{B\text{ on }A}$ and $\vec{F}_{A\text{ on }B}$ are always equal in magnitude and opposite in direction. That is, $\vec{F}_{B\text{ on }A} = -\vec{F}_{A\text{ on }B}$, so $\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \mathbf{0}$. Adding together the two equations in Eq. (8.10), we have

$$\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \frac{d(\vec{p}_A + \vec{p}_B)}{dt} = \mathbf{0} \tag{8.11}$$

The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $\vec{p}_A + \vec{p}_B$ is zero. We now define the **total momentum** $\vec{P}$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

$$\vec{P} = \vec{p}_A + \vec{p}_B \tag{8.12}$$

Then Eq. (8.11) becomes, finally,

$$\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \frac{d\vec{P}}{dt} = \mathbf{0} \tag{8.13}$$

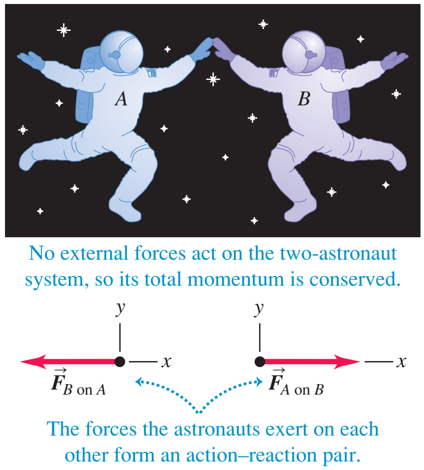
The time rate of change of the *total* momentum $\vec{P}$ is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

If external forces are also present, they must be included on the left side of Eq. (8.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in Fig. 8.9, these forces have no effect on the left side of Eq. (8.13), and $d\vec{P}/dt$ is again zero. Thus we have the following general result:

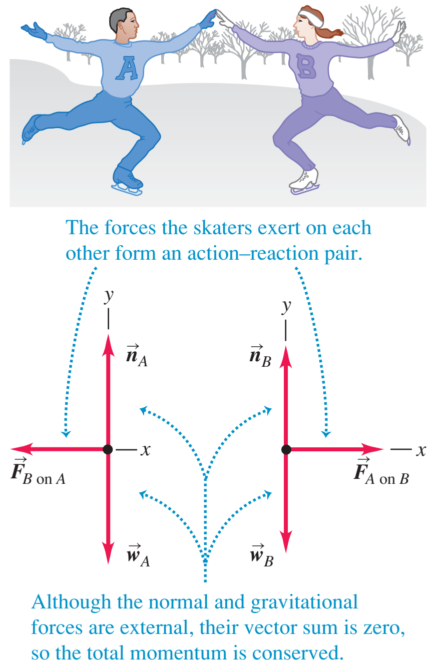
If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

This is the simplest form of the **principle of conservation of momentum.** This principle is a direct consequence of Newton's third law. What makes this principle useful is that it doesn't depend on the detailed nature of the internal forces that

8.8 Two astronauts push each other as they float freely in the zero-gravity environment of space.



8.9 Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.8.)



act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton's second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles A, B, C, . . . interacting only with one another. The total momentum of such a system is

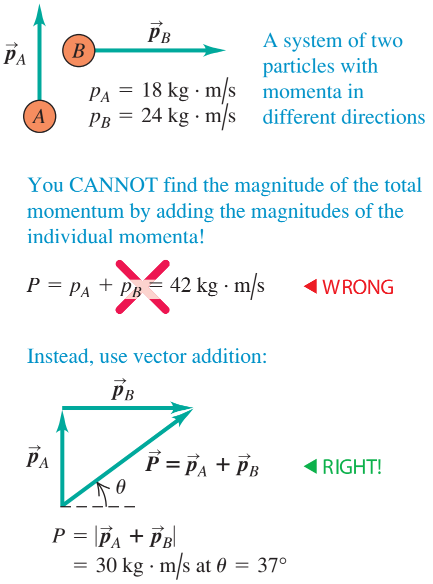
$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots = m_A\vec{v}_A + m_B\vec{v}_B + \cdots \quad \text{(total momentum of a system of particles)} \tag{8.14}$$

We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero whenever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles in the system but not the *total* momentum of the system.

MasteringPHYSICS

ActivPhysics 6.3: Momentum Conservation and Collisions
ActivPhysics 6.7: Explosion Problems
ActivPhysics 6.10: Pendulum Person-Projectile Bowling

8.10 When applying conservation of momentum, remember that momentum is a vector quantity!



CAUTION **Conservation of momentum means conservation of its components** When you apply the conservation of momentum to a system, remember that momentum is a *vector* quantity. Hence you must use vector addition to compute the total momentum of a system (Fig. 8.10). Using components is usually the simplest method. If p_{Ax}, p_{Ay}, and p_{Az} are the components of momentum of particle A, and similarly for the other particles, then Eq. (8.14) is equivalent to the component equations

$$\begin{aligned} P_x &= p_{Ax} + p_{Bx} + \cdots \\ P_y &= p_{Ay} + p_{By} + \cdots \\ P_z &= p_{Az} + p_{Bz} + \cdots \end{aligned} \tag{8.15}$$

If the vector sum of the external forces on the system is zero, then P_x, P_y, and P_z are all constant.

In some ways the principle of conservation of momentum is more general than the principle of conservation of mechanical energy. For example, mechanical energy is conserved only when the internal forces are *conservative*—that is, when the forces allow two-way conversion between kinetic and potential energy—but conservation of momentum is valid even when the internal forces are *not* conservative. In this chapter we will analyze situations in which both momentum and mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we will encounter them throughout our study of physics.

Problem-Solving Strategy 8.1 Conservation of Momentum

MP

IDENTIFY *the relevant concepts:* Confirm that the vector sum of the external forces acting on the system of particles is zero. If it isn't zero, you can't use conservation of momentum.

SET UP *the problem* using the following steps:

1. Treat each body as a particle. Draw "before" and "after" sketches, including velocity vectors. Assign algebraic symbols to each magnitude, angle, and component. Use letters to label each particle and subscripts 1 and 2 for "before" and "after" quantities. Include any given values such as magnitudes, angles, or components.
2. Define a coordinate system and show it in your sketches; define the positive direction for each axis.
3. Identify the target variables.

EXECUTE *the solution:*

1. Write an equation in symbols equating the total initial and final x-components of momentum, using $p_x = mv_x$ for each particle. Write a corresponding equation for the y-components. Velocity components can be positive or negative, so be careful with signs!
2. In some problems, energy considerations (discussed in Section 8.4) give additional equations relating the velocities.
3. Solve your equations to find the target variables.

EVALUATE *your answer:* Does your answer make physical sense? If your target variable is a certain body's momentum, check that the direction of the momentum is reasonable.

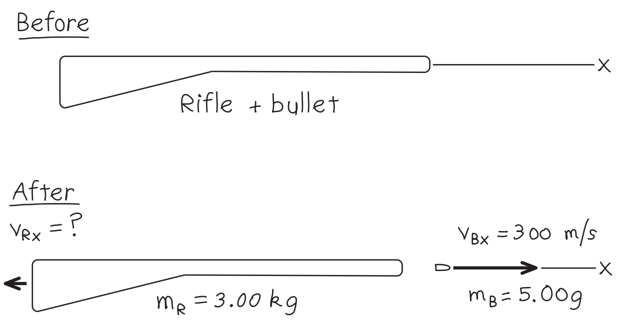
Example 8.4 Recoil of a rifle

A marksman holds a rifle of mass $m_R = 3.00$ kg loosely, so it can recoil freely. He fires a bullet of mass $m_B = 5.00$ g horizontally with a velocity relative to the ground of $v_{Bx} = 300$ m/s. What is the recoil velocity v_{Rx} of the rifle? What are the final momentum and kinetic energy of the bullet and rifle?

SOLUTION

IDENTIFY and SET UP: If the marksman exerts negligible horizontal forces on the rifle, then there is no net horizontal force on the system (the bullet and rifle) during the firing, and the total horizontal momentum of the system is conserved. Figure 8.11 shows our sketch. We take the positive x-axis in the direction of aim. The rifle and the bullet are initially at rest, so the initial x-component of total momentum is zero. After the shot is fired, the bullet's x-momentum is $p_{Bx} = m_B v_{Bx}$ and the rifle's x-momentum is $p_{Rx} = m_R v_{Rx}$. Our target variables are v_{Rx}, p_{Bx}, p_{Rx}, and the final kinetic energies $K_B = \frac{1}{2}m_B v_{Bx}^2$ and $K_R = \frac{1}{2}m_R v_{Rx}^2$.

8.11 Our sketch for this problem.



EXECUTE: Conservation of the x-component of total momentum gives

$$P_x = 0 = m_B v_{Bx} + m_R v_{Rx}$$

$$v_{Rx} = -\frac{m_B}{m_R}v_{Bx} = -\left(\frac{0.00500 \text{ kg}}{3.00 \text{ kg}}\right)(300 \text{ m/s}) = -0.500 \text{ m/s}$$

The negative sign means that the recoil is in the direction opposite to that of the bullet.

The final momenta and kinetic energies are

$$p_{Bx} = m_B v_{Bx} = (0.00500 \text{ kg})(300 \text{ m/s}) = 1.50 \text{ kg}\cdot\text{m/s}$$

$$K_B = \tfrac{1}{2}m_B v_{Bx}^2 = \tfrac{1}{2}(0.00500 \text{ kg})(300 \text{ m/s})^2 = 225 \text{ J}$$

$$p_{Rx} = m_R v_{Rx} = (3.00 \text{ kg})(-0.500 \text{ m/s}) = -1.50 \text{ kg}\cdot\text{m/s}$$

$$K_R = \tfrac{1}{2}m_R v_{Rx}^2 = \tfrac{1}{2}(3.00 \text{ kg})(-0.500 \text{ m/s})^2 = 0.375 \text{ J}$$

EVALUATE: The bullet and rifle have equal and opposite final *momenta* thanks to Newton's third law: They experience equal and opposite interaction forces that act for the same *time,* so the impulses are equal and opposite. But the bullet travels a much greater *distance* than the rifle during the interaction. Hence the force on the bullet does more work than the force on the rifle, giving the bullet much greater *kinetic energy* than the rifle. The 600:1 ratio of the two kinetic energies is the inverse of the ratio of the masses; in fact, you can show that this always happens in recoil situations (see Exercise 8.26).

Example 8.5 Collision along a straight line

Two gliders with different masses move toward each other on a frictionless air track (Fig. 8.12a). After they collide (Fig. 8.12b), glider B has a final velocity of $+2.0$ m/s (Fig. 8.12c). What is the final velocity of glider A? How do the changes in momentum and in velocity compare?

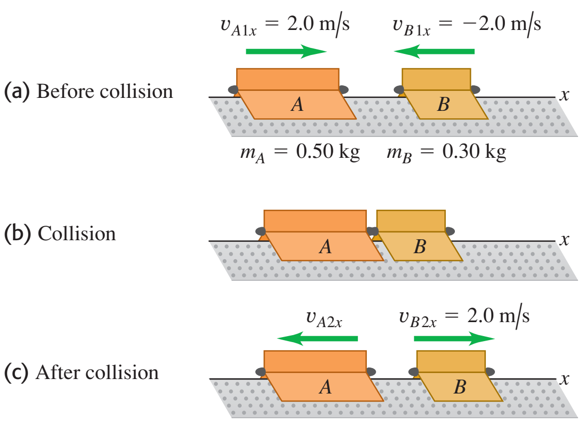
SOLUTION

IDENTIFY and SET UP: As for the skaters in Fig. 8.9, the total vertical force on each glider is zero, and the net force on each individual glider is the horizontal force exerted on it by the other glider. The net external force on the *system* of two gliders is zero, so their total momentum is conserved. We take the positive x-axis to be to the right. We are given the masses and initial velocities of both gliders and the final velocity of glider B. Our target variables are v_{A2x}, the final x-component of velocity of glider A, and the changes in momentum and in velocity of the two gliders (the value *after* the collision minus the value *before* the collision).

EXECUTE: The x-component of total momentum before the collision is

$$\begin{aligned} P_x &= m_A v_{A1x} + m_B v_{B1x} \\ &= (0.50 \text{ kg})(2.0 \text{ m/s}) + (0.30 \text{ kg})(-2.0 \text{ m/s}) \\ &= 0.40 \text{ kg}\cdot\text{m/s} \end{aligned}$$

8.12 Two gliders colliding on an air track.



This is positive (to the right in Fig. 8.12) because A has a greater magnitude of momentum than B. The x-component of total momentum has the same value after the collision, so

$$P_x = m_A v_{A2x} + m_B v_{B2x}$$

Continued

We solve for v_{A2x}:

$$v_{A2x} = \frac{P_x - m_B v_{B2x}}{m_A} = \frac{0.40\ \text{kg} \cdot \text{m/s} - (0.30\ \text{kg})(2.0\ \text{m/s})}{0.50\ \text{kg}}$$
$$= -0.40\ \text{m/s}$$

The changes in the x-momenta are

$$m_A v_{A2x} - m_A v_{A1x} = (0.50\ \text{kg})(-0.40\ \text{m/s}) - (0.50\ \text{kg})(2.0\ \text{m/s}) = -1.2\ \text{kg} \cdot \text{m/s}$$

$$m_B v_{B2x} - m_B v_{B1x} = (0.30\ \text{kg})(2.0\ \text{m/s}) - (0.30\ \text{kg})(-2.0\ \text{m/s}) = +1.2\ \text{kg} \cdot \text{m/s}$$

The changes in x-velocities are

$$v_{A2x} - v_{A1x} = (-0.40\ \text{m/s}) - 2.0\ \text{m/s} = -2.4\ \text{m/s}$$
$$v_{B2x} - v_{B1x} = 2.0\ \text{m/s} - (-2.0\ \text{m/s}) = +4.0\ \text{m/s}$$

EVALUATE: The gliders were subjected to equal and opposite interaction forces for the same time during their collision. By the impulse–momentum theorem, they experienced equal and opposite impulses and therefore equal and opposite changes in momentum. But by Newton's second law, the less massive glider (B) had a greater magnitude of acceleration and hence a greater velocity change.

Example 8.6 Collision in a horizontal plane

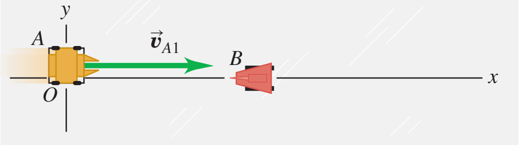
Figure 8.13a shows two battling robots on a frictionless surface. Robot A, with mass 20 kg, initially moves at 2.0 m/s parallel to the x-axis. It collides with robot B, which has mass 12 kg and is initially at rest. After the collision, robot A moves at 1.0 m/s in a direction that makes an angle $\alpha = 30°$ with its initial direction (Fig. 8.13b). What is the final velocity of robot B?

SOLUTION

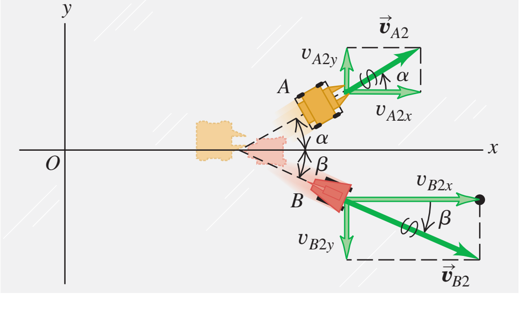
IDENTIFY and SET UP: There are no horizontal external forces, so the x- and y-components of the total momentum of the system are both conserved. Momentum conservation requires that the sum of the x-components of momentum *before* the collision (subscript 1) must equal the sum *after* the collision (subscript 2), and similarly for the sums of the y-components. Our target variable is $\vec{v}_{B2}$, the final velocity of robot B.

8.13 Views from above of the velocities (a) before and (b) after the collision.

(a) Before collision



(b) After collision



EXECUTE: The momentum-conservation equations and their solutions for v_{B2x} and v_{B2y} are

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$
$$v_{B2x} = \frac{m_A v_{A1x} + m_B v_{B1x} - m_A v_{A2x}}{m_B}$$
$$= \frac{\begin{bmatrix} (20\ \text{kg})(2.0\ \text{m/s}) + (12\ \text{kg})(0) \\ - (20\ \text{kg})(1.0\ \text{m/s})(\cos 30°) \end{bmatrix}}{12\ \text{kg}}$$
$$= 1.89\ \text{m/s}$$

$$m_A v_{A1y} + m_B v_{B1y} = m_A v_{A2y} + m_B v_{B2y}$$
$$v_{B2y} = \frac{m_A v_{A1y} + m_B v_{B1y} - m_A v_{A2y}}{m_B}$$
$$= \frac{\begin{bmatrix} (20\ \text{kg})(0) + (12\ \text{kg})(0) \\ - (20\ \text{kg})(1.0\ \text{m/s})(\sin 30°) \end{bmatrix}}{12\ \text{kg}}$$
$$= -0.83\ \text{m/s}$$

Figure 8.13b shows the motion of robot B after the collision. The magnitude of $\vec{v}_{B2}$ is

$$v_{B2} = \sqrt{(1.89\ \text{m/s})^2 + (-0.83\ \text{m/s})^2} = 2.1\ \text{m/s}$$

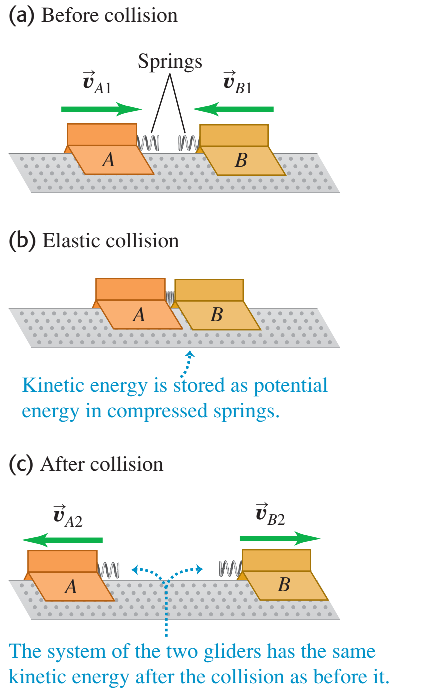
and the angle of its direction from the positive x-axis is

$$\beta = \arctan \frac{-0.83\ \text{m/s}}{1.89\ \text{m/s}} = -24°$$

EVALUATE: We can check our answer by confirming that the components of total momentum before and after the collision are equal. Initially robot A has x-momentum $m_A v_{A1x} = (20\ \text{kg})(2.0\ \text{m/s}) = 40\ \text{kg} \cdot \text{m/s}$ and zero y-momentum; robot B has zero momentum. After the collision, the momentum components are $m_A v_{A2x} = (20\ \text{kg})(1.0\ \text{m/s})(\cos 30°) = 17\ \text{kg} \cdot \text{m/s}$ and $m_B v_{B2x} = (12\ \text{kg})(1.89\ \text{m/s}) = 23\ \text{kg} \cdot \text{m/s}$; the total x-momentum is $40\ \text{kg} \cdot \text{m/s}$, the same as before the collision. The final y-components are $m_A v_{A2y} = (20\ \text{kg})(1.0\ \text{m/s})(\sin 30°) = 10\ \text{kg} \cdot \text{m/s}$ and $m_B v_{B2y} = (12\ \text{kg})(-0.83\ \text{m/s}) = -10\ \text{kg} \cdot \text{m/s}$; the total y-component of momentum is zero, the same as before the collision.

Test Your Understanding of Section 8.2 A spring-loaded toy sits at rest on a horizontal, frictionless surface. When the spring releases, the toy breaks into three equal-mass pieces, *A*, *B*, and *C*, which slide along the surface. Piece *A* moves off in the negative *x*-direction, while piece *B* moves off in the negative *y*-direction. (a) What are the signs of the velocity components of piece *C*? (b) Which of the three pieces is moving the fastest?

8.14 Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider.



8.3 Momentum Conservation and Collisions

To most people the term *collision* is likely to mean some sort of automotive disaster. We'll use it in that sense, but we'll also broaden the meaning to include any strong interaction between bodies that lasts a relatively short time. So we include not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, the impact of a meteor on the Arizona desert, and a close encounter of a spacecraft with the planet Saturn.

If the forces between the bodies are much larger than any external forces, as is the case in most collisions, we can neglect the external forces entirely and treat the bodies as an *isolated* system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

Elastic and Inelastic Collisions

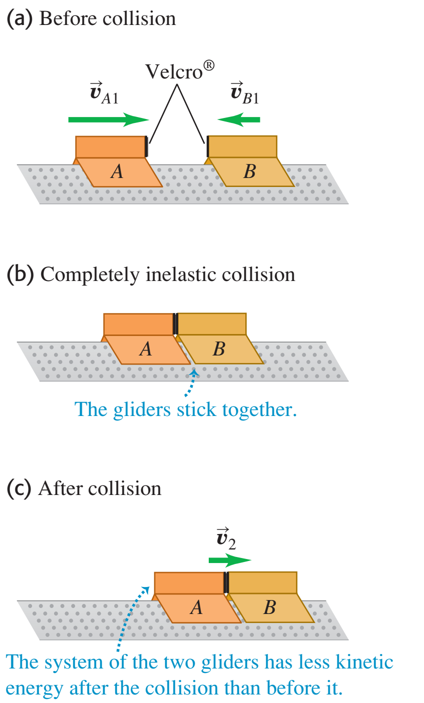
If the forces between the bodies are also *conservative,* so that no mechanical energy is lost or gained in the collision, the total *kinetic* energy of the system is the same after the collision as before. Such a collision is called an **elastic collision.** A collision between two marbles or two billiard balls is almost completely elastic. Figure 8.14 shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

A collision in which the total kinetic energy after the collision is *less* than before the collision is called an **inelastic collision.** A meatball landing on a plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding bodies stick together and move as one body after the collision is often called a **completely inelastic collision.** Figure 8.15 shows an example; we have replaced the spring bumpers in Fig. 8.14 with Velcro®, which sticks the two bodies together.

CAUTION **An inelastic collision doesn't have to be *completely* inelastic** It's a common misconception that the *only* inelastic collisions are those in which the colliding bodies stick together. In fact, inelastic collisions include many situations in which the bodies do *not* stick. If two cars bounce off each other in a "fender bender," the work done to deform the fenders cannot be recovered as kinetic energy of the cars, so the collision is inelastic (Fig. 8.16).

Remember this rule: **In any collision in which external forces can be neglected, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions *only,* the total kinetic energy before equals the total kinetic energy after.**

8.15 Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro®, so the gliders stick together after collision.



Completely Inelastic Collisions

Let's look at what happens to momentum and kinetic energy in a *completely* inelastic collision of two bodies (*A* and *B*), as in Fig. 8.15. Because the two bodies stick together after the collision, they have the same final velocity $\vec{v}_2$:

$$\vec{v}_{A2} = \vec{v}_{B2} = \vec{v}_2$$

8.16 Automobile collisions are intended to be inelastic, so that the structure of the car absorbs as much of the energy of the collision as possible. This absorbed energy cannot be recovered, since it goes into a permanent deformation of the car.

Conservation of momentum gives the relationship

$$m_A\vec{\boldsymbol{v}}_{A1} + m_B\vec{\boldsymbol{v}}_{B1} = (m_A + m_B)\vec{\boldsymbol{v}}_2 \quad \text{(completely inelastic collision)} \tag{8.16}$$

If we know the masses and initial velocities, we can compute the common final velocity $\vec{\boldsymbol{v}}_2$.

Suppose, for example, that a body with mass m_A and initial x-component of velocity v_{A1x} collides inelastically with a body with mass m_B that is initially at rest ($v_{B1x} = 0$). From Eq. (8.16) the common x-component of velocity v_{2x} of both bodies after the collision is

$$v_{2x} = \frac{m_A}{m_A + m_B}v_{A1x} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.17}$$

Let's verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the x-axis, so the kinetic energies K_1 and K_2 before and after the collision, respectively, are

$$K_1 = \tfrac{1}{2}m_A v_{A1x}{}^2$$

$$K_2 = \tfrac{1}{2}(m_A + m_B)v_{2x}{}^2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B}\right)^2 v_{A1x}{}^2$$

The ratio of final to initial kinetic energy is

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.18}$$

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of m_B is not zero, it is not hard to verify that the kinetic energy after a completely inelastic collision is always less than before.

Please note: We don't recommend memorizing Eq. (8.17) or (8.18). We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

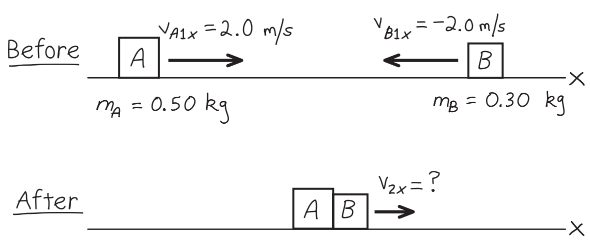
Example 8.7 A completely inelastic collision

We repeat the collision described in Example 8.5 (Section 8.2), but this time equip the gliders so that they stick together when they collide. Find the common final x-velocity, and compare the initial and final kinetic energies of the system.

SOLUTION

IDENTIFY and SET UP: There are no external forces in the x-direction, so the x-component of momentum is conserved. Figure 8.17 shows our sketch. Our target variables are the final x-velocity v_{2x} and the initial and final kinetic energies K_1 and K_2.

8.17 Our sketch for this problem.



EXECUTE: From conservation of momentum,

$$m_A v_{A1x} + m_B v_{B1x} = (m_A + m_B)v_{2x}$$

$$\begin{aligned} v_{2x} &= \frac{m_A v_{A1x} + m_B v_{B1x}}{m_A + m_B} \\ &= \frac{(0.50\text{ kg})(2.0\text{ m/s}) + (0.30\text{ kg})(-2.0\text{ m/s})}{0.50\text{ kg} + 0.30\text{ kg}} \\ &= 0.50\text{ m/s} \end{aligned}$$

Because v_{2x} is positive, the gliders move together to the right after the collision. Before the collision, the kinetic energies are

$$K_A = \tfrac{1}{2}m_A v_{A1x}{}^2 = \tfrac{1}{2}(0.50\text{ kg})(2.0\text{ m/s})^2 = 1.0\text{ J}$$

$$K_B = \tfrac{1}{2}m_B v_{B1x}{}^2 = \tfrac{1}{2}(0.30\text{ kg})(-2.0\text{ m/s})^2 = 0.60\text{ J}$$

The total kinetic energy before the collision is $K_1 = K_A + K_B = 1.6$ J. The kinetic energy after the collision is

$$K_2 = \tfrac{1}{2}(m_A + m_B)v_{2x}{}^2 = \tfrac{1}{2}(0.50\text{ kg} + 0.30\text{ kg})(0.50\text{ m/s})^2 = 0.10\text{ J}$$

EVALUATE: The final kinetic energy is only $\frac{1}{16}$ of the original; $\frac{15}{16}$ is converted from mechanical energy to other forms. If there is a wad of chewing gum between the gliders, it squashes and becomes warmer. If there is a spring between the gliders that is compressed as they lock together, the energy is stored as potential energy of the spring. In both cases the *total* energy of the system is conserved, although *kinetic* energy is not. In an isolated system, however, momentum is *always* conserved whether the collision is elastic or not.

Example 8.8 The ballistic pendulum

Figure 8.18 shows a ballistic pendulum, a simple system for measuring the speed of a bullet. A bullet of mass m_B makes a completely inelastic collision with a block of wood of mass m_W, which is suspended like a pendulum. After the impact, the block swings up to a maximum height y. In terms of y, m_B, and m_W, what is the initial speed v_1 of the bullet?

SOLUTION

IDENTIFY: We'll analyze this event in two stages: (1) the embedding of the bullet in the block and (2) the pendulum swing of the block. During the first stage, the bullet embeds itself in the block so quickly that the block does not move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the bullet–block system, and the horizontal component of momentum is conserved. Mechanical energy is *not* conserved during this stage, however, because a nonconservative force does work (the force of friction between bullet and block).

In the second stage, the block and bullet move together. The only forces acting on this system are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings, *mechanical energy* is conserved. Momentum is *not* conserved during this stage, however, because there is a net external force (the forces of gravity and string tension don't cancel when the strings are inclined).

SET UP: We take the positive x-axis to the right and the positive y-axis upward. Our target variable is v_1. Another unknown quantity is the speed v_2 of the system just after the collision. We'll use momentum conservation in the first stage to relate v_1 to v_2, and we'll use energy conservation in the second stage to relate v_2 to y.

EXECUTE: In the first stage, all velocities are in the $+x$-direction. Momentum conservation gives

$$m_B v_1 = (m_B + m_W)v_2$$
$$v_1 = \frac{m_B + m_W}{m_B}v_2$$

At the beginning of the second stage, the system has kinetic energy $K = \frac{1}{2}(m_B + m_W)v_2^2$. The system swings up and comes to rest for an instant at a height y, where its kinetic energy is zero and the potential energy is $(m_B + m_W)gy$; it then swings back down. Energy conservation gives

$$\tfrac{1}{2}(m_B + m_W)v_2^2 = (m_B + m_W)gy$$
$$v_2 = \sqrt{2gy}$$

We substitute this expression for v_2 into the momentum equation:

$$v_1 = \frac{m_B + m_W}{m_B}\sqrt{2gy}$$

EVALUATE: Let's plug in the realistic numbers $m_B = 5.00\text{ g} = 0.00500\text{ kg}$, $m_W = 2.00\text{ kg}$, and $y = 3.00\text{ cm} = 0.0300\text{ m}$. We then have

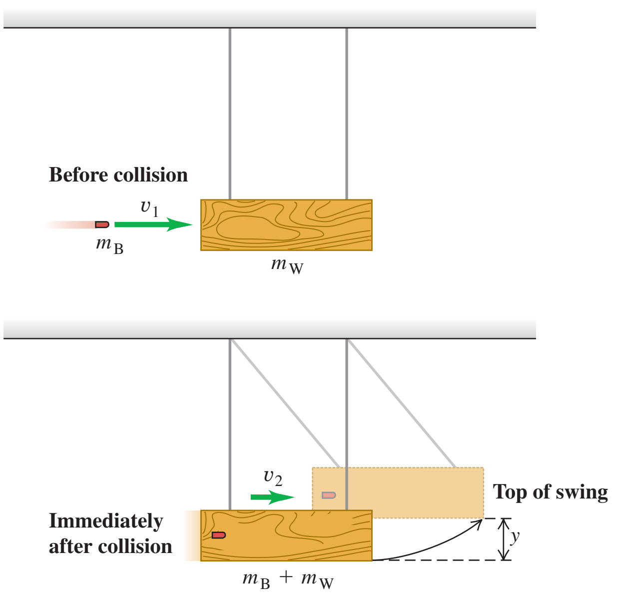
$$v_1 = \frac{0.00500\text{ kg} + 2.00\text{ kg}}{0.00500\text{ kg}}\sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})}$$
$$= 307\text{ m/s}$$

The speed v_2 of the block just after impact is

$$v_2 = \sqrt{2gy} = \sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})}$$
$$= 0.767\text{ m/s}$$

The speeds v_1 and v_2 seem realistic. The kinetic energy of the bullet before impact is $\frac{1}{2}(0.00500\text{ kg})(307\text{ m/s})^2 = 236\text{ J}$. Just after impact the kinetic energy of the system is $\frac{1}{2}(2.005\text{ kg})(0.767\text{ m/s})^2 = 0.590\text{ J}$. Nearly all the kinetic energy disappears as the wood splinters and the bullet and block become warmer.

8.18 A ballistic pendulum.



Example 8.9 An automobile collision

A 1000-kg car traveling north at 15 m/s collides with a 2000-kg truck traveling east at 10 m/s. The occupants, wearing seat belts, are uninjured, but the two vehicles move away from the impact point as one. The insurance adjustor asks you to find the velocity of the wreckage just after impact. What is your answer?

SOLUTION

IDENTIFY and SET UP: We'll treat the cars as an isolated system, so that the momentum of the system is conserved. We can do so because (as we show below) the magnitudes of the horizontal forces that the cars exert on each other during the collision are much larger than any external forces such as friction. Figure 8.19 shows our sketch and the coordinate axes. We can find the total momentum $\vec{P}$ before the collision using Eqs. (8.15). The momentum has the same value just after the collision; hence we can find the velocity $\vec{V}$ just after the collision (our target variable) using $\vec{P} = M\vec{V}$, where $M = m_{\mathrm{C}} + m_{\mathrm{T}} = 3000\ \mathrm{kg}$ is the mass of the wreckage.

EXECUTE: From Eqs. (8.15), the components of $\vec{P}$ are

$$\begin{aligned} P_x &= p_{\mathrm{C}x} + p_{\mathrm{T}x} = m_{\mathrm{C}}v_{\mathrm{C}x} + m_{\mathrm{T}}v_{\mathrm{T}x} \\ &= (1000\ \mathrm{kg})(0) + (2000\ \mathrm{kg})(10\ \mathrm{m/s}) \\ &= 2.0 \times 10^4\ \mathrm{kg \cdot m/s} \\ P_y &= p_{\mathrm{C}y} + p_{\mathrm{T}y} = m_{\mathrm{C}}v_{\mathrm{C}y} + m_{\mathrm{T}}v_{\mathrm{T}y} \\ &= (1000\ \mathrm{kg})(15\ \mathrm{m/s}) + (2000\ \mathrm{kg})(0) \\ &= 1.5 \times 10^4\ \mathrm{kg \cdot m/s} \end{aligned}$$

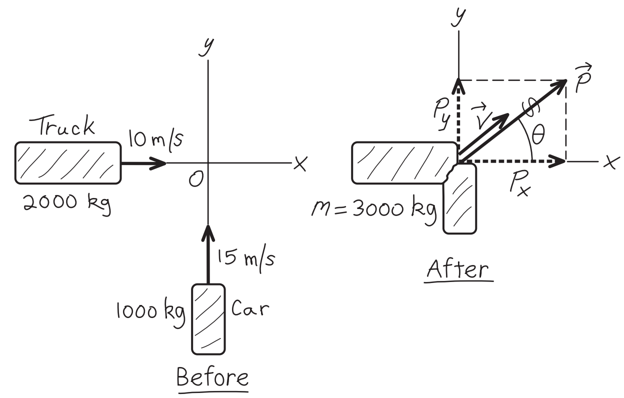
The magnitude of $\vec{P}$ is

$$\begin{aligned} P &= \sqrt{(2.0 \times 10^4\ \mathrm{kg \cdot m/s})^2 + (1.5 \times 10^4\ \mathrm{kg \cdot m/s})^2} \\ &= 2.5 \times 10^4\ \mathrm{kg \cdot m/s} \end{aligned}$$

and its direction is given by the angle θ shown in Fig. 8.19:

$$\tan\theta = \frac{P_y}{P_x} = \frac{1.5 \times 10^4\ \mathrm{kg \cdot m/s}}{2.0 \times 10^4\ \mathrm{kg \cdot m/s}} = 0.75 \qquad \theta = 37^\circ$$

8.19 Our sketch for this problem.



From $\vec{P} = M\vec{V}$, the direction of the velocity $\vec{V}$ just after the collision is also $\theta = 37^\circ$. The velocity magnitude is

$$V = \frac{P}{M} = \frac{2.5 \times 10^4\ \mathrm{kg \cdot m/s}}{3000\ \mathrm{kg}} = 8.3\ \mathrm{m/s}$$

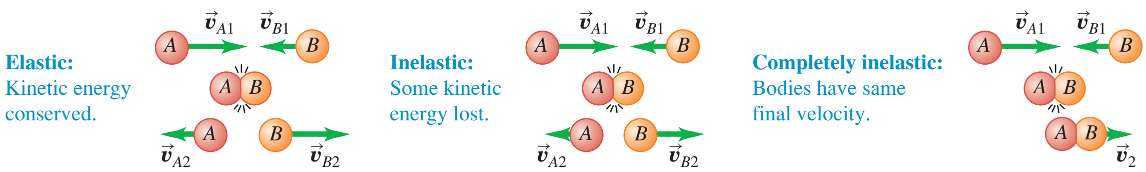
EVALUATE: This is an inelastic collision, so we expect the total kinetic energy to be less after the collision than before. As you can show, the initial kinetic energy is 2.1×10^5 J and the final value is 1.0×10^5 J.

We can now justify our neglect of the external forces on the vehicles during the collision. The car's weight is about 10,000 N; if the coefficient of kinetic friction is 0.5, the friction force on the car during the impact is about 5000 N. The car's initial kinetic energy is $\frac{1}{2}(1000\ \mathrm{kg})(15\ \mathrm{m/s})^2 = 1.1 \times 10^5\ \mathrm{J}$, so $-1.1 \times 10^5\ \mathrm{J}$ of work must be done to stop it. If the car crumples by 0.20 m in stopping, a force of magnitude $(1.1 \times 10^5\ \mathrm{J})/(0.20\ \mathrm{m}) = 5.5 \times 10^5\ \mathrm{N}$ would be needed; that's 110 times the friction force. So it's reasonable to treat the external force of friction as negligible compared with the internal forces the vehicles exert on each other.

Classifying Collisions

It's important to remember that we can classify collisions according to energy considerations (Fig. 8.20). A collision in which kinetic energy is conserved is called *elastic.* (We'll explore these in more depth in the next section.) A collision in which the total kinetic energy decreases is called *inelastic*. When the two bodies have a common final velocity, we say that the collision is *completely inelastic.* There are also cases in which the final kinetic energy is *greater* than the initial value. Rifle recoil, discussed in Example 8.4 (Section 8.2), is an example.

8.20 Collisions are classified according to energy considerations.



Finally, we emphasize again that we can sometimes use momentum conservation even when there are external forces acting on the system, if the net external force acting on the colliding bodies is small in comparison with the internal forces during the collision (as in Example 8.9)

Test Your Understanding of Section 8.3 For each situation, state whether the collision is elastic or inelastic. If it is inelastic, state whether it is completely inelastic. (a) You drop a ball from your hand. It collides with the floor and bounces back up so that it just reaches your hand. (b) You drop a different ball from your hand and let it collide with the ground. This ball bounces back up to half the height from which it was dropped. (c) You drop a ball of clay from your hand. When it collides with the ground, it stops.

8.4 Elastic Collisions

We saw in Section 8.3 that an *elastic collision* in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding bodies are *conservative.* When two billiard balls collide, they squash a little near the surface of contact, but then they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy (Fig. 8.21).

8.21 Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo. Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Let's look at an elastic collision between two bodies A and B. We start with a one-dimensional collision, in which all the velocities lie along the same line; we choose this line to be the x-axis. Each momentum and velocity then has only an x-component. We call the x-velocities before the collision v_{A1x} and v_{B1x}, and those after the collision v_{A2x} and v_{B2x}. From conservation of kinetic energy we have

$$\tfrac{1}{2}m_A v_{A1x}{}^2 + \tfrac{1}{2}m_B v_{B1x}{}^2 = \tfrac{1}{2}m_A v_{A2x}{}^2 + \tfrac{1}{2}m_B v_{B2x}{}^2$$

and conservation of momentum gives

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

If the masses m_A and m_B and the initial velocities v_{A1x} and v_{B1x} are known, we can solve these two equations to find the two final velocities v_{A2x} and v_{B2x}.

Elastic Collisions, One Body Initially at Rest

The general solution to the above equations is a little complicated, so we will concentrate on the particular case in which body B is at rest before the collision (so $v_{B1x} = 0$). Think of body B as a target for body A to hit. Then the kinetic energy and momentum conservation equations are, respectively,

$$\tfrac{1}{2}m_A v_{A1x}{}^2 = \tfrac{1}{2}m_A v_{A2x}{}^2 + \tfrac{1}{2}m_B v_{B2x}{}^2 \tag{8.19}$$

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x} \tag{8.20}$$

We can solve for v_{A2x} and v_{B2x} in terms of the masses and the initial velocity v_{A1x}. This involves some fairly strenuous algebra, but it's worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

First we rearrange Eqs. (8.19) and (8.20) as follows:

$$m_B v_{B2x}{}^2 = m_A(v_{A1x}{}^2 - v_{A2x}{}^2) = m_A(v_{A1x} - v_{A2x})(v_{A1x} + v_{A2x}) \tag{8.21}$$

$$m_B v_{B2x} = m_A(v_{A1x} - v_{A2x}) \tag{8.22}$$

Now we divide Eq. (8.21) by Eq. (8.22) to obtain

$$v_{B2x} = v_{A1x} + v_{A2x} \tag{8.23}$$

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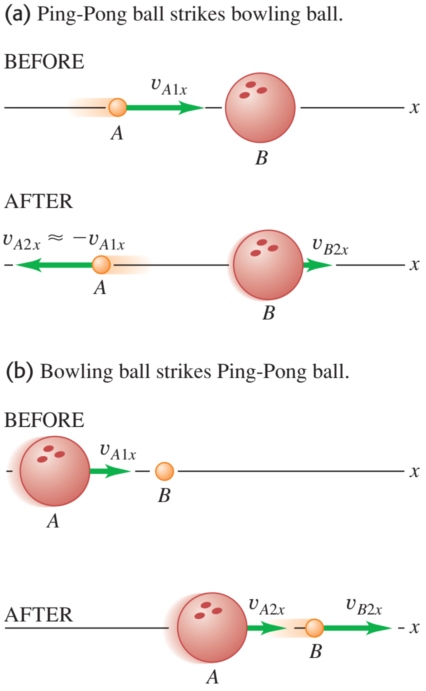
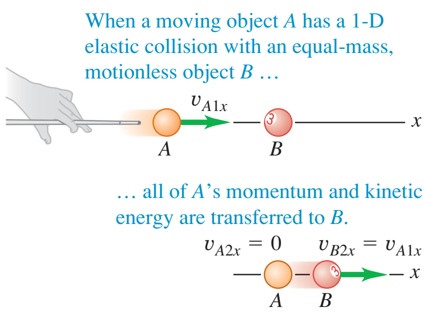
ActivPhysics 6.5: Car Collisions: Two Dimensions
ActivPhysics 6.9: Pendulum Bashes Box

8.22 Collisions between **(a)** a moving Ping-Pong ball and an initially stationary bowling ball, and **(b)** a moving bowling ball and an initially stationary Ping-Pong ball.



8.23 A one-dimensional elastic collision between bodies of equal mass.



We substitute this expression back into Eq. (8.22) to eliminate v_{B2x} and then solve for v_{A2x}:

$$m_B(v_{A1x} + v_{A2x}) = m_A(v_{A1x} - v_{A2x})$$

$$v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x} \qquad (8.24)$$

Finally, we substitute this result back into Eq. (8.23) to obtain

$$v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x} \qquad (8.25)$$

Now we can interpret the results. Suppose body A is a Ping-Pong ball and body B is a bowling ball. Then we expect A to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (Fig. 8.22a), and we expect B's velocity to be much less. That's just what the equations predict. When m_A is much smaller than m_B, the fraction in Eq. (8.24) is approximately equal to (-1), so v_{A2x} is approximately equal to $-v_{A1x}$. The fraction in Eq. (8.25) is much smaller than unity, so v_{B2x} is much less than v_{A1x}. Figure 8.22b shows the opposite case, in which A is the bowling ball and B the Ping-Pong ball and m_A is much larger than m_B. What do you expect to happen then? Check your predictions against Eqs. (8.24) and (8.25).

Another interesting case occurs when the masses are equal (Fig. 8.23). If $m_A = m_B$, then Eqs. (8.24) and (8.25) give $v_{A2x} = 0$ and $v_{B2x} = v_{A1x}$. That is, the body that was moving stops dead; it gives all its momentum and kinetic energy to the body that was at rest. This behavior is familiar to all pool players.

Elastic Collisions and Relative Velocity

Let's return to the more general case in which A and B have different masses. Equation (8.23) can be rewritten as

$$v_{A1x} = v_{B2x} - v_{A2x} \qquad (8.26)$$

Here $v_{B2x} - v_{A2x}$ is the velocity of B relative to A *after* the collision; from Eq. (8.26), this equals v_{A1x}, which is the *negative* of the velocity of B relative to A *before* the collision. (We discussed relative velocity in Section 3.5.) The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because A and B are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the bodies are different but the *relative* velocities are the same. Hence our statement about relative velocities holds for *any* straight-line elastic collision, even when neither body is at rest initially. *In a straight-line elastic collision of two bodies, the relative velocities before and after the collision have the same magnitude but opposite sign.* This means that if B is moving before the collision, Eq. (8.26) becomes

$$v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x}) \qquad (8.27)$$

It turns out that a *vector* relationship similar to Eq. (8.27) is a general property of *all* elastic collisions, even when both bodies are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: *In an elastic collision, the relative velocity of the two bodies has the same magnitude before and after the collision.* Whenever this condition is satisfied, the total kinetic energy is also conserved.

When an elastic two-body collision isn't head-on, the velocities don't all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the x- and y-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

Example 8.10 An elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?

SOLUTION

IDENTIFY and SET UP: The net external force on the system is zero, so the momentum of the system is conserved. Figure 8.24 shows our sketch. We'll find our target variables, v_{A2x} and v_{B2x}, using Eq. (8.27), the relative-velocity relationship for an elastic collision, and the momentum-conservation equation.

EXECUTE: From Eq. (8.27),

$$\begin{aligned} v_{B2x} - v_{A2x} &= -(v_{B1x} - v_{A1x}) \\ &= -(-2.0 \text{ m/s} - 2.0 \text{ m/s}) = 4.0 \text{ m/s} \end{aligned}$$

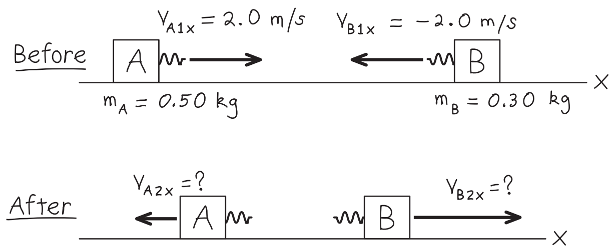
From conservation of momentum,

$$\begin{aligned} m_A v_{A1x} + m_B v_{B1x} &= m_A v_{A2x} + m_B v_{B2x} \\ (0.50 \text{ kg})(2.0 \text{ m/s}) + (0.30 \text{ kg})(-2.0 \text{ m/s}) & \\ &= (0.50 \text{ kg})v_{A2x} + (0.30 \text{ kg})v_{B2x} \\ 0.50v_{A2x} + 0.30v_{B2x} &= 0.40 \text{ m/s} \end{aligned}$$

(To get the last equation we divided both sides of the equation just above it by the quantity 1 kg. This makes the units the same as in the first equation.) Solving these equations simultaneously, we find

$$v_{A2x} = -1.0 \text{ m/s} \qquad v_{B2x} = 3.0 \text{ m/s}$$

8.24 Our sketch for this problem.



EVALUATE: Both bodies reverse their directions of motion; A moves to the left at 1.0 m/s and B moves to the right at 3.0 m/s. This is unlike the result of Example 8.5 because that collision was *not* elastic. The more massive glider A slows down in the collision and so loses kinetic energy. The less massive glider B speeds up and gains kinetic energy. The total kinetic energy before the collision (which we calculated in Example 8.7) is 1.6 J. The total kinetic energy after the collision is

$$\tfrac{1}{2}(0.50 \text{ kg})(-1.0 \text{ m/s})^2 + \tfrac{1}{2}(0.30 \text{ kg})(3.0 \text{ m/s})^2 = 1.6 \text{ J}$$

As expected, the kinetic energies before and after this elastic collision are equal. Kinetic energy is transferred from A to B, but none of it is lost.

CAUTION **Be careful with the elastic collision equations** You could *not* have solved this problem using Eqs. (8.24) and (8.25), which apply only if body B is initially *at rest*. Always be sure that you solve the problem at hand using equations that are applicable!

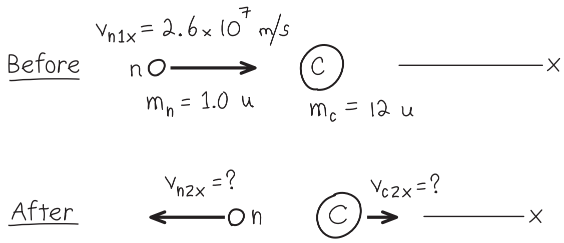
Example 8.11 Moderating fission neutrons in a nuclear reactor

The fission of uranium nuclei in a nuclear reactor produces high-speed neutrons. Before such neutrons can efficiently cause additional fissions, they must be slowed down by collisions with nuclei in the *moderator* of the reactor. The first nuclear reactor (built in 1942 at the University of Chicago) used carbon (graphite) as the moderator. Suppose a neutron (mass 1.0 u) traveling at 2.6×10^7 m/s undergoes a head-on elastic collision with a carbon nucleus (mass 12 u) initially at rest. Neglecting external forces during the collision, find the velocities after the collision. (1 u is the *atomic mass unit,* equal to 1.66×10^{-27} kg.)

SOLUTION

IDENTIFY and SET UP: We neglect external forces, so momentum is conserved in the collision. The collision is elastic, so kinetic energy is also conserved. Figure 8.25 shows our sketch. We take the x-axis to be in the direction in which the neutron is moving initially. The collision is head-on, so both particles move along this same axis after the collision. The carbon nucleus is initially at rest, so we can use Eqs. (8.24) and (8.25); we replace A by n (for the neutron) and B by C (for the carbon nucleus). We have $m_{\text{n}} = 1.0$ u, $m_{\text{C}} = 12$ u, and $v_{\text{n}1x} = 2.6 \times 10^7$ m/s. The target variables are the final velocities $v_{\text{n}2x}$ and $v_{\text{C}2x}$.

8.25 Our sketch for this problem.



EXECUTE: You can do the arithmetic. (*Hint:* There's no reason to convert atomic mass units to kilograms.) The results are

$$v_{\text{n}2x} = -2.2 \times 10^7 \text{ m/s} \qquad v_{\text{C}2x} = 0.4 \times 10^7 \text{ m/s}$$

EVALUATE: The neutron ends up with $|(m_{\text{n}} - m_{\text{C}})/(m_{\text{n}} + m_{\text{C}})| = \frac{11}{13}$ of its initial speed, and the speed of the recoiling carbon nucleus is $|2m_{\text{n}}/(m_{\text{n}} + m_{\text{C}})| = \frac{2}{13}$ of the neutron's initial speed. Kinetic energy is proportional to speed squared, so the neutron's final kinetic energy is $(\frac{11}{13})^2 \approx 0.72$ of its original value. After a second head-on collision, its kinetic energy is $(0.72)^2$, or about half its original value, and so on. After a few dozen collisions (few of which are head-on), the neutron speed will be low enough that it can efficiently cause a fission reaction in a uranium nucleus.

Example 8.12 A two-dimensional elastic collision

Figure 8.26 shows an elastic collision of two pucks (masses $m_A = 0.500$ kg and $m_B = 0.300$ kg) on a frictionless air-hockey table. Puck A has an initial velocity of 4.00 m/s in the positive x-direction and a final velocity of 2.00 m/s in an unknown direction α. Puck B is initially at rest. Find the final speed v_{B2} of puck B and the angles α and β.

SOLUTION

IDENTIFY and SET UP: We'll use the equations for conservation of energy and conservation of x- and y-momentum. These three equations should be enough to solve for the three target variables given in the problem statement.

EXECUTE: The collision is elastic, so the initial and final kinetic energies of the system are equal:

$$\tfrac{1}{2}m_A v_{A1}^2 = \tfrac{1}{2}m_A v_{A2}^2 + \tfrac{1}{2}m_B v_{B2}^2$$

$$v_{B2}^2 = \frac{m_A v_{A1}^2 - m_A v_{A2}^2}{m_B}$$

$$= \frac{(0.500\text{ kg})(4.00\text{ m/s})^2 - (0.500\text{ kg})(2.00\text{ m/s})^2}{0.300\text{ kg}}$$

$$v_{B2} = 4.47\text{ m/s}$$

Conservation of the x- and y-components of total momentum gives

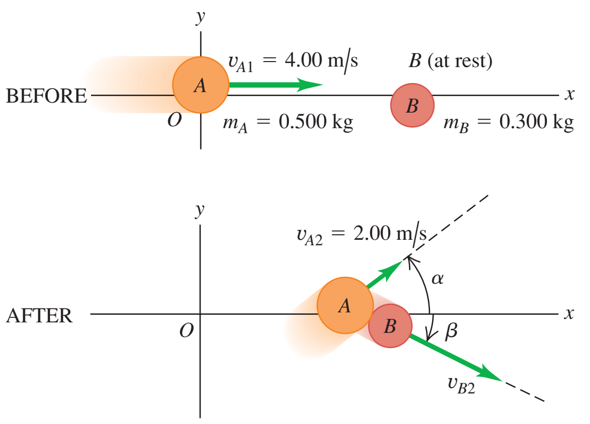
$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$(0.500\text{ kg})(4.00\text{ m/s}) = (0.500\text{ kg})(2.00\text{ m/s})(\cos\alpha) + (0.300\text{ kg})(4.47\text{ m/s})(\cos\beta)$$

$$0 = m_A v_{A2y} + m_B v_{B2y}$$

$$0 = (0.500\text{ kg})(2.00\text{ m/s})(\sin\alpha) - (0.300\text{ kg})(4.47\text{ m/s})(\sin\beta)$$

8.26 An elastic collision that isn't head-on.



These are two simultaneous equations for α and β. We'll leave it to you to supply the details of the solution. (*Hint:* Solve the first equation for $\cos\beta$ and the second for $\sin\beta$; square each equation and add. Since $\sin^2\beta + \cos^2\beta = 1$, this eliminates β and leaves an equation that you can solve for $\cos\alpha$ and hence for α. Substitute this value into either of the two equations and solve for β.) The results are

$$\alpha = 36.9^\circ \qquad \beta = 26.6^\circ$$

EVALUATE: To check the answers we confirm that the y-momentum, which was zero before the collision, is in fact zero after the collision. The y-momenta are

$$p_{A2y} = (0.500\text{ kg})(2.00\text{ m/s})(\sin 36.9^\circ) = +0.600\text{ kg}\cdot\text{m/s}$$

$$p_{B2y} = -(0.300\text{ kg})(4.47\text{ m/s})(\sin 26.6^\circ) = -0.600\text{ kg}\cdot\text{m/s}$$

and their sum is indeed zero.

Test Your Understanding of Section 8.4 Most present-day nuclear reactors use water as a moderator (see Example 8.11). Are water molecules (mass $m_w = 18.0$ u) a better or worse moderator than carbon atoms? (One advantage of water is that it also acts as a coolant for the reactor's radioactive core.)

8.5 Center of Mass

We can restate the principle of conservation of momentum in a useful way by using the concept of **center of mass.** Suppose we have several particles with masses m_1, m_2, and so on. Let the coordinates of m_1 be (x_1, y_1), those of m_2 be (x_2, y_2), and so on. We define the center of mass of the system as the point that has coordinates (x_{cm}, y_{cm}) given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \qquad \text{(center of mass)} \quad (8.28)$$

The position vector $\vec{r}_{cm}$ of the center of mass can be expressed in terms of the position vectors $\vec{r}_1, \vec{r}_2, \ldots$ of the particles as

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad \text{(center of mass)} \tag{8.29}$$

In statistical language, the center of mass is a *mass-weighted average* position of the particles.

Example 8.13 Center of mass of a water molecule

Figure 8.27 shows a simple model of a water molecule. The oxygen-hydrogen separation is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u, and the oxygen atom has mass 16.0 u. Find the position of the center of mass.

SOLUTION

IDENTIFY and SET UP: Nearly all the mass of each atom is concentrated in its nucleus, whose radius is only about 10^{-5} times the overall radius of the atom. Hence we can safely represent each atom as a point particle. Figure 8.27 shows our coordinate system, with the x-axis chosen to lie along the molecule's symmetry axis. We'll use Eqs. (8.28) to find x_{cm} and y_{cm}.

EXECUTE: The oxygen atom is at $x = 0$, $y = 0$. The x-coordinate of each hydrogen atom is $d\cos(105°/2)$; the y-coordinates are $\pm d\sin(105°/2)$. From Eqs. (8.28),

$$x_{cm} = \frac{\left[\begin{array}{l}(1.0\text{ u})(d\cos 52.5°) + (1.0\text{ u}) \\ \times (d\cos 52.5°) + (16.0\text{ u})(0)\end{array}\right]}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0.068d$$

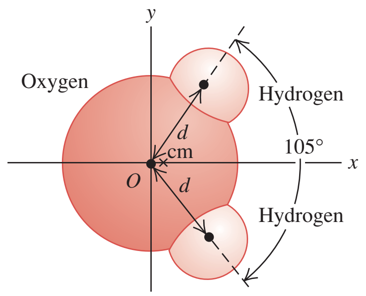
$$y_{cm} = \frac{\left[\begin{array}{l}(1.0\text{ u})(d\sin 52.5°) + (1.0\text{ u}) \\ \times (-d\sin 52.5°) + (16.0\text{ u})(0)\end{array}\right]}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0$$

Substituting $d = 9.57 \times 10^{-11}$ m, we find

$$x_{cm} = (0.068)(9.57 \times 10^{-11}\text{ m}) = 6.5 \times 10^{-12}\text{ m}$$

EVALUATE: The center of mass is much closer to the oxygen atom (located at the origin) than to either hydrogen atom because the oxygen atom is much more massive. The center of mass lies along the molecule's *axis of symmetry*. If the molecule is rotated 180° around this axis, it looks exactly the same as before. The position of the center of mass can't be affected by this rotation, so it *must* lie on the axis of symmetry.

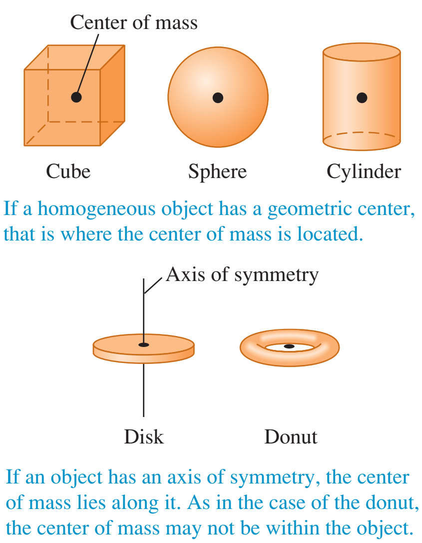
8.27 Where is the center of mass of a water molecule?



For solid bodies, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (8.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (Fig. 8.28). First, whenever a homogeneous body has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever a body has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the body. For example, the center of mass of a donut is right in the middle of the hole.

We'll talk a little more about locating the center of mass in Chapter 11 in connection with the related concept of *center of gravity.*

8.28 Locating the center of mass of a symmetrical object.



Motion of the Center of Mass

To see the significance of the center of mass of a collection of particles, we must ask what happens to the center of mass when the particles move. The x- and y-components of velocity of the center of mass, $v_{cm\text{-}x}$ and $v_{cm\text{-}y}$, are the time derivatives of x_{cm} and y_{cm}. Also, dx_1/dt is the x-component of velocity of particle 1,

8.29 The center of mass of this wrench is marked with a white dot. The net external force acting on the wrench is almost zero. As the wrench spins on a smooth horizontal surface, the center of mass moves in a straight line with nearly constant velocity.

and so on, so $dx_1/dt = v_{1x}$, and so on. Taking time derivatives of Eqs. (8.28), we get

$$v_{\text{cm-}x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$v_{\text{cm-}y} = \frac{m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (8.30)$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2 + m_3\vec{\boldsymbol{v}}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (8.31)$$

We denote the *total* mass $m_1 + m_2 + \cdots$ by M. We can then rewrite Eq. (8.31) as

$$M\vec{\boldsymbol{v}}_{\text{cm}} = m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2 + m_3\vec{\boldsymbol{v}}_3 + \cdots = \vec{\boldsymbol{P}} \quad (8.32)$$

The right side is simply the total momentum $\vec{\boldsymbol{P}}$ of the system. Thus we have proved that *the total momentum is equal to the total mass times the velocity of the center of mass.* When you catch a baseball, you are really catching a collection of a very large number of molecules of masses $m_1, m_2, m_3, \ldots$. The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + \cdots$ moving with velocity $\vec{\boldsymbol{v}}_{\text{cm}}$, the velocity of the collection's center of mass. So Eq. (8.32) helps to justify representing an extended body as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\vec{\boldsymbol{P}}$ is constant, the velocity of the center of mass $\vec{\boldsymbol{v}}_{\text{cm}} = \vec{\boldsymbol{P}}/M$ is also constant. Suppose we mark the center of mass of a wrench and then slide the wrench with a spinning motion across a smooth, horizontal tabletop (Fig. 8.29). The overall motion appears complicated, but the center of mass follows a straight line, as though all the mass were concentrated at that point.

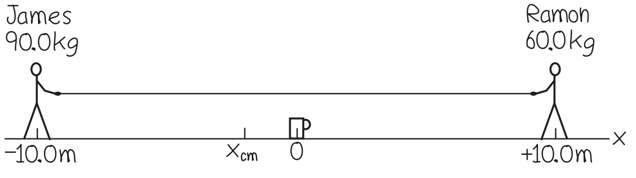
Example 8.14 A tug-of-war on the ice

James (mass 90.0 kg) and Ramon (mass 60.0 kg) are 20.0 m apart on a frozen pond. Midway between them is a mug of their favorite beverage. They pull on the ends of a light rope stretched between them. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

SOLUTION

IDENTIFY and SET UP: The surface is horizontal and (we assume) frictionless, so the net external force on the system of James, Ramon, and the rope is zero; their total momentum is conserved. Initially there is no motion, so the total momentum is zero. The velocity of the center of mass is therefore zero, and it remains at rest. Let's take the origin at the position of the mug and let the $+x$-axis extend from the mug toward Ramon. Figure 8.30 shows our sketch. We use Eq. (8.28) to calculate the position of the center of mass; we neglect the mass of the light rope.

8.30 Our sketch for this problem.



EXECUTE: The initial x-coordinates of James and Ramon are -10.0 m and $+10.0$ m, respectively, so the x-coordinate of the center of mass is

$$x_{\text{cm}} = \frac{(90.0\text{ kg})(-10.0\text{ m}) + (60.0\text{ kg})(10.0\text{ m})}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$

When James moves 6.0 m toward the mug, his new x-coordinate is -4.0 m; we'll call Ramon's new x-coordinate x_2. The center of mass doesn't move, so

$$x_{\text{cm}} = \frac{(90.0\text{ kg})(-4.0\text{ m}) + (60.0\text{ kg})x_2}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$
$$x_2 = 1.0\text{ m}$$

James has moved 6.0 m and is still 4.0 m from the mug, but Ramon has moved 9.0 m and is only 1.0 m from it.

EVALUATE: The ratio of the distances moved, $(6.0\text{ m})/(9.0\text{ m}) = \frac{2}{3}$, is the *inverse* ratio of the masses. Can you see why? Because the surface is frictionless, the two men will keep moving and collide at the center of mass; Ramon will reach the mug first. This is independent of how hard either person pulls; pulling harder just makes them move faster.

External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes. Let's look at the relationship between the motion of the center of mass and the forces acting on the system.

Equations (8.31) and (8.32) give the *velocity* of the center of mass in terms of the velocities of the individual particles. We take the time derivatives of these equations to show that the *accelerations* are related in the same way. Let $\vec{a}_{cm} = d\vec{v}_{cm}/dt$ be the acceleration of the center of mass; then we find

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots \quad (8.33)$$

Now $m_1\vec{a}_1$ is equal to the vector sum of forces on the first particle, and so on, so the right side of Eq. (8.33) is equal to the vector sum $\sum\vec{F}$ of *all* the forces on *all* the particles. Just as we did in Section 8.2, we can classify each force as *external* or *internal.* The sum of all forces on all the particles is then

$$\sum\vec{F} = \sum\vec{F}_{ext} + \sum\vec{F}_{int} = M\vec{a}_{cm}$$

Because of Newton's third law, the internal forces all cancel in pairs, and $\sum\vec{F}_{int} = \mathbf{0}$. What survives on the left side is the sum of only the *external* forces:

$$\sum\vec{F}_{ext} = M\vec{a}_{cm} \qquad \text{(body or collection of particles)} \quad (8.34)$$

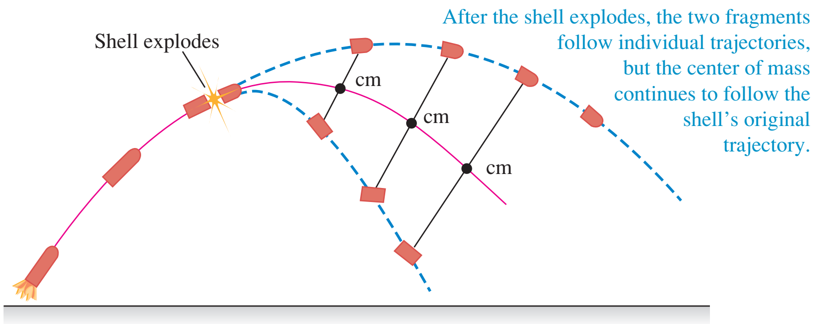
When a body or a collection of particles is acted on by external forces, the center of mass moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.

This result may not sound very impressive, but in fact it is central to the whole subject of mechanics. In fact, we've been using this result all along; without it, we would not be able to represent an extended body as a point particle when we apply Newton's laws. It explains why only *external* forces can affect the motion of an extended body. If you pull upward on your belt, your belt exerts an equal downward force on your hands; these are *internal* forces that cancel and have no effect on the overall motion of your body.

Suppose a cannon shell traveling in a parabolic trajectory (neglecting air resistance) explodes in flight, splitting into two fragments with equal mass (Fig. 8.31a). The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, just as though all the mass were still concentrated at that point. A skyrocket exploding in air (Fig. 8.31b) is a spectacular example of this effect.

8.31 (a) A shell explodes into two fragments in flight. If air resistance is ignored, the center of mass continues on the same trajectory as the shell's path before exploding. (b) The same effect occurs with exploding fireworks.

(a)



(b)

This property of the center of mass is important when we analyze the motion of rigid bodies. We describe the motion of an extended body as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass. We will return to this topic in Chapter 10. This property also plays an important role in the motion of astronomical objects. It's not correct to say that the moon orbits the earth; rather, the earth and moon both move in orbits around their center of mass.

There's one more useful way to describe the motion of a system of particles. Using $\vec{a}_{cm} = d\vec{v}_{cm}/dt$, we can rewrite Eq. (8.33) as

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt} \tag{8.35}$$

The total system mass M is constant, so we're allowed to move it inside the derivative. Substituting Eq. (8.35) into Eq. (8.34), we find

$$\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt} \qquad \text{(extended body or system of particles)} \tag{8.36}$$

This equation looks like Eq. (8.4). The difference is that Eq. (8.36) describes a *system* of particles, such as an extended body, while Eq. (8.4) describes a single particle. The interactions between the particles that make up the system can change the individual momenta of the particles, but the *total* momentum $\vec{P}$ of the system can be changed only by external forces acting from outside the system.

Finally, we note that if the net external force is zero, Eq. (8.34) shows that the acceleration $\vec{a}_{cm}$ of the center of mass is zero. So the center-of-mass velocity $\vec{v}_{cm}$ is constant, as for the wrench in Fig. 8.29. From Eq. (8.36) the total momentum $\vec{P}$ is also constant. This reaffirms our statement in Section 8.3 of the principle of conservation of momentum.

Test Your Understanding of Section 8.5 Will the center of mass in Fig. 8.31a continue on the same parabolic trajectory even after one of the fragments hits the ground? Why or why not?

Application Jet Propulsion in Squids
Both a jet engine and a squid use variations in their mass to provide propulsion: They increase their mass by taking in fluid at low speed (air for a jet engine, water for a squid), then decrease their mass by ejecting that fluid at high speed. The net result is a propulsive force.

8.6 Rocket Propulsion

Momentum considerations are particularly useful for analyzing a system in which the masses of parts of the system change with time. In such cases we can't use Newton's second law $\sum\vec{F} = m\vec{a}$ directly because m changes. Rocket propulsion offers a typical and interesting example of this kind of analysis. A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket (which is why rocket fuel is also called *propellant*). The forward force on the rocket is the reaction to the backward force on the ejected material. The total mass of the system is constant, but the mass of the rocket itself decreases as material is ejected.

As a simple example, consider a rocket fired in outer space, where there is no gravitational force and no air resistance. Let m denote the mass of the rocket, which will change as it expends fuel. We choose our x-axis to be along the rocket's direction of motion. Figure 8.32a shows the rocket at a time t, when its mass is m and its x-velocity relative to our coordinate system is v. (For simplicity, we will drop the subscript x in this discussion.) The x-component of total momentum at this instant is $P_1 = mv$. In a short time interval dt, the mass of the rocket changes by an amount dm. This is an inherently negative quantity because the rocket's mass m *decreases* with time. During dt, a *positive* mass $-dm$ of burned fuel is ejected from the rocket. Let v_{ex} be the exhaust *speed* of this material *relative to the rocket;* the burned fuel is ejected opposite the direction of motion,

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so its x-component of *velocity* relative to the rocket is $-v_{ex}$. The x-velocity v_{fuel} of the burned fuel relative to our coordinate system is then

$$v_{fuel} = v + (-v_{ex}) = v - v_{ex}$$

and the x-component of momentum of the ejected mass $(-dm)$ is

$$(-dm)v_{fuel} = (-dm)(v - v_{ex})$$

Figure 8.32b shows that at the end of the time interval dt, the x-velocity of the rocket and unburned fuel has increased to $v + dv$, and its mass has decreased to $m + dm$ (remember that dm is negative). The rocket's momentum at this time is

$$(m + dm)(v + dv)$$

Thus the *total* x-component of momentum P_2 of the rocket plus ejected fuel at time $t + dt$ is

$$P_2 = (m + dm)(v + dv) + (-dm)(v - v_{ex})$$

According to our initial assumption, the rocket and fuel are an isolated system. Thus momentum is conserved, and the total x-component of momentum of the system must be the same at time t and at time $t + dt$: $P_1 = P_2$. Hence

$$mv = (m + dm)(v + dv) + (-dm)(v - v_{ex})$$

This can be simplified to

$$m\,dv = -dm\,v_{ex} - dm\,dv$$

We can neglect the term $(-dm\,dv)$ because it is a product of two small quantities and thus is much smaller than the other terms. Dropping this term, dividing by dt, and rearranging, we find

$$m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt} \tag{8.37}$$

Now dv/dt is the acceleration of the rocket, so the left side of this equation (mass times acceleration) equals the net force F, or *thrust,* on the rocket:

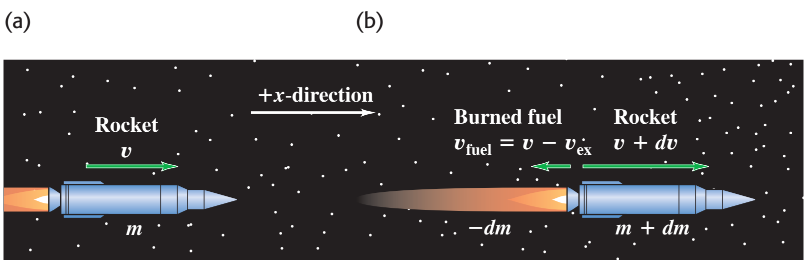
$$F = -v_{ex}\frac{dm}{dt} \tag{8.38}$$

The thrust is proportional both to the relative speed v_{ex} of the ejected fuel and to the mass of fuel ejected per unit time, $-dm/dt$. (Remember that dm/dt is negative because it is the rate of change of the rocket's mass, so F is positive.)

The x-component of acceleration of the rocket is

$$a = \frac{dv}{dt} = -\frac{v_{ex}}{m}\frac{dm}{dt} \tag{8.39}$$

8.32 A rocket moving in gravity-free outer space at (a) time t and (b) time $t + dt$.



At time t, the rocket has mass m and x-component of velocity v.

At time $t + dt$, the rocket has mass $m + dm$ (where dm is inherently *negative*) and x-component of velocity $v + dv$. The burned fuel has x-component of velocity $v_{fuel} = v - v_{ex}$ and mass $-dm$. (The minus sign is needed to make $-dm$ *positive* because dm is negative.)

8.33 To provide enough thrust to lift its payload into space, this Atlas V launch vehicle ejects more than 1000 kg of burned fuel per second at speeds of nearly 4000 m/s.

This is positive because v_{ex} is positive (remember, it's the exhaust *speed*) and dm/dt is negative. The rocket's mass m decreases continuously while the fuel is being consumed. If v_{ex} and dm/dt are constant, the acceleration increases until all the fuel is gone.

Equation (8.38) tells us that an effective rocket burns fuel at a rapid rate (large $-dm/dt$) and ejects the burned fuel at a high relative speed (large v_{ex}), as in Fig. 8.33. In the early days of rocket propulsion, people who didn't understand conservation of momentum thought that a rocket couldn't function in outer space because "it doesn't have anything to push against." On the contrary, rockets work *best* in outer space, where there is no air resistance! The launch vehicle in Fig. 8.33 is *not* "pushing against the ground" to get into the air.

If the exhaust speed v_{ex} is constant, we can integrate Eq. (8.39) to find a relationship between the velocity v at any time and the remaining mass m. At time $t = 0$, let the mass be m_0 and the velocity v_0. Then we rewrite Eq. (8.39) as

$$dv = -v_{ex}\frac{dm}{m}$$

We change the integration variables to v' and m', so we can use v and m as the upper limits (the final speed and mass). Then we integrate both sides, using limits v_0 to v and m_0 to m, and take the constant v_{ex} outside the integral:

$$\int_{v_0}^{v} dv' = -\int_{m_0}^{m} v_{ex}\frac{dm'}{m'} = -v_{ex}\int_{m_0}^{m}\frac{dm'}{m'}$$

$$v - v_0 = -v_{ex}\ln\frac{m}{m_0} = v_{ex}\ln\frac{m_0}{m} \tag{8.40}$$

The ratio m_0/m is the original mass divided by the mass after the fuel has been exhausted. In practical spacecraft this ratio is made as large as possible to maximize the speed gain, which means that the initial mass of the rocket is almost all fuel. The final velocity of the rocket will be greater in magnitude (and is often *much* greater) than the relative speed v_{ex} if $\ln(m_0/m) > 1$—that is, if $m_0/m > e = 2.71828\ldots$

We've assumed throughout this analysis that the rocket is in gravity-free outer space. However, gravity must be taken into account when a rocket is launched from the surface of a planet, as in Fig. 8.33 (see Problem 8.112).

Example 8.15 Acceleration of a rocket

The engine of a rocket in outer space, far from any planet, is turned on. The rocket ejects burned fuel at a constant rate; in the first second of firing, it ejects $\frac{1}{120}$ of its initial mass m_0 at a relative speed of 2400 m/s. What is the rocket's initial acceleration?

SOLUTION

IDENTIFY and SET UP: We are given the rocket's exhaust speed v_{ex} and the fraction of the initial mass lost during the first second of firing, from which we can find dm/dt. We'll use Eq. (8.39) to find the acceleration of the rocket.

EXECUTE: The initial rate of change of mass is

$$\frac{dm}{dt} = -\frac{m_0/120}{1\text{ s}} = -\frac{m_0}{120\text{ s}}$$

From Eq. (8.39),

$$a = -\frac{v_{ex}}{m_0}\frac{dm}{dt} = -\frac{2400\text{ m/s}}{m_0}\left(-\frac{m_0}{120\text{ s}}\right) = 20\text{ m/s}^2$$

EVALUATE: The answer doesn't depend on m_0. If v_{ex} is the same, the initial acceleration is the same for a 120,000-kg spacecraft that ejects 1000 kg/s as for a 60-kg astronaut equipped with a small rocket that ejects 0.5 kg/s.

Example 8.16 Speed of a rocket

Suppose that $\frac{3}{4}$ of the initial mass of the rocket in Example 8.15 is fuel, so that the fuel is completely consumed at a constant rate in 90 s. The final mass of the rocket is $m = m_0/4$. If the rocket starts from rest in our coordinate system, find its speed at the end of this time.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: We are given the initial velocity $v_0 = 0$, the exhaust speed $v_{ex} = 2400\ \text{m/s}$, and the final mass m as a fraction of the initial mass m_0. We'll use Eq. (8.40) to find the final speed v:

$$v = v_0 + v_{ex} \ln \frac{m_0}{m} = 0 + (2400\ \text{m/s})(\ln 4) = 3327\ \text{m/s}$$

EVALUATE: Let's examine what happens as the rocket gains speed. (To illustrate our point, we use more figures than are significant.) At the start of the flight, when the velocity of the rocket is zero, the ejected fuel is moving backward at 2400 m/s relative to our frame of reference. As the rocket moves forward and speeds up, the fuel's speed relative to our system decreases; when the rocket speed reaches 2400 m/s, this relative speed is *zero*. [Knowing the rate of fuel consumption, you can solve Eq. (8.40) to show that this occurs at about $t = 75.6$ s.] After this time the ejected burned fuel moves *forward,* not backward, in our system. Relative to our frame of reference, the last bit of ejected fuel has a forward velocity of $3327\ \text{m/s} - 2400\ \text{m/s} = 927\ \text{m/s}$.

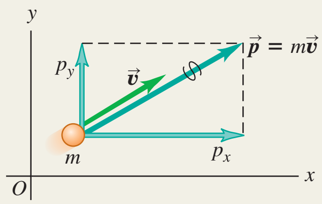
Test Your Understanding of Section 8.6 (a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing? (b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

SUMMARY

Momentum of a particle: The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{v}$. Newton's second law says that the net force on a particle is equal to the rate of change of the particle's momentum.

$$\vec{p} = m\vec{v} \quad (8.2)$$

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \quad (8.4)$$

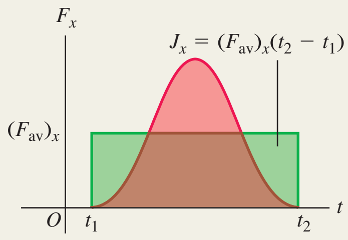


Impulse and momentum: If a constant net force $\sum \vec{F}$ acts on a particle for a time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net force is the product of the net force and the time interval. If $\sum \vec{F}$ varies with time, $\vec{J}$ is the integral of the net force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed. (See Examples 8.1–8.3.)

$$\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\,\Delta t \quad (8.5)$$

$$\vec{J} = \int_{t_1}^{t_2} \sum \vec{F}\,dt \quad (8.7)$$

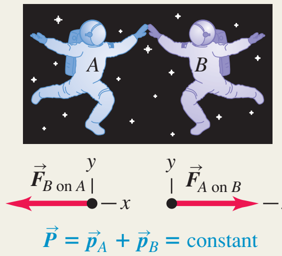
$$\vec{J} = \vec{p}_2 - \vec{p}_1 \quad (8.6)$$



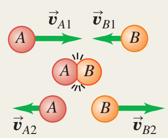
Conservation of momentum: An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved. (See Examples 8.4–8.6.)

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots$$
$$= m_A\vec{v}_A + m_B\vec{v}_B + \cdots \quad (8.14)$$

If $\sum \vec{F} = \mathbf{0}$, then $\vec{P} =$ constant.



Collisions: In collisions of all kinds, the initial and final total momenta are equal. In an elastic collision between two bodies, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-body collision, the total kinetic energy is less after the collision than before. If the two bodies have the same final velocity, the collision is completely inelastic. (See Examples 8.7–8.12.)

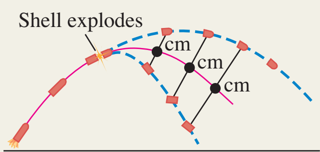


Center of mass: The position vector of the center of mass of a system of particles, $\vec{r}_{cm}$, is a weighted average of the positions $\vec{r}_1, \vec{r}_2, \ldots$ of the individual particles. The total momentum $\vec{P}$ of a system equals its total mass M multiplied by the velocity of its center of mass, $\vec{v}_{cm}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{v}_{cm}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force. (See Examples 8.13 and 8.14.)

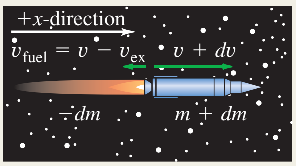
$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$= \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad (8.29)$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$$
$$= M\vec{v}_{cm} \quad (8.32)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \quad (8.34)$$



Rocket propulsion: In rocket propulsion, the mass of a rocket changes as the fuel is used up and ejected from the rocket. Analysis of the motion of the rocket must include the momentum carried away by the spent fuel as well as the momentum of the rocket itself. (See Examples 8.15 and 8.16.)



BRIDGING PROBLEM One Collision After Another

Sphere A of mass 0.600 kg is initially moving to the right at 4.00 m/s. Sphere B, of mass 1.80 kg, is initially to the right of sphere A and moving to the right at 2.00 m/s. After the two spheres collide, sphere B is moving at 3.00 m/s in the same direction as before. (a) What is the velocity (magnitude and direction) of sphere A after this collision? (b) Is this collision elastic or inelastic? (c) Sphere B then has an off-center collision with sphere C, which has mass 1.20 kg and is initially at rest. After this collision, sphere B is moving at 19.0° to its initial direction at 2.00 m/s. What is the velocity (magnitude and direction) of sphere C after this collision? (d) What is the impulse (magnitude and direction) imparted to sphere B by sphere C when they collide? (e) Is this second collision elastic or inelastic? (f) What is the velocity (magnitude and direction) of the center of mass of the system of three spheres (A, B, and C) after the second collision? No external forces act on any of the spheres in this problem.

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY AND SET UP

1. Momentum is conserved in these collisions. Can you explain why?
2. Choose the x- and y-axes, and assign subscripts to values before the first collision, after the first collision but before the second collision, and after the second collision.
3. Make a list of the target variables, and choose the equations that you'll use to solve for these.

EXECUTE

4. Solve for the velocity of sphere A after the first collision. Does A slow down or speed up in the collision? Does this make sense?
5. Now that you know the velocities of both A and B after the first collision, decide whether or not this collision is elastic. (How will you do this?)
6. The second collision is two-dimensional, so you'll have to demand that *both* components of momentum are conserved. Use this to find the speed and direction of sphere C after the second collision. (*Hint:* After the first collision, sphere B maintains the same velocity until it hits sphere C.)
7. Use the definition of impulse to find the impulse imparted to sphere B by sphere C. Remember that impulse is a vector.
8. Use the same technique that you employed in step 5 to decide whether or not the second collision is elastic.
9. Find the velocity of the center of mass after the second collision.

EVALUATE

10. Compare the directions of the vectors you found in steps 6 and 7. Is this a coincidence? Why or why not?
11. Find the velocity of the center of mass before and after the first collision. Compare to your result from step 9. Again, is this a coincidence? Why or why not?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q8.1 In splitting logs with a hammer and wedge, is a heavy hammer more effective than a lighter hammer? Why?

Q8.2 Suppose you catch a baseball and then someone invites you to catch a bowling ball with either the same momentum or the same kinetic energy as the baseball. Which would you choose? Explain.

Q8.3 When rain falls from the sky, what happens to its momentum as it hits the ground? Is your answer also valid for Newton's famous apple?

Q8.4 A car has the same kinetic energy when it is traveling south at 30 m/s as when it is traveling northwest at 30 m/s. Is the momentum of the car the same in both cases? Explain.

Q8.5 A truck is accelerating as it speeds down the highway. One inertial frame of reference is attached to the ground with its origin at a fence post. A second frame of reference is attached to a police car that is traveling down the highway at constant velocity. Is the momentum of the truck the same in these two reference frames? Explain. Is the rate of change of the truck's momentum the same in these two frames? Explain.

Q8.6 (a) When a large car collides with a small car, which one undergoes the greater change in momentum: the large one or the small one? Or is it the same for both? (b) In light of your answer to part (a), why are the occupants of the small car more likely to be hurt than those of the large car, assuming that both cars are equally sturdy?

Q8.7 A woman holding a large rock stands on a frictionless, horizontal sheet of ice. She throws the rock with speed v_0 at an angle α above the horizontal. Consider the system consisting of the woman plus the rock. Is the momentum of the system conserved? Why or why not? Is any component of the momentum of the system conserved? Again, why or why not?

Q8.8 In Example 8.7 (Section 8.3), where the two gliders in Fig. 8.15 stick together after the collision, the collision is inelastic because $K_2 < K_1$. In Example 8.5 (Section 8.2), is the collision inelastic? Explain.

Q8.9 In a completely inelastic collision between two objects, where the objects stick together after the collision, is it possible for the final kinetic energy of the system to be zero? If so, give an example in which this would occur. If the final kinetic energy is zero, what must the initial momentum of the system be? Is the initial kinetic energy of the system zero? Explain.

Q8.10 Since for a particle the kinetic energy is given by $K = \frac{1}{2}mv^2$ and the momentum by $\vec{p} = m\vec{v}$, it is easy to show that $K = p^2/2m$. How, then, is it possible to have an event during which the total momentum of the system is constant but the total kinetic energy changes?

Q8.11 In each of Examples 8.10, 8.11, and 8.12 (Section 8.4), verify that the relative velocity vector of the two bodies has the same magnitude before and after the collision. In each case what happens to the *direction* of the relative velocity vector?

Q8.12 A glass dropped on the floor is more likely to break if the floor is concrete than if it is wood. Why? (Refer to Fig. 8.3b.)

Q8.13 In Fig. 8.22b, the kinetic energy of the Ping-Pong ball is larger after its interaction with the bowling ball than before. From where does the extra energy come? Describe the event in terms of conservation of energy.

Q8.14 A machine gun is fired at a steel plate. Is the average force on the plate from the bullet impact greater if the bullets bounce off or if they are squashed and stick to the plate? Explain.

Q8.15 A net force of 4 N acts on an object initially at rest for 0.25 s and gives it a final speed of 5 m/s. How could a net force of 2 N produce the same final speed?

Q8.16 A net force with x-component $\sum F_x$ acts on an object from time t_1 to time t_2. The x-component of the momentum of the object is the same at t_1 as it is at t_2, but $\sum F_x$ is not zero at all times between t_1 and t_2. What can you say about the graph of $\sum F_x$ versus t?

Q8.17 A tennis player hits a tennis ball with a racket. Consider the system made up of the ball and the racket. Is the total momentum of the system the same just before and just after the hit? Is the total momentum just after the hit the same as 2 s later, when the ball is in midair at the high point of its trajectory? Explain any differences between the two cases.

Q8.18 In Example 8.4 (Section 8.2), consider the system consisting of the rifle plus the bullet. What is the speed of the system's center of mass after the rifle is fired? Explain.

Q8.19 An egg is released from rest from the roof of a building and falls to the ground. As the egg falls, what happens to the momentum of the system of the egg plus the earth?

Q8.20 A woman stands in the middle of a perfectly smooth, frictionless, frozen lake. She can set herself in motion by throwing things, but suppose she has nothing to throw. Can she propel herself to shore *without* throwing anything?

Q8.21 In a zero-gravity environment, can a rocket-propelled spaceship ever attain a speed greater than the relative speed with which the burnt fuel is exhausted?

Q8.22 When an object breaks into two pieces (explosion, radioactive decay, recoil, etc.), the lighter fragment gets more kinetic energy than the heavier one. This is a consequence of momentum conservation, but can you also explain it using Newton's laws of motion?

Q8.23 An apple falls from a tree and feels no air resistance. As it is falling, which of these statements about it are true? (a) Only its momentum is conserved; (b) only its mechanical energy is conserved, (c) both its momentum and its mechanical energy are conserved, (d) its kinetic energy is conserved.

Q8.24 Two pieces of clay collide and stick together. During the collision, which of these statements are true? (a) Only the momentum of the clay is conserved, (b) only the mechanical energy of the clay is conserved, (c) both the momentum and the mechanical energy of the clay are conserved, (d) the kinetic energy of the clay is conserved.

Q8.25 Two marbles are pressed together with a light ideal spring between them, but they are not attached to the spring in any way. They are then released on a frictionless horizontal table and soon move free of the spring. As the marbles are moving away from each other, which of these statements about them are true? (a) Only the momentum of the marbles is conserved, (b) only the mechanical energy of the marbles is conserved, (c) both the momentum and the mechanical energy of the marbles are conserved, (d) the kinetic energy of the marbles is conserved.

Q8.26 A very heavy SUV collides head-on with a very light compact car. Which of these statements about the collision are correct? (a) The amount of kinetic energy lost by the SUV is equal to the amount of kinetic energy gained by the compact, (b) the amount of momentum lost by the SUV is equal to the amount of momentum gained by the compact, (c) the compact feels a considerably greater force during the collision than the SUV does, (d) both cars lose the same amount of kinetic energy.

EXERCISES

Section 8.1 Momentum and Impulse

8.1 • (a) What is the magnitude of the momentum of a 10,000-kg truck whose speed is 12.0 m/s? (b) What speed would a 2000-kg SUV have to attain in order to have (i) the same momentum? (ii) the same kinetic energy?

8.2 • In a certain men's track and field event, the shotput has a mass of 7.30 kg and is released with a speed of 15.0 m/s at 40.0° above the horizontal over a man's straight left leg. What are the initial horizontal and vertical components of the momentum of this shotput?

8.3 •• (a) Show that the kinetic energy K and the momentum magnitude p of a particle with mass m are related by $K = p^2/2m$. (b) A 0.040-kg cardinal (*Richmondena cardinalis*) and a 0.145-kg baseball have the same kinetic energy. Which has the greater magnitude of momentum? What is the ratio of the cardinal's magnitude of momentum to the baseball's? (c) A 700-N man and a 450-N woman have the same momentum. Who has the greater kinetic energy? What is the ratio of the man's kinetic energy to that of the woman?

8.4 • Two vehicles are approaching an intersection. One is a 2500-kg pickup traveling at 14.0 m/s from east to west (the $-x$-direction), and the other is a 1500-kg sedan going from south to north (the $+y$-direction) at 23.0 m/s. (a) Find the x- and y-components of the net momentum of this system. (b) What are the magnitude and direction of the net momentum?

8.5 • One 110-kg football lineman is running to the right at 2.75 m/s while another 125-kg lineman is running directly toward him at 2.60 m/s. What are (a) the magnitude and direction of the net momentum of these two athletes, and (b) their total kinetic energy?

8.6 •• BIO **Biomechanics.** The mass of a regulation tennis ball is 57 g (although it can vary slightly), and tests have shown that the ball is in contact with the tennis racket for 30 ms. (This number can also vary, depending on the racket and swing.) We shall assume a 30.0-ms contact time for this exercise. The fastest-known served tennis ball was served by "Big Bill" Tilden in 1931, and its speed was measured to be 73.14 m/s. (a) What impulse and what force did Big Bill exert on the tennis ball in his record serve? (b) If Big Bill's opponent returned his serve with a speed of 55 m/s, what force and what impulse did he exert on the ball, assuming only horizontal motion?

8.7 • **Force of a Golf Swing.** A 0.0450-kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes. If the club and ball are in contact for 2.00 ms, what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

8.8 • **Force of a Baseball Swing.** A baseball has mass 0.145 kg. (a) If the velocity of a pitched ball has a magnitude of 45.0 m/s and the batted ball's velocity is 55.0 m/s in the opposite direction, find the magnitude of the change in momentum of the ball and of the impulse applied to it by the bat. (b) If the ball remains in contact with the bat for 2.00 ms, find the magnitude of the average force applied by the bat.

8.9 • A 0.160-kg hockey puck is moving on an icy, frictionless, horizontal surface. At $t = 0$, the puck is moving to the right at 3.00 m/s. (a) Calculate the velocity of the puck (magnitude and direction) after a force of 25.0 N directed to the right has been applied for 0.050 s. (b) If, instead, a force of 12.0 N directed to the left is applied from $t = 0$ to $t = 0.050$ s, what is the final velocity of the puck?

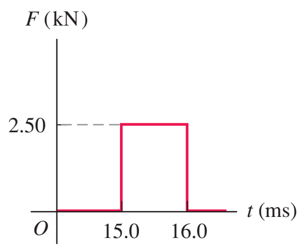
8.10 • An engine of the orbital maneuvering system (OMS) on a space shuttle exerts a force of $(26{,}700\ \text{N})\hat{\jmath}$ for 3.90 s, exhausting a negligible mass of fuel relative to the 95,000-kg mass of the shuttle. (a) What is the impulse of the force for this 3.90 s? (b) What is the shuttle's change in momentum from this impulse? (c) What is the shuttle's change in velocity from this impulse? (d) Why can't we find the resulting change in the kinetic energy of the shuttle?

8.11 • **CALC** At time $t = 0$, a 2150-kg rocket in outer space fires an engine that exerts an increasing force on it in the $+x$-direction. This force obeys the equation $F_x = At^2$, where t is time, and has a magnitude of 781.25 N when $t = 1.25$ s. (a) Find the SI value of the constant A, including its units. (b) What impulse does the engine exert on the rocket during the 1.50-s interval starting 2.00 s after the engine is fired? (c) By how much does the rocket's velocity change during this interval?

8.12 •• A bat strikes a 0.145-kg baseball. Just before impact, the ball is traveling horizontally to the right at 50.0 m/s, and it leaves the bat traveling to the left at an angle of 30° above horizontal with a speed of 65.0 m/s. If the ball and bat are in contact for 1.75 ms, find the horizontal and vertical components of the average force on the ball.

8.13 • A 2.00-kg stone is sliding to the right on a frictionless horizontal surface at 5.00 m/s when it is suddenly struck by an object that exerts a large horizontal force on it for a short period of time. The graph in Fig. E8.13 shows the magnitude of this force as a function of time. (a) What impulse does this force exert on the stone? (b) Just after the force stops acting, find the magnitude and direction of the stone's velocity if the force acts (i) to the right or (ii) to the left.

Figure **E8.13**



8.14 •• **BIO** **Bone Fracture.** Experimental tests have shown that bone will rupture if it is subjected to a force density of $1.03 \times 10^8\ \text{N/m}^2$. Suppose a 70.0-kg person carelessly roller-skates into an overhead metal beam that hits his forehead and completely stops his forward motion. If the area of contact with the person's forehead is $1.5\ \text{cm}^2$, what is the greatest speed with which he can hit the wall without breaking any bone if his head is in contact with the beam for 10.0 ms?

8.15 •• To warm up for a match, a tennis player hits the 57.0-g ball vertically with her racket. If the ball is stationary just before it is hit and goes 5.50 m high, what impulse did she impart to it?

8.16 •• **CALC** Starting at $t = 0$, a horizontal net force $\vec{F} = (0.280\ \text{N/s})t\hat{\imath} + (-0.450\ \text{N/s}^2)t^2\hat{\jmath}$ is applied to a box that has an initial momentum $\vec{p} = (-3.00\ \text{kg}\cdot\text{m/s})\hat{\imath} + (4.00\ \text{kg}\cdot\text{m/s})\hat{\jmath}$. What is the momentum of the box at $t = 2.00$ s?

Section 8.2 Conservation of Momentum

8.17 •• The expanding gases that leave the muzzle of a rifle also contribute to the recoil. A .30-caliber bullet has mass 0.00720 kg and a speed of 601 m/s relative to the muzzle when fired from a rifle that has mass 2.80 kg. The loosely held rifle recoils at a speed of 1.85 m/s relative to the earth. Find the momentum of the propellant gases in a coordinate system attached to the earth as they leave the muzzle of the rifle.

8.18 • A 68.5-kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25-kg tool away from her at 3.20 m/s relative to the space station. With what speed and in what direction will she begin to move?

8.19 • **BIO** **Animal Propulsion.** Squids and octopuses propel themselves by expelling water. They do this by keeping water in a cavity and then suddenly contracting the cavity to force out the water through an opening. A 6.50-kg squid (including the water in the cavity) at rest suddenly sees a dangerous predator. (a) If the squid has 1.75 kg of water in its cavity, at what speed must it expel this water to suddenly achieve a speed of 2.50 m/s to escape the predator? Neglect any drag effects of the surrounding water. (b) How much kinetic energy does the squid create by this maneuver?

8.20 •• You are standing on a sheet of ice that covers the football stadium parking lot in Buffalo; there is negligible friction between your feet and the ice. A friend throws you a 0.400-kg ball that is traveling horizontally at 10.0 m/s. Your mass is 70.0 kg. (a) If you catch the ball, with what speed do you and the ball move afterward? (b) If the ball hits you and bounces off your chest, so afterward it is moving horizontally at 8.0 m/s in the opposite direction, what is your speed after the collision?

8.21 •• On a frictionless, horizontal air table, puck A (with mass 0.250 kg) is moving toward puck B (with mass 0.350 kg), which is initially at rest. After the collision, puck A has a velocity of 0.120 m/s to the left, and puck B has a velocity of 0.650 m/s to the right. (a) What was the speed of puck A before the collision? (b) Calculate the change in the total kinetic energy of the system that occurs during the collision.

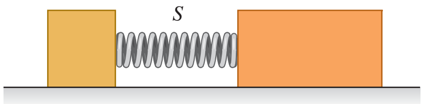
8.22 •• When cars are equipped with flexible bumpers, they will bounce off each other during low-speed collisions, thus causing less damage. In one such accident, a 1750-kg car traveling to the right at 1.50 m/s collides with a 1450-kg car going to the left at 1.10 m/s. Measurements show that the heavier car's speed just after the collision was 0.250 m/s in its original direction. You can ignore any road friction during the collision. (a) What was the speed of the lighter car just after the collision? (b) Calculate the change in the combined kinetic energy of the two-car system during this collision.

8.23 •• Two identical 1.50-kg masses are pressed against opposite ends of a light spring of force constant 1.75 N/cm, compressing the spring by 20.0 cm from its normal length. Find the speed of each mass when it has moved free of the spring on a frictionless horizontal table.

8.24 • Block A in Fig. E8.24 has mass 1.00 kg, and block B has mass 3.00 kg. The blocks are forced together, compressing a spring

S between them; then the system is released from rest on a level, frictionless surface. The spring, which has negligible mass, is not fastened to either block and drops to the surface after it has expanded. Block B acquires a speed of 1.20 m/s. (a) What is the final speed of block A? (b) How much potential energy was stored in the compressed spring?

Figure **E8.24**



8.25 •• A hunter on a frozen, essentially frictionless pond uses a rifle that shoots 4.20-g bullets at 965 m/s. The mass of the hunter (including his gun) is 72.5 kg, and the hunter holds tight to the gun after firing it. Find the recoil velocity of the hunter if he fires the rifle (a) horizontally and (b) at 56.0° above the horizontal.

8.26 • An atomic nucleus suddenly bursts apart (fissions) into two pieces. Piece A, of mass m_A, travels off to the left with speed v_A. Piece B, of mass m_B, travels off to the right with speed v_B. (a) Use conservation of momentum to solve for v_B in terms of m_A, m_B, and v_A. (b) Use the results of part (a) to show that $K_A/K_B = m_B/m_A$, where K_A and K_B are the kinetic energies of the two pieces.

8.27 •• Two ice skaters, Daniel (mass 65.0 kg) and Rebecca (mass 45.0 kg), are practicing. Daniel stops to tie his shoelace and, while at rest, is struck by Rebecca, who is moving at 13.0 m/s before she collides with him. After the collision, Rebecca has a velocity of magnitude 8.00 m/s at an angle of 53.1° from her initial direction. Both skaters move on the frictionless, horizontal surface of the rink. (a) What are the magnitude and direction of Daniel's velocity after the collision? (b) What is the change in total kinetic energy of the two skaters as a result of the collision?

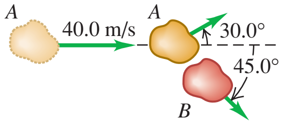
8.28 •• You are standing on a large sheet of frictionless ice and holding a large rock. In order to get off the ice, you throw the rock so it has velocity 12.0 m/s relative to the earth at an angle of 35.0° above the horizontal. If your mass is 70.0 kg and the rock's mass is 15.0 kg, what is your speed after you throw the rock? (See Discussion Question Q8.7.)

8.29 • **Changing Mass.** An open-topped freight car with mass 24,000 kg is coasting without friction along a level track. It is raining very hard, and the rain is falling vertically downward. Originally, the car is empty and moving with a speed of 4.00 m/s. (a) What is the speed of the car after it has collected 3000 kg of rainwater? (b) Since the rain is falling downward, how is it able to affect the horizontal motion of the car?

8.30 • An astronaut in space cannot use a conventional means, such as a scale or balance, to determine the mass of an object. But she does have devices to measure distance and time accurately. She knows her own mass is 78.4 kg, but she is unsure of the mass of a large gas canister in the airless rocket. When this canister is approaching her at 3.50 m/s, she pushes against it, which slows it down to 1.20 m/s (but does not reverse it) and gives her a speed of 2.40 m/s. What is the mass of this canister?

8.31 •• **Asteroid Collision.** Two asteroids of equal mass in the asteroid belt between Mars and Jupiter collide with a glancing blow. Asteroid A, which was initially traveling at 40.0 m/s, is deflected 30.0° from its original direction, while asteroid B, which was initially at rest, travels at 45.0° to the original direction of A (Fig. E8.31). (a) Find the speed of each asteroid after the collision. (b) What fraction of the original kinetic energy of asteroid A dissipates during this collision?

Figure **E8.31**



Section 8.3 Momentum Conservation and Collisions

8.32 • Two skaters collide and grab on to each other on frictionless ice. One of them, of mass 70.0 kg, is moving to the right at 2.00 m/s, while the other, of mass 65.0 kg, is moving to the left at 2.50 m/s. What are the magnitude and direction of the velocity of these skaters just after they collide?

8.33 •• A 15.0-kg fish swimming at 1.10 m/s suddenly gobbles up a 4.50-kg fish that is initially stationary. Neglect any drag effects of the water. (a) Find the speed of the large fish just after it eats the small one. (b) How much mechanical energy was dissipated during this meal?

8.34 • Two fun-loving otters are sliding toward each other on a muddy (and hence frictionless) horizontal surface. One of them, of mass 7.50 kg, is sliding to the left at 5.00 m/s, while the other, of mass 5.75 kg, is slipping to the right at 6.00 m/s. They hold fast to each other after they collide. (a) Find the magnitude and direction of the velocity of these free-spirited otters right after they collide. (b) How much mechanical energy dissipates during this play?

8.35 • **Deep Impact Mission.** In July 2005, NASA's "Deep Impact" mission crashed a 372-kg probe directly onto the surface of the comet Tempel 1, hitting the surface at 37,000 km/h. The original speed of the comet at that time was about 40,000 km/h, and its mass was estimated to be in the range $(0.10 - 2.5) \times 10^{14}$ kg. Use the smallest value of the estimated mass. (a) What change in the comet's velocity did this collision produce? Would this change be noticeable? (b) Suppose this comet were to hit the earth and fuse with it. By how much would it change our planet's velocity? Would this change be noticeable? (The mass of the earth is 5.97×10^{24} kg.)

8.36 • A 1050-kg sports car is moving westbound at 15.0 m/s on a level road when it collides with a 6320-kg truck driving east on the same road at 10.0 m/s. The two vehicles remain locked together after the collision. (a) What is the velocity (magnitude and direction) of the two vehicles just after the collision? (b) At what speed should the truck have been moving so that it and the car are both stopped in the collision? (c) Find the change in kinetic energy of the system of two vehicles for the situations of part (a) and part (b). For which situation is the change in kinetic energy greater in magnitude?

8.37 •• On a very muddy football field, a 110-kg linebacker tackles an 85-kg halfback. Immediately before the collision, the linebacker is slipping with a velocity of 8.8 m/s north and the halfback is sliding with a velocity of 7.2 m/s east. What is the velocity (magnitude and direction) at which the two players move together immediately after the collision?

8.38 •• **Accident Analysis.** Two cars collide at an intersection. Car A, with a mass of 2000 kg, is going from west to east, while car B, of mass 1500 kg, is going from north to south at 15 m/s. As a result of this collision, the two cars become enmeshed and move as one afterward. In your role as an expert witness, you inspect the scene and determine that, after the collision, the enmeshed cars moved at an angle of 65° south of east from the point of impact.

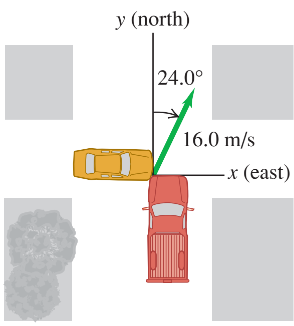
(a) How fast were the enmeshed cars moving just after the collision? (b) How fast was car A going just before the collision?

8.39 • Two cars, one a compact with mass 1200 kg and the other a large gas-guzzler with mass 3000 kg, collide head-on at typical freeway speeds. (a) Which car has a greater magnitude of momentum change? Which car has a greater velocity change? (b) If the larger car changes its velocity by Δv, calculate the change in the velocity of the small car in terms of Δv. (c) Which car's occupants would you expect to sustain greater injuries? Explain.

8.40 •• BIO **Bird Defense.** To protect their young in the nest, peregrine falcons will fly into birds of prey (such as ravens) at high speed. In one such episode, a 600-g falcon flying at 20.0 m/s hit a 1.50-kg raven flying at 9.0 m/s. The falcon hit the raven at right angles to its original path and bounced back at 5.0 m/s. (These figures were estimated by the author as he watched this attack occur in northern New Mexico.) (a) By what angle did the falcon change the raven's direction of motion? (b) What was the raven's speed right after the collision?

8.41 • At the intersection of Texas Avenue and University Drive, a yellow subcompact car with mass 950 kg traveling east on University collides with a red pickup truck with mass 1900 kg that is traveling north on Texas and has run a red light (Fig. E8.41). The two vehicles stick together as a result of the collision, and the wreckage slides at 16.0 m/s in the direction 24.0° east of north. Calculate the speed of each vehicle before the collision. The collision occurs during a heavy rainstorm; you can ignore friction forces between the vehicles and the wet road.

Figure **E8.41**

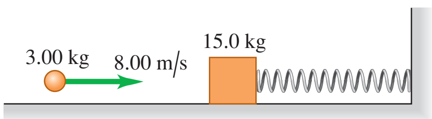


8.42 •• A 5.00-g bullet is fired horizontally into a 1.20-kg wooden block resting on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.20. The bullet remains embedded in the block, which is observed to slide 0.230 m along the surface before stopping. What was the initial speed of the bullet?

8.43 •• **A Ballistic Pendulum.** A 12.0-g rifle bullet is fired with a speed of 380 m/s into a ballistic pendulum with mass 6.00 kg, suspended from a cord 70.0 cm long (see Example 8.8 in Section 8.3). Compute (a) the vertical height through which the pendulum rises, (b) the initial kinetic energy of the bullet, and (c) the kinetic energy of the bullet and pendulum immediately after the bullet becomes embedded in the pendulum.

8.44 •• **Combining Conservation Laws.** A 15.0-kg block is attached to a very light horizontal spring of force constant 500.0 N/m and is resting on a frictionless horizontal table. (Fig. E8.44). Suddenly it is struck by a 3.00-kg stone traveling horizontally at 8.00 m/s to the right, whereupon the stone rebounds at 2.00 m/s horizontally to the left. Find the maximum distance that the block will compress the spring after the collision.

Figure **E8.44**



8.45 •• CP A 5.00-kg ornament is hanging by a 1.50-m wire when it is suddenly hit by a 3.00-kg missile traveling horizontally at 12.0 m/s. The missile embeds itself in the ornament during the collision. What is the tension in the wire immediately after the collision?

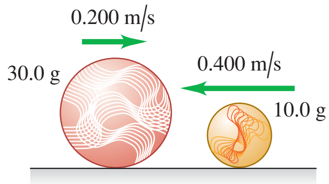
Section 8.4 Elastic Collisions

8.46 •• A 0.150-kg glider is moving to the right on a frictionless, horizontal air track with a speed of 0.80 m/s. It has a head-on collision with a 0.300-kg glider that is moving to the left with a speed of 2.20 m/s. Find the final velocity (magnitude and direction) of each glider if the collision is elastic.

8.47 •• Blocks A (mass 2.00 kg) and B (mass 10.00 kg) move on a frictionless, horizontal surface. Initially, block B is at rest and block A is moving toward it at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10 (Section 8.4). The collision is head-on, so all motion before and after the collision is along a straight line. (a) Find the maximum energy stored in the spring bumpers and the velocity of each block at that time. (b) Find the velocity of each block after they have moved apart.

8.48 • A 10.0-g marble slides to the left with a velocity of magnitude 0.400 m/s on the frictionless, horizontal surface of an icy New York sidewalk and has a head-on, elastic collision with a larger 30.0-g marble sliding to the right with a velocity of magnitude 0.200 m/s (Fig. E8.48). (a) Find the velocity of each marble (magnitude and direction) after the collision. (Since the collision is head-on, all the motion is along a line.) (b) Calculate the *change in momentum* (that is, the momentum after the collision minus the momentum before the collision) for each marble. Compare the values you get for each marble. (c) Calculate the *change in kinetic energy* (that is, the kinetic energy after the collision minus the kinetic energy before the collision) for each marble. Compare the values you get for each marble.

Figure **E8.48**



8.49 •• **Moderators.** Canadian nuclear reactors use *heavy water* moderators in which elastic collisions occur between the neutrons and deuterons of mass 2.0 u (see Example 8.11 in Section 8.4). (a) What is the speed of a neutron, expressed as a fraction of its original speed, after a head-on, elastic collision with a deuteron that is initially at rest? (b) What is its kinetic energy, expressed as a fraction of its original kinetic energy? (c) How many such successive collisions will reduce the speed of a neutron to 1/59,000 of its original value?

8.50 •• You are at the controls of a particle accelerator, sending a beam of 1.50×10^7 m/s protons (mass m) at a gas target of an unknown element. Your detector tells you that some protons bounce straight back after a collision with one of the nuclei of the unknown element. All such protons rebound with a speed of 1.20×10^7 m/s. Assume that the initial speed of the target nucleus is negligible and the collision is elastic. (a) Find the mass of one nucleus of the unknown element. Express your answer in terms of the proton mass m. (b) What is the speed of the unknown nucleus immediately after such a collision?

Section 8.5 Center of Mass

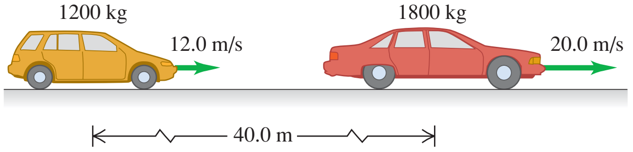
8.51 • Three odd-shaped blocks of chocolate have the following masses and center-of-mass coordinates: (1) 0.300 kg, (0.200 m, 0.300 m); (2) 0.400 kg, (0.100 m, −0.400 m); (3) 0.200 kg, (−0.300 m, 0.600 m). Find the coordinates of the center of mass of the system of three chocolate blocks.

8.52 • Find the position of the center of mass of the system of the sun and Jupiter. (Since Jupiter is more massive than the rest of the planets combined, this is essentially the position of the center of mass of the solar system.) Does the center of mass lie inside or outside the sun? Use the data in Appendix F.

8.53 •• **Pluto and Charon.** Pluto's diameter is approximately 2370 km, and the diameter of its satellite Charon is 1250 km. Although the distance varies, they are often about 19,700 km apart, center to center. Assuming that both Pluto and Charon have the same composition and hence the same average density, find the location of the center of mass of this system relative to the center of Pluto.

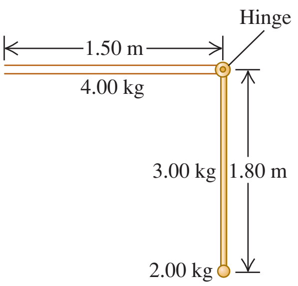
8.54 • A 1200-kg station wagon is moving along a straight highway at 12.0 m/s. Another car, with mass 1800 kg and speed 20.0 m/s, has its center of mass 40.0 m ahead of the center of mass of the station wagon (Fig. E8.54). (a) Find the position of the center of mass of the system consisting of the two automobiles. (b) Find the magnitude of the total momentum of the system from the given data. (c) Find the speed of the center of mass of the system. (d) Find the total momentum of the system, using the speed of the center of mass. Compare your result with that of part (b).

Figure **E8.54**



8.55 • A machine part consists of a thin, uniform 4.00-kg bar that is 1.50 m long, hinged perpendicular to a similar vertical bar of mass 3.00 kg and length 1.80 m. The longer bar has a small but dense 2.00-kg ball at one end (Fig. E8.55). By what distance will the center of mass of this part move horizontally and vertically if the vertical bar is pivoted counterclockwise through 90° to make the entire part horizontal?

Figure **E8.55**



8.56 • At one instant, the center of mass of a system of two particles is located on the x-axis at $x = 2.0$ m and has a velocity of $(5.0 \text{ m/s})\hat{\imath}$. One of the particles is at the origin. The other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 8.0$ m. (a) What is the mass of the particle at the origin? (b) Calculate the total momentum of this system. (c) What is the velocity of the particle at the origin?

8.57 •• In Example 8.14 (Section 8.5), Ramon pulls on the rope to give himself a speed of 0.70 m/s. What is James's speed?

8.58 • **CALC** A system consists of two particles. At $t = 0$ one particle is at the origin; the other, which has a mass of 0.50 kg, is on the y-axis at $y = 6.0$ m. At $t = 0$ the center of mass of the system is on the y-axis at $y = 2.4$ m. The velocity of the center of mass is given by $(0.75 \text{ m/s}^3)t^2\hat{\imath}$. (a) Find the total mass of the system. (b) Find the acceleration of the center of mass at any time t. (c) Find the net external force acting on the system at $t = 3.0$ s.

8.59 • **CALC** A radio-controlled model airplane has a momentum given by $[(-0.75 \text{ kg}\cdot\text{m/s}^3)t^2 + (3.0 \text{ kg}\cdot\text{m/s})]\hat{\imath} + (0.25 \text{ kg}\cdot\text{m/s}^2)t\hat{\jmath}$. What are the x-, y-, and z-components of the net force on the airplane?

8.60 •• **BIO** **Changing Your Center of Mass.** To keep the calculations fairly simple, but still reasonable, we shall model a human leg that is 92.0 cm long (measured from the hip joint) by assuming that the upper leg and the lower leg (which includes the foot) have equal lengths and that each of them is uniform. For a 70.0-kg person, the mass of the upper leg would be 8.60 kg, while that of the lower leg (including the foot) would be 5.25 kg. Find the location of the center of mass of this leg, relative to the hip joint, if it is (a) stretched out horizontally and (b) bent at the knee to form a right angle with the upper leg remaining horizontal.

Section 8.6 Rocket Propulsion

8.61 •• A 70-kg astronaut floating in space in a 110-kg MMU (manned maneuvering unit) experiences an acceleration of 0.029 m/s^2 when he fires one of the MMU's thrusters. (a) If the speed of the escaping N_2 gas relative to the astronaut is 490 m/s, how much gas is used by the thruster in 5.0 s? (b) What is the thrust of the thruster?

8.62 • A small rocket burns 0.0500 kg of fuel per second, ejecting it as a gas with a velocity relative to the rocket of magnitude 1600 m/s. (a) What is the thrust of the rocket? (b) Would the rocket operate in outer space where there is no atmosphere? If so, how would you steer it? Could you brake it?

8.63 • A C6-5 model rocket engine has an impulse of 10.0 N·s while burning 0.0125 kg of propellant in 1.70 s. It has a maximum thrust of 13.3 N. The initial mass of the engine plus propellant is 0.0258 kg. (a) What fraction of the maximum thrust is the average thrust? (b) Calculate the relative speed of the exhaust gases, assuming it is constant. (c) Assuming that the relative speed of the exhaust gases is constant, find the final speed of the engine if it was attached to a very light frame and fired from rest in gravity-free outer space.

8.64 •• Obviously, we can make rockets to go very fast, but what is a reasonable top speed? Assume that a rocket is fired from rest at a space station in deep space, where gravity is negligible. (a) If the rocket ejects gas at a relative speed of 2000 m/s and you want the rocket's speed eventually to be $1.00 \times 10^{-3}c$, where c is the speed of light, what fraction of the initial mass of the rocket and fuel is *not* fuel? (b) What is this fraction if the final speed is to be 3000 m/s?

8.65 •• A single-stage rocket is fired from rest from a deep-space platform, where gravity is negligible. If the rocket burns its fuel in 50.0 s and the relative speed of the exhaust gas is $v_{ex} = 2100$ m/s, what must the mass ratio m_0/m be for a final speed v of 8.00 km/s (about equal to the orbital speed of an earth satellite)?

PROBLEMS

8.66 •• **CP CALC** A young girl with mass 40.0 kg is sliding on a horizontal, frictionless surface with an initial momentum that is due east and that has magnitude 90.0 kg·m/s. Starting at $t = 0$, a net force with magnitude $F = (8.20 \text{ N/s})t$ and direction due west is applied to the girl. (a) At what value of t does the girl have a westward momentum of magnitude 60.0 kg·m/s? (b) How much work has been done on the girl by the force in the time interval from $t = 0$ to the time calculated in part (a)? (c) What is the magnitude of the acceleration of the girl at the time calculated in part (a)?

8.67 •• A steel ball with mass 40.0 g is dropped from a height of 2.00 m onto a horizontal steel slab. The ball rebounds to a height of 1.60 m. (a) Calculate the impulse delivered to the ball during impact. (b) If the ball is in contact with the slab for 2.00 ms, find the average force on the ball during impact.

8.68 • In a volcanic eruption, a 2400-kg boulder is thrown vertically upward into the air. At its highest point, it suddenly explodes (due to trapped gases) into two fragments, one being three times the mass of the other. The lighter fragment starts out with only horizontal velocity and lands 318 m directly north of the point of the explosion. Where will the other fragment land? Neglect any air resistance.

8.69 •• Just before it is struck by a racket, a tennis ball weighing 0.560 N has a velocity of $(20.0\ \text{m/s})\hat{\imath} - (4.0\ \text{m/s})\hat{\jmath}$. During the 3.00 ms that the racket and ball are in contact, the net force on the ball is constant and equal to $-(380\ \text{N})\hat{\imath} + (110\ \text{N})\hat{\jmath}$. (a) What are the *x*- and *y*-components of the impulse of the net force applied to the ball? (b) What are the *x*- and *y*-components of the final velocity of the ball?

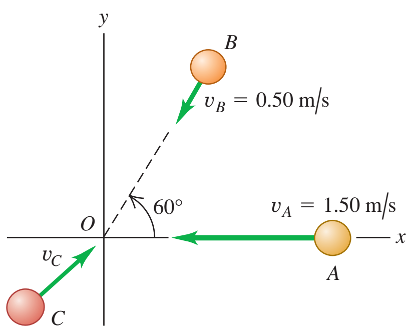
8.70 • Three identical pucks on a horizontal air table have repelling magnets. They are held together and then released simultaneously. Each has the same speed at any instant. One puck moves due west. What is the direction of the velocity of each of the other two pucks?

8.71 •• A 1500-kg blue convertible is traveling south, and a 2000-kg red SUV is traveling west. If the total momentum of the system consisting of the two cars is $7200\ \text{kg}\cdot\text{m/s}$ directed at 60.0° west of south, what is the speed of each vehicle?

8.72 •• A railroad handcar is moving along straight, frictionless tracks with negligible air resistance. In the following cases, the car initially has a total mass (car and contents) of 200 kg and is traveling east with a velocity of magnitude 5.00 m/s. Find the *final velocity* of the car in each case, assuming that the handcar does not leave the tracks. (a) A 25.0-kg mass is thrown sideways out of the car with a velocity of magnitude 2.00 m/s relative to the car's initial velocity. (b) A 25.0-kg mass is thrown backward out of the car with a velocity of 5.00 m/s relative to the initial motion of the car. (c) A 25.0-kg mass is thrown into the car with a velocity of 6.00 m/s relative to the ground and opposite in direction to the initial velocity of the car.

8.73 • Spheres *A* (mass 0.020 kg), *B* (mass 0.030 kg), and *C* (mass 0.050 kg) are approaching the origin as they slide on a frictionless air table (Fig. P8.73). The initial velocities of *A* and *B* are given in the figure. All three spheres arrive at the origin at the same time and stick together. (a) What must the *x*- and *y*-components of the initial velocity of *C* be if all three objects are to end up moving at 0.50 m/s in the $+x$-direction after the collision? (b) If *C* has the velocity found in part (a), what is the change in the kinetic energy of the system of three spheres as a result of the collision?

Figure **P8.73**



8.74 ••• You and your friends are doing physics experiments on a frozen pond that serves as a frictionless, horizontal surface. Sam, with mass 80.0 kg, is given a push and slides eastward. Abigail, with mass 50.0 kg, is sent sliding northward. They collide, and after the collision Sam is moving at 37.0° north of east with a speed of 6.00 m/s and Abigail is moving at 23.0° south of east with a speed of 9.00 m/s. (a) What was the speed of each person before the collision? (b) By how much did the total kinetic energy of the two people decrease during the collision?

8.75 ••• The nucleus of ^{214}Po decays radioactively by emitting an alpha particle (mass 6.65×10^{-27} kg) with kinetic energy 1.23×10^{-12} J, as measured in the laboratory reference frame. Assuming that the Po was initially at rest in this frame, find the recoil velocity of the nucleus that remains after the decay.

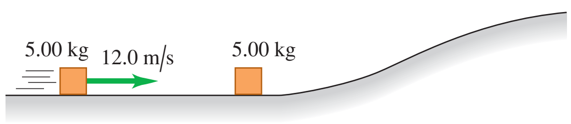
8.76 • **CP** At a classic auto show, a 840-kg 1955 Nash Metropolitan motors by at 9.0 m/s, followed by a 1620-kg 1957 Packard Clipper purring past at 5.0 m/s. (a) Which car has the greater kinetic energy? What is the ratio of the kinetic energy of the Nash to that of the Packard? (b) Which car has the greater magnitude of momentum? What is the ratio of the magnitude of momentum of the Nash to that of the Packard? (c) Let F_N be the net force required to stop the Nash in time *t*, and let F_P be the net force required to stop the Packard in the same time. Which is larger: F_N or F_P? What is the ratio F_N/F_P of these two forces? (d) Now let F_N be the net force required to stop the Nash in a distance *d*, and let F_P be the net force required to stop the Packard in the same distance. Which is larger: F_N or F_P? What is the ratio F_N/F_P?

8.77 •• **CP** An 8.00-kg block of wood sits at the edge of a frictionless table, 2.20 m above the floor. A 0.500-kg blob of clay slides along the length of the table with a speed of 24.0 m/s, strikes the block of wood, and sticks to it. The combined object leaves the edge of the table and travels to the floor. What horizontal distance has the combined object traveled when it reaches the floor?

8.78 ••• **CP** A small wooden block with mass 0.800 kg is suspended from the lower end of a light cord that is 1.60 m long. The block is initially at rest. A bullet with mass 12.0 g is fired at the block with a horizontal velocity v_0. The bullet strikes the block and becomes embedded in it. After the collision the combined object swings on the end of the cord. When the block has risen a vertical height of 0.800 m, the tension in the cord is 4.80 N. What was the initial speed v_0 of the bullet?

8.79 •• **Combining Conservation Laws.** A 5.00-kg chunk of ice is sliding at 12.0 m/s on the floor of an ice-covered valley when it collides with and sticks to another 5.00-kg chunk of ice that is initially at rest. (Fig. P8.79). Since the valley is icy, there is no friction. After the collision, how high above the valley floor will the combined chunks go?

Figure **P8.79**

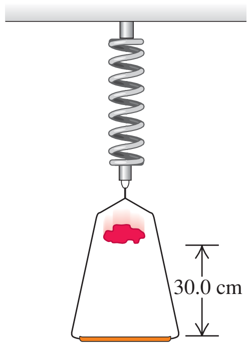


8.80 •• **Automobile Accident Analysis.** You are called as an expert witness to analyze the following auto accident: Car *B*, of mass 1900 kg, was stopped at a red light when it was hit from behind by car *A*, of mass 1500 kg. The cars locked bumpers during the collision and slid to a stop with brakes locked on all wheels. Measurements of the skid marks left by the tires showed them to be 7.15 m long. The coefficient of kinetic friction between the tires and the road was 0.65. (a) What was the speed of car A just before the collision? (b) If the speed limit was 35 mph, was car A speeding, and if so, by how many miles per hour was it *exceeding* the speed limit?

8.81 •• **Accident Analysis.** A 1500-kg sedan goes through a wide intersection traveling from north to south when it is hit by a 2200-kg SUV traveling from east to west. The two cars become enmeshed due to the impact and slide as one thereafter. On-the-scene measurements show that the coefficient of kinetic friction between the tires of these cars and the pavement is 0.75, and the cars slide to a halt at a point 5.39 m west and 6.43 m south of the impact point. How fast was each car traveling just before the collision?

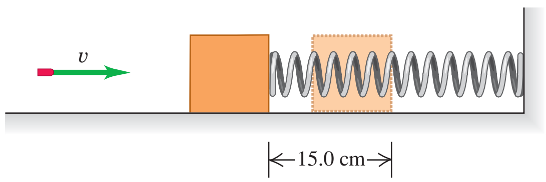
8.82 ••• **CP** A 0.150-kg frame, when suspended from a coil spring, stretches the spring 0.070 m. A 0.200-kg lump of putty is dropped from rest onto the frame from a height of 30.0 cm (Fig. P8.82). Find the maximum distance the frame moves downward from its initial position.

Figure **P8.82**



8.83 • A rifle bullet with mass 8.00 g strikes and embeds itself in a block with mass 0.992 kg that rests on a frictionless, horizontal surface and is attached to a coil spring (Fig. P8.83). The impact compresses the spring 15.0 cm. Calibration of the spring shows that a force of 0.750 N is required to compress the spring 0.250 cm. (a) Find the magnitude of the block's velocity just after impact. (b) What was the initial speed of the bullet?

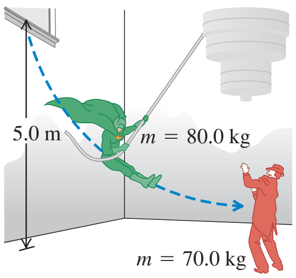
Figure **P8.83**



8.84 •• **A Ricocheting Bullet.** 0.100-kg stone rests on a frictionless, horizontal surface. A bullet of mass 6.00 g, traveling horizontally at 350 m/s, strikes the stone and rebounds horizontally at right angles to its original direction with a speed of 250 m/s. (a) Compute the magnitude and direction of the velocity of the stone after it is struck. (b) Is the collision perfectly elastic?

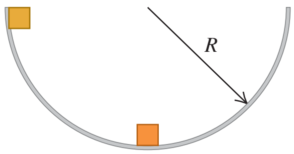
8.85 •• A movie stuntman (mass 80.0 kg) stands on a window ledge 5.0 m above the floor (Fig. P8.85). Grabbing a rope attached to a chandelier, he swings down to grapple with the movie's villain (mass 70.0 kg), who is standing directly under the chandelier. (Assume that the stuntman's center of mass moves downward 5.0 m. He releases the rope just as he reaches the villain.) (a) With what speed do the entwined foes start to slide across the floor? (b) If the coefficient of kinetic friction of their bodies with the floor is $\mu_k = 0.250$, how far do they slide?

Figure **P8.85**



8.86 •• **CP** Two identical masses are released from rest in a smooth hemispherical bowl of radius R from the positions shown in Fig. P8.86. You can ignore friction between the masses and the surface of the bowl. If they stick together when they collide, how high above the bottom of the bowl will the masses go after colliding?

Figure **P8.86**



8.87 •• A ball with mass M, moving horizontally at 4.00 m/s, collides elastically with a block with mass $3M$ that is initially hanging at rest from the ceiling on the end of a 50.0-cm wire. Find the maximum angle through which the block swings after it is hit.

8.88 ••• **CP** A 20.00-kg lead sphere is hanging from a hook by a thin wire 3.50 m long and is free to swing in a complete circle. Suddenly it is struck horizontally by a 5.00-kg steel dart that embeds itself in the lead sphere. What must be the minimum initial speed of the dart so that the combination makes a complete circular loop after the collision?

8.89 ••• **CP** An 8.00-kg ball, hanging from the ceiling by a light wire 135 cm long, is struck in an elastic collision by a 2.00-kg ball moving horizontally at 5.00 m/s just before the collision. Find the tension in the wire just after the collision.

8.90 •• A 7.0-kg shell at rest explodes into two fragments, one with a mass of 2.0 kg and the other with a mass of 5.0 kg. If the heavier fragment gains 100 J of kinetic energy from the explosion, how much kinetic energy does the lighter one gain?

8.91 •• A 4.00-g bullet, traveling horizontally with a velocity of magnitude 400 m/s, is fired into a wooden block with mass 0.800 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its speed reduced to 190 m/s. The block slides a distance of 45.0 cm along the surface from its initial position. (a) What is the coefficient of kinetic friction between block and surface? (b) What is the decrease in kinetic energy of the bullet? (c) What is the kinetic energy of the block at the instant after the bullet passes through it?

8.92 •• A 5.00-g bullet is shot *through* a 1.00-kg wood block suspended on a string 2.00 m long. The center of mass of the block rises a distance of 0.38 cm. Find the speed of the bullet as it emerges from the block if its initial speed is 450 m/s.

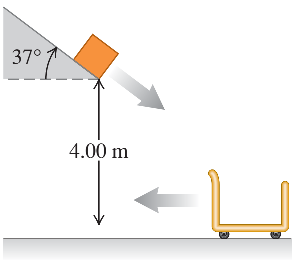
8.93 •• A neutron with mass m makes a head-on, elastic collision with a nucleus of mass M, which is initially at rest. (a) Show that if the neutron's initial kinetic energy is K_0, the kinetic energy that it loses during the collision is $4mMK_0/(M+m)^2$. (b) For what value of M does the incident neutron lose the most energy? (c) When M has the value calculated in part (b), what is the speed of the neutron after the collision?

8.94 •• **Energy Sharing in Elastic Collisions.** A stationary object with mass m_B is struck head-on by an object with mass m_A that is moving initially at speed v_0. (a) If the collision is elastic, what percentage of the original energy does each object have after the collision? (b) What does your answer in part (a) give for the special cases (i) $m_A = m_B$ and (ii) $m_A = 5m_B$? (c) For what values, if any, of the mass ratio m_A/m_B is the original kinetic energy shared equally by the two objects after the collision?

8.95 •• **CP** In a shipping company distribution center, an open cart of mass 50.0 kg is rolling to the left at a speed of 5.00 m/s

(Fig. P8.95). You can ignore friction between the cart and the floor. A 15.0-kg package slides down a chute that is inclined at 37° from the horizontal and leaves the end of the chute with a speed of 3.00 m/s. The package lands in the cart and they roll off together. If the lower end of the chute is a vertical distance of 4.00 m above the bottom of the cart, what are (a) the speed of the package just before it lands in the cart and (b) the final speed of the cart?

Figure **P8.95**



8.96 • A blue puck with mass 0.0400 kg, sliding with a velocity of magnitude 0.200 m/s on a frictionless, horizontal air table, makes a perfectly elastic, head-on collision with a red puck with mass m, initially at rest. After the collision, the velocity of the blue puck is 0.050 m/s in the same direction as its initial velocity. Find (a) the velocity (magnitude and direction) of the red puck after the collision and (b) the mass m of the red puck.

8.97 ••• Jack and Jill are standing on a crate at rest on the frictionless, horizontal surface of a frozen pond. Jack has mass 75.0 kg, Jill has mass 45.0 kg, and the crate has mass 15.0 kg. They remember that they must fetch a pail of water, so each jumps horizontally from the top of the crate. Just after each jumps, that person is moving away from the crate with a speed of 4.00 m/s relative to the crate. (a) What is the final speed of the crate if both Jack and Jill jump simultaneously and in the same direction? (*Hint:* Use an inertial coordinate system attached to the ground.) (b) What is the final speed of the crate if Jack jumps first and then a few seconds later Jill jumps in the same direction? (c) What is the final speed of the crate if Jill jumps first and then Jack, again in the same direction?

8.98 • Suppose you hold a small ball in contact with, and directly over, the center of a large ball. If you then drop the small ball a short time after dropping the large ball, the small ball rebounds with surprising speed. To show the extreme case, ignore air resistance and suppose the large ball makes an elastic collision with the floor and then rebounds to make an elastic collision with the still-descending small ball. Just before the collision between the two balls, the large ball is moving upward with velocity $\vec{v}$ and the small ball has velocity $-\vec{v}$. (Do you see why?) Assume the large ball has a much greater mass than the small ball. (a) What is the velocity of the small ball immediately after its collision with the large ball? (b) From the answer to part (a), what is the ratio of the small ball's rebound distance to the distance it fell before the collision?

8.99 ••• Hockey puck B rests on a smooth ice surface and is struck by a second puck A, which has the same mass. Puck A is initially traveling at 15.0 m/s and is deflected 25.0° from its initial direction. Assume that the collision is perfectly elastic. Find the final speed of each puck and the direction of B's velocity after the collision.

8.100 ••• **Energy Sharing.** An object with mass m, initially at rest, explodes into two fragments, one with mass m_A and the other with mass m_B, where $m_A + m_B = m$. (a) If energy Q is released in the explosion, how much kinetic energy does each fragment have immediately after the explosion? (b) What percentage of the total energy released does each fragment get when one fragment has four times the mass of the other?

8.101 ••• **Neutron Decay.** A neutron at rest decays (breaks up) to a proton and an electron. Energy is released in the decay and appears as kinetic energy of the proton and electron. The mass of a proton is 1836 times the mass of an electron. What fraction of the total energy released goes into the kinetic energy of the proton?

8.102 •• A ^{232}Th (thorium) nucleus at rest decays to a ^{228}Ra (radium) nucleus with the emission of an alpha particle. The total kinetic energy of the decay fragments is 6.54×10^{-13} J. An alpha particle has 1.76% of the mass of a ^{228}Ra nucleus. Calculate the kinetic energy of (a) the recoiling ^{228}Ra nucleus and (b) the alpha particle.

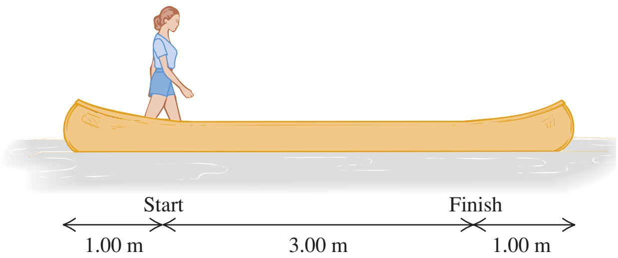
8.103 • **Antineutrino.** In beta decay, a nucleus emits an electron. A ^{210}Bi (bismuth) nucleus at rest undergoes beta decay to ^{210}Po (polonium). Suppose the emitted electron moves to the right with a momentum of 5.60×10^{-22} kg·m/s. The ^{210}Po nucleus, with mass 3.50×10^{-25} kg, recoils to the left at a speed of 1.14×10^{3} m/s. Momentum conservation requires that a second particle, called an antineutrino, must also be emitted. Calculate the magnitude and direction of the momentum of the antineutrino that is emitted in this decay.

8.104 •• Jonathan and Jane are sitting in a sleigh that is at rest on frictionless ice. Jonathan's weight is 800 N, Jane's weight is 600 N, and that of the sleigh is 1000 N. They see a poisonous spider on the floor of the sleigh and immediately jump off. Jonathan jumps to the left with a velocity of 5.00 m/s at 30.0° above the horizontal (relative to the ice), and Jane jumps to the right at 7.00 m/s at 36.9° above the horizontal (relative to the ice). Calculate the sleigh's horizontal velocity (magnitude and direction) after they jump out.

8.105 •• Two friends, Burt and Ernie, are standing at opposite ends of a uniform log that is floating in a lake. The log is 3.0 m long and has mass 20.0 kg. Burt has mass 30.0 kg and Ernie has mass 40.0 kg. Initially the log and the two friends are at rest relative to the shore. Burt then offers Ernie a cookie, and Ernie walks to Burt's end of the log to get it. Relative to the shore, what distance has the log moved by the time Ernie reaches Burt? Neglect any horizontal force that the water exerts on the log and assume that neither Burt nor Ernie falls off the log.

8.106 •• A 45.0-kg woman stands up in a 60.0-kg canoe 5.00 m long. She walks from a point 1.00 m from one end to a point 1.00 m from the other end (Fig. P8.106). If you ignore resistance to motion of the canoe in the water, how far does the canoe move during this process?

Figure **P8.106**



8.107 •• You are standing on a concrete slab that in turn is resting on a frozen lake. Assume there is no friction between the slab and the ice. The slab has a weight five times your weight. If you begin walking forward at 2.00 m/s relative to the ice, with what speed, relative to the ice, does the slab move?

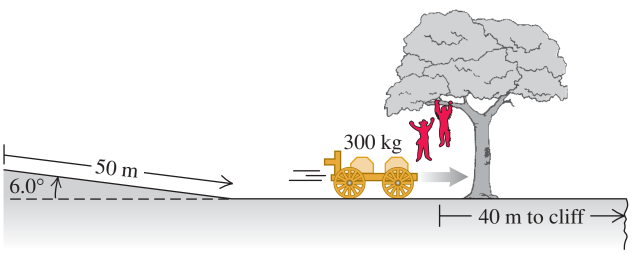
8.108 •• **CP** A 20.0-kg projectile is fired at an angle of 60.0° above the horizontal with a speed of 80.0 m/s. At the highest point of its trajectory, the projectile explodes into two fragments with equal mass, one of which falls vertically with zero initial speed. You can ignore air resistance. (a) How far from the point of firing does the other fragment strike if the terrain is level? (b) How much energy is released during the explosion?

8.109 ••• **CP** A fireworks rocket is fired vertically upward. At its maximum height of 80.0 m, it explodes and breaks into two pieces: one with mass 1.40 kg and the other with mass 0.28 kg. In the explosion, 860 J of chemical energy is converted to kinetic energy of the two fragments. (a) What is the speed of each fragment just after the explosion? (b) It is observed that the two fragments hit the ground at the same time. What is the distance between the points on the ground where they land? Assume that the ground is level and air resistance can be ignored.

8.110 ••• A 12.0-kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. When it is at its highest point, the shell explodes into two fragments, one three times heavier than the other. The two fragments reach the ground at the same time. Assume that air resistance can be ignored. If the heavier fragment lands back at the same point from which the shell was launched, where will the lighter fragment land, and how much energy was released in the explosion?

8.111 • **CP** A wagon with two boxes of gold, having total mass 300 kg, is cut loose from the horses by an outlaw when the wagon is at rest 50 m up a 6.0° slope (Fig. P8.111). The outlaw plans to have the wagon roll down the slope and across the level ground, and then fall into a canyon where his confederates wait. But in a tree 40 m from the canyon edge wait the Lone Ranger (mass 75.0 kg) and Tonto (mass 60.0 kg). They drop vertically into the wagon as it passes beneath them. (a) If they require 5.0 s to grab the gold and jump out, will they make it before the wagon goes over the edge? The wagon rolls with negligible friction. (b) When the two heroes drop into the wagon, is the kinetic energy of the system of the heroes plus the wagon conserved? If not, does it increase or decrease, and by how much?

Figure **P8.111**



8.112 •• **CALC** In Section 8.6, we considered a rocket fired in outer space where there is no air resistance and where gravity is negligible. Suppose instead that the rocket is accelerating vertically upward from rest on the earth's surface. Continue to ignore air resistance and consider only that part of the motion where the altitude of the rocket is small so that g may be assumed to be constant. (a) How is Eq. (8.37) modified by the presence of the gravity force? (b) Derive an expression for the acceleration a of the rocket, analogous to Eq. (8.39). (c) What is the acceleration of the rocket in Example 8.15 (Section 8.6) if it is near the earth's surface rather than in outer space? You can ignore air resistance. (d) Find the speed of the rocket in Example 8.16 (Section 8.6) after 90 s if the rocket is fired from the earth's surface rather than in outer space. You can ignore air resistance. How does your answer compare with the rocket speed calculated in Example 8.16?

8.113 •• **A Multistage Rocket.** Suppose the first stage of a two-stage rocket has total mass 12,000 kg, of which 9000 kg is fuel. The total mass of the second stage is 1000 kg, of which 700 kg is fuel. Assume that the relative speed v_{ex} of ejected material is constant, and ignore any effect of gravity. (The effect of gravity is small during the firing period if the rate of fuel consumption is large.) (a) Suppose the entire fuel supply carried by the two-stage rocket is utilized in a single-stage rocket with the same total mass of 13,000 kg. In terms of v_{ex}, what is the speed of the rocket, starting from rest, when its fuel is exhausted? (b) For the two-stage rocket, what is the speed when the fuel of the first stage is exhausted if the first stage carries the second stage with it to this point? This speed then becomes the initial speed of the second stage. At this point, the second stage separates from the first stage. (c) What is the final speed of the second stage? (d) What value of v_{ex} is required to give the second stage of the rocket a speed of 7.00 km/s?

CHALLENGE PROBLEMS

8.114 • **CALC** **A Variable-Mass Raindrop.** In a rocket-propulsion problem the mass is variable. Another such problem is a raindrop falling through a cloud of small water droplets. Some of these small droplets adhere to the raindrop, thereby *increasing* its mass as it falls. The force on the raindrop is

$$F_{ext} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Suppose the mass of the raindrop depends on the distance x that it has fallen. Then $m = kx$, where k is a constant, and $dm/dt = kv$. This gives, since $F_{ext} = mg$,

$$mg = m\frac{dv}{dt} + v(kv)$$

Or, dividing by k,

$$xg = x\frac{dv}{dt} + v^2$$

This is a differential equation that has a solution of the form $v = at$, where a is the acceleration and is constant. Take the initial velocity of the raindrop to be zero. (a) Using the proposed solution for v, find the acceleration a. (b) Find the distance the raindrop has fallen in $t = 3.00$ s. (c) Given that $k = 2.00$ g/m, find the mass of the raindrop at $t = 3.00$ s. (For many more intriguing aspects of this problem, see K. S. Krane, *American Journal of Physics,* Vol. 49 (1981), pp. 113–117.)

8.115 •• **CALC** In Section 8.5 we calculated the center of mass by considering objects composed of a *finite* number of point masses or objects that, by symmetry, could be represented by a finite number of point masses. For a solid object whose mass distribution does not allow for a simple determination of the center of mass by symmetry, the sums of Eqs. (8.28) must be generalized to integrals

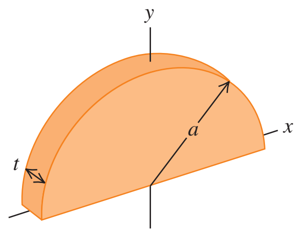
$$x_{cm} = \frac{1}{M}\int x\,dm \qquad y_{cm} = \frac{1}{M}\int y\,dm$$

where x and y are the coordinates of the small piece of the object that has mass dm. The integration is over the whole of the object.

Consider a thin rod of length L, mass M, and cross-sectional area A. Let the origin of the coordinates be at the left end of the rod and the positive x-axis lie along the rod. (a) If the density $\rho = M/V$ of the object is uniform, perform the integration described above to show that the x-coordinate of the center of mass of the rod is at its geometrical center. (b) If the density of the object varies linearly with x—that is, $\rho = \alpha x$, where α is a positive constant—calculate the x-coordinate of the rod's center of mass.

8.116 •• **CALC** Use the methods of Challenge Problem 8.115 to calculate the x- and y-coordinates of the center of mass of a semicircular metal plate with uniform density ρ and thickness t. Let the radius of the plate be a. The mass of the plate is thus $M = \frac{1}{2}\rho\pi a^2 t$. Use the coordinate system indicated in Fig. P8.116.

Figure **P8.116**



Answers

Chapter Opening Question ?

The two bullets have the same magnitude of momentum $p = mv$ (the product of mass and speed), but the faster, lightweight bullet has twice as much kinetic energy $K = \frac{1}{2}mv^2$. Hence, the lightweight bullet can do twice as much work on the carrot (and twice as much damage) in the process of coming to a halt (see Section 8.1).

Test Your Understanding Questions

8.1 Answer: (v), (i) and (ii) (tied for second place), (iii) and (iv) (tied for third place) We use two interpretations of the impulse of the net force: (1) the net force multiplied by the time that the net force acts, and (2) the change in momentum of the particle on which the net force acts. Which interpretation we use depends on what information we are given. We take the positive x-direction to be to the east. (i) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\text{ kg})(0) - (1000\text{ kg})(25\text{ m/s}) = -25{,}000\text{ kg}\cdot\text{m/s}$, so the magnitude of the impulse is $25{,}000\text{ kg}\cdot\text{m/s} = 25{,}000\text{ N}\cdot\text{s}$. (ii) For the same reason as in (i), we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\text{ kg})(0) - (1000\text{ kg})(25\text{ m/s}) = -25{,}000\text{ kg}\cdot\text{m/s}$, and the magnitude of the impulse is again $25{,}000\text{ kg}\cdot\text{m/s} = 25{,}000\text{ N}\cdot\text{s}$. (iii) The final velocity is not given, so we use interpretation 1: $J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (2000\text{ N})(10\text{ s}) = 20{,}000\text{ N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\text{ N}\cdot\text{s}$. (iv) For the same reason as in (iii), we use interpretation 1: $J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (-2000\text{ N})(10\text{ s}) = -20{,}000\text{ N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\text{ N}\cdot\text{s}$. (v) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\text{ kg})(-25\text{ m/s}) - (1000\text{ kg})(25\text{ m/s}) = -50{,}000\text{ kg}\cdot\text{m/s}$, so the magnitude of the impulse is $50{,}000\text{ kg}\cdot\text{m/s} = 50{,}000\text{ N}\cdot\text{s}$.

8.2 Answers: (a) $v_{C2x} > 0$, $v_{C2y} > 0$, **(b) piece C** There are no external horizontal forces, so the x- and y-components of the total momentum of the system are both conserved. Both components of the total momentum are zero before the spring releases, so they must be zero after the spring releases. Hence,

$$P_x = 0 = m_A v_{A2x} + m_B v_{B2x} + m_C v_{C2x}$$
$$P_y = 0 = m_A v_{A2y} + m_B v_{B2y} + m_C v_{C2y}$$

We are given that $m_A = m_B = m_C$, $v_{A2x} < 0$, $v_{A2y} = 0$, $v_{B2x} = 0$, and $v_{B2y} < 0$. You can solve the above equations to show that $v_{C2x} = -v_{A2x} > 0$ and $v_{C2y} = -v_{B2y} > 0$, so the velocity components of piece C are both positive. Piece C has speed $\sqrt{v_{C2x}{}^2 + v_{C2y}{}^2} = \sqrt{v_{A2x}{}^2 + v_{B2y}{}^2}$, which is greater than the speed of either piece A or piece B.

8.3 Answers: (a) elastic, (b) inelastic, (c) completely inelastic In each case gravitational potential energy is converted to kinetic energy as the ball falls, and the collision is between the ball and the ground. In (a) all of the initial energy is converted back to gravitational potential energy, so no kinetic energy is lost in the bounce and the collision is elastic. In (b) there is less gravitational potential energy at the end than at the beginning, so some kinetic energy was lost in the bounce. Hence the collision is inelastic. In (c) the ball loses all the kinetic energy it has to give, the ball and the ground stick together, and the collision is completely inelastic.

8.4 Answer: worse After a collision with a water molecule initially at rest, the speed of the neutron is $|(m_\text{n} - m_\text{w})/(m_\text{n} + m_\text{w})| = |(1.0\text{ u} - 18\text{ u})/(1.0\text{ u} + 18\text{ u})| = \frac{17}{19}$ of its initial speed, and its kinetic energy is $\left(\frac{17}{19}\right)^2 = 0.80$ of the initial value. Hence a water molecule is a worse moderator than a carbon atom, for which the corresponding numbers are $\frac{11}{13}$ and $\left(\frac{11}{13}\right)^2 = 0.72$.

8.5 Answer: no If gravity is the only force acting on the system of two fragments, the center of mass will follow the parabolic trajectory of a freely falling object. Once a fragment lands, however, the ground exerts a normal force on that fragment. Hence the net force on the system has changed, and the trajectory of the center of mass changes in response.

8.6 Answers: (a) increasing, (b) decreasing From Eqs. (8.37) and (8.38), the thrust F is equal to $m(dv/dt)$, where m is the rocket's mass and dv/dt is its acceleration. Because m decreases with time, if the thrust F is constant, then the acceleration must increase with time (the same force acts on a smaller mass); if the acceleration dv/dt is constant, then the thrust must decrease with time (a smaller force is all that's needed to accelerate a smaller mass).

Bridging Problem

Answers: **(a)** 1.00 m/s to the right **(b)** Elastic
(c) 1.93 m/s at $-30.4°$
(d) $2.31\text{ kg}\cdot\text{m/s}$ at $149.6°$ **(e)** Inelastic
(f) 1.67 m/s in the positive x-direction

9 ROTATION OF RIGID BODIES

? All segments of a rotating wind turbine blade have the same angular velocity. Compared to a given blade segment, how many times greater is the linear speed of a second segment twice as far from the axis of rotation? How many times greater is the radial acceleration?

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe the rotation of a rigid body in terms of angular coordinate, angular velocity, and angular acceleration.
- How to analyze rigid-body rotation when the angular acceleration is constant.
- How to relate the rotation of a rigid body to the linear velocity and linear acceleration of a point on the body.
- The meaning of a body's moment of inertia about a rotation axis, and how it relates to rotational kinetic energy.
- How to calculate the moment of inertia of various bodies.

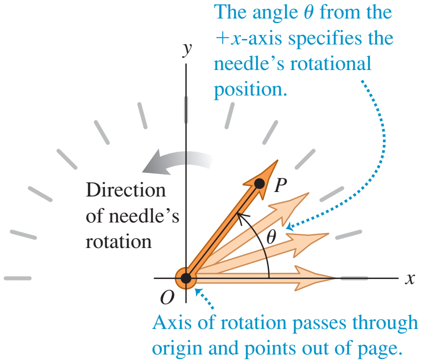
What do the motions of a compact disc, a Ferris wheel, a circular saw blade, and a ceiling fan have in common? None of these can be represented adequately as a moving *point;* each involves a body that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motions of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating body. In this chapter and the next we consider bodies that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world bodies can be very complicated; the forces that act on them can deform them—stretching, twisting, and squeezing them. We'll neglect these deformations for now and assume that the body has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body.** This chapter and the next are mostly about rotational motion of a rigid body.

We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis.



9.1 Angular Velocity and Acceleration

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis*—an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

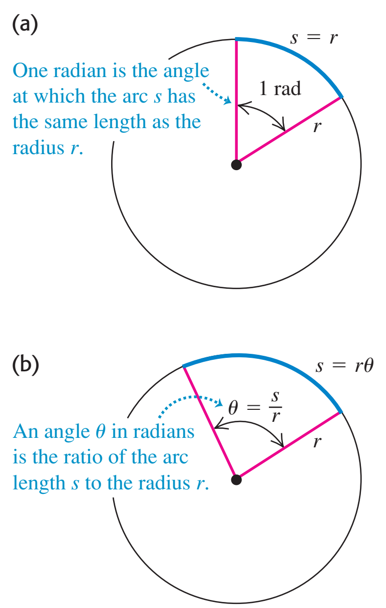
Figure 9.1 shows a rigid body (in this case, the indicator needle of a speedometer) rotating about a fixed axis. The axis passes through point O and is perpendicular to the plane of the diagram, which we choose to call the *xy*-plane. One way to describe the rotation of this body would be to choose a particular point P on the body and to keep track of the x- and y-coordinates of this point. This isn't a terribly convenient method, since it takes two numbers (the two coordinates x and y) to specify the rotational position of the body. Instead, we notice that the line OP is fixed in the body and rotates with it. The angle θ that this line makes with the $+x$-axis describes the rotational position of the body; we will use this single quantity θ as a *coordinate* for rotation.

The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive x-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

To describe rotational motion, the most natural way to measure the angle θ is not in degrees, but in **radians.** As shown in Fig. 9.2a, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta \tag{9.1}$$

9.2 Measuring angles in radians.



An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0$ m and $r = 2.0$ m, then $\theta = 1.5$, but we will often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution $(360°)$. Therefore

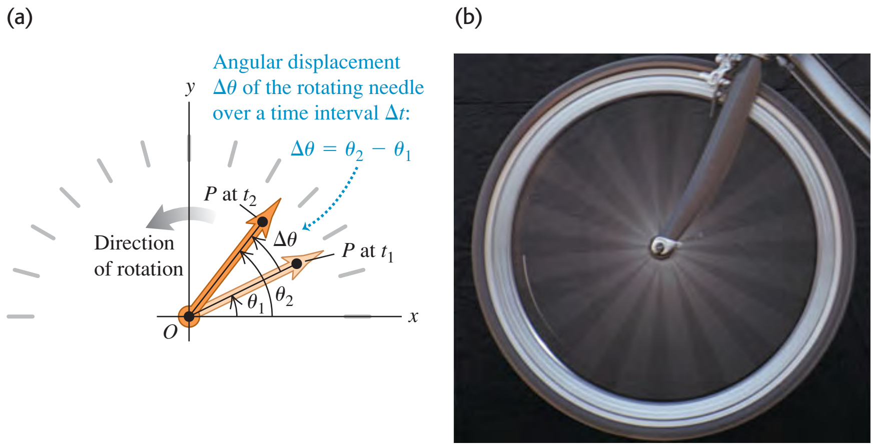
$$1 \text{ rad} = \frac{360°}{2\pi} = 57.3°$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had insisted on measuring the angle θ in degrees, we would have needed to include an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. We'll do this in an analogous way to our description of straight-line motion in Chapter 2. In Fig. 9.3a, a reference line OP in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-}z}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$



9.3 (a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same average angular velocity $\Delta\theta/\Delta t$.

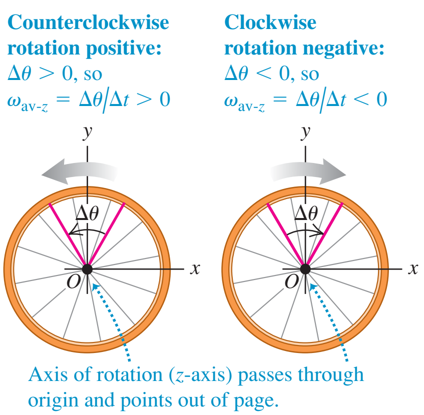
The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{av-}z}$ as Δt approaches zero—that is, the derivative of θ with respect to t:

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(definition of angular velocity)} \qquad (9.3)$$

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (Fig. 9.4). The angular *speed* ω, which we will use extensively in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like ordinary (linear) speed v, the angular speed is never negative.

9.4 A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative.



CAUTION **Angular velocity vs. linear velocity** Keep in mind the distinction between angular velocity ω_z and ordinary velocity, or *linear velocity,* v_x (see Section 2.2). If an object has a velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

Different points on a rotating rigid body move different distances in a given time interval, depending on how far each point lies from the rotation axis. But because the body is rigid, *all* points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity.* The angular velocity is positive if the body is rotating in the direction of increasing θ and negative if it is rotating in the direction of decreasing θ.

If the angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since 1 rev $= 2\pi$ rad, two useful conversions are

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

That is, 1 rad/s is about 10 rpm.

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves over the time interval from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

SOLUTION

IDENTIFY and SET UP: We can find the target variables θ_1 (the angular position at time t_1), θ_2 (the angular position at time t_2), and the angular displacement $\Delta\theta = \theta_2 - \theta_1$ from the given expression. Knowing $\Delta\theta$, we'll find the distance traveled and the average angular velocity between t_1 and t_2 using Eqs. (9.1) and (9.2), respectively. To find the instantaneous angular velocities ω_{1z} (at time t_1) and ω_{2z} (at time t_2), we'll take the derivative of the given equation for θ with respect to time, as in Eq. (9.3).

EXECUTE: (a) We substitute the values of t into the equation for θ:

$$\begin{aligned}\theta_1 &= (2.0\ \text{rad/s}^3)(2.0\ \text{s})^3 = 16\ \text{rad}\\ &= (16\ \text{rad})\frac{360°}{2\pi\ \text{rad}} = 920°\\ \theta_2 &= (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}\\ &= (250\ \text{rad})\frac{360°}{2\pi\ \text{rad}} = 14{,}000°\end{aligned}$$

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250\ \text{rad} - 16\ \text{rad} = 234\ \text{rad}$. The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18\ \text{m})(234\ \text{rad}) = 42\ \text{m}$$

We can drop "radians" from the unit for s because θ is a pure, dimensionless number; the distance s is measured in meters, the same as r.

(c) From Eq. (9.2),

$$\begin{aligned}\omega_{\text{av-}z} &= \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\ \text{rad} - 16\ \text{rad}}{5.0\ \text{s} - 2.0\ \text{s}} = 78\ \text{rad/s}\\ &= \left(78\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 740\ \text{rev/min}\end{aligned}$$

(d) From Eq. (9.3),

$$\begin{aligned}\omega_z &= \frac{d\theta}{dt} = \frac{d}{dt}[(2.0\ \text{rad/s}^3)t^3] = (2.0\ \text{rad/s}^3)(3t^2)\\ &= (6.0\ \text{rad/s}^3)t^2\end{aligned}$$

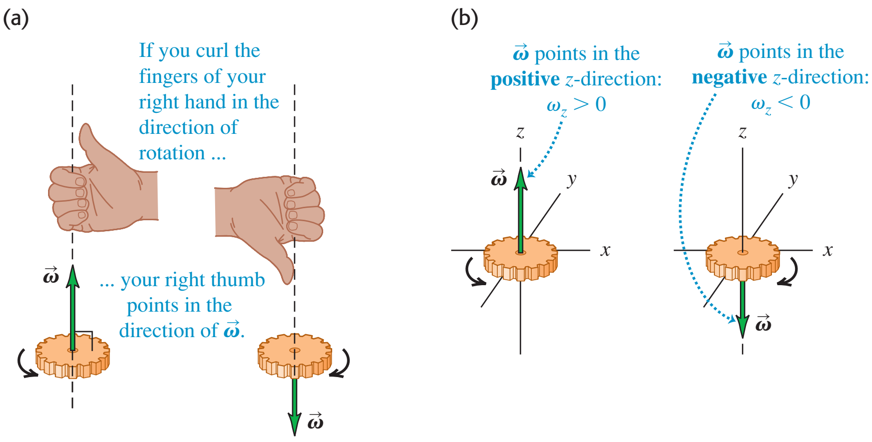
At times $t_1 = 2.0$ s and $t_2 = 5.0$ s we have

$$\begin{aligned}\omega_{1z} &= (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}\\ \omega_{2z} &= (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}\end{aligned}$$

EVALUATE: The angular velocity $\omega_z = (6.0\ \text{rad/s}^3)t^2$ increases with time. Our results are consistent with this; the instantaneous angular velocity at the end of the interval ($\omega_{2z} = 150$ rad/s) is greater than at the beginning ($\omega_{1z} = 24$ rad/s), and the average angular velocity $\omega_{\text{av-}z} = 78$ rad/s over the interval is intermediate between these two values.

Angular Velocity As a Vector

As we have seen, our notation for the angular velocity ω_z about the z-axis is reminiscent of the notation v_x for the ordinary velocity along the x-axis (see Section 2.2). Just as v_x is the x-component of the velocity vector $\vec{v}$, ω_z is the z-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As Fig. 9.5a shows, the direction of $\vec{\omega}$ is given by the right-hand rule that we used to define the vector



9.5 (a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the z-axis.

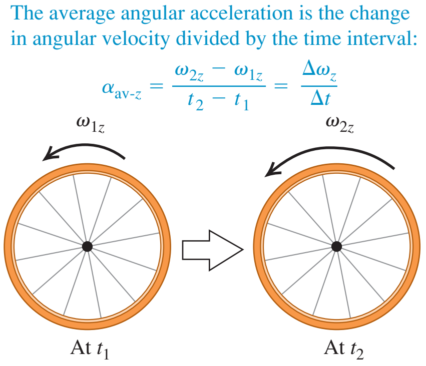
product in Section 1.10. If the rotation is about the z-axis, then $\vec{\omega}$ has only a z-component; this component is positive if $\vec{\omega}$ is along the positive z-axis and negative if $\vec{\omega}$ is along the negative z-axis (Fig. 9.5b).

The vector formulation is especially useful in situations in which the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of the angular velocity vector $\vec{\omega}$ along the axis.

Angular Acceleration

When the angular velocity of a rigid body changes, it has an *angular acceleration.* When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration. You also impart an angular acceleration whenever you change the rotation speed of a piece of spinning machinery such as an automobile engine's crankshaft.

9.6 Calculating the average angular acceleration of a rotating rigid body.



If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (Fig. 9.6):

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{9.4}$$

The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av-}z}$ as $\Delta t \to 0$:

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \qquad \text{(definition of angular acceleration)} \tag{9.5}$$

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we will use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$

You have probably noticed that we are using Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, v_x for velocity, and a_x for acceleration, respectively, in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We will sometimes use the terms "*linear* velocity" and "*linear* acceleration" for the familiar quantities we defined in Chapters 2 and 3 to distinguish clearly between these and the *angular* quantities introduced in this chapter.

In rotational motion, if the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and *linear* velocity v_x for straight-line motion; see Section 2.3.)

Example 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

SOLUTION

IDENTIFY and SET UP: We use Eq. (9.4) for the average angular acceleration $\alpha_{\text{av-}z}$ and Eq. (9.5) for the instantaneous angular acceleration α_z.

EXECUTE: (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24 \text{ rad/s} \qquad \omega_{2z} = 150 \text{ rad/s}$$

From Eq. (9.4), the average angular acceleration is

$$\alpha_{\text{av-}z} = \frac{150 \text{ rad/s} - 24 \text{ rad/s}}{5.0 \text{ s} - 2.0 \text{ s}} = 42 \text{ rad/s}^2$$

(b) From Eq. (9.5), the value of α_z at any time t is

$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0 \text{ rad/s}^3)(t^2)] = (6.0 \text{ rad/s}^3)(2t)$$
$$= (12 \text{ rad/s}^3)t$$

Hence

$$\alpha_{1z} = (12 \text{ rad/s}^3)(2.0 \text{ s}) = 24 \text{ rad/s}^2$$
$$\alpha_{2z} = (12 \text{ rad/s}^3)(5.0 \text{ s}) = 60 \text{ rad/s}^2$$

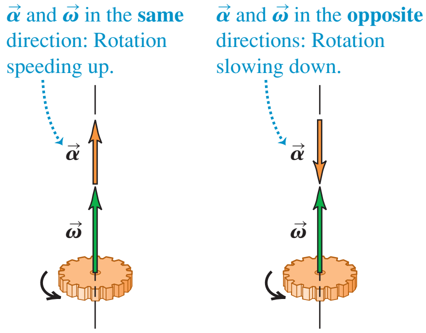
EVALUATE: Note that the angular acceleration is *not* constant in this situation. The angular velocity ω_z is always increasing because α_z is always positive. Furthermore, the rate at which the angular velocity increases is itself increasing, since α_z increases with time.

Angular Acceleration As a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\alpha}$. Mathematically, $\vec{\alpha}$ is the time derivative of the angular velocity vector $\vec{\omega}$. If the object rotates around the fixed z-axis, then $\vec{\alpha}$ has only a z-component; the quantity α_z is just that component. In this case, $\vec{\alpha}$ is in the same direction as $\vec{\omega}$ if the rotation is speeding up and opposite to $\vec{\omega}$ if the rotation is slowing down (Fig. 9.7).

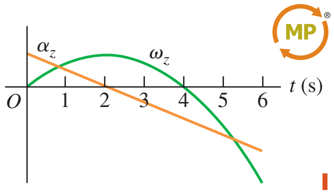
The angular acceleration vector will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis can change direction. In this chapter, however, the rotation axis will always be fixed and we need use only the z-component α_z.

9.7 When the rotation axis is fixed, the angular acceleration and angular velocity vectors both lie along that axis.



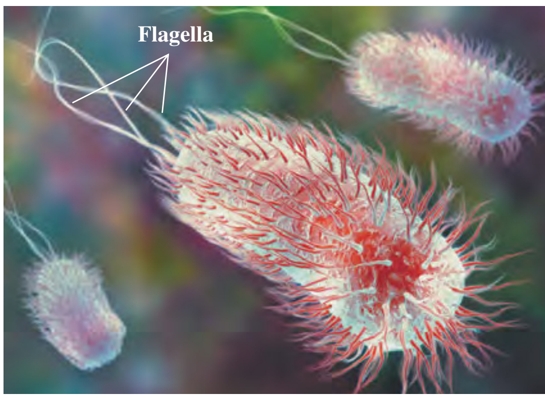
Test Your Understanding of Section 9.1

The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) $2 \text{ s} < t < 4$ s; (iii) $4 \text{ s} < t < 6$ s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) $2 \text{ s} < t < 4$ s; (iii) $4 \text{ s} < t < 6$ s.



Application Rotational Motion in Bacteria

Escherichia coli bacteria (about 2 μm by 0.5 μm) are found in the lower intestines of humans and other warm-blooded animals. The bacteria swim by rotating their long, corkscrew-shaped flagella, which act like the blades of a propeller. Each flagellum is powered by a remarkable protein motor at its base. The motor can rotate the flagellum at angular speeds from 200 to 1000 rev/min (about 20 to 100 rad/s) and can vary its speed to give the flagellum an angular acceleration.



9.2 Rotation with Constant Angular Acceleration

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position using exactly the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace x with θ, v_x with ω_z, and a_x with α_z. We suggest that you review Section 2.4 before continuing.

Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$, and let ω_z be its angular velocity at any later time t. The angular acceleration α_z is constant and equal to the average value for any interval. Using Eq. (9.4) with the interval from 0 to t, we find

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \qquad \text{or}$$

$$\omega_z = \omega_{0z} + \alpha_z t \qquad \text{(constant angular acceleration only)} \qquad (9.7)$$

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time t; the angular velocity ω_z at time t is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2} \tag{9.8}$$

We also know that $\omega_{\text{av-}z}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

$$\omega_{\text{av-}z} = \frac{\theta - \theta_0}{t - 0} \tag{9.9}$$

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad \text{(constant angular acceleration only)} \tag{9.10}$$

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \tfrac{1}{2}[\omega_{0z} + (\omega_{0z} + \alpha_z t)]t \qquad \text{or}$$

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad \text{(constant angular acceleration only)} \tag{9.11}$$

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z}, then its angular position θ at any later time t is the sum of three terms: its initial angular position θ_0, plus the rotation $\omega_{0z}t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\alpha_z t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (2.13). (See Exercise 9.12.) In fact, because of the perfect analogy between straight-line and rotational quantities, we can simply take Eq. (2.13) and replace each straight-line quantity by its rotational analog. We get

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad \text{(constant angular acceleration only)} \tag{9.12}$$

CAUTION **Constant angular acceleration** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* be careful not to try to apply them to problems in which α_z is *not* constant. Table 9.1 shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

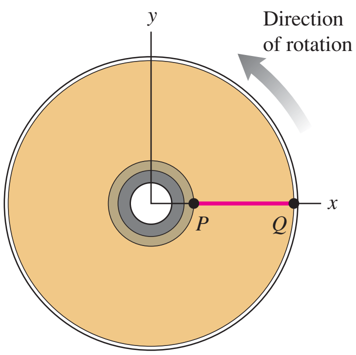
Example 9.3 Rotation with constant angular acceleration

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

SOLUTION

IDENTIFY and SET UP: The angular acceleration of the disc is constant, so we can use any of the equations derived in this section (Table 9.1). Our target variables are the angular velocity ω_z and the angular displacement θ at $t = 0.300$ s. Given $\omega_{0z} = 27.5$ rad/s, $\theta_0 = 0$, and $\alpha_z = -10.0\ \text{rad/s}^2$, it's easiest to use Eqs. (9.7) and (9.11) to find the target variables.

9.8 A line PQ on a rotating Blu-ray disc at $t = 0$.



EXECUTE: (a) From Eq. (9.7), at $t = 0.300$ s we have

$$\begin{aligned}\omega_z &= \omega_{0z} + \alpha_z t = 27.5\ \text{rad/s} + (-10.0\ \text{rad/s}^2)(0.300\ \text{s}) \\ &= 24.5\ \text{rad/s}\end{aligned}$$

(b) From Eq. (9.11),

$$\begin{aligned}\theta &= \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \\ &= 0 + (27.5\ \text{rad/s})(0.300\ \text{s}) + \tfrac{1}{2}(-10.0\ \text{rad/s}^2)(0.300\ \text{s})^2 \\ &= 7.80\ \text{rad} = 7.80\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 1.24\ \text{rev}\end{aligned}$$

The disc has turned through one complete revolution plus an additional 0.24 revolution—that is, through 360° plus (0.24 rev)(360°/rev) = 87°. Hence the line PQ makes an angle of 87° with the $+x$-axis.

EVALUATE: Our answer to part (a) tells us that the disc's angular velocity has decreased, as it should since $\alpha_z < 0$. We can use our result for ω_z from part (a) with Eq. (9.12) to check our result for θ from part (b). To do so, we solve Eq. (9.12) for θ:

$$\begin{aligned}\omega_z^2 &= \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \\ \theta &= \theta_0 + \left(\frac{\omega_z^2 - \omega_{0z}^2}{2\alpha_z}\right) \\ &= 0 + \frac{(24.5\ \text{rad/s})^2 - (27.5\ \text{rad/s})^2}{2(-10.0\ \text{rad/s}^2)} = 7.80\ \text{rad}\end{aligned}$$

This agrees with our previous result from part (b).

Test Your Understanding of Section 9.2 Suppose the disc in Example 9.3 was initially spinning at twice the rate (55.0 rad/s rather than 27.5 rad/s) and slowed down at twice the rate ($-20.0\ \text{rad/s}^2$ rather than $-10.0\ \text{rad/s}^2$). (a) Compared to the situation in Example 9.3, how long would it take the disc to come to a stop? (i) the same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) $\frac{1}{2}$ as much time; (v) $\frac{1}{4}$ as much time. (b) Compared to the situation in Example 9.3, through how many revolutions would the disc rotate before coming to a stop? (i) the same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) $\frac{1}{2}$ as many revolutions; (v) $\frac{1}{4}$ as many revolutions. |

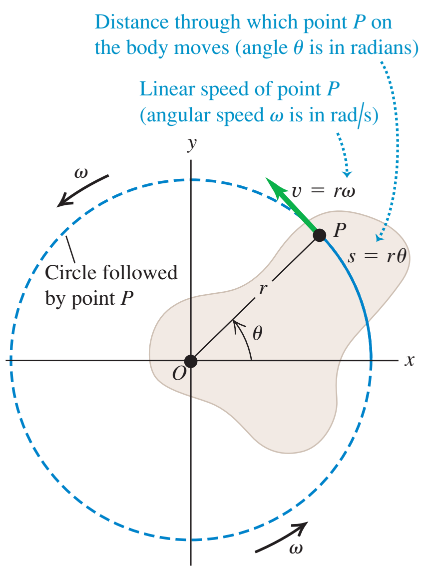
9.3 Relating Linear and Angular Kinematics

How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires knowing the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path. The circle lies in a plane perpendicular to the axis and is centered on the axis. The speed of a particle is directly proportional to the body's angular

9.9 A rigid body rotating about a fixed axis through point O.



PhET: Ladybug Revolution

velocity; the faster the body rotates, the greater the speed of each particle. In Fig. 9.9, point P is a constant distance r from the axis of rotation, so it moves in a circle of radius r. At any time, the angle θ (in radians) and the arc length s are related by

$$s = r\theta$$

We take the time derivative of this, noting that r is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. Analogously, $|d\theta/dt|$, the absolute value of the rate of change of the angle, is the instantaneous **angular speed** ω—that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

$$v = r\omega \qquad \text{(relationship between linear and angular speeds)} \tag{9.13}$$

The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The corresponding quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

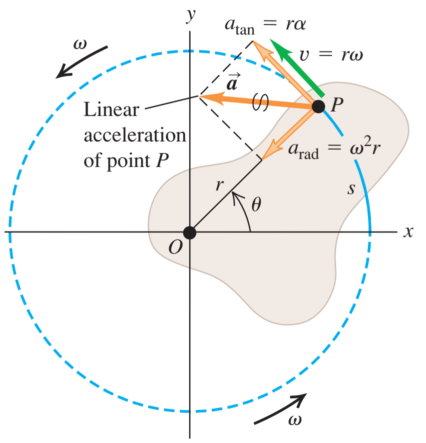
Linear Acceleration in Rigid-Body Rotation

We can represent the acceleration of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and a_{tan} (Fig. 9.10), as we did in Section 3.4. It would be a good idea to review that section now. We found that the **tangential component of acceleration** a_{tan}, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \text{(tangential acceleration of a point on a rotating body)} \tag{9.14}$$

9.10 A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{v}$) and a component a_{tan} along the circle that point P follows (parallel to $\vec{v}$).

Radial and tangential acceleration components:
- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



This component of a particle's acceleration is always tangent to the circular path of the particle.

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed*. It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity*. For example, consider a body rotating so that its angular velocity vector points in the $-z$-direction (see Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10\ \text{rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha_z = -10\ \text{rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of the particle's acceleration directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the particle's velocity. In Section 3.4 we worked out the relationship $a_{\text{rad}} = v^2/r$. We can express this in terms of ω by using Eq. (9.13):

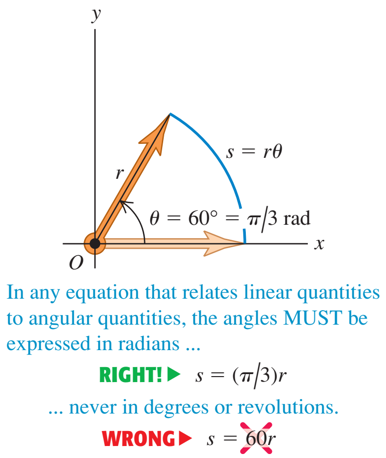
$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \qquad \text{(centripetal acceleration of a point on a rotating body)} \qquad (9.15)$$

This is true at each instant, *even when ω and v are not constant.* The centripetal component always points toward the axis of rotation.

The vector sum of the centripetal and tangential components of acceleration of a particle in a rotating body is the linear acceleration $\vec{a}$ (Fig. 9.10).

CAUTION **Use angles in radians in all equations** It's important to remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (Fig. 9.11).

9.11 Always use radians when relating linear and angular quantities.



Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We will have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

Example 9.4 Throwing a discus

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

SOLUTION

IDENTIFY and SET UP: We treat the discus as a particle traveling in a circular path (Fig. 9.12a), so we can use the ideas developed in this section. We are given $r = 0.800$ m, $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2 (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15), respectively, to find the acceleration components a_{tan} and a_{rad}; we'll then find the magnitude a using the Pythagorean theorem.

EXECUTE: From Eqs. (9.14) and (9.15),

$$a_{\text{tan}} = r\alpha = (0.800 \text{ m})(50.0 \text{ rad/s}^2) = 40.0 \text{ m/s}^2$$

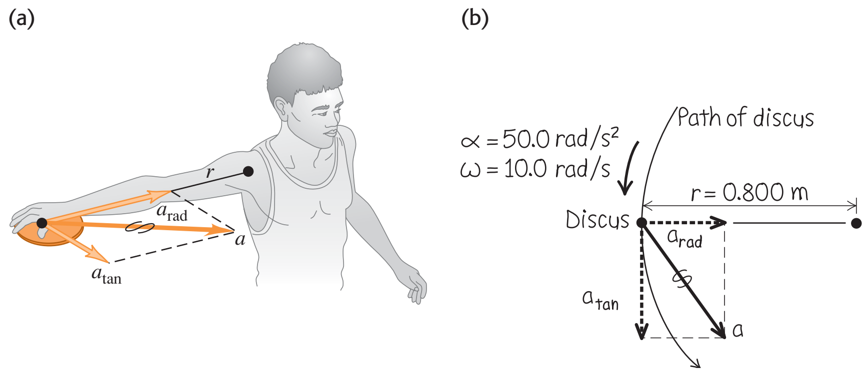
$$a_{\text{rad}} = \omega^2 r = (10.0 \text{ rad/s})^2(0.800 \text{ m}) = 80.0 \text{ m/s}^2$$

Then

$$a = \sqrt{a_{\text{tan}}^2 + a_{\text{rad}}^2} = 89.4 \text{ m/s}^2$$

EVALUATE: Note that we dropped the unit "radian" from our results for a_{tan}, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s^2?

9.12 (a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.



Example 9.5 Designing a propeller

You are designing an airplane propeller that is to turn at 2400 rpm (Fig. 9.13a). The forward airspeed of the plane is to be 75.0 m/s, and the speed of the tips of the propeller blades through the air must not exceed 270 m/s. (This is about 80% of the speed of sound in air. If the speed of the propeller tips were greater than this, they would produce a lot of noise.) (a) What is the maximum possible propeller radius? (b) With this radius, what is the acceleration of the propeller tip?

SOLUTION

IDENTIFY and SET UP: We consider a particle at the tip of the propeller; our target variables are the particle's distance from the axis and its acceleration. The speed of this particle through the air, which cannot exceed 270 m/s, is due to both the propeller's rotation and the forward motion of the airplane. Figure 9.13b shows that the particle's velocity $\vec{v}_{tip}$ is the vector sum of its tangential velocity due to the propeller's rotation of magnitude $v_{tan} = \omega r$, given by Eq. (9.13), and the forward velocity of the airplane of magnitude $v_{plane} = 75.0$ m/s. The propeller rotates in a plane perpendicular to the direction of flight, so $\vec{v}_{tan}$ and $\vec{v}_{plane}$ are perpendicular to each other, and we can use the Pythagorean theorem to obtain an expression for v_{tip} from v_{tan} and v_{plane}. We will then set $v_{tip} = 270$ m/s and solve for the radius r. The angular speed of the propeller is constant, so the acceleration of the propeller tip has only a radial component; we'll find it using Eq. (9.15).

EXECUTE: We first convert ω to rad/s (see Fig. 9.11):

$$\omega = 2400\text{ rpm} = \left(2400\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)$$
$$= 251\text{ rad/s}$$

(a) From Fig. 9.13b and Eq. (9.13),

$$v_{tip}{}^2 = v_{plane}{}^2 + v_{tan}{}^2 = v_{plane}{}^2 + r^2\omega^2 \quad \text{so}$$
$$r^2 = \frac{v_{tip}{}^2 - v_{plane}{}^2}{\omega^2} \quad \text{and} \quad r = \frac{\sqrt{v_{tip}{}^2 - v_{plane}{}^2}}{\omega}$$

If $v_{tip} = 270$ m/s, the maximum propeller radius is

$$r = \frac{\sqrt{(270\text{ m/s})^2 - (75.0\text{ m/s})^2}}{251\text{ rad/s}} = 1.03\text{ m}$$

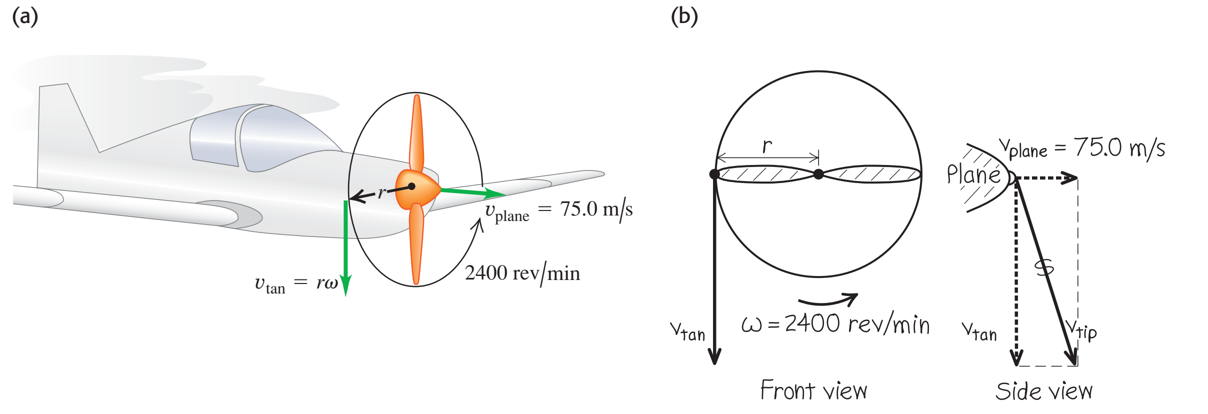
(b) The centripetal acceleration of the particle is

$$a_{rad} = \omega^2 r = (251\text{ rad/s})^2(1.03\text{ m})$$
$$= 6.5 \times 10^4\text{ m/s}^2 = 6600g$$

The tangential acceleration a_{rad} is zero because the angular speed is constant.

EVALUATE: From $\sum\vec{F} = m\vec{a}$, the propeller must exert a force of 6.5×10^4 N on each kilogram of material at its tip! This is why propellers are made out of tough material, usually aluminum alloy.

9.13 (a) A propeller-driven airplane in flight. (b) Our sketch showing the velocity components for the propeller tip.



Test Your Understanding of Section 9.3 Information is stored on a disc (see Fig. 9.8) in a coded pattern of tiny pits. The pits are arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a player, the track is scanned at a constant *linear* speed. How must the rotation speed of the disc change as the player's scanning head moves over the track? (i) The rotation speed must increase. (ii) The rotation speed must decrease. (iii) The rotation speed must stay the same.

9.4 Energy in Rotational Motion

A rotating rigid body consists of mass in motion, so it has kinetic energy. As we will see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia,* that depends on the body's mass and how the mass is distributed.

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ActivPhysics 7.7: Rotational Inertia

To begin, we think of a body as being made up of a large number of particles, with masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis of rotation. We label the particles with the index i: The mass of the ith particle is m_i and its distance from the axis of rotation is r_i. The particles don't necessarily all lie in the same plane, so we specify that r_i is the *perpendicular* distance from the axis to the ith particle.

When a rigid body rotates about a fixed axis, the speed v_i of the ith particle is given by Eq. (9.13), $v_i = r_i\omega$, where ω is the body's angular speed. Different particles have different values of r, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the ith particle can be expressed as

$$\tfrac{1}{2}m_i v_i{}^2 = \tfrac{1}{2}m_i r_i{}^2\omega^2$$

The *total* kinetic energy of the body is the sum of the kinetic energies of all its particles:

$$K = \tfrac{1}{2}m_1 r_1{}^2\omega^2 + \tfrac{1}{2}m_2 r_2{}^2\omega^2 + \cdots = \sum_i \tfrac{1}{2}m_i r_i{}^2\omega^2$$

Taking the common factor $\omega^2/2$ out of this expression, we get

$$K = \tfrac{1}{2}(m_1 r_1{}^2 + m_2 r_2{}^2 + \cdots)\omega^2 = \tfrac{1}{2}\Big(\sum_i m_i r_i{}^2\Big)\omega^2$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is denoted by I and is called the **moment of inertia** of the body for this rotation axis:

$$I = m_1 r_1{}^2 + m_2 r_2{}^2 + \cdots = \sum_i m_i r_i{}^2 \qquad \text{(definition of moment of inertia)} \qquad (9.16)$$

The word "moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distance from the axis to the particles that make up the body, the greater the moment of inertia. In a rigid body, the distances r_i are all constant and I is independent of how the body rotates around the given axis. The SI unit of moment of inertia is the kilogram-meter^2 ($\text{kg}\cdot\text{m}^2$).

In terms of moment of inertia I, the **rotational kinetic energy** K of a rigid body is

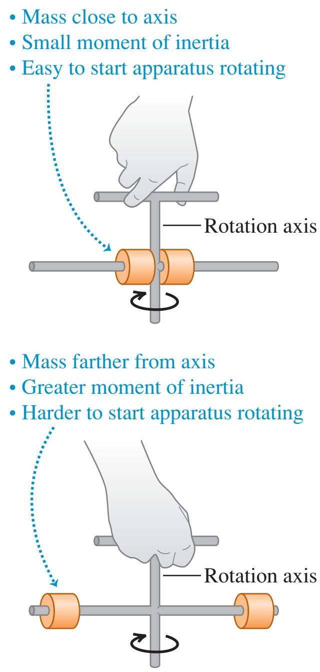
$$K = \tfrac{1}{2}I\omega^2 \qquad \text{(rotational kinetic energy of a rigid body)} \qquad (9.17)$$

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $v_i = r_i\omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of a body equals the amount of work done to accelerate that body from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (Fig. 9.14). For this reason, I is also called the *rotational inertia.*

The next example shows how *changing* the rotation axis can affect the value of I.

9.14 An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft.



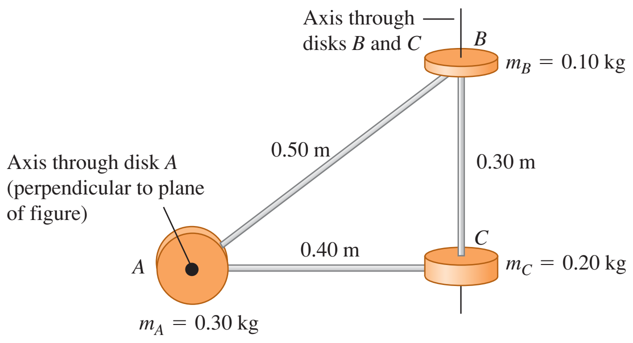
Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about an axis through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about an axis through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about the axis through A with angular speed $\omega = 4.0$ rad/s?

SOLUTION

IDENTIFY and SET UP: We'll consider the disks as massive particles located at the centers of the disks, and consider the struts as massless. In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia. Given the moment of inertia about axis A, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

9.15 An oddly shaped machine part.



EXECUTE: (a) The particle at point A lies *on* the axis through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute, and Eq. (9.16) gives

$$I_A = \sum m_i r_i^2 = (0.10\ \text{kg})(0.50\ \text{m})^2 + (0.20\ \text{kg})(0.40\ \text{m})^2 = 0.057\ \text{kg}\cdot\text{m}^2$$

(b) The particles at B and C both lie on axis BC, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$I_{BC} = \sum m_i r_i^2 = (0.30\ \text{kg})(0.40\ \text{m})^2 = 0.048\ \text{kg}\cdot\text{m}^2$$

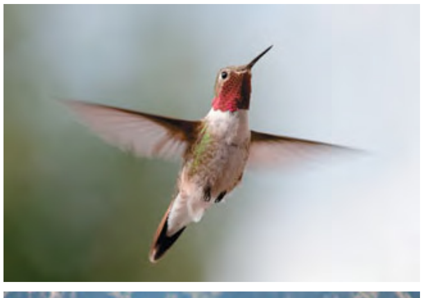
(c) From Eq. (9.17),

$$K_A = \tfrac{1}{2}I_A\omega^2 = \tfrac{1}{2}(0.057\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}$$

EVALUATE: The moment of inertia about axis A is greater than that about axis BC. Hence of the two axes it's easier to make the machine part rotate about axis BC.

Application Moment of Inertia of a Bird's Wing

When a bird flaps its wings, it rotates the wings up and down around the shoulder. A hummingbird has small wings with a small moment of inertia, so the bird can make its wings move rapidly (up to 70 beats per second). By contrast, the Andean condor (*Vultur gryphus*) has immense wings that are hard to move due to their large moment of inertia. Condors flap their wings at about one beat per second on takeoff, but at most times prefer to soar while holding their wings steady.



CAUTION Moment of inertia depends on the choice of axis The results of parts (a) and (b) of Example 9.6 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is $0.048\ \text{kg}\cdot\text{m}^2$." We have to be specific and say, "The moment of inertia of this body *about the axis through B and C* is $0.048\ \text{kg}\cdot\text{m}^2$."

In Example 9.6 we represented the body as several point masses, and we evaluated the sum in Eq. (9.16) directly. When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. We will give several examples of such calculations in Section 9.6; meanwhile, Table 9.2 gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform;* that is, the density has the same value at all points within the solid parts of the body.

CAUTION Computing the moment of inertia You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. Resist that temptation; it doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion. Here are some points of strategy and some examples.

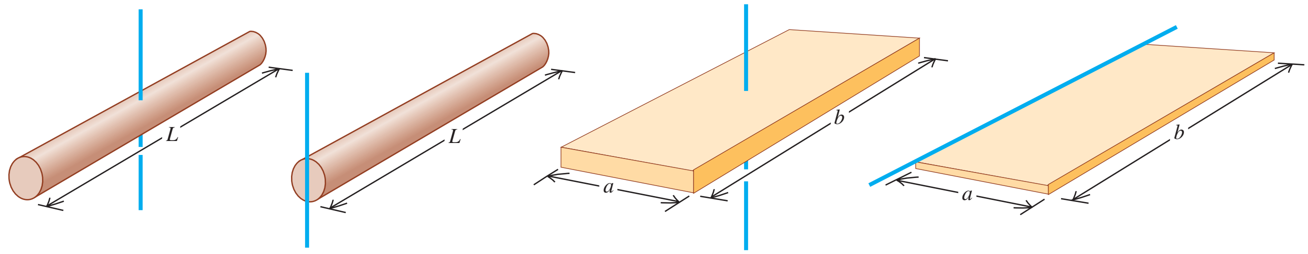
Table 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center: $I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center: $I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



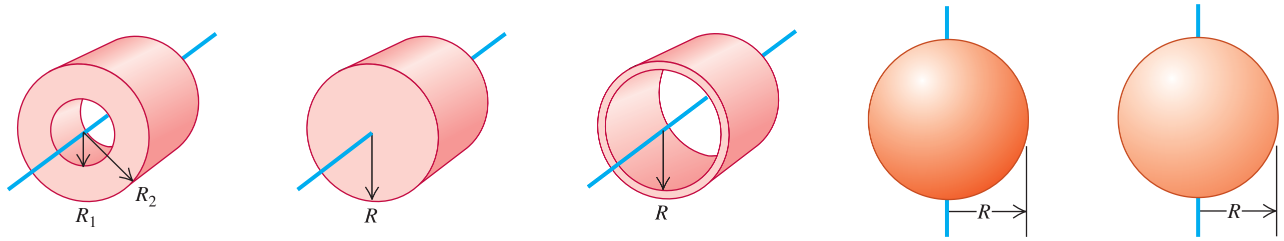
(e) Hollow cylinder: $I = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$

(f) Solid cylinder: $I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder: $I = MR^2$

(h) Solid sphere: $I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$



Problem-Solving Strategy 9.1 Rotational Energy

IDENTIFY *the relevant concepts:* You can use work–energy relationships and conservation of energy to find relationships involving the position and motion of a rigid body rotating around a fixed axis. The energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind.

SET UP *the problem* using Problem-Solving Strategy 7.1 (Section 7.1), with the following additions:

5. You can use Eqs. (9.13) and (9.14) in problems involving a rope (or the like) wrapped around a rotating rigid body, if the rope doesn't slip. These equations relate the linear speed and tangential acceleration of a point on the body to the body's angular velocity and angular acceleration. (See Examples 9.7 and 9.8.)
6. Use Table 9.2 to find moments of inertia. Use the parallel-axis theorem, Eq. (9.19) (to be derived in Section 9.5), to find moments of inertia for rotation about axes parallel to those shown in the table.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, U_1, and U_2 and for the nonconservative work W_{other} (if any), where K_1 and K_2 must now include any rotational kinetic energy $K = \frac{1}{2}I\omega^2$. Substitute these expressions into Eq. (7.14), $K_1 + U_1 + W_{other} = K_2 + U_2$ (if nonconservative work is done), or Eq. (7.11), $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done), and solve for the target variables. It's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

EVALUATE *your answer:* Check whether your answer makes physical sense.

Example 9.7 An unwinding cable I

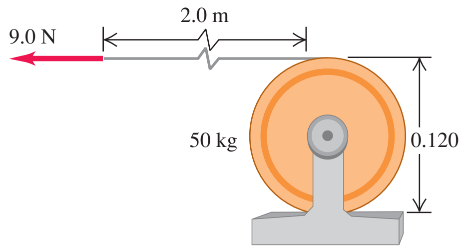
We wrap a light, nonstretching cable around a solid cylinder of mass 50 kg and diameter 0.120 m, which rotates in frictionless bearings about a stationary horizontal axis (Fig. 9.16). We pull the free end of the cable with a constant 9.0-N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its final angular speed and the final speed of the cable.

SOLUTION

IDENTIFY: We'll solve this problem using energy methods. We'll assume that the cable is massless, so only the cylinder has kinetic energy. There are no changes in gravitational potential energy. There is friction between the cable and the cylinder, but because the cable doesn't slip, there is no motion of the cable relative to the

Continued

9.16 A cable unwinds from a cylinder (side view).



cylinder and no mechanical energy is lost in frictional work. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

SET UP: Point 1 is when the cable begins to move. The cylinder starts at rest, so $K_1 = 0$. Point 2 is when the cable has moved a distance $s = 2.0$ m and the cylinder has kinetic energy $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll use Eq. (9.13) to find v from ω.

EXECUTE: The work done on the cylinder is $W_{\text{other}} = Fs = (9.0\ \text{N})(2.0\ \text{m}) = 18$ J. From Table 9.2 the moment of inertia is

$$I = \tfrac{1}{2}mR^2 = \tfrac{1}{2}(50\ \text{kg})(0.060\ \text{m})^2 = 0.090\ \text{kg}\cdot\text{m}^2$$

(The radius R is half the diameter.) From Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$, so

$$0 + 0 + W_{\text{other}} = \tfrac{1}{2}I\omega^2 + 0$$

$$\omega = \sqrt{\frac{2W_{\text{other}}}{I}} = \sqrt{\frac{2(18\ \text{J})}{0.090\ \text{kg}\cdot\text{m}^2}} = 20\ \text{rad/s}$$

From Eq. (9.13), the final tangential speed of the cylinder, and hence the final speed of the cable, is

$$v = R\omega = (0.060\ \text{m})(20\ \text{rad/s}) = 1.2\ \text{m/s}$$

EVALUATE: If the cable mass is not negligible, some of the 18 J of work would go into the kinetic energy of the cable. Then the cylinder would have less kinetic energy and a lower angular speed than we calculated here.

Example 9.8 An unwinding cable II

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

SOLUTION

IDENTIFY: As in Example 9.7, the cable doesn't slip and so friction does no work. We assume that the cable is massless, so that the forces it exerts on the cylinder and the block have equal magnitudes. At its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions, so the cable does no *net* work and $W_{\text{other}} = 0$. Only gravity does work, and mechanical energy is conserved.

SET UP: Figure 9.17a shows the situation before the block begins to fall (point 1). The initial kinetic energy is $K_1 = 0$. We take the gravitational potential energy to be zero when the block is at floor level (point 2), so $U_1 = mgh$ and $U_2 = 0$. (We ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.17b), both the block and the cylinder have kinetic energy, so

$$K_2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Also, $v = R\omega$ since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder.

EXECUTE: We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in Eq. (7.4), $K_1 + U_1 = K_2 + U_2$, and solve for v:

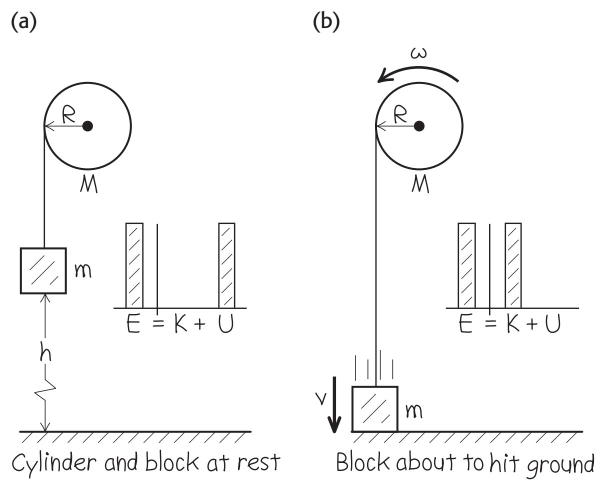
$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \tfrac{1}{2}\left(m + \tfrac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.

EVALUATE: When M is much larger than m, v is very small; when M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

9.17 Our sketches for this problem.



Gravitational Potential Energy for an Extended Body

In Example 9.8 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

$$U = Mgy_{cm} \quad \text{(gravitational potential energy for an extended body)} \quad (9.18)$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not (Fig. 9.18).

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots)g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots)y_{cm} = My_{cm}$$

where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = Mgy_{cm}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. We'll make use of this relationship in Chapter 10 in the analysis of rigid-body problems in which the axis of rotation moves.

9.18 In a technique called the "Fosbury flop" after its innovator, this athlete arches her body as she passes over the bar in the high jump. As a result, her center of mass actually passes *under* the bar. This technique requires a smaller increase in gravitational potential energy [Eq. (9.18)] than the older method of straddling the bar.

Test Your Understanding of Section 9.4 Suppose the cylinder and block in Example 9.8 have the same mass, so $m = M$. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy of the falling block and the rotational kinetic energy of the cylinder? (i) The block has more kinetic energy than the cylinder. (ii) The block has less kinetic energy than the cylinder. (iii) The block and the cylinder have equal amounts of kinetic energy.

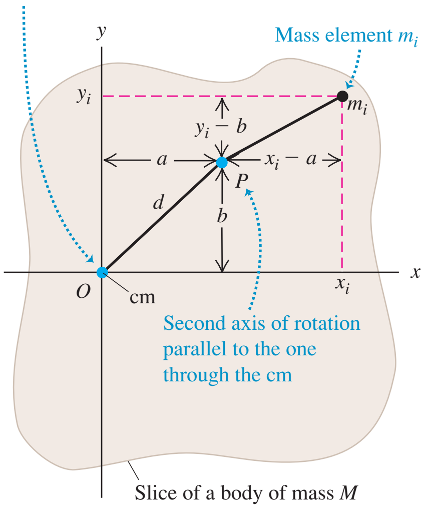
9.5 Parallel-Axis Theorem

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship between the moment of inertia I_{cm} of a body of mass M about an axis through its center of mass and the moment of inertia I_P about any other axis parallel to the original one but displaced from it by a distance d. This relationship, called the **parallel-axis theorem,** states that

$$I_P = I_{cm} + Md^2 \quad \text{(parallel-axis theorem)} \quad (9.19)$$

To prove this theorem, we consider two axes, both parallel to the z-axis: one through the center of mass and the other through a point P (Fig. 9.19). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{cm} = y_{cm} = z_{cm} = 0$. The axis through the center of mass passes through this thin slice at point O, and the parallel axis passes through point P, whose x- and y-coordinates are (a, b). The distance of this axis from the axis through the center of mass is d, where $d^2 = a^2 + b^2$.

9.19 The mass element m_i has coordinates (x_i, y_i) with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P.



We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at O) is

$$I_{cm} = \sum_i m_i(x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through P. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

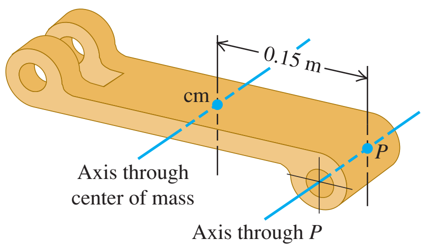
The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{cm} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

Example 9.9 Using the parallel-axis theorem

A part of a mechanical linkage (Fig. 9.20) has a mass of 3.6 kg. Its moment of inertia I_P about an axis 0.15 m from its center of mass is $I_P = 0.132\ \text{kg}\cdot\text{m}^2$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?

9.20 Calculating I_{cm} from a measurement of I_P.



SOLUTION

IDENTIFY, SET UP, and EXECUTE: We'll determine the target variable I_{cm} using the parallel-axis theorem, Eq. (9.19). Rearranging the equation, we obtain

$$\begin{aligned} I_{cm} &= I_P - Md^2 = 0.132\ \text{kg}\cdot\text{m}^2 - (3.6\ \text{kg})(0.15\ \text{m})^2 \\ &= 0.051\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

EVALUATE: As we expect, I_{cm} is less than I_P; the moment of inertia for an axis through the center of mass is lower than for any other parallel axis.

Test Your Understanding of Section 9.5 A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Use the parallel-axis theorem to decide whether a pool cue has a larger moment of inertia (i) for an axis through the thicker end of the rod and perpendicular to the length of the rod, or (ii) for an axis through the thinner end of the rod and perpendicular to the length of the rod.

9.6 Moment-of-Inertia Calculations

If a rigid body is a continuous distribution of mass—like a solid cylinder or a solid sphere—it cannot be represented by a few point masses. In this case the *sum* of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an *integral*. Imagine dividing the body into elements of mass dm that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance r, as before. Then the moment of inertia is

$$I = \int r^2\, dm \tag{9.20}$$

To evaluate the integral, we have to represent r and dm in terms of the same integration variable. When the object is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate x along the length and relate dm to an increment dx. For a three-dimensional object it is usually easiest to express dm in terms of an element of volume dV and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may also write Eq. (9.20) as

$$I = \int r^2 \rho\, dV$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume (Fig. 9.21). If the body is uniform in density, then we may take ρ outside the integral:

$$I = \rho \int r^2\, dV \tag{9.21}$$

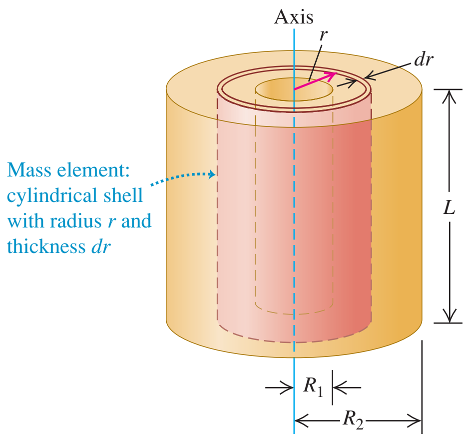
To use this equation, we have to express the volume element dV in terms of the differentials of the integration variables, such as $dV = dx\, dy\, dz$. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

9.21 By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers.

Example 9.10 Hollow or solid cylinder, rotating about axis of symmetry

Figure 9.22 shows a hollow cylinder of uniform mass density ρ with length L, inner radius R_1, and outer radius R_2. (It might be a steel cylinder in a printing press.) Using integration, find its moment of inertia about its axis of symmetry.

9.22 Finding the moment of inertia of a hollow cylinder about its symmetry axis.



SOLUTION

IDENTIFY and SET UP: We choose as a volume element a thin cylindrical shell of radius r, thickness dr, and length L. All parts of this shell are at very nearly the same distance r from the axis. The volume of the shell is very nearly that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell). The mass of the shell is

$$dm = \rho\, dV = \rho(2\pi r L\, dr)$$

We'll use this expression in Eq. (9.20), integrating from $r = R_1$ to $r = R_2$.

EXECUTE: From Eq. (9.20), the moment of inertia is

$$\begin{aligned} I &= \int r^2\, dm = \int_{R_1}^{R_2} r^2 \rho(2\pi r L\, dr) \\ &= 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr \\ &= \frac{2\pi\rho L}{4}(R_2^4 - R_1^4) \\ &= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2) \end{aligned}$$

(In the last step we used the identity $a^2 - b^2 = (a - b)(a + b)$.) Let's express this result in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The cylinder's volume is

$$V = \pi L(R_2^2 - R_1^2)$$

so its total mass M is

$$M = \rho V = \pi L\rho(R_2^2 - R_1^2)$$

Continued

Comparing with the above expression for I, we see that

$$I = \tfrac{1}{2}M(R_1^2 + R_2^2)$$

EVALUATE: Our result agrees with Table 9.2, case (e). If the cylinder is solid, with outer radius $R_2 = R$ and inner radius $R_1 = 0$, its moment of inertia is

$$I = \tfrac{1}{2}MR^2$$

in agreement with case (f). If the cylinder wall is very thin, we have $R_1 \approx R_2 = R$ and the moment of inertia is

$$I = MR^2$$

in agreement with case (g). We could have predicted this last result without calculation; in a thin-walled cylinder, all the mass is at the same distance $r = R$ from the axis, so $I = \int r^2\,dm = R^2 \int dm = MR^2$.

Example 9.11 Uniform sphere with radius R, axis through center

Find the moment of inertia of a solid sphere of uniform mass density ρ (like a billiard ball) about an axis through its center.

SOLUTION

IDENTIFY and SET UP: We divide the sphere into thin, solid disks of thickness dx (Fig. 9.23), whose moment of inertia we know from Table 9.2, case (f). We'll integrate over these to find the total moment of inertia.

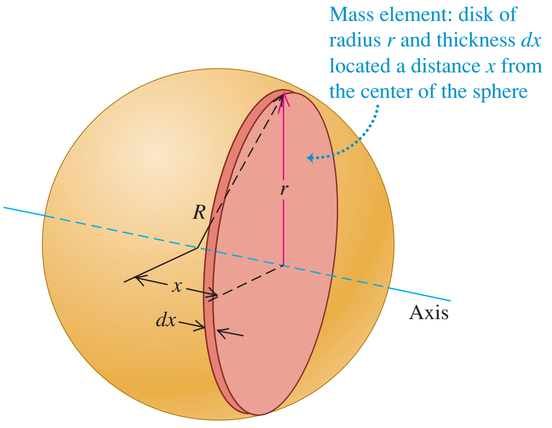
EXECUTE: The radius and hence the volume and mass of a disk depend on its distance x from the center of the sphere. The radius r of the disk shown in Fig. 9.23 is

$$r = \sqrt{R^2 - x^2}$$

Its volume is

$$dV = \pi r^2\,dx = \pi(R^2 - x^2)\,dx$$

9.23 Finding the moment of inertia of a sphere about an axis through its center.



and so its mass is

$$dm = \rho\,dV = \pi\rho(R^2 - x^2)\,dx$$

From Table 9.2, case (f), the moment of inertia of a disk of radius r and mass dm is

$$dI = \tfrac{1}{2}r^2\,dm = \tfrac{1}{2}(R^2 - x^2)[\pi\rho(R^2 - x^2)\,dx] = \frac{\pi\rho}{2}(R^2 - x^2)^2\,dx$$

Integrating this expression from $x = 0$ to $x = R$ gives the moment of inertia of the right hemisphere. The total I for the entire sphere, including both hemispheres, is just twice this:

$$I = (2)\frac{\pi\rho}{2}\int_0^R (R^2 - x^2)^2\,dx$$

Carrying out the integration, we find

$$I = \frac{8\pi\rho R^5}{15}$$

The volume of the sphere is $V = 4\pi R^3/3$, so in terms of its mass M its density is

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

Hence our expression for I becomes

$$I = \left(\frac{8\pi R^5}{15}\right)\left(\frac{3M}{4\pi R^3}\right) = \tfrac{2}{5}MR^2$$

EVALUATE: This is just as in Table 9.2, case (h). Note that the moment of inertia $I = \frac{2}{5}MR^2$ of a solid sphere of mass M and radius R is less than the moment of inertia $I = \frac{1}{2}MR^2$ of a solid *cylinder* of the same mass and radius, because more of the sphere's mass is located close to the axis.

Test Your Understanding of Section 9.6 Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry? (i) the wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

CHAPTER 9 SUMMARY

Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the z-axis), its position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. (See Examples 9.1 and 9.2.) If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration. (See Example 9.3.)

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.3}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \qquad (9.5), (9.6)$$

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \tag{9.11}$$

(constant α_z only)

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \tag{9.10}$$

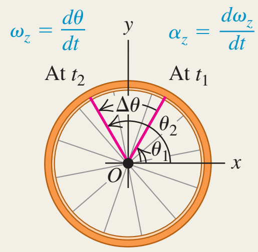
(constant α_z only)

$$\omega_z = \omega_{0z} + \alpha_z t \tag{9.7}$$

(constant α_z only)

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \tag{9.12}$$

(constant α_z only)

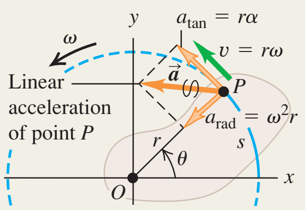


Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of its angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α. (See Examples 9.4 and 9.5.)

$$v = r\omega \tag{9.13}$$

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{9.14}$$

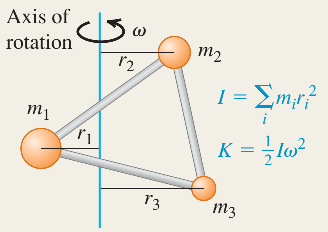
$$a_{rad} = \frac{v^2}{r} = \omega^2 r \tag{9.15}$$



Moment of inertia and rotational kinetic energy: The moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body's rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis. (See Examples 9.6–9.8.)

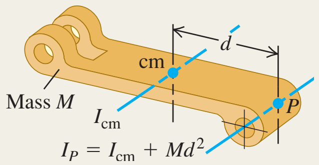
$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \tag{9.16}$$

$$K = \tfrac{1}{2}I\omega^2 \tag{9.17}$$



Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). (See Example 9.9.) If the body has a continuous mass distribution, the moment of inertia can be calculated by integration. (See Examples 9.10 and 9.11.)

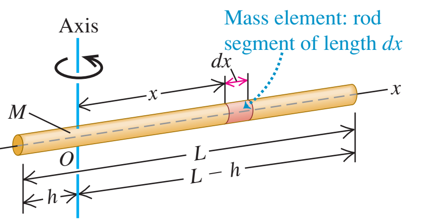
$$I_P = I_{cm} + Md^2 \tag{9.19}$$



BRIDGING PROBLEM A Rotating, Uniform Thin Rod

Figure 9.24 shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (less the rubber end caps). (a) Use integration to compute its moment of inertia about an axis through O, at an arbitrary distance h from one end. (b) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t. (c) At time t, what is the *linear* acceleration of the point on the rod farthest from the axis?

9.24 A thin rod with an axis through O.



SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Make a list of the target variables for this problem.
2. To calculate the moment of inertia of the rod, you'll have to divide the rod into infinitesimal elements of mass. If an element has length dx, what is the mass of the element? What are the limits of integration?
3. What is the angular speed of the rod at time t? How does the work required to accelerate the rod from rest to this angular speed compare to the rod's kinetic energy at time t?
4. At time t, does the point on the rod farthest from the axis have a centripetal acceleration? A tangential acceleration? Why or why not?

EXECUTE

5. Do the integration required to find the moment of inertia.
6. Use your result from step 5 to calculate the work done in time t to accelerate the rod from rest.
7. Find the linear acceleration components for the point in question at time t. Use these to find the magnitude of the acceleration.

EVALUATE

8. Check your results for the special cases $h = 0$ (the axis passes through one end of the rod) and $h = L/2$ (the axis passes through the middle of the rod). Are these limits consistent with Table 9.2? With the parallel-axis theorem?
9. Is the acceleration magnitude from step 7 constant? Would you expect it to be?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

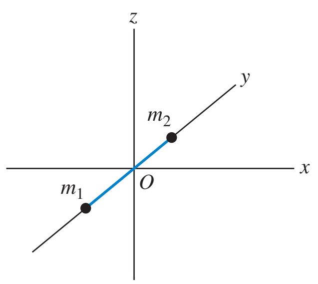
•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q9.1 Which of the following formulas is valid if the angular acceleration of an object is *not* constant? Explain your reasoning in each case. (a) $v = r\omega$; (b) $a_{\tan} = r\alpha$; (c) $\omega = \omega_0 + \alpha t$; (d) $a_{\tan} = r\omega^2$; (e) $K = \frac{1}{2}I\omega^2$.

Q9.2 A diatomic molecule can be modeled as two point masses, m_1 and m_2, slightly separated (Fig. Q9.2). If the molecule is oriented along the y-axis, it has kinetic energy K when it spins about the x-axis. What will its kinetic energy (in terms of K) be if it spins at the same angular speed about (a) the z-axis and (b) the y-axis?

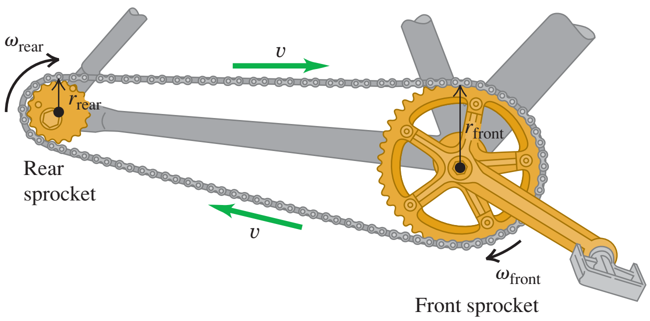
Figure **Q9.2**



Q9.3 What is the difference between tangential and radial acceleration for a point on a rotating body?

Q9.4 In Fig. Q9.4, all points on the chain have the same linear speed. Is the magnitude of the linear acceleration also the same for all points on the chain? How are the angular accelerations of the two sprockets related? Explain.

Figure **Q9.4**



Q9.5 In Fig. Q9.4, how are the radial accelerations of points at the teeth of the two sprockets related? Explain the reasoning behind your answer.

Q9.6 A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? A radial acceleration? Are these accelerations constant in magnitude? In direction? In each case give the reasoning behind your answer.

Q9.7 What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.

Q9.8 Although angular velocity and angular acceleration can be treated as vectors, the angular displacement θ, despite having a magnitude and a direction, cannot. This is because θ does not follow the commutative law of vector addition (Eq. 1.3). Prove this to yourself in the following way: Lay your physics textbook flat on the desk in front of you with the cover side up so you can read the writing on it. Rotate it through 90° about a horizontal axis so that the farthest edge comes toward you. Call this angular displacement θ_1. Then rotate it by 90° about a vertical axis so that the left edge comes toward you. Call this angular displacement θ_2. The spine of the book should now face you, with the writing on it oriented so that you can read it. Now start over again but carry out the two rotations in the reverse order. Do you get a different result? That is, does $\theta_1 + \theta_2$ equal $\theta_2 + \theta_1$? Now repeat this experiment but this time with an angle of 1° rather than 90°. Do you think that the infinitesimal displacement $d\vec{\boldsymbol{\theta}}$ obeys the commutative law of addition and hence qualifies as a vector? If so, how is the direction of $d\vec{\boldsymbol{\theta}}$ related to the direction of $\vec{\boldsymbol{\omega}}$?

Q9.9 Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

Q9.10 To maximize the moment of inertia of a flywheel while minimizing its weight, what shape and distribution of mass should it have? Explain.

Q9.11 How might you determine experimentally the moment of inertia of an irregularly shaped body about a given axis?

Q9.12 A cylindrical body has mass M and radius R. Can the mass be distributed within the body in such a way that its moment of inertia about its axis of symmetry is greater than MR^2? Explain.

Q9.13 Describe how you could use part (b) of Table 9.2 to derive the result in part (d).

Q9.14 A hollow spherical shell of radius R that is rotating about an axis through its center has rotational kinetic energy K. If you want to modify this sphere so that it has three times as much kinetic energy at the same angular speed while keeping the same mass, what should be its radius in terms of R?

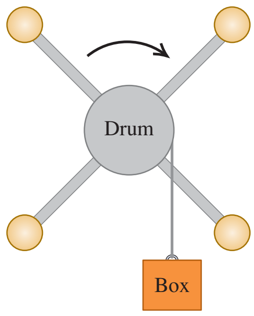
Q9.15 For the equations for I given in parts (a) and (b) of Table 9.2 to be valid, must the rod have a circular cross section? Is there any restriction on the size of the cross section for these equations to apply? Explain.

Q9.16 In part (d) of Table 9.2, the thickness of the plate must be much less than a for the expression given for I to apply. But in part (c), the expression given for I applies no matter how thick the plate is. Explain.

Q9.17 Two identical balls, A and B, are each attached to very light string, and each string is wrapped around the rim of a frictionless pulley of mass M. The only difference is that the pulley for ball A is a solid disk, while the one for ball B is a hollow disk, like part (e) in Table 9.2. If both balls are released from rest and fall the same distance, which one will have more kinetic energy, or will they have the same kinetic energy? Explain your reasoning.

Q9.18 An elaborate pulley consists of four identical balls at the ends of spokes extending out from a rotating drum (Fig. Q9.18). A box is connected to a light thin rope wound around the rim of the drum. When it is released from rest, the box acquires a speed V after having fallen a distance d. Now the four balls are moved inward closer to the drum, and the box is again released from rest. After it has fallen a distance d, will its speed be equal to V, greater than V, or less than V? Show or explain why.

Figure **Q9.18**



Q9.19 You can use any angular measure—radians, degrees, or revolutions—in some of the equations in Chapter 9, but you can use only radian measure in others. Identify those for which using radians is necessary and those for which it is not, and in each case give the reasoning behind your answer.

Q9.20 When calculating the moment of inertia of an object, can we treat all its mass as if it were concentrated at the center of mass of the object? Justify your answer.

Q9.21 A wheel is rotating about an axis perpendicular to the plane of the wheel and passing through the center of the wheel. The angular speed of the wheel is increasing at a constant rate. Point A is on the rim of the wheel and point B is midway between the rim and center of the wheel. For each of the following quantities, is its magnitude larger at point A or at point B, or is it the same at both points? (a) angular speed; (b) tangential speed; (c) angular acceleration; (d) tangential acceleration; (e) radial acceleration. Justify each of your answers.

Q9.22 Estimate your own moment of inertia about a vertical axis through the center of the top of your head when you are standing up straight with your arms outstretched. Make reasonable approximations and measure or estimate necessary quantities.

EXERCISES

Section 9.1 Angular Velocity and Acceleration

9.1 • (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

9.2 • An airplane propeller is rotating at 1900 rpm (rev/min). (a) Compute the propeller's angular velocity in rad/s. (b) How many seconds does it take for the propeller to turn through 35°?

9.3 • **CP CALC** The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω_z is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0.00$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s? (*Hint:* See Section 2.6.)

9.4 •• **CALC** A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3. (a) Calculate the angular acceleration as a function of time. (b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{av\text{-}z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why are they different?

9.5 •• **CALC** A child is pushing a merry-go-round. The angle through which the merry-go-round has turned varies with time according to $\theta(t) = \gamma t + \beta t^3$, where $\gamma = 0.400$ rad/s and $\beta = 0.0120$ rad/s^3. (a) Calculate the angular velocity of the merry-go-round as a function of time. (b) What is the initial value of the angular velocity? (c) Calculate the instantaneous value of the angular velocity ω_z at $t = 5.00$ s and the average angular velocity $\omega_{av\text{-}z}$ for the time interval $t = 0$ to $t = 5.00$ s. Show that $\omega_{av\text{-}z}$ is *not* equal to the average of the instantaneous angular velocities at $t = 0$ and $t = 5.00$ s, and explain why it is not.

9.6 • **CALC** At $t = 0$ the current to a dc electric motor is reversed, resulting in an angular displacement of the motor shaft given by $\theta(t) = (250\text{ rad/s})t - (20.0\text{ rad/s}^2)t^2 - (1.50\text{ rad/s}^3)t^3$. (a) At what time is the angular velocity of the motor shaft zero? (b) Calculate the angular acceleration at the instant that the motor shaft has zero angular velocity. (c) How many revolutions does the motor shaft turn through between the time when the current is reversed and the instant when the angular velocity is zero? (d) How fast was the motor shaft rotating at $t = 0$, when the current was reversed? (e) Calculate the average angular velocity for the time period from $t = 0$ to the time calculated in part (a).

9.7 • **CALC** The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is 2.00 rad/s, and when $t = 1.50$ s, the angular acceleration is 1.25 rad/s^2. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is 3.50 rad/s^2?

9.8 • A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is -6.00 rad/s at $t = 0$, increases linearly with time, and is $+8.00$ rad/s at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Section 9.2 Rotation with Constant Angular Acceleration

9.9 • A bicycle wheel has an initial angular velocity of 1.50 rad/s. (a) If its angular acceleration is constant and equal to 0.300 rad/s^2, what is its angular velocity at $t = 2.50$ s? (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?

9.10 •• An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4.00 s. (a) Find the angular acceleration in rev/s^2 and the number of revolutions made by the motor in the 4.00-s interval. (b) How many more seconds are required for the fan to come to rest if the angular acceleration remains constant at the value calculated in part (a)?

9.11 •• The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

9.12 • (a) Derive Eq. (9.12) by combining Eqs. (9.7) and (9.11) to eliminate t. (b) The angular velocity of an airplane propeller increases from 12.0 rad/s to 16.0 rad/s while turning through 7.00 rad. What is the angular acceleration in rad/s^2?

9.13 •• A turntable rotates with a constant 2.25 rad/s^2 angular acceleration. After 4.00 s it has rotated through an angle of 60.0 rad. What was the angular velocity of the wheel at the beginning of the 4.00-s interval?

9.14 • A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

9.15 •• A high-speed flywheel in a motor is spinning at 500 rpm when a power failure suddenly occurs. The flywheel has mass 40.0 kg and diameter 75.0 cm. The power is off for 30.0 s, and during this time the flywheel slows due to friction in its axle bearings. During the time the power is off, the flywheel makes 200 complete revolutions. (a) At what rate is the flywheel spinning when the power comes back on? (b) How long after the beginning of the power failure would it have taken the flywheel to stop if the power had not come back on, and how many revolutions would the wheel have made during this time?

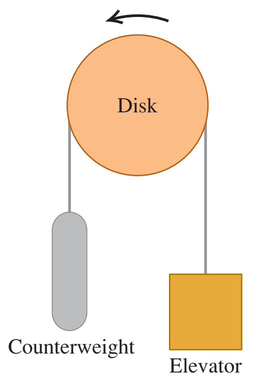
9.16 •• At $t = 0$ a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. (a) Through what total angle did the wheel turn between $t = 0$ and the time it stopped? (b) At what time did it stop? (c) What was its acceleration as it slowed down?

9.17 •• A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_3 that was three times as great, $\omega_3 = 3\omega_1$?

Section 9.3 Relating Linear and Angular Kinematics

9.18 • In a charming 19th-century hotel, an old-style elevator is connected to a counterweight by a cable that passes over a rotating disk 2.50 m in diameter (Fig. E9.18). The elevator is raised and lowered by turning the disk, and the cable does not slip on the rim of the disk but turns with it. (a) At how many rpm must the disk turn to raise the elevator at 25.0 cm/s? (b) To start the elevator moving, it must be accelerated at $\frac{1}{8}g$. What must be the angular acceleration of the disk, in rad/s^2? (c) Through what angle (in radians and degrees) has the disk turned when it has raised the elevator 3.25 m between floors?

Figure **E9.18**



9.19 • Using astronomical data from Appendix F, along with the fact that the earth spins on its axis once per day, calculate (a) the earth's orbital angular speed (in rad/s) due to its motion around the sun, (b) its angular speed (in rad/s) due to its axial spin, (c) the tangential speed of the earth around the sun (assuming a circular orbit), (d) the tangential speed of a point on the earth's equator due to the planet's axial spin, and (e) the radial and tangential acceleration components of the point in part (d).

9.20 • **Compact Disc.** A compact disc (CD) stores music in a coded pattern of tiny pits 10^{-7} m deep. The pits are arranged in a track that spirals outward toward the rim of the disc; the inner and outer radii of this spiral are 25.0 mm and 58.0 mm, respectively. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of 1.25 m/s. (a) What is the angular speed of the CD when the innermost part of the track is scanned? The outermost part of the track? (b) The maximum playing time of a CD is 74.0 min. What would be the length of the track on such a maximum-duration CD if it were stretched out in a straight line? (c) What is the average angular acceleration of a maximum-duration CD during its 74.0-min playing time? Take the direction of rotation of the disc to be positive.

9.21 •• A wheel of diameter 40.0 cm starts from rest and rotates with a constant angular acceleration of 3.00 rad/s^2. At the instant the wheel has computed its second revolution, compute the radial acceleration of a point on the rim in two ways: (a) using the relationship $a_{rad} = \omega^2 r$ and (b) from the relationship $a_{rad} = v^2/r$.

9.22 •• You are to design a rotating cylindrical axle to lift 800-N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to a hook on the free end of a cable that wraps around the rim of the axle; as the axle turns, the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

9.23 • A flywheel with a radius of 0.300 m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s^2. Compute the magnitude of the tangential acceleration, the radial acceleration, and the resultant acceleration of a point on its rim (a) at the start; (b) after it has turned through 60.0°; (c) after it has turned through 120.0°.

9.24 •• An electric turntable 0.750 m in diameter is rotating about a fixed axis with an initial angular velocity of 0.250 rev/s and a constant angular acceleration of 0.900 rev/s^2. (a) Compute the angular velocity of the turntable after 0.200 s. (b) Through how many revolutions has the turntable spun in this time interval? (c) What is the tangential speed of a point on the rim of the turntable at $t = 0.200$ s? (d) What is the magnitude of the *resultant* acceleration of a point on the rim at $t = 0.200$ s?

9.25 •• **Centrifuge.** An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of 3000g at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

9.26 • (a) Derive an equation for the radial acceleration that includes v and ω, but not r. (b) You are designing a merry-go-round for which a point on the rim will have a radial acceleration of 0.500 m/s^2 when the tangential velocity of that point has magnitude 2.00 m/s. What angular velocity is required to achieve these values?

9.27 • **Electric Drill.** According to the shop manual, when drilling a 12.7-mm-diameter hole in wood, plastic, or aluminum, a drill should have a speed of 1250 rev/min. For a 12.7-mm-diameter drill bit turning at a constant 1250 rev/min, find (a) the maximum linear speed of any part of the bit and (b) the maximum radial acceleration of any part of the bit.

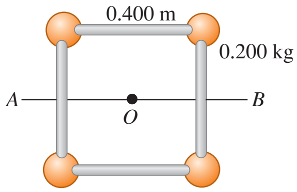
9.28 • At $t = 3.00$ s a point on the rim of a 0.200-m-radius wheel has a tangential speed of 50.0 m/s as the wheel slows down with a tangential acceleration of constant magnitude 10.0 m/s^2. (a) Calculate the wheel's constant angular acceleration. (b) Calculate the angular velocities at $t = 3.00$ s and $t = 0$. (c) Through what angle did the wheel turn between $t = 0$ and $t = 3.00$ s? (d) At what time will the radial acceleration equal g?

9.29 • The spin cycles of a washing machine have two angular speeds, 423 rev/min and 640 rev/min. The internal diameter of the drum is 0.470 m. (a) What is the ratio of the maximum radial force on the laundry for the higher angular speed to that for the lower speed? (b) What is the ratio of the maximum tangential speed of the laundry for the higher angular speed to that for the lower speed? (c) Find the laundry's maximum tangential speed and the maximum radial acceleration, in terms of g.

Section 9.4 Energy in Rotational Motion

9.30 • Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods (Fig. E9.30). Find the moment of inertia of the system about an axis (a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); (b) bisecting two opposite sides of the square (an axis along the line AB in the figure); (c) that passes through the centers of the upper left and lower right spheres and through point O.

Figure **E9.30**



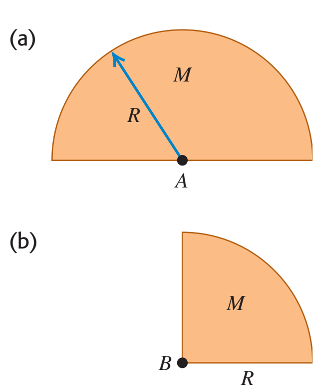
9.31 • Calculate the moment of inertia of each of the following uniform objects about the axes indicated. Consult Table 9.2 as needed. (a) A thin 2.50-kg rod of length 75.0 cm, about an axis perpendicular to it and passing through (i) one end and (ii) its center, and (iii) about an axis parallel to the rod and passing through it. (b) A 3.00-kg sphere 38.0 cm in diameter, about an axis through its center, if the sphere is (i) solid and (ii) a thin-walled hollow shell. (c) An 8.00-kg cylinder, of length 19.5 cm and diameter 12.0 cm, about the central axis of the cylinder, if the cylinder is (i) thin-walled and hollow, and (ii) solid.

9.32 •• Small blocks, each with mass m, are clamped at the ends and at the center of a rod of length L and negligible mass. Compute the moment of inertia of the system about an axis perpendicular to the rod and passing through (a) the center of the rod and (b) a point one-fourth of the length from one end.

9.33 • A uniform bar has two small balls glued to its ends. The bar is 2.00 m long and has mass 4.00 kg, while the balls each have mass 0.500 kg and can be treated as point masses. Find the moment of inertia of this combination about each of the following axes: (a) an axis perpendicular to the bar through its center; (b) an axis perpendicular to the bar through one of the balls; (c) an axis parallel to the bar through both balls; (d) an axis parallel to the bar and 0.500 m from it.

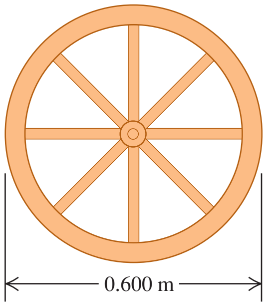
9.34 • A uniform disk of radius R is cut in half so that the remaining half has mass M (Fig. E9.34a). (a) What is the moment of inertia of this half about an axis perpendicular to its plane through point A? (b) Why did your answer in part (a) come out the same as if this were a complete disk of mass M? (c) What would be the moment of inertia of a quarter disk of mass M and radius R about an axis perpendicular to its plane passing through point B (Fig. E9.34b)?

Figure **E9.34**



9.35 •• A wagon wheel is constructed as shown in Fig. E9.35. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along a diameter and are 0.300 m long has mass 0.280 kg. What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel? (Use the formulas given in Table 9.2.)

Figure **E9.35**



9.36 •• An airplane propeller is 2.08 m in length (from tip to tip) with mass 117 kg and is rotating at 2400 rpm (rev/min) about an axis through its center. You can model the propeller as a slender rod. (a) What is its rotational kinetic energy? (b) Suppose that, due to weight constraints, you had to reduce the propeller's mass to 75.0% of its original mass, but you still needed to keep the same size and kinetic energy. What would its angular speed have to be, in rpm?

9.37 •• A compound disk of outside diameter 140.0 cm is made up of a uniform solid disk of radius 50.0 cm and area density 3.00 g/cm^2 surrounded by a concentric ring of inner radius 50.0 cm, outer radius 70.0 cm, and area density 2.00 g/cm^2. Find the moment of inertia of this object about an axis perpendicular to the plane of the object and passing through its center.

9.38 • A wheel is turning about an axis through its center with constant angular acceleration. Starting from rest, at $t = 0$, the wheel turns through 8.20 revolutions in 12.0 s. At $t = 12.0$ s the kinetic energy of the wheel is 36.0 J. For an axis through its center, what is the moment of inertia of the wheel?

9.39 • A uniform sphere with mass 28.0 kg and radius 0.380 m is rotating at constant angular velocity about a stationary axis that lies along a diameter of the sphere. If the kinetic energy of the sphere is 176 J, what is the tangential velocity of a point on the rim of the sphere?

9.40 •• A hollow spherical shell has mass 8.20 kg and radius 0.220 m. It is initially at rest and then rotates about a stationary axis that lies along a diameter with a constant acceleration of 0.890 rad/s^2. What is the kinetic energy of the shell after it has turned through 6.00 rev?

9.41 • **Energy from the Moon?** Suppose that some time in the future we decide to tap the moon's rotational energy for use on earth. In additional to the astronomical data in Appendix F, you may need to know that the moon spins on its axis once every 27.3 days. Assume that the moon is uniform throughout. (a) How much total energy could we get from the moon's rotation? (b) The world presently uses about 4.0×10^{20} J of energy per year. If in the future the world uses five times as much energy yearly, for how many years would the moon's rotation provide us energy? In light of your answer, does this seem like a cost-effective energy source in which to invest?

9.42 •• You need to design an industrial turntable that is 60.0 cm in diameter and has a kinetic energy of 0.250 J when turning at 45.0 rpm (rev/min). (a) What must be the moment of inertia of the turntable about the rotation axis? (b) If your workshop makes this turntable in the shape of a uniform solid disk, what must be its mass?

9.43 •• The flywheel of a gasoline engine is required to give up 500 J of kinetic energy while its angular velocity decreases from 650 rev/min to 520 rev/min. What moment of inertia is required?

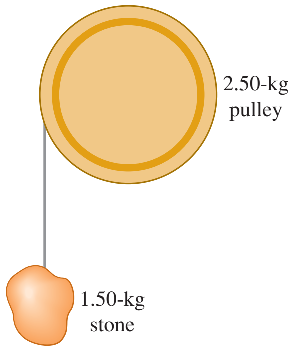
9.44 • A light, flexible rope is wrapped several times around a *hollow* cylinder, with a weight of 40.0 N and a radius of 0.25 m, that rotates without friction about a fixed horizontal axis. The cylinder is attached to the axle by spokes of a negligible moment of inertia. The cylinder is initially at rest. The free end of the rope is pulled with a constant force P for a distance of 5.00 m, at which point the end of the rope is moving at 6.00 m/s. If the rope does not slip on the cylinder, what is the value of P?

9.45 •• Energy is to be stored in a 70.0-kg flywheel in the shape of a uniform solid disk with radius $R = 1.20$ m. To prevent structural failure of the flywheel, the maximum allowed radial acceleration of a point on its rim is 3500 m/s^2. What is the maximum kinetic energy that can be stored in the flywheel?

9.46 •• Suppose the solid cylinder in the apparatus described in Example 9.8 (Section 9.4) is replaced by a thin-walled, hollow cylinder with the same mass M and radius R. The cylinder is attached to the axle by spokes of a negligible moment of inertia. (a) Find the speed of the hanging mass m just as it strikes the floor. (b) Use energy concepts to explain why the answer to part (a) is different from the speed found in Example 9.8.

9.47 •• A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50-kg stone is attached to a very light wire that is wrapped around the rim of the pulley (Fig. E9.47), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

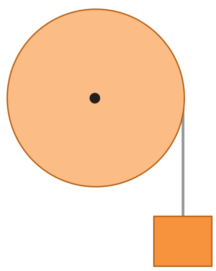
Figure **E9.47**



9.48 •• A bucket of mass m is tied to a massless cable that is wrapped around the outer rim of a frictionless uniform pulley of radius R, similar to the system shown in Fig. E9.47. In terms of the stated variables, what must be the moment of inertia of the pulley so that it always has half as much kinetic energy as the bucket?

9.49 •• **CP** A thin, light wire is wrapped around the rim of a wheel, as shown in Fig. E9.49. The wheel rotates without friction about a stationary horizontal axis that passes through the center of the wheel. The wheel is a uniform disk with radius $R = 0.280$ m. An object of mass $m = 4.20$ kg is suspended from the free end of the wire. The system is released from rest and the suspended object descends with constant acceleration. If the suspended object moves downward a distance of 3.00 m in 2.00 s, what is the mass of the wheel?

Figure **E9.49**



9.50 •• A uniform 2.00-m ladder of mass 9.00 kg is leaning against a vertical wall while making an angle of 53.0° with the floor. A worker pushes the ladder up against the wall until it is vertical. What is the increase in the gravitational potential energy of the ladder?

9.51 •• **How *I* Scales.** If we multiply all the design dimensions of an object by a scaling factor f, its volume and mass will be multiplied by f^3. (a) By what factor will its moment of inertia be multiplied? (b) If a $\frac{1}{48}$-scale model has a rotational kinetic energy of 2.5 J, what will be the kinetic energy for the full-scale object of the same material rotating at the same angular velocity?

9.52 •• A uniform 3.00-kg rope 24.0 m long lies on the ground at the top of a vertical cliff. A mountain climber at the top lets down half of it to help his partner climb up the cliff. What was the change in potential energy of the rope during this maneuver?

Section 9.5 Parallel-Axis Theorem

9.53 •• About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, hollow, lead sphere of the same mass and radius, with the axis along a diameter?

9.54 •• Find the moment of inertia of a hoop (a thin-walled, hollow ring) with mass M and radius R about an axis perpendicular to the hoop's plane at an edge.

9.55 •• A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

9.56 • (a) For the thin rectangular plate shown in part (d) of Table 9.2, find the moment of inertia about an axis that lies in the plane of the plate, passes through the center of the plate, and is parallel to the axis shown in the figure. (b) Find the moment of inertia of the plate for an axis that lies in the plane of the plate, passes through the center of the plate, and is perpendicular to the axis in part (a).

9.57 •• A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through (a) the point where the two segments meet and (b) the midpoint of the line connecting its two ends.

Section 9.6 Moment-of-Inertia Calculations

9.58 • **CALC** Use Eq. (9.20) to calculate the moment of inertia of a slender, uniform rod with mass M and length L about an axis at one end, perpendicular to the rod.

9.59 •• **CALC** Use Eq. (9.20) to calculate the moment of inertia of a uniform, solid disk with mass M and radius R for an axis perpendicular to the plane of the disk and passing through its center.

9.60 •• **CALC** A slender rod with length L has a mass per unit length that varies with distance from the left end, where $x = 0$, according to $dm/dx = \gamma x$, where γ has units of kg/m^2. (a) Calculate the total mass of the rod in terms of γ and L. (b) Use Eq. (9.20) to calculate the moment of inertia of the rod for an axis at the left end, perpendicular to the rod. Use the expression you derived in part (a) to express I in terms of M and L. How does your result compare to that for a uniform rod? Explain this comparison. (c) Repeat part (b) for an axis at the right end of the rod. How do the results for parts (b) and (c) compare? Explain this result.

PROBLEMS

9.61 • **CP CALC** A flywheel has angular acceleration $\alpha_z(t) = 8.60\ \text{rad/s}^2 - (2.30\ \text{rad/s}^3)t$, where counterclockwise rotation is positive. (a) If the flywheel is at rest at $t = 0$, what is its angular velocity at 5.00 s? (b) Through what angle (in radians) does the flywheel turn in the time interval from $t = 0$ to $t = 5.00$ s?

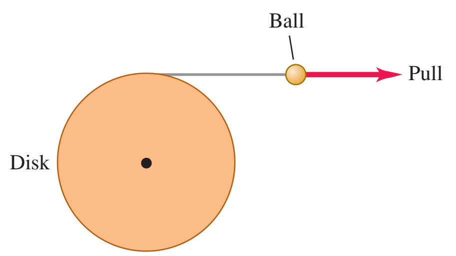
9.62 •• **CALC** A uniform disk with radius $R = 0.400$ m and mass 30.0 kg rotates in a horizontal plane on a frictionless vertical axle that passes through the center of the disk. The angle through which the disk has turned varies with time according to $\theta(t) = (1.10\ \text{rad/s})t + (8.60\ \text{rad/s}^2)t^2$. What is the resultant linear acceleration of a point on the rim of the disk at the instant when the disk has turned through 0.100 rev?

9.63 •• **CP** A circular saw blade with radius 0.120 m starts from rest and turns in a vertical plane with a constant angular acceleration of $3.00\ \text{rev/s}^2$. After the blade has turned through 155 rev, a small piece of the blade breaks loose from the top of the blade. After the piece breaks loose, it travels with a velocity that is initially horizontal and equal to the tangential velocity of the rim of the blade. The piece travels a vertical distance of 0.820 m to the floor. How far does the piece travel horizontally, from where it broke off the blade until it strikes the floor?

9.64 • **CALC** A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20\ \text{rad/s}^2$ and $\beta = 0.500\ \text{rad/s}^3$. (a) Calculate the angular velocity of the roller as a function of time. (b) Calculate the angular acceleration of the roller as a function of time. (c) What is the maximum positive angular velocity, and at what value of t does it occur?

9.65 •• **CP CALC** A disk of radius 25.0 cm is free to turn about an axle perpendicular to it through its center. It has very thin but strong string wrapped around its rim, and the string is attached to a ball that is pulled tangentially away from the rim of the disk (Fig. P9.65). The pull increases in magnitude and produces an acceleration of the ball that obeys the equation $a(t) = At$, where t is in seconds and A is a constant. The cylinder starts from rest, and at the end of the third second, the ball's acceleration is $1.80\ \text{m/s}^2$. (a) Find A. (b) Express the angular acceleration of the disk as a function of time. (c) How much time after the disk has begun to turn does it reach an angular speed of 15.0 rad/s? (d) Through what angle has the disk turned just as it reaches 15.0 rad/s? (*Hint:* See Section 2.6.)

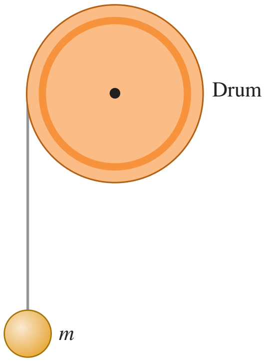
Figure **P9.65**



9.66 •• When a toy car is rapidly scooted across the floor, it stores energy in a flywheel. The car has mass 0.180 kg, and its flywheel has moment of inertia $4.00 \times 10^{-5}\ \text{kg} \cdot \text{m}^2$. The car is 15.0 cm long. An advertisement claims that the car can travel at a scale speed of up to 700 km/h (440 mi/h). The scale speed is the speed of the toy car multiplied by the ratio of the length of an actual car to the length of the toy. Assume a length of 3.0 m for a real car. (a) For a scale speed of 700 km/h, what is the actual translational speed of the car? (b) If all the kinetic energy that is initially in the flywheel is converted to the translational kinetic energy of the toy, how much energy is originally stored in the flywheel? (c) What initial angular velocity of the flywheel was needed to store the amount of energy calculated in part (b)?

9.67 • A classic 1957 Chevrolet Corvette of mass 1240 kg starts from rest and speeds up with a constant tangential acceleration of $2.00\ \text{m/s}^2$ on a circular test track of radius 60.0 m. Treat the car as a particle. (a) What is its angular acceleration? (b) What is its angular speed 6.00 s after it starts? (c) What is its radial acceleration at this time? (d) Sketch a view from above showing the circular track, the car, the velocity vector, and the acceleration component vectors 6.00 s after the car starts. (e) What are the magnitudes of the total acceleration and net force for the car at this time? (f) What angle do the total acceleration and net force make with the car's velocity at this time?

Figure **P9.68**



9.68 •• Engineers are designing a system by which a falling mass m imparts kinetic energy to a rotating uniform drum to which it is attached by thin, very light wire wrapped around the rim of the drum (Fig. P9.68). There is no appreciable friction in the axle of the drum, and everything starts from rest. This system is being tested on earth, but it is to be used on Mars, where the acceleration due to gravity is 3.71 m/s^2. In the earth tests, when m is set to 15.0 kg and allowed to fall through 5.00 m, it gives 250.0 J of kinetic energy to the drum. (a) If the system is operated on Mars, through what distance would the 15.0-kg mass have to fall to give the same amount of kinetic energy to the drum? (b) How fast would the 15.0-kg mass be moving on Mars just as the drum gained 250.0 J of kinetic energy?

9.69 • A vacuum cleaner belt is looped over a shaft of radius 0.45 cm and a wheel of radius 1.80 cm. The arrangement of the belt, shaft, and wheel is similar to that of the chain and sprockets in Fig. Q9.4. The motor turns the shaft at 60.0 rev/s and the moving belt turns the wheel, which in turn is connected by another shaft to the roller that beats the dirt out of the rug being vacuumed. Assume that the belt doesn't slip on either the shaft or the wheel. (a) What is the speed of a point on the belt? (b) What is the angular velocity of the wheel, in rad/s?

9.70 •• The motor of a table saw is rotating at 3450 rev/min. A pulley attached to the motor shaft drives a second pulley of half the diameter by means of a V-belt. A circular saw blade of diameter 0.208 m is mounted on the same rotating shaft as the second pulley. (a) The operator is careless and the blade catches and throws back a small piece of wood. This piece of wood moves with linear speed equal to the tangential speed of the rim of the blade. What is this speed? (b) Calculate the radial acceleration of points on the outer edge of the blade to see why sawdust doesn't stick to its teeth.

9.71 ••• While riding a multispeed bicycle, the rider can select the radius of the rear sprocket that is fixed to the rear axle. The front sprocket of a bicycle has radius 12.0 cm. If the angular speed of the front sprocket is 0.600 rev/s, what is the radius of the rear sprocket for which the tangential speed of a point on the rim of the rear wheel will be 5.00 m/s? The rear wheel has radius 0.330 m.

9.72 ••• A computer disk drive is turned on starting from rest and has constant angular acceleration. If it took 0.750 s for the drive to make its *second* complete revolution, (a) how long did it take to make the first complete revolution, and (b) what is its angular acceleration, in rad/s^2?

9.73 • A wheel changes its angular velocity with a constant angular acceleration while rotating about a fixed axis through its center. (a) Show that the change in the magnitude of the radial acceleration during any time interval of a point on the wheel is twice the product of the angular acceleration, the angular displacement, and the perpendicular distance of the point from the axis. (b) The radial acceleration of a point on the wheel that is 0.250 m from the axis changes from 25.0 m/s^2 to 85.0 m/s^2 as the wheel rotates through 20.0 rad. Calculate the tangential acceleration of this point. (c) Show that the change in the wheel's kinetic energy during any time interval is the product of the moment of inertia about the axis, the angular acceleration, and the angular displacement. (d) During the 20.0-rad angular displacement of part (b), the kinetic energy of the wheel increases from 20.0 J to 45.0 J. What is the moment of inertia of the wheel about the rotation axis?

9.74 •• A sphere consists of a solid wooden ball of uniform density 800 kg/m^3 and radius 0.30 m and is covered with a thin coating of lead foil with area density 20 kg/m^2. Calculate the moment of inertia of this sphere about an axis passing through its center.

9.75 ••• It has been argued that power plants should make use of off-peak hours (such as late at night) to generate mechanical energy and store it until it is needed during peak load times, such as the middle of the day. One suggestion has been to store the energy in large flywheels spinning on nearly frictionless ball bearings. Consider a flywheel made of iron (density 7800 kg/m^3) in the shape of a 10.0-cm-thick uniform disk. (a) What would the diameter of such a disk need to be if it is to store 10.0 megajoules of kinetic energy when spinning at 90.0 rpm about an axis perpendicular to the disk at its center? (b) What would be the centripetal acceleration of a point on its rim when spinning at this rate?

9.76 •• While redesigning a rocket engine, you want to reduce its weight by replacing a solid spherical part with a hollow spherical shell of the same size. The parts rotate about an axis through their center. You need to make sure that the new part always has the same rotational kinetic energy as the original part had at any given rate of rotation. If the original part had mass M, what must be the mass of the new part?

9.77 • The earth, which is not a uniform sphere, has a moment of inertia of $0.3308MR^2$ about an axis through its north and south poles. It takes the earth 86,164 s to spin once about this axis. Use Appendix F to calculate (a) the earth's kinetic energy due to its rotation about this axis and (b) the earth's kinetic energy due to its orbital motion around the sun. (c) Explain how the value of the earth's moment of inertia tells us that the mass of the earth is concentrated toward the planet's center.

9.78 ••• A uniform, solid disk with mass m and radius R is pivoted about a horizontal axis through its center. A small object of the same mass m is glued to the rim of the disk. If the disk is released from rest with the small object at the end of a horizontal radius, find the angular speed when the small object is directly below the axis.

9.79 •• **CALC** A metal sign for a car dealership is a thin, uniform right triangle with base length b and height h. The sign has mass M. (a) What is the moment of inertia of the sign for rotation about the side of length h? (b) If $M = 5.40$ kg, $b = 1.60$ m, and $h = 1.20$ m, what is the kinetic energy of the sign when it is rotating about an axis along the 1.20-m side at 2.00 rev/s?

9.80 •• **Measuring *I*.** As an intern with an engineering firm, you are asked to measure the moment of inertia of a large wheel, for rotation about an axis through its center. Since you were a good physics student, you know what to do. You measure the diameter of the wheel to be 0.740 m and find that it weighs 280 N. You mount the wheel, using frictionless bearings, on a horizontal axis through the wheel's center. You wrap a light rope around the wheel and hang an 8.00-kg mass from the free end of the rope, as shown in Fig. 9.17. You release the mass from rest; the mass descends and the wheel turns as the rope unwinds. You find that the mass has speed 5.00 m/s after it has descended 2.00 m. (a) What is the moment of inertia of the wheel for an axis perpendicular to the wheel at its center? (b) Your boss tells you that a larger I is needed. He asks you to design a wheel of the same mass and radius that has $I = 19.0\ kg \cdot m^2$. How do you reply?

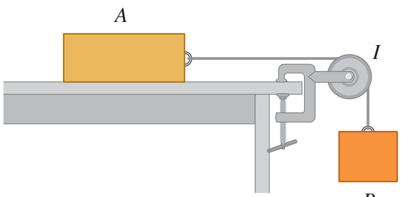
9.81 •• **CP** A meter stick with a mass of 0.180 kg is pivoted about one end so it can rotate without friction about a horizontal axis.

The meter stick is held in a horizontal position and released. As it swings through the vertical, calculate (a) the change in gravitational potential energy that has occurred; (b) the angular speed of the stick; (c) the linear speed of the end of the stick opposite the axis. (d) Compare the answer in part (c) to the speed of a particle that has fallen 1.00 m, starting from rest.

9.82 •• Exactly one turn of a flexible rope with mass m is wrapped around a uniform cylinder with mass M and radius R. The cylinder rotates without friction about a horizontal axle along the cylinder axis. One end of the rope is attached to the cylinder. The cylinder starts with angular speed ω_0. After one revolution of the cylinder the rope has unwrapped and, at this instant, hangs vertically down, tangent to the cylinder. Find the angular speed of the cylinder and the linear speed of the lower end of the rope at this time. You can ignore the thickness of the rope. [*Hint:* Use Eq. (9.18).]

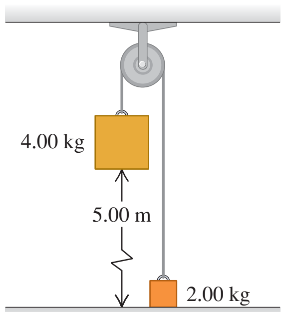
9.83 • The pulley in Fig. P9.83 has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

Figure **P9.83**



9.84 •• The pulley in Fig. P9.84 has radius 0.160 m and moment of inertia $0.560\ \text{kg}\cdot\text{m}^2$. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.00-kg block just before it strikes the floor.

Figure **P9.84**

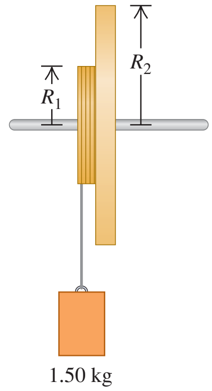


9.85 •• You hang a thin hoop with radius R over a nail at the rim of the hoop. You displace it to the side (within the plane of the hoop) through an angle β from its equilibrium position and let it go. What is its angular speed when it returns to its equilibrium position? [*Hint:* Use Eq. (9.18).]

9.86 •• A passenger bus in Zurich, Switzerland, derived its motive power from the energy stored in a large flywheel. The wheel was brought up to speed periodically, when the bus stopped at a station, by an electric motor, which could then be attached to the electric power lines. The flywheel was a solid cylinder with mass 1000 kg and diameter 1.80 m; its top angular speed was 3000 rev/min. (a) At this angular speed, what is the kinetic energy of the flywheel? (b) If the average power required to operate the bus is 1.86×10^4 W, how long could it operate between stops?

9.87 •• Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (Fig. P9.87). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50-kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat the calculation of part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block greater? Explain why this is so.

Figure **P9.87**

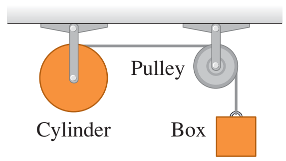


9.88 •• A thin, light wire is wrapped around the rim of a wheel, as shown in Fig. E9.49. The wheel rotates about a stationary horizontal axle that passes through the center of the wheel. The wheel has radius 0.180 m and moment of inertia for rotation about the axle of $I = 0.480\ \text{kg}\cdot\text{m}^2$. A small block with mass 0.340 kg is suspended from the free end of the wire. When the system is released from rest, the block descends with constant acceleration. The bearings in the wheel at the axle are rusty, so friction there does -6.00 J of work as the block descends 3.00 m. What is the magnitude of the angular velocity of the wheel after the block has descended 3.00 m?

9.89 ••• In the system shown in Fig. 9.17, a 12.0-kg mass is released from rest and falls, causing the uniform 10.0-kg cylinder of diameter 30.0 cm to turn about a frictionless axle through its center. How far will the mass have to descend to give the cylinder 480 J of kinetic energy?

9.90 • In Fig. P9.90, the cylinder and pulley turn without friction about stationary horizontal axles that pass through their centers. A light rope is wrapped around the cylinder, passes over the pulley, and has a 3.00-kg box suspended from its free end. There is no slipping between the rope and the pulley surface. The uniform cylinder has mass 5.00 kg and radius 40.0 cm. The pulley is a uniform disk with mass 2.00 kg and radius 20.0 cm. The box is released from rest and descends as the rope unwraps from the cylinder. Find the speed of the box when it has fallen 2.50 m.

Figure **P9.90**



9.91 •• A thin, flat, uniform disk has mass M and radius R. A circular hole of radius $R/4$, centered at a point $R/2$ from the disk's center, is then punched in the disk. (a) Find the moment of inertia of the disk with the hole about an axis through the original center of the disk, perpendicular to the plane of the disk. (*Hint:* Find the moment of inertia of the piece punched from the disk.) (b) Find the moment of inertia of the disk with the hole about an axis through the center of the hole, perpendicular to the plane of the disk.

Figure **P9.92**

9.92 •• BIO **Human Rotational Energy.** A dancer is spinning at 72 rpm about an axis through her center with her arms outstretched, as shown in Fig. P9.92. From biomedical measurements, the typical distribution of mass in a human body is as follows:

Head: 7.0%
Arms: 13% (for both)
Trunk and legs: 80.0%

Suppose you are this dancer. Using this information plus length measurements on your own body, calculate (a) your moment of inertia about your spin axis and (b) your rotational kinetic energy. Use the figures in Table 9.2 to model reasonable approximations for the pertinent parts of your body.

9.93 •• BIO The Kinetic Energy of Walking. If a person of mass M simply moved forward with speed V, his kinetic energy would be $\frac{1}{2}MV^2$. However, in addition to possessing a forward motion, various parts of his body (such as the arms and legs) undergo rotation. Therefore, his total kinetic energy is the sum of the energy from his forward motion plus the rotational kinetic energy of his arms and legs. The purpose of this problem is to see how much this rotational motion contributes to the person's kinetic energy. Biomedical measurements show that the arms and hands together typically make up 13% of a person's mass, while the legs and feet together account for 37%. For a rough (but reasonable) calculation, we can model the arms and legs as thin uniform bars pivoting about the shoulder and hip, respectively. In a brisk walk, the arms and legs each move through an angle of about $\pm 30°$ (a total of 60°) from the vertical in approximately 1 second. We shall assume that they are held straight, rather than being bent, which is not quite true. Let us consider a 75-kg person walking at 5.0 km/h, having arms 70 cm long and legs 90 cm long. (a) What is the average angular velocity of his arms and legs? (b) Using the average angular velocity from part (a), calculate the amount of rotational kinetic energy in this person's arms and legs as he walks. (c) What is the total kinetic energy due to both his forward motion and his rotation? (d) What percentage of his kinetic energy is due to the rotation of his legs and arms?

9.94 •• BIO The Kinetic Energy of Running. Using Problem 9.93 as a guide, apply it to a person running at 12 km/h, with his arms and legs each swinging through $\pm 30°$ in $\frac{1}{2}$ s. As before, assume that the arms and legs are kept straight.

9.95 •• Perpendicular-Axis Theorem. Consider a rigid body that is a thin, plane sheet of arbitrary shape. Take the body to lie in the xy-plane and let the origin O of coordinates be located at any point within or outside the body. Let I_x and I_y be the moments of inertia about the x- and y-axes, and let I_O be the moment of inertia about an axis through O perpendicular to the plane. (a) By considering mass elements m_i with coordinates (x_i, y_i), show that $I_x + I_y = I_O$. This is called the perpendicular-axis theorem. Note that point O does not have to be the center of mass. (b) For a thin washer with mass M and with inner and outer radii R_1 and R_2, use the perpendicular-axis theorem to find the moment of inertia about an axis that is in the plane of the washer and that passes through its center. You may use the information in Table 9.2. (c) Use the perpendicular-axis theorem to show that for a thin, square sheet with mass M and side L, the moment of inertia about *any* axis in the plane of the sheet that passes through the center of the sheet is $\frac{1}{12}ML^2$. You may use the information in Table 9.2.

9.96 ••• A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (*Hint:* Use the parallel-axis theorem.)

9.97 • CALC A cylinder with radius R and mass M has density that increases linearly with distance r from the cylinder axis, $\rho = \alpha r$, where α is a positive constant. (a) Calculate the moment of inertia of the cylinder about a longitudinal axis through its center in terms of M and R. (b) Is your answer greater or smaller than the moment of inertia of a cylinder of the same mass and radius but of uniform density? Explain why this result makes qualitative sense.

9.98 •• CALC Neutron Stars and Supernova Remnants. The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light-years from the earth (Fig. P9.98). It is the remnant of a star that underwent a *supernova explosion,* seen on earth in 1054 A.D. Energy is released by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock $(3000\ \text{kg/m}^3)$ and to the density of an atomic nucleus (about $10^{17}\ \text{kg/m}^3$). Justify the statement that a neutron star is essentially a large atomic nucleus.

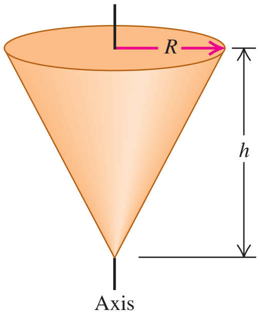
Figure **P9.98**

9.99 •• CALC A sphere with radius $R = 0.200$ m has density ρ that decreases with distance r from the center of the sphere according to $\rho = 3.00 \times 10^3\ \text{kg/m}^3 - (9.00 \times 10^3\ \text{kg/m}^4)r$. (a) Calculate the total mass of the sphere. (b) Calculate the moment of inertia of the sphere for an axis along a diameter.

CHALLENGE PROBLEMS

9.100 ••• CALC Calculate the moment of inertia of a uniform solid cone about an axis through its center (Fig. P9.100). The cone has mass M and altitude h. The radius of its circular base is R.

Figure **P9.100**



9.101 ••• CALC On a compact disc (CD), music is coded in a pattern of tiny pits arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of $v = 1.25$ m/s. Because the radius of the track varies as it spirals outward, the *angular* speed of the disc must change as the CD is played. (See Exercise 9.20.) Let's see what angular acceleration is required to keep v constant. The equation of a spiral is $r(\theta) = r_0 + \beta\theta$, where r_0 is the radius of the spiral at $\theta = 0$ and β is a constant. On a CD, r_0 is the inner radius of the spiral track. If we take the rotation direction of the CD to be positive, β must be positive so that r increases as the disc turns and θ increases. (a) When the disc rotates through a small angle $d\theta$, the distance scanned along the track is $ds = r\,d\theta$. Using the above expression for $r(\theta)$, integrate ds to find the total distance s scanned along the track as a function of the total angle θ through which the disc has rotated. (b) Since the track is scanned at a constant linear speed v, the distance s found in part (a) is equal to vt. Use this to find θ as a function of time. There will be two solutions for θ; choose the positive one, and explain why this is the solution to choose. (c) Use your expression for $\theta(t)$ to find the angular velocity ω_z and the angular acceleration α_z as functions of time. Is α_z constant? (d) On a CD, the inner radius of the track is 25.0 mm, the track radius increases by 1.55 μm per revolution, and the playing time is 74.0 min. Find the values of r_0 and β, and find the total number of revolutions made during the playing time. (e) Using your results from parts (c) and (d), make graphs of ω_z (in rad/s) versus t and α_z (in rad/s^2) versus t between $t = 0$ and $t = 74.0$ min.

Answers

Chapter Opening Question ?

Both segments of the rigid blade have the same angular speed ω. From Eqs. (9.13) and (9.15), doubling the distance r for the same ω doubles the linear speed $v = r\omega$ and doubles the radial acceleration $a_{rad} = \omega^2 r$.

Test Your Understanding Questions

9.1 Answers: (a) (i) and (iii), (b) (ii) The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (ω_z and α_z are both positive) and for $4\text{ s} < t < 6$ s (ω_z and α_z are both negative), but is slowing down for $2\text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

9.2 Answers: (a) (i), (b) (ii) When the disc comes to rest, $\omega_z = 0$. From Eq. (9.7), the *time* when this occurs is $t = (\omega_z - \omega_{0z})/\alpha_z = -\omega_{0z}/\alpha_z$ (this is a positive time because α_z is negative). If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z, their ratio is unchanged and the rotation stops in the same amount of time. The *angle* through which the disc rotates is given by Eq. (9.10): $\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$ (since the final angular velocity is $\omega_z = 0$). The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled. You can also come to the same conclusion using Eq. (9.12).

9.3 Answer: (ii) From Eq. (9.13), $v = r\omega$. To maintain a constant linear speed v, the angular speed ω must decrease as the scanning head moves outward (greater r).

9.4 Answer: (i) The kinetic energy in the falling block is $\frac{1}{2}mv^2$, and the kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4}mv^2$. Hence the total kinetic energy of the system is $\frac{3}{4}mv^2$, of which two-thirds is in the block and one-third is in the cylinder.

9.5 Answer: (ii) More of the mass of the pool cue is concentrated at the thicker end, so the center of mass is closer to that end. The moment of inertia through a point P at either end is $I_P = I_{cm} + Md^2$; the thinner end is farther from the center of mass, so the distance d and the moment of inertia I_P are greater for the thinner end.

9.6 Answer: (iii) Our result from Example 9.10 does *not* depend on the cylinder length L. The moment of inertia depends only on the *radial* distribution of mass, not on its distribution along the axis.

Bridging Problem

Answers: (a) $I = \left[\dfrac{M}{L}\left(\dfrac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$

(b) $W = \frac{1}{6}M(L^2 - 3Lh + 3h^2)\alpha^2t^2$

(c) $a = (L - h)\alpha\sqrt{1 + \alpha^2t^4}$

10 DYNAMICS OF ROTATIONAL MOTION

? If you stand at the north pole, the north star, Polaris, is almost directly overhead, and the other stars appear to trace circles around it. But 5000 years ago a different star, Thuban, was directly above the north pole and was the north star. What caused this change?

LEARNING GOALS

By studying this chapter, you will learn:

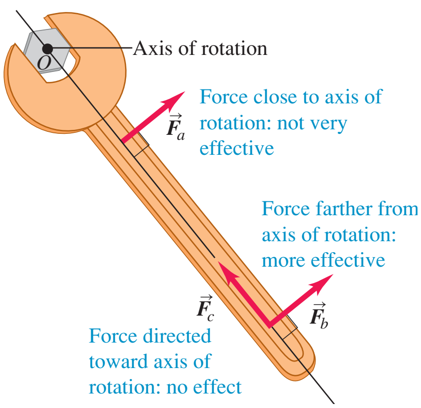
- What is meant by the torque produced by a force.
- How the net torque on a body affects the rotational motion of the body.
- How to analyze the motion of a body that both rotates and moves as a whole through space.
- How to solve problems that involve work and power for rotating bodies.
- What is meant by the angular momentum of a particle or of a rigid body.
- How the angular momentum of a system changes with time.
- Why a spinning gyroscope goes through the curious motion called precession.

We learned in Chapters 4 and 5 that a net force applied to a body gives that body an acceleration. But what does it take to give a body an *angular* acceleration? That is, what does it take to start a stationary body rotating or to bring a spinning body to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we will define a new physical quantity, *torque,* that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on a body determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand such problems as how energy is transmitted by the rotating drive shaft in a car. Finally, we will develop a new conservation principle, *conservation of angular momentum,* that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes,* rotating devices that seemingly defy common sense and don't fall over when you might think they should—but that actually behave in perfect accordance with the dynamics of rotational motion.

10.1 Torque

10.1 Which of these three equal-magnitude forces is most likely to loosen the tight bolt?



We know that forces acting on a body can affect its **translational motion**—that is, the motion of the body as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the body where the force is applied. In Fig. 10.1 a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ doesn't do any good at all; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change a body's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point O to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

Figure 10.2 shows three examples of how to calculate torque. The body in the figure can rotate about an axis that is perpendicular to the plane of the figure and passes through point O. Three forces, $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, act on the body in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about O depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point O and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment arm**) of force $\vec{F}_1$ about O. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or *moment*) of the force $\vec{F}_1$ with respect to O as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. In general, for a force of magnitude F whose line of action is a perpendicular distance l from O, the torque is

$$\tau = Fl \tag{10.1}$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft). Both groups use the term "lever arm" or "moment arm" for the distance l.

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point O, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION **Torque is always measured about a point** Note that torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may also change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O, but the torque of $\vec{F}_3$ is *not* zero about point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative,* the torques of $\vec{F}_1$ and $\vec{F}_2$ about O are

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

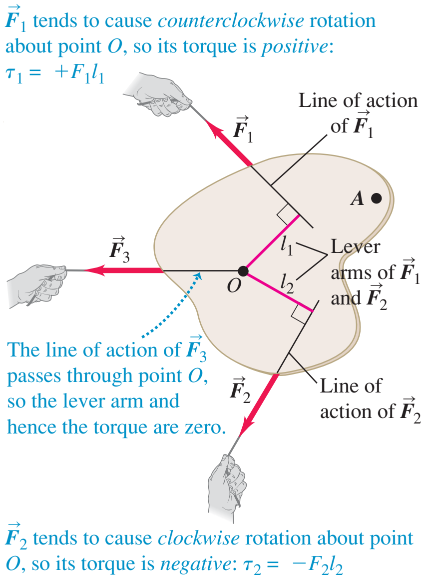
Figure 10.2 shows this choice for the sign of torque. We will often use the symbol (+↺) to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

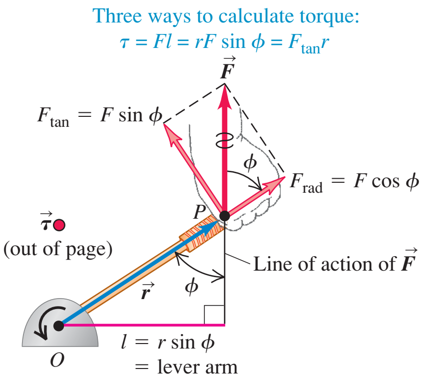
Figure 10.3 shows a force $\vec{F}$ applied at a point P described by a position vector $\vec{r}$ with respect to the chosen point O. There are three ways to calculate the torque of this force:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component F_{rad} along the direction of $\vec{r}$ and a tangential component F_{tan} at right angles, perpendicular to $\vec{r}$. (We call this a tangential component because if the body rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.) Then

10.2 The torque of a force about a point is the product of the force magnitude and the lever arm of the force.



10.3 Three ways to calculate the torque of the force $\vec{F}$ about the point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.

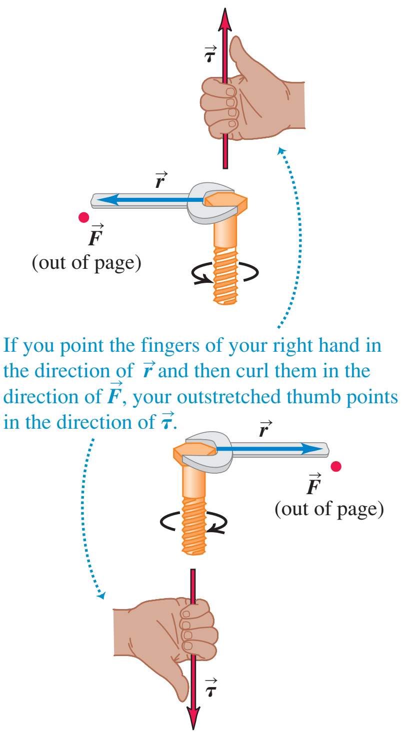


$F_{tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{tan}r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

$$\tau = Fl = rF\sin\phi = F_{tan}r \qquad \text{(magnitude of torque)} \qquad (10.2)$$

10.4 The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.



Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (You should go back and review that definition.) We now generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to O is the *vector* quantity

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition of torque vector)} \qquad (10.3)$$

The torque as defined in Eq. (10.2) is just the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (Fig. 1.29). Figure 10.4 shows the direction relationships.

In diagrams that involve $\vec{r}$, $\vec{F}$, and $\vec{\tau}$, it's common to have one of the vectors oriented perpendicular to the page. (Indeed, by the very nature of the cross product, $\vec{\tau} = \vec{r} \times \vec{F}$ *must* be perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$.) We use a dot (•) to represent a vector that points out of the page (see Fig. 10.3) and a cross (×) to represent a vector that points into the page.

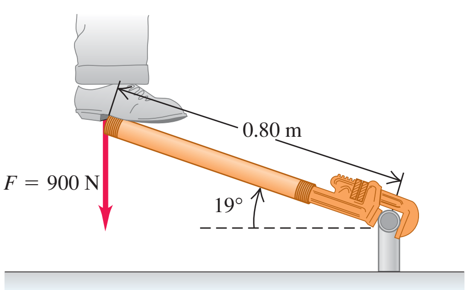
In the following sections we will usually be concerned with rotation of a body about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis is of interest, and we often call that component the torque with respect to the specified *axis*.

Example 10.1 Applying a torque

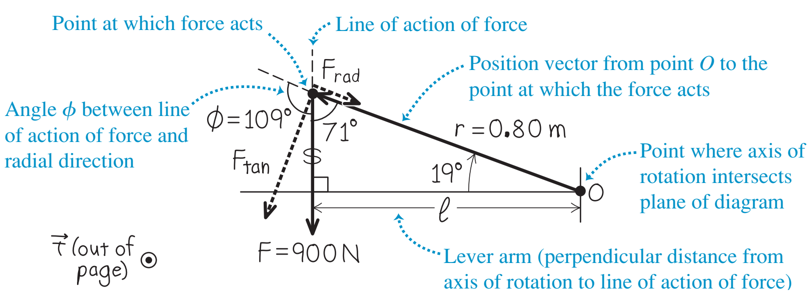
To loosen a pipe fitting, a weekend plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his full 900-N weight at a point 0.80 m from the center of the fitting (Fig. 10.5a). The wrench handle and cheater make an angle of 19° with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

10.5 (a) A weekend plumber tries to loosen a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about O.

(a) Diagram of situation



(b) Free-body diagram



SOLUTION

IDENTIFY and SET UP: Figure 10.5b shows the vectors $\vec{r}$ and $\vec{F}$ and the angle between them ($\phi = 109°$). Equation (10.1) or (10.2) will tell us the magnitude of the torque. The right-hand rule with Eq. (10.3), $\vec{\tau} = \vec{r} \times \vec{F}$, will tell us the direction of the torque.

EXECUTE: To use Eq. (10.1), we first calculate the lever arm l. As Fig. 10.5b shows,

$$l = r\sin\phi = (0.80\ \text{m})\sin 109° = (0.80\ \text{m})\sin 71° = 0.76\ \text{m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

$$\tau = Fl = (900\ \text{N})(0.76\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

We get the same result from Eq. (10.2):

$$\tau = rF\sin\phi = (0.80\ \text{m})(900\ \text{N})(\sin 109°) = 680\ \text{N}\cdot\text{m}$$

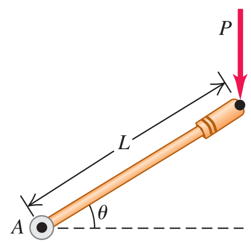
Alternatively, we can find $F_{\tan}$, the tangential component of $\vec{F}$ that acts perpendicular to $\vec{r}$. Figure 10.5b shows that this component is at an angle of $109° - 90° = 19°$ from $\vec{F}$, so $F_{\tan} = F\sin\phi = F(\cos 19°) = (900\ \text{N})(\cos 19°) = 851$ N. Then, from Eq. 10.2,

$$\tau = F_{\tan}r = (851\ \text{N})(0.80\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

Curl the fingers of your right hand from the direction of $\vec{r}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{F}$ (straight down). Then your right thumb points out of the plane of the figure: This is the direction of $\vec{\tau}$.

EVALUATE: To check the direction of $\vec{\tau}$, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about O. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

Test Your Understanding of Section 10.1 The figure shows a force P being applied to one end of a lever of length L. What is the magnitude of the torque of this force about point A? (i) $PL\sin\theta$; (ii) $PL\cos\theta$; (iii) $PL\tan\theta$.



10.2 Torque and Angular Acceleration for a Rigid Body

We are now ready to develop the fundamental relationship for the rotational dynamics of a rigid body. We will show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, we again imagine the body as being made up of a large number of particles. We choose the axis of rotation to be the z-axis; the first particle has mass m_1 and distance r_1 from this axis (Fig. 10.6). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $F_{1,\tan}$ that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

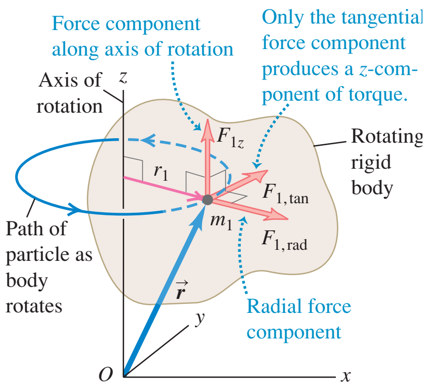
$$F_{1,\tan} = m_1 a_{1,\tan} \tag{10.4}$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\tan}r_1 = m_1 r_1^2 \alpha_z \tag{10.5}$$

From Eq. (10.2), $F_{1,\tan}r_1$ is just the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript z is a reminder that the torque affects rotation around the z-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the z-axis.

10.6 As a rigid body rotates around the z-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about O with a z-component (along the rotation axis).



Neither of the components $F_{1,\mathrm{rad}}$ or F_{1z} contributes to the torque about the z-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\tan} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1\alpha_z = m_1 r_1^2 \alpha_z$$

We write an equation like this for every particle in the body and then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1 r_1^2\alpha_z + m_2 r_2^2 \alpha_z + \cdots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right)\alpha_z \tag{10.6}$$

MasteringPHYSICS

ActivPhysics 7.8: Rotoride—Dynamics Approach
ActivPhysics 7.9: Falling Ladder
ActivPhysics 7.10: Woman and Flywheel Elevator—Dynamics Approach

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus for the rigid body as a whole, Eq. (10.6) is the *rotational analog of Newton's second law:*

$$\sum \tau_z = I\alpha_z \tag{10.7}$$

(rotational analog of Newton's second law for a rigid body)

10.7 Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. This is made easier by using a screwdriver with a large-radius handle, which provides a large lever arm for the force you apply with your hand.

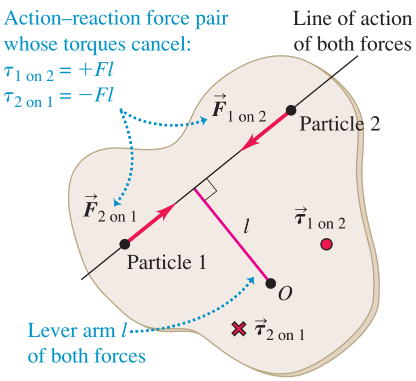
Just as Newton's second law says that the net force on a particle equals the particle's mass times its acceleration, Eq. (10.7) says that the net torque on a rigid body equals the body's moment of inertia about the rotation axis times its angular acceleration (Fig. 10.7).

Note that because our derivation assumed that the angular acceleration α_z is the same for all particles in the body, Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Also note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z$, α_z must be measured in $\mathrm{rad/s^2}$.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal and opposite (Fig. 10.8). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a body is its *weight.* This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, it turns out that if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We will prove this statement in Chapter 11, but meanwhile we will use it for some of the problems in this chapter.

10.8 Two particles in a rigid body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through O are the same and the torques due to the two forces are equal and opposite. Only *external* torques affect the body's rotation.



Problem-Solving Strategy 10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

IDENTIFY *the relevant concepts:* Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

SET UP *the problem* using the following steps:

1. Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
2. For each body, draw a free-body diagram that shows the *shape* of each body, including all dimensions and angles that you will need for torque calculations. Label pertinent quantities with algebraic symbols.
3. Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of α_z, pick that as the positive sense of rotation.

EXECUTE *the solution:*

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.
3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

EVALUATE *your answer:* Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back on to the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

Example 10.2 An unwinding cable I

Figure 10.9a shows the situation analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

SOLUTION

IDENTIFY and SET UP: We can't use the energy method of Section 9.4, which doesn't involve acceleration. Instead we'll apply rotational dynamics to find the angular acceleration of the cylinder (Fig. 10.9b). We'll then find a relationship between the motion of the cable and the motion of the cylinder rim, and use this to find the acceleration of the cable. The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest. The force F exerted by the cable produces a torque about the rotation axis. The weight (magnitude Mg) and the normal force (magnitude n) exerted by the cylinder's bearings produce *no* torque about the rotation axis because they both act along lines through that axis.

EXECUTE: The lever arm of F is equal to the radius $R = 0.060$ m of the cylinder, so the torque is $\tau_z = FR$. (This torque is positive, as it tends to cause a counterclockwise rotation.) From Table 9.2, case (f), the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Then Eq. (10.7) tells us that

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\ \text{N})}{(50\ \text{kg})(0.060\ \text{m})} = 6.0\ \text{rad/s}^2$$

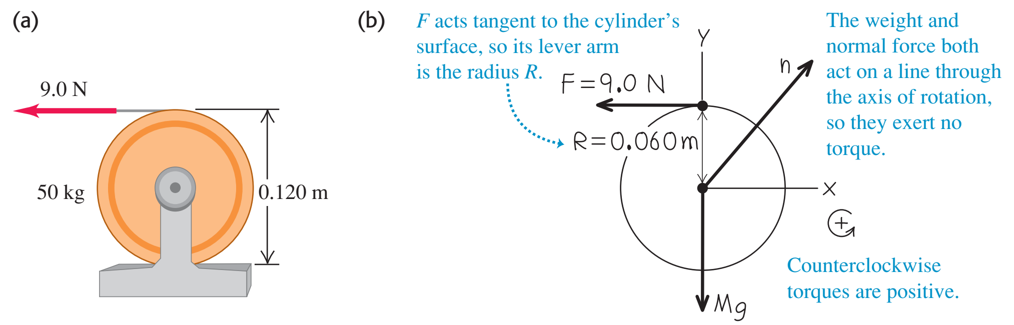
(We can add "rad" to our result because radians are dimensionless.)

To get the linear acceleration of the cable, recall from Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_{\tan} = R\alpha_z = (0.060\ \text{m})(6.0\ \text{rad/s}^2) = 0.36\ \text{m/s}^2$$

EVALUATE: Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Does your result agree with that of Example 9.7?

10.9 (a) Cylinder and cable. (b) Our free-body diagram for the cylinder.



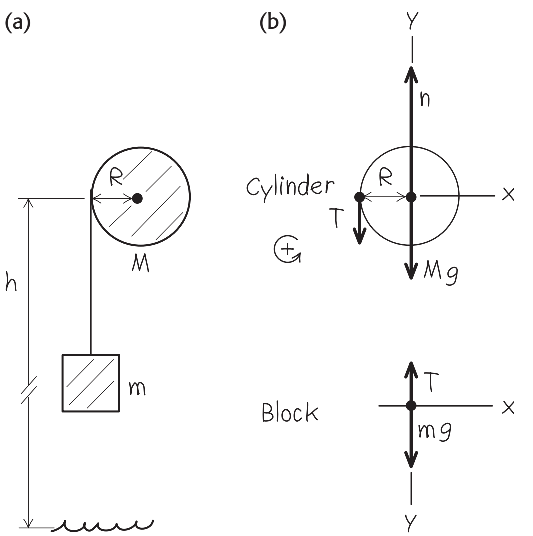
Example 10.3 An unwinding cable II

In Example 9.8 (Section 9.4), what are the acceleration of the falling block and the tension in the cable?

SOLUTION

IDENTIFY and SET UP: We'll apply translational dynamics to the block and rotational dynamics to the cylinder. As in Example 10.2, we'll relate the linear acceleration of the block (our target variable) to the angular acceleration of the cylinder. Figure 10.10 shows our sketch of the situation and a free-body diagram for each body. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the block to be downward.

10.10 (a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.



EXECUTE: For the block, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the only torque about its axis is that due to the cable tension T. Hence Eq. (10.7) gives

$$\sum \tau_z = RT = I\alpha_z = \tfrac{1}{2}MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. From Eq. (9.14), this acceleration is $a_y = a_{\tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with a_y in the cylinder equation above, and then divide by R. The result is $T = \tfrac{1}{2}Ma_y$. Now we substitute this expression for T into Newton's second law for the block and solve for the acceleration a_y:

$$mg - \tfrac{1}{2}Ma_y = ma_y$$

$$a_y = \frac{g}{1 + M/2m}$$

To find the cable tension T, we substitute our expression for a_y into the block equation:

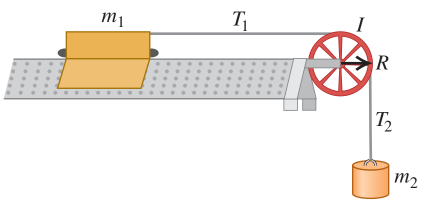
$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

EVALUATE: The acceleration is positive (in the downward direction) and less than g, as it should be, since the cable is holding back the block. The cable tension is *not* equal to the block's weight mg; if it were, the block could not accelerate.

Let's check some particular cases. When M is much larger than m, the tension is nearly equal to mg and the acceleration is correspondingly much less than g. When M is zero, $T = 0$ and $a_y = g$; the object falls freely. If the object starts from rest $(v_{0y} = 0)$ a height h above the floor, its y-velocity when it strikes the ground is given by $v_y^2 = v_{0y}^2 + 2a_yh = 2a_yh$, so

$$v_y = \sqrt{2a_yh} = \sqrt{\frac{2gh}{1 + M/2m}}$$

We found this same result from energy considerations in Example 9.8.

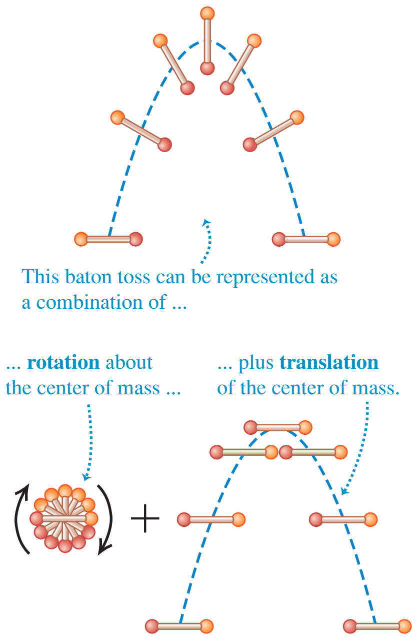


Test Your Understanding of Section 10.2 The figure shows a glider of mass m_1 that can slide without friction on a horizontal air track. It is attached to an object of mass m_2 by a massless string. The pulley has radius R and moment of inertia I about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) the tension force (magnitude T_1) in the horizontal part of the string; (ii) the tension force (magnitude T_2) in the vertical part of the string; (iii) the weight m_2g of the hanging object.

10.3 Rigid-Body Rotation About a Moving Axis

We can extend our analysis of the dynamics of rotational motion to some cases in which the axis of rotation moves. When that happens, the motion of the body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so that it is not at rest in any inertial frame. Figure 10.11 illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. Other examples of combined translational and rotational motions include a ball rolling down a hill and a yo-yo unwinding at the end of a string.

10.11 The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.



Combined Translation and Rotation: Energy Relationships

It's beyond the scope of this book to prove that the motion of a rigid body can always be divided into translation of the center of mass and rotation about the center of mass. But we can show that this is true for the *kinetic energy* of a rigid body that has both translational and rotational motions. In this case, the body's kinetic energy is the sum of a part $\frac{1}{2}Mv_{\mathrm{cm}}^2$ associated with motion of the center of mass and a part $\frac{1}{2}I_{\mathrm{cm}}\omega^2$ associated with rotation about an axis through the center of mass:

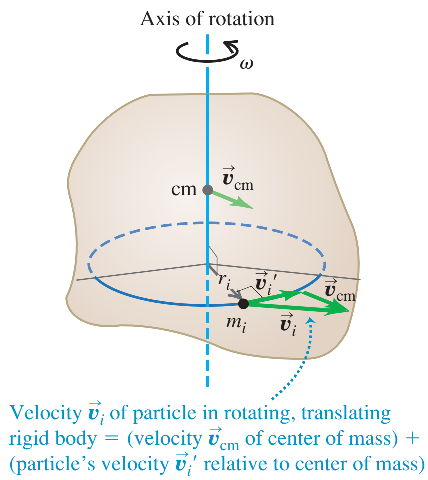
$$K = \tfrac{1}{2}Mv_{\mathrm{cm}}^2 + \tfrac{1}{2}I_{\mathrm{cm}}\omega^2 \tag{10.8}$$

(rigid body with both translation and rotation)

To prove this relationship, we again imagine the rigid body to be made up of particles. Consider a typical particle with mass m_i as shown in Fig. 10.12. The velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{\mathrm{cm}}$ of the center of mass and the velocity $\vec{v}_i'$ of the particle *relative to* the center of mass:

$$\vec{v}_i = \vec{v}_{\mathrm{cm}} + \vec{v}_i' \tag{10.9}$$

10.12 A rigid body with both translation and rotation.



The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}m_i v_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{v}_i \cdot \vec{v}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}m_i(\vec{v}_{\mathrm{cm}} + \vec{v}_i') \cdot (\vec{v}_{\mathrm{cm}} + \vec{v}_i') \\ &= \tfrac{1}{2}m_i(\vec{v}_{\mathrm{cm}} \cdot \vec{v}_{\mathrm{cm}} + 2\vec{v}_{\mathrm{cm}} \cdot \vec{v}_i' + \vec{v}_i' \cdot \vec{v}_i') \\ &= \tfrac{1}{2}m_i(v_{\mathrm{cm}}^2 + 2\vec{v}_{\mathrm{cm}} \cdot \vec{v}_i' + v_i'^2) \end{aligned}$$

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the body. Expressing the three terms in this equation as separate sums, we get

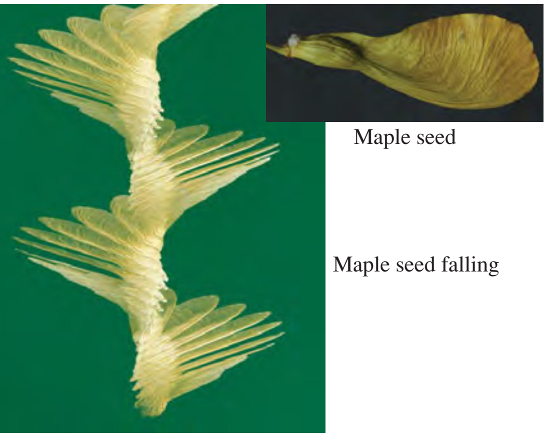
$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_i v_{\mathrm{cm}}^2\right) + \sum\left(m_i \vec{v}_{\mathrm{cm}} \cdot \vec{v}_i'\right) + \sum\left(\tfrac{1}{2}m_i v_i'^2\right)$$

The first and second terms have common factors that can be taken outside the sum:

$$K = \tfrac{1}{2}\left(\sum m_i\right)v_{\mathrm{cm}}^2 + \vec{v}_{\mathrm{cm}} \cdot \left(\sum m_i \vec{v}_i'\right) + \sum\left(\tfrac{1}{2}m_i v_i'^2\right) \tag{10.10}$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i \vec{v}_i'$ is M times the velocity of the center of mass *relative to the center of mass,* and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation

Application Combined Translation and Rotation
A maple seed consists of a pod attached to a much lighter, flattened wing. Airflow around the wing slows the fall to about 1 m/s and causes the seed to rotate about its center of mass. The seed's slow fall means that a breeze can carry the seed some distance from the parent tree. In the absence of wind, the seed's center of mass falls straight down.



around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{cm}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2$$

Rolling Without Slipping

An important case of combined translation and rotation is **rolling without slipping,** such as the motion of the wheel shown in Fig. 10.13. The wheel is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{cm}$. If the radius of the wheel is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_1'$ is $R\omega$; hence we must have

$$v_{cm} = R\omega \quad \text{(condition for rolling without slipping)} \tag{10.11}$$

As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

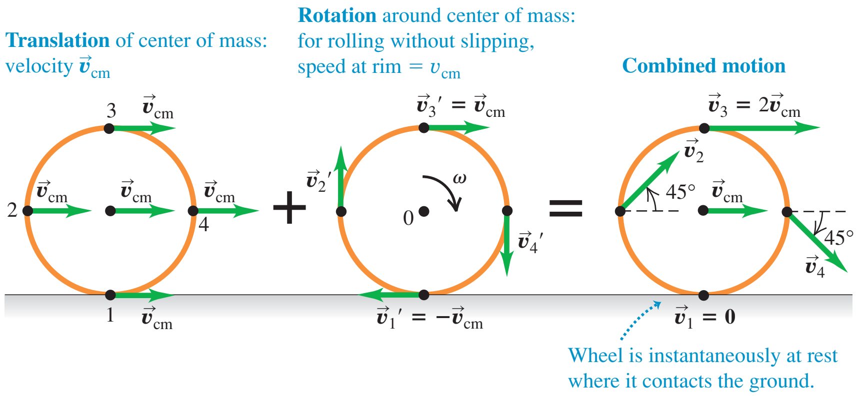
At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2}I_1\omega^2$, where I_1 is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), the kinetic energy of the wheel is

$$K = \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}^2$$

which is the same as Eq. (10.8).

MasteringPHYSICS
ActivPhysics 7.11: Race Between a Block and a Disk

10.13 The motion of a rolling wheel is the sum of the translational motion of the center of mass plus the rotational motion of the wheel around the center of mass.



CAUTION **Rolling without slipping** Note that the relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (Fig. 10.14). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

10.14 The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

If a rigid body changes height as it moves, we must also consider gravitational potential energy. As we discussed in Section 9.4, the gravitational potential energy associated with any extended body of mass M, rigid or not, is the same as if we replace the body by a particle of mass M located at the body's center of mass. That is,

$$U = Mgy_{cm}$$

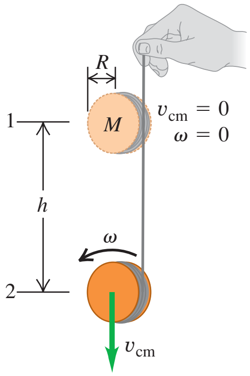
Example 10.4 Speed of a primitive yo-yo

You make a primitive yo-yo by wrapping a massless string around a solid cylinder with mass M and radius R (Fig. 10.15). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed v_{cm} of the center of mass of the cylinder after it has descended a distance h.

SOLUTION

IDENTIFY and SET UP: The upper end of the string is held fixed, not pulled upward, so your hand does no work on the string–cylinder system. There is friction between the string and the cylinder, but the string doesn't slip so no mechanical energy is lost. Hence we can use conservation of mechanical energy. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy K_2 is given by Eq. (10.8); the massless string has no kinetic energy. The moment of inertia is $I = \frac{1}{2}MR^2$, and by Eq. (9.13) $\omega = v_{cm}/R$ because the string doesn't slip. The potential energies are $U_1 = Mgh$ and $U_2 = 0$.

10.15 Calculating the speed of a primitive yo-yo.



EXECUTE: From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2}Mv_{cm}{}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \tfrac{3}{4}Mv_{cm}{}^2$$

The kinetic energy is $1\frac{1}{2}$ times what it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy $(\frac{1}{2}Mv_{cm}{}^2)$ is translational and one-third $(\frac{1}{4}Mv_{cm}{}^2)$ is rotational. Using conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$

$$0 + Mgh = \tfrac{3}{4}Mv_{cm}{}^2 + 0$$

$$v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

EVALUATE: No mechanical energy was lost or gained, so from the energy standpoint the string is merely a way to convert some of the gravitational potential energy (which is released as the cylinder falls) into rotational kinetic energy rather than translational kinetic energy. Because not all of the released energy goes into translation, v_{cm} is less than the speed $\sqrt{2gh}$ of an object dropped from height h with no strings attached.

Example 10.5 Race of the rolling bodies

In a physics demonstration, an instructor "races" various bodies that roll without slipping from rest down an inclined plane (Fig. 10.16). What shape should a body have to reach the bottom of the incline first?

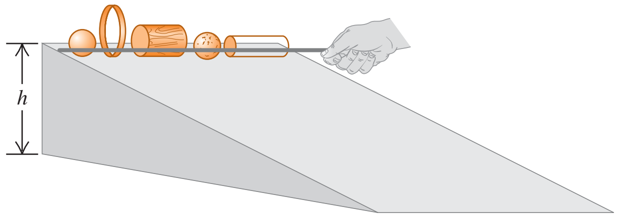
SOLUTION

IDENTIFY and SET UP: Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction,* introduced in Section 5.3, if the bodies and the surface of the

Continued

incline are rigid. (Later in this section we'll explain why this is so.) We can therefore use conservation of energy. Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. Equation (10.8) gives the kinetic energy at the bottom of the incline; since the bodies roll without slipping, $\omega = v_{cm}/R$. We can express the moments of inertia of the four round bodies in Table 9.2, cases (f)–(i), as $I_{cm} = cMR^2$, where c is a number less than or equal to 1 that depends on the shape of the body. Our goal is to find the value of c that gives the body the greatest speed v_{cm} after its center of mass has descended a vertical distance h.

10.16 Which body rolls down the incline fastest, and why?



EXECUTE: From conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$

$$0 + Mgh = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}cMR^2\left(\frac{v_{cm}}{R}\right)^2 + 0$$

$$Mgh = \tfrac{1}{2}(1 + c)Mv_{cm}^2$$

$$v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

EVALUATE: For a given value of c, the speed v_{cm} after descending a distance h is *independent* of the body's mass M and radius R. Hence *all* uniform solid cylinders $(c = \tfrac{1}{2})$ have the same speed at the bottom, regardless of their mass and radii. The values of c tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $(c = \tfrac{2}{5})$, (2) any solid cylinder $(c = \tfrac{1}{2})$, (3) any thin-walled, hollow sphere $(c = \tfrac{2}{3})$, and (4) any thin-walled, hollow cylinder $(c = 1)$. Small-c bodies always beat large-c bodies because less of their kinetic energy is tied up in rotation and so more is available for translation.

Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for a body with total mass M, the acceleration $\vec{a}_{cm}$ of the center of mass is the same as that of a point mass M acted on by all the external forces on the actual body:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

$$\sum \tau_z = I_{cm}\alpha_z \qquad (10.13)$$

where I_{cm} is the moment of inertia with respect to an axis through the center of mass and the sum $\sum \tau_z$ includes all external torques with respect to this axis. It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But in fact, Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (Fig. 10.17). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same body.* One of these, Eq. (10.12), describes the translational motion of the center of mass. The other equation of motion, Eq. (10.13), describes the rotational motion about the axis through the center of mass.

10.17 The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

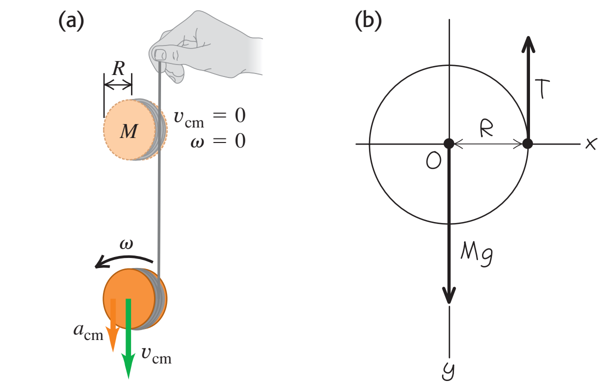
Example 10.6 Acceleration of a primitive yo-yo

For the primitive yo-yo in Example 10.4 (Fig. 10.18a), find the downward acceleration of the cylinder and the tension in the string.

SOLUTION

IDENTIFY and SET UP: Figure 10.18b shows our free-body diagram for the yo-yo, including our choice of positive coordinate directions. Our target variables are $a_{\text{cm-}y}$ and T. We'll use Eq. (10.12) for the translational motion of the center of mass and Eq. (10.13) for rotational motion around the center of mass. We'll also use Eq. (10.11), which says that the string unwinds without slipping. As in Example 10.4, the moment of inertia of the yo-yo for an axis through its center of mass is $I_{\text{cm}} = \frac{1}{2}MR^2$.

10.18 Dynamics of a primitive yo-yo (see Fig. 10.15).



EXECUTE: From Eq. (10.12),

$$\sum F_y = Mg + (-T) = Ma_{\text{cm-}y} \tag{10.14}$$

From Eq. (10.13),

$$\sum \tau_z = TR = I_{\text{cm}}\alpha_z = \tfrac{1}{2}MR^2\alpha_z \tag{10.15}$$

From Eq. (10.11), $v_{\text{cm-}z} = R\omega_z$; the derivative of this expression with respect to time gives us

$$a_{\text{cm-}y} = R\alpha_z \tag{10.16}$$

We now use Eq. (10.16) to eliminate α_z from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for T and $a_{\text{cm-}y}$. The results are

$$a_{\text{cm-}y} = \tfrac{2}{3}g \qquad T = \tfrac{1}{3}Mg$$

EVALUATE: The string slows the fall of the yo-yo, but not enough to stop it completely. Hence $a_{\text{cm-}y}$ is less than the free-fall value g and T is less than the yo-yo weight Mg.

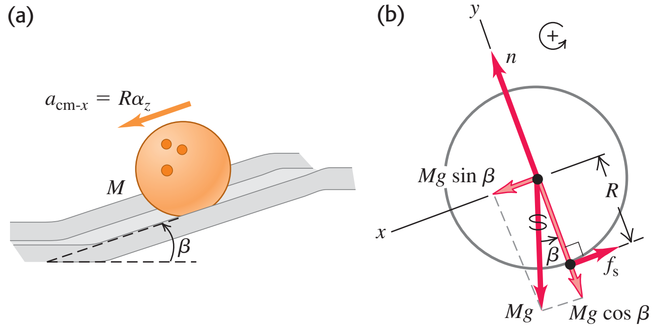
Example 10.7 Acceleration of a rolling sphere

A bowling ball rolls without slipping down a ramp, which is inclined at an angle β to the horizontal (Fig. 10.19a). What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

SOLUTION

IDENTIFY and SET UP: The free-body diagram (Fig. 10.19b) shows that only the friction force exerts a torque about the center of mass. Our target variables are the acceleration $a_{\text{cm-}x}$ of the ball's center of mass and the magnitude f of the friction force. (Because the ball does not slip at the instantaneous point of contact with the ramp, this is a *static* friction force; it prevents slipping and gives the ball its angular acceleration.) We use Eqs. (10.12) and (10.13) as in Example 10.6.

10.19 A bowling ball rolling down a ramp.



EXECUTE: The ball's moment of inertia is $I_{\text{cm}} = \frac{2}{5}MR^2$. The equations of motion are

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{\text{cm-}x} \tag{10.17}$$

$$\sum \tau_z = fR = I_{\text{cm}}\alpha_z = \left(\tfrac{2}{5}MR^2\right)\alpha_z \tag{10.18}$$

The ball rolls without slipping, so as in Example 10.6 we use $a_{\text{cm-}x} = R\alpha_z$ to eliminate α_z from Eq. (10.18):

$$fR = \tfrac{2}{5}MRa_{\text{cm-}x}$$

This equation and Eq. (10.17) are two equations for the unknowns $a_{\text{cm-}x}$ and f. We solve Eq. (10.17) for f, substitute that expression into the above equation to eliminate f, and solve for $a_{\text{cm-}x}$:

$$a_{\text{cm-}x} = \tfrac{5}{7}g\sin\beta$$

Finally, we substitute this acceleration into Eq. (10.17) and solve for f:

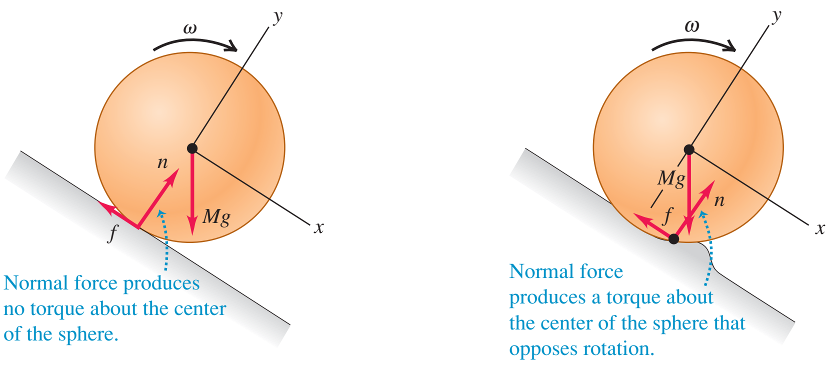
$$f = \tfrac{2}{7}Mg\sin\beta$$

Continued

EVALUATE: The ball's acceleration is just $\frac{5}{7}$ as large as that of an object *sliding* down the slope without friction. If the ball descends a vertical distance h as it rolls down the ramp, its displacement along the ramp is $h/\sin\beta$. You can show that the speed of the ball at the bottom of the ramp is $v_{cm} = \sqrt{\frac{10}{7}gh}$, the same as our result from Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill* without slipping, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

10.20 Rolling down **(a)** a perfectly rigid surface and **(b)** a deformable surface. The deformation in part **(b)** is greatly exaggerated.



Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In Fig. 10.20a a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point, but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction.* Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

Test Your Understanding of Section 10.3 Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same? MP

10.4 Work and Power in Rotational Motion

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. We can express this work in terms of torque and angular displacement.

Suppose a tangential force $\vec{F}_{tan}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (Fig. 10.21a). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt (Fig. 10.21b). The work dW done by the force $\vec{F}_{tan}$ while a point on the rim moves a distance ds is $dW = F_{tan}\,ds$. If $d\theta$ is measured in radians, then $ds = R\,d\theta$ and

$$dW = F_{tan}R\,d\theta$$

Now $F_{tan}R$ is the *torque* τ_z due to the force $\vec{F}_{tan}$, so

$$dW = \tau_z\,d\theta \qquad (10.19)$$

The total work W done by the torque during an angular displacement from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta \qquad \text{(work done by a torque)} \qquad (10.20)$$

If the torque remains *constant* while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$, then

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z\Delta\theta \qquad \text{(work done by a constant torque)} \qquad (10.21)$$

The work done by a *constant* torque is the product of torque and the angular displacement. If torque is expressed in newton-meters $(\mathrm{N \cdot m})$ and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\,dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\,d\theta = (I\alpha_z)\,d\theta = I\frac{d\omega_z}{dt}\,d\theta = I\frac{d\theta}{dt}\,d\omega_z = I\omega_z\,d\omega_z$$

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

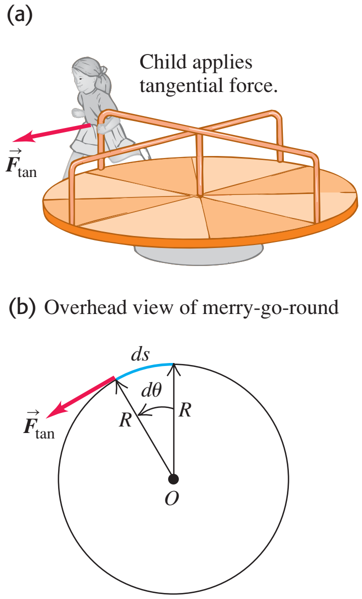
$$W_{tot} = \int_{\omega_1}^{\omega_2} I\omega_z\,d\omega_z = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (Fig. 10.22). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

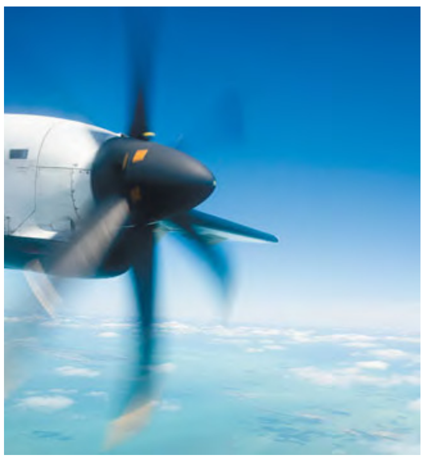
What about the *power* associated with work done by a torque acting on a rotating body? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z\frac{d\theta}{dt}$$

10.21 A tangential force applied to a rotating body does work.



10.22 The rotational kinetic energy of an airplane propeller is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the propeller by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.



But dW/dt is the rate of doing work, or *power P,* and $d\theta/dt$ is angular velocity ω_z, so

$$P = \tau_z \omega_z \tag{10.23}$$

When a torque τ_z (with respect to the axis of rotation) acts on a body that rotates with angular velocity ω_z, its power (rate of doing work) is the product of τ_z and ω_z. This is the analog of the relationship $P = \vec{F} \cdot \vec{v}$ that we developed in Section 6.4 for particle motion.

Example 10.8 Calculating power from torque

An electric motor exerts a constant 10-N·m torque on a grindstone, which has a moment of inertia of 2.0 kg·m^2 about its shaft. The system starts from rest. Find the work W done by the motor in 8.0 s and the grindstone kinetic energy K at this time. What average power P_{av} is delivered by the motor?

SOLUTION

IDENTIFY and SET UP: The only torque acting is that due to the motor. Since this torque is constant, the grindstone's angular acceleration α_z is constant. We'll use Eq. (10.7) to find α_z, and then use this in the kinematics equations from Section 9.2 to calculate the angle $\Delta\theta$ through which the grindstone rotates in 8.0 s and its final angular velocity ω_z. From these we'll calculate W, K, and P_{av}.

EXECUTE: We have $\sum \tau_z = 10\ \text{N}\cdot\text{m}$ and $I = 2.0\ \text{kg}\cdot\text{m}^2$, so $\sum \tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \text{rad/s}^2$. From Eq. (9.11),

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \text{rad/s}^2)(8.0\ \text{s})^2 = 160\ \text{rad}$$

$$W = \tau_z \Delta\theta = (10\ \text{N}\cdot\text{m})(160\ \text{rad}) = 1600\ \text{J}$$

From Eqs. (9.7) and (9.17),

$$\omega_z = \alpha_z t = (5.0\ \text{rad/s}^2)(8.0\ \text{s}) = 40\ \text{rad/s}$$

$$K = \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}(2.0\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})^2 = 1600\ \text{J}$$

The average power is the work done divided by the time interval:

$$P_{av} = \frac{1600\ \text{J}}{8.0\ \text{s}} = 200\ \text{J/s} = 200\ \text{W}$$

EVALUATE: The initial kinetic energy was zero, so the work done W must equal the final kinetic energy K [Eq. (10.22)]. This is just as we calculated. We can check our result $P_{av} = 200\ \text{W}$ by considering the *instantaneous* power $P = \tau_z\omega_z$. Because ω_z increases continuously, P increases continuously as well; its value increases from zero at $t = 0$ to $(10\ \text{N}\cdot\text{m})(40\ \text{rad/s}) = 400\ \text{W}$ at $t = 8.0\ \text{s}$. Both ω_z and P increase *uniformly* with time, so the *average* power is just half this maximum value, or 200 W.

Test Your Understanding of Section 10.4 You apply equal torques to two different cylinders, one of which has a moment of inertia twice as large as the other cylinder. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) the cylinder with the larger moment of inertia; (ii) the cylinder with the smaller moment of inertia; (iii) both cylinders have the same kinetic energy.

10.5 Angular Momentum

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we will often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass m, velocity $\vec{v}$, momentum $\vec{p} = m\vec{v}$, and position vector $\vec{r}$ relative to the origin O of an inertial frame, we define angular momentum $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad \text{(angular momentum of a particle)} \tag{10.24}$$

The value of $\vec{L}$ depends on the choice of origin O, since it involves the particle's position vector relative to O. The units of angular momentum are $\text{kg} \cdot \text{m}^2/\text{s}$.

In Fig. 10.23 a particle moves in the xy-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

$$L = mvr\sin\phi = mvl \tag{10.25}$$

where l is the perpendicular distance from the line of $\vec{v}$ to O. This distance plays the role of "lever arm" for the momentum vector.

When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$, obtaining

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad \text{(for a particle acted on by net force } \vec{F}\text{)} \tag{10.26}$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.4), which states that the rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the z-axis with angular speed ω. First consider a thin slice of the body lying in the xy-plane (Fig. 10.24). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{v}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass m_i at a distance r_i from O has a speed v_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

$$L_i = m_i(r_i\omega)r_i = m_i r_i^2\omega \tag{10.27}$$

The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

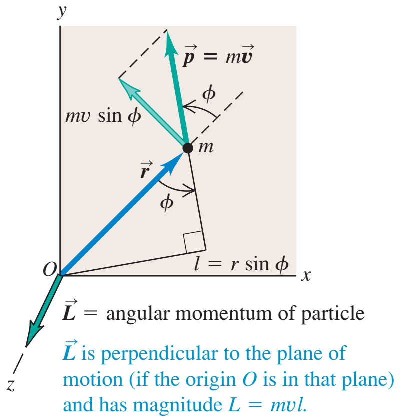
The *total* angular momentum of the slice of the body lying in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of the particles. Summing Eq. (10.27), we have

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

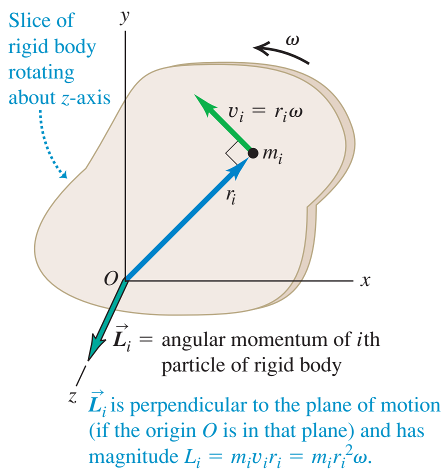
where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But *if the z-axis is an axis of symmetry,* the perpendicular components for particles on opposite sides of this axis add up to zero (Fig. 10.25). So when a body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

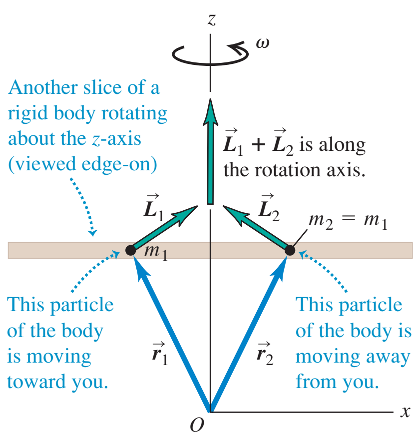
10.23 Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass m moving in the xy-plane.



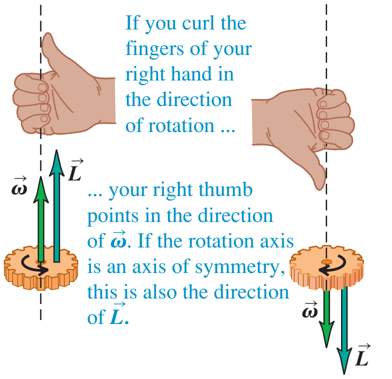
10.24 Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω. (Compare Fig. 10.23.)



10.25 Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does.



10.26 For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).



The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we discussed at the end of Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (Fig. 10.26). So we have the *vector* relationship

$$\vec{L} = I\vec{\omega} \qquad \text{(for a rigid body rotating around a symmetry axis)} \qquad (10.28)$$

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (A similar cancellation occurred in our discussion of center-of-mass motion in Section 8.5.) If the total angular momentum of the system of particles is $\vec{L}$ and the sum of the external torques is $\sum\vec{\tau}$, then

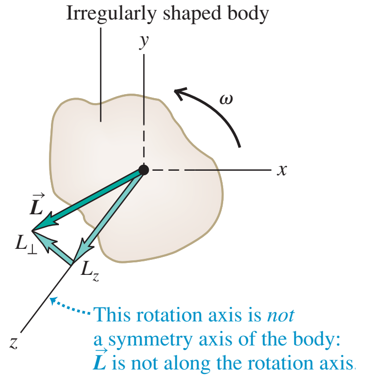
$$\sum\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \text{(for any system of particles)} \qquad (10.29)$$

Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then the vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

$$\sum\tau_z = I\alpha_z$$

which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change, and in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

10.27 If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.



When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (Fig. 10.27). As the body turns, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

Example 10.9 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$. (a) Find the fan's angular momentum as a function of time, and find its value at $t = 3.0$ s. (b) Find the net torque on the fan as a function of time, and find its value at $t = 3.0$ s.

SOLUTION

IDENTIFY and SET UP: The fan rotates about its axis of symmetry (the z-axis). Hence the angular momentum vector has only a z-component L_z, which we can determine from the angular velocity ω_z. Since the direction of angular momentum is constant, the net torque likewise has only a component τ_z along the rotation axis. We'll use Eq. (10.28) to find L_z from ω_z and then use Eq. (10.29) to find τ_z.

EXECUTE: (a) From Eq. (10.28),

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2 = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

(b) From Eq. (10.29),

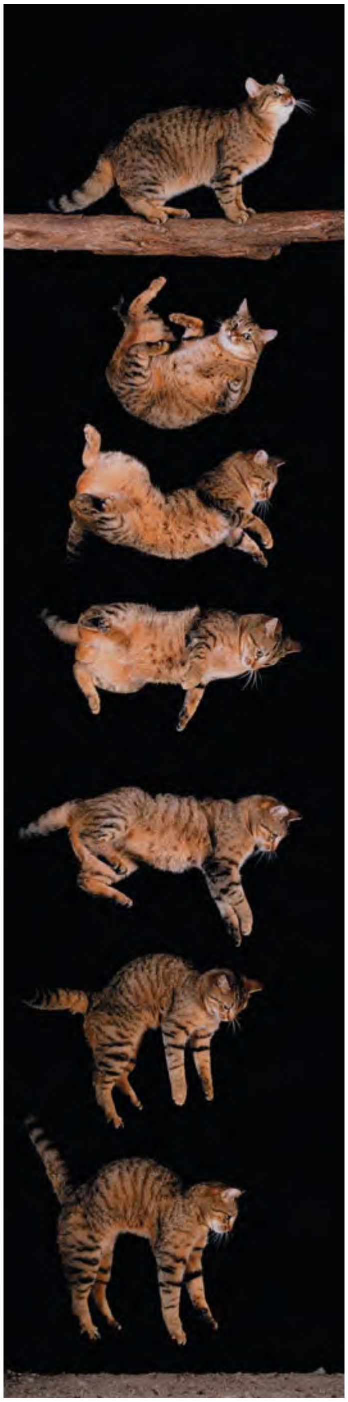
$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s}) = 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

EVALUATE: As a check on our expression for τ_z, note that the angular acceleration of the turbine is $\alpha_z = d\omega_z/dt = (40\ \text{rad/s}^3)(2t) = (80\ \text{rad/s}^3)t$. Hence from Eq. (10.7), the torque on the fan is $\tau_z = I\alpha_z = (2.5\ \text{kg}\cdot\text{m}^2)(80\ \text{rad/s}^3)t = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$, just as we calculated.

Test Your Understanding of Section 10.5 A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?



10.28 A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

10.6 Conservation of Angular Momentum

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the **principle of conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum\vec{\tau} = d\vec{L}/dt$. If $\sum\vec{\tau} = \mathbf{0}$, then $d\vec{L}/dt = \mathbf{0}$, and $\vec{L}$ is constant.

When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

A circus acrobat, a diver, and an ice skater pirouetting on the toe of one skate all take advantage of this principle. Suppose an acrobat has just left a swing with arms and legs extended and rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \qquad \text{(zero net external torque)} \qquad (10.30)$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two bodies A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.8). Suppose body A exerts a force $\vec{F}_{A\text{ on }B}$ on body B; the corresponding torque (with respect to whatever point we choose) is $\vec{\tau}_{A\text{ on }B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\tau}_{A\text{ on }B} = \frac{d\vec{L}_B}{dt}$$

At the same time, body B exerts a force $\vec{F}_{B\text{ on }A}$ on body A, with a corresponding torque $\vec{\tau}_{B\text{ on }A}$, and

$$\vec{\tau}_{B\text{ on }A} = \frac{d\vec{L}_A}{dt}$$

From Newton's third law, $\vec{F}_{B \text{ on } A} = -\vec{F}_{A \text{ on } B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\tau}_{B \text{ on } A} = -\vec{\tau}_{A \text{ on } B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = \mathbf{0}$$

or, because $\vec{L}_A + \vec{L}_B$ is the *total* angular momentum $\vec{L}$ of the system,

$$\frac{d\vec{L}}{dt} = \mathbf{0} \qquad \text{(zero net external torque)} \tag{10.31}$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one body to the other, but they can't change the *total* angular momentum of the system (Fig. 10.28).

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Example 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0-kg dumbbell in each hand (Fig. 10.29). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells in to his stomach. His moment of inertia (without the dumbbells) is 3.0 kg·m^2 with arms outstretched and 2.2 kg·m^2 with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: No external torques act about the z-axis, so L_z is constant. We'll use Eq. (10.30) to find the final angular velocity ω_{2z}. The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. We treat each dumbbell as a particle of mass m that contributes mr^2 to $I_{\text{dumbbells}}$, where r is the perpendicular distance from the axis to the dumbbell. Initially we have

$$I_1 = 3.0 \text{ kg}\cdot\text{m}^2 + 2(5.0 \text{ kg})(1.0 \text{ m})^2 = 13 \text{ kg}\cdot\text{m}^2$$

$$\omega_{1z} = \frac{1 \text{ rev}}{2.0 \text{ s}} = 0.50 \text{ rev/s}$$

The final moment of inertia is

$$I_2 = 2.2 \text{ kg}\cdot\text{m}^2 + 2(5.0 \text{ kg})(0.20 \text{ m})^2 = 2.6 \text{ kg}\cdot\text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13 \text{ kg}\cdot\text{m}^2}{2.6 \text{ kg}\cdot\text{m}^2}(0.50 \text{ rev/s}) = 2.5 \text{ rev/s} = 5\omega_{1z}$$

Can you see why we didn't have to change "revolutions" to "radians" in this calculation?

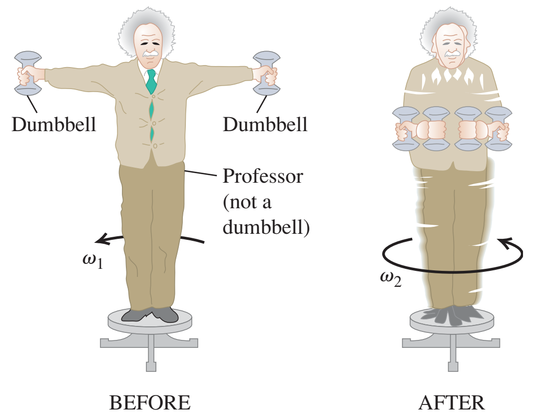
EVALUATE: The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50 \text{ rev/s})(2\pi \text{ rad/rev}) = 3.14$ rad/s to $\omega_{2z} = (2.5 \text{ rev/s})(2\pi \text{ rad/rev}) = 15.7$ rad/s. The initial and final kinetic energies are then

$$K_1 = \tfrac{1}{2}I_1\omega_{1z}^2 = \tfrac{1}{2}(13 \text{ kg}\cdot\text{m}^2)(3.14 \text{ rad/s})^2 = 64 \text{ J}$$

$$K_2 = \tfrac{1}{2}I_2\omega_{2z}^2 = \tfrac{1}{2}(2.6 \text{ kg}\cdot\text{m}^2)(15.7 \text{ rad/s})^2 = 320 \text{ J}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

10.29 Fun with conservation of angular momentum.



Example 10.11 A rotational "collision"

Figure 10.30 shows two disks: an engine flywheel (A) and a clutch plate (B) attached to a transmission shaft. Their moments of inertia are I_A and I_B; initially, they are rotating with constant angular speeds ω_A and ω_B, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed ω. Derive an expression for ω.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: There are no external torques, so the only torque acting on either disk is the torque applied by the other disk. Hence the total angular momentum of the system of two disks is conserved. At the end they rotate together as one body with total moment of inertia $I = I_A + I_B$ and angular speed ω.

10.30 When the net external torque is zero, angular momentum is conserved.

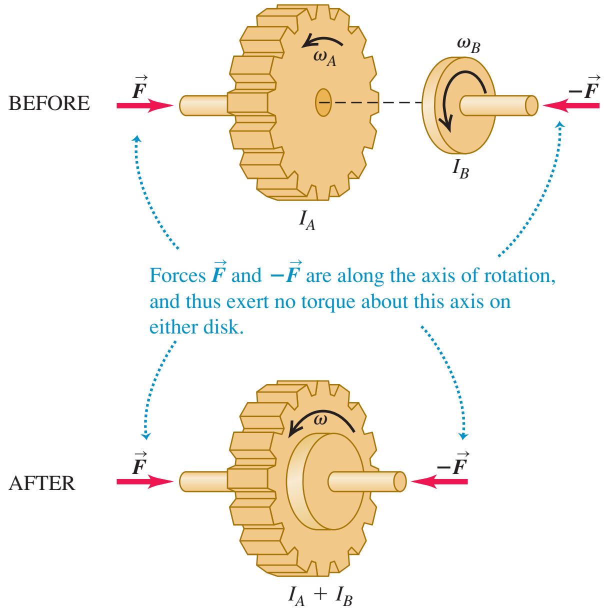


Figure 10.30 shows that all angular velocities are in the same direction, so we can regard ω_A, ω_B, and ω as components of angular velocity along the rotation axis. Conservation of angular momentum gives

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$

EVALUATE: This "collision" is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the same axis collide and stick, the linear momentum of the system is conserved. Here two objects in *rotational* motion around the same axis "collide" and stick, and the *angular* momentum of the system is conserved.

The kinetic energy of a system decreases in a completely inelastic collision. Here kinetic energy is lost because nonconservative (frictional) internal forces act while the two disks rub together. Suppose flywheel A has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm), and clutch plate B has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Can you show that the final kinetic energy is only two-thirds of the initial kinetic energy?

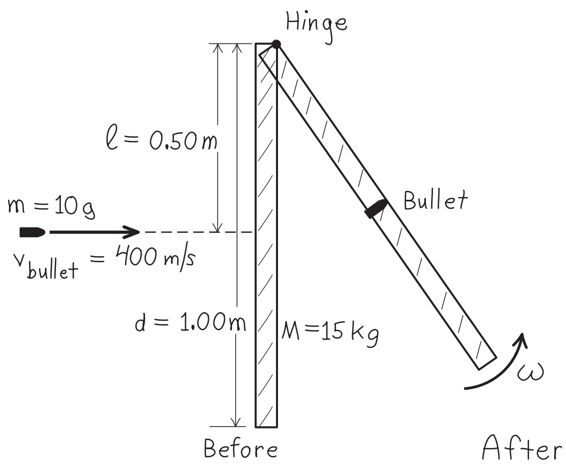
Example 10.12 Angular momentum in a crime bust

A door 1.00 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door, in a direction perpendicular to the plane of the door, and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

SOLUTION

IDENTIFY and SET UP: We consider the door and bullet as a system. There is no external torque about the hinge axis, so angular momentum about this axis is conserved. Figure 10.31 shows our sketch. The initial angular momentum is that of the bullet, as given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll equate these quantities and solve for the resulting angular speed ω of the door and bullet.

10.31 Our sketch for this problem.



EXECUTE: From Eq. (10.25), the initial angular momentum of the bullet is

$$L = mvl = (0.010\ \text{kg})(400\ \text{m/s})(0.50\ \text{m}) = 2.0\ \text{kg}\cdot\text{m}^2/\text{s}$$

The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\ \text{kg})(1.00\ \text{m})^2}{3} = 5.0\ \text{kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\ \text{kg})(0.50\ \text{m})^2 = 0.0025\ \text{kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

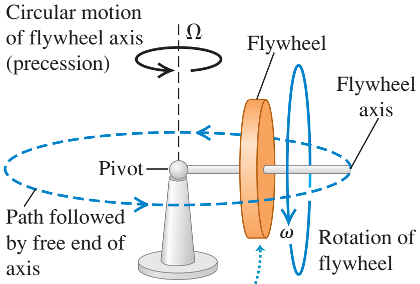
$$\omega = \frac{mvl}{I} = \frac{2.0\ \text{kg}\cdot\text{m}^2/\text{s}}{5.0\ \text{kg}\cdot\text{m}^2 + 0.0025\ \text{kg}\cdot\text{m}^2} = 0.40\ \text{rad/s}$$

The initial and final kinetic energies are

$$K_1 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.010\ \text{kg})(400\ \text{m/s})^2 = 800\ \text{J}$$

$$K_2 = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(5.0025\ \text{kg}\cdot\text{m}^2)(0.40\ \text{rad/s})^2 = 0.40\ \text{J}$$

EVALUATE: The final kinetic energy is only $\frac{1}{2000}$ of the initial value! We did not expect kinetic energy to be conserved: The collision is inelastic because nonconservative friction forces act during the impact. The door's final angular speed is quite slow: At 0.40 rad/s, it takes 3.9 s to swing through 90° ($\pi/2$ radians).

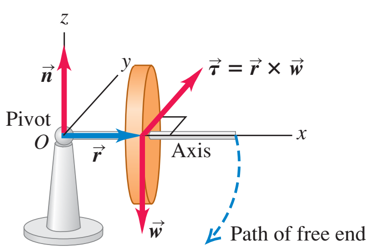


10.32 A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω.

When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

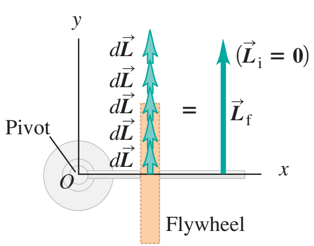
10.33 (a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero (b) In each successive time interval dt, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

(a) Nonrotating flywheel falls



When the flywheel is not rotating, its weight creates a torque around the pivot, causing it to fall along a circular path until its axis rests on the table surface.

(b) View from above as flywheel falls



In falling, the flywheel rotates about the pivot and thus acquires an angular momentum $\vec{L}$. The *direction* of $\vec{L}$ stays constant.

Test Your Understanding of Section 10.6 If the polar ice caps were to completely melt due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

10.7 Gyroscopes and Precession

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation can change direction. For example, consider a toy gyroscope that's supported at one end (Fig. 10.32). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

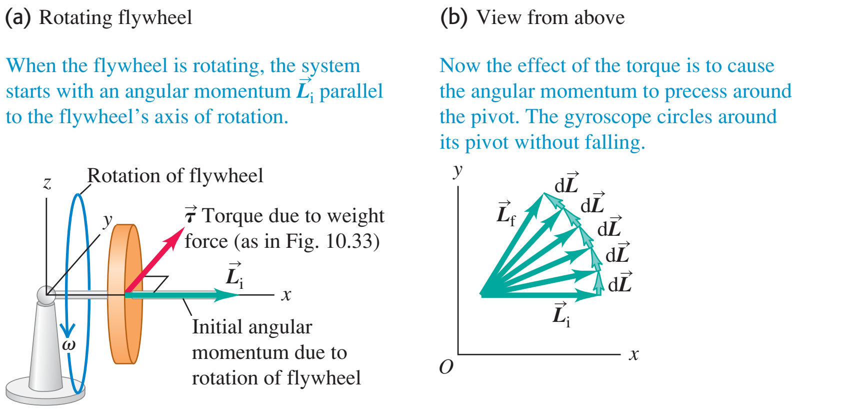
To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on a body and the rate of change of the body's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (Fig. 10.33a). We take the origin O at the pivot and assume that the flywheel is symmetrical, with mass M and moment of inertia I about the flywheel axis. The flywheel axis is initially along the x-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance r from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the y-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval dt following this is

$$d\vec{L} = \vec{\tau}\,dt \qquad (10.32)$$

This change is in the y-direction because $\vec{\tau}$ is. As each additional time interval dt elapses, the angular momentum changes by additional increments $d\vec{L}$ in the y-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the y-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (Fig. 10.34a). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along the axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to the axis (Fig. 10.34b). This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal xy-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. In other words, the axis doesn't fall—it just precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the



10.34 (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\,dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time dt, the ball acquires a momentum $d\vec{p} = \vec{F}\,dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval dt later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in Fig. 10.35 shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by Ω, we find

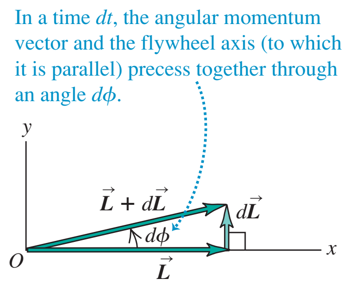
$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega} \qquad (10.33)$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases!* The precession angular speed of the earth is very slow (1 rev/26,000 yr) because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small. ?

As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated only with the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

10.35 Detailed view of part of Fig. 10.34b.



Example 10.13 A precessing gyroscope

Figure 10.36a shows a top view of a spinning, cylindrical gyroscope wheel. The pivot is at O, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyroscope takes 4.0 s for one revolution of precession, what is the angular speed of the wheel?

SOLUTION

IDENTIFY and SET UP: We'll determine the direction of precession using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed Ω and spin angular speed ω, Eq. (10.33), to find ω.

EXECUTE: (a) The right-hand rule shows that $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{L}}$ are to the left in Fig. 10.36b. The weight $\vec{\boldsymbol{w}}$ points into the page in this top view and acts at the center of mass (denoted by × in the figure). The torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{w}}$ is toward the top of the page, so $d\vec{\boldsymbol{L}}/dt$ is also toward the top of the page. Adding a small $d\vec{\boldsymbol{L}}$ to the initial vector $\vec{\boldsymbol{L}}$ changes the direction of $\vec{\boldsymbol{L}}$ as shown, so the precession is clockwise as seen from above.

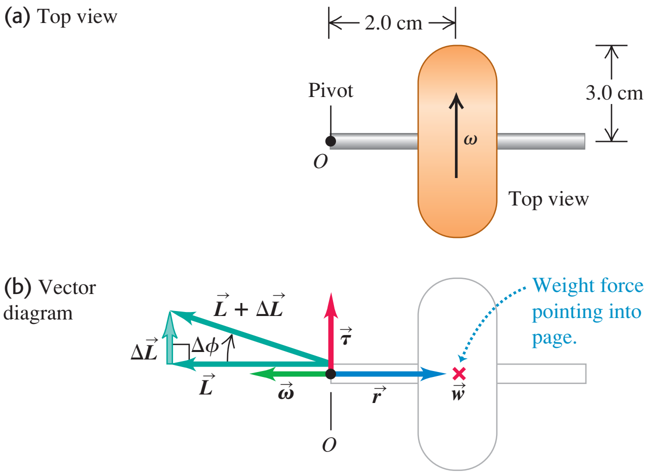
(b) Be careful not to confuse ω and Ω! The precession angular speed is $\Omega = (1\text{ rev})/(4.0\text{ s}) = (2\pi\text{ rad})/(4.0\text{ s}) = 1.57\text{ rad/s}$. The weight is mg, and if the wheel is a solid, uniform cylinder, its moment of inertia about its symmetry axis is $I = \frac{1}{2}mR^2$. From Eq. (10.33),

$$\omega = \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega}$$

$$= \frac{2(9.8\text{ m/s}^2)(2.0 \times 10^{-2}\text{ m})}{(3.0 \times 10^{-2}\text{ m})^2(1.57\text{ rad/s})} = 280\text{ rad/s} = 2600\text{ rev/min}$$

EVALUATE: The precession angular speed Ω is only about 0.6% of the spin angular speed ω, so this is an example of slow precession.

10.36 In which direction and at what speed does this gyroscope precess?



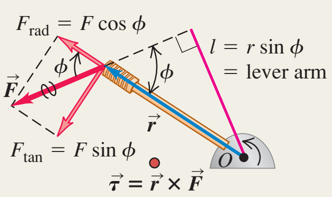
Test Your Understanding of Section 10.7 Suppose the mass of the flywheel in Fig. 10.34 were doubled but all other dimensions and the spin angular speed remained the same. What effect would this change have on the precession angular speed Ω? (i) Ω would increase by a factor of 4; (ii) Ω would double; (iii) Ω would be unaffected; (iv) Ω would be one-half as much; (v) Ω would be one-quarter as much.

CHAPTER 10 SUMMARY

Torque: When a force $\vec{F}$ acts on a body, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1.)

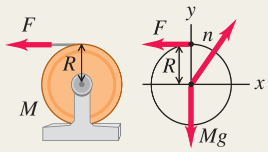
$$\tau = Fl \qquad (10.2)$$

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (10.3)$$



Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on a body equals the product of the body's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

$$\sum \tau_z = I\alpha_z \qquad (10.7)$$



Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.7.)

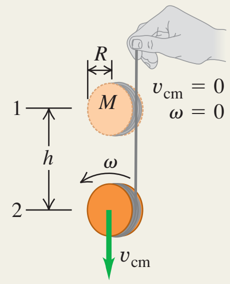
$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2 \qquad (10.8)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

$$\sum \tau_z = I_{cm}\alpha_z \qquad (10.13)$$

$$v_{cm} = R\omega \qquad (10.11)$$

(rolling without slipping)



Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.8.)

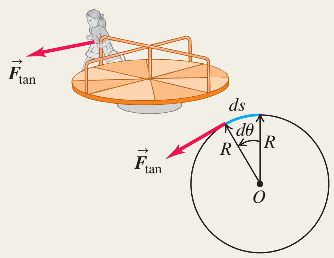
$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta \qquad (10.20)$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \qquad (10.21)$$

(constant torque only)

$$W_{tot} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \qquad (10.22)$$

$$P = \tau_z \omega_z \qquad (10.23)$$



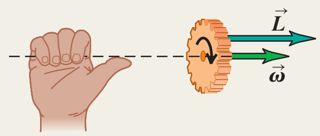
Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical body rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the body is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.9.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad (10.24)$$

(particle)

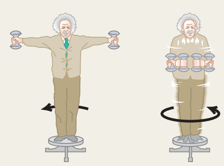
$$\vec{L} = I\vec{\omega} \qquad (10.28)$$

(rigid body rotating about axis of symmetry)



Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.10–10.13.)

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \qquad (10.29)$$



BRIDGING PROBLEM Billiard Physics

A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (Fig. 10.37). The force of the hit is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a short time Δt. (a) For what value of h will the ball roll without slipping? (b) If you hit the ball dead center $(h = 0)$, the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then?

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Draw a free-body diagram for the ball for the situation in part (a), including your choice of coordinate axes. Note that the cue exerts both an impulsive force on the ball and an impulsive torque around the center of mass.
2. The cue force applied for a time Δt gives the ball's center of mass a speed v_{cm}, and the cue torque applied for that same time gives the ball an angular speed ω. What must be the relationship between v_{cm} and ω for the ball to roll without slipping?
3. Draw two free-body diagrams for the ball in part (b): one showing the forces during the hit and the other showing the forces after the hit but before the ball is rolling without slipping.
4. What is the angular speed of the ball in part (b) just after the hit? While the ball is sliding, does v_{cm} increase or decrease? Does ω increase or decrease? What is the relationship between v_{cm} and ω when the ball is finally rolling without slipping?

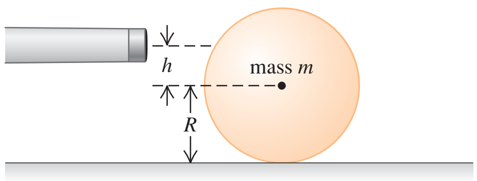
EXECUTE

5. In part (a), use the impulse–momentum theorem to find the speed of the ball's center of mass immediately after the hit. Then use the rotational version of the impulse–momentum theorem to find the angular speed immediately after the hit. (*Hint:* To write down the rotational version of the impulse–momentum theorem, remember that the relationship between torque and angular momentum is the same as that between force and linear momentum.)
6. Use your results from step 5 to find the value of h that will cause the ball to roll without slipping immediately after the hit.
7. In part (b), again find the ball's center-of-mass speed and angular speed immediately after the hit. Then write Newton's second law for the translational motion and rotational motion of the ball as it is sliding. Use these equations to write expressions for v_{cm} and ω as functions of the elapsed time t since the hit.
8. Using your results from step 7, find the time t when v_{cm} and ω have the correct relationship for rolling without slipping. Then find the value of v_{cm} at this time.

EVALUATE

9. If you have access to a pool table, test out the results of parts (a) and (b) for yourself!
10. Can you show that if you used a hollow cylinder rather than a solid ball, you would have to hit the top of the cylinder to cause rolling without slipping as in part (a)?

10.37



Problems

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•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q10.1 When cylinder-head bolts in an automobile engine are tightened, the critical quantity is the *torque* applied to the bolts. Why is the torque more important than the actual *force* applied to the wrench handle?

Q10.2 Can a single force applied to a body change both its translational and rotational motion? Explain.

Q10.3 Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

Q10.4 A four-wheel-drive car is accelerating forward from rest. Show the direction the car's wheels turn and how this causes a friction force due to the pavement that accelerates the car forward.

Q10.5 Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

Q10.6 The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

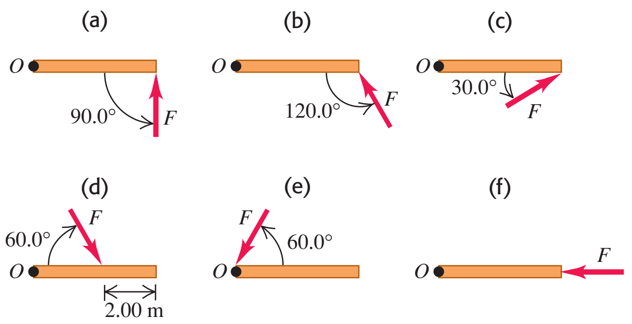
Q10.7 When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]
Q10.8 When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?
Q10.9 Experienced cooks can tell whether an egg is raw or hard-boiled by rolling it down a slope (taking care to catch it at the bottom). How is this possible? What are they looking for?
Q10.10 The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.
Q10.11 A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.
Q10.12 You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration α, what will be the angular acceleration of the larger version in terms of α?
Q10.13 Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.
Q10.14 The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?
Q10.15 A certain solid uniform ball reaches a maximum height h_0 when it rolls up a hill without slipping. What maximum height (in terms of h_0) will it reach if you (a) double its diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?
Q10.16 A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?
Q10.17 Part of the kinetic energy of a moving automobile is in the rotational motion of its wheels. When the brakes are applied hard on an icy street, the wheels "lock" and the car starts to slide. What becomes of the rotational kinetic energy?
Q10.18 A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.
Q10.19 A ball is rolling along at speed v without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answers in both cases in terms of energy conservation and in terms of Newton's second law.
Q10.20 You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain your answer.
Q10.21 A certain uniform turntable of diameter D_0 has an angular momentum L_0. If you want to redesign it so it retains the same mass but has twice as much angular momentum at the same angular velocity as before, what should be its diameter in terms of D_0?
Q10.22 A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance l. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?
Q10.23 In Example 10.10 (Section 10.6) the angular speed ω changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), α_z must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.
Q10.24 In Example 10.10 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning that leads to this apparent contradiction. Where *does* the extra kinetic energy come from?
Q10.25 As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?
Q10.26 If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.
Q10.27 A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?
Q10.28 In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.
Q10.29 A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.
Q10.30 A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?
Q10.31 A bullet spins on its axis as it emerges from a rifle. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

EXERCISES

Section 10.1 Torque

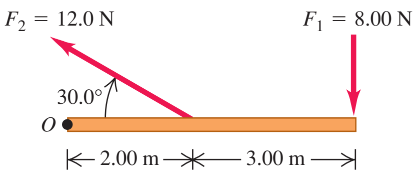
10.1 • Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the cases sketched in Fig. E10.1. In each case, the force $\vec{F}$ and the rod both lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure **E10.1**



10.2 • Calculate the net torque about point O for the two forces applied as in Fig. E10.2. The rod and both forces are in the plane of the page.

Figure **E10.2**



10.3 •• A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (Fig. E10.3). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure **E10.3**

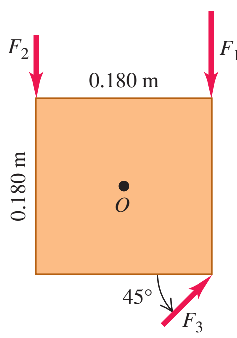
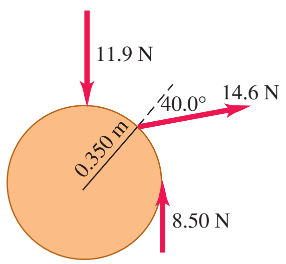


Figure **E10.4**



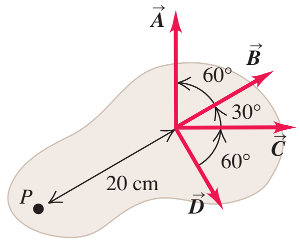
10.4 • Three forces are applied to a wheel of radius 0.350 m, as shown in Fig. E10.4. One force is perpendicular to the rim, one is tangent to it, and the other one makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

10.5 • One force acting on a machine part is $\vec{F} = (-5.00\text{ N})\hat{\imath} + (4.00\text{ N})\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\text{ m})\hat{\imath} + (0.150\text{ m})\hat{\jmath}$. (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

10.6 • A metal bar is in the xy-plane with one end of the bar at the origin. A force $\vec{F} = (7.00\text{ N})\hat{\imath} + (-3.00\text{ N})\hat{\jmath}$ is applied to the bar at the point $x = 3.00$ m, $y = 4.00$ m. (a) In terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$, what is the position vector $\vec{r}$ for the point where the force is applied? (b) What are the magnitude and direction of the torque with respect to the origin produced by $\vec{F}$?

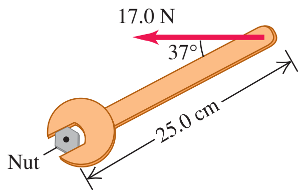
10.7 • In Fig. E10.7, forces $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ each have magnitude 50 N and act at the same point on the object. (a) What torque (magnitude and direction) does each of these forces exert on the object about point P? (b) What is the total torque about point P?

Figure **E10.7**



10.8 • A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0-N force at the end of the handle at 37° with the handle (Fig. E10.8). (a) What torque does the machinist exert about the center of the nut? (b) What is the maximum torque he could exert with this force, and how should the force be oriented?

Figure **E10.8**



Section 10.2 Torque and Angular Acceleration for a Rigid Body

10.9 •• The flywheel of an engine has moment of inertia $2.50\text{ kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

10.10 •• A uniform disk with mass 40.0 kg and radius 0.200 m is pivoted at its center about a horizontal, frictionless axle that is stationary. The disk is initially at rest, and then a constant force $F = 30.0$ N is applied tangent to the rim of the disk. (a) What is the magnitude v of the tangential velocity of a point on the rim of the disk after the disk has turned through 0.200 revolution? (b) What is the magnitude a of the resultant acceleration of a point on the rim of the disk after the disk has turned through 0.200 revolution?

10.11 •• A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

10.12 • A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

10.13 •• **CP** A 2.00-kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

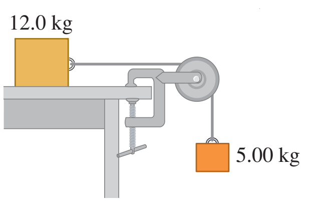
10.14 •• **CP** A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 50.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

10.15 • A wheel rotates without friction about a stationary horizontal axis at the center of the wheel. A constant tangential force equal to 80.0 N is applied to the rim of the wheel. The wheel has radius 0.120 m. Starting from rest, the wheel has an angular speed of 12.0 rev/s after 2.00 s. What is the moment of inertia of the wheel?

10.16 •• **CP** A 15.0-kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

10.17 •• A 12.0-kg box resting on a horizontal, frictionless surface is attached to a 5.00-kg weight by a thin, light wire that passes over a frictionless pulley (Fig. E10.17). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure **E10.17**



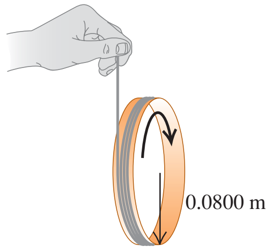
Section 10.3 Rigid-Body Rotation About a Moving Axis

10.18 • **BIO** **Gymnastics.** We can roughly model a gymnastic tumbler as a uniform solid cylinder of mass 75 kg and diameter 1.0 m. If this tumbler rolls forward at 0.50 rev/s, (a) how much total kinetic energy does he have, and (b) what percent of his total kinetic energy is rotational?

10.19 • A 2.20-kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 3.00 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop; (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), but this time as viewed by someone moving along with the same velocity as the hoop.

10.20 •• A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (Fig. E10.20). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure **E10.20**



10.21 • What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) a uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius R and inner radius $R/2$.

10.22 •• A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

10.23 •• A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

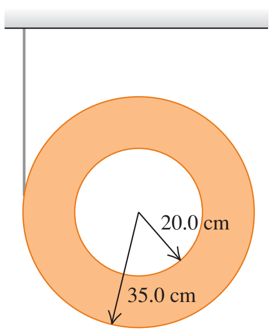
10.24 •• A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

10.25 •• A 392-N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height h above the bottom of the hill; this work has absolute value 3500 J. Calculate h.

10.26 •• **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill.* (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

10.27 •• A thin, light string is wrapped around the outer rim of a uniform hollow cylinder of mass 4.75 kg having inner and outer radii as shown in Fig. E10.27. The cylinder is then released from rest.

Figure **E10.27**



(a) How far must the cylinder fall before its center is moving at 6.66 m/s? (b) If you just dropped this cylinder without any string, how fast would its center be moving when it had fallen the distance in part (a)? (c) Why do you get two different answers when the cylinder falls the same distance in both cases?

10.28 •• A bicycle racer is going downhill at 11.0 m/s when, to his horror, one of his 2.25-kg wheels comes off as he is 75.0 m above the foot of the hill. We can model the wheel as a thin-walled cylinder 85.0 cm in diameter and neglect the small mass of the spokes. (a) How fast is the wheel moving when it reaches the foot of the hill if it rolled without slipping all the way down? (b) How much total kinetic energy does the wheel have when it reaches the bottom of the hill?

10.29 •• A size-5 soccer ball of diameter 22.6 cm and mass 426 g rolls up a hill without slipping, reaching a maximum height of 5.00 m above the base of the hill. We can model this ball as a thin-walled hollow sphere. (a) At what rate was it rotating at the base of the hill? (b) How much rotational kinetic energy did it have then?

Section 10.4 Work and Power in Rotational Motion

10.30 • An engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

10.31 • A playground merry-go-round has radius 2.40 m and moment of inertia 2100 $kg \cdot m^2$ about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0-N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0-s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

10.32 •• An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

10.33 • A 1.50-kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

10.34 •• An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant torque of 1950 $N \cdot m$ to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

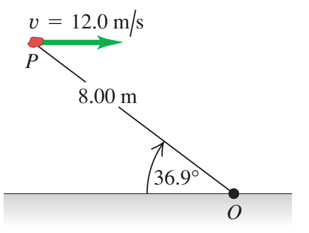
10.35 • (a) Compute the torque developed by an industrial motor whose output is 150 kW at an angular speed of 4000 rev/min. (b) A drum with negligible mass, 0.400 m in diameter, is attached to the motor shaft, and the power output of the motor is used to raise a weight hanging from a rope wrapped around the drum. How heavy a weight can the motor lift at constant speed? (c) At what constant speed will the weight rise?

Section 10.5 Angular Momentum

10.36 •• A woman with mass 50 kg is standing on the rim of a large disk that is rotating at 0.50 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman–disk system. (Assume that you can treat the woman as a point.)

10.37 • A 2.00-kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point P in Fig. E10.37. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point O? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure **E10.37**



10.38 •• (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

10.39 •• Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

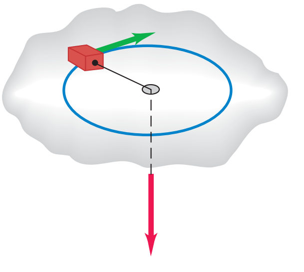
10.40 •• **CALC** A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where A has numerical value 1.50 and B has numerical value 1.10. (a) What are the units of the constants A and B? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

Section 10.6 Conservation of Angular Momentum

10.41 •• Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star*. The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

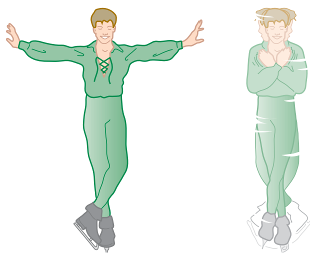
10.42 • **CP** A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (Fig. E10.42). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 1.75 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is the angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure **E10.42**



10.43 •• **The Spinning Figure Skater.** The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (Fig. E10.43). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to $0.40\ \text{kg} \cdot \text{m}^2$. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure **E10.43**



10.44 •• A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of $18\ \text{kg} \cdot \text{m}^2$. She then tucks into a small ball, decreasing this moment of inertia to $3.6\ \text{kg} \cdot \text{m}^2$. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

10.45 •• A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0-kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

10.46 •• A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

10.47 •• A small 10.0-g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

10.48 •• **Asteroid Collision!** Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

10.49 •• A thin, uniform metal bar, 2.00 m long and weighing 90.0 N, is hanging vertically from the ceiling by a frictionless pivot. Suddenly it is struck 1.50 m below the ceiling by a small 3.00-kg ball, initially traveling horizontally at 10.0 m/s. The ball rebounds in the opposite direction with a speed of 6.00 m/s. (a) Find the angular speed of the bar just after the collision. (b) During the collision, why is the angular momentum conserved but not the linear momentum?

10.50 •• A thin uniform rod has a length of 0.500 m and is rotating in a circle on a frictionless table. The axis of rotation is perpendicular to the length of the rod at one end and is stationary. The rod has an angular velocity of 0.400 rad/s and a moment of inertia about the axis of $3.00 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$. A bug initially standing on the rod at the axis of rotation decides to crawl out to the other end of the rod. When the bug has reached the end of the rod and sits there, its tangential speed is 0.160 m/s. The bug can be treated as a point mass. (a) What is the mass of the rod? (b) What is the mass of the bug?

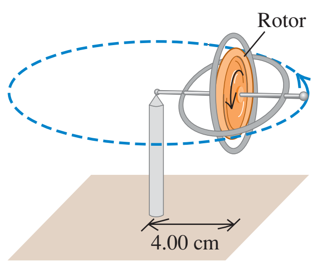
10.51 •• A uniform, 4.5-kg, square, solid wooden gate 1.5 m on each side hangs vertically from a frictionless pivot at the center of its upper edge. A 1.1-kg raven flying horizontally at 5.0 m/s flies into this door at its center and bounces back at 2.0 m/s in the opposite direction. (a) What is the angular speed of the gate just after it is struck by the unfortunate raven? (b) During the collision, why is the angular momentum conserved, but not the linear momentum?

10.52 •• **Sedna.** In November 2003, the now-most-distant-known object in the solar system was discovered by observation with a telescope on Mt. Palomar. This object, known as Sedna, is approximately 1700 km in diameter, takes about 10,500 years to orbit our sun, and reaches a maximum speed of 4.64 km/s. Calculations of its complete path, based on several measurements of its position, indicate that its orbit is highly elliptical, varying from 76 AU to 942 AU in its distance from the sun, where AU is the astronomical unit, which is the average distance of the earth from the sun (1.50×10^8 km). (a) What is Sedna's minimum speed? (b) At what points in its orbit do its maximum and minimum speeds occur? (c) What is the ratio of Sedna's maximum kinetic energy to its minimum kinetic energy?

Section 10.7 Gyroscopes and Precession

10.53 •• The rotor (flywheel) of a toy gyroscope has mass 0.140 kg. Its moment of inertia about its axis is $1.20 \times 10^{-4}\ \text{kg} \cdot \text{m}^2$. The mass of the frame is 0.0250 kg. The gyroscope is supported on a single pivot (Fig. E10.53) with its center of mass a horizontal distance of 4.00 cm from the pivot. The gyroscope is precessing in a horizontal plane at the rate of one revolution in 2.20 s. (a) Find the upward force exerted by the pivot. (b) Find the angular speed with which the rotor is spinning about its axis, expressed in rev/min. (c) Copy the diagram and draw vectors to show the angular momentum of the rotor and the torque acting on it.

Figure **E10.53**



10.54 • **A Gyroscope on the Moon.** A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is $0.165g$, what would be its precession rate?

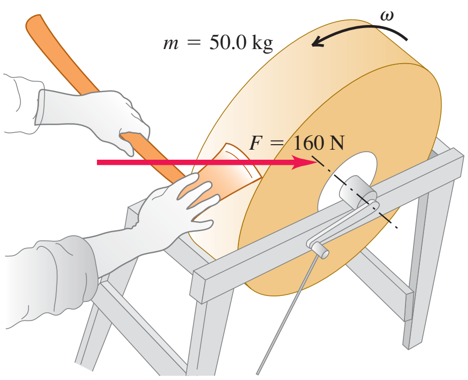
10.55 • A gyroscope is precessing about a vertical axis. Describe what happens to the precession angular speed if the following changes in the variables are made, with all other variables remaining the same: (a) the angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all four of the variables in parts (a) through (d) are doubled?

10.56 • **Stabilization of the Hubble Space Telescope.** The Hubble Space Telescope is stabilized to within an angle of about 2-millionths of a degree by means of a series of gyroscopes that spin at 19,200 rpm. Although the structure of these gyroscopes is actually quite complex, we can model each of the gyroscopes as a thin-walled cylinder of mass 2.0 kg and diameter 5.0 cm, spinning about its central axis. How large a torque would it take to cause these gyroscopes to precess through an angle of 1.0×10^{-6} degree during a 5.0-hour exposure of a galaxy?

PROBLEMS

10.57 •• A 50.0-kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (Fig. P10.57). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of 6.50 N · m between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

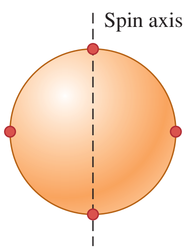
Figure **P10.57**



10.58 •• An experimental bicycle wheel is placed on a test stand so that it is free to turn on its axle. If a constant net torque of 7.00 N · m is applied to the tire for 2.00 s, the angular speed of the tire increases from 0 to 100 rev/min. The external torque is then removed, and the wheel is brought to rest by friction in its bearings in 125 s. Compute (a) the moment of inertia of the wheel about the rotation axis; (b) the friction torque; (c) the total number of revolutions made by the wheel in the 125-s time interval.

10.59 ••• A grindstone in the shape of a solid disk with diameter 0.520 m and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. P10.57), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

Figure **P10.60**

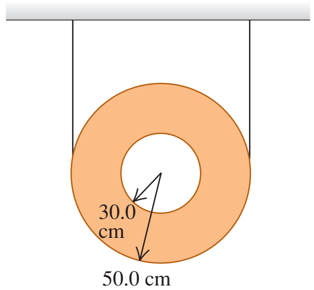


10.60 ••• A uniform, 8.40-kg, spherical shell 50.0 cm in diameter has four small 2.00-kg masses attached to its outer surface and equally spaced around it. This combination is spinning about an axis running through the center of the sphere and two of the small masses (Fig. P10.60). What friction torque is needed to reduce its angular speed from 75.0 rpm to 50.0 rpm in 30.0 s?

10.61 ••• A solid uniform cylinder with mass 8.25 kg and diameter 15.0 cm is spinning at 220 rpm on a thin, frictionless axle that passes along the cylinder axis. You design a simple friction brake to stop the cylinder by pressing the brake against the outer rim with a normal force. The coefficient of kinetic friction between the brake and rim is 0.333. What must the applied normal force be to bring the cylinder to rest after it has turned through 5.25 revolutions?

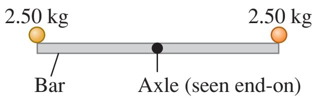
10.62 ••• A uniform hollow disk has two pieces of thin, light wire wrapped around its outer rim and is supported from the ceiling (Fig. P10.62). Suddenly one of the wires breaks, and the remaining wire does not slip as the disk rolls down. Use energy conservation to find the speed of the center of this disk after it has fallen a distance of 2.20 m.

Figure **P10.62**



10.63 ••• A thin, uniform, 3.80-kg bar, 80.0 cm long, has very small 2.50-kg balls glued on at either end (Fig. P10.63). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure **P10.63**



10.64 ••• While exploring a castle, Exena the Exterminator is spotted by a dragon that chases her down a hallway. Exena runs into a room and attempts to swing the heavy door shut before the dragon gets her. The door is initially perpendicular to the wall, so it must be turned through 90° to close. The door is 3.00 m tall and 1.25 m wide, and it weighs 750 N. You can ignore the friction at the hinges. If Exena applies a force of 220 N at the edge of the door and perpendicular to it, how much time does it take her to close the door?

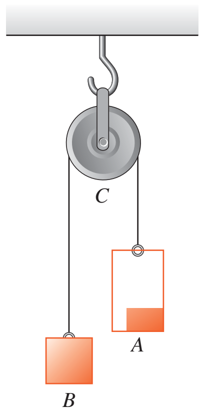
10.65 •• **CALC** You connect a light string to a point on the edge of a uniform vertical disk with radius R and mass M. The disk is free to rotate without friction about a stationary horizontal axis through its center. Initially, the disk is at rest with the string connection at the highest point on the disk. You pull the string with a constant horizontal force $\vec{F}$ until the wheel has made exactly one-quarter revolution about a horizontal axis through its center, and then you let go. (a) Use Eq. (10.20) to find the work done by the string. (b) Use Eq. (6.14) to find the work done by the string. Do you obtain the same result as in part (a)? (c) Find the final angular speed of the disk. (d) Find the maximum tangential acceleration of a point on the disk. (e) Find the maximum radial (centripetal) acceleration of a point on the disk.

10.66 ••• **Balancing Act.** Attached to one end of a long, thin, uniform rod of length L and mass M is a small blob of clay of the same mass M. (a) Locate the position of the center of mass of the system of rod and clay. Note this position on a drawing of the rod.

(b) You carefully balance the rod on a frictionless tabletop so that it is standing vertically, with the end without the clay touching the table. If the rod is now tipped so that it is a small angle θ away from the vertical, determine its angular acceleration at this instant. Assume that the end without the clay remains in contact with the tabletop. (*Hint:* See Table 9.2.) (c) You again balance the rod on the frictionless tabletop so that it is standing vertically, but now the end of the rod *with* the clay is touching the table. If the rod is again tipped so that it is a small angle θ away from the vertical, determine its angular acceleration at this instant. Assume that the end with the clay remains in contact with the tabletop. How does this compare to the angular acceleration in part (b)? (d) A pool cue is a tapered wooden rod that is thick at one end and thin at the other. You can easily balance a pool cue vertically on one finger if the thin end is in contact with your finger; this is quite a bit harder to do if the thick end is in contact with your finger. Explain why there is a difference.

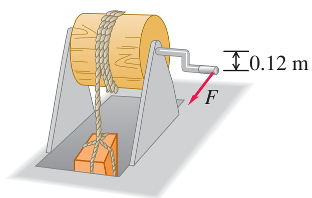
10.67 •• **Atwood's Machine.** Figure P10.67 illustrates an Atwood's machine. Find the linear accelerations of blocks A and B, the angular acceleration of the wheel C, and the tension in each side of the cord if there is no slipping between the cord and the surface of the wheel. Let the masses of blocks A and B be 4.00 kg and 2.00 kg, respectively, the moment of inertia of the wheel about its axis be $0.300\ \text{kg} \cdot \text{m}^2$, and the radius of the wheel be 0.120 m.

Figure **P10.67**



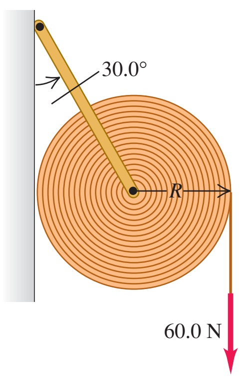
10.68 ••• The mechanism shown in Fig. P10.68 is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9\ \text{kg} \cdot \text{m}^2$ about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of $1.40\ \text{m/s}^2$? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

Figure **P10.68**



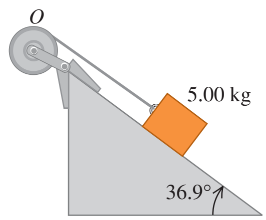
10.69 •• A large 16.0-kg roll of paper with radius $R = 18.0$ cm rests against the wall and is held in place by a bracket attached to a rod through the center of the roll (Fig. P10.69). The rod turns without friction in the bracket, and the moment of inertia of the paper and rod about the axis is $0.260\ \text{kg} \cdot \text{m}^2$. The other end of the bracket is attached by a frictionless hinge to the wall such that the bracket makes an angle of 30.0° with the wall. The weight of the bracket is negligible. The coefficient of kinetic friction between the paper and the wall is $\mu_k = 0.25$. A constant vertical force $F = 60.0$ N is applied to the paper, and the paper unrolls. (a) What is the magnitude of the force that the rod exerts on the paper as it unrolls? (b) What is the magnitude of the angular acceleration of the roll?

Figure **P10.69**



10.70 •• A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (Fig. P10.70). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has mass 25.0 kg and moment of inertia $0.500\ \text{kg} \cdot \text{m}^2$ with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200 m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

Figure **P10.70**

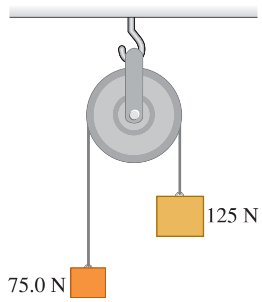


10.71 ••• Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center, as in Problem 9.87. (a) A light string is wrapped around the edge of the smaller disk, and a 1.50-kg block is suspended from the free end of the string. What is the magnitude of the downward acceleration of the block after it is released? (b) Repeat the calculation of part (a), this time with the string wrapped around the edge of the larger disk. In which case is the acceleration of the block greater? Does your answer make sense?

10.72 •• A lawn roller in the form of a thin-walled, hollow cylinder with mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

10.73 • Two weights are connected by a very light, flexible cord that passes over an 80.0-N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (Fig. P10.73). What force does the ceiling exert on the hook?

Figure **P10.73**

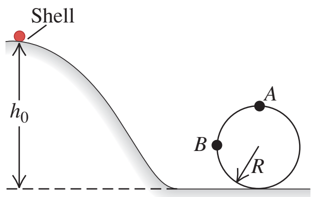


10.74 •• A solid disk is rolling without slipping on a level surface at a constant speed of 3.60 m/s. (a) If the disk rolls up a 30.0° ramp, how far along the ramp will it move before it stops? (b) Explain why your answer in part (a) does not depend on either the mass or the radius of the disk.

10.75 • **The Yo-yo.** A yo-yo is made from two uniform disks, each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

10.76 •• **CP** A thin-walled, hollow spherical shell of mass m and radius r starts from rest and rolls without slipping down the track shown in Fig. P10.76. Points A and B are on a circular part of the track having radius R. The diameter of the shell is very small compared to h_0 and R, and the work done by rolling friction is negligible. (a) What is the minimum height h_0 for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point B, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height h_0 you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point A, the top of the circle? How hard did it push on the shell in part (a)?

Figure **P10.76**

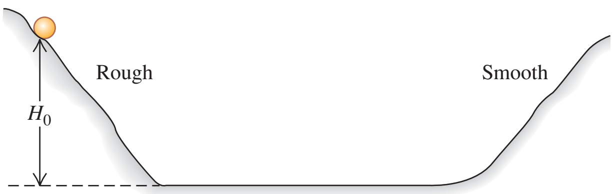


10.77 • Starting from rest, a constant force $F = 100$ N is applied to the free end of a 50-m cable wrapped around the outer rim of a uniform solid cylinder, similar to the situation shown in Fig. 10.9(a). The cylinder has mass 4.00 kg and diameter 30.0 cm and is free to turn about a fixed, frictionless axle through its center. (a) How long does it take to unwrap all the cable, and how fast is the cable moving just as the last bit comes off? (b) Now suppose that the cylinder is replaced by a uniform hoop, with all other quantities remaining unchanged. In this case, would the answers in part (a) be larger or smaller? Explain.

10.78 •• As shown in Fig. E10.20, a string is wrapped several times around the rim of a small hoop with radius 0.0800 m and mass 0.180 kg. The free end of the string is pulled upward in just the right way so that the hoop does not move vertically as the string unwinds. (a) Find the tension in the string as the string unwinds. (b) Find the angular acceleration of the hoop as the string unwinds. (c) Find the upward acceleration of the hand that pulls on the free end of the string. (d) How would your answers be different if the hoop were replaced by a solid disk of the same mass and radius?

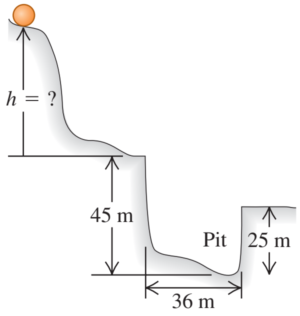
10.79 •• A basketball (which can be closely modeled as a hollow spherical shell) rolls down a mountainside into a valley and then up the opposite side, starting from rest at a height H_0 above the bottom. In Fig. P10.79, the rough part of the terrain prevents slipping while the smooth part has no friction. (a) How high, in terms of H_0, will the ball go up the other side? (b) Why doesn't the ball return to height H_0? Has it lost any of its original potential energy?

Figure **P10.79**



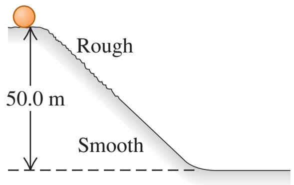
10.80 • **CP** A uniform marble rolls without slipping down the path shown in Fig. P10.80, starting from rest. (a) Find the minimum height h required for the marble not to fall into the pit. (b) The moment of inertia of the marble depends on its radius. Explain why the answer to part (a) does not depend on the radius of the marble. (c) Solve part (a) for a block that slides without friction instead of the rolling marble. How does the minimum h in this case compare to the answer in part (a)?

Figure **P10.80**



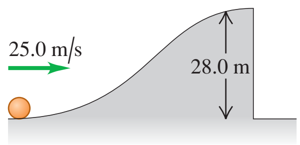
10.81 •• **Rolling Stones.** A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in Fig. P10.81. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill?

Figure **P10.81**



10.82 •• **CP** A solid uniform ball rolls without slipping up a hill, as shown in Fig. P10.82. At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. (a) How far from the foot of the cliff does the ball land, and how fast is it moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

Figure **P10.82**



10.83 •• A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

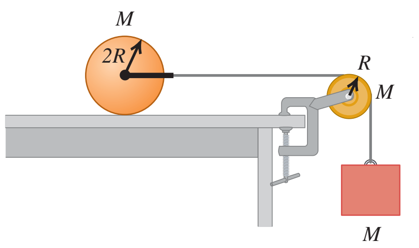
10.84 •• A child rolls a 0.600-kg basketball up a long ramp. The basketball can be considered a thin-walled, hollow sphere. When the child releases the basketball at the bottom of the ramp, it has a speed of 8.0 m/s. When the ball returns to her after rolling up the ramp and then rolling back down, it has a speed of 4.0 m/s. Assume the work done by friction on the basketball is the same when the ball moves up or down the ramp and that the basketball rolls without slipping. Find the maximum vertical height increase of the ball as it rolls up the ramp.

10.85 •• **CP** In a lab experiment you let a uniform ball roll down a curved track. The ball starts from rest and rolls without slipping. While on the track, the ball descends a vertical distance h. The lower end of the track is horizontal and extends over the edge of the lab table; the ball leaves the track traveling horizontally. While free-falling after leaving the track, the ball moves a horizontal distance x and a vertical distance y. (a) Calculate x in terms of h and y, ignoring the work done by friction. (b) Would the answer to part (a) be any different on the moon? (c) Although you do the experiment very carefully, your measured value of x is consistently a bit smaller than the value calculated in part (a). Why? (d) What would x be for the same h and y as in part (a) if you let a silver dollar roll down the track? You can ignore the work done by friction.

10.86 •• A uniform drawbridge 8.00 m long is attached to the roadway by a frictionless hinge at one end, and it can be raised by a cable attached to the other end. The bridge is at rest, suspended at 60.0° above the horizontal, when the cable suddenly breaks. (a) Find the angular acceleration of the drawbridge just after the cable breaks. (Gravity behaves as though it all acts at the center of mass.) (b) Could you use the equation $\omega = \omega_0 + \alpha t$ to calculate the angular speed of the drawbridge at a later time? Explain why. (c) What is the angular speed of the drawbridge as it becomes horizontal?

10.87 • A uniform solid cylinder with mass M and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass M and radius R that is mounted on a frictionless axle through its center. A block of mass M is suspended from the free end of the string (Fig. P10.87). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure **P10.87**



10.88 ••• A uniform, 0.0300-kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 30.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. (a) What is the angular speed of the system at the instant when the rings reach the ends of the rod? (b) What is the angular speed of the rod after the rings leave it?

10.89 ••• A 5.00-kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00-kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

10.90 •• **Tarzan and Jane in the 21st Century.** Tarzan has foolishly gotten himself into another scrape with the animals and must be rescued once again by Jane. The 60.0-kg Jane starts from rest at a height of 5.00 m in the trees and swings down to the ground using a thin, but very rigid, 30.0-kg vine 8.00 m long. She arrives just in time to snatch the 72.0-kg Tarzan from the jaws of an angry hippopotamus. What is Jane's (and the vine's) angular speed (a) just before she grabs Tarzan and (b) just after she grabs him? (c) How high will Tarzan and Jane go on their first swing after this daring rescue?

10.91 •• A uniform rod of length L rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

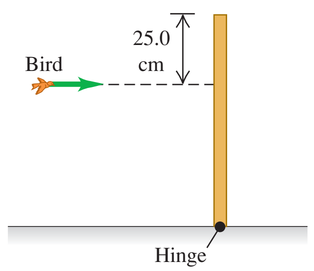
10.92 •• The solid wood door of a gymnasium is 1.00 m wide and 2.00 m high, has total mass 35.0 kg, and is hinged along one side. The door is open and at rest when a stray basketball hits the center of the door head-on, applying an average force of 1500 N to the door for 8.00 ms. Find the angular speed of the door after the impact. [*Hint:* Integrating Eq. (10.29) yields $\Delta L_z = \int_{t_1}^{t_2}(\sum\tau_z)dt = (\sum\tau_z)_{\text{av}}\Delta t$. The quantity $\int_{t_1}^{t_2}(\sum\tau_z)dt$ is called the angular impulse.]

10.93 ••• A target in a shooting gallery consists of a vertical square wooden board, 0.250 m on a side and with mass 0.750 kg, that pivots on a horizontal axis along its top edge. The board is struck face-on at its center by a bullet with mass 1.90 g that is traveling at 360 m/s and that remains embedded in the board. (a) What is the angular speed of the board just after the bullet's impact? (b) What maximum height above the equilibrium position does the center of the board reach before starting to swing down again? (c) What minimum bullet speed would be required for the board to swing all the way over after impact?

10.94 •• **Neutron Star Glitches.** Occasionally, a rotating neutron star (see Exercise 10.41) undergoes a sudden and unexpected speedup called a *glitch*. One explanation is that a glitch occurs when the crust of the neutron star settles slightly, decreasing the moment of inertia about the rotation axis. A neutron star with angular speed $\omega_0 = 70.4$ rad/s underwent such a glitch in October 1975 that increased its angular speed to $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega/\omega_0 = 2.01 \times 10^{-6}$. If the radius of the neutron star before the glitch was 11 km, by how much did its radius decrease in the starquake? Assume that the neutron star is a uniform sphere.

10.95 ••• A 500.0-g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (Fig. P10.95). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

Figure **P10.95**



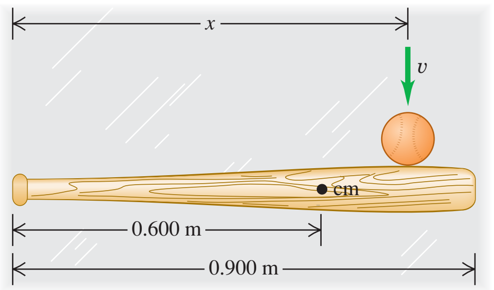
10.96 ••• **CP** A small block with mass 0.250 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. E10.42). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

10.97 • A horizontal plywood disk with mass 7.00 kg and diameter 1.00 m pivots on frictionless bearings about a vertical axis through its center. You attach a circular model-railroad track of negligible mass and average diameter 0.95 m to the disk. A 1.20-kg, battery-driven model train rests on the tracks. To demonstrate conservation of angular momentum, you switch on the train's engine. The train moves counterclockwise, soon attaining a constant speed of 0.600 m/s relative to the tracks. Find the magnitude and direction of the angular velocity of the disk relative to the earth.

10.98 • A 55-kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 $\mathrm{kg \cdot m^2}$. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

10.99 •• **Center of Percussion.** A baseball bat rests on a frictionless, horizontal surface. The bat has a length of 0.900 m, a mass of 0.800 kg, and its center of mass is 0.600 m from the handle end of the bat (Fig. P10.99). The moment of inertia of the bat about its center of mass is 0.0530 $\mathrm{kg \cdot m^2}$. The bat is struck by a baseball traveling perpendicular to the bat. The impact applies an impulse $J = \int_{t_1}^{t_2} F\,dt$ at a point a distance x from the handle end of the bat. What must x be so that the handle end of the bat remains at rest as the bat begins to move? [*Hint:* Consider the motion of the center of mass and the rotation about the center of mass. Find x so that these two motions combine to give $v = 0$ for the end of the bat just after the collision. Also, note that integration of Eq. (10.29) gives $\Delta L = \int_{t_1}^{t_2} (\Sigma\tau)\,dt$ (see Problem 10.92).] The point on the bat you have located is called the *center of percussion.* Hitting a pitched ball at the center of percussion of the bat minimizes the "sting" the batter experiences on the hands.

Figure **P10.99**



CHALLENGE PROBLEMS

10.100 ••• A uniform ball of radius R rolls without slipping between two rails such that the horizontal distance is d between the two contact points of the rails to the ball. (a) In a sketch, show that at any instant $v_{cm} = \omega\sqrt{R^2 - d^2/4}$. Discuss this expression in the limits $d = 0$ and $d = 2R$. (b) For a uniform ball starting from rest and descending a vertical distance h while rolling without slipping down a ramp, $v_{cm} = \sqrt{10gh/7}$. Replacing the ramp with the two rails, show that

$$v_{cm} = \sqrt{\frac{10gh}{5 + 2/(1 - d^2/4R^2)}}$$

In each case, the work done by friction has been ignored. (c) Which speed in part (b) is smaller? Why? Answer in terms of how the loss of potential energy is shared between the gain in translational and rotational kinetic energies. (d) For which value of the ratio d/R do the two expressions for the speed in part (b) differ by 5.0%? By 0.50%?

10.101 ••• When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that a_x and α_z are approximately zero and v_x and ω_z are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_k. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations a_x of the center of mass and α_z of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = R\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

10.102 ••• A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

10.103 ••• **CP CALC** A block with mass m is revolving with linear speed v_1 in a circle of radius r_1 on a frictionless horizontal surface (see Fig. E10.42). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to r_2. (a) Calculate the tension T in the string as a function of r, the distance of the block from the hole. Your answer will be in terms of the initial velocity v_1 and the radius r_1. (b) Use $W = \int_{r_1}^{r_2} \vec{T}(r) \cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when r changes from r_1 to r_2. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

Answers

Chapter Opening Question ?

The earth precesses like a top due to torques exerted on it by the sun and moon. As a result, its rotation axis (which passes through the earth's north and south poles) slowly changes its orientation relative to the distant stars, taking 26,000 years for a complete cycle of precession. Today the rotation axis points toward Polaris, but 5000 years ago it pointed toward Thuban, and 12,000 years from now it will point toward the bright star Vega.

Test Your Understanding Questions

10.1 Answer: (ii) The force P acts along a vertical line, so the lever arm is the horizontal distance from A to the line of action. This is the horizontal component of the distance L, which is $L\cos\theta$. Hence the magnitude of the torque is the product of the force magnitude P and the lever arm $L\cos\theta$, or $\tau = PL\cos\theta$.

10.2 Answer: (iii), (ii), (i) In order for the hanging object of mass m_2 to accelerate downward, the net force on it must be downward. Hence the magnitude $m_2 g$ of the downward weight force must be greater than the magnitude T_2 of the upward tension force. In order for the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. The tension T_2 tends to rotate the pulley clockwise, while the tension T_1 tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm R, so there is a clockwise torque T_2R and a counterclockwise torque T_1R. In order for the net torque to be clockwise, T_2 must be greater than T_1. Hence $m_2 g > T_2 > T_1$.

10.3 Answers: (a) (ii), (b) (i) If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{cm} = MR^2$) instead of a solid cylinder (moment of inertia $I_{cm} = \frac{1}{2}MR^2$), you will find $a_{cm\text{-}y} = \frac{1}{2}g$ and $T = \frac{1}{2}Mg$ (instead of $a_{cm\text{-}y} = \frac{2}{3}g$ and $T = \frac{1}{3}Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. In order to slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

10.4 Answer: (iii) You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

10.5 Answers: (a) no, (b) yes As the ball goes around the circle, the magnitude of $\vec{p} = m\vec{v}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{L} = \vec{r} \times \vec{p}$ *is* constant: It has a constant magnitude (the speed and the perpendicular distance from your hand to the ball are both constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{F}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque;* the vector $\vec{r}$ points from your hand to the ball and the force $\vec{F}$ on the ball is directed toward your hand, so the vector product $\vec{\tau} = \vec{r} \times \vec{F}$ is zero.

10.6 Answer: (i) In the absence of any external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia I would increase slightly. Hence the angular velocity ω_z would decrease slightly and the day would be slightly longer.

10.7 Answer: (iii) Doubling the flywheel mass would double both its moment of inertia I and its weight w, so the ratio I/w would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of Ω.

Bridging Problem

Answers: (a) $h = \dfrac{2R}{5}$

(b) $\frac{5}{7}$ of the speed it had just after the hit

11 EQUILIBRIUM AND ELASTICITY

? This Roman aqueduct uses the principle of the arch to sustain the weight of the structure and the water it carries. Are the blocks that make up the arch being compressed, stretched, or a combination?

LEARNING GOALS

By studying this chapter, you will learn:

- The conditions that must be satisfied for a body or structure to be in equilibrium.
- What is meant by the center of gravity of a body, and how it relates to the body's stability.
- How to solve problems that involve rigid bodies in equilibrium.
- How to analyze situations in which a body is deformed by tension, compression, pressure, or shear.
- What happens when a body is stretched so much that it deforms or breaks.

We've devoted a good deal of effort to understanding why and how bodies accelerate in response to the forces that act on them. But very often we're interested in making sure that bodies *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

A body that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended body, an additional requirement must be satisfied to ensure that the body has no tendency to *rotate:* The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics developed in Chapter 10. We can compute the torque due to the weight of a body using the concept of center of gravity, which we introduce in this chapter.

Rigid bodies don't bend, stretch, or squash when forces act on them. But the rigid body is an idealization; all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. The steel frame of an earthquake-resistant building has to be able to flex, but not too much. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress, strain,* and *elastic modulus* and a simple principle called *Hooke's law* that helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) body.

11.1 Conditions for Equilibrium

We learned in Sections 4.2 and 5.1 that a particle is in *equilibrium*—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum \vec{F} = \mathbf{0}$. For an *extended* body, the equivalent statement is that the center of mass of the body has zero acceleration if the vector sum of all external forces acting on the body is zero, as discussed in Section 8.5. This is often called the **first condition for equilibrium.** In vector and component forms,

$$\sum \vec{F} = \mathbf{0}$$
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \qquad \text{(first condition for equilibrium)} \qquad (11.1)$$

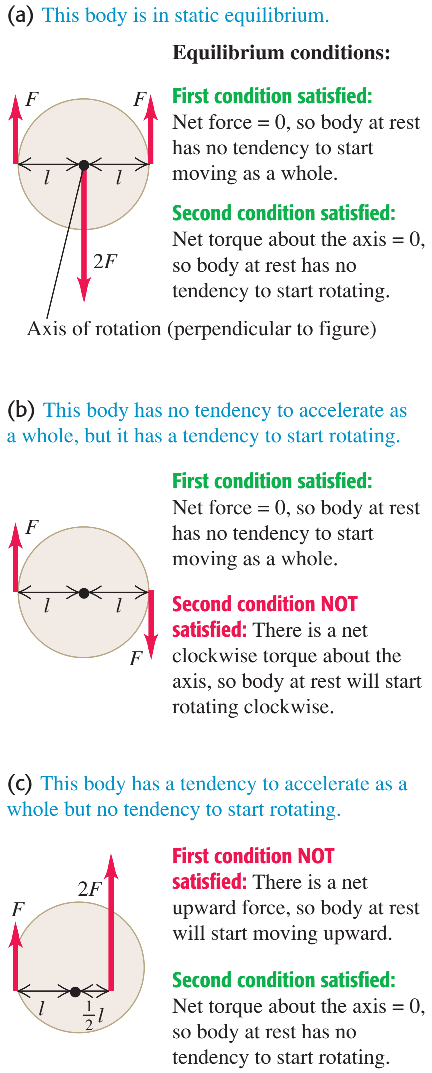
A second condition for an extended body to be in equilibrium is that the body must have no tendency to *rotate*. This condition is based on the dynamics of rotational motion in exactly the same way that the first condition is based on Newton's first law. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the body must be zero. A rigid body in equilibrium can't have any tendency to start rotating about *any* point, so the sum of external torques must be zero about any point. This is the **second condition for equilibrium:**

$$\sum \vec{\tau} = \mathbf{0} \quad \text{about any point} \qquad \text{(second condition for equilibrium)} \qquad (11.2)$$

The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero.

In this chapter we will apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a body is said to be in **static equilibrium** (Fig. 11.1). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a body is in equilibrium but is not static.

11.1 To be in static equilibrium, a body at rest must satisfy *both* conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating.



Test Your Understanding of Section 11.1 Which situation satisfies both the first and second conditions for equilibrium? (i) a seagull gliding at a constant angle below the horizontal and at a constant speed; (ii) an automobile crankshaft turning at an increasing angular speed in the engine of a parked car; (iii) a thrown baseball that does not rotate as it sails through the air.

11.2 Center of Gravity

In most equilibrium problems, one of the forces acting on the body is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire body. But we can always calculate the torque due to the body's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the body, then the body's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

First let's review the definition of the center of mass. For a collection of particles with masses $m_1, m_2, \ldots$ and coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$, the coordinates x_{cm}, y_{cm}, and z_{cm} of the center of mass are given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

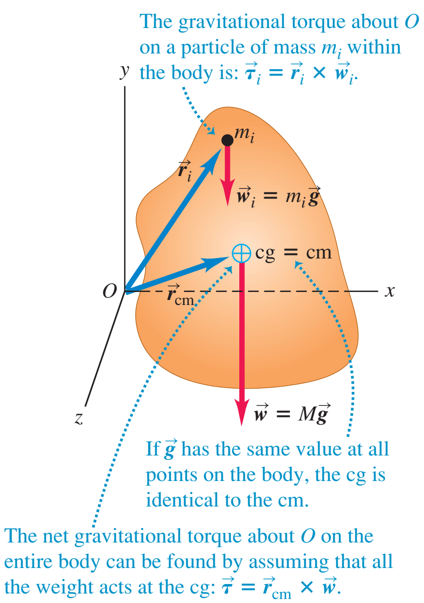
$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass)} \qquad (11.3)$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Also, x_{cm}, y_{cm}, and z_{cm} are the components of the position vector $\vec{r}_{cm}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad (11.4)$$

11.2 The center of gravity (cg) and center of mass (cm) of an extended body.



Now consider the gravitational torque on a body of arbitrary shape (Fig. 11.2). We assume that the acceleration due to gravity $\vec{g}$ is the same at every point in the body. Every particle in the body experiences a gravitational force, and the total weight of the body is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i \vec{g}$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin O, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to O is, from Eq. (10.3),

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i \vec{g}$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}\vec{\tau} &= \sum_i \vec{\tau}_i = \vec{r}_1 \times m_1 \vec{g} + \vec{r}_2 \times m_2 \vec{g} + \cdots \\ &= (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots) \times \vec{g} \\ &= \left(\sum_i m_i \vec{r}_i\right) \times \vec{g}\end{aligned}$$

When we multiply and divide this by the total mass of the body,

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

we get

$$\vec{\tau} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots}{m_1 + m_2 + \cdots} \times M\vec{g} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \times M\vec{g}$$

The fraction in this equation is just the position vector $\vec{r}_{cm}$ of the center of mass, with components x_{cm}, y_{cm}, and z_{cm}, as given by Eq. (11.4), and $M\vec{g}$ is equal to the total weight $\vec{w}$ of the body. Thus

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} \qquad (11.5)$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{w}$ were acting on the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity.* **If $\vec{g}$ has the same value at all points on a body, its center of gravity is identical to its center of mass.** Note, however, that the center of mass is defined independently of any gravitational effect.

While the value of $\vec{g}$ does vary somewhat with elevation, the variation is extremely slight (Fig. 11.3). Hence we will assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

11.3 The acceleration due to gravity at the bottom of the 452-m-tall Petronas Towers in Malaysia is only 0.014% greater than at the top. The center of gravity of the towers is only about 2 cm below the center of mass.

Finding and Using the Center of Gravity

We can often use symmetry considerations to locate the center of gravity of a body, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, circular sheet, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

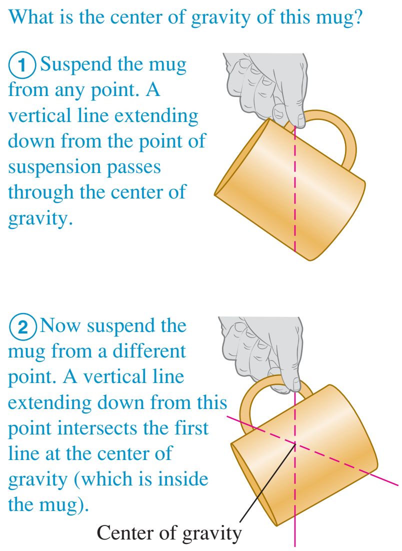
For a body with a more complex shape, we can sometimes locate the center of gravity by thinking of the body as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can locate the center of gravity of the combination with Eqs. (11.3), letting $m_1, m_2, \ldots$ be the masses of the individual pieces and $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ be the coordinates of their centers of gravity.

When a body acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the body could not be in rotational equilibrium. Figure 11.4 shows how to use this fact to determine experimentally the location of the center of gravity of an irregular body.

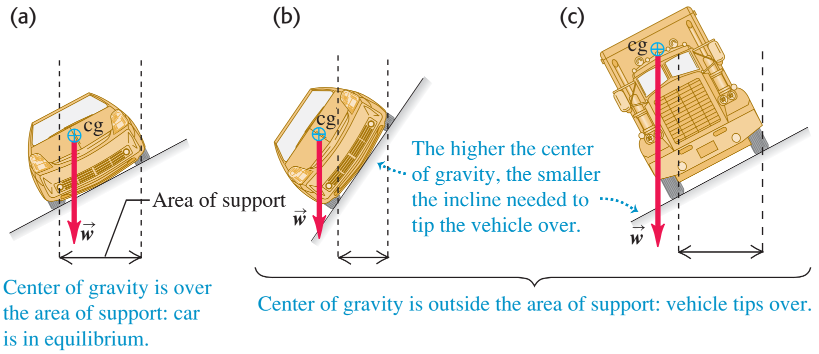
Using the same reasoning, we can see that a body supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (Fig. 11.5a) but will tip over if the angle is too steep (Fig. 11.5b). The truck in Fig. 11.5c has a higher center of gravity than the car and will tip over on a shallower incline. When a truck overturns on a highway and blocks traffic for hours, it's the high center of gravity that's to blame.

The lower the center of gravity and the larger the area of support, the more difficult it is to overturn a body. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk erect on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of

11.4 Finding the center of gravity of an irregularly shaped body—in this case, a coffee mug.



11.5 In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.



support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex,* it must perform a delicate balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

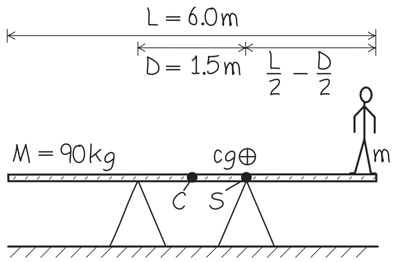
Example 11.1 Walking the plank

A uniform plank of length $L = 6.0$ m and mass $M = 90$ kg rests on sawhorses separated by $D = 1.5$ m and equidistant from the center of the plank. Cousin Throckmorton wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

SOLUTION

IDENTIFY and SET UP: To just balance, Throckmorton's mass m must be such that the center of gravity of the plank–Throcky system is directly over the right-hand sawhorse (Fig. 11.6). We take the origin at C, the geometric center and center of gravity of the plank, and take the positive x-axis horizontally to the right. Then the centers of gravity of the plank and Throcky are at $x_P = 0$ and $x_T = L/2 = 3.0$ m, respectively, and the right-hand sawhorse is at $x_S = D/2$. We'll use Eqs. (11.3) to locate the center of gravity x_{cg} of the plank–Throcky system.

11.6 Our sketch for this problem.



EXECUTE: From the first of Eqs. (11.3),

$$x_{cg} = \frac{M(0) + m(L/2)}{M + m} = \frac{m}{M + m}\frac{L}{2}$$

We set $x_{cg} = x_S$ and solve for m:

$$\frac{m}{M + m}\frac{L}{2} = \frac{D}{2}$$

$$mL = (M + m)D$$

$$m = M\frac{D}{L - D} = (90\text{ kg})\frac{1.5\text{ m}}{6.0\text{ m} - 1.5\text{ m}} = 30\text{ kg}$$

EVALUATE: As a check, let's repeat the calculation with the origin at the right-hand sawhorse. Now $x_S = 0$, $x_P = -D/2$, and $x_T = (L/2) - (D/2)$, and we require $x_{cg} = x_S = 0$:

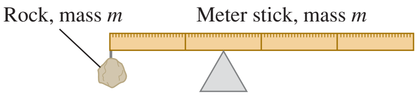
$$x_{cg} = \frac{M(-D/2) + m[(L/2) - (D/2)]}{M + m} = 0$$

$$m = \frac{MD/2}{(L/2) - (D/2)} = M\frac{D}{L - D} = 30\text{ kg}$$

The result doesn't depend on our choice of origin.

A 60-kg adult could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

11.7 At what point will the meter stick with rock attached be in balance?



Test Your Understanding of Section 11.2 A rock is attached to the left end of a uniform meter stick that has the same mass as the rock. In order for the combination of rock and meter stick to balance atop the triangular object in Fig. 11.7, how far from the left end of the stick should the triangular object be placed? (i) less than 0.25 m; (ii) 0.25 m; (iii) between 0.25 m and 0.50 m; (iv) 0.50 m; (v) more than 0.50 m.

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11.3 Solving Rigid-Body Equilibrium Problems

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the body must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the xy-plane. Then we can ignore the condition $\sum F_z = 0$ in Eqs. (11.1), and in Eq. (11.2) we need consider only the z-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \text{(first condition for equilibrium, forces in } xy\text{-plane)}$$

$$\sum \tau_z = 0 \quad \text{(second condition for equilibrium, forces in } xy\text{-plane)} \qquad (11.6)$$

CAUTION **Choosing the reference point for calculating torques** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on a body. Choose the point so as to simplify the calculations as much as possible.

The challenge is to apply these simple conditions to specific problems. Problem-Solving Strategy 11.1 is very similar to the suggestions given in Section 5.2 for the equilibrium of a particle. You should compare it with Problem-Solving Strategy 10.1 (Section 10.2) for rotational dynamics problems.

Problem-Solving Strategy 11.1 Equilibrium of a Rigid Body

IDENTIFY *the relevant concepts:* The first and second conditions for equilibrium $(\sum F_x = 0, \sum F_y = 0, \text{and } \sum \tau_z = 0)$ are applicable to any rigid body that is not accelerating in space and not rotating.

SET UP *the problem* using the following steps:

1. Sketch the physical situation and identify the body in equilibrium to be analyzed. Sketch the body accurately; do *not* represent it as a point. Include dimensions.
2. Draw a free-body diagram showing all forces acting *on* the body. Show the point on the body at which each force acts.
3. Choose coordinate axes and specify their direction. Specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the chosen axes.
4. Choose a reference point about which to compute torques. Choose wisely; you can eliminate from your torque equation any force whose line of action goes through the point you choose. The body doesn't actually have to be pivoted about an axis through the reference point.

EXECUTE *the solution* as follows:

1. Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are *separate* equations. You can compute the torque of a force by finding the torque of each of its components separately, each with its appropriate lever arm and sign, and adding the results.
2. To obtain as many equations as you have unknowns, you may need to compute torques with respect to two or more reference points; choose them wisely, too.

EVALUATE *your answer:* Check your results by writing $\sum \tau_z = 0$ with respect to a different reference point. You should get the same answers.

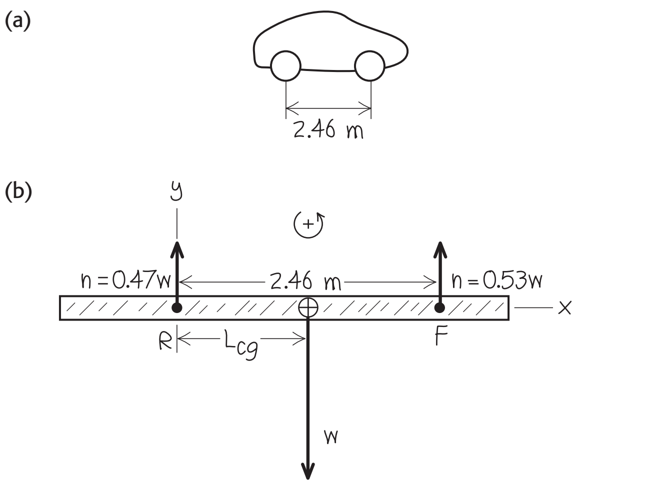
Example 11.2 Weight distribution for a car

An auto magazine reports that a certain sports car has 53% of its weight on the front wheels and 47% on its rear wheels. (That is, the total normal forces on the front and rear wheels are $0.53w$ and $0.47w$, respectively, where w is the car's weight.) The distance between the axles is 2.46 m. How far in front of the rear axle is the car's center of gravity?

SOLUTION

IDENTIFY and SET UP: We can use the two conditions for equilibrium, Eqs. (11.6), for a car at rest (or traveling in a straight line at constant speed), since the net force and net torque on the car are zero. Figure 11.8 shows our sketch and a free-body diagram, including x- and y-axes and our convention that counterclockwise torques are positive. The weight w acts at the center of gravity. Our target variable is the distance L_{cg}, the lever arm of the weight with respect to the rear axle R, so it is wise to take torques with respect to R. The torque due to the weight is negative because it tends to cause a clockwise rotation about R. The torque due to the upward normal force at the front axle F is positive because it tends to cause a counterclockwise rotation about R.

11.8 Our sketches for this problem.



EXECUTE: The first condition for equilibrium is satisfied (see Fig. 11.8b): $\sum F_x = 0$ because there are no x-components of force and $\sum F_y = 0$ because $0.47w + 0.53w + (-w) = 0$. We write the torque equation and solve for L_{cg}:

$$\sum \tau_R = 0.47w(0) - wL_{cg} + 0.53w(2.46 \text{ m}) = 0$$
$$L_{cg} = 1.30 \text{ m}$$

EVALUATE: The center of gravity is between the two supports, as it must be (see Section 11.2). You can check our result by writing the torque equation about the front axle F. You'll find that the center of gravity is 1.16 m behind the front axle, or $(2.46 \text{ m}) - (1.16 \text{ m}) = 1.30$ m in front of the rear axle.

Example 11.3 Will the ladder slip?

Sir Lancelot, who weighs 800 N, is assaulting a castle by climbing a uniform ladder that is 5.0 m long and weighs 180 N (Fig. 11.9a). The bottom of the ladder rests on a ledge and leans across the moat in equilibrium against a frictionless, vertical castle wall. The ladder makes an angle of 53.1° with the horizontal. Lancelot pauses one-third of the way up the ladder. (a) Find the normal and friction forces on the base of the ladder. (b) Find the minimum coefficient of static friction needed to prevent slipping at the base. (c) Find the magnitude and direction of the contact force on the base of the ladder.

SOLUTION

IDENTIFY and SET UP: The ladder–Lancelot system is stationary, so we can use the two conditions for equilibrium to solve part (a). In part (b), we need the relationship among the static friction force, the coefficient of static friction, and the normal force (see Section 5.3). In part (c), the contact force is the vector sum of the normal and friction forces acting at the base of the ladder, found in part (a). Figure 11.9b shows the free-body diagram, with *x*- and *y*-directions as shown and with counterclockwise torques taken to be positive. The ladder's center of gravity is at its geometric center. Lancelot's 800-N weight acts at a point one-third of the way up the ladder.

The wall exerts only a normal force n_1 on the top of the ladder. The forces on the base are an upward normal force n_2 and a static friction force f_s, which must point to the right to prevent slipping. The magnitudes n_2 and f_s are the target variables in part (a). From Eq. (5.6), these magnitudes are related by $f_s \le \mu_s n_2$; the coefficient of static friction μ_s is the target variable in part (b).

EXECUTE: (a) From Eqs. (11.6), the first condition for equilibrium gives

$$\sum F_x = f_s + (-n_1) = 0$$
$$\sum F_y = n_2 + (-800\ \text{N}) + (-180\ \text{N}) = 0$$

These are two equations for the three unknowns n_1, n_2, and f_s. The second equation gives $n_2 = 980$ N. To obtain a third equation, we use the second condition for equilibrium. We take torques about point B, about which n_2 and f_s have no torque. The 53.1° angle creates a 3-4-5 right triangle, so from Fig. 11.9b the lever arm for the ladder's weight is 1.5 m, the lever arm for Lancelot's weight is 1.0 m, and the lever arm for n_1 is 4.0 m. The torque equation for point B is then

$$\sum \tau_B = n_1(4.0\ \text{m}) - (180\ \text{N})(1.5\ \text{m}) - (800\ \text{N})(1.0\ \text{m}) + n_2(0) + f_s(0) = 0$$

Solving for n_1, we get $n_1 = 268$ N. We substitute this into the $\sum F_x = 0$ equation and get $f_s = 268$ N.

(b) The static friction force f_s cannot exceed $\mu_s n_2$, so the *minimum* coefficient of static friction to prevent slipping is

$$(\mu_s)_{\text{min}} = \frac{f_s}{n_2} = \frac{268\ \text{N}}{980\ \text{N}} = 0.27$$

(c) The components of the contact force $\vec{F}_B$ at the base are the static friction force f_s and the normal force n_2, so

$$\vec{F}_B = f_s\hat{\imath} + n_2\hat{\jmath} = (268\ \text{N})\hat{\imath} + (980\ \text{N})\hat{\jmath}$$

The magnitude and direction of $\vec{F}_B$ (Fig. 11.9c) are

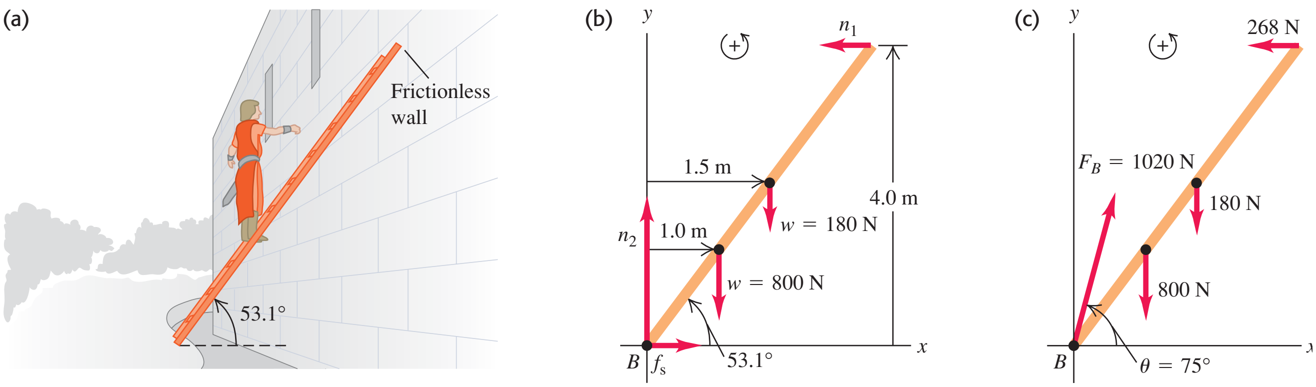
$$F_B = \sqrt{(268\ \text{N})^2 + (980\ \text{N})^2} = 1020\ \text{N}$$
$$\theta = \arctan\frac{980\ \text{N}}{268\ \text{N}} = 75°$$

EVALUATE: As Fig. 11.9c shows, the contact force $\vec{F}_B$ is *not* directed along the length of the ladder. Can you show that if $\vec{F}_B$ were directed along the ladder, there would be a net counterclockwise torque with respect to the top of the ladder, and equilibrium would be impossible?

As Lancelot climbs higher on the ladder, the lever arm and torque of his weight about B increase. This increases the values of n_1, f_s, and the required friction coefficient $(\mu_s)_{\text{min}}$, so the ladder is more and more likely to slip as he climbs (see Problem 11.10). A simple way to make slipping less likely is to use a larger ladder angle (say, 75° rather than 53.1°). This decreases the lever arms with respect to B of the weights of the ladder and Lancelot and increases the lever arm of n_1, all of which decrease the required friction force.

If we had assumed friction on the wall as well as on the floor, the problem would be impossible to solve by using the equilibrium conditions alone. (Try it!) The difficulty is that it's no longer adequate to treat the body as being perfectly rigid. Another problem of this kind is a four-legged table; there's no way to use the equilibrium conditions alone to find the force on each separate leg.

11.9 (a) Sir Lancelot pauses a third of the way up the ladder, fearing it will slip. (b) Free-body diagram for the system of Sir Lancelot and the ladder. (c) The contact force at B is the superposition of the normal force and the static friction force.



Example 11.4 Equilibrium and pumping iron

Figure 11.10a shows a horizontal human arm lifting a dumbbell. The forearm is in equilibrium under the action of the weight $\vec{w}$ of the dumbbell, the tension $\vec{T}$ in the tendon connected to the biceps muscle, and the force $\vec{E}$ exerted on the forearm by the upper arm at the elbow joint. We neglect the weight of the forearm itself. (For clarity, the point A where the tendon is attached is drawn farther from the elbow than its actual position.) Given the weight w and the angle θ between the tension force and the horizontal, find T and the two components of $\vec{E}$ (three unknown scalar quantities in all).

SOLUTION

IDENTIFY and SET UP: The system is at rest, so we use the conditions for equilibrium. We represent $\vec{T}$ and $\vec{E}$ in terms of their components (Fig. 11.10b). We guess that the directions of E_x and E_y are as shown; the signs of E_x and E_y as given by our solution will tell us the actual directions. Our target variables are T, E_x, and E_y.

EXECUTE: To find T, we take torques about the elbow joint so that the torque equation does not contain E_x, E_y, or T_x:

$$\sum \tau_{\text{elbow}} = Lw - DT_y = 0$$

From this we find

$$T_y = \frac{Lw}{D} \quad \text{and} \quad T = \frac{Lw}{D\sin\theta}$$

To find E_x and E_y, we use the first conditions for equilibrium:

$$\sum F_x = T_x + (-E_x) = 0$$

$$E_x = T_x = T\cos\theta = \frac{Lw}{D\sin\theta}\cos\theta$$

$$= \frac{Lw}{D}\cot\theta = \frac{Lw}{D}\frac{D}{h} = \frac{Lw}{h}$$

$$\sum F_y = T_y + E_y + (-w) = 0$$

$$E_y = w - \frac{Lw}{D} = -\frac{(L-D)w}{D}$$

The negative sign for E_y tells us that it should actually point *down* in Fig. 11.10b.

EVALUATE: We can check our results for E_x and E_y by taking torques about points A and B, about both of which T has zero torque:

$$\sum \tau_A = (L-D)w + DE_y = 0 \qquad \text{so} \qquad E_y = -\frac{(L-D)w}{D}$$

$$\sum \tau_B = Lw - hE_x = 0 \qquad \text{so} \qquad E_x = \frac{Lw}{h}$$

As a realistic example, take $w = 200$ N, $D = 0.050$ m, $L = 0.30$ m, and $\theta = 80°$, so that $h = D\tan\theta = (0.050\text{ m})(5.67) = 0.28$ m. Using our results for T, E_x, and E_y, we find

$$T = \frac{Lw}{D\sin\theta} = \frac{(0.30\text{ m})(200\text{ N})}{(0.050\text{ m})(0.98)} = 1220\text{ N}$$

$$E_y = -\frac{(L-D)w}{D} = -\frac{(0.30\text{ m} - 0.050\text{ m})(200\text{ N})}{0.050\text{ m}}$$

$$= -1000\text{ N}$$

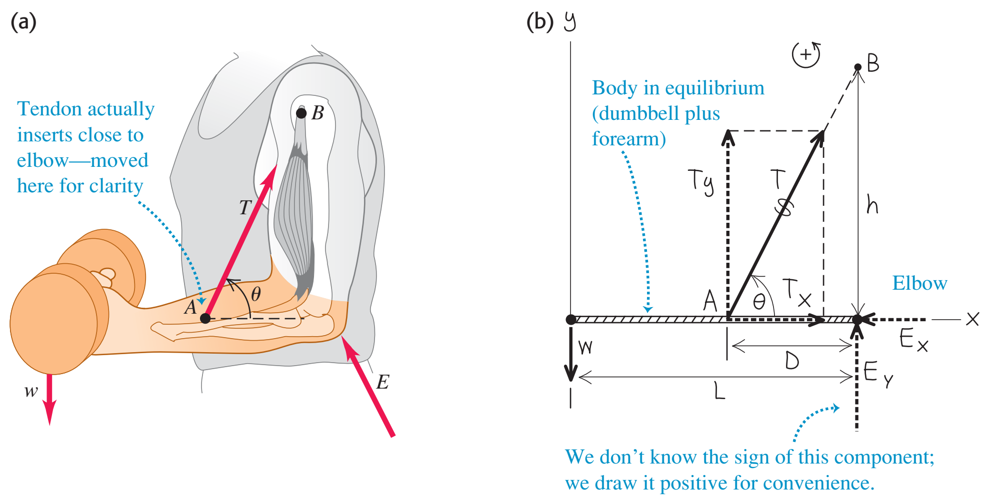
$$E_x = \frac{Lw}{h} = \frac{(0.30\text{ m})(200\text{ N})}{0.28\text{ m}} = 210\text{ N}$$

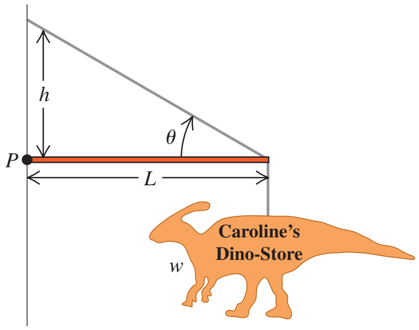
The magnitude of the force at the elbow is

$$E = \sqrt{E_x^2 + E_y^2} = 1020\text{ N}$$

The large values of T and E suggest that it was reasonable to neglect the weight of the forearm itself, which may be 20 N or so.

11.10 (a) The situation. (b) Our free-body diagram for the forearm. The weight of the forearm is neglected, and the distance D is greatly exaggerated for clarity.





11.11 What are the tension in the diagonal cable and the force exerted by the hinge at P?

Test Your Understanding of Section 11.3 A metal advertising sign (weight w) for a specialty shop is suspended from the end of a horizontal rod of length L and negligible mass (Fig. 11.11). The rod is supported by a cable at an angle θ from the horizontal and by a hinge at point P. Rank the following force magnitudes in order from greatest to smallest: (i) the weight w of the sign; (ii) the tension in the cable; (iii) the vertical component of force exerted on the rod by the hinge at P.

11.4 Stress, Strain, and Elastic Moduli

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real bodies when forces are applied are often too important to ignore. Figure 11.12 shows three examples. We want to study the relationship between the forces and deformations for each case.

For each kind of deformation we will introduce a quantity called **stress** that characterizes the strength of the forces causing the deformation, on a "force per unit area" basis. Another quantity, **strain,** describes the resulting deformation. When the stress and strain are small enough, we often find that the two are directly proportional, and we call the proportionality constant an **elastic modulus.** The harder you pull on something, the more it stretches; the more you squeeze it, the more it compresses. In equation form, this says

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad \text{(Hooke's law)} \tag{11.7}$$

The proportionality of stress and strain (under certain conditions) is called **Hooke's law,** after Robert Hooke (1635–1703), a contemporary of Newton. We used one form of Hooke's law in Sections 6.3 and 7.2: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's "law" is not really a general law; it is valid over only a limited range. The last section of this chapter discusses what this limited range is.

Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.12a). Figure 11.13 shows an object that initially has uniform cross-sectional area A and length l_0. We then apply forces of equal

11.12 Three types of stress. **(a)** Bridge cables under *tensile stress,* being stretched by forces acting at their ends. **(b)** A diver under *bulk stress,* being squeezed from all sides by forces due to water pressure. **(c)** A ribbon under *shear stress,* being deformed and eventually cut by forces exerted by the scissors.

magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension.** We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross section.

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area A:

$$\text{Tensile stress} = \frac{F_\perp}{A} \tag{11.8}$$

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter (N/m^2):

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

In the British system the logical unit of stress would be the pound per square foot, but the pound per square inch (lb/in.^{2} or psi) is more commonly used. The conversion factors are

$$1 \text{ psi} = 6895 \text{ Pa} \qquad \text{and} \qquad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi}$$

The units of stress are the same as those of *pressure,* which we will encounter often in later chapters. Air pressure in automobile tires is typically around 3×10^5 Pa = 300 kPa, and steel cables are commonly required to withstand tensile stresses of the order of 10^8 Pa.

The object shown in Fig. 11.13 stretches to a length $l = l_0 + \Delta l$ when under tension. The elongation Δl does not occur only at the ends; every part of the bar stretches in the same proportion. The **tensile strain** of the object is equal to the fractional change in length, which is the ratio of the elongation Δl to the original length l_0:

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} \tag{11.9}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus,** denoted by Y:

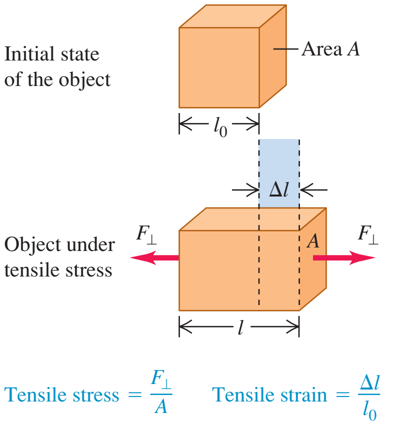
$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \quad \text{(Young's modulus)} \tag{11.10}$$

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. Some typical values are listed in Table 11.1.

Table 11.1 Approximate Elastic Moduli

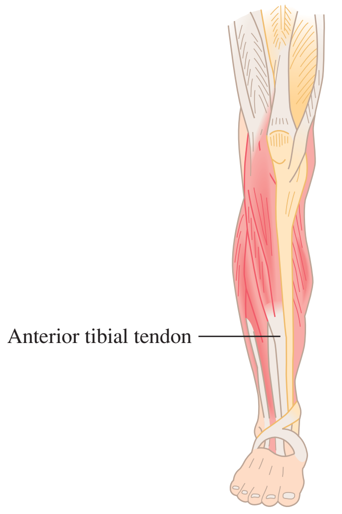
Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Crown glass	6.0×10^{10}	5.0×10^{10}	2.5×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}

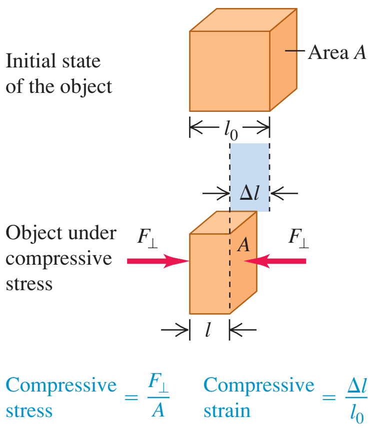
11.13 An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation Δl is exaggerated for clarity.



Application Young's Modulus of a Tendon

The anterior tibial tendon connects your foot to the large muscle that runs along the side of your shinbone. (You can feel this tendon at the front of your ankle.) Measurements show that this tendon has a Young's modulus of 1.2×10^9 Pa, much less than for the solid materials listed in Table 11.1. Hence this tendon stretches substantially (up to 2.5% of its length) in response to the stresses experienced in walking and running.



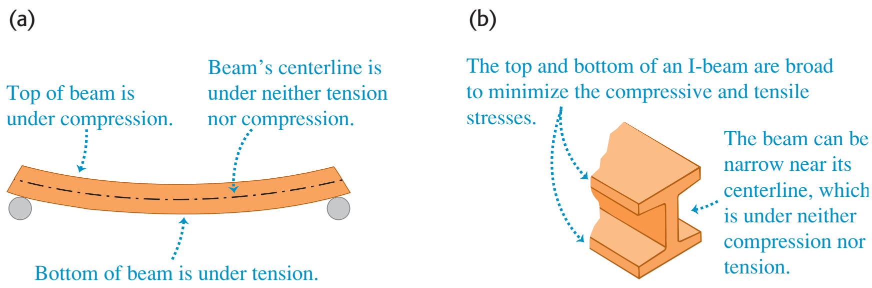


11.14 An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig. 11.13), except that Δl now denotes the distance that the object contracts.

(This table also gives values of two other elastic moduli that we will discuss later in this chapter.) A material with a large value of Y is relatively unstretchable; a large stress is required for a given strain. For example, the value of Y for cast steel (2×10^{11} Pa) is much larger than that for rubber (5×10^{8} Pa).

When the forces on the ends of a bar are pushes rather than pulls (Fig. 11.14), the bar is in **compression** and the stress is a **compressive stress.** The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but Δl has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used by ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. Hence they used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

In many situations, bodies can experience both tensile and compressive stresses at the same time. As an example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression, while the bottom of the beam is under tension (Fig. 11.15a). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps to keep the weight of the bar to a minimum and further helps to reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.15b).



11.15 (a) A beam supported at both ends is under both compression and tension. (b) The cross-sectional shape of an I-beam minimizes both stress and weight.

Example 11.5 Tensile stress and strain

A steel rod 2.0 m long has a cross-sectional area of 0.30 cm^2. It is hung by one end from a support, and a 550-kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: The rod is under tension, so we can use Eq. (11.8) to find the tensile stress; Eq. (11.9), with the value of Young's modulus Y for steel from Table 11.1, to find the corresponding strain; and Eq. (11.10) to find the elongation Δl:

$$\text{Tensile stress} = \frac{F_\perp}{A} = \frac{(550\ \text{kg})(9.8\ \text{m/s}^2)}{3.0 \times 10^{-5}\ \text{m}^2} = 1.8 \times 10^{8}\ \text{Pa}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^{8}\ \text{Pa}}{20 \times 10^{10}\ \text{Pa}} = 9.0 \times 10^{-4}$$

$$\begin{aligned}\text{Elongation} &= \Delta l = (\text{Strain}) \times l_0 \\ &= (9.0 \times 10^{-4})(2.0\ \text{m}) = 0.0018\ \text{m} = 1.8\ \text{mm}\end{aligned}$$

EVALUATE: This small elongation, resulting from a load of over half a ton, is a testament to the stiffness of steel.

Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (see Fig. 11.12b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The stress is now a uniform pressure on all sides, and the resulting deformation is a volume change. We use the terms **bulk stress** (or **volume stress**) and **bulk strain** (or **volume strain**) to describe these quantities.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort.) The force $F_\perp$ per unit area that the fluid exerts on the surface of an immersed object is called the **pressure** p in the fluid:

$$p = \frac{F_\perp}{A} \qquad \text{(pressure in a fluid)} \qquad (11.11)$$

The pressure in a fluid increases with depth. For example, the pressure of the air is about 21% greater at sea level than in Denver (at an elevation of 1.6 km, or 1.0 mi). If an immersed object is relatively small, however, we can ignore pressure differences due to depth for the purpose of calculating bulk stress. Hence we will treat the pressure as having the same value at all points on an immersed object's surface.

Pressure has the same units as stress; commonly used units include 1 Pa $(=1\ \text{N/m}^2)$ and 1 lb/in.^2 (1 psi). Also in common use is the **atmosphere,** abbreviated atm. One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1 \text{ atmosphere} = 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} = 14.7 \text{ lb/in.}^2$$

CAUTION **Pressure vs. force** Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar* quantity, not a vector quantity.

Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume (Fig. 11.16)—that is, the ratio of the volume change ΔV to the original volume V_0:

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0} \qquad (11.12)$$

Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus,** denoted by B. When the pressure on a body changes by a small amount Δp, from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke's law takes the form

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad \text{(bulk modulus)} \qquad (11.13)$$

We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if Δp is positive, ΔV is negative. The bulk modulus B itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider B to be constant. The bulk modulus of a *gas,* however, depends on the initial pressure p_0. Table 11.1 includes values of the bulk modulus for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

Application **Bulk Stress on an Anglerfish**

The anglerfish (*Melanocetus johnsoni*) is found in oceans throughout the world at depths as great as 1000 m, where the pressure (that is, the bulk stress) is about 100 atmospheres. Anglerfish are able to withstand such stress because they have no internal air spaces, unlike fish found in the upper ocean where pressures are lower. The largest anglerfish are about 12 cm (5 in.) long.

11.16 An object under bulk stress. Without the stress, the cube has volume V_0; when the stress is applied, the cube has a smaller volume V. The volume change ΔV is exaggerated for clarity.

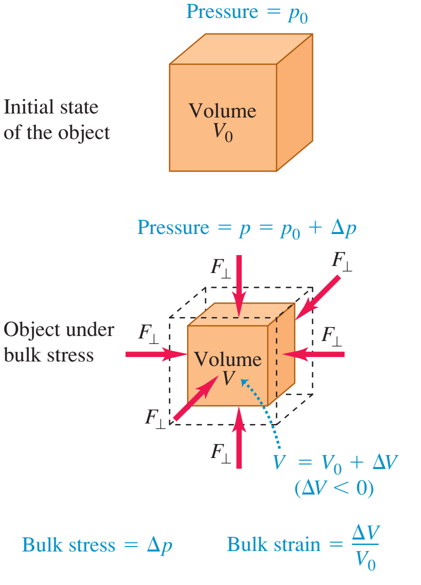


Table 11.2 Compressibilities of Liquids

	Compressibility, k	
Liquid	$\mathbf{Pa^{-1}}$	$\mathbf{atm^{-1}}$
Carbon disulfide	93×10^{-11}	94×10^{-6}
Ethyl alcohol	110×10^{-11}	111×10^{-6}
Glycerine	21×10^{-11}	21×10^{-6}
Mercury	3.7×10^{-11}	3.8×10^{-6}
Water	45.8×10^{-11}	46.4×10^{-6}

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k. From Eq. (11.13),

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \qquad \text{(compressibility)} \tag{11.14}$$

Compressibility is the fractional decrease in volume, $-\Delta V/V_0$, per unit increase Δp in pressure. The units of compressibility are those of *reciprocal pressure,* Pa^{-1} or atm^{-1}.

Table 11.2 lists the values of compressibility k for several liquids. For example, the compressibility of water is $46.4 \times 10^{-6}\ \text{atm}^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1-atmosphere increase in pressure. Materials with small bulk modulus and large compressibility are easier to compress.

Example 11.6 Bulk stress and strain

A hydraulic press contains $0.25\ \text{m}^3$ (250 L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about 160 atm or 2300 psi). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa (about 5.0×10^4 atm) and its compressibility is $k = 1/B = 20 \times 10^{-6}\ \text{atm}^{-1}$.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: This example uses the ideas of bulk stress and strain. We are given both the bulk modulus and the compressibility, and our target variable is ΔV. Solving Eq. (11.13) for ΔV, we find

$$\Delta V = -\frac{V_0 \Delta p}{B} = -\frac{(0.25\ \text{m}^3)(1.6 \times 10^7\ \text{Pa})}{5.0 \times 10^9\ \text{Pa}}$$
$$= -8.0 \times 10^{-4}\ \text{m}^3 = -0.80\ \text{L}$$

Alternatively, we can use Eq. (11.14) with the approximate unit conversions given above:

$$\Delta V = -kV_0\Delta p = -(20 \times 10^{-6}\ \text{atm}^{-1})(0.25\ \text{m}^3)(160\ \text{atm})$$
$$= -8.0 \times 10^{-4}\ \text{m}^3$$

EVALUATE: The negative value of ΔV means that the volume decreases when the pressure increases. Even though the 160-atm pressure increase is large, the *fractional* change in volume is very small:

$$\frac{\Delta V}{V_0} = \frac{-8.0 \times 10^{-4}\ \text{m}^3}{0.25\ \text{m}^3} = -0.0032 \qquad \text{or} \qquad -0.32\%$$

Shear Stress and Strain

The third kind of stress-strain situation is called *shear.* The ribbon in Fig. 11.12c is under **shear stress:** One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. Figure 11.17 shows a body being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_\parallel$ acting tangent to the surface divided by the area A on which it acts:

$$\text{Shear stress} = \frac{F_\parallel}{A} \tag{11.15}$$

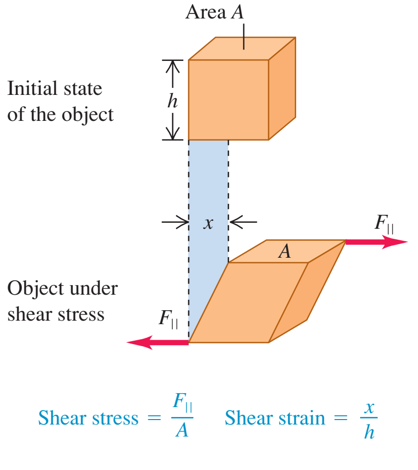
Shear stress, like the other two types of stress, is a force per unit area.

Figure 11.17 shows that one face of the object under shear stress is displaced by a distance x relative to the opposite face. We define **shear strain** as the ratio of the displacement x to the transverse dimension h:

$$\text{Shear strain} = \frac{x}{h} \tag{11.16}$$

In real-life situations, x is nearly always much smaller than h. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

11.17 An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.13, in which the forces act perpendicular to the surfaces). The deformation x is exaggerated for clarity.



If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus,** denoted by S:

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{\parallel}/A}{x/h} = \frac{F_{\parallel}}{A}\frac{h}{x} \quad \text{(shear modulus)} \tag{11.17}$$

with x and h defined as in Fig. 11.17.

Table 11.1 gives several values of shear modulus. For a given material, S is usually one-third to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that *shear* refers to deforming an object that has a definite shape (see Fig. 11.17). This concept doesn't apply to gases and liquids, which do not have definite shapes.

Example 11.7 Shear stress and strain

Suppose the object in Fig. 11.17 is the brass base plate of an outdoor sculpture that experiences shear forces in an earthquake. The plate is 0.80 m square and 0.50 cm thick. What is the force exerted on each of its edges if the resulting displacement x is 0.16 mm?

SOLUTION

IDENTIFY and SET UP: This example uses the relationship among shear stress, shear strain, and shear modulus. Our target variable is the force $F_{\parallel}$ exerted parallel to each edge, as shown in Fig. 11.17. We'll find the shear strain using Eq. (11.16), the shear stress using Eq. (11.17), and $F_{\parallel}$ using Eq. (11.15). Table 11.1 gives the shear modulus of brass. In Fig. 11.17, h represents the 0.80-m length of each side of the plate. The area A in Eq. (11.15) is the product of the 0.80-m length and the 0.50-cm thickness.

EXECUTE: From Eq. (11.16),

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

From Eq. (11.17),

$$\begin{aligned}\text{Shear stress} &= (\text{Shear strain}) \times S \\ &= (2.0 \times 10^{-4})(3.5 \times 10^{10}\ \text{Pa}) = 7.0 \times 10^{6}\ \text{Pa}\end{aligned}$$

Finally, from Eq. (11.15),

$$\begin{aligned}F_{\parallel} &= (\text{Shear stress}) \times A \\ &= (7.0 \times 10^{6}\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^{4}\ \text{N}\end{aligned}$$

EVALUATE: The shear force supplied by the earthquake is more than 3 tons! The large shear modulus of brass makes it hard to deform. Further, the plate is relatively thick (0.50 cm), so the area A is relatively large and a substantial force $F_{\parallel}$ is needed to provide the necessary stress $F_{\parallel}/A$.

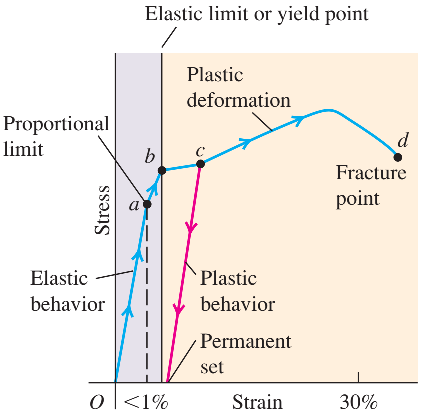
Test Your Understanding of Section 11.4 A copper rod of cross-sectional area 0.500 cm^2 and length 1.00 m is elongated by 2.00×10^{-2} mm, and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by 2.00×10^{-3} mm. (a) Which rod has greater tensile *strain*? (i) the copper rod; (ii) the steel rod; (iii) the strain is the same for both. (b) Which rod is under greater tensile *stress*? (i) the copper rod; (ii) the steel rod; (iii) the stress is the same for both.

11.5 Elasticity and Plasticity

Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity. In the preceding section we used phrases such as "provided that the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? We know that if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

Let's look at tensile stress and strain again. Suppose we plot a graph of stress as a function of strain. If Hooke's law is obeyed, the graph is a straight line with a

11.18 Typical stress-strain diagram for a ductile metal under tension.



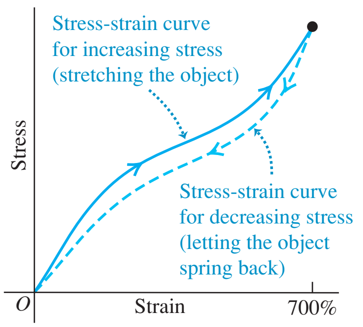
slope equal to Young's modulus. Figure 11.18 shows a typical stress-strain graph for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point a; the stress at this point is called the *proportional limit.*

From a to b, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. If the load is gradually removed, starting at any point between O and b, the curve is retraced until the material returns to its original length. The deformation is *reversible,* and the forces are conservative; the energy put into the material to cause the deformation is recovered when the stress is removed. In region Ob we say that the material shows *elastic behavior.* Point b, the end of this region, is called the *yield point;* the stress at the yield point is called the *elastic limit.*

When we increase the stress beyond point b, the strain continues to increase. But now when we remove the load at some point beyond b, say c, the material does not come back to its original length. Instead, it follows the red line in Fig. 11.18. The length at zero stress is now greater than the original length; the material has undergone an irreversible deformation and has acquired what we call a *permanent set.* Further increase of load beyond c produces a large increase in strain for a relatively small increase in stress, until a point d is reached at which *fracture* takes place. The behavior of the material from b to d is called *plastic flow* or *plastic deformation.* A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state.

For some materials, such as the one whose properties are graphed in Fig. 11.18, a large amount of plastic deformation takes place between the elastic limit and the fracture point. Such a material is said to be *ductile.* But if fracture occurs soon after the elastic limit is passed, the material is said to be *brittle.* A soft iron wire that can have considerable permanent stretch without breaking is ductile, while a steel piano string that breaks soon after its elastic limit is reached is brittle.

11.19 Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.



Something very curious can happen when an object is stretched and then allowed to relax. An example is shown in Fig. 11.19, which is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis.* The work done by the material when it returns to its original shape is less than the work required to deform it; there are nonconservative forces associated with internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars.

Table 11.3 Approximate Breaking Stresses

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Phosphor bronze	5.6×10^8
Steel	$5–20 \times 10^8$

The stress required to cause actual fracture of a material is called the *breaking stress,* the *ultimate strength,* or (for tensile stress) the *tensile strength.* Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses. Table 11.3 gives typical values of breaking stress for several materials in tension. The conversion factor 6.9×10^8 Pa = 100,000 psi may help put these numbers in perspective. For example, if the breaking stress of a particular steel is 6.9×10^8 Pa, then a bar with a 1-in.^{2} cross section has a breaking strength of 100,000 lb.

Test Your Understanding of Section 11.5 While parking your car on a crowded street, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage. What does your rear bumper look like if the strain in the impact was (a) less than at the proportional limit; (b) greater than at the proportional limit, but less than at the yield point; (c) greater than at the yield point, but less than at the fracture point; and (d) greater than at the fracture point?

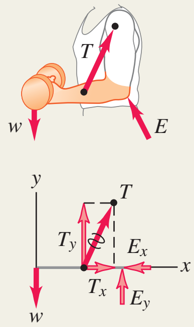
CHAPTER 11 SUMMARY

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of a body can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points. (See Examples 11.1–11.4.)

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \tag{11.1}$$

$$\sum \vec{\tau} = \mathbf{0} \quad \text{about } any \text{ point} \tag{11.2}$$

$$\vec{r}_{\mathrm{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{11.4}$$

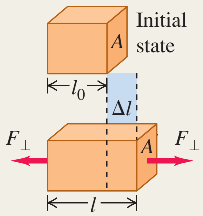


Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \tag{11.7}$$

Tensile and compressive stress: Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus is called Young's modulus Y. Compressive stress and strain are defined in the same way. (See Example 11.5.)

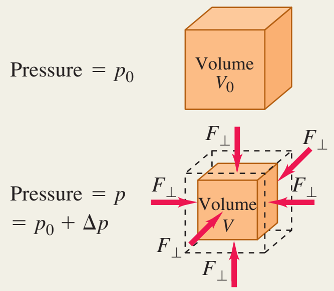
$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \tag{11.10}$$



Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: $k = 1/B$. (See Example 11.6.)

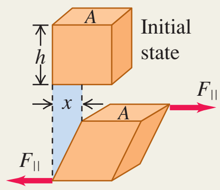
$$p = \frac{F_\perp}{A} \tag{11.11}$$

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \tag{11.13}$$



Shear stress: Shear stress is force per unit area, $F_\parallel/A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus is called the shear modulus, S. (See Example 11.7.)

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \tag{11.17}$$



The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.

BRIDGING PROBLEM In Equilibrium and Under Stress

A horizontal, uniform, solid copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, bulk modulus B, shear modulus S, and mass m. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (Fig. 11.20). Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle θ with the rod and compresses it. (a) Find the tension in the cable. (b) Find the magnitude and direction of the force exerted by the pivot on the right end of the rod. How does this magnitude compare to the cable tension? How does this angle compare to θ? (c) Find the change in length of the rod due to the stresses exerted by the cable and pivot on the rod. (d) By what factor would your answer in part (c) increase if the solid copper rod were twice as long but had the same cross-sectional area?

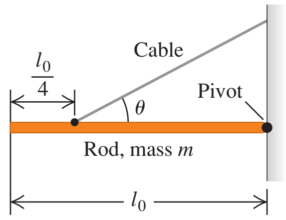
SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Draw a free-body diagram for the rod. Be careful to place each force in the correct location.
2. Make a list of the unknown quantities, and decide which are the target variables.
3. What are the conditions that must be met so that the rod remains at rest? What kind of stress (and resulting strain) is involved? Use your answers to select the appropriate equations.

11.20 What are the forces on the rod? What are the stress and strain?



EXECUTE

4. Use your equations to solve for the target variables. (*Hint:* You can make the solution easier by carefully choosing the point around which you calculate torques.)
5. Use your knowledge of trigonometry to decide whether the pivot force or the cable tension has the greater magnitude, as well as to decide whether the angle of the pivot force is greater than, less than, or equal to θ.

EVALUATE

6. Check whether your answers are reasonable. Which force, the cable tension or the pivot force, holds up more of the weight of the rod? Does this make sense?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q11.1 Does a rigid object in uniform rotation about a fixed axis satisfy the first and second conditions for equilibrium? Why? Does it then follow that every particle in this object is in equilibrium? Explain.

Q11.2 (a) Is it possible for an object to be in translational equilibrium (the first condition) but *not* in rotational equilibrium (the second condition)? Illustrate your answer with a simple example. (b) Can an object be in rotational equilibrium yet *not* in translational equilibrium? Justify your answer with a simple example.

Q11.3 Car tires are sometimes "balanced" on a machine that pivots the tire and wheel about the center. Weights are placed around the wheel rim until it does not tip from the horizontal plane. Discuss this procedure in terms of the center of gravity.

Q11.4 Does the center of gravity of a solid body always lie within the material of the body? If not, give a counterexample.

Q11.5 In Section 11.2 we always assumed that the value of g was the same at all points on the body. This is *not* a good approximation if the dimensions of the body are great enough, because the value of g decreases with altitude. If this is taken into account, will the center of gravity of a long, vertical rod be above, below, or at its center of mass? Explain how this can be used to keep the long axis of an orbiting spacecraft pointed toward the earth. (This would be useful for a weather satellite that must always keep its camera lens trained on the earth.) The moon is not exactly spherical but is somewhat elongated. Explain why this same effect is responsible for keeping the same face of the moon pointed toward the earth at all times.

Q11.6 You are balancing a wrench by suspending it at a single point. Is the equilibrium stable, unstable, or neutral if the point is above, at, or below the wrench's center of gravity? In each case give the reasoning behind your answer. (For rotation, a rigid body is in *stable* equilibrium if a small rotation of the body produces a torque that tends to return the body to equilibrium; it is in *unstable* equilibrium if a small rotation produces a torque that tends to take the body farther from equilibrium; and it is in *neutral* equilibrium if a small rotation produces no torque.)

Q11.7 You can probably stand flatfooted on the floor and then rise up and balance on your tiptoes. Why are you unable do it if your toes are touching the wall of your room? (Try it!)

Q11.8 You freely pivot a horseshoe from a horizontal nail through one of its nail holes. You then hang a long string with a weight at its bottom from the same nail, so that the string hangs vertically in front of the horseshoe without touching it. How do you know that the horseshoe's center of gravity is along the line behind the string? How can you locate the center of gravity by repeating the process at another nail hole? Will the center of gravity be within the solid material of the horseshoe?

Q11.9 An object consists of a ball of weight W glued to the end of a uniform bar also of weight W. If you release it from rest, with the bar horizontal, what will its behavior be as it falls if air resistance is negligible? Will it (a) remain horizontal; (b) rotate about its center of gravity; (c) rotate about the ball; or (d) rotate so that the ball swings downward? Explain your reasoning.

Q11.10 Suppose that the object in Question 11.9 is released from rest with the bar tilted at 60° above the horizontal with the ball at the upper end. As it is falling, will it (a) rotate about its center of gravity until it is horizontal; (b) rotate about its center of gravity until it is vertical with the ball at the bottom; (c) rotate about the ball until it is vertical with the ball at the bottom; or (d) remain at 60° above the horizontal?

Q11.11 Why must a water skier moving with constant velocity lean backward? What determines how far back she must lean? Draw a free-body diagram for the water skier to justify your answers.

Q11.12 In pioneer days, when a Conestoga wagon was stuck in the mud, people would grasp the wheel spokes and try to turn the wheels, rather than simply pushing the wagon. Why?

Q11.13 The mighty Zimbo claims to have leg muscles so strong that he can stand flat on his feet and lean forward to pick up an apple on the floor with his teeth. Should you pay to see him perform, or do you have any suspicions about his claim? Why?

Q11.14 Why is it easier to hold a 10-kg dumbbell in your hand at your side than it is to hold it with your arm extended horizontally?

Q11.15 Certain features of a person, such as height and mass, are fixed (at least over relatively long periods of time). Are the following features also fixed? (a) location of the center of gravity of the body; (b) moment of inertia of the body about an axis through the person's center of mass. Explain your reasoning.

Q11.16 During pregnancy, women often develop back pains from leaning backward while walking. Why do they have to walk this way?

Q11.17 Why is a tapered water glass with a narrow base easier to tip over than a glass with straight sides? Does it matter whether the glass is full or empty?

Q11.18 When a tall, heavy refrigerator is pushed across a rough floor, what factors determine whether it slides or tips?

Q11.19 If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulus change?

Q11.20 Why is concrete with steel reinforcing rods embedded in it stronger than plain concrete?

Q11.21 A metal wire of diameter D stretches by 0.100 mm when supporting a weight W. If the same-length wire is used to support a weight three times as heavy, what would its diameter have to be (in terms of D) so it still stretches only 0.100 mm?

Q11.22 Compare the mechanical properties of a steel cable, made by twisting many thin wires together, with the properties of a solid steel rod of the same diameter. What advantages does each have?

Q11.23 The material in human bones and elephant bones is essentially the same, but an elephant has much thicker legs. Explain why, in terms of breaking stress.

Q11.24 There is a small but appreciable amount of elastic hysteresis in the large tendon at the back of a horse's leg. Explain how this can cause damage to the tendon if a horse runs too hard for too long a time.

Q11.25 When rubber mounting blocks are used to absorb machine vibrations through elastic hysteresis, as mentioned in Section 11.5, what becomes of the energy associated with the vibrations?

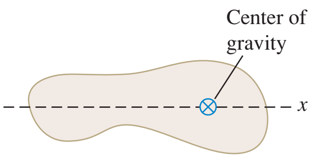
EXERCISES

Section 11.2 Center of Gravity

11.1 •• A 0.120-kg, 50.0-cm-long uniform bar has a small 0.055-kg mass glued to its left end and a small 0.110-kg mass glued to the other end. The two small masses can each be treated as point masses. You want to balance this system horizontally on a fulcrum placed just under its center of gravity. How far from the left end should the fulcrum be placed?

11.2 •• The center of gravity of a 5.00-kg irregular object is shown in Fig. E11.2. You need to move the center of gravity 2.20 cm to the left by gluing on a 1.50-kg mass, which will then be considered as part of the object. Where should the center of gravity of this additional mass be located?

Figure **E11.2**



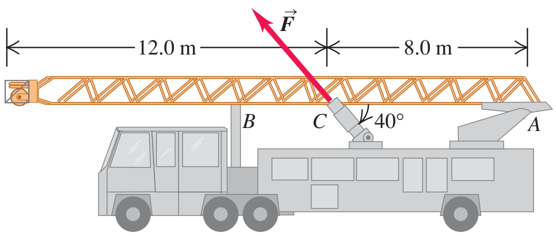
11.3 • A uniform rod is 2.00 m long and has mass 1.80 kg. A 2.40-kg clamp is attached to the rod. How far should the center of gravity of the clamp be from the left-hand end of the rod in order for the center of gravity of the composite object to be 1.20 m from the left-hand end of the rod?

Section 11.3 Solving Rigid-Body Equilibrium Problems

11.4 • A uniform 300-N trapdoor in a floor is hinged at one side. Find the net upward force needed to begin to open it and the total force exerted on the door by the hinges (a) if the upward force is applied at the center and (b) if the upward force is applied at the center of the edge opposite the hinges.

11.5 •• **Raising a Ladder.** A ladder carried by a fire truck is 20.0 m long. The ladder weighs 2800 N and its center of gravity is at its center. The ladder is pivoted at one end (A) about a pin (Fig. E11.5); you can ignore the friction torque at the pin. The ladder is raised into position by a force applied by a hydraulic piston at C. Point C is 8.0 m from A, and the force $\vec{F}$ exerted by the piston makes an angle of 40° with the ladder. What magnitude must $\vec{F}$ have to just lift the ladder off the support bracket at B? Start with a free-body diagram of the ladder.

Figure **E11.5**



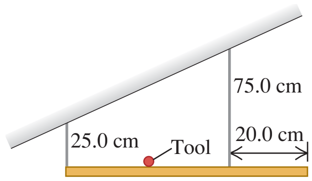
11.6 •• Two people are carrying a uniform wooden board that is 3.00 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.

11.7 •• Two people carry a heavy electric motor by placing it on a light board 2.00 m long. One person lifts at one end with a force of 400 N, and the other lifts the opposite end with a force of 600 N.

(a) What is the weight of the motor, and where along the board is its center of gravity located? (b) Suppose the board is not light but weighs 200 N, with its center of gravity at its center, and the two people each exert the same forces as before. What is the weight of the motor in this case, and where is its center of gravity located?

11.8 •• A 60.0-cm, uniform, 50.0-N shelf is supported horizontally by two vertical wires attached to the sloping ceiling (Fig. E11.8). A very small 25.0-N tool is placed on the shelf midway between the points where the wires are attached to it. Find the tension in each wire. Begin by making a free-body diagram of the shelf.

Figure **E11.8**

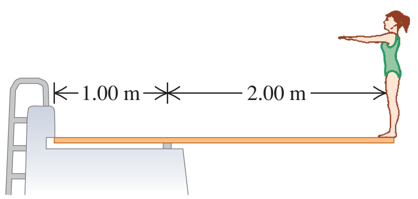


11.9 •• A 350-N, uniform, 1.50-m bar is suspended horizontally by two vertical cables at each end. Cable *A* can support a maximum tension of 500.0 N without breaking, and cable *B* can support up to 400.0 N. You want to place a small weight on this bar. (a) What is the heaviest weight you can put on without breaking either cable, and (b) where should you put this weight?

11.10 •• A uniform ladder 5.0 m long rests against a frictionless, vertical wall with its lower end 3.0 m from the wall. The ladder weighs 160 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 740 N climbs slowly up the ladder. Start by drawing a free-body diagram of the ladder. (a) What is the maximum frictional force that the ground can exert on the ladder at its lower end? (b) What is the actual frictional force when the man has climbed 1.0 m along the ladder? (c) How far along the ladder can the man climb before the ladder starts to slip?

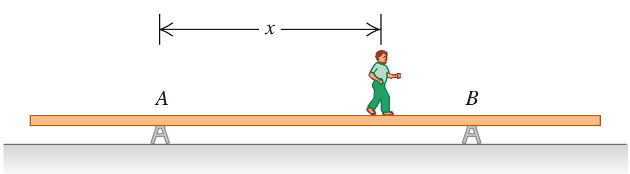
11.11 • A diving board 3.00 m long is supported at a point 1.00 m from the end, and a diver weighing 500 N stands at the free end (Fig. E11.11). The diving board is of uniform cross section and weighs 280 N. Find (a) the force at the support point and (b) the force at the left-hand end.

Figure **E11.11**



11.12 • A uniform aluminum beam 9.00 m long, weighing 300 N, rests symmetrically on two supports 5.00 m apart (Fig. E11.12). A boy weighing 600 N starts at point *A* and walks toward the right. (a) In the same diagram construct two graphs showing the upward forces F_A and F_B exerted on the beam at points *A* and *B*, as functions of the coordinate x of the boy. Let 1 cm = 100 N vertically, and 1 cm = 1.00 m horizontally. (b) From your diagram, how far beyond point *B* can the boy walk before the beam tips? (c) How far

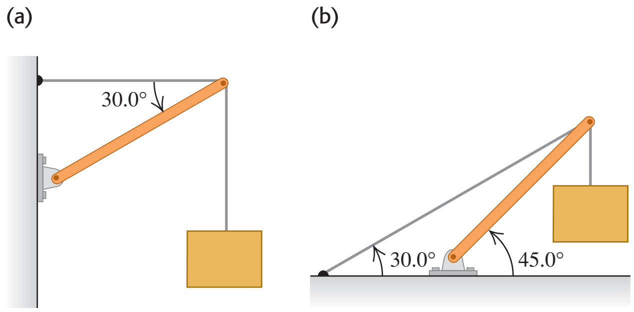
Figure **E11.12**



from the right end of the beam should support *B* be placed so that the boy can walk just to the end of the beam without causing it to tip?

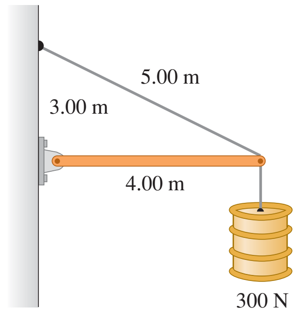
11.13 • Find the tension T in each cable and the magnitude and direction of the force exerted on the strut by the pivot in each of the arrangements in Fig. E11.13. In each case let w be the weight of the suspended crate full of priceless art objects. The strut is uniform and also has weight w. Start each case with a free-body diagram of the strut.

Figure **E11.13**



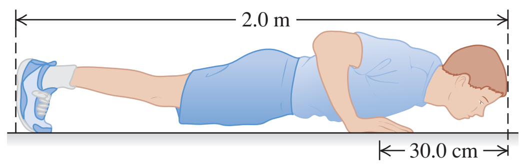
11.14 • The horizontal beam in Fig. E11.14 weighs 150 N, and its center of gravity is at its center. Find (a) the tension in the cable and (b) the horizontal and vertical components of the force exerted on the beam at the wall.

Figure **E11.14**



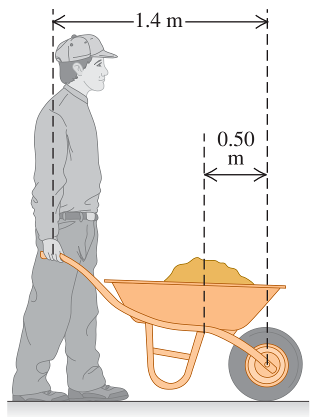
11.15 •• **BIO Push-ups.** To strengthen his arm and chest muscles, an 82-kg athlete who is 2.0 m tall is doing push-ups as shown in Fig. E11.15. His center of mass is 1.15 m from the bottom of his feet, and the centers of his palms are 30.0 cm from the top of his head. Find the force that the floor exerts on each of his feet and on each hand, assuming that both feet exert the same force and both palms do likewise. Begin with a free-body diagram of the athlete.

Figure **E11.15**



11.16 •• Suppose that you can lift no more than 650 N (around 150 lb) unaided. (a) How much can you lift using a 1.4-m-long wheelbarrow that weighs 80.0 N and whose center of gravity is 0.50 m from the center of the wheel (Fig. E11.16)? The center of gravity of the load carried in

Figure **E11.16**

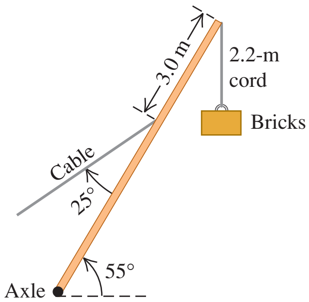


the wheelbarrow is also 0.50 m from the center of the wheel. (b) Where does the force come from to enable you to lift more than 650 N using the wheelbarrow?

11.17 •• You take your dog Clea to the vet, and the doctor decides he must locate the little beast's center of gravity. It would be awkward to hang the pooch from the ceiling, so the vet must devise another method. He places Clea's front feet on one scale and her hind feet on another. The front scale reads 157 N, while the rear scale reads 89 N. The vet next measures Clea and finds that her rear feet are 0.95 m behind her front feet. How much does Clea weigh, and where is her center of gravity?

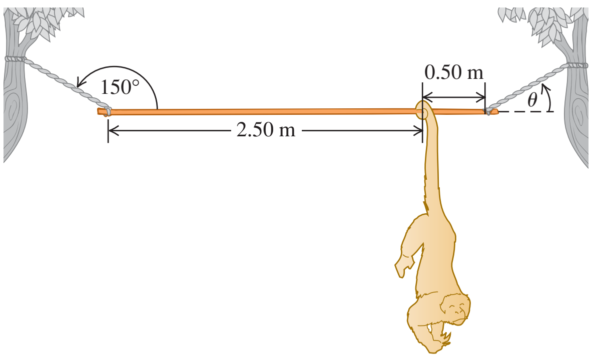
11.18 •• A 15,000-N crane pivots around a friction-free axle at its base and is supported by a cable making a 25° angle with the crane (Fig. E11.18). The crane is 16 m long and is not uniform, its center of gravity being 7.0 m from the axle as measured along the crane. The cable is attached 3.0 m from the upper end of the crane. When the crane is raised to 55° above the horizontal holding an 11,000-N pallet of bricks by a 2.2-m, very light cord, find (a) the tension in the cable and (b) the horizontal and vertical components of the force that the axle exerts on the crane. Start with a free-body diagram of the crane.

Figure **E11.18**



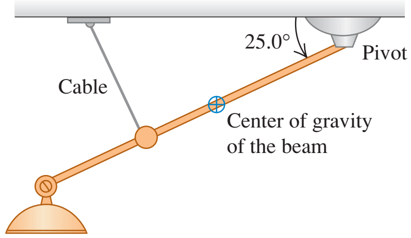
11.19 •• A 3.00-m-long, 240-N, uniform rod at the zoo is held in a horizontal position by two ropes at its ends (Fig. E11.19). The left rope makes an angle of 150° with the rod and the right rope makes an angle θ with the horizontal. A 90-N howler monkey (*Alouatta seniculus*) hangs motionless 0.50 m from the right end of the rod as he carefully studies you. Calculate the tensions in the two ropes and the angle θ. First make a free-body diagram of the rod.

Figure **E11.19**



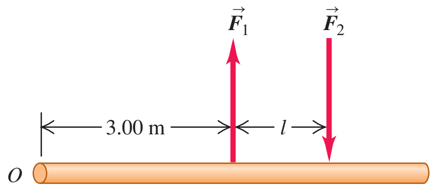
11.20 •• A nonuniform beam 4.50 m long and weighing 1.00 kN makes an angle of 25.0° below the horizontal. It is held in position by a frictionless pivot at its upper right end and by a cable 3.00 m farther down the beam and perpendicular to it (Fig. E11.20). The center of gravity of the beam is 2.00 m down the beam from the pivot. Lighting equipment exerts a 5.00-kN downward force on the lower left end of the beam. Find the tension T in the cable and the horizontal and vertical components of the force exerted on the beam by the pivot. Start by sketching a free-body diagram of the beam.

Figure **E11.20**



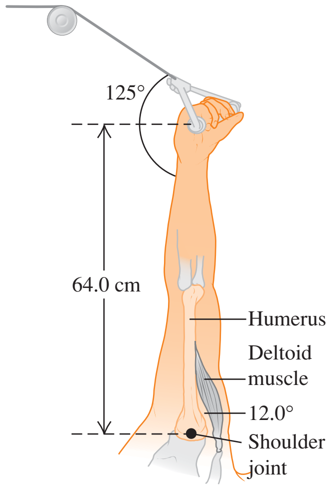
11.21 • **A Couple.** Two forces equal in magnitude and opposite in direction, acting on an object at two different points, form what is called a *couple*. Two antiparallel forces with equal magnitudes $F_1 = F_2 = 8.00\ N$ are applied to a rod as shown in Fig. E11.21. (a) What should the distance l between the forces be if they are to provide a net torque of 6.40 N · m about the left end of the rod? (b) Is the sense of this torque clockwise or counterclockwise? (c) Repeat parts (a) and (b) for a pivot at the point on the rod where $\vec{F}_2$ is applied.

Figure **E11.21**



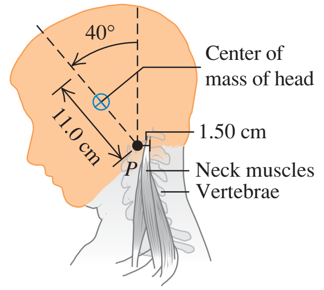
11.22 •• BIO **A Good Workout.** You are doing exercises on a Nautilus machine in a gym to strengthen your deltoid (shoulder) muscles. Your arms are raised vertically and can pivot around the shoulder joint, and you grasp the cable of the machine in your hand 64.0 cm from your shoulder joint. The deltoid muscle is attached to the humerus 15.0 cm from the shoulder joint and makes a 12.0° angle with that bone (Fig. E11.22). If you have set the tension in the cable of the machine to 36.0 N on each arm, what is the tension in each deltoid muscle if you simply hold your outstretched arms in place? (*Hint:* Start by making a clear free-body diagram of your arm.)

Figure **E11.22**



11.23 •• BIO **Neck Muscles.** A student bends her head at 40.0° from the vertical while intently reading her physics book, pivoting the head around the upper vertebra (point P in Fig. E11.23). Her head has a mass of 4.50 kg (which is typical), and its center of mass is 11.0 cm from the pivot point P. Her neck muscles are 1.50 cm from point P, as measured *perpendicular* to these muscles. The neck itself and the vertebrae are held vertical. (a) Draw a free-body diagram of the student's head. (b) Find the tension in her neck muscles.

Figure **E11.23**



Section 11.4 Stress, Strain, and Elastic Moduli

11.24 • BIO **Biceps Muscle.** A relaxed biceps muscle requires a force of 25.0 N for an elongation of 3.0 cm; the same muscle under maximum tension requires a force of 500 N for the same elongation. Find Young's modulus for the muscle tissue under each of these conditions if the muscle is assumed to be a uniform cylinder with length 0.200 m and cross-sectional area 50.0 cm^2.

11.25 •• A circular steel wire 2.00 m long must stretch no more than 0.25 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire?

11.26 •• Two circular rods, one steel and the other copper, are joined end to end. Each rod is 0.750 m long and 1.50 cm in diameter. The combination is subjected to a tensile force with magnitude 4000 N. For each rod, what are (a) the strain and (b) the elongation?

11.27 •• A metal rod that is 4.00 m long and 0.50 cm^2 in cross-sectional area is found to stretch 0.20 cm under a tension of 5000 N. What is Young's modulus for this metal?

11.28 •• **Stress on a Mountaineer's Rope.** A nylon rope used by mountaineers elongates 1.10 m under the weight of a 65.0-kg climber. If the rope is 45.0 m in length and 7.0 mm in diameter, what is Young's modulus for nylon?

11.29 •• In constructing a large mobile, an artist hangs an aluminum sphere of mass 6.0 kg from a vertical steel wire 0.50 m long and 2.5×10^{-3} cm^2 in cross-sectional area. On the bottom of the sphere he attaches a similar steel wire, from which he hangs a brass cube of mass 10.0 kg. For each wire, compute (a) the tensile strain and (b) the elongation.

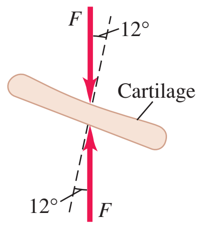
11.30 •• A vertical, solid steel post 25 cm in diameter and 2.50 m long is required to support a load of 8000 kg. You can ignore the weight of the post. What are (a) the stress in the post; (b) the strain in the post; and (c) the change in the post's length when the load is applied?

11.31 •• BIO **Compression of Human Bone.** The bulk modulus for bone is 15 GPa. (a) If a diver-in-training is put into a pressurized suit, by how much would the pressure have to be raised (in atmospheres) above atmospheric pressure to compress her bones by 0.10% of their original volume? (b) Given that the pressure in the ocean increases by 1.0×10^4 Pa for every meter of depth below the surface, how deep would this diver have to go for her bones to compress by 0.10%? Does it seem that bone compression is a problem she needs to be concerned with when diving?

11.32 • A solid gold bar is pulled up from the hold of the sunken RMS *Titanic*. (a) What happens to its volume as it goes from the pressure at the ship to the lower pressure at the ocean's surface? (b) The pressure difference is proportional to the depth. How many times greater would the volume change have been had the ship been twice as deep? (c) The bulk modulus of lead is one-fourth that of gold. Find the ratio of the volume change of a solid lead bar to that of a gold bar of equal volume for the same pressure change.

11.33 • BIO **Downhill Hiking.** During vigorous downhill hiking, the force on the knee cartilage (the medial and lateral meniscus) can be up to eight times body weight. Depending on the angle of descent, this force can cause a large shear force on the cartilage and deform it. The cartilage has an area of about 10 cm^2 and a shear modulus of 12 MPa. If the hiker plus his pack have a combined mass of 110 kg (not unreasonable), and if the maximum force at impact is 8 times his body weight (which, of course, includes the weight of his pack) at an angle of 12° with the cartilage (Fig. E11.33), through what angle (in degrees) will his knee cartilage be deformed? (Recall that the bone below the cartilage pushes upward with the same force as the downward force.)

Figure **E11.33**



11.34 •• In the Challenger Deep of the Marianas Trench, the depth of seawater is 10.9 km and the pressure is 1.16×10^8 Pa (about 1.15×10^3 atm). (a) If a cubic meter of water is taken from the surface to this depth, what is the change in its volume? (Normal atmospheric pressure is about 1.0×10^5 Pa. Assume that k for seawater is the same as the freshwater value given in Table 11.2.) (b) What is the density of seawater at this depth? (At the surface, seawater has a density of $1.03 \times 10^3\ \text{kg/m}^3$.)

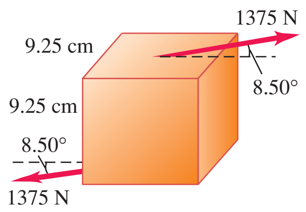
11.35 • A specimen of oil having an initial volume of 600 cm^3 is subjected to a pressure increase of 3.6×10^6 Pa, and the volume is found to decrease by 0.45 cm^3. What is the bulk modulus of the material? The compressibility?

11.36 •• A square steel plate is 10.0 cm on a side and 0.500 cm thick. (a) Find the shear strain that results if a force of magnitude 9.0×10^5 N is applied to each of the four sides, parallel to the side. (b) Find the displacement x in centimeters.

11.37 •• A copper cube measures 6.00 cm on each side. The bottom face is held in place by very strong glue to a flat horizontal surface, while a horizontal force F is applied to the upper face parallel to one of the edges. (Consult Table 11.1.) (a) Show that the glue exerts a force F on the bottom face that is equal but opposite to the force on the top face. (b) How large must F be to cause the cube to deform by 0.250 mm? (c) If the same experiment were performed on a lead cube of the same size as the copper one, by what distance would it deform for the same force as in part (b)?

11.38 • In lab tests on a 9.25-cm cube of a certain material, a force of 1375 N directed at 8.50° to the cube (Fig. E11.38) causes the cube to deform through an angle of 1.24°. What is the shear modulus of the material?

Figure **E11.38**



Section 11.5 Elasticity and Plasticity

11.39 •• In a materials testing laboratory, a metal wire made from a new alloy is found to break when a tensile force of 90.8 N is applied perpendicular to each end. If the diameter of the wire is 1.84 mm, what is the breaking stress of the alloy?

11.40 • A 4.0-m-long steel wire has a cross-sectional area of 0.050 cm^2. Its proportional limit has a value of 0.0016 times its Young's modulus (see Table 11.1). Its breaking stress has a value of 0.0065 times its Young's modulus. The wire is fastened at its upper end and hangs vertically. (a) How great a weight can be hung from the wire without exceeding the proportional limit?

(b) How much will the wire stretch under this load? (c) What is the maximum weight that the wire can support?

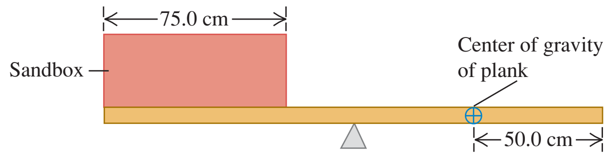
11.41 •• **CP** A steel cable with cross-sectional area 3.00 cm^2 has an elastic limit of 2.40×10^8 Pa. Find the maximum upward acceleration that can be given a 1200-kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

11.42 •• A brass wire is to withstand a tensile force of 350 N without breaking. What minimum diameter must the wire have?

PROBLEMS

11.43 ••• A box of negligible mass rests at the left end of a 2.00-m, 25.0-kg plank (Fig. P11.43). The width of the box is 75.0 cm, and sand is to be distributed uniformly throughout it. The center of gravity of the nonuniform plank is 50.0 cm from the right end. What mass of sand should be put into the box so that the plank balances horizontally on a fulcrum placed just below its midpoint?

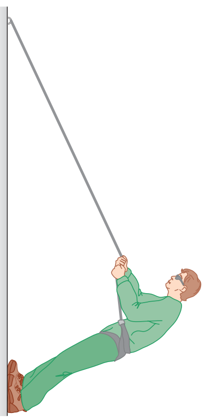
Figure **P11.43**



11.44 ••• A door 1.00 m wide and 2.00 m high weighs 280 N and is supported by two hinges, one 0.50 m from the top and the other 0.50 m from the bottom. Each hinge supports half the total weight of the door. Assuming that the door's center of gravity is at its center, find the horizontal components of force exerted on the door by each hinge.

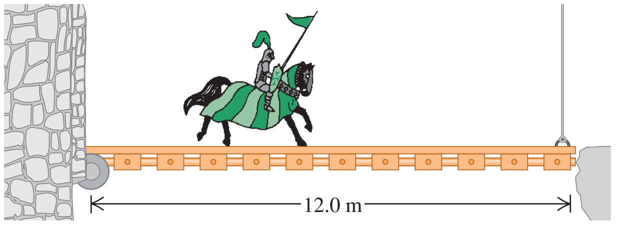
11.45 ••• **Mountain Climbing.** Mountaineers often use a rope to lower themselves down the face of a cliff (this is called *rappelling*). They do this with their body nearly horizontal and their feet pushing against the cliff (Fig. P11.45). Suppose that an 82.0-kg climber, who is 1.90 m tall and has a center of gravity 1.1 m from his feet, rappels down a vertical cliff with his body raised 35.0° above the horizontal. He holds the rope 1.40 m from his feet, and it makes a 25.0° angle with the cliff face. (a) What tension does his rope need to support? (b) Find the horizontal and vertical components of the force that the cliff face exerts on the climber's feet. (c) What minimum coefficient of static friction is needed to prevent the climber's feet from slipping on the cliff face if he has one foot at a time against the cliff?

Figure **P11.45**



11.46 • Sir Lancelot rides slowly out of the castle at Camelot and onto the 12.0-m-long drawbridge that passes over the moat (Fig. P11.46). Unbeknownst to him, his enemies have partially severed the vertical cable holding up the front end of the bridge so that it will break under a tension of 5.80×10^3 N. The bridge has mass 200 kg and its center of gravity is at its center. Lancelot, his lance, his armor, and his horse together have a combined mass of 600 kg. Will the cable break before Lancelot reaches the end of the drawbridge? If so, how far from the castle end of the bridge will the center of gravity of the horse plus rider be when the cable breaks?

Figure **P11.46**

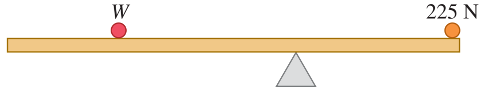


11.47 • Three vertical forces act on an airplane when it is flying at a constant altitude and with a constant velocity. These are the weight of the airplane, an aerodynamic force on the wing of the airplane, and an aerodynamic force on the airplane's horizontal tail. (The aerodynamic forces are exerted by the surrounding air and are reactions to the forces that the wing and tail exert on the air as the airplane flies through it.) For a particular light airplane with a weight of 6700 N, the center of gravity is 0.30 m in front of the point where the wing's vertical aerodynamic force acts and 3.66 m in front of the point where the tail's vertical aerodynamic force acts. Determine the magnitude and direction (upward or downward) of each of the two vertical aerodynamic forces.

11.48 •• A pickup truck has a wheelbase of 3.00 m. Ordinarily, 10,780 N rests on the front wheels and 8820 N on the rear wheels when the truck is parked on a level road. (a) A box weighing 3600 N is now placed on the tailgate, 1.00 m behind the rear axle. How much total weight now rests on the front wheels? On the rear wheels? (b) How much weight would need to be placed on the tailgate to make the front wheels come off the ground?

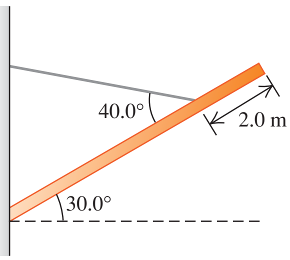
11.49 •• A uniform, 255-N rod that is 2.00 m long carries a 225-N weight at its right end and an unknown weight W toward the left end (Fig. P11.49). When W is placed 50.0 cm from the left end of the rod, the system just balances horizontally when the fulcrum is located 75.0 cm from the right end. (a) Find W. (b) If W is now moved 25.0 cm to the right, how far and in what direction must the fulcrum be moved to restore balance?

Figure **P11.49**



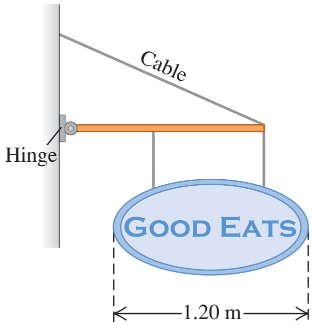
11.50 •• A uniform, 8.0-m, 1500-kg beam is hinged to a wall and supported by a thin cable attached 2.0 m from the free end of the beam, (Fig. P11.50). The beam is supported at an angle of 30.0° above the horizontal. (a) Draw a free-body diagram of the beam. (b) Find the tension in the cable. (c) How hard does the beam push inward on the wall?

Figure **P11.50**



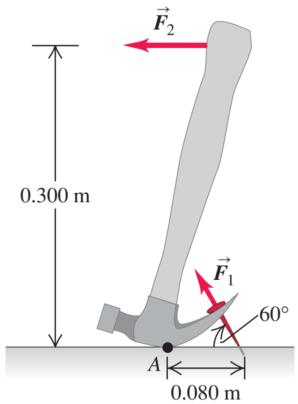
11.51 •• You open a restaurant and hope to entice customers by hanging out a sign (Fig. P11.51). The uniform horizontal beam supporting the sign is 1.50 m long, has a mass of 12.0 kg, and is hinged to the wall. The sign itself is uniform with a mass of 28.0 kg and overall length of 1.20 m. The two wires supporting the sign are each 32.0 cm long, are 90.0 cm apart, and are equally spaced from the middle of the sign. The cable supporting the beam is 2.00 m long. (a) What minimum tension must your cable be able to support without having your sign come crashing down? (b) What minimum vertical force must the hinge be able to support without pulling out of the wall?

Figure **P11.51**



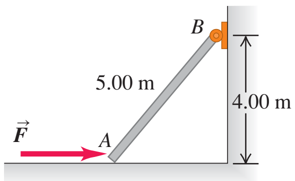
11.52 ••• A claw hammer is used to pull a nail out of a board (Fig. P11.52). The nail is at an angle of 60° to the board, and a force $\vec{F}_1$ of magnitude 400 N applied to the nail is required to pull it from the board. The hammer head contacts the board at point A, which is 0.080 m from where the nail enters the board. A horizontal force $\vec{F}_2$ is applied to the hammer handle at a distance of 0.300 m above the board. What magnitude of force $\vec{F}_2$ is required to apply the required 400-N force (F_1) to the nail? (You can ignore the weight of the hammer.)

Figure **P11.52**



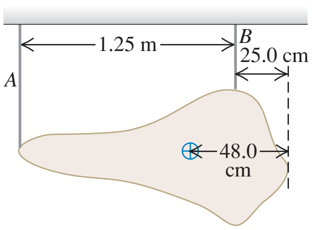
11.53 • End A of the bar AB in Fig. P11.53 rests on a frictionless horizontal surface, and end B is hinged. A horizontal force $\vec{F}$ of magnitude 160 N is exerted on end A. You can ignore the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at B?

Figure **P11.53**



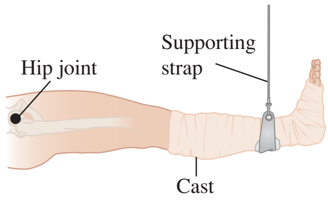
11.54 • A museum of modern art is displaying an irregular 426-N sculpture by hanging it from two thin vertical wires, A and B, that are 1.25 m apart (Fig. P11.54). The center of gravity of this piece of art is located 48.0 cm from its extreme right tip. Find the tension in each wire.

Figure **P11.54**



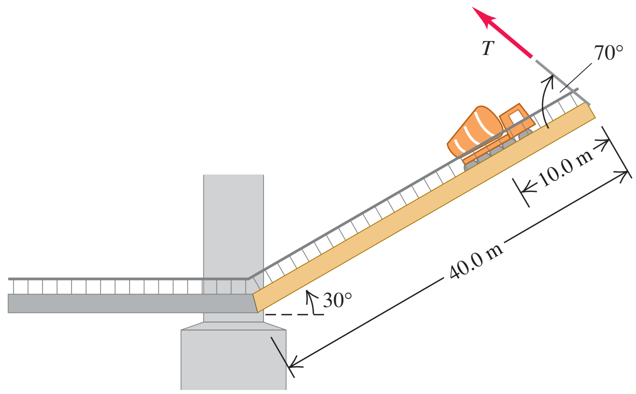
11.55 •• BIO **Supporting a Broken Leg.** A therapist tells a 74-kg patient with a broken leg that he must have his leg in a cast suspended horizontally. For minimum discomfort, the leg should be supported by a vertical strap attached at the center of mass of the leg–cast system. (Fig. P11.55). In order to comply with these instructions, the patient consults a table of typical mass distributions and finds that both upper legs (thighs) together typically account for 21.5% of body weight and the center of mass of each thigh is 18.0 cm from the hip joint. The patient also reads that the two lower legs (including the feet) are 14.0% of body weight, with a center of mass 69.0 cm from the hip joint. The cast has a mass of 5.50 kg, and its center of mass is 78.0 cm from the hip joint. How far from the hip joint should the supporting strap be attached to the cast?

Figure **P11.55**



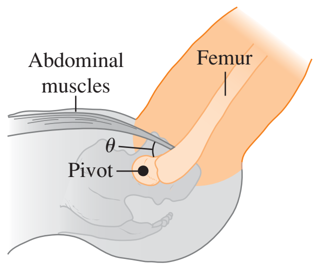
11.56 • **A Truck on a Drawbridge.** A loaded cement mixer drives onto an old drawbridge, where it stalls with its center of gravity three-quarters of the way across the span. The truck driver radios for help, sets the handbrake, and waits. Meanwhile, a boat approaches, so the drawbridge is raised by means of a cable attached to the end opposite the hinge (Fig. P11.56). The drawbridge is 40.0 m long and has a mass of 18,000 kg; its center of gravity is at its midpoint. The cement mixer, with driver, has mass 30,000 kg. When the drawbridge has been raised to an angle of 30° above the horizontal, the cable makes an angle of 70° with the surface of the bridge. (a) What is the tension T in the cable when the drawbridge is held in this position? (b) What are the horizontal and vertical components of the force the hinge exerts on the span?

Figure **P11.56**



11.57 •• BIO **Leg Raises.** In a simplified version of the musculature action in leg raises, the abdominal muscles pull on the femur (thigh bone) to raise the leg by pivoting it about one end (Fig. P11.57). When you are lying horizontally, these muscles make an angle of approximately 5° with the femur,

Figure **P11.57**



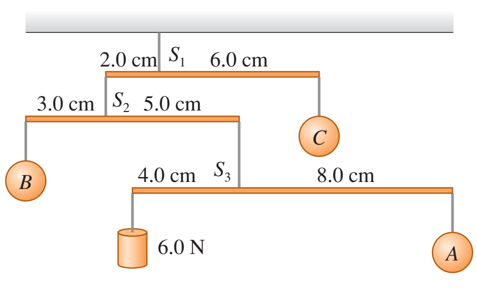
and if you raise your legs, the muscles remain approximately horizontal, so the angle θ increases. We shall assume for simplicity that these muscles attach to the femur in only one place, 10 cm from the hip joint (although, in reality, the situation is more complicated). For a certain 80-kg person having a leg 90 cm long, the mass of the leg is 15 kg and its center of mass is 44 cm from his hip joint as measured along the leg. If the person raises his leg to 60° above the horizontal, the angle between the abdominal muscles and his femur would also be about 60°. (a) With his leg raised to 60°, find the tension in the abdominal muscle on each leg. As usual, begin your solution with a free-body diagram. (b) When is the tension in this muscle greater: when the leg is raised to 60° or when the person just starts to raise it off the ground? Why? (Try this yourself to check your answer.) (c) If the abdominal muscles attached to the femur were perfectly horizontal when a person was lying down, could the person raise his leg? Why or why not?

11.58 • A nonuniform fire escape ladder is 6.0 m long when extended to the icy alley below. It is held at the top by a frictionless pivot, and there is negligible frictional force from the icy surface at the bottom. The ladder weighs 250 N, and its center of gravity is 2.0 m along the ladder from its bottom. A mother and child of total weight 750 N are on the ladder 1.5 m from the pivot. The ladder makes an angle θ with the horizontal. Find the magnitude and direction of (a) the force exerted by the icy alley on the ladder and (b) the force exerted by the ladder on the pivot. (c) Do your answers in parts (a) and (b) depend on the angle θ?

11.59 •• A uniform strut of mass m makes an angle θ with the horizontal. It is supported by a frictionless pivot located at one-third its length from its lower left end and a horizontal rope at its upper right end. A cable and package of total weight w hang from its upper right end. (a) Find the vertical and horizontal components V and H of the pivot's force on the strut as well as the tension T in the rope. (b) If the maximum safe tension in the rope is 700 N and the mass of the strut is 30.0 kg, find the maximum safe weight of the cable and package when the strut makes an angle of 55.0° with the horizontal. (c) For what angle θ can no weight be safely suspended from the right end of the strut?

11.60 • You are asked to design the decorative mobile shown in Fig. P11.60. The strings and rods have negligible weight, and the rods are to hang horizontally. (a) Draw a free-body diagram for each rod. (b) Find the weights of the balls A, B, and C. Find the tensions in the strings S_1, S_2, and S_3. (c) What can you say about the horizontal location of the mobile's center of gravity? Explain.

Figure **P11.60**

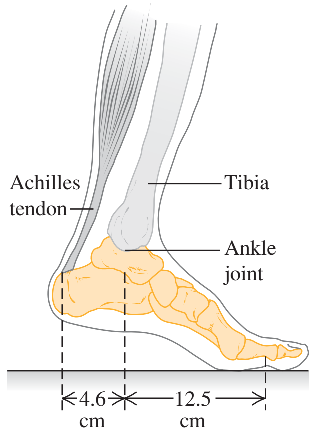


11.61 •• A uniform, 7.5-m-long beam weighing 5860 N is hinged to a wall and supported by a thin cable attached 1.5 m from the free end of the beam. The cable runs between the beam and the wall and makes a 40° angle with the beam. What is the tension in the cable when the beam is at an angle of 30° above the horizontal?

11.62 •• **CP** A uniform drawbridge must be held at a 37° angle above the horizontal to allow ships to pass underneath. The drawbridge weighs 45,000 N and is 14.0 m long. A cable is connected 3.5 m from the hinge where the bridge pivots (measured along the bridge) and pulls horizontally on the bridge to hold it in place. (a) What is the tension in the cable? (b) Find the magnitude and direction of the force the hinge exerts on the bridge. (c) If the cable suddenly breaks, what is the magnitude of the angular acceleration of the drawbridge just after the cable breaks? (d) What is the angular speed of the drawbridge as it becomes horizontal?

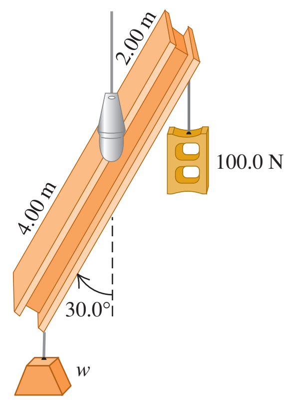
11.63 •• **BIO** **Tendon-Stretching Exercises.** As part of an exercise program, a 75-kg person does toe raises in which he raises his entire body weight on the ball of one foot (Fig. P11.63). The Achilles tendon pulls straight upward on the heel bone of his foot. This tendon is 25 cm long and has a cross-sectional area of 78 mm^2 and a Young's modulus of 1470 MPa. (a) Make a free-body diagram of the person's foot (everything below the ankle joint). You can neglect the weight of the foot. (b) What force does the Achilles tendon exert on the heel during this exercise? Express your answer in newtons and in multiples of his weight. (c) By how many millimeters does the exercise stretch his Achilles tendon?

Figure **P11.63**



11.64 •• (a) In Fig. P11.64 a 6.00-m-long, uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100.0-N weight is hung; an unknown weight w hangs at the left end. If the system is in equilibrium, what is w? You can ignore the thickness of the beam. (b) If the beam makes, instead, an angle of 45.0° with the vertical, what is w?

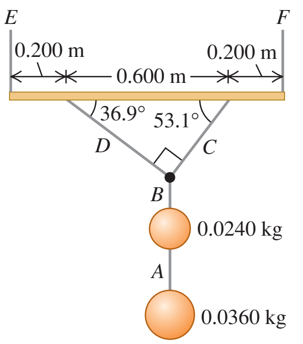
Figure **P11.64**



11.65 ••• A uniform, horizontal flagpole 5.00 m long with a weight of 200 N is hinged to a vertical wall at one end. A 600-N stuntwoman hangs from its other end. The flagpole is supported by a guy wire running from its outer end to a point on the wall directly above the pole. (a) If the tension in this wire is not to exceed 1000 N, what is the minimum height above the pole at which it may be fastened to the wall? (b) If the flagpole remains horizontal, by how many newtons would the tension be increased if the wire were fastened 0.50 m below this point?

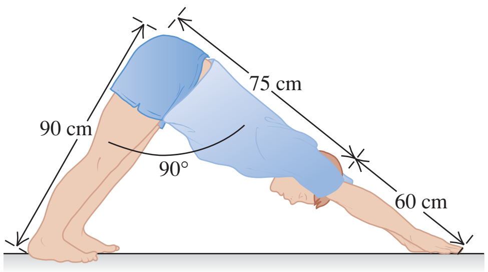
11.66 • A holiday decoration consists of two shiny glass spheres with masses 0.0240 kg and 0.0360 kg suspended from a uniform rod with mass 0.120 kg and length 1.00 m (Fig. P11.66). The rod is suspended from the ceiling by a vertical cord at each end, so that it is horizontal. Calculate the tension in each of the cords *A* through *F*.

Figure **P11.66**



11.67 •• BIO **Downward-Facing Dog.** One yoga exercise, known as the "Downward-Facing Dog," requires stretching your hands straight out above your head and bending down to lean against the floor. This exercise is performed by a 750-N person, as shown in Fig. P11.67. When he bends his body at the hip to a 90° angle between his legs and trunk, his legs, trunk, head, and arms have the dimensions indicated. Furthermore, his legs and feet weigh a total of 277 N, and their center of mass is 41 cm from his hip, measured along his legs. The person's trunk, head, and arms weigh 473 N, and their center of gravity is 65 cm from his hip, measured along the upper body. (a) Find the normal force that the floor exerts on each foot and on each hand, assuming that the person does not favor either hand or either foot. (b) Find the friction force on each foot and on each hand, assuming that it is the same on both feet and on both hands (but not necessarily the same on the feet as on the hands). [*Hint:* First treat his entire body as a system; then isolate his legs (or his upper body).]

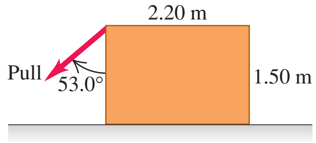
Figure **P11.67**



11.68 • When you stretch a wire, rope, or rubber band, it gets thinner as well as longer. When Hooke's law holds, the fractional decrease in width is proportional to the tensile strain. If w_0 is the original width and Δw is the change in width, then $\Delta w/w_0 = -\sigma \Delta l/l_0$, where the minus sign reminds us that width decreases when length increases. The dimensionless constant σ, different for different materials, is called *Poisson's ratio.* (a) If the steel rod of Example 11.5 (Section 11.4) has a circular cross section and a Poisson's ratio of 0.23, what is its change in diameter when the milling machine is hung from it? (b) A cylinder made of nickel (Poisson's ratio $= 0.42$) has radius 2.0 cm. What tensile force $F_{\perp}$ must be applied perpendicular to each end of the cylinder to cause its radius to decrease by 0.10 mm? Assume that the breaking stress and proportional limit for the metal are extremely large and are not exceeded.

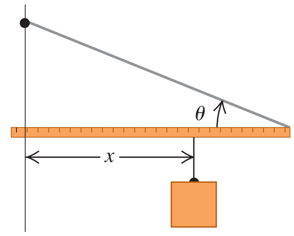
11.69 • A worker wants to turn over a uniform, 1250-N, rectangular crate by pulling at 53.0° on one of its vertical sides (Fig. P11.69). The floor is rough enough to prevent the crate from slipping. (a) What pull is needed to just start the crate to tip? (b) How hard does the floor push upward on the crate? (c) Find the friction force on the crate. (d) What is the minimum coefficient of static friction needed to prevent the crate from slipping on the floor?

Figure **P11.69**



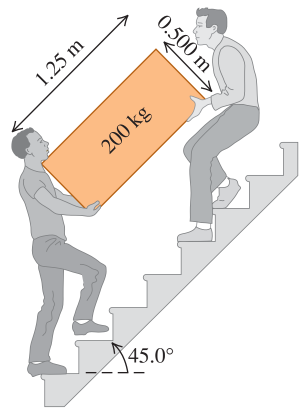
11.70 ••• One end of a uniform meter stick is placed against a vertical wall (Fig. P11.70). The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium? (b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium? (c) When $\theta = 15°$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure **P11.70**



11.71 •• Two friends are carrying a 200-kg crate up a flight of stairs. The crate is 1.25 m long and 0.500 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (Fig. P11.71). If the force each person applies is vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?

Figure **P11.71**



11.72 •• BIO **Forearm.** In the human arm, the forearm and hand pivot about the elbow joint. Consider a simplified model in which the biceps muscle is attached to the forearm 3.80 cm from the elbow joint. Assume that the person's hand and forearm together weigh 15.0 N and that their center of gravity is 15.0 cm from the elbow (not quite halfway to the hand). The forearm is held horizontally at a right angle to the upper arm, with the biceps muscle exerting its force perpendicular to the forearm. (a) Draw a free-body diagram for the forearm, and find the force exerted by the biceps when the hand is empty. (b) Now the person holds a 80.0-N weight in his hand, with the forearm still horizontal. Assume that the center of gravity of this weight is 33.0 cm from the elbow. Construct a free-body diagram for the forearm, and find the force now exerted by the biceps. Explain why the biceps muscle needs to be very strong. (c) Under the conditions of part (b), find the magnitude and direction of the force that the elbow joint exerts on the forearm. (d) While holding the 80.0-N weight, the person raises his forearm until it is at an angle of 53.0° above the horizontal. If the biceps muscle continues to exert its force perpendicular to the forearm, what is this force when the forearm is in this position? Has the force increased or decreased from its value in part (b)? Explain why this is so, and test your answer by actually doing this with your own arm.

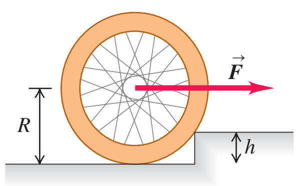
11.73 •• **BIO** **CALC** Refer to the discussion of holding a dumbbell in Example 11.4 (Section 11.3). The maximum weight that can be held in this way is limited by the maximum allowable tendon tension T (determined by the strength of the tendons) and by the distance D from the elbow to where the tendon attaches to the forearm. (a) Let T_{max} represent the maximum value of the tendon tension. Use the results of Example 11.4 to express w_{max} (the maximum weight that can be held) in terms of T_{max}, L, D, and h. Your expression should *not* include the angle θ. (b) The tendons of different primates are attached to the forearm at different values of D. Calculate the derivative of w_{max} with respect to D, and determine whether the derivative is positive or negative. (c) A chimpanzee tendon is attached to the forearm at a point farther from the elbow than for humans. Use this to explain why chimpanzees have stronger arms than humans. (The disadvantage is that chimpanzees have less flexible arms than do humans.)

11.74 •• A uniform, 90.0-N table is 3.6 m long, 1.0 m high, and 1.2 m wide. A 1500-N weight is placed 0.50 m from one end of the table, a distance of 0.60 m from each side of the table. Draw a free-body diagram for the table and find the force that each of the four legs exerts on the floor.

11.75 ••• **Flying Buttress.** (a) A symmetric building has a roof sloping upward at 35.0° above the horizontal on each side. If each side of the uniform roof weighs 10,000 N, find the horizontal force that this roof exerts at the top of the wall, which tends to push out the walls. Which type of building would be more in danger of collapsing: one with tall walls or one with short walls? Explain. (b) As you saw in part (a), tall walls are in danger of collapsing from the weight of the roof. This problem plagued the ancient builders of large structures. A solution used in the great Gothic cathedrals during the 1200s was the flying buttress, a stone support running between the walls and the ground that helped to hold in the walls. A Gothic church has a uniform roof weighing a total of 20,000 N and rising at 40° above the horizontal at each wall. The walls are 40 m tall, and a flying buttress meets each wall 10 m below the base of the roof. What horizontal force must this flying buttress apply to the wall?

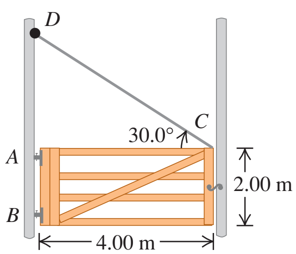
11.76 •• You are trying to raise a bicycle wheel of mass m and radius R up over a curb of height h. To do this, you apply a horizontal force $\vec{F}$ (Fig. P11.76). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel and (b) at the top of the wheel? (c) In which case is less force required?

Figure **P11.76**



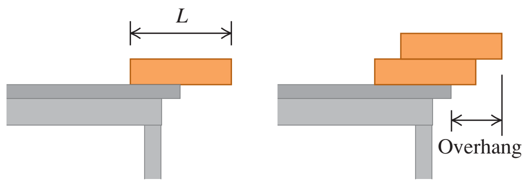
11.77 • **The Farmyard Gate.** A gate 4.00 m wide and 2.00 m high weighs 500 N. Its center of gravity is at its center, and it is hinged at A and B. To relieve the strain on the top hinge, a wire CD is connected as shown in Fig. P11.77. The tension in CD is increased until the horizontal force at hinge A is zero. (a) What is the tension in the wire CD? (b) What is the magnitude of the horizontal component of the force at hinge B? (c) What is the combined vertical force exerted by hinges A and B?

Figure **P11.77**



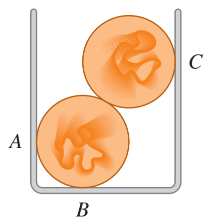
11.78 • If you put a uniform block at the edge of a table, the center of the block must be over the table for the block not to fall off. (a) If you stack two identical blocks at the table edge, the center of the top block must be over the bottom block, and the center of gravity of the two blocks together must be over the table. In terms of the length L of each block, what is the maximum overhang possible (Fig. P11.78)? (b) Repeat part (a) for three identical blocks and for four identical blocks. (c) Is it possible to make a stack of blocks such that the uppermost block is not directly over the table at all? How many blocks would it take to do this? (Try this with your friends using copies of this book.)

Figure **P11.78**



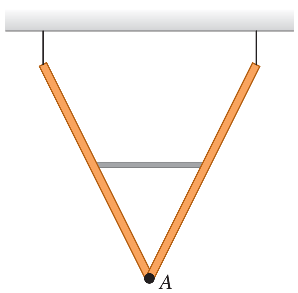
11.79 ••• Two uniform, 75.0-g marbles 2.00 cm in diameter are stacked as shown in Fig. P11.79 in a container that is 3.00 cm wide. (a) Find the force that the container exerts on the marbles at the points of contact A, B, and C. (b) What force does each marble exert on the other?

Figure **P11.79**



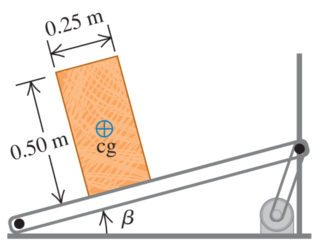
11.80 •• Two identical, uniform beams weighing 260 N each are connected at one end by a frictionless hinge. A light horizontal crossbar attached at the midpoints of the beams maintains an angle of 53.0° between the beams. The beams are suspended from the ceiling by vertical wires such that they form a "V," as shown in Fig. P11.80. (a) What force does the crossbar exert on each beam? (b) Is the crossbar under tension or compression? (c) What force (magnitude and direction) does the hinge at point A exert on each beam?

Figure **P11.80**



11.81 • An engineer is designing a conveyor system for loading hay bales into a wagon (Fig. P11.81). Each bale is 0.25 m wide, 0.50 m high, and 0.80 m long (the dimension perpendicular to the plane of the figure), with mass 30.0 kg. The center of

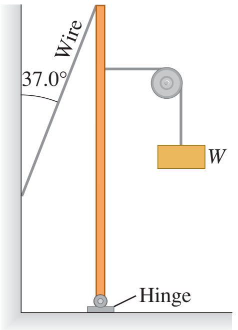
Figure **P11.81**



gravity of each bale is at its geometrical center. The coefficient of static friction between a bale and the conveyor belt is 0.60, and the belt moves with constant speed. (a) The angle β of the conveyor is slowly increased. At some critical angle a bale will tip (if it doesn't slip first), and at some different critical angle it will slip (if it doesn't tip first). Find the two critical angles and determine which happens at the smaller angle. (b) Would the outcome of part (a) be different if the coefficient of friction were 0.40?

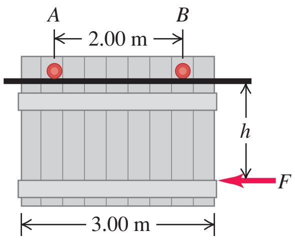
11.82 • A weight W is supported by attaching it to a vertical uniform metal pole by a thin cord passing over a pulley having negligible mass and friction. The cord is attached to the pole 40.0 cm below the top and pulls horizontally on it (Fig. P11.82). The pole is pivoted about a hinge at its base, is 1.75 m tall, and weighs 55.0 N. A thin wire connects the top of the pole to a vertical wall. The nail that holds this wire to the wall will pull out if an *outward* force greater than 22.0 N acts on it. (a) What is the greatest weight W that can be supported this way without pulling out the nail? (b) What is the *magnitude* of the force that the hinge exerts on the pole?

Figure **P11.82**



11.83 •• A garage door is mounted on an overhead rail (Fig. P11.83). The wheels at A and B have rusted so that they do not roll, but rather slide along the track. The coefficient of kinetic friction is 0.52. The distance between the wheels is 2.00 m, and each is 0.50 m from the vertical sides of the door. The door is uniform and weighs 950 N. It is pushed to the left at constant speed by a horizontal force $\vec{F}$. (a) If the distance h is 1.60 m, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value h can have without causing one wheel to leave the track.

Figure **P11.83**

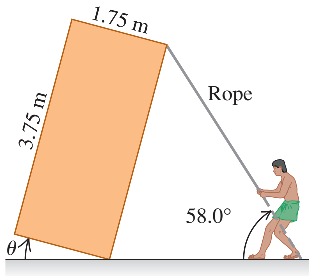


11.84 •• A horizontal boom is supported at its left end by a frictionless pivot. It is held in place by a cable attached to the right-hand end of the boom. A chain and crate of total weight w hang from somewhere along the boom. The boom's weight w_b cannot be ignored and the boom may or may not be uniform. (a) Show that the tension in the cable is the same whether the cable makes an angle θ or an angle $180° - \theta$ with the horizontal, and that the horizontal force component exerted on the boom by the pivot has equal magnitude but opposite direction for the two angles. (b) Show that the cable cannot be horizontal. (c) Show that the tension in the cable is a minimum when the cable is vertical, pulling upward on the right end of the boom. (d) Show that when the cable is vertical, the force exerted by the pivot on the boom is vertical.

11.85 •• Prior to being placed in its hole, a 5700-N, 9.0-m-long, uniform utility pole makes some nonzero angle with the vertical. A vertical cable attached 2.0 m below its upper end holds it in place while its lower end rests on the ground. (a) Find the tension in the cable and the magnitude and direction of the force exerted by the ground on the pole. (b) Why don't we need to know the angle the pole makes with the vertical, as long as it is not zero?

11.86 ••• **Pyramid Builders.** Ancient pyramid builders are balancing a uniform rectangular slab of stone tipped at an angle θ above the horizontal using a rope (Fig. P11.86). The rope is held by five workers who share the force equally. (a) If $\theta = 20.0°$, what force does each worker exert on the rope? (b) As θ increases, does each worker have to exert more or less force than in part (a), assuming they do not change the angle of the rope? Why? (c) At what angle do the workers need to exert *no force* to balance the slab? What happens if θ exceeds this value?

Figure **P11.86**



11.87 • You hang a floodlamp from the end of a vertical steel wire. The floodlamp stretches the wire 0.18 mm and the stress is proportional to the strain. How much would it have stretched (a) if the wire were twice as long? (b) if the wire had the same length but twice the diameter? (c) for a copper wire of the original length and diameter?

11.88 •• **Hooke's Law for a Wire.** A wire of length l_0 and cross-sectional area A supports a hanging weight W. (a) Show that if the wire obeys Eq. (11.7), it behaves like a spring of force constant AY/l_0, where Y is Young's modulus for the material of which the wire is made. (b) What would the force constant be for a 75.0-cm length of 16-gauge (diameter = 1.291 mm) copper wire? See Table 11.1. (c) What would W have to be to stretch the wire in part (b) by 1.25 mm?

11.89 ••• **CP** A 12.0-kg mass, fastened to the end of an aluminum wire with an unstretched length of 0.50 m, is whirled in a vertical circle with a constant angular speed of 120 rev/min. The cross-sectional area of the wire is 0.014 cm^2. Calculate the elongation of the wire when the mass is (a) at the lowest point of the path and (b) at the highest point of its path.

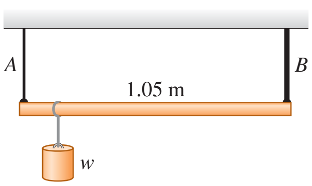
11.90 • A metal wire 3.50 m long and 0.70 mm in diameter was given the following test. A load weighing 20 N was originally hung from the wire to keep it taut. The position of the lower end of the wire was read on a scale as load was added.

Added Load (N)	Scale Reading (cm)
0	3.02
10	3.07
20	3.12
30	3.17
40	3.22
50	3.27
60	3.32
70	4.27

(a) Graph these values, plotting the increase in length horizontally and the added load vertically. (b) Calculate the value of Young's modulus. (c) The proportional limit occurred at a scale reading of 3.34 cm. What was the stress at this point?

11.91 ••• A 1.05-m-long rod of negligible weight is supported at its ends by wires A and B of equal length (Fig. P11.91). The cross-sectional area of A is

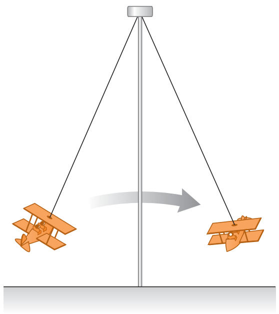
Figure **P11.91**



2.00 mm^2 and that of B is 4.00 mm^2. Young's modulus for wire A is 1.80×10^{11} Pa; that for B is 1.20×10^{11} Pa. At what point along the rod should a weight w be suspended to produce (a) equal stresses in A and B and (b) equal strains in A and B?

11.92 ••• CP An amusement park ride consists of airplane-shaped cars attached to steel rods (Fig. P11.92). Each rod has a length of 15.0 m and a cross-sectional area of 8.00 cm^2. (a) How much is the rod stretched when the ride is at rest? (Assume that each car plus two people seated in it has a total weight of 1900 N.) (b) When operating, the ride has a maximum angular speed of 8.0 rev/min. How much is the rod stretched then?

Figure **P11.92**



11.93 • A brass rod with a length of 1.40 m and a cross-sectional area of 2.00 cm^2 is fastened end to end to a nickel rod with length L and cross-sectional area 1.00 cm^2. The compound rod is subjected to equal and opposite pulls of magnitude 4.00×10^4 N at its ends. (a) Find the length L of the nickel rod if the elongations of the two rods are equal. (b) What is the stress in each rod? (c) What is the strain in each rod?

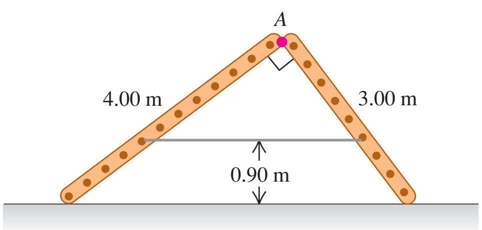
11.94 ••• CP BIO Stress on the Shin Bone. The compressive strength of our bones is important in everyday life. Young's modulus for bone is about 1.4×10^{10} Pa. Bone can take only about a 1.0% change in its length before fracturing. (a) What is the maximum force that can be applied to a bone whose minimum cross-sectional area is 3.0 cm^2? (This is approximately the cross-sectional area of a tibia, or shin bone, at its narrowest point.) (b) Estimate the maximum height from which a 70-kg man could jump and not fracture the tibia. Take the time between when he first touches the floor and when he has stopped to be 0.030 s, and assume that the stress is distributed equally between his legs.

11.95 ••• A moonshiner produces pure ethanol (ethyl alcohol) late at night and stores it in a stainless steel tank in the form of a cylinder 0.300 m in diameter with a tight-fitting piston at the top. The total volume of the tank is 250 L (0.250 m^3). In an attempt to squeeze a little more into the tank, the moonshiner piles 1420 kg of lead bricks on top of the piston. What additional volume of ethanol can the moonshiner squeeze into the tank? (Assume that the wall of the tank is perfectly rigid.)

CHALLENGE PROBLEMS

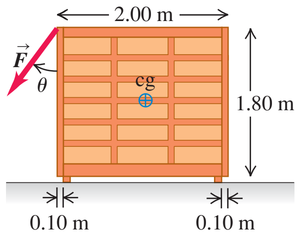
11.96 ••• Two ladders, 4.00 m and 3.00 m long, are hinged at point A and tied together by a horizontal rope 0.90 m above the floor (Fig. P11.96). The ladders weigh 480 N and 360 N, respectively, and the center of gravity of each is at its center. Assume that the floor is freshly waxed and frictionless. (a) Find the upward force at the bottom of each ladder. (b) Find the tension in the rope. (c) Find the magnitude of the force one ladder exerts on the other at point A. (d) If an 800-N painter stands at point A, find the tension in the horizontal rope.

Figure **P11.96**



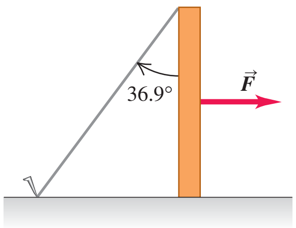
11.97 ••• A bookcase weighing 1500 N rests on a horizontal surface for which the coefficient of static friction is $\mu_s = 0.40$. The bookcase is 1.80 m tall and 2.00 m wide; its center of gravity is at its geometrical center. The bookcase rests on four short legs that are each 0.10 m from the edge of the bookcase. A person pulls on a rope attached to an upper corner of the bookcase with a force $\vec{F}$ that makes an angle θ with the bookcase (Fig. P11.97). (a) If $\theta = 90°$, so $\vec{F}$ is horizontal, show that as F is increased from zero, the bookcase will start to slide before it tips, and calculate the magnitude of $\vec{F}$ that will start the bookcase sliding. (b) If $\theta = 0°$, so $\vec{F}$ is vertical, show that the bookcase will tip over rather than slide, and calculate the magnitude of $\vec{F}$ that will cause the bookcase to start to tip. (c) Calculate as a function of θ the magnitude of $\vec{F}$ that will cause the bookcase to start to slide and the magnitude that will cause it to start to tip. What is the smallest value that θ can have so that the bookcase will still start to slide before it starts to tip?

Figure **P11.97**



11.98 ••• Knocking Over a Post. One end of a post weighing 400 N and with height h rests on a rough horizontal surface with $\mu_s = 0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (Fig. P11.98). A horizontal force $\vec{F}$ is exerted on the post as shown. (a) If the force $\vec{F}$ is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? (b) How large can the force be without causing the post to slip if its point of application is $\frac{6}{10}$ of the way from the ground to the top of the post? (c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

Figure **P11.98**



11.99 ••• CALC Minimizing the Tension. A heavy horizontal girder of length L has several objects suspended from it. It is supported by a frictionless pivot at its left end and a cable of negligible weight that is attached to an I-beam at a point a distance h directly above the girder's center. Where should the other end of the cable be attached to the girder so that the cable's tension is a minimum? (*Hint:* In evaluating and presenting your answer, don't forget that the maximum distance of the point of attachment from the pivot is the length L of the beam.)

11.100 ••• Bulk Modulus of an Ideal Gas. The equation of state (the equation relating pressure, volume, and temperature) for an ideal gas is $pV = nRT$, where n and R are constants. (a) Show that if the gas is compressed while the temperature T is held constant, the bulk modulus is equal to the pressure. (b) When an ideal gas is compressed without the transfer of any heat into or out of it, the pressure and volume are related by pV^{γ} = constant, where γ is a constant having different values for different gases. Show that, in this case, the bulk modulus is given by $B = \gamma p$.

11.101 ••• **CP** An angler hangs a 4.50-kg fish from a vertical steel wire 1.50 m long and 5.00×10^{-3} cm^2 in cross-sectional area. The upper end of the wire is securely fastened to a support. (a) Calculate the amount the wire is stretched by the hanging fish. The angler now applies a force $\vec{F}$ to the fish, pulling it very slowly downward by 0.500 mm from its equilibrium position. For this downward motion, calculate (b) the work done by gravity; (c) the work done by the force $\vec{F}$; (d) the work done by the force the wire exerts on the fish; and (e) the change in the elastic potential energy (the potential energy associated with the tensile stress in the wire). Compare the answers in parts (d) and (e).

Answers

Chapter Opening Question ?

Each stone in the arch is under compression, not tension. This is because the forces on the stones tend to push them inward toward the center of the arch and thus squeeze them together. Compared to a solid supporting wall, a wall with arches is just as strong yet much more economical to build.

Test Your Understanding Questions

11.1 Answer: (i) Situation (i) satisfies both equilibrium conditions because the seagull has zero acceleration (so $\sum \vec{F} = \mathbf{0}$) and no tendency to start rotating (so $\sum \vec{\tau} = \mathbf{0}$). Situation (ii) satisfies the first condition because the crankshaft as a whole does not accelerate through space, but it does not satisfy the second condition; the crankshaft has an angular acceleration, so $\sum \vec{\tau}$ is not zero. Situation (iii) satisfies the second condition (there is no tendency to rotate) but not the first one; the baseball accelerates in its flight (due to gravity), so $\sum \vec{F}$ is not zero.

11.2 Answer: (ii) In equilibrium, the center of gravity must be at the point of support. Since the rock and meter stick have the same mass and hence the same weight, the center of gravity of the system is midway between their respective centers. The center of gravity of the meter stick alone is 0.50 m from the left end (that is, at the middle of the meter stick), so the center of gravity of the combination of rock and meter stick is 0.25 m from the left end.

11.3 Answer: (ii), (i), (iii) This is the same situation described in Example 11.4, with the rod replacing the forearm, the hinge replacing the elbow, and the cable replacing the tendon. The only difference is that the cable attachment point is at the end of the rod, so the distances D and L are identical. From Example 11.4, the tension is

$$T = \frac{Lw}{L\sin\theta} = \frac{w}{\sin\theta}$$

Since $\sin\theta$ is less than 1, the tension T is greater than the weight w. The vertical component of the force exerted by the hinge is

$$E_y = -\frac{(L - L)w}{L} = 0$$

In this situation, the hinge exerts *no* vertical force. You can see this easily if you calculate torques around the right end of the horizontal rod: The only force that exerts a torque around this point is the vertical component of the hinge force, so this force component must be zero.

11.4 Answers: (a) (iii), (b) (ii) In (a), the copper rod has 10 times the elongation Δl of the steel rod, but it also has 10 times the original length l_0. Hence the tensile strain $\Delta l/l_0$ is the same for both rods. In (b), the stress is equal to Young's modulus Y multiplied by the strain. From Table 11.1, steel has a larger value of Y, so a greater stress is required to produce the same strain.

11.5 In (a) and (b), the bumper will have sprung back to its original shape (although the paint may be scratched). In (c), the bumper will have a permanent dent or deformation. In (d), the bumper will be torn or broken.

Bridging Problem

Answers:

(a) $T = \dfrac{2mg}{3\sin\theta}$

(b) $F = \dfrac{2mg}{3\sin\theta}\sqrt{\cos^2\theta + \tfrac{1}{4}\sin^2\theta}$, $\phi = \arctan\left(\tfrac{1}{2}\tan\theta\right)$

(c) $\Delta l = \dfrac{2mgl_0}{3AY\tan\theta}$ **(d)** 4

12 FLUID MECHANICS

? This shark must swim constantly to keep from sinking to the bottom of the ocean, yet the orange tropical fish can remain at the same level in the water with little effort. Why is there a difference?

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of the density of a material and the average density of a body.
- What is meant by the pressure in a fluid, and how it is measured.
- How to calculate the buoyant force that a fluid exerts on a body immersed in it.
- The significance of laminar versus turbulent fluid flow, and how the speed of flow in a tube depends on the tube size.
- How to use Bernoulli's equation to relate pressure and flow speed at different points in certain types of flow.

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. Airplanes fly through them; ships float in them. A fluid is any substance that can flow; we use the term for both liquids and gases. We usually think of a gas as easily compressed and a liquid as nearly incompressible, although there are exceptional cases.

We begin our study with **fluid statics,** the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We will explore the key concepts of density, pressure, and buoyancy. **Fluid dynamics,** the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we will barely scratch the surface of this broad and interesting topic.

12.1 Density

An important property of any material is its **density,** defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ (the Greek letter rho) for density. If a mass m of homogeneous material has volume V, the density ρ is

$$\rho = \frac{m}{V} \qquad \text{(definition of density)} \tag{12.1}$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. That's because the *ratio* of mass to volume is the same for both objects (Fig. 12.1).

12.1 Two objects with different masses and different volumes but the same density.

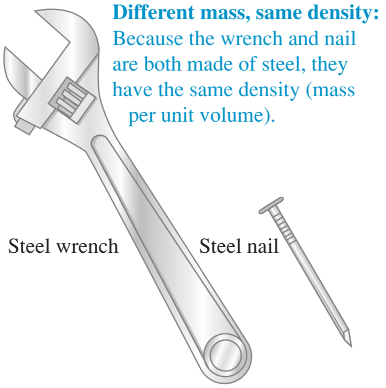


Table 12.1 Densities of Some Common Substances

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerine	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

The SI unit of density is the kilogram per cubic meter $(1\ kg/m^3)$. The cgs unit, the gram per cubic centimeter $(1\ g/cm^3)$, is also widely used:

$$1\ g/cm^3 = 1000\ kg/m^3$$

The densities of some common substances at ordinary temperatures are given in Table 12.1. Note the wide range of magnitudes. The densest material found on earth is the metal osmium ($\rho = 22{,}500\ kg/m^3$), but its density pales by comparison to the densities of exotic astronomical objects such as white dwarf stars and neutron stars.

The **specific gravity** of a material is the ratio of its density to the density of water at 4.0°C, $1000\ kg/m^3$; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. "Specific gravity" is a poor term, since it has nothing to do with gravity; "relative density" would have been better.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about $940\ kg/m^3$) and high-density bone (from 1700 to $2500\ kg/m^3$). Two others are the earth's atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (12.1) describes the **average density.** In general, the density of a material depends on environmental factors such as temperature and pressure.

Measuring density is an important analytical technique. For example, we can determine the charge condition of a storage battery by measuring the density of its electrolyte, a sulfuric acid solution. As the battery discharges, the sulfuric acid (H_2SO_4) combines with lead in the battery plates to form insoluble lead sulfate ($PbSO_4$), decreasing the concentration of the solution. The density decreases from about $1.30 \times 10^3\ kg/m^3$ for a fully charged battery to $1.15 \times 10^3\ kg/m^3$ for a discharged battery.

Another automotive example is permanent-type antifreeze, which is usually a solution of ethylene glycol ($\rho = 1.12 \times 10^3\ kg/m^3$) and water. The freezing point of the solution depends on the glycol concentration, which can be determined by measuring the specific gravity. Such measurements can be performed by using a device called a hydrometer, which we'll discuss in Section 12.3.

Example 12.1 The weight of a roomful of air

Find the mass and weight of the air at 20°C in a living room with a 4.0 m × 5.0 m floor and a ceiling 3.0 m high, and the mass and weight of an equal volume of water.

SOLUTION

IDENTIFY and SET UP: We assume that the air density is the same throughout the room. (Air is less dense at high elevations than near sea level, but the density varies negligibly over the room's 3.0-m height; see Section 12.2.) We use Eq. (12.1) to relate the mass m_{air} to the room's volume V (which we'll calculate) and the air density ρ_{air} (given in Table 12.1).

EXECUTE: We have $V = (4.0\text{ m})(5.0\text{ m})(3.0\text{ m}) = 60\text{ m}^3$, so from Eq. (12.1),

$$m_{\text{air}} = \rho_{\text{air}}V = (1.20\text{ kg/m}^3)(60\text{ m}^3) = 72\text{ kg}$$
$$w_{\text{air}} = m_{\text{air}}g = (72\text{ kg})(9.8\text{ m/s}^2) = 700\text{ N} = 160\text{ lb}$$

The mass and weight of an equal volume of water are

$$m_{\text{water}} = \rho_{\text{water}}V = (1000\text{ kg/m}^3)(60\text{ m}^3) = 6.0 \times 10^4\text{ kg}$$
$$w_{\text{water}} = m_{\text{water}}g = (6.0 \times 10^4\text{ kg})(9.8\text{ m/s}^2)$$
$$= 5.9 \times 10^5\text{ N} = 1.3 \times 10^5\text{ lb} = 66\text{ tons}$$

EVALUATE: A roomful of air weighs about the same as an average adult. Water is nearly a thousand times denser than air, so its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

Test Your Understanding of Section 12.1 Rank the following objects in order from highest to lowest average density: (i) mass 4.00 kg, volume $1.60 \times 10^{-3}\text{ m}^3$; (ii) mass 8.00 kg, volume $1.60 \times 10^{-3}\text{ m}^3$; (iii) mass 8.00 kg, volume $3.20 \times 10^{-3}\text{ m}^3$; (iv) mass 2560 kg, volume 0.640 m^3; (v) mass 2560 kg, volume 1.28 m^3.

12.2 Pressure in a Fluid

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. While the fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

If we think of an imaginary surface *within* the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface. (Otherwise, the surface would accelerate and the fluid would not remain at rest.) Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_\perp$ (Fig. 12.2). We define the **pressure** p at that point as the normal force per unit area—that is, the ratio of $dF_\perp$ to dA (Fig. 12.3):

$$p = \frac{dF_\perp}{dA} \qquad \text{(definition of pressure)} \tag{12.2}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_\perp}{A} \tag{12.3}$$

where $F_\perp$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal,** where

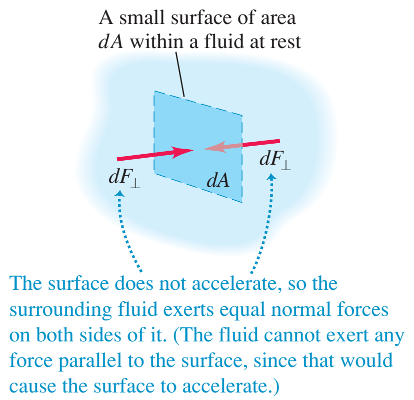
$$1\text{ pascal} = 1\text{ Pa} = 1\text{ N/m}^2$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar,* equal to 10^5 Pa, and the *millibar,* equal to 100 Pa.

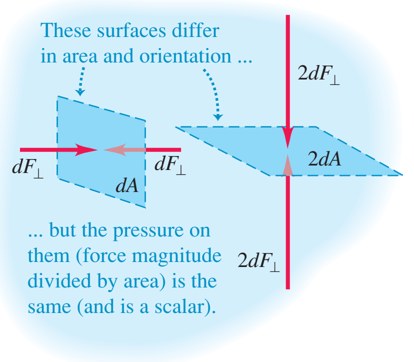
Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$(p_a)_{\text{av}} = 1\text{ atm} = 1.013 \times 10^5\text{ Pa}$$
$$= 1.013\text{ bar} = 1013\text{ millibar} = 14.70\text{ lb/in.}^2$$

12.2 Forces acting on a small surface within a fluid at rest.



12.3 The pressure on either side of a surface is force divided by area. Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons.



CAUTION **Don't confuse pressure and force** In everyday language the words "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe distinct quantities with different characteristics. Fluid pressure acts perpendicular to any surface in the fluid, no matter how that surface is oriented (Fig. 12.3). Hence pressure has no intrinsic direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 12.3 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the pressure is the same.

Example 12.2 The force of air

In the room described in Example 12.1, what is the total downward force on the floor due to an air pressure of 1.00 atm?

SOLUTION

IDENTIFY and SET UP: This example uses the relationship among the pressure p of a fluid (air), the area A subjected to that pressure, and the resulting normal force $F_\perp$ the fluid exerts. The pressure is uniform, so we use Eq. (12.3), $F_\perp = pA$, to determine $F_\perp$. The floor is horizontal, so $F_\perp$ is vertical (downward).

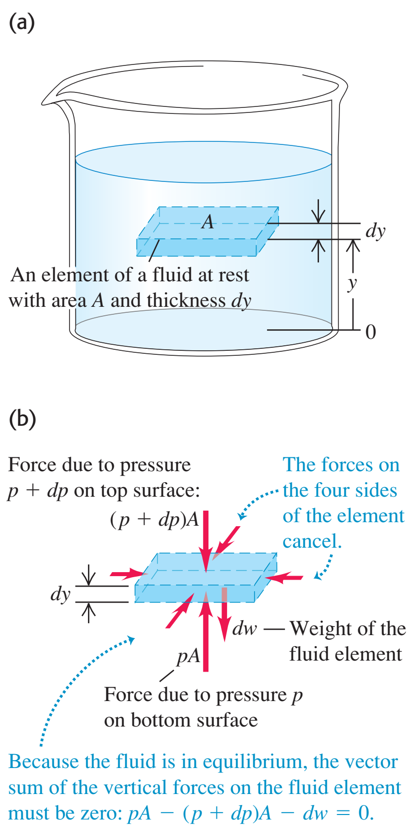
EXECUTE: We have $A = (4.0\text{ m})(5.0\text{ m}) = 20\text{ m}^2$, so from Eq. (12.3),

$$\begin{aligned} F_\perp &= pA = (1.013 \times 10^5\text{ N/m}^2)(20\text{ m}^2) \\ &= 2.0 \times 10^6\text{ N} = 4.6 \times 10^5\text{ lb} = 230\text{ tons} \end{aligned}$$

EVALUATE: Unlike the water in Example 12.1, $F_\perp$ will not collapse the floor here, because there is an *upward* force of equal magnitude on the floor's underside. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we neglect the thickness of the floor, the *net* force due to air pressure is zero.

Pressure, Depth, and Pascal's Law

12.4 The forces on an element of fluid in equilibrium.



If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible. Atmospheric pressure is less at high altitude than at sea level, which is why an airplane cabin has to be pressurized when flying at 35,000 feet. When you dive into deep water, your ears tell you that the pressure increases rapidly with increasing depth below the surface.

We can derive a general relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, every volume element is in equilibrium. Consider a thin element of fluid with thickness dy (Fig. 12.4a). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The volume of the fluid element is $dV = A\,dy$, its mass is $dm = \rho\,dV = \rho A\,dy$, and its weight is $dw = dm\,g = \rho g A\,dy$.

What are the other forces on this fluid element (Fig 12.4b)? Let's call the pressure at the bottom surface p; then the total y-component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total y-component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\sum F_y = 0 \qquad \text{so} \qquad pA - (p + dp)A - \rho g A\,dy = 0$$

When we divide out the area A and rearrange, we get

$$\frac{dp}{dy} = -\rho g \tag{12.4}$$

This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p_1 and p_2 are the pressures at elevations y_1 and y_2, respectively, and if ρ and g are constant, then

$$p_2 - p_1 = -\rho g(y_2 - y_1) \quad \text{(pressure in a fluid of uniform density)} \quad (12.5)$$

It's often convenient to express Eq. (12.5) in terms of the *depth* below the surface of a fluid (Fig. 12.5). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (12.5) becomes

$$p_0 - p = -\rho g(y_2 - y_1) = -\rho gh \qquad \text{or}$$

$$p = p_0 + \rho gh \quad \text{(pressure in a fluid of uniform density)} \quad (12.6)$$

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount ρgh. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (Fig. 12.6).

Equation (12.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This fact was recognized in 1653 by the French scientist Blaise Pascal (1623–1662) and is called *Pascal's law.*

Pascal's law: Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

The hydraulic lift shown schematically in Fig. 12.7 illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad \text{and} \qquad F_2 = \frac{A_2}{A_1}F_1 \quad (12.7)$$

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic only over short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density 1.2 kg/m^3, the difference in pressure between floor and ceiling, given by Eq. (12.6), is

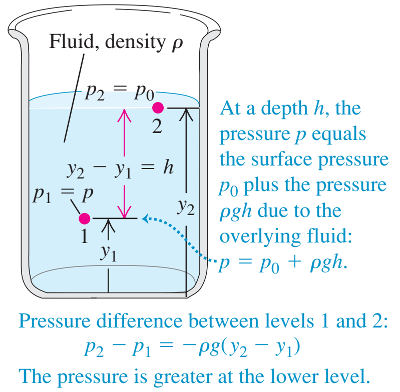
$$\rho gh = (1.2 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(3.0 \text{ m}) = 35 \text{ Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (12.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure. A pressure of several hundred atmospheres will cause only a few percent increase in the density of most liquids.

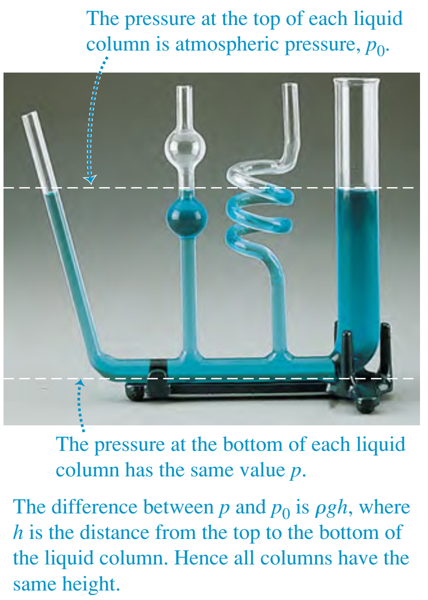
Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "32 pounds" (actually 32 lb/in.^{2}, equal to 220 kPa or 2.2 × 10^5 Pa), we mean that it is *greater* than atmospheric pressure

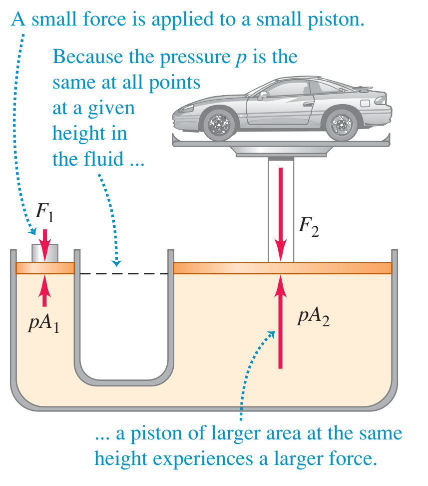
12.5 How pressure varies with depth in a fluid with uniform density.



12.6 Each fluid column has the same height, no matter what its shape.



12.7 The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity.



($14.7\ \text{lb/in.}^2$ or $1.01 \times 10^5\ \text{Pa}$) by this amount. The *total* pressure in the tire is then $47\ \text{lb/in.}^2$ or 320 kPa. The excess pressure above atmospheric pressure is usually called **gauge pressure,** and the total pressure is called **absolute pressure.** Engineers use the abbreviations psig and psia for "pounds per square inch gauge" and "pounds per square inch absolute," respectively. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

Example 12.3 Finding absolute and gauge pressures

Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressures at the bottom of the tank?

SOLUTION

IDENTIFY and SET UP: Table 11.2 indicates that water is nearly incompressible, so we can treat it as having uniform density. The level of the top of the tank corresponds to point 2 in Fig. 12.5, and the level of the bottom of the tank corresponds to point 1. Our target variable is p in Eq. (12.6). We have $h = 12.0\ \text{m}$ and $p_0 = 1\ \text{atm} = 1.01 \times 10^5\ \text{Pa}$.

EXECUTE: From Eq. (12.6), the pressures are

absolute:

$$\begin{aligned} p &= p_0 + \rho g h \\ &= (1.01 \times 10^5\ \text{Pa}) + (1000\ \text{kg/m}^3)(9.80\ \text{m/s}^2)(12.0\ \text{m}) \\ &= 2.19 \times 10^5\ \text{Pa} = 2.16\ \text{atm} = 31.8\ \text{lb/in.}^2 \end{aligned}$$

gauge:

$$\begin{aligned} p - p_0 &= (2.19 - 1.01) \times 10^5\ \text{Pa} \\ &= 1.18 \times 10^5\ \text{Pa} = 1.16\ \text{atm} = 17.1\ \text{lb/in.}^2 \end{aligned}$$

EVALUATE: A pressure gauge at the bottom of such a tank would probably be calibrated to read gauge pressure rather than absolute pressure.

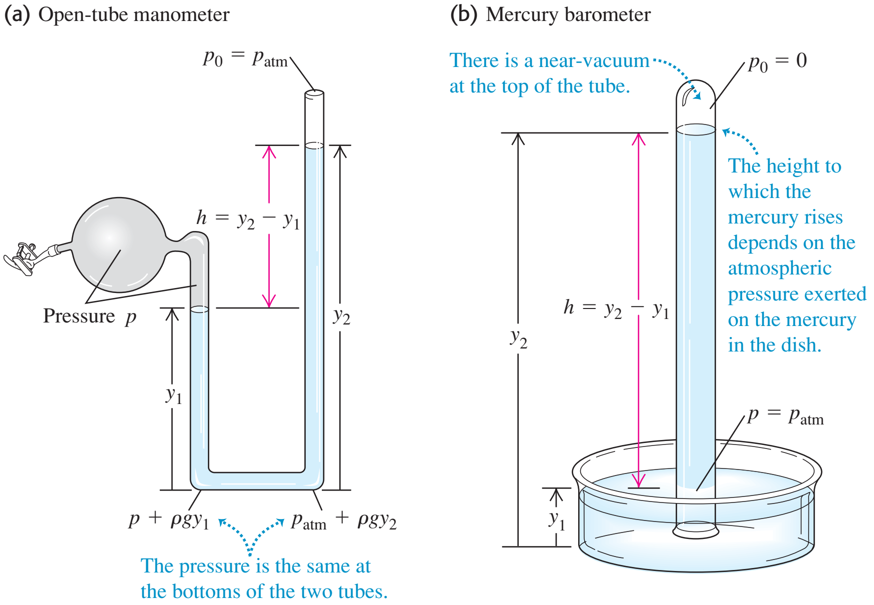
Pressure Gauges

The simplest pressure gauge is the open-tube *manometer* (Fig. 12.8a). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - p_{\text{atm}} &= \rho g (y_2 - y_1) = \rho g h \end{aligned} \tag{12.8}$$

In Eq. (12.8), p is the *absolute pressure,* and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

12.8 Two types of pressure gauge.



Another common pressure gauge is the **mercury barometer.** It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 12.8b). The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (12.6),

$$p_{atm} = p = 0 + \rho g(y_2 - y_1) = \rho g h \qquad (12.9)$$

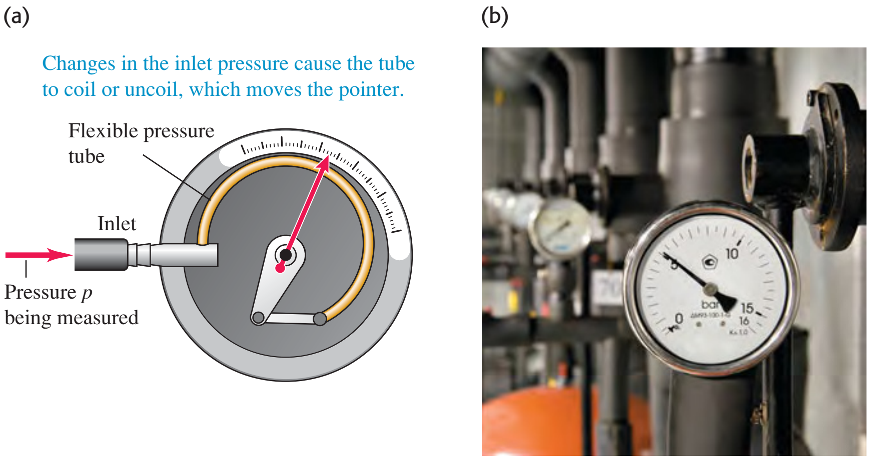
Thus the mercury barometer reads the atmospheric pressure p_{atm} directly from the height of the mercury column.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr,* after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Many types of pressure gauges use a flexible sealed tube (Fig. 12.9). A change in the pressure either inside or outside the tube causes a change in its dimensions. This change is detected optically, electrically, or mechanically.

Application Gauge Pressure of Blood

Blood-pressure readings, such as 130/80, give the maximum and minimum gauge pressures in the arteries, measured in mm Hg or torr. Blood pressure varies with vertical position within the body; the standard reference point is the upper arm, level with the heart.



12.9 (a) A Bourdon pressure gauge. When the pressure inside the flexible tube increases, the tube straightens out a little, deflecting the attached pointer. (b) This Bourdon-type pressure gauge is connected to a high-pressure gas line. The gauge pressure shown is just over 5 bars (1 bar = 10^5 Pa).

Example 12.4 A tale of two fluids

A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil–water interface is at the midpoint of the tube as shown. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

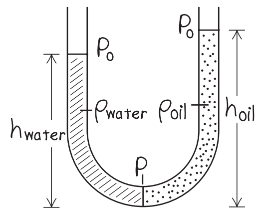
SOLUTION

IDENTIFY and SET UP: Figure 12.10 shows our sketch. The relationship between pressure and depth given by Eq. (12.6) applies only to fluids of uniform density; we have two fluids of different densities, so we must write a separate pressure–depth relationship for each. Both fluid columns have pressure p at the bottom (where they are in contact and in equilibrium) and are both at atmospheric pressure p_0 at the top (where both are in contact with and in equilibrium with the air).

EXECUTE: Writing Eq. (12.6) for each fluid gives

$$p = p_0 + \rho_{water} g h_{water}$$
$$p = p_0 + \rho_{oil} g h_{oil}$$

12.10 Our sketch for this problem.



Since the pressure p at the bottom of the tube is the same for both fluids, we set these two expressions equal to each other and solve for h_{oil} in terms of h_{water}. You can show that the result is

$$h_{oil} = \frac{\rho_{water}}{\rho_{oil}} h_{water}$$

EVALUATE: Water ($\rho_{water} = 1000\ \mathrm{kg/m^3}$) is denser than oil ($\rho_{oil} \approx 850\ \mathrm{kg/m^3}$), so h_{oil} is greater than h_{water} as Fig. 12.10 shows. It takes a greater height of low-density oil to produce the same pressure p at the bottom of the tube.

Test Your Understanding of Section 12.2 Mercury is less dense at high temperatures than at low temperatures. Suppose you move a mercury barometer from the cold interior of a tightly sealed refrigerator to outdoors on a hot summer day. You find that the column of mercury remains at the same height in the tube. Compared to the air pressure inside the refrigerator, is the air pressure outdoors (i) higher, (ii) lower, or (iii) the same? (Ignore the very small change in the dimensions of the glass tube due to the temperature change.)

MasteringPHYSICS
PhET: Balloons & Buoyancy

12.3 Buoyancy

Buoyancy is a familiar phenomenon: A body immersed in water seems to weigh less than when it is in air. When the body is less dense than the fluid, it floats. The human body usually floats in water, and a helium-filled balloon floats in air.

Archimedes's principle: When a body is completely or partially immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.

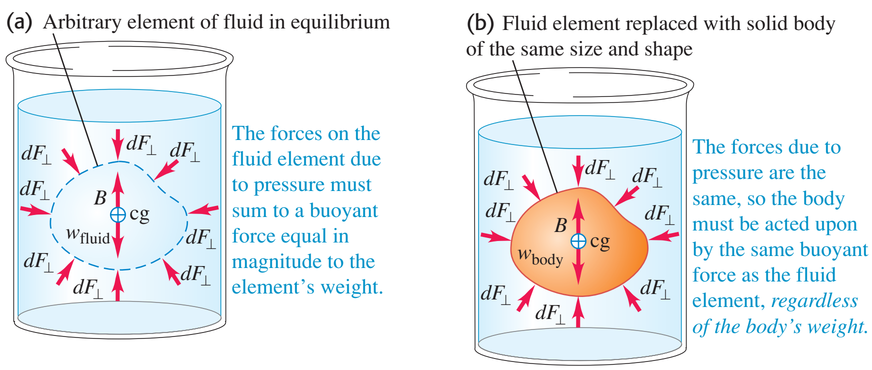
To prove this principle, we consider an arbitrary element of fluid at rest. In Fig. 12.11a the irregular outline is the surface boundary of this element of fluid. The arrows represent the forces exerted on the boundary surface by the surrounding fluid.

The entire fluid is in equilibrium, so the sum of all the y-components of force on this element of fluid is zero. Hence the sum of the y-components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant y-component of surface force must pass through the center of gravity of this element of fluid.

Now we remove the fluid inside the surface and replace it with a solid body having exactly the same shape (Fig. 12.11b). The pressure at every point is exactly the same as before. So the total upward force exerted on the body by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the body. We call this upward force the **buoyant force** on the solid body. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the body).

When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. A fish's flesh is denser than water, yet a fish can float while

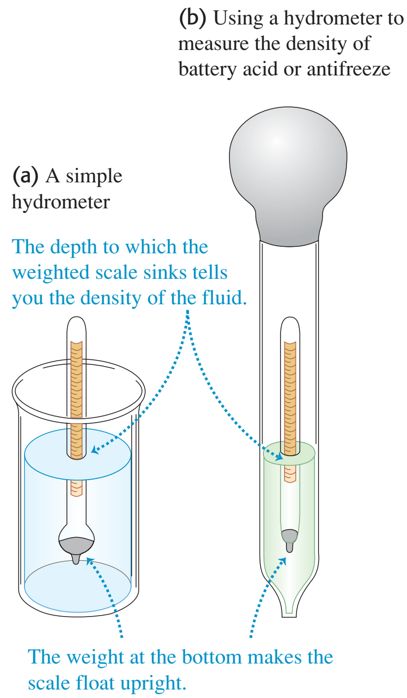
12.11 Archimedes's principle.



submerged because it has a gas-filled cavity within its body. This makes the fish's *average* density the same as water's, so its net weight is the same as the weight of the water it displaces. A body whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. The greater the density of the liquid, the less of the body is submerged. When you swim in seawater (density $1030\ \mathrm{kg/m^3}$), your body floats higher than in fresh water ($1000\ \mathrm{kg/m^3}$).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (Fig. 12.12a). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Figure 12.12b shows a type of hydrometer that is commonly used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this sample of the liquid.

12.12 Measuring the density of a fluid.



Example 12.5 Buoyancy

A 15.0-kg solid gold statue is raised from the sea bottom (Fig. 12.13a). What is the tension in the hoisting cable (assumed massless) when the statue is (a) at rest and completely underwater and (b) at rest and completely out of the water?

SOLUTION

IDENTIFY and SET UP: In both cases the statue is in equilibrium and experiences three forces: its weight, the cable tension, and a buoyant force equal in magnitude to the weight of the fluid displaced by the statue (seawater in part (a), air in part (b)). Figure 12.13b shows the free-body diagram for the statue. Our target variables are the values of the tension in seawater (T_{sw}) and in air (T_{air}). We are given the mass m_{statue}, and we can calculate the buoyant force in seawater (B_{sw}) and in air (B_{air}) using Archimedes's principle.

EXECUTE: (a) To find B_{sw}, we first find the statue's volume V using the density of gold from Table 12.1:

$$V = \frac{m_{statue}}{\rho_{gold}} = \frac{15.0\ \mathrm{kg}}{19.3 \times 10^3\ \mathrm{kg/m^3}} = 7.77 \times 10^{-4}\ \mathrm{m^3}$$

The buoyant force B_{sw} equals the weight of this same volume of seawater. Using Table 12.1 again:

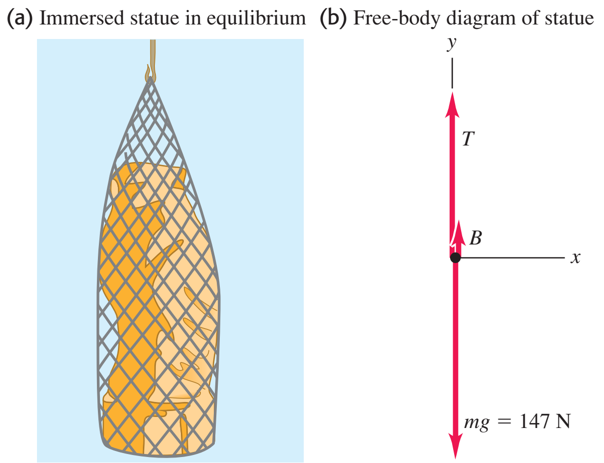
$$\begin{aligned} B_{sw} &= w_{sw} = m_{sw}g = \rho_{sw}Vg \\ &= (1.03 \times 10^3\ \mathrm{kg/m^3})(7.77 \times 10^{-4}\ \mathrm{m^3})(9.80\ \mathrm{m/s^2}) \\ &= 7.84\ \mathrm{N} \end{aligned}$$

The statue is at rest, so the net external force acting on it is zero. From Fig. 12.13b,

$$\sum F_y = B_{sw} + T_{sw} + (-m_{statue}g) = 0$$

$$\begin{aligned} T_{sw} &= m_{statue}g - B_{sw} = (15.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2}) - 7.84\ \mathrm{N} \\ &= 147\ \mathrm{N} - 7.84\ \mathrm{N} = 139\ \mathrm{N} \end{aligned}$$

12.13 What is the tension in the cable hoisting the statue?



A spring scale attached to the upper end of the cable will indicate a tension 7.84 N less than the statue's actual weight $m_{statue}g = 147\ \mathrm{N}$.

(b) The density of air is about $1.2\ \mathrm{kg/m^3}$, so the buoyant force of air on the statue is

$$\begin{aligned} B_{air} &= \rho_{air}Vg = (1.2\ \mathrm{kg/m^3})(7.77 \times 10^{-4}\ \mathrm{m^3})(9.80\ \mathrm{m/s^2}) \\ &= 9.1 \times 10^{-3}\ \mathrm{N} \end{aligned}$$

This is negligible compared to the statue's actual weight $m_{statue}g = 147\ \mathrm{N}$. So within the precision of our data, the tension in the cable with the statue in air is $T_{air} = m_{statue}g = 147\ \mathrm{N}$.

EVALUATE: Note that the buoyant force is proportional to the density of the *fluid* in which the statue is immersed, *not* the density of

Continued

the statue. The denser the fluid, the greater the buoyant force and the smaller the cable tension. If the fluid had the same density as the statue, the buoyant force would be equal to the statue's weight and the tension would be zero (the cable would go slack). If the fluid were denser than the statue, the tension would be *negative:* The buoyant force would be greater than the statue's weight, and a downward force would be required to keep the statue from rising upward.

12.14 The surface of the water acts like a membrane under tension, allowing this water strider to literally "walk on water."

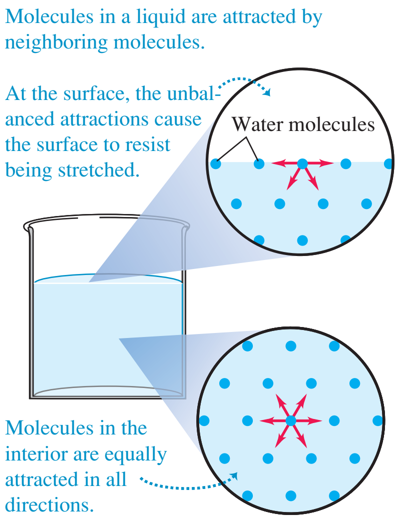
Surface Tension

An object less dense than water, such as an air-filled beach ball, floats with part of its volume below the surface. Conversely, a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension:** The surface of the liquid behaves like a membrane under tension (Fig. 12.14). Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid, but a surface molecule is drawn into the volume (Fig. 12.15). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why freely falling raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (Fig. 12.16). To do so requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius.) For large quantities of liquid, however, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we will consider only fluids in bulk and hence will ignore the effects of surface tension.

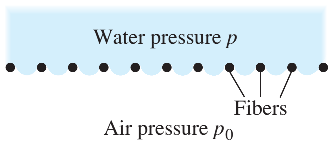
12.15 A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area.



Test Your Understanding of Section 12.3 You place a container of seawater on a scale and note the reading on the scale. You now suspend the statue of Example 12.5 in the water (Fig. 12.17). How does the scale reading change? (i) It increases by 7.84 N; (ii) it decreases by 7.84 N; (iii) it remains the same; (iv) none of these.

12.4 Fluid Flow

12.16 Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension.



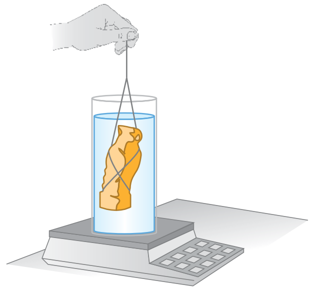
We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But some situations can be represented by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can neglect these shear forces in comparison with forces arising from gravitation and pressure differences.

The path of an individual particle in a moving fluid is called a **flow line.** If the overall flow pattern does not change with time, the flow is called **steady flow.** In steady flow, every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, the streamlines do not coincide with the flow lines. We will consider only steady-flow situations, for which flow lines and streamlines are identical.

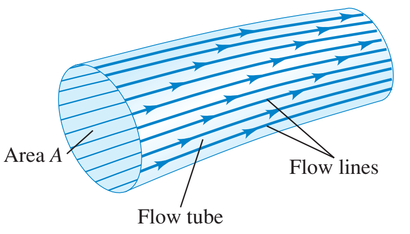
The flow lines passing through the edge of an imaginary element of area, such as the area A in Fig. 12.18, form a tube called a **flow tube.** From the definition of a flow line, in steady flow no fluid can cross the side walls of a flow tube; the fluids in different flow tubes cannot mix.

Figure 12.19 shows patterns of fluid flow from left to right around three different obstacles. The photographs were made by injecting dye into water flowing between two closely spaced glass plates. These patterns are typical of **laminar flow,** in which adjacent layers of fluid slide smoothly past each other and the flow is steady. (A *lamina* is a thin sheet.) At sufficiently high flow rates, or when boundary surfaces cause abrupt changes in velocity, the flow can become irregular and chaotic. This is called **turbulent flow** (Fig. 12.20). In turbulent flow there is no steady-state pattern; the flow pattern changes continuously.

12.17 How does the scale reading change when the statue is immersed in water?



12.18 A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.



The Continuity Equation

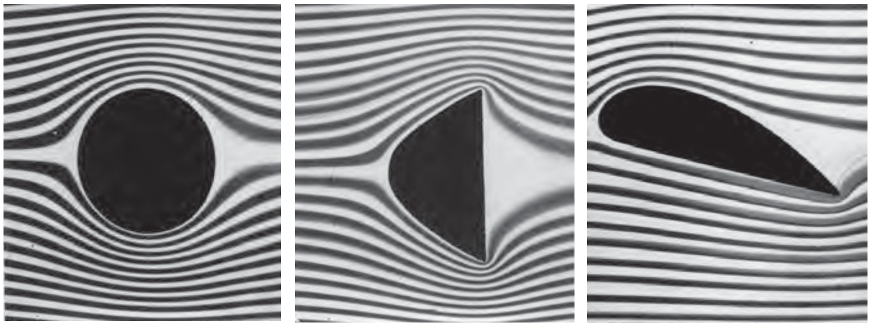
The mass of a moving fluid doesn't change as it flows. This leads to an important quantitative relationship called the **continuity equation.** Consider a portion of a flow tube between two stationary cross sections with areas A_1 and A_2 (Fig. 12.21). The fluid speeds at these sections are v_1 and v_2, respectively. No fluid flows in or out across the sides of the tube because the fluid velocity is tangent to the wall at every point on the wall. During a small time interval dt, the fluid at A_1 moves a distance $v_1\,dt$, so a cylinder of fluid with height $v_1\,dt$ and volume $dV_1 = A_1 v_1\,dt$ flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2 v_2\,dt$ flows out of the tube across A_2.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1 v_1\,dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2 v_2\,dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

$$\rho A_1 v_1\,dt = \rho A_2 v_2\,dt \qquad \text{or}$$

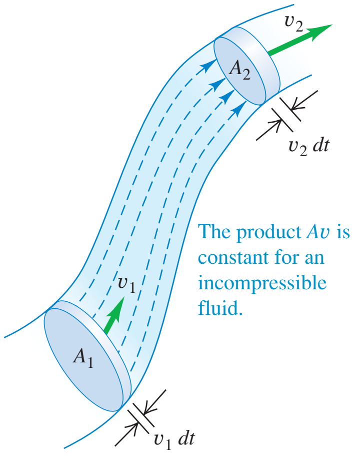
$$A_1 v_1 = A_2 v_2 \qquad \text{(continuity equation, incompressible fluid)} \qquad (12.10)$$

12.19 Laminar flow around obstacles of different shapes.



12.20 The flow of smoke rising from these incense sticks is laminar up to a certain point, and then becomes turbulent.

12.21 A flow tube with changing cross-sectional area. If the fluid is incompressible, the product Av has the same value at all points along the tube.



The product Av is the *volume flow rate* dV/dt, the rate at which volume crosses a section of the tube:

$$\frac{dV}{dt} = Av \qquad \text{(volume flow rate)} \tag{12.11}$$

The *mass* flow rate is the mass flow per unit time through a cross section. This is equal to the density ρ times the volume flow rate dV/dt.

Equation (12.10) shows that the volume flow rate has the same value at all points along any flow tube. When the cross section of a flow tube decreases, the speed increases, and vice versa. A broad, deep part of a river has larger cross section and slower current than a narrow, shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, “Still waters run deep.” The stream of water from a faucet narrows as it gains speed during its fall, but dV/dt is the same everywhere along the stream. If a water pipe with 2-cm diameter is connected to a pipe with 1-cm diameter, the flow speed is four times as great in the 1-cm part as in the 2-cm part.

We can generalize Eq. (12.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \qquad \text{(continuity equation, compressible fluid)} \tag{12.12}$$

If the fluid is denser at point 2 than at point 1 $(\rho_2 > \rho_1)$, the volume flow rate at point 2 will be less than at point 1 $(A_2 v_2 < A_1 v_1)$. We leave the details to you. If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (12.12) reduces to Eq. (12.10).

Example 12.6 Flow of an incompressible fluid

Incompressible oil of density 850 kg/m^3 is pumped through a cylindrical pipe at a rate of 9.5 liters per second. (a) The first section of the pipe has a diameter of 8.0 cm. What is the flow speed of the oil? What is the mass flow rate? (b) The second section of the pipe has a diameter of 4.0 cm. What are the flow speed and mass flow rate in that section?

SOLUTION

IDENTIFY and SET UP: Since the oil is incompressible, the volume flow rate has the *same* value (9.5 L/s) in both sections of pipe. The mass flow rate (the density times the volume flow rate) also has the same value in both sections. (This is just the statement that no fluid is lost or added anywhere along the pipe.) We use the volume flow rate equation, Eq. (12.11), to determine the speed v_1 in the 8.0-cm-diameter section and the continuity equation for incompressible flow, Eq. (12.10), to find the speed v_2 in the 4.0-cm-diameter section.

EXECUTE: (a) From Eq. (12.11) the volume flow rate in the first section is $dV/dt = A_1 v_1$, where A_1 is the cross-sectional area of the pipe of diameter 8.0 cm and radius 4.0 cm. Hence

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5\ \text{L/s})(10^{-3}\ \text{m}^3/\text{L})}{\pi(4.0 \times 10^{-2}\ \text{m})^2} = 1.9\ \text{m/s}$$

The mass flow rate is $\rho\, dV/dt = (850\ \text{kg/m}^3)(9.5 \times 10^{-3}\ \text{m}^3/\text{s}) = 8.1\ \text{kg/s}$.

(b) From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0 \times 10^{-2}\ \text{m})^2}{\pi(2.0 \times 10^{-2}\ \text{m})^2}(1.9\ \text{m/s}) = 7.6\ \text{m/s} = 4v_1$$

The volume and mass flow rates are the same as in part (a).

EVALUATE: The second section of pipe has one-half the diameter and one-fourth the cross-sectional area of the first section. Hence the speed must be four times greater in the second section, which is just what our result shows.

Test Your Understanding of Section 12.4 A maintenance crew is working on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam). Do cars on a highway behave like (i) the molecules of an incompressible fluid or (ii) the molecules of a compressible fluid?

12.5 Bernoulli's Equation

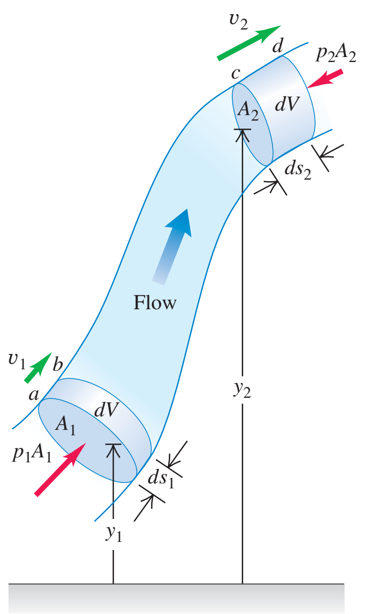
According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation (see Section 12.2), and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation* that relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is an essential tool in analyzing plumbing systems, hydroelectric generating stations, and the flight of airplanes.

The dependence of pressure on speed follows from the continuity equation, Eq. (12.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

Deriving Bernoulli's Equation

To derive Bernoulli's equation, we apply the work–energy theorem to the fluid in a section of a flow tube. In Fig. 12.22 we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval dt, the fluid that is initially at a moves to b, a distance $ds_1 = v_1\,dt$, and the fluid that is initially at c moves to d, a distance $ds_2 = v_2\,dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (12.10), the volume of fluid dV passing *any* cross section during time dt is the same. That is, $dV = A_1\,ds_1 = A_2\,ds_2$.

12.22 Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy.



Let's compute the *work* done on this fluid element during dt. We assume that there is negligible internal friction in the fluid (i.e., no viscosity), so the only nongravitational forces that do work on the fluid element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2; the force on the cross section at a is p_1A_1, and the force at c is p_2A_2. The net work dW done on the element by the surrounding fluid during this displacement is therefore

$$dW = p_1A_1\,ds_1 - p_2A_2\,ds_2 = (p_1 - p_2)dV \qquad (12.13)$$

The second term has a negative sign because the force at c opposes the displacement of the fluid.

The work dW is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections b and c does not change. At the beginning of dt the fluid between a and b has volume $A_1\,ds_1$, mass $\rho A_1\,ds_1$, and kinetic energy $\frac{1}{2}\rho(A_1\,ds_1)v_1^2$. At the end of dt the fluid between c and d has kinetic energy $\frac{1}{2}\rho(A_2\,ds_2)v_2^2$. The net change in kinetic energy dK during time dt is

$$dK = \tfrac{1}{2}\rho\,dV(v_2^2 - v_1^2) \qquad (12.14)$$

What about the change in gravitational potential energy? At the beginning of dt, the potential energy for the mass between a and b is $dm\,gy_1 = \rho\,dV\,gy_1$. At the end of dt, the potential energy for the mass between c and d is $dm\, gy_2 = \rho\, dV\, gy_2$. The net change in potential energy dU during dt is

$$dU = \rho\, dV\, g(y_2 - y_1) \tag{12.15}$$

Combining Eqs. (12.13), (12.14), and (12.15) in the energy equation $dW = dK + dU$, we obtain

$$\begin{aligned} (p_1 - p_2)dV &= \tfrac{1}{2}\rho\, dV(v_2^2 - v_1^2) + \rho\, dV\, g(y_2 - y_1) \\ p_1 - p_2 &= \tfrac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1) \end{aligned} \tag{12.16}$$

This is **Bernoulli's equation.** It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (12.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

We can also express Eq. (12.16) in a more convenient form as

$$p_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2 \qquad \text{(Bernoulli's equation)} \tag{12.17}$$

The subscripts 1 and 2 refer to *any* two points along the flow tube, so we can also write

$$p + \rho g y + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{12.18}$$

Note that when the fluid is *not* moving (so $v_1 = v_2 = 0$), Eq. (12.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (12.5).

CAUTION **Bernoulli's principle applies only in certain situations** We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation that's easy to use; don't let this tempt you to use it in situations in which it doesn't apply!

Problem-Solving Strategy 12.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so much of Problem-Solving Strategy 7.1 (Section 7.1) is applicable here.

IDENTIFY *the relevant concepts:* Bernoulli's equation is applicable to steady flow of an incompressible fluid that has no internal friction (see Section 12.6). It is generally applicable to flows through large pipes and to flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* using the following steps:

1. Identify the points 1 and 2 referred to in Bernoulli's equation, Eq. (12.17).
2. Define your coordinate system, particularly the level at which $y = 0$. Take the positive y-direction to be upward.
3. Make lists of the unknown and known quantities in Eq. (12.17). Decide which unknowns are the target variables.

EXECUTE *the solution* as follows: Write Bernoulli's equation and solve for the unknowns. You may need the continuity equation, Eq. (12.10), to get a relationship between the two speeds in terms of cross-sectional areas of pipes or containers. You may also need Eq. (12.11) to find the volume flow rate.

EVALUATE *your answer:* Verify that the results make physical sense. Check that you have used consistent units: In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. Also note that the pressures must be either *all* absolute pressures or *all* gauge pressures.

Example 12.7 Water pressure in the home

Water enters a house (Fig. 12.23) through a pipe with an inside diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0-cm-diameter pipe leads to the second-floor bathroom 5.0 m above. When the flow speed at the inlet pipe is 1.5 m/s, find the flow speed, pressure, and volume flow rate in the bathroom.

SOLUTION

IDENTIFY and SET UP: We assume that the water flows at a steady rate. Water is effectively incompressible, so we can use the continuity equation. It's reasonable to ignore internal friction because the pipe has a relatively large diameter, so we can also use Bernoulli's equation. Let points 1 and 2 be at the inlet pipe and at the bathroom, respectively. We are given the pipe diameters at points 1 and 2, from which we calculate the areas A_1 and A_2, as well as the speed $v_1 = 1.5$ m/s and pressure $p_1 = 4.0 \times 10^5$ Pa at the inlet pipe. We take $y_1 = 0$ and $y_2 = 5.0$ m. We find the speed v_2 using the continuity equation and the pressure p_2 using Bernoulli's equation. Knowing v_2, we calculate the volume flow rate $v_2 A_2$.

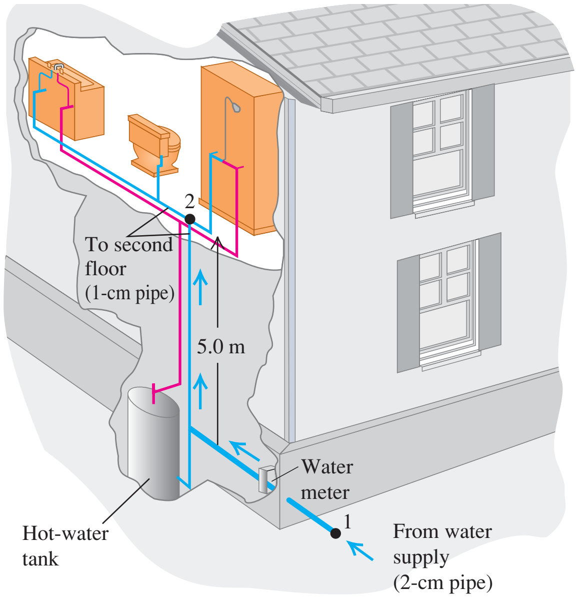
EXECUTE: From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(1.0\text{ cm})^2}{\pi(0.50\text{ cm})^2}(1.5\text{ m/s}) = 6.0\text{ m/s}$$

From Bernoulli's equation, Eq. (12.16),

$$\begin{aligned} p_2 &= p_1 - \tfrac{1}{2}\rho(v_2^2 - v_1^2) - \rho g(y_2 - y_1) \\ &= 4.0 \times 10^5\text{ Pa} \\ &\quad - \tfrac{1}{2}(1.0 \times 10^3\text{ kg/m}^3)(36\text{ m}^2/\text{s}^2 - 2.25\text{ m}^2/\text{s}^2) \\ &\quad - (1.0 \times 10^3\text{ kg/m}^3)(9.8\text{ m/s}^2)(5.0\text{ m}) \\ &= 4.0 \times 10^5\text{ Pa} - 0.17 \times 10^5\text{ Pa} - 0.49 \times 10^5\text{ Pa} \\ &= 3.3 \times 10^5\text{ Pa} = 3.3\text{ atm} = 48\text{ lb/in.}^2 \end{aligned}$$

12.23 What is the water pressure in the second-story bathroom of this house?



The volume flow rate is

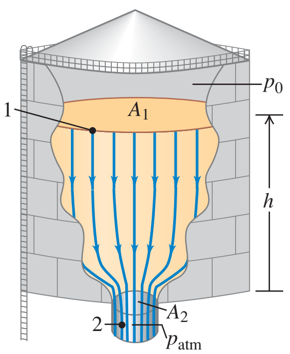
$$\begin{aligned} \frac{dV}{dt} &= A_2 v_2 = \pi(0.50 \times 10^{-2}\text{ m})^2(6.0\text{ m/s}) \\ &= 4.7 \times 10^{-4}\text{ m}^3/\text{s} = 0.47\text{ L/s} \end{aligned}$$

EVALUATE: This is a reasonable flow rate for a bathroom faucet or shower. Note that if the water is turned off, v_1 and v_2 are both zero, the term $\frac{1}{2}\rho(v_2^2 - v_1^2)$ in Bernoulli's equation vanishes, and p_2 rises from 3.3×10^5 Pa to 3.5×10^5 Pa.

Example 12.8 Speed of efflux

Figure 12.24 shows a gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0, and the gasoline flows out the bottom of the tank through a short pipe with cross-sectional area A_2. Derive expressions for the flow speed in the pipe and the volume flow rate.

12.24 Calculating the speed of efflux for gasoline flowing out the bottom of a storage tank.



SOLUTION

IDENTIFY and SET UP: We consider the entire volume of moving liquid as a single flow tube of an incompressible fluid with negligible internal friction. Hence, we can use Bernoulli's equation. Points 1 and 2 are at the surface of the gasoline and at the exit pipe, respectively. At point 1 the pressure is p_0, which we assume to be fixed; at point 2 it is atmospheric pressure p_{atm}. We take $y = 0$ at the exit pipe, so $y_1 = h$ and $y_2 = 0$. Because A_1 is very much larger than A_2, the upper surface of the gasoline will drop very slowly and we can regard v_1 as essentially equal to zero. We find v_2 from Eq. (12.17) and the volume flow rate from Eq. (12.11).

EXECUTE: We apply Bernoulli's equation to points 1 and 2:

$$p_0 + \tfrac{1}{2}\rho v_1^2 + \rho g h = p_{atm} + \tfrac{1}{2}\rho v_2^2 + \rho g(0)$$

$$v_2^2 = v_1^2 + 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

Continued

Using $v_1 = 0$, we find

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{\text{atm}}}{\rho}\right) + 2gh}$$

From Eq. (12.11), the volume flow rate is $dV/dt = v_2 A_2$.

EVALUATE: The speed v_2, sometimes called the *speed of efflux,* depends on both the pressure difference $(p_0 - p_{\text{atm}})$ and the height h of the liquid level in the tank. If the top of the tank is vented to the atmosphere, $p_0 = p_{\text{atm}}$ and $p_0 - p_{\text{atm}} = 0$. Then

$$v_2 = \sqrt{2gh}$$

That is, the speed of efflux from an opening at a distance h below the top surface of the liquid is the *same* as the speed a body would acquire in falling freely through a height h. This result is called *Torricelli's theorem.* It is valid not only for an opening in the bottom of a container, but also for a hole in a side wall at a depth h below the surface. In this case the volume flow rate is

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

Example 12.9 The Venturi meter

Figure 12.25 shows a *Venturi meter,* used to measure flow speed in a pipe. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.

SOLUTION

IDENTIFY and SET UP: The flow is steady, and we assume the fluid is incompressible and has negligible internal friction. Hence we can use Bernoulli's equation. We apply that equation to the wide part (point 1) and narrow part (point 2, the *throat*) of the pipe. Equation (12.6) relates h to the pressure difference $p_1 - p_2$.

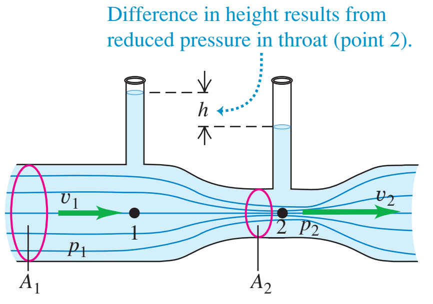
EXECUTE: Points 1 and 2 have the same vertical coordinate $y_1 = y_2$, so Eq. (12.17) says

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this and rearranging, we get

$$p_1 - p_2 = \tfrac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

12.25 The Venturi meter.



From Eq. (12.6), the pressure difference $p_1 - p_2$ is also equal to $\rho g h$. Substituting this and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

EVALUATE: Because A_1 is greater than A_2, v_2 is greater than v_1 and the pressure p_2 in the throat is *less* than p_1. Those pressure differences produce a net force to the right that makes the fluid speed up as it enters the throat, and a net force to the left that slows it as it leaves.

Conceptual Example 12.10 Lift on an airplane wing

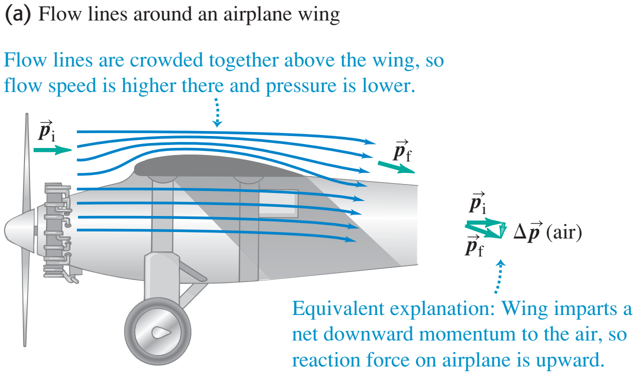
Figure 12.26a shows flow lines around a cross section of an airplane wing. The flow lines crowd together above the wing, corresponding to increased flow speed and reduced pressure, just as in the Venturi throat in Example 12.9. Hence the downward force of the air on the top side of the wing is less than the upward force of the air on the underside of the wing, and there is a net upward force or *lift.* Lift is not simply due to the impulse of air striking the underside of the wing; in fact, the reduced pressure on the upper wing surface makes the greatest contribution to the lift. (This simplified discussion ignores the formation of vortices.)

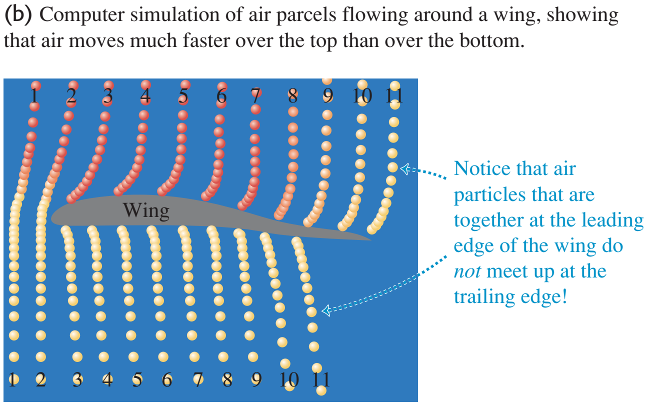
We can also understand the lift force on the basis of momentum changes. The vector diagram in Fig. 12.26a shows that there is a net *downward* change in the vertical component of momentum of the air flowing past the wing, corresponding to the downward force the wing exerts on the air. The reaction force *on* the wing is *upward,* as we concluded above.

Similar flow patterns and lift forces are found in the vicinity of any humped object in a wind. A moderate wind makes an umbrella "float"; a strong wind can turn it inside out. At high speed, lift can reduce traction on a car's tires; a "spoiler" at the car's tail, shaped like an upside-down wing, provides a compensating downward force.

CAUTION **A misconception about wings** Some discussions of lift claim that air travels faster over the top of a wing because "it has farther to travel." This claim assumes that air molecules that part company at the front of the wing, one traveling over the wing and one under it, must meet again at the wing's trailing edge. Not so! Figure 12.26b shows a computer simulation of parcels of air flowing around an airplane wing. Parcels that are adjacent at the front of the wing do *not* meet at the trailing edge; the flow over the top of the wing is much faster than if the parcels had to meet. In accordance with Bernoulli's equation, this faster speed means that there is even lower pressure above the wing (and hence greater lift) than the "farther-to-travel" claim would suggest.

12.26 Flow around an airplane wing.





Test Your Understanding of Section 12.5 Which is the most accurate statement of Bernoulli's principle? (i) Fast-moving air causes lower pressure; (ii) lower pressure causes fast-moving air; (iii) both (i) and (ii) are equally accurate.

12.6 Viscosity and Turbulence

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

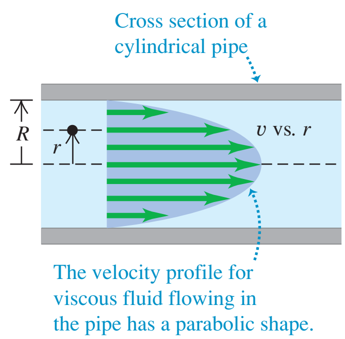
Fluids that flow readily, such as water or gasoline, have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (Fig. 12.27). Oils for engine lubrication must flow equally well in cold and warm conditions, and so are designed to have as *little* temperature variation of viscosity as possible.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (12.17). If the two ends of a long cylindrical pipe are at the same height $(y_1 = y_2)$ and the flow speed is the same at both ends (so $v_1 = v_2$), Bernoulli's equation tells us that the pressure is the same at both ends of the pipe. But this result simply isn't true if we take viscosity into account. To see why, consider Fig. 12.28, which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to

12.27 Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

12.28 Velocity profile for a viscous fluid in a cylindrical pipe.



Application Listening for Turbulent Flow
Normal blood flow in the human aorta is laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4. If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a high-cholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence.** Figure 12.20 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equation is *not* applicable to regions where there is turbulence because the flow is not steady.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets or lamina and the more likely the flow is to be laminar. (When we discussed Bernoulli's equation in Section 12.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar.)

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature (Fig. 12.29a). When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern (Fig. 12.29b).

12.29 The flow of water from a faucet is (a) laminar at low speeds but (b) turbulent at sufficiently high speeds.

Conceptual Example 12.11 The curve ball

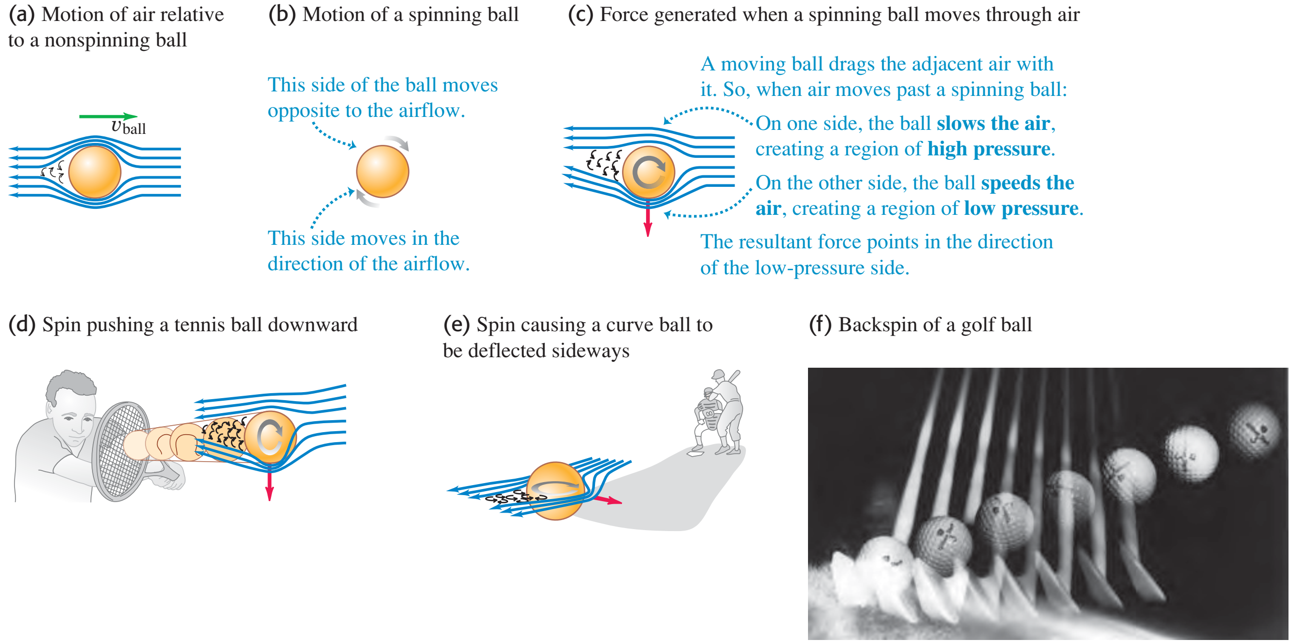
Does a curve ball *really* curve? Yes, it certainly does, and the reason is turbulence. Figure 12.30a shows a nonspinning ball moving through the air from left to right. The flow lines show that to an observer moving with the ball, the air stream appears to move from right to left. Because of the high speeds that are ordinarily involved (near 35 m/s, or 75 mi/h), there is a region of *turbulent* flow behind the ball.

Figure 12.30b shows a *spinning* ball with "top spin." Layers of air near the ball's surface are pulled around in the direction of the spin by friction between the ball and air and by the air's internal friction (viscosity). Hence air moves relative to the ball's surface more slowly at the top of the ball than at the bottom, and turbulence occurs farther forward on the top side than on the bottom. This asymmetry causes a pressure difference; the average pressure at the top of the ball is now greater than that at the bottom. As Fig. 12.30c shows, the resulting net force deflects the ball downward. "Top spin" is used in tennis to keep a fast serve in the court (Fig. 12.30d).

In baseball, a curve ball spins about a nearly *vertical* axis and the resulting deflection is sideways. In that case, Fig. 12.30c is a *top* view of the situation. A curve ball thrown by a left-handed pitcher spins as shown in Fig. 12.30e and will curve *toward* a right-handed batter, making it harder to hit.

A similar effect occurs with golf balls, which acquire "back spin" from impact with the grooved, slanted club face. Figure 12.30f shows the backspin of a golf ball just after impact. The resulting pressure difference between the top and bottom of the ball causes a *lift* force that keeps the ball in the air longer than would be possible without spin. A well-hit drive appears, from the tee, to "float" or even curve *upward* during the initial portion of its flight. This is a real effect, not an illusion. The dimples on the golf ball play an essential role; the viscosity of air gives a dimpled ball a much longer trajectory than an undimpled one with the same initial velocity and spin.

12.30 (a)–(e) Analyzing the motion of a spinning ball through the air. (f) Stroboscopic photograph of a golf ball being struck by a club. The picture was taken at 1000 flashes per second. The ball rotates about once in eight pictures, corresponding to an angular speed of 125 rev/s, or 7500 rpm.



Test Your Understanding of Section 12.6 How much more thumb pressure must a nurse use to administer an injection with a hypodermic needle of inside diameter 0.30 mm compared to one with inside diameter 0.60 mm? Assume that the two needles have the same length and that the volume flow rate is the same in both cases. (i) twice as much; (ii) 4 times as much; (iii) 8 times as much; (iv) 16 times as much; (v) 32 times as much.

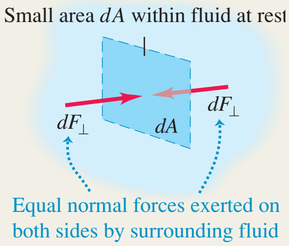
CHAPTER 12 SUMMARY

Density and pressure: Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density ρ is the ratio m/V. Specific gravity is the ratio of the density of a material to the density of water. (See Example 12.1.)

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa $=$ 1 N/m^2. (See Example 12.2.)

$$\rho = \frac{m}{V} \quad (12.1)$$

$$p = \frac{dF_\perp}{dA} \quad (12.2)$$



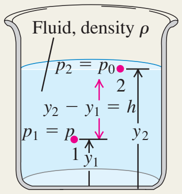
Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is p_0, then the pressure at a depth h is greater by an amount ρgh. (See Examples 12.3 and 12.4.)

$$p_2 - p_1 = -\rho g(y_2 - y_1) \quad (12.5)$$

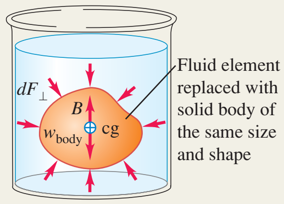
(pressure in a fluid of uniform density)

$$p = p_0 + \rho gh \quad (12.6)$$

(pressure in a fluid of uniform density)



Buoyancy: Archimedes's principle states that when a body is immersed in a fluid, the fluid exerts an upward buoyant force on the body equal to the weight of the fluid that the body displaces. (See Example 12.5.)



Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern.

Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds v_1 and v_2 for two cross sections A_1 and A_2 in a flow tube. The product Av equals the volume flow rate, dV/dt, the rate at which volume crosses a section of the tube. (See Example 12.6.)

Bernoulli's equation relates the pressure p, flow speed v, and elevation y for any two points, assuming steady flow in an ideal fluid. (See Examples 12.7–12.10.)

$$A_1v_1 = A_2v_2 \quad (12.10)$$

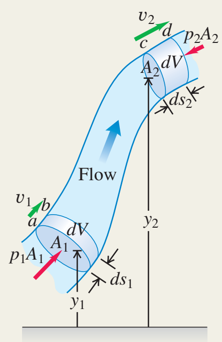
(continuity equation, incompressible fluid)

$$\frac{dV}{dt} = Av \quad (12.11)$$

(volume flow rate)

$$p_1 + \rho gy_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho gy_2 + \tfrac{1}{2}\rho v_2^2 \quad (12.17)$$

(Bernoulli's equation)



BRIDGING PROBLEM How Long to Drain?

A large cylindrical tank with diameter D is open to the air at the top. The tank contains water to a height H. A small circular hole with diameter d, where d is very much less than D, is then opened at the bottom of the tank. Ignore any effects of viscosity. (a) Find y, the height of water in the tank a time t after the hole is opened, as a function of t. (b) How long does it take to drain the tank completely? (c) If you double the initial height of water in the tank, by what factor does the time to drain the tank increase?

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Draw a sketch of the situation that shows all of the relevant dimensions.
2. Make a list of the unknown quantities, and decide which of these are the target variables.
3. What is the speed at which water flows out of the bottom of the tank? How is this related to the volume flow rate of water out of the tank? How is the volume flow rate related to the rate of change of y?

EXECUTE

4. Use your results from step 3 to write an equation for dy/dt.
5. Your result from step 4 is a relatively simple differential equation. With your knowledge of calculus, you can integrate it to find y as a function of t. (*Hint:* Once you've done the integration, you'll still have to do a little algebra.)
6. Use your result from step 5 to find the time when the tank is empty. How does your result depend on the initial height H?

EVALUATE

7. Check whether your answers are reasonable. A good check is to draw a graph of y versus t. According to your graph, what is the algebraic sign of dy/dt at different times? Does this make sense?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q12.1 A cube of oak wood with very smooth faces normally floats in water. Suppose you submerge it completely and press one face flat against the bottom of a tank so that no water is under that face. Will the block float to the surface? Is there a buoyant force on it? Explain.

Q12.2 A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, even though the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

Q12.3 Comparing Example 12.1 (Section 12.1) and Example 12.2 (Section 12.2), it seems that 700 N of air is exerting a downward force of 2.0×10^6 N on the floor. How is this possible?

Q12.4 Equation (12.7) shows that an area ratio of 100 to 1 can give 100 times more output force than input force. Doesn't this violate conservation of energy? Explain.

Q12.5 You have probably noticed that the lower the tire pressure, the larger the contact area between the tire and the road. Why?

Q12.6 In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

Q12.7 In describing the size of a large ship, one uses such expressions as "it displaces 20,000 tons." What does this mean? Can the weight of the ship be obtained from this information?

Q12.8 You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

Q12.9 A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

Q12.10 Air pressure decreases with increasing altitude. So why is air near the surface not continuously drawn upward toward the lower-pressure regions above?

Q12.11 The purity of gold can be tested by weighing it in air and in water. How? Do you think you could get away with making a fake gold brick by gold-plating some cheaper material?

Q12.12 During the Great Mississippi Flood of 1993, the levees in St. Louis tended to rupture first at the bottom. Why?

Q12.13 A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain why.

Q12.14 You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

Q12.15 An old question is "Which weighs more, a pound of feathers or a pound of lead?" If the weight in pounds is the gravitational force, will a pound of feathers balance a pound of lead on opposite pans of an equal-arm balance? Explain, taking into account buoyant forces.

Q12.16 Suppose the door of a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure on one side were standard atmospheric pressure and the air pressure on the other side differed from standard by 1%? Explain.

Q12.17 At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

Q12.18 A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain your answers.

Q12.19 You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain your answer.

Q12.20 You are floating in a canoe in the middle of a swimming pool. Your friend is at the edge of the pool, carefully noting the level of the water on the side of the pool. You have a bowling ball with you in the canoe. If you carefully drop the bowling ball over the side of the canoe and it sinks to the bottom of the pool, does the water level in the pool rise or fall?

Q12.21 You are floating in a canoe in the middle of a swimming pool. A large bird flies up and lights on your shoulder. Does the water level in the pool rise or fall?

Q12.22 At a certain depth in the incompressible ocean the gauge pressure is p_g. At three times this depth, will the gauge pressure be greater than $3p_g$, equal to $3p_g$, or less than $3p_g$? Justify your answer.

Q12.23 An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

Q12.24 You are told, "Bernoulli's equation tells us that where there is higher fluid speed, there is lower fluid pressure, and vice versa." Is this statement always true, even for an idealized fluid? Explain.

Q12.25 If the velocity at each point in space in steady-state fluid flow is constant, how can a fluid particle accelerate?

Q12.26 In a store-window vacuum cleaner display, a table-tennis ball is suspended in midair in a jet of air blown from the outlet hose of a tank-type vacuum cleaner. The ball bounces around a little but always moves back toward the center of the jet, even if the jet is tilted from the vertical. How does this behavior illustrate Bernoulli's equation?

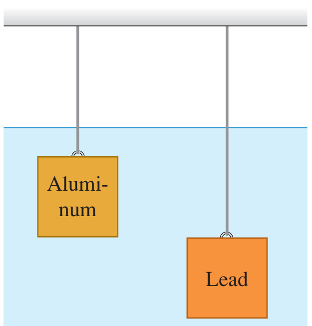
Q12.27 A tornado consists of a rapidly whirling air vortex. Why is the pressure always much lower in the center than at the outside? How does this condition account for the destructive power of a tornado?

Q12.28 Airports at high elevations have longer runways for takeoffs and landings than do airports at sea level. One reason is that aircraft engines develop less power in the thin air well above sea level. What is another reason?

Q12.29 When a smooth-flowing stream of water comes out of a faucet, it narrows as it falls. Explain why this happens.

Q12.30 Identical-size lead and aluminum cubes are suspended at different depths by two wires in a large vat of water (Fig. Q12.30). (a) Which cube experiences a greater buoyant force? (b) For which cube is the tension in the wire greater? (c) Which cube experiences a greater force on its lower face? (d) For which cube is the difference in pressure between the upper and lower faces greater?

Figure **Q12.30**



EXERCISES

Section 12.1 Density

12.1 •• On a part-time job, you are asked to bring a cylindrical iron rod of length 85.8 cm and diameter 2.85 cm from a storage room to a machinist. Will you need a cart? (To answer, calculate the weight of the rod.)

12.2 •• A cube 5.0 cm on each side is made of a metal alloy. After you drill a cylindrical hole 2.0 cm in diameter all the way through and perpendicular to one face, you find that the cube weighs 7.50 N. (a) What is the density of this metal? (b) What did the cube weigh before you drilled the hole in it?

12.3 • You purchase a rectangular piece of metal that has dimensions $5.0 \times 15.0 \times 30.0$ mm and mass 0.0158 kg. The seller tells you that the metal is gold. To check this, you compute the average density of the piece. What value do you get? Were you cheated?

12.4 •• **Gold Brick.** You win the lottery and decide to impress your friends by exhibiting a million-dollar cube of gold. At the time, gold is selling for $426.60 per troy ounce, and 1.0000 troy ounce equals 31.1035 g. How tall would your million-dollar cube be?

12.5 •• A uniform lead sphere and a uniform aluminum sphere have the same mass. What is the ratio of the radius of the aluminum sphere to the radius of the lead sphere?

12.6 • (a) What is the average density of the sun? (b) What is the average density of a neutron star that has the same mass as the sun but a radius of only 20.0 km?

12.7 •• A hollow cylindrical copper pipe is 1.50 m long and has an outside diameter of 3.50 cm and an inside diameter of 2.50 cm. How much does it weigh?

Section 12.2 Pressure in a Fluid

12.8 •• **Black Smokers.** Black smokers are hot volcanic vents that emit smoke deep in the ocean floor. Many of them teem with exotic creatures, and some biologists think that life on earth may have begun around such vents. The vents range in depth from about 1500 m to 3200 m below the surface. What is the gauge pressure at a 3200-m deep vent, assuming that the density of water does not vary? Express your answer in pascals and atmospheres.

12.9 •• **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is $3.71\ \mathrm{m/s^2}$. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

12.10 •• BIO (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What *additional* outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

12.11 • BIO In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density $1050\ \mathrm{kg/m^3}$) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity (see Section 12.6) of the fluid.

12.12 • A barrel contains a 0.120-m layer of oil floating on water that is 0.250 m deep. The density of the oil is 600 kg/m^3. (a) What is the gauge pressure at the oil–water interface? (b) What is the gauge pressure at the bottom of the barrel?

12.13 • BIO **Standing on Your Head.** (a) What is the *difference* between the pressure of the blood in your brain when you stand on your head and the pressure when you stand on your feet? Assume that you are 1.85 m tall. The density of blood is 1060 kg/m^3. (b) What effect does the increased pressure have on the blood vessels in your brain?

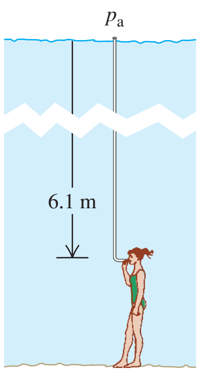
12.14 •• You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. (a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) (b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (You can ignore the small variation of pressure over the surface of the window.)

12.15 •• BIO **Ear Damage from Diving.** If the force on the tympanic membrane (eardrum) increases by about 1.5 N above the force from atmospheric pressure, the membrane can be damaged. When you go scuba diving in the ocean, below what depth could damage to your eardrum start to occur? The eardrum is typically 8.2 mm in diameter. (Consult Table 12.1.)

12.16 •• The liquid in the open-tube manometer in Fig. 12.8a is mercury, $y_1 = 3.00$ cm, and $y_2 = 7.00$ cm. Atmospheric pressure is 980 millibars. (a) What is the absolute pressure at the bottom of the U-shaped tube? (b) What is the absolute pressure in the open tube at a depth of 4.00 cm below the free surface? (c) What is the absolute pressure of the gas in the container? (d) What is the gauge pressure of the gas in pascals?

12.17 • BIO There is a maximum depth at which a diver can breathe through a snorkel tube (Fig. E12.17) because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

Figure **E12.17**



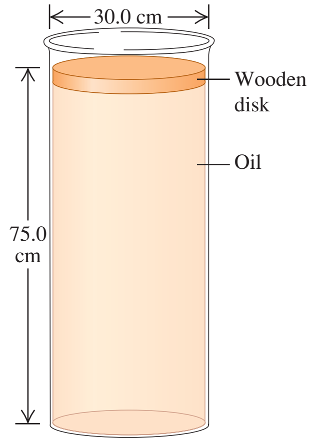
12.18 •• A tall cylinder with a cross-sectional area 12.0 cm^2 is partially filled with mercury; the surface of the mercury is 5.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

12.19 •• An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area 0.75 m^2 and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12.20 •• A closed container is partially filled with water. Initially, the air above the water is at atmospheric pressure $(1.01 \times 10^5 \text{ Pa})$ and the gauge pressure at the bottom of the water is 2500 Pa. Then additional air is pumped in, increasing the pressure of the air above the water by 1500 Pa. (a) What is the gauge pressure at the bottom of the water? (b) By how much must the water level in the container be reduced, by drawing some water out through a valve at the bottom of the container, to return the gauge pressure at the bottom of the water to its original value of 2500 Pa? The pressure of the air above the water is maintained at 1500 Pa above atmospheric pressure.

12.21 •• A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density 0.850 g/cm^3 (Fig. E12.21). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the *change* in pressure at (i) the bottom of the oil and (ii) halfway down in the oil?

Figure **E12.21**



12.22 •• **Exploring Venus.** The surface pressure on Venus is 92 atm, and the acceleration due to gravity there is 0.894g. In a future exploratory mission, an upright cylindrical tank of benzene is sealed at the top but still pressurized at 92 atm just above the benzene. The tank has a diameter of 1.72 m, and the benzene column is 11.50 m tall. Ignore any effects due to the very high temperature on Venus. (a) What total force is exerted on the inside surface of the bottom of the tank? (b) What force does the Venusian atmosphere exert on the outside surface of the bottom of the tank? (c) What total inward force does the atmosphere exert on the vertical walls of the tank?

12.23 •• **Hydraulic Lift I.** For the hydraulic lift shown in Fig. 12.7, what must be the ratio of the diameter of the vessel at the car to the diameter of the vessel where the force F_1 is applied so that a 1520-kg car can be lifted with a force F_1 of just 125 N?

12.24 • **Hydraulic Lift II.** The piston of a hydraulic automobile lift is 0.30 m in diameter. What gauge pressure, in pascals, is required to lift a car with a mass of 1200 kg? Also express this pressure in atmospheres.

Section 12.3 Buoyancy

12.25 • A 950-kg cylindrical can buoy floats vertically in salt water. The diameter of the buoy is 0.900 m. Calculate the additional distance the buoy will sink when a 70.0-kg man stands on top of it.

12.26 •• A slab of ice floats on a freshwater lake. What minimum volume must the slab have for a 45.0-kg woman to be able to stand on it without getting her feet wet?

12.27 •• An ore sample weighs 17.50 N in air. When the sample is suspended by a light cord and totally immersed in water, the tension in the cord is 11.20 N. Find the total volume and the density of the sample.

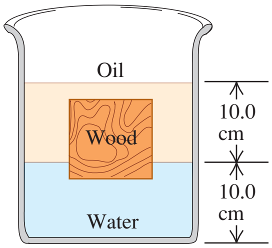
12.28 •• You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerine and a surface acceleration due to gravity of 4.15 m/s^2. If your apparatus floats in the oceans on earth with 25.0% of its volume submerged, what percentage will be submerged in the glycerine oceans of Caasi?

12.29 •• An object of average density ρ floats at the surface of a fluid of density ρ_{fluid}. (a) How must the two densities be related? (b) In view of the answer to part (a), how can steel ships float in water? (c) In terms of ρ and ρ_{fluid}, what fraction of the object is submerged and what fraction is above the fluid? Check that your answers give the correct limiting behavior as $\rho \rightarrow \rho_{fluid}$ and as $\rho \rightarrow 0$. (d) While on board your yacht, your cousin Throckmorton cuts a rectangular piece (dimensions $5.0 \times 4.0 \times 3.0$ cm) out of a life preserver and throws it into the ocean. The piece has a mass of 42 g. As it floats in the ocean, what percentage of its volume is above the surface?

12.30 • A hollow plastic sphere is held below the surface of a freshwater lake by a cord anchored to the bottom of the lake. The sphere has a volume of 0.650 m^3 and the tension in the cord is 900 N. (a) Calculate the buoyant force exerted by the water on the sphere. (b) What is the mass of the sphere? (c) The cord breaks and the sphere rises to the surface. When the sphere comes to rest, what fraction of its volume will be submerged?

12.31 •• A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. E12.31). The density of the oil is 790 kg/m^3. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure **E12.31**



12.32 • A solid aluminum ingot weighs 89 N in air. (a) What is its volume? (b) The ingot is suspended from a rope and totally immersed in water. What is the tension in the rope (the *apparent* weight of the ingot in water)?

12.33 •• A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 18.6 N. What is the density of the unknown liquid?

Section 12.4 Fluid Flow

12.34 •• Water runs into a fountain, filling all the pipes, at a steady rate of 0.750 m^3/s. (a) How fast will it shoot out of a hole 4.50 cm in diameter? (b) At what speed will it shoot out if the diameter of the hole is three times as large?

12.35 •• A shower head has 20 circular openings, each with radius 1.0 mm. The shower head is connected to a pipe with radius 0.80 cm. If the speed of water in the pipe is 3.0 m/s, what is its speed as it exits the shower-head openings?

12.36 • Water is flowing in a pipe with a varying cross-sectional area, and at all points the water completely fills the pipe. At point 1 the cross-sectional area of the pipe is 0.070 m^2, and the magnitude of the fluid velocity is 3.50 m/s. (a) What is the fluid speed at points in the pipe where the cross-sectional area is (a) 0.105 m^2 and (b) 0.047 m^2? (c) Calculate the volume of water discharged from the open end of the pipe in 1.00 hour.

12.37 • Water is flowing in a pipe with a circular cross section but with varying cross-sectional area, and at all points the water completely fills the pipe. (a) At one point in the pipe the radius is 0.150 m. What is the speed of the water at this point if water is flowing into this pipe at a steady rate of 1.20 m^3/s? (b) At a second point in the pipe the water speed is 3.80 m/s. What is the radius of the pipe at this point?

12.38 • **Home Repair.** You need to extend a 2.50-inch-diameter pipe, but you have only a 1.00-inch-diameter pipe on hand. You make a fitting to connect these pipes end to end. If the water is flowing at 6.00 cm/s in the wide pipe, how fast will it be flowing through the narrow one?

12.39 • At a point where an irrigation canal having a rectangular cross section is 18.5 m wide and 3.75 m deep, the water flows at 2.50 cm/s. At a point downstream, but on the same level, the canal is 16.5 m wide, but the water flows at 11.0 cm/s. How deep is the canal at this point?

12.40 •• BIO **Artery Blockage.** A medical technician is trying to determine what percentage of a patient's artery is blocked by plaque. To do this, she measures the blood pressure just before the region of blockage and finds that it is 1.20×10^4 Pa, while in the region of blockage it is 1.15×10^4 Pa. Furthermore, she knows that blood flowing through the normal artery just before the point of blockage is traveling at 30.0 cm/s, and the specific gravity of this patient's blood is 1.06. What percentage of the cross-sectional area of the patient's artery is blocked by the plaque?

Section 12.5 Bernoulli's Equation

12.41 •• A sealed tank containing seawater to a height of 11.0 m also contains air above the water at a gauge pressure of 3.00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

12.42 • A small circular hole 6.00 mm in diameter is cut in the side of a large water tank, 14.0 m below the water level in the tank. The top of the tank is open to the air. Find (a) the speed of efflux of the water and (b) the volume discharged per second.

12.43 • What gauge pressure is required in the city water mains for a stream from a fire hose connected to the mains to reach a vertical height of 15.0 m? (Assume that the mains have a much larger diameter than the fire hose.)

12.44 •• At one point in a pipeline the water's speed is 3.00 m/s and the gauge pressure is 5.00×10^4 Pa. Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

12.45 • At a certain point in a horizontal pipeline, the water's speed is 2.50 m/s and the gauge pressure is 1.80×10^4 Pa. Find the gauge pressure at a second point in the line if the cross-sectional area at the second point is twice that at the first.

12.46 • A soft drink (mostly water) flows in a pipe at a beverage plant with a mass flow rate that would fill 220 0.355-L cans per minute. At point 2 in the pipe, the gauge pressure is 152 kPa and the cross-sectional area is 8.00 cm^2. At point 1, 1.35 m above point 2, the cross-sectional area is 2.00 cm^2. Find the (a) mass flow rate; (b) volume flow rate; (c) flow speeds at points 1 and 2; (d) gauge pressure at point 1.

12.47 •• A golf course sprinkler system discharges water from a horizontal pipe at the rate of 7200 cm^3/s. At one point in the pipe, where the radius is 4.00 cm, the water's absolute pressure is 2.40×10^5 Pa. At a second point in the pipe, the water passes through a constriction where the radius is 2.00 cm. What is the water's absolute pressure as it flows through this constriction?

Section 12.6 Viscosity and Turbulence

12.48 • A pressure difference of 6.00×10^4 Pa is required to maintain a volume flow rate of 0.800 m^3/s for a viscous fluid flowing through a section of cylindrical pipe that has radius 0.210 m.

What pressure difference is required to maintain the same volume flow rate if the radius of the pipe is decreased to 0.0700 m?

12.49 •• **BIO** **Clogged Artery.** Viscous blood is flowing through an artery partially clogged by cholesterol. A surgeon wants to remove enough of the cholesterol to double the flow rate of blood through this artery. If the original diameter of the artery is D, what should be the new diameter (in terms of D) to accomplish this for the same pressure gradient?

PROBLEMS

12.50 •• **CP** The deepest point known in any of the earth's oceans is in the Marianas Trench, 10.92 km deep. (a) Assuming water is incompressible, what is the pressure at this depth? Use the density of seawater. (b) The actual pressure is 1.16×10^8 Pa; your calculated value will be less because the density actually varies with depth. Using the compressibility of water and the actual pressure, find the density of the water at the bottom of the Marianas Trench. What is the percent change in the density of the water?

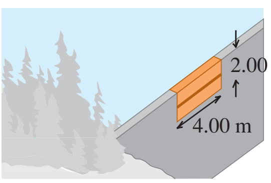
12.51 ••• In a lecture demonstration, a professor pulls apart two hemispherical steel shells (diameter D) with ease using their attached handles. She then places them together, pumps out the air to an absolute pressure of p, and hands them to a bodybuilder in the back row to pull apart. (a) If atmospheric pressure is p_0, how much force must the bodybuilder exert on each shell? (b) Evaluate your answer for the case $p = 0.025$ atm, $D = 10.0$ cm.

12.52 •• **BIO** **Fish Navigation.** (a) As you can tell by watching them in an aquarium, fish are able to remain at any depth in water with no effort. What does this ability tell you about their density? (b) Fish are able to inflate themselves using a sac (called the *swim bladder*) located under their spinal column. These sacs can be filled with an oxygen–nitrogen mixture that comes from the blood. If a 2.75-kg fish in freshwater inflates itself and increases its volume by 10%, find the *net* force that the *water* exerts on it. (c) What is the net *external* force on it? Does the fish go up or down when it inflates itself?

12.53 ••• **CALC** A swimming pool is 5.0 m long, 4.0 m wide, and 3.0 m deep. Compute the force exerted by the water against (a) the bottom and (b) either end. (*Hint:* Calculate the force on a thin, horizontal strip at a depth h, and integrate this over the end of the pool.) Do not include the force due to air pressure.

12.54 ••• **CP** **CALC** The upper edge of a gate in a dam runs along the water surface. The gate is 2.00 m high and 4.00 m wide and is hinged along a horizontal line through its center (Fig. P12.54). Calculate the torque about the hinge arising from the force due to the water. (*Hint:* Use a procedure similar to that used in Problem 12.53; calculate the torque on a thin, horizontal strip at a depth h and integrate this over the gate.)

Figure **P12.54**



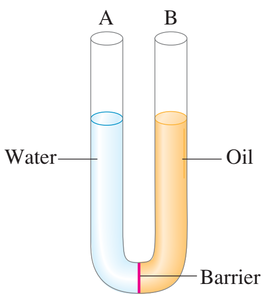
12.55 ••• **CP** **CALC** **Force and Torque on a Dam.** A dam has the shape of a rectangular solid. The side facing the lake has area A and height H. The surface of the freshwater lake behind the dam is at the top of the dam. (a) Show that the net horizontal force exerted by the water on the dam equals $\frac{1}{2}\rho gHA$—that is, the average gauge pressure across the face of the dam times the area (see Problem 12.53). (b) Show that the torque exerted by the water about an axis along the bottom of the dam is $\rho gH^2A/6$. (c) How do the force and torque depend on the size of the lake?

12.56 •• **Ballooning on Mars.** It has been proposed that we could explore Mars using inflated balloons to hover just above the surface. The buoyancy of the atmosphere would keep the balloon aloft. The density of the Martian atmosphere is 0.0154 kg/m^3 (although this varies with temperature). Suppose we construct these balloons of a thin but tough plastic having a density such that each square meter has a mass of 5.00 g. We inflate them with a very light gas whose mass we can neglect. (a) What should be the radius and mass of these balloons so they just hover above the surface of Mars? (b) If we released one of the balloons from part (a) on earth, where the atmospheric density is 1.20 kg/m^3, what would be its initial acceleration assuming it was the same size as on Mars? Would it go up or down? (c) If on Mars these balloons have five times the radius found in part (a), how heavy an instrument package could they carry?

12.57 •• A 0.180-kg cube of ice (frozen water) is floating in glycerine. The gylcerine is in a tall cylinder that has inside radius 3.50 cm. The level of the glycerine is well below the top of the cylinder. If the ice completely melts, by what distance does the height of liquid in the cylinder change? Does the level of liquid rise or fall? That is, is the surface of the water above or below the original level of the gylcerine before the ice melted?

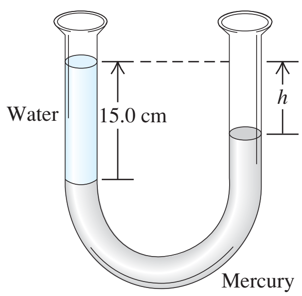
12.58 •• A narrow, U-shaped glass tube with open ends is filled with 25.0 cm of oil (of specific gravity 0.80) and 25.0 cm of water on opposite sides, with a barrier separating the liquids (Fig. P12.58). (a) Assume that the two liquids do not mix, and find the final heights of the columns of liquid in each side of the tube after the barrier is removed. (b) For the following cases, arrive at your answer by simple physical reasoning, not by calculations: (i) What would be the height on each side if the oil and water had equal densities? (ii) What would the heights be if the oil's density were much less than that of water?

Figure **P12.58**



12.59 • A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm (Fig. P12.59). (a) What is the gauge pressure at the water–mercury interface? (b) Calculate the vertical distance h from the top of the mercury in the right-hand arm of the tube to the top of the water in the left-hand arm.

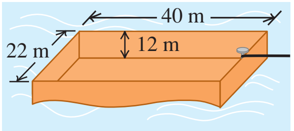
Figure **P12.59**



12.60 •• **CALC** **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7-m-high, 27.4-m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 5-m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of 1600 kg/m^3. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (*Hint:* Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

12.61 • An open barge has the dimensions shown in Fig. P12.61. If the barge is made out of 4.0-cm-thick steel plate on each of its four sides and its bottom, what mass of coal can the barge carry in freshwater without sinking? Is there enough room in the barge to hold this amount of coal? (The density of coal is about 1500 kg/m^3.)

Figure **P12.61**



12.62 ••• A hot-air balloon has a volume of 2200 m^3. The balloon fabric (the envelope) weighs 900 N. The basket with gear and full propane tanks weighs 1700 N. If the balloon can barely lift an additional 3200 N of passengers, breakfast, and champagne when the outside air density is 1.23 kg/m^3, what is the average density of the heated gases in the envelope?

12.63 •• Advertisements for a certain small car claim that it floats in water. (a) If the car's mass is 900 kg and its interior volume is 3.0 m^3, what fraction of the car is immersed when it floats? You can ignore the volume of steel and other materials. (b) Water gradually leaks in and displaces the air in the car. What fraction of the interior volume is filled with water when the car sinks?

12.64 • A single ice cube with mass 9.70 g floats in a glass completely full of 420 cm^3 of water. You can ignore the water's surface tension and its variation in density with temperature (as long as it remains a liquid). (a) What volume of water does the ice cube displace? (b) When the ice cube has completely melted, has any water overflowed? If so, how much? If not, explain why this is so. (c) Suppose the water in the glass had been very salty water of density 1050 kg/m^3. What volume of salt water would the 9.70-g ice cube displace? (d) Redo part (b) for the freshwater ice cube in the salty water.

12.65 ••• A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is 700 kg/m^3. What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

12.66 •• A hydrometer consists of a spherical bulb and a cylindrical stem with a cross-sectional area of 0.400 cm^2 (see Fig. 12.12a). The total volume of bulb and stem is 13.2 cm^3. When immersed in water, the hydrometer floats with 8.00 cm of the stem above the water surface. When the hydrometer is immersed in an organic fluid, 3.20 cm of the stem is above the surface. Find the density of the organic fluid. (*Note:* This illustrates the precision of such a hydrometer. Relatively small density differences give rise to relatively large differences in hydrometer readings.)

12.67 •• The densities of air, helium, and hydrogen (at p = 1.0 atm and T = 20°C) are 1.20 kg/m^3, 0.166 kg/m^3, and 0.0899 kg/m^3, respectively. (a) What is the volume in cubic meters displaced by a hydrogen-filled airship that has a total "lift" of 90.0 kN? (The "lift" is the amount by which the buoyant force exceeds the weight of the gas that fills the airship.) (b) What would be the "lift" if helium were used instead of hydrogen? In view of your answer, why is helium used in modern airships like advertising blimps?

12.68 •• When an open-faced boat has a mass of 5750 kg, including its cargo and passengers, it floats with the water just up to the top of its gunwales (sides) on a freshwater lake. (a) What is the volume of this boat? (b) The captain decides that it is too dangerous to float with his boat on the verge of sinking, so he decides to throw some cargo overboard so that 20% of the boat's volume will be above water. How much mass should he throw out?

12.69 •• **CP** An open cylindrical tank of acid rests at the edge of a table 1.4 m above the floor of the chemistry lab. If this tank springs a small hole in the side at its base, how far from the foot of the table will the acid hit the floor if the acid in the tank is 75 cm deep?

12.70 •• **CP** A firehose must be able to shoot water to the top of a building 28.0 m tall when aimed straight up. Water enters this hose at a steady rate of 0.500 m^3/s and shoots out of a round nozzle. (a) What is the maximum diameter this nozzle can have? (b) If the only nozzle available has a diameter twice as great, what is the highest point the water can reach?

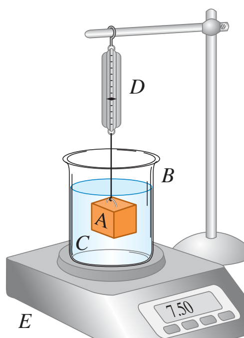
12.71 •• **CP** You drill a small hole in the side of a vertical cylindrical water tank that is standing on the ground with its top open to the air. (a) If the water level has a height H, at what height above the base should you drill the hole for the water to reach its greatest distance from the base of the cylinder when it hits the ground? (b) What is the greatest distance the water will reach?

12.72 ••• **CALC** A closed and elevated vertical cylindrical tank with diameter 2.00 m contains water to a depth of 0.800 m. A worker accidently pokes a circular hole with diameter 0.0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of 5.00×10^3 Pa at the surface of the water. Ignore any effects of viscosity. (a) Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air? (b) How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

12.73 •• A block of balsa wood placed in one scale pan of an equal-arm balance is exactly balanced by a 0.115-kg brass mass in the other scale pan. Find the true mass of the balsa wood if its density is 150 kg/m^3. Explain why it is accurate to ignore the buoyancy in air of the brass but *not* the buoyancy in air of the balsa wood.

12.74 •• Block A in Fig. P12.74 hangs by a cord from spring balance D and is submerged in a liquid C contained in beaker B. The mass of the beaker is 1.00 kg; the mass of the liquid is 1.80 kg. Balance D reads 3.50 kg, and balance E reads 7.50 kg. The volume of block A is 3.80×10^{-3} m^3. (a) What is the density of the liquid? (b) What will each balance read if block A is pulled up out of the liquid?

Figure **P12.74**



12.75 •• A hunk of aluminum is completely covered with a gold shell to form an ingot of weight 45.0 N. When you suspend the ingot from a spring balance and submerge the ingot in water, the balance reads 39.0 N. What is the weight of the gold in the shell?

12.76 •• A plastic ball has radius 12.0 cm and floats in water with 24.0% of its volume submerged. (a) What force must you apply to the ball to hold it at rest totally below the surface of the water? (b) If you let go of the ball, what is its acceleration the instant you release it?

12.77 •• The weight of a king's solid crown is w. When the crown is suspended by a light rope and completely immersed in water, the tension in the rope (the crown's apparent weight) is fw. (a) Prove that the crown's relative density (specific gravity) is $1/(1 - f)$. Discuss the meaning of the limits as f approaches 0 and 1. (b) If the crown is solid gold and weighs 12.9 N in air, what is its apparent weight when completely immersed in water? (c) Repeat part (b) if the crown is solid lead with a very thin gold plating, but still has a weight in air of 12.9 N.

12.78 •• A piece of steel has a weight w, an apparent weight (see Problem 12.77) w_{water} when completely immersed in water, and an apparent weight w_{fluid} when completely immersed in an unknown fluid. (a) Prove that the fluid's density relative to water (specific gravity) is $(w - w_{fluid})/(w - w_{water})$. (b) Is this result reasonable for the three cases of w_{fluid} greater than, equal to, or less than w_{water}? (c) The apparent weight of the piece of steel in water of density $1000\ kg/m^3$ is 87.2% of its weight. What percentage of its weight will its apparent weight be in formic acid (density $1220\ kg/m^3$)?

12.79 ••• You cast some metal of density ρ_m in a mold, but you are worried that there might be cavities within the casting. You measure the weight of the casting to be w, and the buoyant force when it is completely surrounded by water to be B. (a) Show that $V_0 = B/(\rho_{water} g) - w/(\rho_m g)$ is the total volume of any enclosed cavities. (b) If your metal is copper, the casting's weight is 156 N, and the buoyant force is 20 N, what is the total volume of any enclosed cavities in your casting? What fraction is this of the total volume of the casting?

12.80 • A cubical block of wood 0.100 m on a side and with a density of $550\ kg/m^3$ floats in a jar of water. Oil with a density of $750\ kg/m^3$ is poured on the water until the top of the oil layer is 0.035 m below the top of the block. (a) How deep is the oil layer? (b) What is the gauge pressure at the block's lower face?

12.81 •• **Dropping Anchor.** An iron anchor with mass 35.0 kg and density $7860\ kg/m^3$ lies on the deck of a small barge that has vertical sides and floats in a freshwater river. The area of the bottom of the barge is $8.00\ m^2$. The anchor is thrown overboard but is suspended above the bottom of the river by a rope; the mass and volume of the rope are small enough to ignore. After the anchor is overboard and the barge has finally stopped bobbing up and down, has the barge risen or sunk down in the water? By what vertical distance?

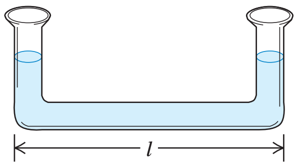
12.82 •• Assume that crude oil from a supertanker has density $750\ kg/m^3$. The tanker runs aground on a sandbar. To refloat the tanker, its oil cargo is pumped out into steel barrels, each of which has a mass of 15.0 kg when empty and holds $0.120\ m^3$ of oil. You can ignore the volume occupied by the steel from which the barrel is made. (a) If a salvage worker accidentally drops a filled, sealed barrel overboard, will it float or sink in the seawater? (b) If the barrel floats, what fraction of its volume will be above the water surface? If it sinks, what minimum tension would have to be exerted by a rope to haul the barrel up from the ocean floor? (c) Repeat parts (a) and (b) if the density of the oil is $910\ kg/m^3$ and the mass of each empty barrel is 32.0 kg.

12.83 ••• A cubical block of density ρ_B and with sides of length L floats in a liquid of greater density ρ_L. (a) What fraction of the block's volume is above the surface of the liquid? (b) The liquid is denser than water (density ρ_W) and does not mix with it. If water is poured on the surface of the liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of L, ρ_B, ρ_L, and ρ_W. (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and the side length is 10.0 cm.

12.84 •• A barge is in a rectangular lock on a freshwater river. The lock is 60.0 m long and 20.0 m wide, and the steel doors on each end are closed. With the barge floating in the lock, a 2.50×10^6 N load of scrap metal is put onto the barge. The metal has density $9000\ kg/m^3$. (a) When the load of scrap metal, initially on the bank, is placed onto the barge, what vertical distance does the water in the lock rise? (b) The scrap metal is now pushed overboard into the water. Does the water level in the lock rise, fall, or remain the same? If it rises or falls, by what vertical distance does it change?

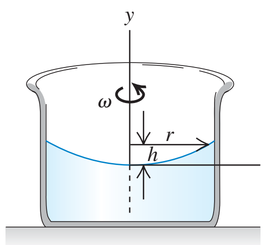
12.85 • **CP CALC** A U-shaped tube with a horizontal portion of length l (Fig. P12.85) contains a liquid. What is the difference in height between the liquid columns in the vertical arms (a) if the tube has an acceleration a toward the right and (b) if the tube is mounted on a horizontal turntable rotating with an angular speed ω with one of the vertical arms on the axis of rotation? (c) Explain why the difference in height does not depend on the density of the liquid or on the cross-sectional area of the tube. Would it be the same if the vertical tubes did not have equal cross-sectional areas? Would it be the same if the horizontal portion were tapered from one end to the other? Explain.

Figure **P12.85**



12.86 • **CP CALC** A cylindrical container of an incompressible liquid with density ρ rotates with constant angular speed ω about its axis of symmetry, which we take to be the y-axis (Fig. P12.86). (a) Show that the pressure at a given height within the fluid increases in the radial direction (outward from the axis of rotation) according to $\partial p/\partial r = \rho\omega^2 r$. (b) Integrate this partial differential equation to find the pressure as a function of distance from the axis of rotation along a horizontal line at $y = 0$. (c) Combine the result of part (b) with Eq. (12.5) to show that the surface of the rotating liquid has a *parabolic* shape; that is, the height of the liquid is given by $h(r) = \omega^2 r^2/2g$. (This technique is used for making parabolic telescope mirrors; liquid glass is rotated and allowed to solidify while rotating.)

Figure **P12.86**

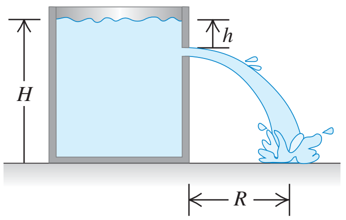


12.87 •• **CP CALC** An incompressible fluid with density ρ is in a horizontal test tube of inner cross-sectional area A. The test tube spins in a horizontal circle in an ultracentrifuge at an angular speed ω. Gravitational forces are negligible. Consider a volume element of the fluid of area A and thickness dr' a distance r' from the rotation axis. The pressure on its inner surface is p and on its outer surface is $p + dp$. (a) Apply Newton's second law to the volume element to show that $dp = \rho\omega^2 r'\,dr'$. (b) If the surface of the fluid is at a radius r_0 where the pressure is p_0, show that the pressure p at a distance $r \geq r_0$ is $p = p_0 + \rho\omega^2(r^2 - r_0^2)/2$. (c) An object of volume V and density ρ_{ob} has its center of mass at a distance R_{cmob} from the axis. Show that the net horizontal force on the object is $\rho V\omega^2 R_{cm}$, where R_{cm} is the distance from the axis to the center of mass of the displaced fluid. (d) Explain why the object will move inward if $\rho R_{cm} > \rho_{ob} R_{cmob}$ and outward if $\rho R_{cm} < \rho_{ob} R_{cmob}$. (e) For small objects of uniform density, $R_{cm} = R_{cmob}$. What happens to a mixture of small objects of this kind with different densities in an ultracentrifuge?

12.88 ••• **CALC** Untethered helium balloons, floating in a car that has all the windows rolled up and outside air vents closed, move in the direction of the car's acceleration, but loose balloons filled with air move in the opposite direction. To show why, consider only the horizontal forces acting on the balloons. Let a be the magnitude of the car's forward acceleration. Consider a horizontal tube of air with a cross-sectional area A that extends from the windshield, where $x = 0$ and $p = p_0$, back along the x-axis. Now consider a volume element of thickness dx in this tube. The pressure on its front surface is p and the pressure on its rear surface is $p + dp$. Assume the air has a constant density ρ. (a) Apply Newton's second law to the volume element to show that $dp = \rho a\, dx$. (b) Integrate the result of part (a) to find the pressure at the front surface in terms of a and x. (c) To show that considering ρ constant is reasonable, calculate the pressure difference in atm for a distance as long as 2.5 m and a large acceleration of $5.0\ \text{m/s}^2$. (d) Show that the net horizontal force on a balloon of volume V is ρVa. (e) For negligible friction forces, show that the acceleration of the balloon (average density ρ_{bal}) is $(\rho/\rho_{\text{bal}})a$, so that the acceleration relative to the car is $a_{\text{rel}} = [(\rho/\rho_{\text{bal}}) - 1]a$. (f) Use the expression for a_{rel} in part (e) to explain the movement of the balloons.

12.89 • **CP** Water stands at a depth H in a large, open tank whose side walls are vertical (Fig. P12.89). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

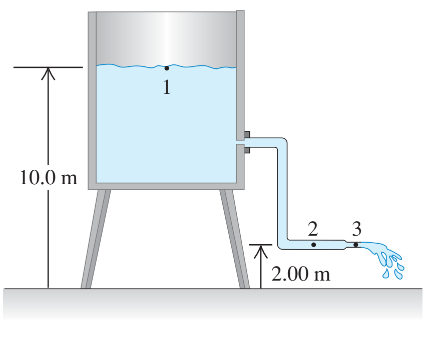
Figure **P12.89**



12.90 ••• A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area $1.50\ \text{cm}^2$ is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of $2.40 \times 10^{-4}\ \text{m}^3/\text{s}$. How high will the water in the bucket rise?

12.91 • Water flows steadily from an open tank as in Fig. P12.91. The elevation of point 1 is 10.0 m, and the elevation of points 2 and 3 is 2.00 m. The cross-sectional area at point 2 is $0.0480\ \text{m}^2$; at point 3 it is $0.0160\ \text{m}^2$. The area of the tank is very large compared with the cross-sectional area of the pipe. Assuming that Bernoulli's equation applies, compute (a) the discharge rate in cubic meters per second and (b) the gauge pressure at point 2.

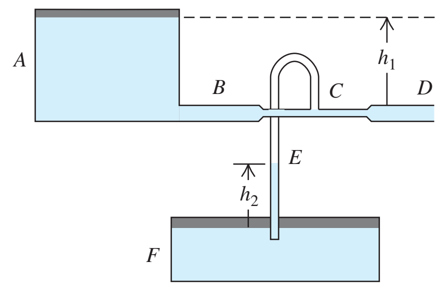
Figure **P12.91**



12.92 •• **CP** In 1993 the radius of Hurricane Emily was about 350 km. The wind speed near the center ("eye") of the hurricane, whose radius was about 30 km, reached about 200 km/h. As air swirled in from the rim of the hurricane toward the eye, its angular momentum remained roughly constant. (a) Estimate the wind speed at the rim of the hurricane. (b) Estimate the pressure difference at the earth's surface between the eye and the rim. (*Hint:* See Table 12.1.) Where is the pressure greater? (c) If the kinetic energy of the swirling air in the eye could be converted completely to gravitational potential energy, how high would the air go? (d) In fact, the air in the eye is lifted to heights of several kilometers. How can you reconcile this with your answer to part (c)?

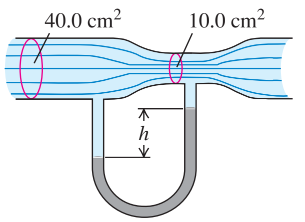
12.93 •• Two very large open tanks A and F (Fig. P12.93) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.

Figure **P12.93**



12.94 •• The horizontal pipe shown in Fig. P12.94 has a cross-sectional area of $40.0\ \text{cm}^2$ at the wider portions and $10.0\ \text{cm}^2$ at the constriction. Water is flowing in the pipe, and the discharge from the pipe is $6.00 \times 10^{-3}\ \text{m}^3/\text{s}$ $(6.00\ \text{L/s})$. Find (a) the flow speeds at the wide and the narrow portions; (b) the pressure difference between these portions; (c) the difference in height between the mercury columns in the U-shaped tube.

Figure **P12.94**



12.95 • A liquid flowing from a vertical pipe has a definite shape as it flows from the pipe. To get the equation for this shape, assume that the liquid is in free fall once it leaves the pipe. Just as it leaves the pipe, the liquid has speed v_0 and the radius of the stream of liquid is r_0. (a) Find an equation for the speed of the liquid as a function of the distance y it has fallen. Combining this with the equation of continuity, find an expression for the radius of the stream as a function of y. (b) If water flows out of a vertical pipe at a speed of 1.20 m/s, how far below the outlet will the radius be one-half the original radius of the stream?

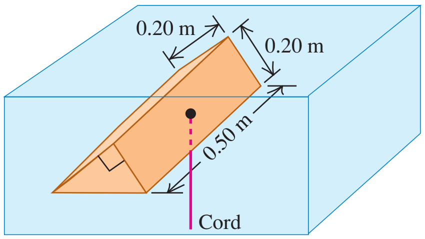
Challenge Problems

12.96 ••• A rock with mass $m = 3.00$ kg is suspended from the roof of an elevator by a light cord. The rock is totally immersed in a bucket of water that sits on the floor of the elevator, but the rock doesn't touch the bottom or sides of the bucket. (a) When the elevator is at rest, the tension in the cord is 21.0 N. Calculate the volume of the rock. (b) Derive an expression for the tension in the cord when the elevator is accelerating *upward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50\ \text{m/s}^2$

upward. (c) Derive an expression for the tension in the cord when the elevator is accelerating *downward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50\ \text{m/s}^2$ downward. (d) What is the tension when the elevator is in free fall with a downward acceleration equal to g?

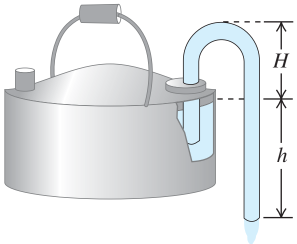
12.97 ••• **CALC** Suppose a piece of styrofoam, $\rho = 180\ \text{kg/m}^3$, is held completely submerged in water (Fig. P12.97). (a) What is the tension in the cord? Find this using Archimedes's principle. (b) Use $p = p_0 + \rho g h$ to calculate directly the force exerted by the water on the two sloped sides and the bottom of the styrofoam; then show that the vector sum of these forces is the buoyant force.

Figure **P12.97**



12.98 ••• A *siphon,* as shown in Fig. P12.98, is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density ρ, and let the atmospheric pressure be p_{atm}. Assume that the cross-sectional area of the tube is the same at all points along it. (a) If the lower end of the siphon is at a distance h below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.) (b) A curious feature of a siphon is that the fluid initially flows "uphill." What is the greatest height H that the high point of the tube can have if flow is still to occur?

Figure **P12.98**



Answers

Chapter Opening Question ?

The flesh of both the shark and the tropical fish is denser than seawater, so left to themselves they would sink. However, a tropical fish has a gas-filled body cavity called a swimbladder, so that the *average* density of the fish's body is the same as that of seawater and the fish neither sinks nor rises. Sharks have no such cavity. Hence they must swim constantly to keep from sinking, using their pectoral fins to provide lift much like the wings of an airplane (see Section 12.5).

Test Your Understanding Questions

12.1 Answer: (ii), (iv), (i) and (iii) (tie), (v) In each case the average density equals the mass divided by the volume. Hence we have (i) $\rho = (4.00\ \text{kg})/(1.60 \times 10^{-3}\ \text{m}^3) = 2.50 \times 10^3\ \text{kg/m}^3$; (ii) $\rho = (8.00\ \text{kg})/(1.60 \times 10^{-3}\ \text{m}^3) = 5.00 \times 10^3\ \text{kg/m}^3$; (iii) $\rho = (8.00\ \text{kg})/(3.20 \times 10^{-3}\ \text{m}^3) = 2.50 \times 10^3\ \text{kg/m}^3$; (iv) $\rho = (2560\ \text{kg})/(0.640\ \text{m}^3) = 4.00 \times 10^3\ \text{kg/m}^3$; (v) $\rho = (2560\ \text{kg})/(1.28\ \text{m}^3) = 2.00 \times 10^3\ \text{kg/m}^3$. Note that compared to object (i), object (ii) has double the mass but the same volume and so has double the average density. Object (iii) has double the mass and double the volume of object (i), so (i) and (iii) have the same average density. Finally, object (v) has the same mass as object (iv) but double the volume, so (v) has half the average density of (iv).

12.2 Answer: (ii) From Eq. (12.9), the pressure outside the barometer is equal to the product $\rho g h$. When the barometer is taken out of the refrigerator, the density ρ decreases while the height h of the mercury column remains the same. Hence the air pressure must be lower outdoors than inside the refrigerator.

12.3 Answer: (i) Consider the water, the statue, and the container together as a system; the total weight of the system does not depend on whether the statue is immersed. The total supporting force, including the tension T and the upward force F of the scale on the container (equal to the scale reading), is the same in both cases. But we saw in Example 12.5 that T decreases by 7.84 N when the statue is immersed, so the scale reading F must *increase* by 7.84 N. An alternative viewpoint is that the water exerts an upward buoyant force of 7.84 N on the statue, so the statue must exert an equal downward force on the water, making the scale reading 7.84 N greater than the weight of water and container.

12.4 Answer: (ii) A highway that narrows from three lanes to one is like a pipe whose cross-sectional area narrows to one-third of its value. If cars behaved like the molecules of an incompressible fluid, then as the cars encountered the one-lane section, the spacing between cars (the "density") would stay the same but the cars would triple their speed. This would keep the "volume flow rate" (number of cars per second passing a point on the highway) the same. In real life cars behave like the molecules of a *compressible* fluid: They end up packed closer (the "density" increases) and fewer cars per second pass a point on the highway (the "volume flow rate" decreases).

12.5 Answer: (ii) Newton's second law tells us that a body accelerates (its velocity changes) in response to a net force. In fluid flow, a pressure difference between two points means that fluid particles moving between those two points experience a force, and this force causes the fluid particles to accelerate and change speed.

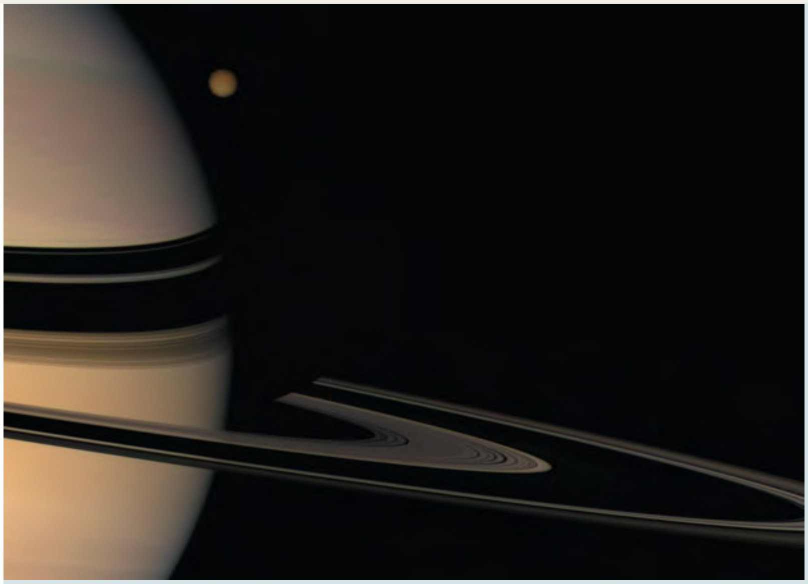
12.6 Answer: (iv) The required pressure is proportional to $1/R^4$, where R is the inside radius of the needle (half the inside diameter). With the smaller-diameter needle, the pressure is greater by a factor of $[(0.60\ \text{mm})/(0.30\ \text{mm})]^4 = 2^4 = 16$.

Bridging Problem

Answers: (a) $y = H - \left(\frac{d}{D}\right)^2 \sqrt{2gH}\, t + \left(\frac{d}{D}\right)^4 \frac{gt^2}{2}$

(b) $T = \sqrt{\frac{2H}{g}} \left(\frac{D}{d}\right)^2$ **(c)** $\sqrt{2}$

13 GRAVITATION



? The rings of Saturn are made of countless individual orbiting particles. Do all the ring particles orbit at the same speed, or do the inner particles orbit faster or slower than the outer ones?

LEARNING GOALS

By studying this chapter, you will learn:

- How to calculate the gravitational forces that any two bodies exert on each other.
- How to relate the weight of an object to the general expression for gravitational force.
- How to use and interpret the generalized expression for gravitational potential energy.
- How to relate the speed, orbital period, and mechanical energy of a satellite in a circular orbit.
- The laws that describe the motions of planets, and how to work with these laws.
- What black holes are, how to calculate their properties, and how they are discovered.

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics,* the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you will learn the basic law that governs gravitational interactions. This law is *universal:* Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

13.1 Newton's Law of Gravitation

The example of gravitational attraction that's probably most familiar to you is your *weight,* the force that attracts you toward the earth. During his study of the motions of the planets and of the moon, Newton discovered the fundamental character of the gravitational attraction between *any* two bodies. Along with his three laws of motion, Newton published the **law of gravitation** in 1687. It may be stated as follows:

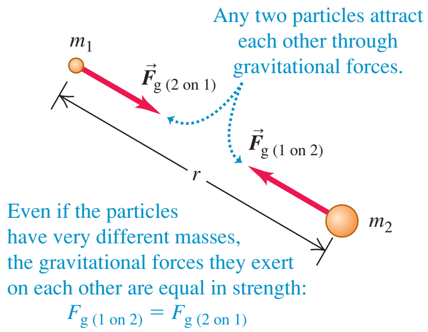
Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Translating this into an equation, we have

$$F_g = \frac{Gm_1 m_2}{r^2} \quad \text{(law of gravitation)} \tag{13.1}$$

where F_g is the magnitude of the gravitational force on either particle, m_1 and m_2 are their masses, r is the distance between them (Fig. 13.1), and G is a fundamental physical constant called the **gravitational constant.** The numerical value of G depends on the system of units used.

Equation (13.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.



13.1 The gravitational forces between two particles of masses m_1 and m_2.

CAUTION **Don't confuse g and G** Because the symbols g and G are so similar, it's common to confuse the two very different gravitational quantities that these symbols represent. Lowercase g is the acceleration due to gravity, which relates the weight w of a body to its mass m: $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of different planets. By contrast, capital G relates the gravitational force between any two bodies to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two bodies, no matter where in space they are located. In the next section we'll see how the values of g and G are related.

Gravitational forces always act along the line joining the two particles, and they form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 13.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours.)

Gravitation and Spherically Symmetric Bodies

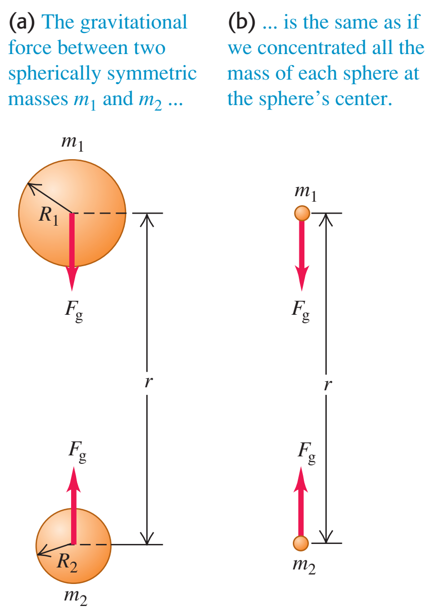
We have stated the law of gravitation in terms of the interaction between two *particles.* It turns out that the gravitational interaction of any two bodies having *spherically symmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in Fig. 13.2. Thus, if we model the earth as a spherically symmetric body with mass m_E, the force it exerts on a particle or a spherically symmetric body with mass m, at a distance r between centers, is

$$F_g = \frac{Gm_E m}{r^2} \tag{13.2}$$

provided that the body lies outside the earth. A force of the same magnitude is exerted *on* the earth by the body. (We will prove these statements in Section 13.6.)

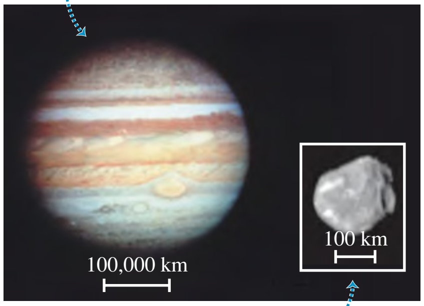
At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on a body at various depths, we would find that toward the center of the earth the force *decreases,*

13.2 The gravitational effect *outside* any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center.



13.3 Spherical and nonspherical bodies: the planet Jupiter and one of Jupiter's small moons, Amalthea.

Jupiter's mass is very large (1.90×10^{27} kg), so the mutual gravitational attraction of its parts has pulled it into a nearly spherical shape.



Amalthea, one of Jupiter's small moons, has a relatively tiny mass (7.17×10^{18} kg, only about 3.8×10^{-9} the mass of Jupiter) and weak mutual gravitation, so it has an irregular shape.

rather than increasing as $1/r^2$. As the body enters the interior of the earth (or other spherical body), some of the earth's mass is on the side of the body opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the body is zero.

Spherically symmetric bodies are an important case because moons, planets, and stars all tend to be spherical. Since all particles in a body gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the body naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial bodies of low mass, since the gravitational attraction is less, and these bodies tend *not* to be spherical (Fig. 13.3).

Determining the Value of G

To determine the value of the gravitational constant G, we have to *measure* the gravitational force between two bodies of known masses m_1 and m_2 at a known distance r. The force is extremely small for bodies that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance,* which Sir Henry Cavendish used in 1798 to determine G.

Figure 13.4 shows a modern version of the Cavendish torsion balance. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1, are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass m_2, to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

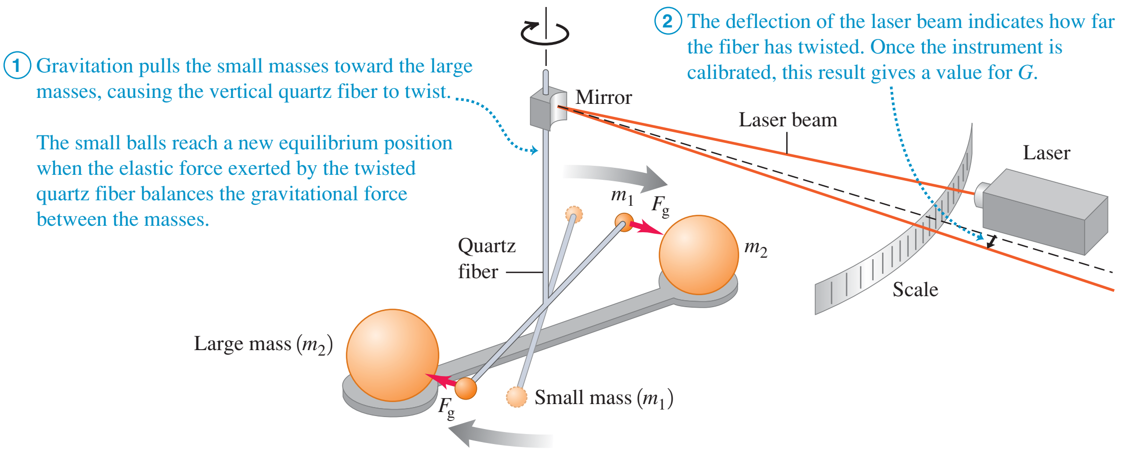
After calibrating the Cavendish balance, we can measure gravitational forces and thus determine G. The presently accepted value is

$$G = 6.67428(67) \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

To three significant figures, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. Because $1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}$, the units of G can also be expressed as $\mathrm{m^3/(kg \cdot s^2)}$.

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 13.3 makes use of this property, which is often called *superposition of forces.*

13.4 The principle of the Cavendish balance, used for determining the value of G. The angle of deflection has been exaggerated here for clarity.



Example 13.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of the nearest large sphere is 0.500 kg, and the center-to-center distance between them is 0.0500 m. Find the gravitational force F_g on each sphere due to the other.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: Because the spheres are spherically symmetric, we can calculate F_g by treating them as *particles* separated by 0.0500 m, as in Fig. 13.2. Each sphere experiences the same magnitude of force from the other sphere. We use Newton's law of gravitation, Eq. (13.1), to determine F_g:

$$F_g = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(0.0100\ \text{kg})(0.500\ \text{kg})}{(0.0500\ \text{m})^2}$$
$$= 1.33 \times 10^{-10}\ \text{N}$$

EVALUATE: It's remarkable that such a small force could be measured—or even detected—more than 200 years ago. Only a very massive object such as the earth exerts a gravitational force we can feel.

Example 13.2 Acceleration due to gravitational attraction

Suppose the two spheres in Example 13.1 are placed with their centers 0.0500 m apart at a point in space far removed from all other bodies. What is the magnitude of the acceleration of each, relative to an inertial system?

SOLUTION

IDENTIFY, SET UP, and EXECUTE: Each sphere exerts on the other a gravitational force of the same magnitude F_g, which we found in Example 13.1. We can neglect any other forces. The *acceleration* magnitudes a_1 and a_2 are different because the masses are different. To determine these we'll use Newton's second law:

$$a_1 = \frac{F_g}{m_1} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.0100\ \text{kg}} = 1.33 \times 10^{-8}\ \text{m/s}^2$$
$$a_2 = \frac{F_g}{m_2} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.500\ \text{kg}} = 2.66 \times 10^{-10}\ \text{m/s}^2$$

EVALUATE: The larger sphere has 50 times the mass of the smaller one and hence has $\frac{1}{50}$ the acceleration. These accelerations are *not* constant; the gravitational forces increase as the spheres move toward each other.

Example 13.3 Superposition of gravitational forces

Many stars belong to *systems* of two or more stars held together by their mutual gravitational attraction. Figure 13.5 shows a three-star system at an instant when the stars are at the vertices of a 45° right triangle. Find the total gravitational force exerted on the small star by the two large ones.

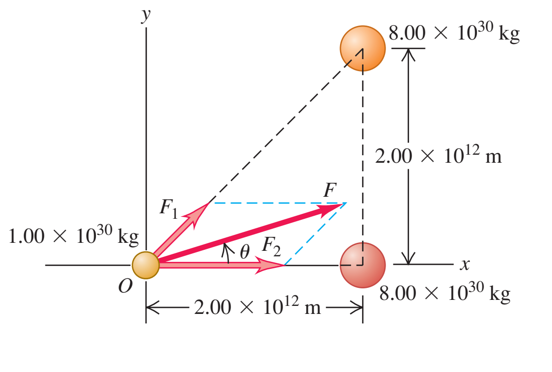
SOLUTION

IDENTIFY, SET UP, and EXECUTE: We use the principle of superposition: The total force $\vec{F}$ on the small star is the vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$ due to each large star, as Fig. 13.5 shows. We assume that the stars are spheres as in Fig. 13.2. We first calculate the magnitudes F_1 and F_2 using Eq. (13.1) and then compute the vector sum using components:

$$F_1 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2) \\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \text{m})^2 + (2.00 \times 10^{12}\ \text{m})^2}$$
$$= 6.67 \times 10^{25}\ \text{N}$$

$$F_2 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2) \\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \text{m})^2}$$
$$= 1.33 \times 10^{26}\ \text{N}$$

13.5 The total gravitational force on the small star (at O) is the vector sum of the forces exerted on it by the two larger stars. (For comparison, the mass of the sun—a rather ordinary star—is 1.99×10^{30} kg and the earth–sun distance is 1.50×10^{11} m.)



The x- and y-components of these forces are

$$F_{1x} = (6.67 \times 10^{25}\ \text{N})(\cos 45°) = 4.72 \times 10^{25}\ \text{N}$$
$$F_{1y} = (6.67 \times 10^{25}\ \text{N})(\sin 45°) = 4.72 \times 10^{25}\ \text{N}$$
$$F_{2x} = 1.33 \times 10^{26}\ \text{N}$$
$$F_{2y} = 0$$

Continued

The components of the total force $\vec{F}$ on the small star are

$$F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\ \text{N}$$
$$F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\ \text{N}$$

The magnitude of $\vec{F}$ and its angle θ (see Fig. 13.5) are

$$F = \sqrt{F_x{}^2 + F_y{}^2} = \sqrt{(1.81 \times 10^{26}\ \text{N})^2 + (4.72 \times 10^{25}\ \text{N})^2}$$
$$= 1.87 \times 10^{26}\ \text{N}$$
$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \text{N}}{1.81 \times 10^{26}\ \text{N}} = 14.6°$$

EVALUATE: While the force magnitude F is tremendous, the magnitude of the resulting acceleration is not: $a = F/m = (1.87 \times 10^{26}\ \text{N})/(1.00 \times 10^{30}\ \text{kg}) = 1.87 \times 10^{-4}\ \text{m/s}^2$. Furthermore, the force $\vec{F}$ is *not* directed toward the center of mass of the two large stars.

Why Gravitational Forces Are Important

Comparing Examples 13.1 and 13.3 shows that gravitational forces are negligible between ordinary household-sized objects, but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies (Fig. 13.6). It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

The gravitational force is so important on the cosmic scale because it acts *at a distance,* without any direct contact between bodies. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field.* One body sets up a disturbance or field at all points in space, and the force that acts on a second body at a particular point is its response to the first body's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

13.6 Our solar system is part of a spiral galaxy like this one, which contains roughly 10^{11} stars as well as gas, dust, and other matter. The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

Test Your Understanding of Section 13.1 The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation? (i) 100 times greater; (ii) 10 times greater; (iii) the same; (iv) $\frac{1}{10}$ as great; (v) $\frac{1}{100}$ as great.

13.2 Weight

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PhET: Lunar Lander

We defined the *weight* of a body in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition:

The weight of a body is the total gravitational force exerted on the body by all other bodies in the universe.

When the body is near the surface of the earth, we can neglect all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider a body's weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric body with radius R_E and mass m_E, the weight w of a small body of mass m at the earth's surface (a distance R_E from its center) is

$$w = F_g = \frac{Gm_E m}{R_E^2} \quad \text{(weight of a body of mass } m \text{ at the earth's surface)} \tag{13.3}$$

But we also know from Section 4.4 that the weight w of a body is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (13.3) and dividing by m, we find

$$g = \frac{Gm_E}{R_E^2} \quad \text{(acceleration due to gravity at the earth's surface)} \tag{13.4}$$

The acceleration due to gravity g is independent of the mass m of the body because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (13.4) except for m_E, so this relationship allows us to compute the mass of the earth. Solving Eq. (13.4) for m_E and using $R_E = 6380 \text{ km} = 6.38 \times 10^6 \text{ m}$ and $g = 9.80 \text{ m/s}^2$, we find

$$m_E = \frac{gR_E^2}{G} = 5.98 \times 10^{24} \text{ kg}$$

This is very close to the currently accepted value of 5.974×10^{24} kg. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance $r - R_E$ above the surface), the weight of a body is given by Eq. (13.3) with R_E replaced by r:

$$w = F_g = \frac{Gm_E m}{r^2} \tag{13.5}$$

The weight of a body decreases inversely with the square of its distance from the earth's center (Fig. 13.7). Figure 13.8 shows how the weight varies with height above the earth for an astronaut who weighs 700 N at the earth's surface.

The *apparent* weight of a body on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We have ignored this effect in our earlier discussion and have assumed that the earth *is* an inertial system. We will return to the effect of the earth's rotation in Section 13.7.

While the earth is an approximately spherically symmetric distribution of mass, it is *not* uniform throughout its volume. To demonstrate this, let's first calculate the average *density,* or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

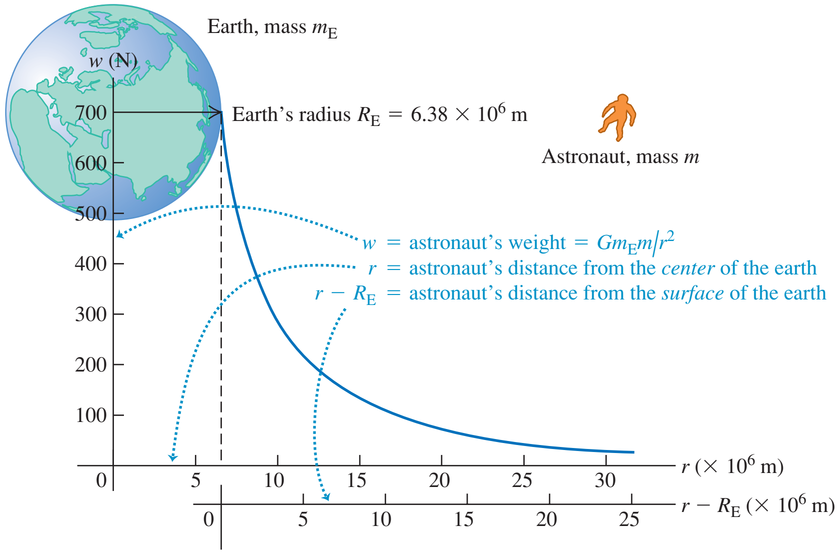
$$V_E = \tfrac{4}{3}\pi R_E^3 = \tfrac{4}{3}\pi(6.38 \times 10^6 \text{ m})^3 = 1.09 \times 10^{21} \text{ m}^3$$

Application Walking and Running on the Moon

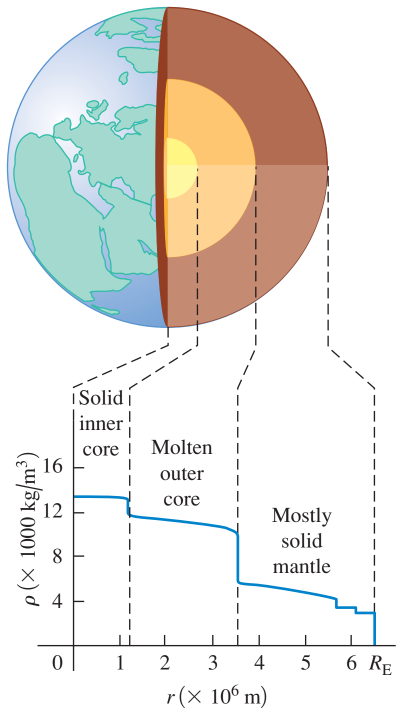
You automatically transition from a walk to a run when the vertical force you exert on the ground—which, by Newton's third law, equals the vertical force the ground exerts on you—exceeds your weight. This transition from walking to running happens at much lower speeds on the moon, where objects weigh only 17% as much as on earth. Hence, the Apollo astronauts found themselves running even when moving relatively slowly during their moon "walks."

13.7 In an airliner at high altitude, you are farther from the center of the earth than when on the ground and hence weigh slightly less. Can you show that at an altitude of 10 km above the surface, you weigh 0.3% less than you do on the ground?

13.8 An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface).



13.9 The density of the earth decreases with increasing distance from its center.



The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\rho = \frac{m_E}{V_E} = \frac{5.97 \times 10^{24}\ \text{kg}}{1.09 \times 10^{21}\ \text{m}^3}$$

$$= 5500\ \text{kg/m}^3 = 5.5\ \text{g/cm}^3$$

(For comparison, the density of water is $1000\ \text{kg/m}^3 = 1.00\ \text{g/cm}^3$.) If the earth were uniform, we would expect rocks near the earth's surface to have this same density. In fact, the density of surface rocks is substantially lower, ranging from about $2000\ \text{kg/m}^3$ for sedimentary rocks to about $3300\ \text{kg/m}^3$ for basalt. So the earth *cannot* be uniform, and the interior of the earth must be much more dense than the surface in order that the *average* density be $5500\ \text{kg/m}^3$. According to geophysical models of the earth's interior, the maximum density at the center is about $13{,}000\ \text{kg/m}^3$. Figure 13.9 is a graph of density as a function of distance from the center.

Example 13.4 Gravity on Mars

A robotic lander with an earth weight of 3430 N is sent to Mars, which has radius $R_M = 3.40 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg (see Appendix F). Find the weight F_g of the lander on the Martian surface and the acceleration there due to gravity, g_M.

SOLUTION

IDENTIFY and SET UP: To find F_g we use Eq. (13.3), replacing m_E and R_E with m_M and R_M. We determine the lander mass m from the lander's earth weight w and then find g_M from $F_g = mg_M$.

EXECUTE: The lander's earth weight is $w = mg$, so

$$m = \frac{w}{g} = \frac{3430\ \text{N}}{9.80\ \text{m/s}^2} = 350\ \text{kg}$$

The mass is the same no matter where the lander is. From Eq. (13.3), the lander's weight on Mars is

$$F_g = \frac{Gm_Mm}{R_M^2}$$

$$= \frac{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})(350\ \text{kg})}{(3.40 \times 10^6\ \text{m})^2}$$

$$= 1.30 \times 10^3\ \text{N}$$

The acceleration due to gravity on Mars is

$$g_M = \frac{F_g}{m} = \frac{1.30 \times 10^3\ \text{N}}{350\ \text{kg}} = 3.7\ \text{m/s}^2$$

EVALUATE: Even though Mars has just 11% of the earth's mass $(6.42 \times 10^{23}\ \text{kg versus } 5.98 \times 10^{24}\ \text{kg})$, the acceleration due to gravity g_M (and hence an object's weight F_g) is roughly 40% as large as on earth. That's because g_M is also inversely proportional to the square of the planet's radius, and Mars has only 53% the radius of earth $(3.40 \times 10^6\ \text{m versus } 6.38 \times 10^6\ \text{m})$.

You can check our result for g_M by using Eq. (13.4), with appropriate replacements. Do you get the same answer?

Test Your Understanding of Section 13.2 Rank the following hypothetical planets in order from highest to lowest value of g at the surface:
(i) mass $=$ 2 times the mass of the earth, radius $=$ 2 times the radius of the earth;
(ii) mass $=$ 4 times the mass of the earth, radius $=$ 4 times the radius of the earth;
(iii) mass $=$ 4 times the mass of the earth, radius $=$ 2 times the radius of the earth;
(iv) mass $=$ 2 times the mass of the earth, radius $=$ 4 times the radius of the earth.

13.3 Gravitational Potential Energy

When we first introduced gravitational potential energy in Section 7.1, we assumed that the gravitational force on a body is constant in magnitude and direction. This led to the expression $U = mgy$. But the earth's gravitational force on a body of mass m at any point outside the earth is given more generally by Eq. (13.2), $F_g = Gm_E m/r^2$, where m_E is the mass of the earth and r is the distance of the body from the earth's center. For problems in which r changes enough that the gravitational force can't be considered constant, we need a more general expression for gravitational potential energy.

To find this expression, we follow the same steps as in Section 7.1. We consider a body of mass m outside the earth, and first compute the work W_{grav} done by the gravitational force when the body moves directly away from or toward the center of the earth from $r = r_1$ to $r = r_2$, as in Fig. 13.10. This work is given by

$$W_{grav} = \int_{r_1}^{r_2} F_r\, dr \qquad (13.6)$$

where F_r is the radial component of the gravitational force $\vec{F}$—that is, the component in the direction *outward* from the center of the earth. Because $\vec{F}$ points directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

$$F_r = -\frac{Gm_E m}{r^2} \qquad (13.7)$$

Substituting Eq. (13.7) into Eq. (13.6), we see that W_{grav} is given by

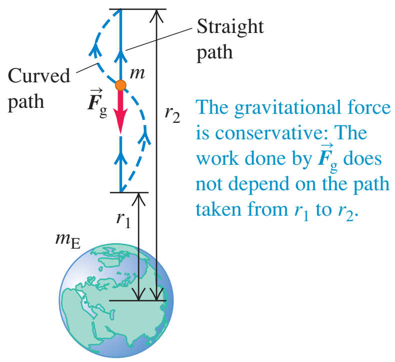
$$W_{grav} = -Gm_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_E m}{r_2} - \frac{Gm_E m}{r_1} \qquad (13.8)$$

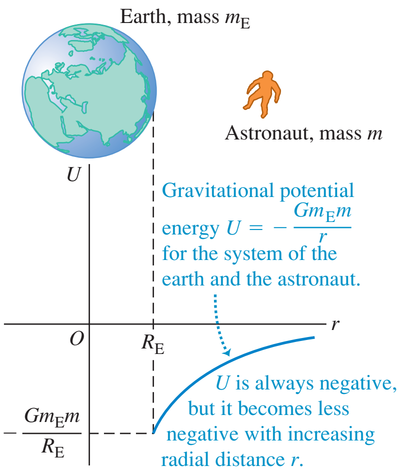
The path doesn't have to be a straight line; it could also be a curve like the one in Fig. 13.10. By an argument similar to that in Section 7.1, this work depends only on the initial and final values of r, not on the path taken. This also proves that the gravitational force is always *conservative.*

We now define the corresponding potential energy U so that $W_{grav} = U_1 - U_2$, as in Eq. (7.3). Comparing this with Eq. (13.8), we see that the appropriate definition for **gravitational potential energy** is

$$U = -\frac{Gm_E m}{r} \quad \text{(gravitational potential energy)} \qquad (13.9)$$

13.10 Calculating the work done on a body by the gravitational force as the body moves from radial coordinate r_1 to r_2.





13.11 A graph of the gravitational potential energy U for the system of the earth (mass m_E) and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.

Figure 13.11 shows how the gravitational potential energy depends on the distance r between the body of mass m and the center of the earth. When the body moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the body "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

You may be troubled by Eq. (13.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula $U = mgy$ in Section 7.1, we found that U was negative whenever the body of mass m was at a value of y below the arbitrary height we chose to be $y = 0$—that is, whenever the body and the earth were closer together than some certain arbitrary distance. (See, for instance, Example 7.2 in Section 7.1.) In defining U by Eq. (13.9), we have chosen U to be zero when the body of mass m is infinitely far from the earth $(r = \infty)$. As the body moves toward the earth, gravitational potential energy decreases and so becomes negative.

If we wanted, we could make $U = 0$ at the surface of the earth, where $r = R_E$, by simply adding the quantity $Gm_E m/R_E$ to Eq. (13.9). This would make U positive when $r > R_E$. We won't do this for two reasons: One, it would make the expression for U more complicated; and two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

CAUTION **Gravitational force vs. gravitational potential energy** Be careful not to confuse the expressions for gravitational force, Eq. (13.7), and gravitational potential energy, Eq. (13.9). The force F_r is proportional to $1/r^2$, while potential energy U is proportional to $1/r$.

Armed with Eq. (13.9), we can now use general energy relationships for problems in which the $1/r^2$ behavior of the earth's gravitational force has to be included. If the gravitational force on the body is the only force that does work, the total mechanical energy of the system is constant, or *conserved.* In the following example we'll use this principle to calculate **escape speed,** the speed required for a body to escape completely from a planet.

Example 13.5 "From the earth to the moon"

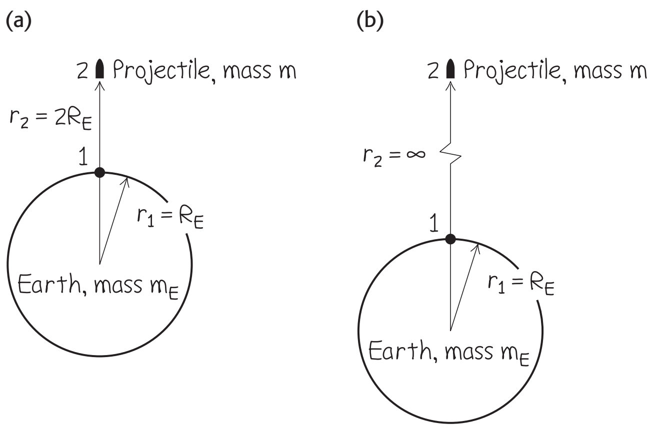
In Jules Verne's 1865 story with this title, three men went to the moon in a shell fired from a giant cannon sunk in the earth in Florida. (a) Find the minimum muzzle speed needed to shoot a shell straight up to a height above the earth equal to the earth's radius R_E. (b) Find the minimum muzzle speed that would allow a shell to escape from the earth completely (the *escape speed*). Neglect air resistance, the earth's rotation, and the gravitational pull of the moon. The earth's radius and mass are $R_E = 6.38 \times 10^6$ m and $m_E = 5.97 \times 10^{24}$ kg.

SOLUTION

IDENTIFY and SET UP: Once the shell leaves the cannon muzzle, only the (conservative) gravitational force does work. Hence we can use conservation of mechanical energy to find the speed at which the shell must leave the muzzle so as to come to a halt (a) at two earth radii from the earth's center and (b) at an infinite distance from earth. The energy-conservation equation is $K_1 + U_1 = K_2 + U_2$, with U given by Eq. (13.9).

Figure 13.12 shows our sketches. Point 1 is at $r_1 = R_E$, where the shell leaves the cannon with speed v_1 (the target variable). Point 2 is where the shell reaches its maximum height; in part (a) $r_2 = 2R_E$ (Fig. 13.12a), and in part (b) $r_2 = \infty$ (Fig 13.12b). In both cases $v_2 = 0$ and $K_2 = 0$. Let m be the mass of the shell (with passengers).

13.12 Our sketches for this problem.



EXECUTE: (a) We solve the energy-conservation equation for v_1:

$$K_1 + U_1 = K_2 + U_2$$

$$\tfrac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + \left(-\frac{Gm_E m}{2R_E}\right)$$

$$v_1 = \sqrt{\frac{Gm_E}{R_E}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

$$= 7900\ \mathrm{m/s}\ (= 28{,}400\ \mathrm{km/h} = 17{,}700\ \mathrm{mi/h})$$

(b) Now $r_2 = \infty$ so $U_2 = 0$ (see Fig. 13.11). Since $K_2 = 0$, the total mechanical energy $K_2 + U_2$ is zero in this case. Again we solve the energy-conservation equation for v_1:

$$\tfrac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + 0$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

$$= 1.12 \times 10^4\ \mathrm{m/s}\ (= 40{,}200\ \mathrm{km/h} = 25{,}000\ \mathrm{mi/h})$$

EVALUATE: Our result in part (b) doesn't depend on the mass of the shell or the direction of launch. A modern spacecraft launched from Florida must attain essentially the speed found in part (b) to escape the earth; however, before launch it's already moving at 410 m/s to the east because of the earth's rotation. Launching to the east takes advantage of this "free" contribution toward escape speed.

To generalize, the initial speed v_1 needed for a body to escape from the surface of a spherical body of mass M and radius R (ignoring air resistance) is $v_1 = \sqrt{2GM/R}$ (escape speed). This equation yields escape speeds of 5.02×10^3 m/s for Mars, 5.95×10^4 m/s for Jupiter, and 6.18×10^5 m/s for the sun.

More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (13.9) reduces to the familiar $U = mgy$ from Chapter 7. We first rewrite Eq. (13.8) as

$$W_{\mathrm{grav}} = Gm_E m \frac{r_1 - r_2}{r_1 r_2}$$

If the body stays close to the earth, then in the denominator we may replace r_1 and r_2 by R_E, the earth's radius, so

$$W_{\mathrm{grav}} = Gm_E m \frac{r_1 - r_2}{R_E^2}$$

According to Eq. (13.4), $g = Gm_E/R_E^2$, so

$$W_{\mathrm{grav}} = mg(r_1 - r_2)$$

If we replace the r's by y's, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), $U = mgy$, so we may consider Eq. (7.2) for gravitational potential energy to be a special case of the more general Eq. (13.9).

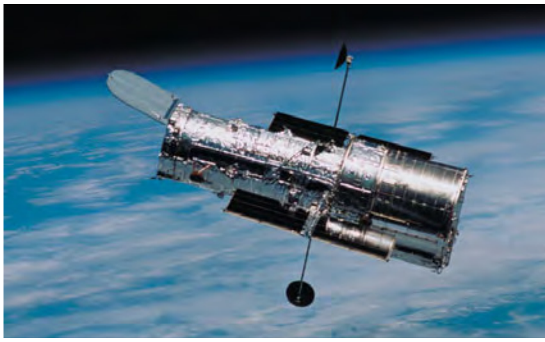
Test Your Understanding of Section 13.3 Is it possible for a planet to have the same surface gravity as the earth (that is, the same value of g at the surface) and yet have a greater escape speed?

13.4 The Motion of Satellites

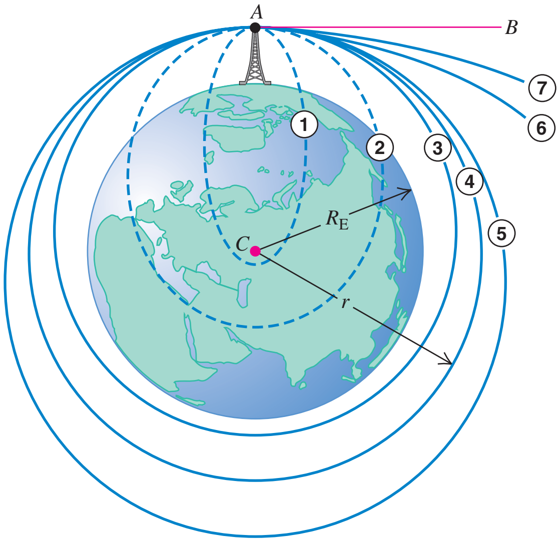
Artificial satellites orbiting the earth are a familiar part of modern technology (Fig. 13.13). But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. We'll see in the next section that the motion of planets can be analyzed in the same way.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his

13.13 With a length of 13.2 m and a mass of 11,000 kg, the Hubble Space Telescope is among the largest satellites placed in orbit.



13.14 Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at C. (This illustration is based on one in Isaac Newton's *Principia.*)



launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 13.14 shows a variation on this theme. We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.

Trajectories 1 through 5 close on themselves and are called **closed orbits.** All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 13.5.) Trajectories 6 and 7 are **open orbits.** For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

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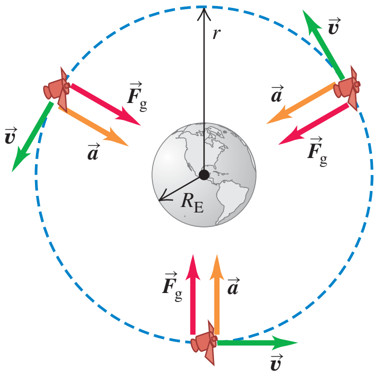
Satellites: Circular Orbits

A *circular* orbit, like trajectory 4 in Fig. 13.14, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (Fig. 13.15). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

13.15 The force $\vec{F}_g$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.28.



The satellite is in a circular orbit: Its acceleration $\vec{a}$ is always perpendicular to its velocity $\vec{v}$, so its speed v is constant.

Let's see how to find the constant speed v of a satellite in a circular orbit. ? The radius of the orbit is r, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{rad} = v^2/r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass m has magnitude $F_g = Gm_Em/r^2$ and is in the same direction as the acceleration. Newton's second law ($\sum\vec{F} = m\vec{a}$) then tells us that

$$\frac{Gm_Em}{r^2} = \frac{mv^2}{r}$$

Solving this for v, we find

$$v = \sqrt{\frac{Gm_E}{r}} \quad \text{(circular orbit)} \tag{13.10}$$

This relationship shows that we can't choose the orbit radius r and the speed v independently; for a given radius r, the speed v for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (13.10), which shows that the motion of a satellite does not depend on its mass. If we could cut a satellite in half without changing its speed, each half would continue on with the original motion. An astronaut on board a space shuttle is herself a satellite of the earth, held by the earth's gravitational attraction in the same orbit as the shuttle. The astronaut has the same velocity and acceleration as the shuttle, so nothing is pushing her against the floor or walls of the shuttle. She is in a state of *apparent weightlessness,* as in a freely falling elevator; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero.) Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders (Fig. 13.16).

13.16 These space shuttle astronauts are in a state of apparent weightlessness. Which are right side up and which are upside down?

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories 6 and 7 in Fig. 13.14.

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

$$v = \frac{2\pi r}{T} \tag{13.11}$$

To get an expression for T, we solve Eq. (13.11) for T and substitute v from Eq. (13.10):

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_{\mathrm{E}}}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_{\mathrm{E}}}} \qquad \text{(circular orbit)} \tag{13.12}$$

Equations (13.10) and (13.12) show that larger orbits correspond to slower speeds and longer periods. As an example, the International Space Station orbits 6800 km from the center of the earth (400 km above the earth's surface) with an orbital speed of 7.7 km/s and an orbital period of 93 minutes. The moon orbits the earth in a much larger orbit of radius 384,000 km, and so has a much slower orbital speed (1.0 km/s) and a much longer orbital period (27.3 days).

It's interesting to compare Eq. (13.10) to the calculation of escape speed in Example 13.5. We see that the escape speed from a spherical body with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

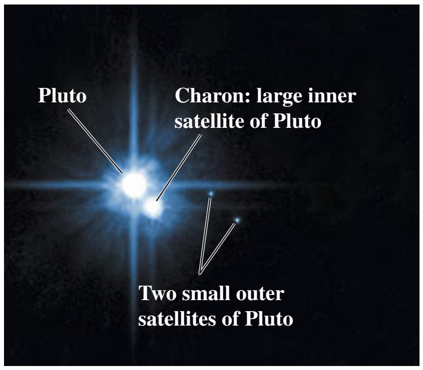
Since the speed v in a circular orbit is determined by Eq. (13.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (13.9) and (13.10), we have

$$E = K + U = \tfrac{1}{2}mv^2 + \left(-\frac{Gm_{\mathrm{E}}m}{r}\right) = \tfrac{1}{2}m\left(\frac{Gm_{\mathrm{E}}}{r}\right) - \frac{Gm_{\mathrm{E}}m}{r}$$
$$= -\frac{Gm_{\mathrm{E}}m}{2r} \qquad \text{(circular orbit)} \tag{13.13}$$

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere.

We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* body under its gravitational attraction to a stationary body. Other examples include the earth's moon and the moons of other worlds (Fig. 13.17).

13.17 The two small satellites of the minor planet Pluto were discovered in 2005. In accordance with Eqs. (13.10) and (13.12), the satellite in the larger orbit has a slower orbital speed and a longer orbital period than the satellite in the smaller orbit.



Example 13.6 A satellite orbit

You wish to put a 1000-kg satellite into a circular orbit 300 km above the earth's surface. (a) What speed, period, and radial acceleration will it have? (b) How much work must be done to the satellite to put it in orbit? (c) How much additional work would have to be done to make the satellite escape the earth? The earth's radius and mass are given in Example 13.5 (Section 13.3).

SOLUTION

IDENTIFY and SET UP: The satellite is in a circular orbit, so we can use the equations derived in this section. In part (a), we first find the radius r of the satellite's orbit from its altitude. We then calculate the speed v and period T using Eqs. (13.10) and (13.12) and the acceleration from $a_{rad} = v^2/r$. In parts (b) and (c), the work required is the difference between the initial and final mechanical energy, which for a circular orbit is given by Eq. (13.13).

EXECUTE: (a) The radius of the satellite's orbit is $r = 6380\text{ km} + 300\text{ km} = 6680\text{ km} = 6.68 \times 10^6\text{ m}$. From Eq. (13.10), the orbital speed is

$$v = \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.68 \times 10^6\text{ m}}}$$
$$= 7720\text{ m/s}$$

We find the orbital period from Eq. (13.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.68 \times 10^6\text{ m})}{7720\text{ m/s}} = 5440\text{ s} = 90.6\text{ min}$$

Finally, the radial acceleration is

$$a_{rad} = \frac{v^2}{r} = \frac{(7720\text{ m/s})^2}{6.68 \times 10^6\text{ m}} = 8.92\text{ m/s}^2$$

This is the value of g at a height of 300 km above the earth's surface; it is about 10% less than the value of g at the surface.

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the total mechanical energy when the satellite was at rest on the launch pad. From Eq. (13.13), the energy in orbit is

$$E_2 = -\frac{Gm_E m}{2r}$$
$$= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{2(6.68 \times 10^6\text{ m})}$$
$$= -2.98 \times 10^{10}\text{ J}$$

The satellite's kinetic energy is zero on the launch pad ($r = R_E$), so

$$E_1 = K_1 + U_1 = 0 + \left(-\frac{Gm_E m}{R_E}\right)$$
$$= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{6.38 \times 10^6\text{ m}}$$
$$= -6.24 \times 10^{10}\text{ J}$$

Hence the work required is

$$W_{required} = E_2 - E_1 = (-2.98 \times 10^{10}\text{ J}) - (-6.24 \times 10^{10}\text{ J})$$
$$= 3.26 \times 10^{10}\text{ J}$$

(c) We saw in part (b) of Example 13.5 that the minimum total mechanical energy for a satellite to escape to infinity is zero. Here, the total mechanical energy in the circular orbit is $E_2 = -2.98 \times 10^{10}$ J; to increase this to zero, an amount of work equal to 2.98×10^{10} J would have to be done on the satellite, presumably by rocket engines attached to it.

EVALUATE: In part (b) we ignored the satellite's initial kinetic energy (while it was still on the launch pad) due to the rotation of the earth. How much difference does this make? (See Example 13.5 for useful data.)

Test Your Understanding of Section 13.4 Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the orbital radius to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

13.5 Kepler's Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning "wanderer," and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to precisely determine their orbits.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using a voluminous set of precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. **A line from the sun to a given planet sweeps out equal areas in equal times.**
3. **The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.**

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived;* they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

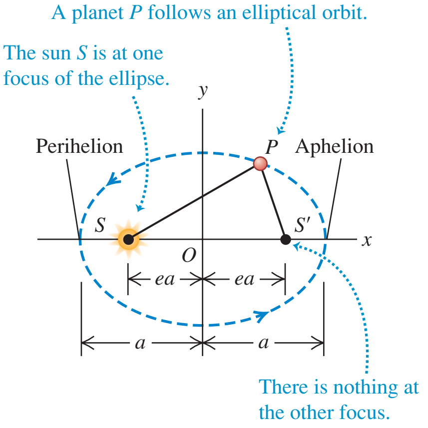
Kepler's First Law

First consider the elliptical orbits described in Kepler's first law. Figure 13.18 shows the geometry of an ellipse. The longest dimension is the *major axis,* with half-length a; this half-length is called the **semi-major axis.** The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S, and the planet is at P; we think of them both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus S'.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity.** If $e = 0$, the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$.) The point in the planet's orbit closest to the sun is the *perihelion,* and the point most distant from the sun is the *aphelion.*

Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 13.14) must be parabolas or hyperbolas. These results can be derived by a straightforward application of Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

13.18 Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.



Kepler's Second Law

Figure 13.19 shows Kepler's second law. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the colored triangle with height r, base length $r\,d\theta$, and area $dA = \frac{1}{2}r^2\,d\theta$ in Fig. 13.19b. The rate at which area is swept out, dA/dt, is called the *sector velocity:*

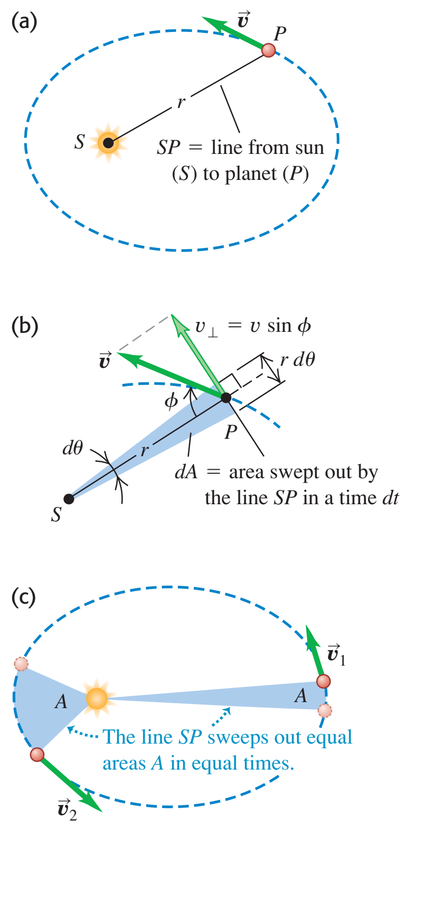
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \qquad (13.14)$$

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

To see how Kepler's second law follows from Newton's laws, we express dA/dt in terms of the velocity vector $\vec{v}$ of the planet P. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v\sin\phi$. From Fig. 13.19b the displacement along the direction of $v_\perp$ during time dt is $r\,d\theta$, so we also have $v_\perp = r\,d\theta/dt$. Using this relationship in Eq. (13.14), we find

$$\frac{dA}{dt} = \frac{1}{2}rv\sin\phi \qquad \text{(sector velocity)} \qquad (13.15)$$

13.19 (a) The planet (P) moves about the sun (S) in an elliptical orbit. (b) In a time dt the line SP sweeps out an area $dA = \frac{1}{2}(r\,d\theta)r = \frac{1}{2}r^2\,d\theta$. (c) The planet's speed varies so that the line SP sweeps out the same area A in a given time t regardless of the planet's position in its orbit.



Now $rv \sin\phi$ is the magnitude of the vector product $\vec{r} \times \vec{v}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r} \times m\vec{v}| = \frac{L}{2m} \tag{13.16}$$

Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant. According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

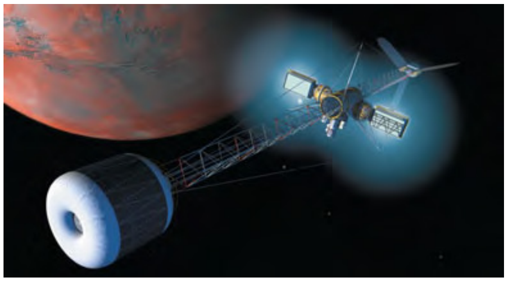
$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun. So these vectors always lie along the same line, and their vector product $\vec{r} \times \vec{F}$ is zero. Hence $d\vec{L}/dt = \mathbf{0}$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force.* (Kepler's first and third laws are valid *only* for a $1/r^2$ force.)

Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{v}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{v}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is just the plane of the planet's orbit.

Application Biological Hazards of Interplanetary Travel

A spacecraft sent from earth to another planet spends most of its journey coasting along an elliptical orbit with the sun at one focus. Rockets are used only at the start and end of the journey, and even the trip to a nearby planet like Mars takes several months. During its journey, the spacecraft is exposed to cosmic rays—radiation that emanates from elsewhere in our galaxy. (On earth we're shielded from this radiation by our planet's magnetic field, as we'll describe in Chapter 27.) This poses no problem for a robotic spacecraft, but would be a severe medical hazard for astronauts undertaking such a voyage.



Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (13.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius r replaced by the semi-major axis a:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} \quad \text{(elliptical orbit around the sun)} \tag{13.17}$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_E in Eq. (13.12) with the sun's mass m_S. Note that the period does not depend on the eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 13.19c), while the planet's speed is constant around its circular orbit.

Conceptual Example 13.7 Orbital speeds

At what point in an elliptical orbit (see Fig. 13.19) does a planet move the fastest? The slowest?

SOLUTION

Mechanical energy is conserved as a planet moves in its orbit. The planet's kinetic energy $K = \frac{1}{2}mv^2$ is maximum when the potential energy $U = -Gm_Sm/r$ is minimum (that is, most negative; see Fig. 13.11), which occurs when the sun–planet distance r is a minimum. Hence the speed v is greatest at perihelion. Similarly, K is minimum when r is maximum, so the speed is slowest at aphelion.

Your intuition about falling bodies is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. The planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

Example 13.8 Kepler's third law

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

SOLUTION

IDENTIFY and SET UP: This example uses Kepler's third law, which relates the period T and the semi-major axis a for an orbiting object (such as an asteroid). We use Eq. (13.17) to determine a; from Appendix F we have $m_S = 1.99 \times 10^{30}$ kg, and a conversion factor from Appendix E gives $T = (4.62\text{ yr})(3.156 \times 10^7\text{ s/yr}) = 1.46 \times 10^8$ s. Note that we don't need the value of the eccentricity.

EXECUTE: From Eq. (13.17), $a^{3/2} = [(Gm_S)^{1/2}T]/2\pi$. To solve for a, we raise both sides of this expression to the $\frac{2}{3}$ power and then substitute the values of G, m_S, and T:

$$a = \left(\frac{Gm_S T^2}{4\pi^2}\right)^{1/3} = 4.15 \times 10^{11}\text{ m}$$

(Plug in the numbers yourself to check.)

EVALUATE: Our result is intermediate between the semi-major axes of Mars and Jupiter (see Appendix F). Most known asteroids orbit in an "asteroid belt" between the orbits of these two planets.

Example 13.9 Comet Halley

Comet Halley moves in an elongated elliptical orbit around the sun (Fig. 13.20). Its distances from the sun at perihelion and aphelion are 8.75×10^7 km and 5.26×10^9 km, respectively. Find the orbital semi-major axis, eccentricity, and period.

SOLUTION

IDENTIFY and SET UP: We are to find the semi-major axis a, eccentricity e, and orbital period T. We can use Fig. 13.18 to find a and e from the given perihelion and aphelion distances. Knowing a, we can find T from Kepler's third law, Eq. (13.17).

EXECUTE: From Fig. 13.18, the length $2a$ of the major axis equals the sum of the comet–sun distance at perihelion and the comet–sun distance at aphelion. Hence

$$a = \frac{(8.75 \times 10^7\text{ km}) + (5.26 \times 10^9\text{ km})}{2} = 2.67 \times 10^9\text{ km}$$

Figure 13.19 also shows that the comet–sun distance at perihelion is $a - ea = a(1 - e)$. This distance is 8.75×10^7 km, so

$$e = 1 - \frac{8.75 \times 10^7\text{ km}}{a} = 1 - \frac{8.75 \times 10^7\text{ km}}{2.67 \times 10^9\text{ km}} = 0.967$$

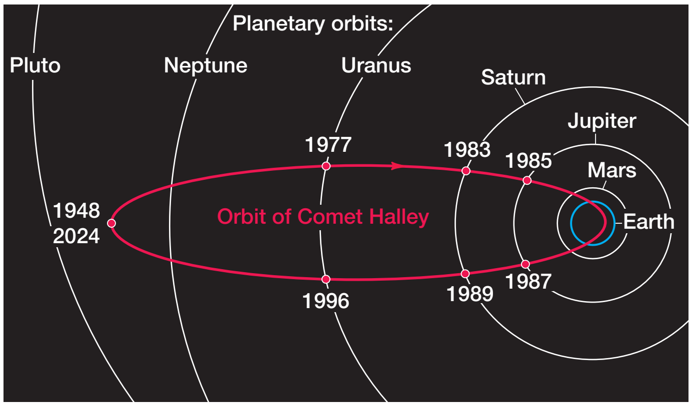
From Eq. (13.17), the period is

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} = \frac{2\pi(2.67 \times 10^{12}\text{ m})^{3/2}}{\sqrt{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\text{ kg})}}$$
$$= 2.38 \times 10^9\text{ s} = 75.5\text{ years}$$

EVALUATE: The eccentricity is close to 1, so the orbit is very elongated (see Fig. 13.20a). Comet Halley was at perihelion in early 1986 (Fig. 13.20b); it will next reach perihelion one period later, in 2061.

13.20 (a) The orbit of Comet Halley. (b) Comet Halley as it appeared in 1986. At the heart of the comet is an icy body, called the nucleus, that is about 10 km across. When the comet's orbit carries it close to the sun, the heat of sunlight causes the nucleus to partially evaporate. The evaporated material forms the tail, which can be tens of millions of kilometers long.

(a)

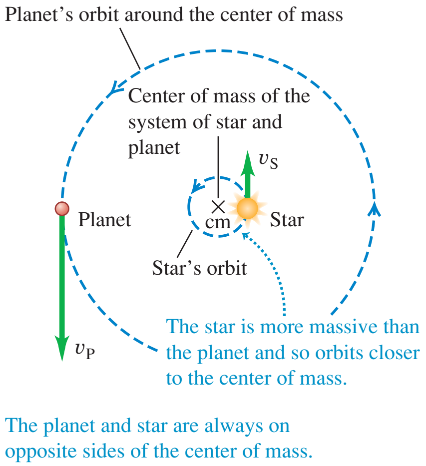


(b)

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. Of course, this can't be correct; because the sun exerts a

13.21 A star and its planet both orbit about their common center of mass.



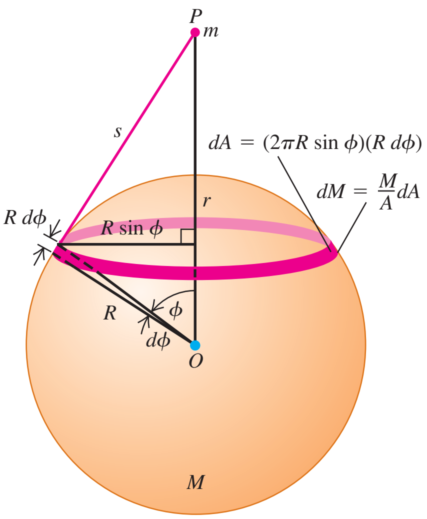
gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common center of mass (Fig. 13.21). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly.) By analyzing these "wobbles," astronomers have discovered planets in orbit around hundreds of other stars.

Newton's analysis of planetary motions is used on a daily basis by modern-day astronomers. But the most remarkable result of Newton's work is that the motions of bodies in the heavens obey the *same* laws of motion as do bodies on the earth. This *Newtonian synthesis,* as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe—not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

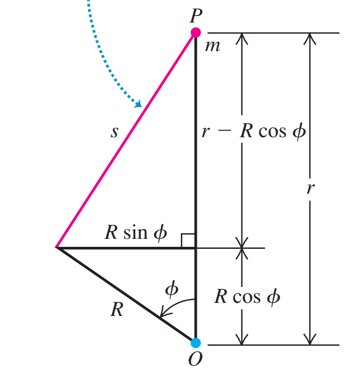
Test Your Understanding of Section 13.5 The orbit of Comet X has a semi-major axis that is four times longer than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y? (i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

13.22 Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell.

(a) Geometry of the situation



(b) The distance s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$.



13.6 Spherical Mass Distributions

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

Here's our program. Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass m interacting with a thin spherical shell with total mass M. We will show that when m is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though M were all concentrated at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on m is also the same as for a point mass M. Any spherically symmetric mass distribution can be thought of as being made up of many concentric spherical shells, so our result will also hold for *any* spherically symmetric M.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of the shell (Fig. 13.22a), centered on the line from the center of the shell to m. We do this because all of the particles that make up the ring are the same distance s from the point mass m. From Eq. (13.9) the potential energy of interaction between the earth (mass m_E) and a point mass m, separated by a distance r, is $U = -Gm_E m/r$. By changing notation in this expression, we see that the potential energy of interaction between the point mass m and a particle of mass m_i within the ring is given by

$$U_i = -\frac{Gmm_i}{s}$$

To find the potential energy of interaction between m and the entire ring of mass $dM = \sum_i m_i$, we sum this expression for U_i over all particles in the ring. Calling this potential energy dU, we find

$$dU = \sum_i U_i = \sum_i \left(-\frac{Gmm_i}{s} \right) = -\frac{Gm}{s} \sum_i m_i = -\frac{Gm\,dM}{s} \tag{13.18}$$

To proceed, we need to know the mass dM of the ring. We can find this with the aid of a little geometry. The radius of the shell is R, so in terms of the angle ϕ shown in the figure, the radius of the ring is $R\sin\phi$, and its circumference is $2\pi R\sin\phi$. The width of the ring is $R\,d\phi$, and its area dA is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin\phi\,d\phi$$

The ratio of the ring mass dM to the total mass M of the shell is equal to the ratio of the area dA of the ring to the total area $A = 4\pi R^2$ of the shell:

$$\frac{dM}{M} = \frac{2\pi R^2 \sin\phi\,d\phi}{4\pi R^2} = \tfrac{1}{2}\sin\phi\,d\phi \tag{13.19}$$

Now we solve Eq. (13.19) for dM and substitute the result into Eq. (13.18) to find the potential energy of interaction between the point mass m and the ring:

$$dU = -\frac{GMm\sin\phi\,d\phi}{2s} \tag{13.20}$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (13.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π!) and s varies from $r - R$ to $r + R$. To carry out the integration, we have to express the integrand in terms of a single variable; we choose s. To express ϕ and $d\phi$ in terms of s, we have to do a little more geometry. Figure 13.22b shows that s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$, so the Pythagorean theorem gives

$$\begin{aligned} s^2 &= (r - R\cos\phi)^2 + (R\sin\phi)^2 \\ &= r^2 - 2rR\cos\phi + R^2 \end{aligned} \tag{13.21}$$

We take differentials of both sides:

$$2s\,ds = 2rR\sin\phi\,d\phi$$

Next we divide this by $2rR$ and substitute the result into Eq. (13.20):

$$dU = -\frac{GMm}{2s}\frac{s\,ds}{rR} = -\frac{GMm}{2rR}\,ds \tag{13.22}$$

We can now integrate Eq. (13.22), recalling that s varies from $r - R$ to $r + R$:

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}\big[(r + R) - (r - R)\big] \tag{13.23}$$

Finally, we have

$$U = -\frac{GMm}{r} \qquad \text{(point mass } m \text{ outside spherical shell } M\text{)} \tag{13.24}$$

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of the spherical shell M and the point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

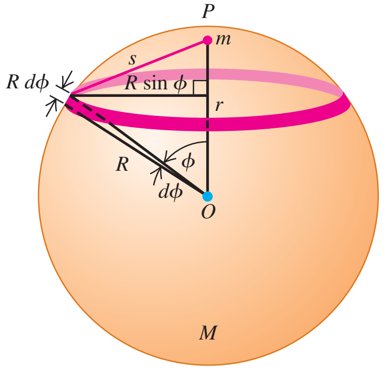
The Gravitational Force Between Spherical Mass Distributions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though they were both points. That's easier. In Fig. 13.22a the forces the two bodies exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts *on* the sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

13.23 When a point mass m is *inside* a uniform spherical shell of mass M, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero.



We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in Fig. 13.23. The entire analysis goes just as before; Eqs. (13.18) through (13.22) are still valid. But when we get to Eq. (13.23), the limits of integration have to be changed to $R - r$ and $R + r$. We then have

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}[(R + r) - (R - r)] \qquad (13.25)$$

and the final result is

$$U = -\frac{GMm}{R} \qquad \text{(point mass } m \text{ inside spherical shell } M) \qquad (13.26)$$

Compare this result to Eq. (13.24): Instead of having r, the distance between m and the center of M, in the denominator, we have R, the radius of the shell. This means that U in Eq. (13.26) doesn't depend on r and thus has the same value everywhere inside the shell. When m moves around inside the shell, no work is done on it, so the force on m at any point inside the shell must be zero.

More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance r from its center, the gravitational force on a point mass m is the same as though we removed all the mass at points farther than r from the center and concentrated all the remaining mass at the center.

Example 13.10 "Journey to the center of the earth"

Imagine that we drill a hole through the earth along a diameter and drop a mail pouch down the hole. Derive an expression for the gravitational force F_g on the pouch as a function of its distance from the earth's center. Assume that the earth's density is uniform (not a very realistic model; see Fig. 13.9).

SOLUTION

IDENTIFY and SET UP: From the discussion immediately above, the value of F_g at a distance r from the earth's center is determined only by the mass M within a spherical region of radius r (Fig. 13.24). Hence F_g is the same as if all the mass within radius r were concentrated at the center of the earth. The mass of a uniform sphere is proportional to the volume of the sphere, which is $\frac{4}{3}\pi r^3$ for a sphere of arbitrary radius r and $\frac{4}{3}\pi R_E^3$ for the entire earth.

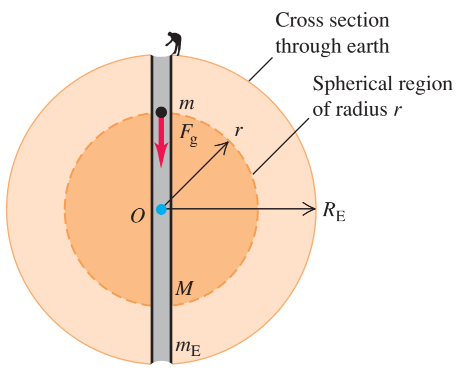
EXECUTE: The ratio of the mass M of the sphere of radius r to the mass m_E of the earth is

$$\frac{M}{m_E} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3} = \frac{r^3}{R_E^3} \qquad \text{so} \qquad M = m_E\frac{r^3}{R_E^3}$$

The magnitude of the gravitational force on m is then

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left(m_E \frac{r^3}{R_E^3}\right) = \frac{Gm_E m}{R_E^3} r$$

EVALUATE: Inside this uniform-density sphere, F_g is *directly proportional* to the distance r from the center, rather than to $1/r^2$ as it is outside the sphere. At the surface $r = R_E$, we have $F_g = Gm_E m/R_E^2$, as we should. In the next chapter we'll learn how to compute the time it would take for the mail pouch to emerge on the other side of the earth.

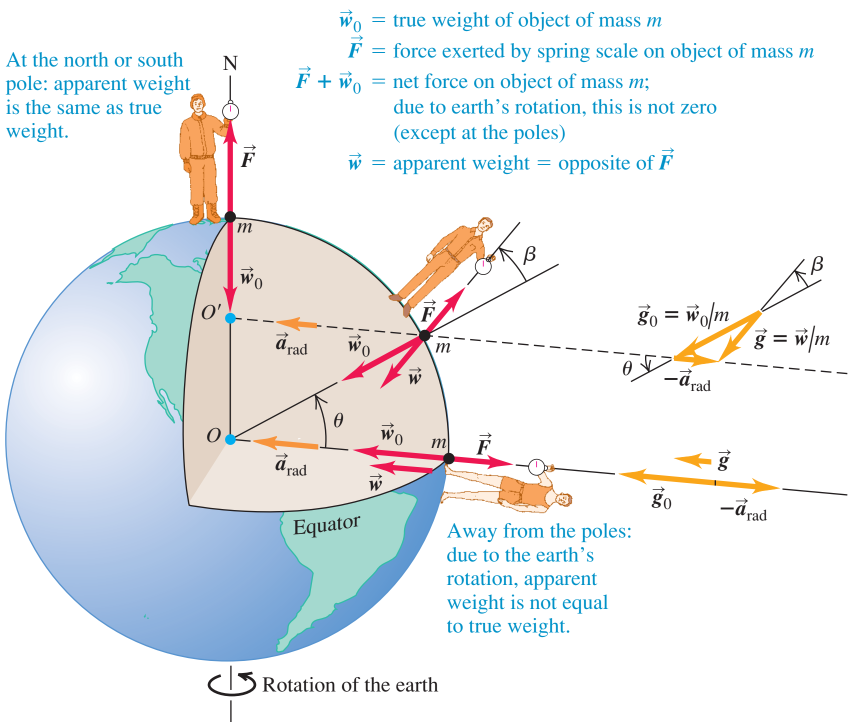


13.24 A hole through the center of the earth (assumed to be uniform). When an object is a distance r from the center, only the mass inside a sphere of radius r exerts a net gravitational force on it.

Test Your Understanding of Section 13.6 In the classic 1913 science-fiction novel *At the Earth's Core* by Edgar Rice Burroughs, explorers discover that the earth is a hollow sphere and that an entire civilization lives on the inside of the sphere. Would it be possible to stand and walk on the inner surface of a hollow, nonrotating planet?

13.7 Apparent Weight and the Earth's Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of a body on earth is not precisely equal to the earth's gravitational attraction, which we will call the **true weight** $\vec{w}_0$ of the body. Figure 13.25 is a cutaway view of the earth, showing three observers. Each one holds a spring scale with a body of mass m hanging from it. Each scale applies a tension force $\vec{F}$ to the body hanging from it, and the reading on each scale is the magnitude F of this force. If the observers are unaware of the earth's



13.25 Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors.

rotation, each one *thinks* that the scale reading equals the weight of the body because he thinks the body on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight.** But if the bodies are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{w}_0$.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude $Gm_E m/R_E^2$, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the body at the north pole really *is* in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the body at the equator is moving in a circle of radius R_E with speed v, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{mv^2}{R_E}$$

So the magnitude of the apparent weight (equal to the magnitude of F) is

$$w = w_0 - \frac{mv^2}{R_E} \quad \text{(at the equator)} \tag{13.27}$$

If the earth were not rotating, the body when released would have a free-fall acceleration $g_0 = w_0/m$. Since the earth *is* rotating, the falling body's actual acceleration relative to the observer at the equator is $g = w/m$. Dividing Eq. (13.27) by m and using these relationships, we find

$$g = g_0 - \frac{v^2}{R_E} \quad \text{(at the equator)}$$

To evaluate v^2/R_E, we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_E = 2\pi(6.38 \times 10^6\ \text{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun.) Thus we find

$$v = \frac{2\pi(6.38 \times 10^6\ \text{m})}{86{,}164\ \text{s}} = 465\ \text{m/s}$$

$$\frac{v^2}{R_E} = \frac{(465\ \text{m/s})^2}{6.38 \times 10^6\ \text{m}} = 0.0339\ \text{m/s}^2$$

So for a spherically symmetric earth the acceleration due to gravity should be about 0.03 m/s^2 less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight $\vec{w}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (13.27). From Fig. 13.25 we see that the appropriate equation is

$$\vec{w} = \vec{w}_0 - m\vec{a}_{\text{rad}} = m\vec{g}_0 - m\vec{a}_{\text{rad}} \tag{13.28}$$

The difference in the magnitudes of g and g_0 lies between zero and 0.0339 m/s^2. As shown in Fig. 13.25, the *direction* of the apparent weight differs from the direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 13.1 gives the values of g at several locations, showing variations with latitude. There are also small additional variations due to the lack of perfect spherical symmetry of the earth, local variations in density, and differences in elevation.

Table 13.1 Variations of g with Latitude and Elevation

Station	North Latitude	Elevation (m)	$g(\text{m/s}^2)$
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

Test Your Understanding of Section 13.7 Imagine a planet that has the same mass and radius as the earth, but that makes 10 rotations during the time the earth makes one rotation. What would be the difference between the acceleration due to gravity at the planet's equator and the acceleration due to gravity at its poles? (i) $0.00339\ \text{m/s}^2$; (ii) $0.0339\ \text{m/s}^2$; (iii) $0.339\ \text{m/s}^2$; (iv) $3.39\ \text{m/s}^2$.

13.8 Black Holes

The concept of a black hole is one of the most interesting and startling products of modern gravitational theory, yet the basic idea can be understood on the basis of Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass $M = 1.99 \times 10^{30}$ kg and radius $R = 6.96 \times 10^8$ m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 13.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(6.96 \times 10^8\ \text{m})^3} = 1410\ \text{kg/m}^3$$

The sun's temperatures range from 5800 K (about 5500°C or 10,000°F) at the surface up to 1.5×10^7 K (about 2.7×10^7°F) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for a body at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass M with radius R is $v = \sqrt{2GM/R}$. We can relate this to the average density. Substituting $M = \rho V = \rho(\frac{4}{3}\pi R^3)$ into the expression for escape speed gives

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R \tag{13.29}$$

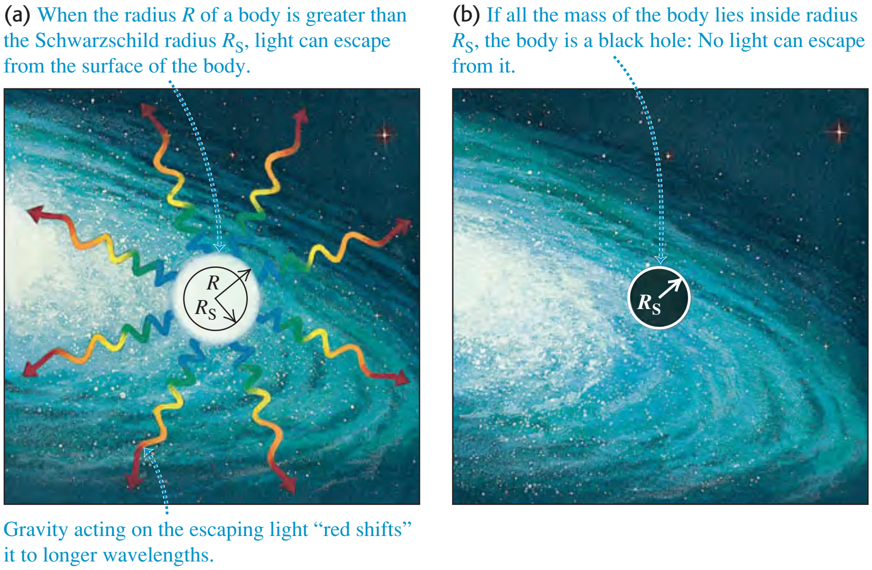
Using either form of this equation, you can show that the escape speed for a body at the surface of our sun is $v = 6.18 \times 10^5$ m/s (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly $\frac{1}{500}$ the speed of light, is independent of the mass of the escaping body; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii R. Equation (13.29) shows that for a given value of density ρ, the escape speed v is directly proportional to R. In 1783 the Rev. John Mitchell, an amateur astronomer, noted that if a body with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light c. With his statement that "all light emitted from such a body would be made to return toward it," Mitchell became the first person to suggest the existence of what we now call a **black hole**—an object that exerts a gravitational force on other bodies but cannot emit any light of its own.

Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (13.29) suggests that a body of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $v = c$ in Eq. (13.29). As a matter of fact, this does give the correct result, but only because of two compensating errors.

13.26 (a) A body with a radius R greater than the Schwarzschild radius R_S. (b) If the body collapses to a radius smaller than R_S, it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole.



The kinetic energy of light is *not* $mc^2/2$, and the gravitational potential energy near a black hole is *not* given by Eq. (13.9). In 1916, Karl Schwarzschild used Einstein's general theory of relativity (in part a generalization and extension of Newtonian gravitation theory) to derive an expression for the critical radius R_S, now called the **Schwarzschild radius.** The result turns out to be the same as though we had set $v = c$ in Eq. (13.29), so

$$c = \sqrt{\frac{2GM}{R_S}}$$

Solving for the Schwarzschild radius R_S, we find

$$R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)} \tag{13.30}$$

If a spherical, nonrotating body with mass M has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the body, and the body is a black hole (Fig. 13.26). In this case, any other body within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon:** Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the body is irretrievably lost when it collapses inside its event horizon.

Example 13.11 Black hole calculations

Astrophysical theory suggests that a burned-out star whose mass is at least three solar masses will collapse under its own gravity to form a black hole. If it does, what is the radius of its event horizon?

SOLUTION

IDENTIFY, SET UP, and EXECUTE: The radius in question is the Schwarzschild radius. We use Eq. (13.30) with a value of M equal to three solar masses, or $M = 3(1.99 \times 10^{30}\ \text{kg}) = 6.0 \times 10^{30}\ \text{kg}$:

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(6.0 \times 10^{30}\ \text{kg})}{(3.00 \times 10^8\ \text{m/s})^2}$$

$$= 8.9 \times 10^3\ \text{m} = 8.9\ \text{km} = 5.5\ \text{mi}$$

EVALUATE: The average density of such an object is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{6.0 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(8.9 \times 10^3\ \text{m})^3} = 2.0 \times 10^{18}\ \text{kg/m}^3$$

This is about 10^{15} times as great as the density of familiar matter on earth and is comparable to the densities of atomic nuclei. In fact, once the body collapses to a radius of R_S, nothing can prevent it from collapsing further. All of the mass ends up being crushed down to a single point called a *singularity* at the center of the event horizon. This point has zero volume and so has *infinite* density.

A Visit to a Black Hole

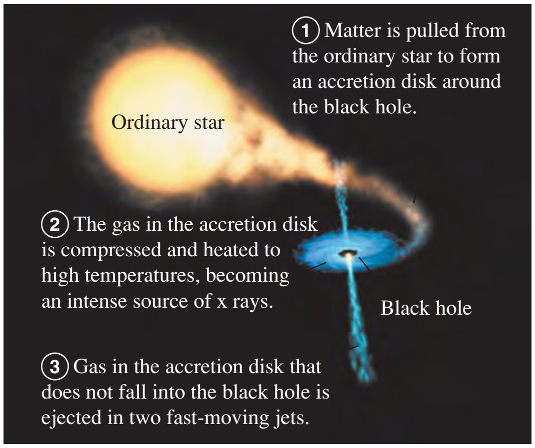
At points far from a black hole, its gravitational effects are the same as those of any normal body with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift.* Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation.* In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Detecting Black Holes

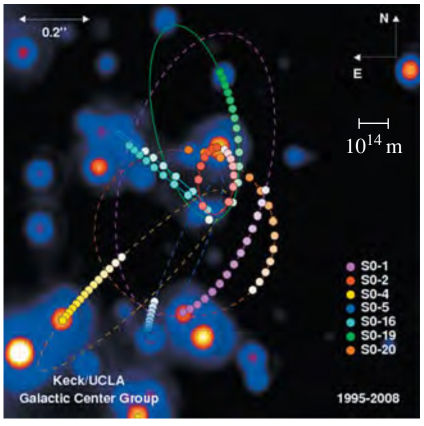
If light cannot escape from a black hole and if black holes are as small as Example 13.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpool (Fig. 13.27). Friction within the accretion disk's material causes it to lose mechanical energy



13.27 A binary star system in which an ordinary star and a black hole orbit each other. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected.

and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the material, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 10^6 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do bodies that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

13.28 This false-color image shows the motions of stars at the center of our galaxy over a 13-year period. Analyzing these orbits using Kepler's third law indicates that the stars are moving about an unseen object that is some 4.1×10^6 times the mass of the sun. The scale bar indicates a length of 10^{14} m (670 times the distance from the earth to the sun) at the distance of the galactic center.



Black holes in binary star systems like the one depicted in Fig. 13.27 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes*. One example is thought to lie at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio waves called Sgr A* (Fig. 13.28). By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can then be calculated using Kepler's third law in the form given in Eq. (13.17), with the mass of the sun m_S replaced by m_X:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \quad \text{so} \quad m_X = \frac{4\pi^2 a^3}{GT^2}$$

The conclusion is that the mysterious dark object at the galactic center has a mass of 8.2×10^{36} kg, or 4.1 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than 4.4×10^{10} m, about one-third of the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

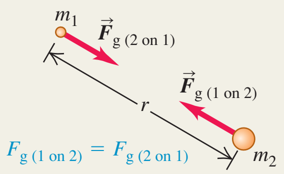
Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

Test Your Understanding of Section 13.8 If the sun somehow collapsed to form a black hole, what effect would this event have on the orbit of the earth? (i) The orbit would shrink; (ii) the orbit would expand; (iii) the orbit would remain the same size.

CHAPTER 13 SUMMARY

Newton's law of gravitation: *Any* two bodies with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more bodies exert gravitational forces on a particular body, the total gravitational force on that individual body is the vector sum of the forces exerted by the other bodies. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center. (See Examples 13.1–13.3 and 13.10.)

$$F_g = \frac{Gm_1m_2}{r^2} \qquad (13.1)$$



Gravitational force, weight, and gravitational potential energy: The weight w of a body is the total gravitational force exerted on it by all other bodies in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two bodies are infinitely far apart. (See Examples 13.4 and 13.5.)

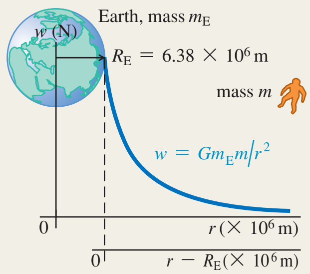
$$w = F_g = \frac{Gm_Em}{R_E^2} \qquad (13.3)$$

(weight at earth's surface)

$$g = \frac{Gm_E}{R_E^2} \qquad (13.4)$$

(acceleration due to gravity at earth's surface)

$$U = -\frac{Gm_Em}{r} \qquad (13.9)$$



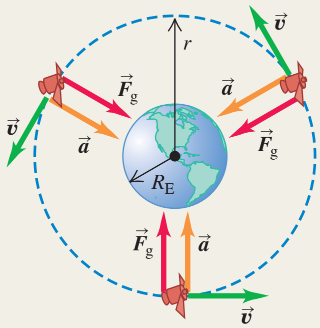
Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet. (See Examples 13.6–13.9.)

$$v = \sqrt{\frac{Gm_E}{r}}$$

(speed in circular orbit) (13.10)

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$$

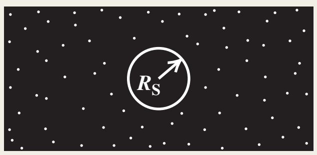
(period in circular orbit) (13.12)



Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S. (See Example 13.11.)

$$R_S = \frac{2GM}{c^2} \qquad (13.30)$$

(Schwarzschild radius)



If all of the body is inside its Schwarzschild radius $R_S = 2GM/c^2$, the body is a black hole.

BRIDGING PROBLEM Speeds in an Elliptical Orbit

A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. (a) Find expressions for the speeds of the comet at perihelion and aphelion. (b) Evaluate these expressions for Comet Halley (see Example 13.9).

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Sketch the situation; show all relevant dimensions. Label the perihelion and aphelion.
2. List the unknown quantities, and identify the target variables.
3. Just as for a satellite orbiting the earth, the mechanical energy is conserved for a comet orbiting the sun. (Why?) What other quantity is conserved as the comet moves in its orbit? (*Hint:* See Section 13.5.)

EXECUTE

4. You'll need at least two equations that involve the two unknown speeds, and you'll need expressions for the sun–comet distances at perihelion and aphelion. (*Hint:* See Fig. 13.18.)
5. Solve the equations for your target variables. Compare your expressions: Which speed is lower? Does this make sense?
6. Use your expressions from step 5 to find the perihelion and aphelion speeds for Comet Halley. (*Hint:* See Appendix F.)

EVALUATE

7. Check whether your results make sense for the special case of a circular orbit ($e = 0$).

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q13.1 A student wrote: "The only reason an apple falls downward to meet the earth instead of the earth rising upward to meet the apple is that the earth is much more massive and so exerts a much greater pull." Please comment.

Q13.2 A planet makes a circular orbit with period T around a star. If it were to orbit, at the same distance, a star with three times the mass of the original star, would the new period (in terms of T) be (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.3 If all planets had the same average density, how would the acceleration due to gravity at the surface of a planet depend on its radius?

Q13.4 Is a pound of butter on the earth the same amount as a pound of butter on Mars? What about a kilogram of butter? Explain.

Q13.5 Example 13.2 (Section 13.1) shows that the acceleration of each sphere caused by the gravitational force is inversely proportional to the mass of that sphere. So why does the force of gravity give all masses the same acceleration when they are dropped near the surface of the earth?

Q13.6 When will you attract the sun more: today at noon, or tonight at midnight? Explain.

Q13.7 Since the moon is constantly attracted toward the earth by the gravitational interaction, why doesn't it crash into the earth?

Q13.8 A planet makes a circular orbit with period T around a star. If the planet were to orbit at the same distance around this star, but had three times as much mass, what would the new period (in terms of T) be: (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.9 The sun pulls on the moon with a force that is more than twice the magnitude of the force with which the earth attracts the moon. Why, then, doesn't the sun take the moon away from the earth?

Q13.10 As defined in Chapter 7, gravitational potential energy is $U = mgy$ and is positive for a body of mass m above the earth's surface (which is at $y = 0$). But in this chapter, gravitational potential energy is $U = -Gm_E m/r$, which is *negative* for a body of mass m above the earth's surface (which is at $r = R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?

Q13.11 A planet is moving at constant speed in a circular orbit around a star. In one complete orbit, what is the net amount of work done on the planet by the star's gravitational force: positive, negative, or zero? What if the planet's orbit is an ellipse, so that the speed is not constant? Explain your answers.

Q13.12 Does the escape speed for an object at the earth's surface depend on the direction in which it is launched? Explain. Does your answer depend on whether or not you include the effects of air resistance?

Q13.13 If a projectile is fired straight up from the earth's surface, what would happen if the total mechanical energy (kinetic plus potential) is (a) less than zero, and (b) greater than zero? In each case, ignore air resistance and the gravitational effects of the sun, the moon, and the other planets.

Q13.14 Discuss whether this statement is correct: "In the absence of air resistance, the trajectory of a projectile thrown near the earth's surface is an *ellipse,* not a parabola."

Q13.15 The earth is closer to the sun in November than in May. In which of these months does it move faster in its orbit? Explain why.

Q13.16 A communications firm wants to place a satellite in orbit so that it is always directly above the earth's 45th parallel (latitude 45° north). This means that the plane of the orbit will not pass through the center of the earth. Is such an orbit possible? Why or why not?

Q13.17 At what point in an elliptical orbit is the acceleration maximum? At what point is it minimum? Justify your answers.

Q13.18 Which takes more fuel: a voyage from the earth to the moon or from the moon to the earth? Explain.

Q13.19 What would Kepler's third law be for circular orbits if an amendment to Newton's law of gravitation made the gravitational force inversely proportional to r^3? Would this change affect Kepler's other two laws? Explain.

Q13.20 In the elliptical orbit of Comet Halley shown in Fig. 13.20a, the sun's gravity is responsible for making the comet fall inward from aphelion to perihelion. But what is responsible for making the comet move from perihelion back outward to aphelion?

Q13.21 Many people believe that orbiting astronauts feel weightless because they are "beyond the pull of the earth's gravity." How far from the earth would a spacecraft have to travel to be truly beyond the earth's gravitational influence? If a spacecraft were really unaffected by the earth's gravity, would it remain in orbit? Explain. What is the real reason astronauts in orbit feel weightless?

Q13.22 As part of their training before going into orbit, astronauts ride in an airliner that is flown along the same parabolic trajectory as a freely falling projectile. Explain why this gives the same experience of apparent weightlessness as being in orbit.

EXERCISES

Section 13.1 Newton's Law of Gravitation

13.1 • What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun?

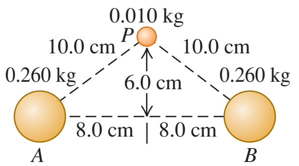
13.2 •• **CP** **Cavendish Experiment.** In the Cavendish balance apparatus shown in Fig. 13.4, suppose that $m_1 = 1.10$ kg, $m_2 = 25.0$ kg, and the rod connecting the m_1 pairs is 30.0 cm long. If, in each pair, m_1 and m_2 are 12.0 cm apart center to center, find (a) the net force and (b) the net torque (about the rotation axis) on the rotating part of the apparatus. (c) Does it seem that the torque in part (b) would be enough to easily rotate the rod? Suggest some ways to improve the sensitivity of this experiment.

13.3 • **Rendezvous in Space!** A couple of astronauts agree to rendezvous in space after hours. Their plan is to let gravity bring them together. One of them has a mass of 65 kg and the other a mass of 72 kg, and they start from rest 20.0 m apart. (a) Make a free-body diagram of each astronaut, and use it to find his or her initial acceleration. As a rough approximation, we can model the astronauts as uniform spheres. (b) If the astronauts' acceleration remained constant, how many days would they have to wait before reaching each other? (Careful! They *both* have acceleration toward each other.) (c) Would their acceleration, in fact, remain constant? If not, would it increase or decrease? Why?

13.4 •• Two uniform spheres, each with mass M and radius R, touch each other. What is the magnitude of their gravitational force of attraction?

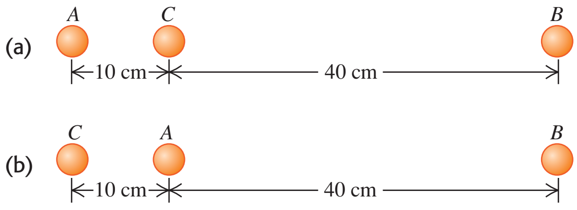
13.5 • Two uniform spheres, each of mass 0.260 kg, are fixed at points A and B (Fig. E13.5). Find the magnitude and direction of the initial acceleration of a uniform sphere with mass 0.010 kg if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

Figure **E13.5**



13.6 •• Find the magnitude and direction of the net gravitational force on mass A due to masses B and C in Fig. E13.6. Each mass is 2.00 kg.

Figure **E13.6**



13.7 • A typical adult human has a mass of about 70 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its center 378,000 km away? (b) Compare this force with the force exerted on the human by the earth.

13.8 •• An 8.00-kg point mass and a 15.0-kg point mass are held in place 50.0 cm apart. A particle of mass m is released from a point between the two masses 20.0 cm from the 8.00-kg mass along the line connecting the two fixed masses. Find the magnitude and direction of the acceleration of the particle.

13.9 •• A particle of mass $3m$ is located 1.00 m from a particle of mass m. (a) Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero? (b) Is the equilibrium of M at this point stable or unstable (i) for points along the line connecting m and $3m$, and (ii) for points along the line passing through M and perpendicular to the line connecting m and $3m$?

13.10 •• The point masses m and $2m$ lie along the x-axis, with m at the origin and $2m$ at $x = L$. A third point mass M is moved along the x-axis. (a) At what point is the net gravitational force on M due to the other two masses equal to zero? (b) Sketch the x-component of the net force on M due to m and $2m$, taking quantities to the right as positive. Include the regions $x < 0$, $0 < x < L$, and $x > L$. Be especially careful to show the behavior of the graph on either side of $x = 0$ and $x = L$.

Section 13.2 Weight

13.11 •• At what distance above the surface of the earth is the acceleration due to the earth's gravity 0.980 m/s^2 if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

13.12 • The mass of Venus is 81.5% that of the earth, and its radius is 94.9% that of the earth. (a) Compute the acceleration due to gravity on the surface of Venus from these data. (b) If a rock weighs 75.0 N on earth, what would it weigh at the surface of Venus?

13.13 • Titania, the largest moon of the planet Uranus, has $\frac{1}{8}$ the radius of the earth and $\frac{1}{1700}$ the mass of the earth. (a) What is the acceleration due to gravity at the surface of Titania? (b) What is the average density of Titania? (This is less than the density of rock, which is one piece of evidence that Titania is made primarily of ice.)

13.14 • Rhea, one of Saturn's moons, has a radius of 765 km and an acceleration due to gravity of 0.278 m/s^2 at its surface. Calculate its mass and average density.

13.15 •• Calculate the earth's gravity force on a 75-kg astronaut who is repairing the Hubble Space Telescope 600 km above the earth's surface, and then compare this value with his weight at the earth's surface. In view of your result, explain why we say astronauts are weightless when they orbit the earth in a satellite such as a space shuttle. Is it because the gravitational pull of the earth is negligibly small?

Section 13.3 Gravitational Potential Energy

13.16 •• **Volcanoes on Io.** Jupiter's moon Io has active volcanoes (in fact, it is the most volcanically active body in the solar system) that eject material as high as 500 km (or even higher) above the surface. Io has a mass of 8.94×10^{22} kg and a radius of 1815 km. Ignore any variation in gravity over the 500-km range of the debris. How high would this material go on earth if it were ejected with the same speed as on Io?

13.17 • Use the results of Example 13.5 (Section 13.3) to calculate the escape speed for a spacecraft (a) from the surface of Mars and (b) from the surface of Jupiter. Use the data in Appendix F. (c) Why is the escape speed for a spacecraft independent of the spacecraft's mass?

13.18 •• Ten days after it was launched toward Mars in December 1998, the *Mars Climate Orbiter* spacecraft (mass 629 kg) was 2.87×10^6 km from the earth and traveling at 1.20×10^4 km/h relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth–spacecraft system?

Section 13.4 The Motion of Satellites

13.19 • For a satellite to be in a circular orbit 780 km above the surface of the earth, (a) what orbital speed must it be given, and (b) what is the period of the orbit (in hours)?

13.20 •• **Aura Mission.** On July 15, 2004, NASA launched the Aura spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface. Assume a circular orbit. (a) How many hours does it take this satellite to make one orbit? (b) How fast (in km/s) is the Aura spacecraft moving?

13.21 •• Two satellites are in circular orbits around a planet that has radius 9.00×10^6 m. One satellite has mass 68.0 kg, orbital radius 5.00×10^7 m, and orbital speed 4800 m/s. The second satellite has mass 84.0 kg and orbital radius 3.00×10^7 m. What is the orbital speed of this second satellite?

13.22 •• **International Space Station.** The International Space Station makes 15.65 revolutions per day in its orbit around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

13.23 • Deimos, a moon of Mars, is about 12 km in diameter with mass 2.0×10^{15} kg. Suppose you are stranded alone on Deimos and want to play a one-person game of baseball. You would be the pitcher, and you would be the batter! (a) With what speed would you have to throw a baseball so that it would go into a circular orbit just above the surface and return to you so you could hit it? Do you think you could actually throw it at this speed? (b) How long (in hours) after throwing the ball should you be ready to hit it? Would this be an action-packed baseball game?

Section 13.5 Kepler's Laws and the Motion of Planets

13.24 •• **Planet Vulcan.** Suppose that a planet were discovered between the sun and Mercury, with a circular orbit of radius equal to $\frac{2}{3}$ of the average orbit radius of Mercury. What would be the orbital period of such a planet? (Such a planet was once postulated, in part to explain the precession of Mercury's orbit. It was even given the name Vulcan, although we now have no evidence that it actually exists. Mercury's precession has been explained by general relativity.)

13.25 •• The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are (a) the orbital speed and (b) the orbital period of the planet of Rho^1 Cancri?

13.26 •• In March 2006, two small satellites were discovered orbiting Pluto, one at a distance of 48,000 km and the other at 64,000 km. Pluto already was known to have a large satellite Charon, orbiting at 19,600 km with an orbital period of 6.39 days. Assuming that the satellites do not affect each other, find the orbital periods of the two small satellites *without* using the mass of Pluto.

13.27 • (a) Use Fig. 13.18 to show that the sun–planet distance at perihelion is $(1 - e)a$, the sun–planet distance at aphelion is $(1 + e)a$, and therefore the sum of these two distances is $2a$. (b) When the dwarf planet Pluto was at perihelion in 1989, it was almost 100 million km closer to the sun than Neptune. The semi-major axes of the orbits of Pluto and Neptune are 5.92×10^{12} m and 4.50×10^{12} m, respectively, and the eccentricities are 0.248 and 0.010. Find Pluto's closest distance and Neptune's farthest distance from the sun. (c) How many years after being at perihelion in 1989 will Pluto again be at perihelion?

13.28 •• **Hot Jupiters.** In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving?

13.29 •• **Planets Beyond the Solar System.** On October 15, 2001, a planet was discovered orbiting around the star HD 68988. Its orbital distance was measured to be 10.5 million kilometers from the center of the star, and its orbital period was estimated at 6.3 days. What is the mass of HD 68988? Express your answer in kilograms and in terms of our sun's mass. (Consult Appendix F.)

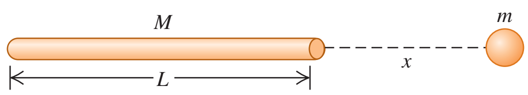
Section 13.6 Spherical Mass Distributions

13.30 • A uniform, spherical, 1000.0-kg shell has a radius of 5.00 m. (a) Find the gravitational force this shell exerts on a 2.00-kg point mass placed at the following distances from the center of the shell: (i) 5.01 m, (ii) 4.99 m, (iii) 2.72 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

13.31 •• A uniform, solid, 1000.0-kg sphere has a radius of 5.00 m. (a) Find the gravitational force this sphere exerts on a 2.00-kg point mass placed at the following distances from the center of the sphere: (i) 5.01 m, (ii) 2.50 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

13.32 • **CALC** A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (Fig. E13.32). (a) Calculate

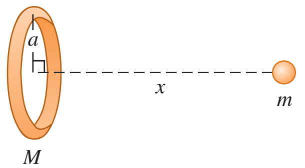
Figure **E13.32**



the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L. (*Hint:* Use the power series expansion for $\ln(1 + x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected result when x is much larger than L.

13.33 • **CALC** Consider the ring-shaped body of Fig. E13.33. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane. (a) Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart. (b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring. (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the particle (see Section 7.4). (d) Show that your answer to part (c) reduces to the expected result when x is much larger than a. (e) What are the values of U and F_x when $x = 0$? Explain why these results make sense.

Figure **E13.33**



Section 13.7 Apparent Weight and the Earth's Rotation

13.34 •• **A Visit to Santa.** You decide to visit Santa Claus at the north pole to put in a good word about your splendid behavior throughout the year. While there, you notice that the elf Sneezy, when hanging from a rope, produces a tension of 475.0 N in the rope. If Sneezy hangs from a similar rope while delivering presents at the earth's equator, what will the tension in it be? (Recall that the earth is rotating about an axis through its north and south poles.) Consult Appendix F and start with a free-body diagram of Sneezy at the equator.

13.35 • The acceleration due to gravity at the north pole of Neptune is approximately 10.7 m/s^2. Neptune has mass 1.0×10^{26} kg and radius 2.5×10^4 km and rotates once around its axis in about 16 h. (a) What is the gravitational force on a 5.0-kg object at the north pole of Neptune? (b) What is the apparent weight of this same object at Neptune's equator? (Note that Neptune's "surface" is gaseous, not solid, so it is impossible to stand on it.)

Section 13.8 Black Holes

13.36 •• **Mini Black Holes.** Cosmologists have speculated that black holes the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be 1.0×10^{-15} m, what would be the mass of a mini black hole?

13.37 •• **At the Galaxy's Core.** Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 13.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

13.38 • (a) Show that a black hole attracts an object of mass m with a force of $mc^2 R_S/(2r^2)$, where r is the distance between the object and the center of the black hole. (b) Calculate the magnitude of the gravitational force exerted by a black hole of Schwarzschild radius 14.0 mm on a 5.00-kg mass 3000 km from it. (c) What is the mass of this black hole?

13.39 • In 2005 astronomers announced the discovery of a large black hole in the galaxy Markarian 766 having clumps of matter orbiting around once every 27 hours and moving at 30,000 km/s. (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? Express your answer in kilograms and as a multiple of our sun's mass. (c) What is the radius of its event horizon?

PROBLEMS

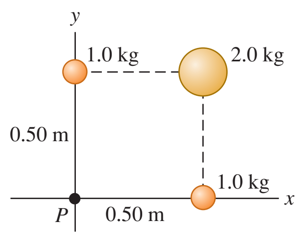
13.40 ••• Four identical masses of 800 kg each are placed at the corners of a square whose side length is 10.0 cm. What is the net gravitational force (magnitude and direction) on one of the masses, due to the other three?

13.41 ••• Neutron stars, such as the one at the center of the Crab Nebula, have about the same mass as our sun but have a *much* smaller diameter. If you weigh 675 N on the earth, what would you weigh at the surface of a neutron star that has the same mass as our sun and a diameter of 20 km?

13.42 ••• **CP Exploring Europa.** There is strong evidence that Europa, a satellite of Jupiter, has a liquid ocean beneath its icy surface. Many scientists think we should land a vehicle there to search for life. Before launching it, we would want to test such a lander under the gravity conditions at the surface of Europa. One way to do this is to put the lander at the end of a rotating arm in an orbiting earth satellite. If the arm is 4.25 m long and pivots about one end, at what angular speed (in rpm) should it spin so that the acceleration of the lander is the same as the acceleration due to gravity at the surface of Europa? The mass of Europa is 4.8×10^{22} kg and its diameter is 3138 km.

13.43 • Three uniform spheres are fixed at the positions shown in Fig. P13.43. (a) What are the magnitude and direction of the force on a 0.0150-kg particle placed at P? (b) If the spheres are in deep outer space and a 0.0150-kg particle is released from rest 300 m from the origin along a line 45° below the $-x$-axis, what will the particle's speed be when it reaches the origin?

Figure **P13.43**



13.44 •• A uniform sphere with mass 60.0 kg is held with its center at the origin, and a second uniform sphere with mass 80.0 kg is held with its center at the point $x = 0$, $y = 3.00$ m. (a) What are the magnitude and direction of the net gravitational force due to these objects on a third uniform sphere with mass 0.500 kg placed at the point $x = 4.00$ m, $y = 0$? (b) Where, other than infinitely far away, could the third sphere be placed such that the net gravitational force acting on it from the other two spheres is equal to zero?

13.45 •• **CP BIO** **Hip Wear on the Moon.** (a) Use data from Appendix F to calculate the acceleration due to gravity on the moon. (b) Calculate the friction force on a walking 65-kg astronaut carrying a 43-kg instrument pack on the moon if the coefficient of kinetic friction at her hip joint is 0.0050. (c) What would be the friction force on earth for this astronaut?

13.46 •• **Mission to Titan.** On December 25, 2004, the *Huygens* probe separated from the *Cassini* spacecraft orbiting Saturn and began a 22-day journey to Saturn's giant moon Titan, on whose surface it landed. Besides the data in Appendix F, it is useful to know that Titan is 1.22×10^6 km from the center of Saturn and has a mass of 1.35×10^{23} kg and a diameter of 5150 km. At what distance from Titan should the gravitational pull of Titan just balance the gravitational pull of Saturn?

13.47 •• The asteroid Toro has a radius of about 5.0 km. Consult Appendix F as necessary. (a) Assuming that the density of Toro is the same as that of the earth ($5.5\ \text{g/cm}^3$), find its total mass and find the acceleration due to gravity at its surface. (b) Suppose an object is to be placed in a circular orbit around Toro, with a radius just slightly larger than the asteroid's radius. What is the speed of the object? Could you launch yourself into orbit around Toro by running?

13.48 ••• At a certain instant, the earth, the moon, and a stationary 1250-kg spacecraft lie at the vertices of an equilateral triangle whose sides are 3.84×10^5 km in length. (a) Find the magnitude and direction of the net gravitational force exerted on the spacecraft by the earth and moon. State the direction as an angle measured from a line connecting the earth and the spacecraft. In a sketch, show the earth, the moon, the spacecraft, and the force vector. (b) What is the minimum amount of work that you would have to do to move the spacecraft to a point far from the earth and moon? You can ignore any gravitational effects due to the other planets or the sun.

13.49 ••• **CP** An experiment is performed in deep space with two uniform spheres, one with mass 50.0 kg and the other with mass 100.0 kg. They have equal radii, $r = 0.20$ m. The spheres are released from rest with their centers 40.0 m apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres. (a) Explain why linear momentum is conserved. (b) When their centers are 20.0 m apart, find (i) the speed of each sphere and (ii) the magnitude of the relative velocity with which one sphere is approaching the other. (c) How far from the initial position of the center of the 50.0-kg sphere do the surfaces of the two spheres collide?

13.50 •• **CP** **Submarines on Europa.** Some scientists are eager to send a remote-controlled submarine to Jupiter's moon Europa to search for life in its oceans below an icy crust. Europa's mass has been measured to be 4.8×10^{22} kg, its diameter is 3138 km, and it has no appreciable atmosphere. Assume that the layer of ice at the surface is not thick enough to exert substantial force on the water. If the windows of the submarine you are designing are 25.0 cm square and can stand a maximum inward force of 9750 N per window, what is the greatest depth to which this submarine can safely dive?

13.51 • **Geosynchronous Satellites.** Many satellites are moving in a circle in the earth's equatorial plane. They are at such a height above the earth's surface that they always remain above the same point. (a) Find the altitude of these satellites above the earth's surface. (Such an orbit is said to be *geosynchronous*.) (b) Explain, with a sketch, why the radio signals from these satellites cannot directly reach receivers on earth that are north of 81.3° N latitude.

13.52 ••• A landing craft with mass 12,500 kg is in a circular orbit 5.75×10^5 m above the surface of a planet. The period of the orbit is 5800 s. The astronauts in the lander measure the diameter of the planet to be 9.60×10^6 m. The lander sets down at the north pole of the planet. What is the weight of an 85.6-kg astronaut as he steps out onto the planet's surface?

13.53 ••• What is the escape speed from a 300-km-diameter asteroid with a density of $2500\ \text{kg/m}^3$?

13.54 •• (a) Asteroids have average densities of about $2500\ \text{kg/m}^3$ and radii from 470 km down to less than a kilometer. Assuming that the asteroid has a spherically symmetric mass distribution, estimate the radius of the largest asteroid from which you could escape simply by jumping off. (*Hint:* You can estimate your jump speed by relating it to the maximum height that you can jump on earth.) (b) Europa, one of Jupiter's four large moons, has a radius of 1570 km. The acceleration due to gravity at its surface is $1.33\ \text{m/s}^2$. Calculate its average density.

13.55 ••• (a) Suppose you are at the earth's equator and observe a satellite passing directly overhead and moving from west to east in the sky. Exactly 12.0 hours later, you again observe this satellite to be directly overhead. How far above the earth's surface is the satellite's orbit? (b) You observe another satellite directly overhead and traveling east to west. This satellite is again overhead in 12.0 hours. How far is this satellite's orbit above the surface of the earth?

13.56 •• Planet X rotates in the same manner as the earth, around an axis through its north and south poles, and is perfectly spherical. An astronaut who weighs 943.0 N on the earth weighs 915.0 N at the north pole of Planet X and only 850.0 N at its equator. The distance from the north pole to the equator is 18,850 km, measured along the surface of Planet X. (a) How long is the day on Planet X? (b) If a 45,000-kg satellite is placed in a circular orbit 2000 km above the surface of Planet X, what will be its orbital period?

13.57 •• There are two equations from which a change in the gravitational potential energy U of the system of a mass m and the earth can be calculated. One is $U = mgy$ (Eq. 7.2). The other is $U = -Gm_{\text{E}}m/r$ (Eq. 13.9). As shown in Section 13.3, the first equation is correct only if the gravitational force is a constant over the change in height Δy. The second is always correct. Actually, the gravitational force is never exactly constant over any change in height, but if the variation is small, we can ignore it. Consider the difference in U between a mass at the earth's surface and a distance h above it using both equations, and find the value of h for which Eq. (7.2) is in error by 1%. Express this value of h as a fraction of the earth's radius, and also obtain a numerical value for it.

13.58 ••• **CP** Your starship, the *Aimless Wanderer,* lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 6.00 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: (a) What is the mass of Mongo? (b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

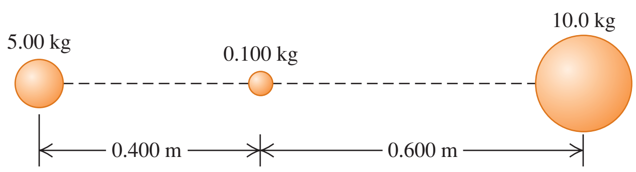
13.59 •• **CP** An astronaut, whose mission is to go where no one has gone before, lands on a spherical planet in a distant galaxy. As she stands on the surface of the planet, she releases a small rock from rest and finds that it takes the rock 0.480 s to fall 1.90 m. If the radius of the planet is 8.60×10^7 m, what is the mass of the planet?

13.60 •• In Example 13.5 (Section 13.3) we ignored the gravitational effects of the moon on a spacecraft en route from the earth to the moon. In fact, we must include the gravitational potential energy due to the moon as well. For this problem, you can ignore the motion of the earth and moon. (a) If the moon has radius R_M and the distance between the centers of the earth and the moon is R_{EM}, find the total gravitational potential energy of the particle–earth and particle–moon systems when a particle with mass m is between the earth and the moon, and a distance r from the center of the earth. Take the gravitational potential energy to be zero when the objects are far from each other. (b) There is a point along a line between the earth and the moon where the net gravitational force is zero. Use the expression derived in part (a) and numerical values from Appendix F to find the distance of this point from the center of the earth. With what speed must a spacecraft be launched from the surface of the earth just barely to reach this point? (c) If a spacecraft were launched from the earth's surface toward the moon with an initial speed of 11.2 km/s, with what speed would it impact the moon?

13.61 •• Calculate the percent difference between your weight in Sacramento, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.

13.62 •• The 0.100-kg sphere in Fig. P13.62 is released from rest at the position shown in the sketch, with its center 0.400 m from the center of the 5.00-kg mass. Assume that the only forces on the 0.100-kg sphere are the gravitational forces exerted by the other two spheres and that the 5.00-kg and 10.0-kg spheres are held in place at their initial positions. What is the speed of the 0.100-kg sphere when it has moved 0.400 m to the right from its initial position?

Figure **P13.62**



13.63 ••• An unmanned spacecraft is in a circular orbit around the moon, observing the lunar surface from an altitude of 50.0 km (see Appendix F). To the dismay of scientists on earth, an electrical fault causes an on-board thruster to fire, decreasing the speed of the spacecraft by 20.0 m/s. If nothing is done to correct its orbit, with what speed (in km/h) will the spacecraft crash into the lunar surface?

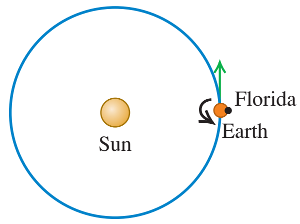
13.64 ••• **Mass of a Comet.** On July 4, 2005, the NASA spacecraft Deep Impact fired a projectile onto the surface of Comet Tempel 1. This comet is about 9.0 km across. Observations of surface debris released by the impact showed that dust with a speed as low as 1.0 m/s was able to escape the comet. (a) Assuming a spherical shape, what is the mass of this comet? (*Hint:* See Example 13.5 in Section 13.3.) (b) How far from the comet's center will this debris be when it has lost (i) 90.0% of its initial kinetic energy at the surface and (ii) all of its kinetic energy at the surface?

13.65 • **Falling Hammer.** A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. If you ignore air resistance, derive an expression for the speed v of the hammer when it reaches the surface of the earth. Your expression should involve h, R_E, and m_E, the mass of the earth.

13.66 • (a) Calculate how much work is required to launch a spacecraft of mass m from the surface of the earth (mass m_E, radius R_E) and place it in a circular *low earth orbit*—that is, an orbit whose altitude above the earth's surface is much less than R_E. (As an example, the International Space Station is in low earth orbit at an altitude of about 400 km, much less than $R_E = 6380$ km.) You can ignore the kinetic energy that the spacecraft has on the ground due to the earth's rotation. (b) Calculate the minimum amount of additional work required to move the spacecraft from low earth orbit to a very great distance from the earth. You can ignore the gravitational effects of the sun, the moon, and the other planets. (c) Justify the statement: "In terms of energy, low earth orbit is halfway to the edge of the universe."

13.67 • A spacecraft is to be launched from the surface of the earth so that it will escape from the solar system altogether. (a) Find the speed relative to the center of the earth with which the spacecraft must be launched. Take into consideration the gravitational effects of both the earth and the sun, and include the effects of the earth's orbital speed, but ignore air resistance. (b) The rotation of the earth can help this spacecraft achieve escape speed. Find the speed that the spacecraft must have relative to the earth's *surface* if the spacecraft is launched from Florida at the point shown in Fig. P13.67. The rotation and orbital motions of the earth are in the same direction. The launch facilities in Florida are 28.5° north of the equator. (c) The European Space Agency (ESA) uses launch facilities in French Guiana (immediately north of Brazil), 5.15° north of the equator. What speed relative to the earth's surface would a spacecraft need to escape the solar system if launched from French Guiana?

Figure **P13.67**



13.68 • **Gravity Inside the Earth.** Find the gravitational force that the earth exerts on a 10.0-kg mass if it is placed at the following locations. Consult Fig. 13.9, and assume a constant density through each of the interior regions (mantle, outer core, inner core), but *not* the same density in each of these regions. Use the graph to estimate the average density for each region: (a) at the surface of the earth; (b) at the outer surface of the molten outer core; (c) at the surface of the solid inner core; (d) at the center of the earth.

13.69 • **Kirkwood Gaps.** Hundreds of thousands of asteroids orbit the sun within the *asteroid belt,* which extends from about 3×10^8 km to about 5×10^8 km from the sun. (a) Find the orbital period (in years) of (i) an asteroid at the inside of the belt and (ii) an asteroid at the outside of the belt. Assume circular orbits. (b) In 1867 the American astronomer Daniel Kirkwood pointed out that several gaps exist in the asteroid belt where relatively few asteroids are found. It is now understood that these *Kirkwood gaps* are caused by the gravitational attraction of Jupiter, the largest planet, which orbits the sun once every 11.86 years. As an example, if an asteroid has an orbital period half that of Jupiter, or 5.93 years, on every other orbit this asteroid would be at its closest to Jupiter and feel a strong attraction toward the planet. This attraction, acting over and over on successive orbits, could sweep asteroids out of the Kirkwood gap. Use this hypothesis to determine the orbital radius for this Kirkwood gap. (c) One of several other Kirkwood gaps appears at a distance from the sun where the orbital period is 0.400 that of Jupiter. Explain why this happens, and find the orbital radius for this Kirkwood gap.

13.70 ••• If a satellite is in a sufficiently low orbit, it will encounter air drag from the earth's atmosphere. Since air drag does negative work (the force of air drag is directed opposite the motion), the mechanical energy will decrease. According to Eq. (13.13), if E decreases (becomes more negative), the radius r of the orbit will decrease. If air drag is relatively small, the satellite can be considered to be in a circular orbit of continually decreasing radius. (a) According to Eq. (13.10), if the radius of a satellite's circular orbit decreases, the satellite's orbital speed v *increases.* How can you reconcile this with the statement that the mechanical energy *decreases?* (*Hint:* Is air drag the only force that does work on the satellite as the orbital radius decreases?) (b) Due to air drag, the radius of a satellite's circular orbit decreases from r to $r - \Delta r$, where the positive quantity Δr is much less than r. The mass of the satellite is m. Show that the increase in orbital speed is $\Delta v = +(\Delta r/2)\sqrt{Gm_E/r^3}$; that the change in kinetic energy is $\Delta K = +(Gm_E m/2r^2)\,\Delta r$; that the change in gravitational potential energy is $\Delta U = -2\,\Delta K = -(Gm_E m/r^2)\,\Delta r$; and that the amount of work done by the force of air drag is $W = -(Gm_E m/2r^2)\,\Delta r$. Interpret these results in light of your comments in part (a). (c) A satellite with mass 3000 kg is initially in a circular orbit 300 km above the earth's surface. Due to air drag, the satellite's altitude decreases to 250 km. Calculate the initial orbital speed; the increase in orbital speed; the initial mechanical energy; the change in kinetic energy; the change in gravitational potential energy; the change in mechanical energy; and the work done by the force of air drag. (d) Eventually a satellite will descend to a low enough altitude in the atmosphere that the satellite burns up and the debris falls to the earth. What becomes of the initial mechanical energy?

13.71 • **Binary Star—Equal Masses.** Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

13.72 •• CP **Binary Star—Different Masses.** Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses—that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (see Fig. 13.27). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object's orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

13.73 ••• Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the center of the sun, what is its speed when at a distance of 5.0×10^{10} m?

13.74 •• CP An astronaut is standing at the north pole of a newly discovered, spherically symmetric planet of radius R. In his hands he holds a container full of a liquid with mass m and volume V. At the surface of the liquid, the pressure is p_0; at a depth d below the surface, the pressure has a greater value p. From this information, determine the mass of the planet.

13.75 •• CALC The earth does not have a uniform density; it is most dense at its center and least dense at its surface. An approximation of its density is $\rho(r) = A - Br$, where $A = 12{,}700$ kg/m^3 and $B = 1.50 \times 10^{-3}$ kg/m^4. Use $R = 6.37 \times 10^6$ m for the radius of the earth approximated as a sphere. (a) Geological evidence indicates that the densities are 13,100 kg/m^3 and 2400 kg/m^3 at the earth's center and surface, respectively. What values does the linear approximation model give for the densities at these two locations? (b) Imagine dividing the earth into concentric, spherical shells. Each shell has radius r, thickness dr, volume $dV = 4\pi r^2\,dr$, and mass $dm = \rho(r)dV$. By integrating from $r = 0$ to $r = R$, show that the mass of the earth in this model is $M = \frac{4}{3}\pi R^3(A - \frac{3}{4}BR)$. (c) Show that the given values of A and B give the correct mass of the earth to within 0.4%. (d) We saw in Section 13.6 that a uniform spherical shell gives no contribution to g inside it. Show that $g(r) = \frac{4}{3}\pi Gr(A - \frac{3}{4}Br)$ inside the earth in this model. (e) Verify that the expression of part (d) gives $g = 0$ at the center of the earth and $g = 9.85$ m/s^2 at the surface. (f) Show that in this model g does *not* decrease uniformly with depth but rather has a maximum of $4\pi GA^2/9B = 10.01$ m/s^2 at $r = 2A/3B = 5640$ km.

13.76 •• CP CALC In Example 13.10 (Section 13.6) we saw that inside a planet of uniform density (not a realistic assumption for the earth) the acceleration due to gravity increases uniformly with distance from the center of the planet. That is, $g(r) = g_s r/R$, where g_s is the acceleration due to gravity at the surface, r is the distance from the center of the planet, and R is the radius of the planet. The interior of the planet can be treated approximately as an incompressible fluid of density ρ. (a) Replace the height y in Eq. (12.4) with the radial coordinate r and integrate to find the pressure inside a uniform planet as a function of r. Let the pressure at the surface be zero. (This means ignoring the pressure of the planet's atmosphere.) (b) Using this model, calculate the pressure at the center of the earth. (Use a value of ρ equal to the average density of the earth, calculated from the mass and radius given in Appendix F.) (c) Geologists estimate the pressure at the center of the earth to be approximately 4×10^{11} Pa. Does this agree with your calculation for the pressure at $r = 0$? What might account for any differences?

13.77 ••• CP Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee, of its orbit, it is 400 km above the earth's surface; at the high point, or apogee, it is 4000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

13.78 • The planet Uranus has a radius of 25,560 km and a surface acceleration due to gravity of 11.1 m/s^2 at its poles. Its moon Miranda (discovered by Kuiper in 1948) is in a circular orbit about Uranus at an altitude of 104,000 km above the planet's surface. Miranda has a mass of 6.6×10^{19} kg and a radius of 235 km. (a) Calculate the mass of Uranus from the given data. (b) Calculate the magnitude of Miranda's acceleration due to its orbital motion about Uranus. (c) Calculate the acceleration due to Miranda's gravity at the surface of Miranda. (d) Do the answers to parts (b) and (c) mean that an object released 1 m above Miranda's surface on the side toward Uranus will fall *up* relative to Miranda? Explain.

13.79 ••• A 5000-kg spacecraft is in a circular orbit 2000 km above the surface of Mars. How much work must the spacecraft engines perform to move the spacecraft to a circular orbit that is 4000 km above the surface?

13.80 •• One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years. Find the semi-major axis of this comet's orbit. Compare it to the average sun–Pluto distance and to the distance to Alpha Centauri, the nearest star to the sun, which is 4.3 light-years distant.

13.81 ••• **CALC** Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be $15.0 \times 10^3\ \text{kg/m}^3$ at the center and $2.0 \times 10^3\ \text{kg/m}^3$ at the surface. What is the acceleration due to gravity at the surface of this planet?

13.82 •• **CALC** A uniform wire with mass M and length L is bent into a semicircle. Find the magnitude and direction of the gravitational force this wire exerts on a point with mass m placed at the center of curvature of the semicircle.

13.83 ••• **CALC** An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (see Fig. E13.33). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

13.84 ••• **CALC** A thin, uniform rod has length L and mass M. Calculate the magnitude of the gravitational force the rod exerts on a particle with mass m that is at a point along the axis of the rod a distance x from one end (see Fig. E13.32). Show that your result reduces to the expected result when x is much larger than L.

13.85 • **CALC** A shaft is drilled from the surface to the center of the earth (see Fig. 13.24). As in Example 13.10 (Section 13.6), make the unrealistic assumption that the density of the earth is uniform. With this approximation, the gravitational force on an object with mass m, that is inside the earth at a distance r from the center, has magnitude $F_g = Gm_E mr/R_E^3$ (as shown in Example 13.10) and points toward the center of the earth. (a) Derive an expression for the gravitational potential energy $U(r)$ of the object–earth system as a function of the object's distance from the center of the earth. Take the potential energy to be zero when the object is at the center of the earth. (b) If an object is released in the shaft at the earth's surface, what speed will it have when it reaches the center of the earth?

CHALLENGE PROBLEMS

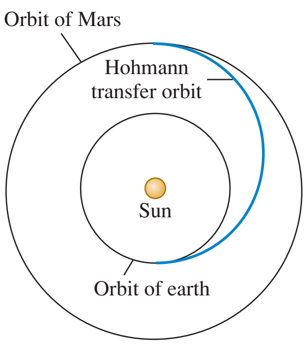
13.86 ••• (a) When an object is in a circular orbit of radius r around the earth (mass m_E), the period of the orbit is T, given by Eq. (13.12), and the orbital speed is v, given by Eq. (13.10). Show that when the object is moved into a circular orbit of slightly larger radius $r + \Delta r$, where $\Delta r \ll r$, its new period is $T + \Delta T$ and its new orbital speed is $v - \Delta v$, where Δr, ΔT, and Δv are all positive quantities and

$$\Delta T = \frac{3\pi\,\Delta r}{v} \qquad \text{and} \qquad \Delta v = \frac{\pi\,\Delta r}{T}$$

[*Hint:* Use the expression $(1 + x)^n \approx 1 + nx$, valid for $|x| \ll 1$.] (b) The International Space Station (ISS) is in a nearly circular orbit at an altitude of 398.00 km above the surface of the earth. A maintenance crew is about to arrive on the space shuttle that is also in a circular orbit in the same orbital plane as the ISS, but with an altitude of 398.10 km. The crew has come to remove a faulty 125-m electrical cable, one end of which is attached to the ISS and the other end of which is floating free in space. The plan is for the shuttle to snag the free end just at the moment that the shuttle, the ISS, and the center of the earth all lie along the same line. The cable will then break free from the ISS when it becomes taut. How long after the free end is caught by the space shuttle will it detach from the ISS? Give your answer in minutes. (c) If the shuttle misses catching the cable, show that the crew must wait a time $t \approx T^2/\Delta T$ before they have a second chance. Find the numerical value of t and explain whether it would be worth the wait.

13.87 ••• **Interplanetary Navigation.** The most efficient way to send a spacecraft from the earth to another planet is by using a *Hohmann transfer orbit* (Fig. P13.87). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion, or opposite the direction of motion? What about for a flight from Mars to the earth? (b) How long does a one-way trip from the the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun–Mars line and a sun–earth line be? Use data from Appendix F.

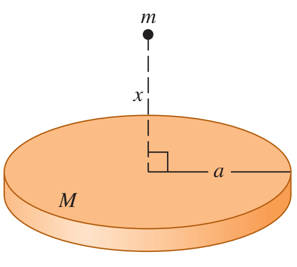
Figure **P13.87**



13.88 ••• **CP** **Tidal Forces near a Black Hole.** An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030-kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

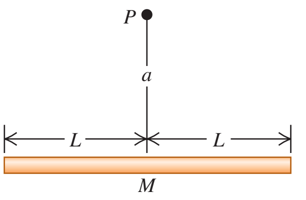
13.89 ••• CALC Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the center of the disk (Fig. P13.89). Does your result reduce to the correct expression as x becomes very large? (*Hint:* Divide the disk into infinitesimally thin concentric rings, use the expression derived in Exercise 13.33 for the gravitational force due to each ring, and integrate to find the total force.)

Figure **P13.89**



13.90 ••• CALC Mass M is distributed uniformly along a line of length $2L$. A particle with mass m is at a point that is a distance a above the center of the line on its perpendicular bisector (point P in Fig. P13.90). For the gravitational force that the line exerts on the particle, calculate the components perpendicular and parallel to the line. Does your result reduce to the correct expression as a becomes very large?

Figure **P13.90**



Answers

Chapter Opening Question ?

The smaller the orbital radius r of a satellite, the faster its orbital speed v [see Eq. (13.10)]. Hence a particle near the inner edge of Saturn's rings has a faster speed than a particle near the outer edge of the rings.

Test Your Understanding Questions

13.1 Answer: (v) From Eq. (13.1), the gravitational force of the sun (mass m_1) on a planet (mass m_2) a distance r away has magnitude $F_g = Gm_1m_2/r^2$. Compared to the earth, Saturn has a value of r^2 that is $10^2 = 100$ times greater and a value of m_2 that is also 100 times greater. Hence the *force* that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The *acceleration* of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth, its acceleration is $\frac{1}{100}$ as great as that of the earth.

13.2 Answer: (iii), (i), (ii), (iv) From Eq. (13.4), the acceleration due to gravity at the surface of a planet of mass m_P and radius R_P is $g_P = Gm_P/R_P^2$. That is, g_P is directly proportional to the planet's mass and inversely proportional to the square of its radius. It follows that compared to the value of g at the earth's surface, the value of g_P on each planet is (i) $2/2^2 = \frac{1}{2}$ as great; (ii) $4/4^2 = \frac{1}{4}$ as great; (iii) $4/2^2 = 1$ time as great—that is, the same as on earth; and (iv) $2/4^2 = \frac{1}{8}$ as great.

13.3 Answer: yes This is possible because surface gravity and escape speed depend in different ways on the planet's mass m_P and radius R_P: The value of g at the surface is Gm_P/R_P^2, while the escape speed is $\sqrt{2Gm_P/R_P}$. For the planet Saturn, for example, m_P is about 100 times the earth's mass and R_P is about 10 times the earth's radius. The value of g is different than on earth by a factor of $(100)/(10)^2 = 1$ (i.e., it is the same as on earth), while the escape speed is greater by a factor of $\sqrt{100/10} = 3.2$. It may help to remember that the surface gravity tells you about conditions right next to the planet's surface, while the escape speed (which tells you how fast you must travel to escape to infinity) depends on conditions at *all* points between the planet's surface and infinity.

13.4 Answer: (ii) Equation (13.10) shows that in a smaller-radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the *total* mechanical energy $E = K + U$; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount.

13.5 Answer: (iii) Equation (13.17) shows that the orbital period T is proportional to the $\frac{3}{2}$ power of the semi-major axis a. Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

13.6 Answer: no Our analysis shows that there is *zero* gravitational force inside a hollow spherical shell. Hence visitors to the interior of a hollow planet would find themselves weightless, and they could not stand or walk on the planet's inner surface.

13.7 Answer: (iv) The discussion following Eq. (13.27) shows that the difference between the acceleration due to gravity at the equator and at the poles is v^2/R_E. Since this planet has the same radius and hence the same circumference as the earth, the speed v at its equator must be 10 times the speed of the earth's equator. Hence v^2/R_E is $10^2 = 100$ times greater than for the earth, or $100(0.0339\ \text{m/s}^2) = 3.39\ \text{m/s}^2$. The acceleration due to gravity at the poles is $9.80\ \text{m/s}^2$, while at the equator it is dramatically less, $9.80\ \text{m/s}^2 - 3.39\ \text{m/s}^2 = 6.41\ \text{m/s}^2$. You can show that if this planet were to rotate 17.0 times faster than the earth, the acceleration due to gravity at the equator would be *zero* and loose objects would fly off the equator's surface!

13.8 Answer: (iii) If the sun collapsed into a black hole (which, according to our understanding of stars, it cannot do), the sun would have the same mass but a much smaller radius. Because the gravitational attraction of the sun on the earth does not depend on the sun's radius, the earth's orbit would be unaffected.

Bridging Problem

Answers: (a) Perihelion: $v_P = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1+e)}{(1-e)}}$

aphelion: $v_A = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1-e)}{(1+e)}}$

(b) $v_P = 54.4\ \text{km/s}$, $v_A = 0.913\ \text{km/s}$

14 PERIODIC MOTION

? Dogs walk with much quicker strides than do humans. Is this primarily because dogs' legs are shorter than human legs, less massive than human legs, or both?

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
- How to do calculations with simple harmonic motion, an important type of oscillation.
- How to use energy concepts to analyze simple harmonic motion.
- How to apply the ideas of simple harmonic motion to different physical situations.
- How to analyze the motions of a simple pendulum.
- What a physical pendulum is, and how to calculate the properties of its motion.
- What determines how rapidly an oscillation dies out.
- How a driving force applied to an oscillator at the right frequency can cause a very large response, or resonance.

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation,** is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

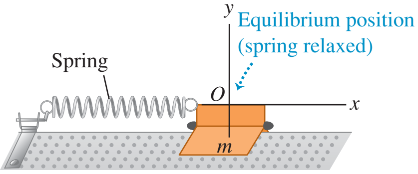
A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we will concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We will also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

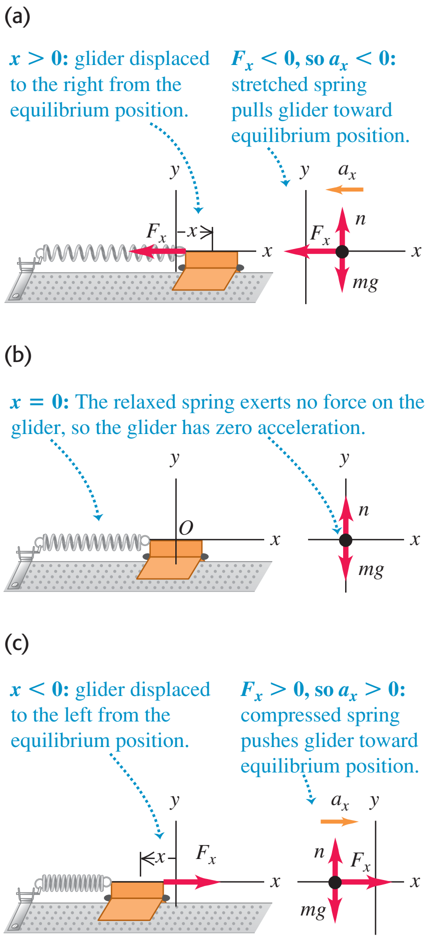
14.1 Describing Oscillation

Figure 14.1 shows one of the simplest systems that can have periodic motion. A body with mass m rests on a frictionless horizontal guide system, such as a linear air track, so it can move only along the x-axis. The body is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed and the right end is attached to the body. The spring force is the only horizontal force acting on the body; the vertical normal and gravitational forces always add to zero.

14.1 A system that can have periodic motion.



14.2 Model for periodic motion. When the body is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position.



Application Wing Frequencies

The ruby-throated hummingbird (*Archilochus colubris*) normally flaps its wings at about 50 Hz, producing the characteristic sound that gives hummingbirds their name. Insects can flap their wings at even faster rates, from 330 Hz for a house fly and 600 Hz for a mosquito to an amazing 1040 Hz for the tiny biting midge.

It's simplest to define our coordinate system so that the origin O is at the equilibrium position, where the spring is neither stretched nor compressed. Then x is the x-component of the **displacement** of the body from equilibrium and is also the change in the length of the spring. The x-component of the force that the spring exerts on the body is F_x, and the x-component of acceleration a_x is given by $a_x = F_x/m$.

Figure 14.2 shows the body for three different displacements of the spring. Whenever the body is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force.** Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Let's analyze how oscillation occurs in this system. If we displace the body to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 14.2a). The speed increases as the body approaches the equilibrium position O. When the body is at O, the net force acting on it is zero (Fig. 14.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the body is still moving to the left, but the net force and the acceleration are to the right (Fig. 14.2c); hence the speed decreases until the body comes to a stop. We will show later that with an ideal spring, the stopping point is at $x = -A$. The body then accelerates to the right, overshoots equilibrium again, and stops at the starting point $x = A$, ready to repeat the whole process. The body is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the body back toward the equilibrium position, only to have the body overshoot time after time.

In different situations the force may depend on the displacement x from equilibrium in different ways. But oscillation *always* occurs if the force is a *restoring* force that tends to return the system to equilibrium.

Amplitude, Period, Frequency, and Angular Frequency

Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by A, is the maximum magnitude of displacement from equilibrium—that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 14.2 is an ideal one, the total overall range of the motion is $2A$. The SI unit of A is the meter. A complete vibration, or **cycle,** is one complete round trip—say, from A to $-A$ and back to A, or from O to A, back through O to $-A$, and back to O. Note that motion from one side to the other (say, $-A$ to A) is a half-cycle, not a whole cycle.

The **period,** T, is the time for one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency,** f, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

This unit is named in honor of the German physicist Heinrich Hertz (1857–1894), a pioneer in investigating electromagnetic waves.

The **angular frequency,** ω, is 2π times the frequency:

$$\omega = 2\pi f$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since f is in cycle/s, we may regard the number 2π as having units rad/cycle.

From the definitions of period T and frequency f we see that each is the reciprocal of the other:

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \qquad \text{(relationships between frequency and period)} \qquad (14.1)$$

Also, from the definition of ω,

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{(angular frequency)} \tag{14.2}$$

Example 14.1 Period, frequency, and angular frequency

An ultrasonic transducer used for medical diagnosis oscillates at 6.7 MHz $= 6.7 \times 10^6$ Hz. How long does each oscillation take, and what is the angular frequency?

SOLUTION

IDENTIFY and SET UP: The target variables are the period T and the angular frequency ω. We can find these using the given frequency f in Eqs. (14.1) and (14.2).

EXECUTE: From Eqs. (14.1) and (14.2),

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \text{Hz}} = 1.5 \times 10^{-7}\ \text{s} = 0.15\ \mu\text{s}$$

$$\begin{aligned}\omega &= 2\pi f = 2\pi(6.7 \times 10^6\ \text{Hz}) \\ &= (2\pi\ \text{rad/cycle})(6.7 \times 10^6\ \text{cycle/s}) \\ &= 4.2 \times 10^7\ \text{rad/s}\end{aligned}$$

EVALUATE: This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

Test Your Understanding of Section 14.1 A body like that shown in Fig. 14.2 oscillates back and forth. For each of the following values of the body's x-velocity v_x and x-acceleration a_x, state whether its displacement x is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.

14.2 Simple Harmonic Motion

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x. This happens if the spring in Figs. 14.1 and 14.2 is an ideal one that obeys Hooke's law. The constant of proportionality between F_x and x is the force constant k. (You may want to review Hooke's law and the definition of the force constant in Section 6.3.) On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as $F_x = kx$. The x-component of force the spring exerts *on the body* is the negative of this, so the x-component of force F_x on the body is

$$F_x = -kx \quad \text{(restoring force exerted by an ideal spring)} \tag{14.3}$$

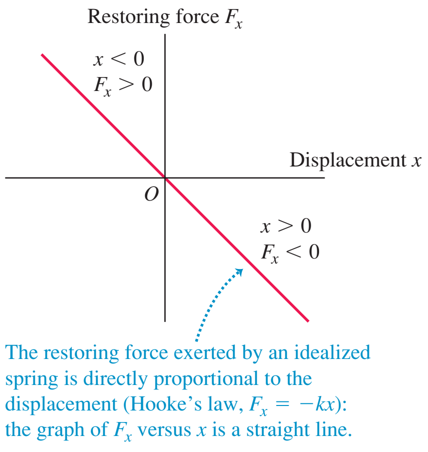
This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (Fig. 14.3). The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, so Eq. (14.3) gives the *net* force on the body.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), the oscillation is called **simple harmonic motion,** *abbreviated* **SHM.** The acceleration $a_x = d^2x/dt^2 = F_x/m$ of a body in SHM is given by

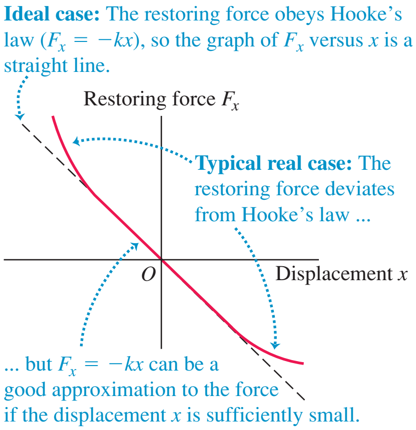
$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(simple harmonic motion)} \tag{14.4}$$

The minus sign means the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constant-acceleration equations from Chapter 2. We'll see shortly how to solve this equation to find the displacement x as a function of time. A body that undergoes simple harmonic motion is called a **harmonic oscillator.**

14.3 An idealized spring exerts a restoring force that obeys Hooke's law, $F_x = -kx$. Oscillation with such a restoring force is called simple harmonic motion.



14.4 In most real oscillations Hooke's law applies provided the body doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic.



Why is simple harmonic motion important? Keep in mind that not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (14.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (Fig. 14.4). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (14.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of the quartz crystal in a watch, the motion of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

Circular Motion and the Equations of SHM

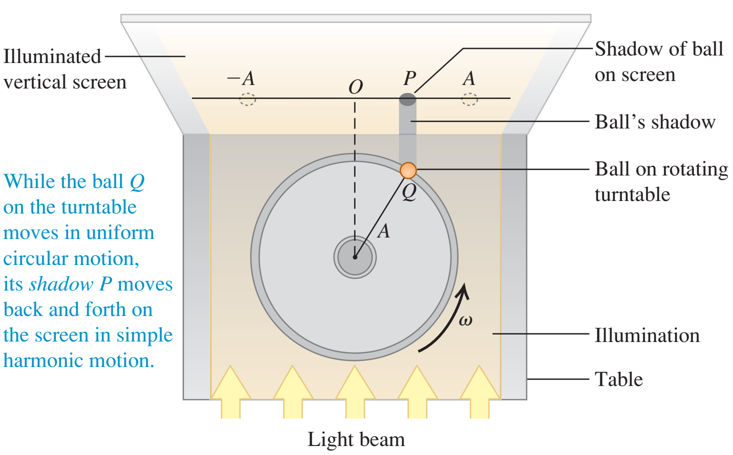
To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating body as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required by Eq. (14.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noticing a striking similarity between SHM and another form of motion that we've already studied.

Figure 14.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam shines on the rotating disk and casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange a body attached to an ideal spring, like the combination shown in Figs. 14.1 and 14.2, so that the body oscillates parallel to the shadow. We will prove that the motion of the body and the motion of the ball's shadow are *identical* if the amplitude of the body's oscillation is equal to the disk radius A, and if the angular frequency $2\pi f$ of the oscillating body is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter.*

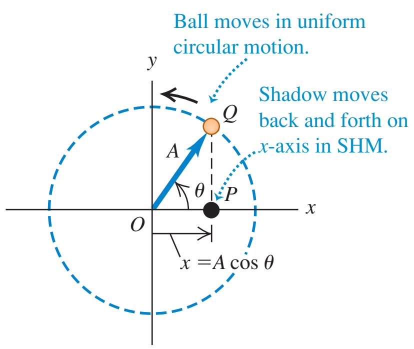
We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of a body undergoing SHM, given by Eq. (14.4). The circle in which the ball moves so that its projection matches the motion of the oscillating body is called the **reference circle;** we will call the point Q the *reference point.* We take the reference circle to lie in the

14.5 (a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like a body oscillating on an ideal spring.

(a) Apparatus for creating the reference circle



(b) An abstract representation of the motion in (a)



xy-plane, with the origin O at the center of the circle (Fig. 14.5b). At time t the vector OQ from the origin to the reference point Q makes an angle θ with the positive x-axis. As the point Q moves around the reference circle with constant angular speed ω, the vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor.** (This term was in use long before the invention of the Star Trek stun gun with a similar name. The phasor method for analyzing oscillations is useful in many areas of physics. We'll use phasors when we study alternating-current circuits in Chapter 31 and the interference of light in Chapters 35 and 36.)

The x-component of the phasor at time t is just the x-coordinate of the point Q:

$$x = A\cos\theta \tag{14.5}$$

This is also the x-coordinate of the shadow P, which is the *projection* of Q onto the x-axis. Hence the x-velocity of the shadow P along the x-axis is equal to the x-component of the velocity vector of point Q (Fig. 14.6a), and the x-acceleration of P is equal to the x-component of the acceleration vector of Q (Fig. 14.6b). Since point Q is in uniform circular motion, its acceleration vector $\vec{a}_Q$ is always directed toward O. Furthermore, the magnitude of $\vec{a}_Q$ is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

$$a_Q = \omega^2 A \tag{14.6}$$

Figure 14.6b shows that the x-component of $\vec{a}_Q$ is $a_x = -a_Q\cos\theta$. Combining this with Eqs. (14.5) and (14.6), we get that the acceleration of point P is

$$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta \qquad \text{or} \tag{14.7}$$

$$a_x = -\omega^2 x \tag{14.8}$$

The acceleration of point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

Equation (14.8) is *exactly* the same as Eq. (14.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating body by

$$\omega^2 = \frac{k}{m} \qquad \text{or} \qquad \omega = \sqrt{\frac{k}{m}} \tag{14.9}$$

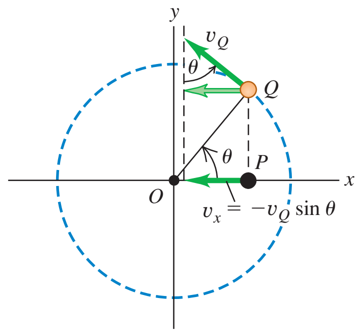
We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P. The reason is that these quantities are equal! If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi/T$. But this is just the same as Eq. (14.2) for the angular frequency of the point P, which verifies our statement about the two interpretations of ω. This is why we introduced angular frequency in Section 14.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (14.9) as an expression for the angular frequency of simple harmonic motion for a body of mass m, acted on by a restoring force with force constant k:

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(simple harmonic motion)} \tag{14.10}$$

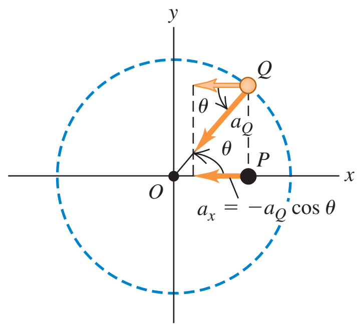
When you start a body oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m. The units of k are N/m or kg/s^2, so k/m is in $(\text{kg/s}^2)/\text{kg} = \text{s}^{-2}$. When we take the square root in Eq. (14.10), we get s^{-1}, or more properly rad/s because this is an *angular* frequency (recall that a radian is not a true unit).

14.6 The **(a)** x-velocity and **(b)** x-acceleration of the ball's shadow P (see Fig. 14.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q.

(a) Using the reference circle to determine the x-velocity of point P



(b) Using the reference circle to determine the x-acceleration of point P



According to Eqs. (14.1) and (14.2), the frequency f and period T are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)} \tag{14.11}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \text{(simple harmonic motion)} \tag{14.12}$$

14.7 The greater the mass m in a tuning fork's tines, the lower the frequency of oscillation $f = (1/2\pi)\sqrt{k/m}$ and the lower the pitch of the sound that the tuning fork produces.

Tines with large mass m: low frequency $f = 128$ Hz

Tines with small mass m: high frequency $f = 4096$ Hz

We see from Eq. (14.12) that a larger mass m, with its greater inertia, will have less acceleration, move more slowly, and take a longer time for a complete cycle (Fig. 14.7). In contrast, a stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration, higher speeds, and a shorter time T per cycle.

CAUTION Don't confuse frequency and angular frequency You can run into trouble if you don't make the distinction between frequency f and angular frequency $\omega = 2\pi f$. Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω.

Period and Amplitude in SHM

Equations (14.11) and (14.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k. *In simple harmonic motion the period and frequency do not depend on the amplitude A*. For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation (14.3) shows why we should expect this. Larger A means that the body reaches larger values of $|x|$ and is subjected to larger restoring forces. This increases the average speed of the body over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, which means that it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to make familiar types of mechanical and electronic clocks run accurately or to play most musical instruments in tune. If you encounter an oscillating body with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

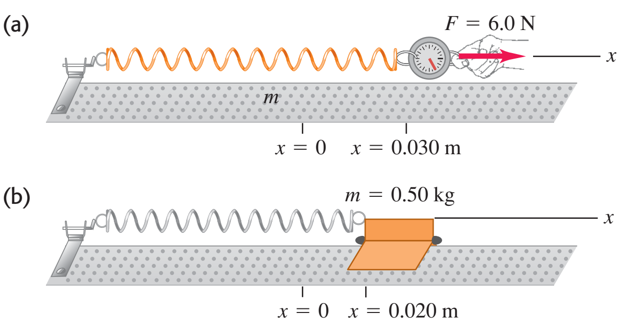
Example 14.2 Angular frequency, frequency, and period in SHM

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (Fig. 14.8a) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50-kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (Fig. 14.8b). (a) Find the force constant k of the spring. (b) Find the angular frequency ω, frequency f, and period T of the resulting oscillation.

SOLUTION

IDENTIFY and SET UP: Because the spring force (equal in magnitude to the stretching force) is proportional to the displacement, the motion is simple harmonic. We find k using Hooke's law, Eq. (14.3), and ω, f, and T using Eqs. (14.10), (14.11), and (14.12), respectively.

14.8 (a) The force exerted *on* the spring (shown by the vector F) has x-component $F_x = +6.0$ N. The force exerted *by* the spring has x-component $F_x = -6.0$ N. (b) A glider is attached to the same spring and allowed to oscillate.



EXECUTE: (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (14.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\ \text{N}}{0.030\ \text{m}} = 200\ \text{N/m} = 200\ \text{kg/s}^2$$

(b) From Eq. (14.10), with $m = 0.50$ kg,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{20\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 3.2\ \text{cycle/s} = 3.2\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{3.2\ \text{cycle/s}} = 0.31\ \text{s}$$

EVALUATE: The amplitude of the oscillation is 0.020 m, the distance that we pulled the glider before releasing it. In SHM the angular frequency, frequency, and period are all independent of the amplitude. Note that a period is usually stated in "seconds" rather than "seconds per cycle."

Displacement, Velocity, and Acceleration in SHM

We still need to find the displacement x as a function of time for a harmonic oscillator. Equation (14.4) for a body in simple harmonic motion along the x-axis is identical to Eq. (14.8) for the x-coordinate of the reference point in uniform circular motion with constant angular speed $\omega = \sqrt{k/m}$. Hence Eq. (14.5), $x = A\cos\theta$, describes the x-coordinate for both of these situations. If at $t = 0$ the phasor OQ makes an angle ϕ (the Greek letter phi) with the positive x-axis, then at any later time t this angle is $\theta = \omega t + \phi$. We substitute this into Eq. (14.5) to obtain

$$x = A\cos(\omega t + \phi) \qquad \text{(displacement in SHM)} \qquad (14.13)$$

where $\omega = \sqrt{k/m}$. Figure 14.9 shows a graph of Eq. (14.13) for the particular case $\phi = 0$. The displacement x is a periodic function of time, as expected for SHM. We could also have written Eq. (14.13) in terms of a sine function rather than a cosine by using the identity $\cos\alpha = \sin(\alpha + \pi/2)$. *In simple harmonic motion the position is a periodic, sinusoidal function of time.* There are many other periodic functions, but none so simple as a sine or cosine function.

The value of the cosine function is always between -1 and 1, so in Eq. (14.13), x is always between $-A$ and A. This confirms that A is the amplitude of the motion.

The period T is the time for one complete cycle of oscillation, as Fig. 14.9 shows. The cosine function repeats itself whenever the quantity in parentheses in Eq. (14.13) increases by 2π radians. Thus, if we start at time $t = 0$, the time T to complete one cycle is given by

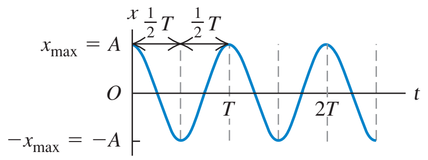
$$\omega T = \sqrt{\frac{k}{m}}\,T = 2\pi \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

which is just Eq. (14.12). Changing either m or k changes the period of oscillation, as shown in Figs. 14.10a and 14.10b. The period does not depend on the amplitude A (Fig. 14.10c).

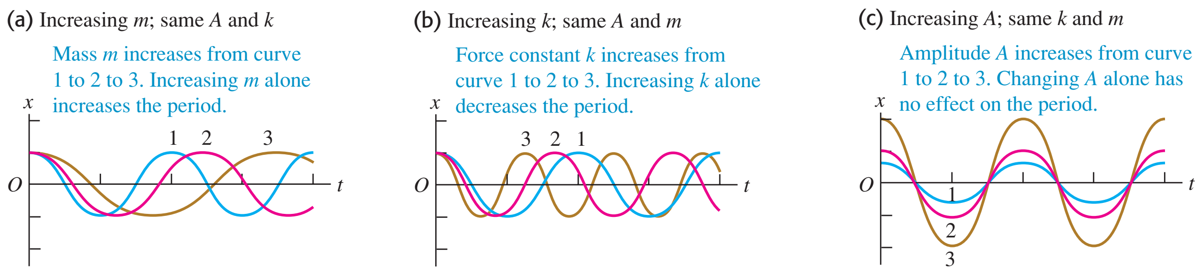
Mastering PHYSICS

PhET: Motion in 2D
ActivPhysics 9.1: Position Graphs and Equations
ActivPhysics 9.2: Describing Vibrational Motion
ActivPhysics 9.5: Age Drops Tarzan

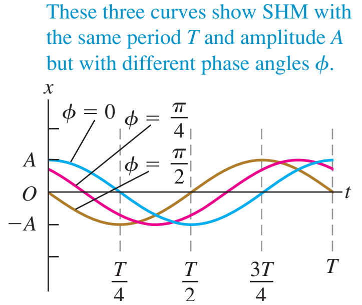
14.9 Graph of x versus t [see Eq. (14.13)] for simple harmonic motion. The case shown has $\phi = 0$.



14.10 Variations of simple harmonic motion. All cases shown have $\phi = 0$ [see Eq. (14.13)].



14.11 Variations of SHM: displacement versus time for the same harmonic oscillator with different phase angles ϕ.



The constant ϕ in Eq. (14.13) is called the **phase angle.** It tells us at what point in the cycle the motion was at $t = 0$ (equivalent to where around the circle the point Q was at $t = 0$). We denote the position at $t = 0$ by x_0. Putting $t = 0$ and $x = x_0$ in Eq. (14.13), we get

$$x_0 = A\cos\phi \tag{14.14}$$

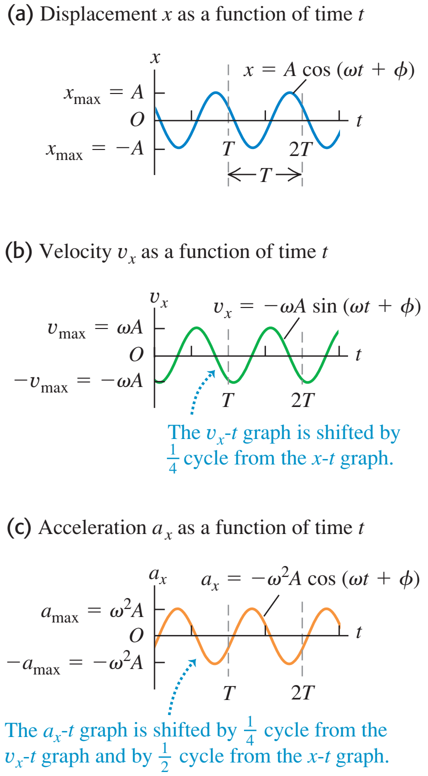
If $\phi = 0$, then $x_0 = A\cos 0 = A$, and the body starts at its maximum positive displacement. If $\phi = \pi$, then $x_0 = A\cos\pi = -A$, and the particle starts at its maximum *negative* displacement. If $\phi = \pi/2$, then $x_0 = A\cos(\pi/2) = 0$, and the particle is initially at the origin. Figure 14.11 shows the displacement x versus time for three different phase angles.

We find the velocity v_x and acceleration a_x as functions of time for a harmonic oscillator by taking derivatives of Eq. (14.13) with respect to time:

$$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad \text{(velocity in SHM)} \tag{14.15}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) \qquad \text{(acceleration in SHM)} \tag{14.16}$$

14.12 Graphs of (a) x versus t, (b) v_x versus t, and (c) a_x versus t for a body in SHM. For the motion depicted in these graphs, $\phi = \pi/3$.



The velocity v_x oscillates between $v_{max} = +\omega A$ and $-v_{max} = -\omega A$, and the acceleration a_x oscillates between $a_{max} = +\omega^2 A$ and $-a_{max} = -\omega^2 A$ (Fig. 14.12). Comparing Eq. (14.16) with Eq. (14.13) and recalling that $\omega^2 = k/m$ from Eq. (14.9), we see that

$$a_x = -\omega^2 x = -\frac{k}{m}x$$

which is just Eq. (14.4) for simple harmonic motion. This confirms that Eq. (14.13) for x as a function of time is correct.

We actually derived Eq. (14.16) earlier in a geometrical way by taking the x-component of the acceleration vector of the reference point Q. This was done in Fig. 14.6b and Eq. (14.7) (recall that $\theta = \omega t + \phi$). In the same way, we could have derived Eq. (14.15) by taking the x-component of the velocity vector of Q, as shown in Fig. 14.6b. We'll leave the details for you to work out.

Note that the sinusoidal graph of displacement versus time (Fig. 14.12a) is shifted by one-quarter period from the graph of velocity versus time (Fig. 14.12b) and by one-half period from the graph of acceleration versus time (Fig. 14.12c). Figure 14.13 shows why this is so. When the body is passing through the equilibrium position so that the displacement is zero, the velocity equals either v_{max} or $-v_{max}$ (depending on which way the body is moving) and the acceleration is zero. When the body is at either its maximum positive displacement, $x = +A$, or its maximum negative displacement, $x = -A$, the velocity is zero and the body is instantaneously at rest. At these points, the restoring force $F_x = -kx$ and the acceleration of the body have their maximum magnitudes. At $x = +A$ the acceleration is negative and equal to $-a_{max}$. At $x = -A$ the acceleration is positive: $a_x = +a_{max}$.

If we are given the initial position x_0 and initial velocity v_{0x} for the oscillating body, we can determine the amplitude A and the phase angle ϕ. Here's how to do it. The initial velocity v_{0x} is the velocity at time $t = 0$; putting $v_x = v_{0x}$ and $t = 0$ in Eq. (14.15), we find

$$v_{0x} = -\omega A\sin\phi \tag{14.17}$$

To find ϕ, we divide Eq. (14.17) by Eq. (14.14). This eliminates A and gives an equation that we can solve for ϕ:

$$\frac{v_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega\tan\phi$$

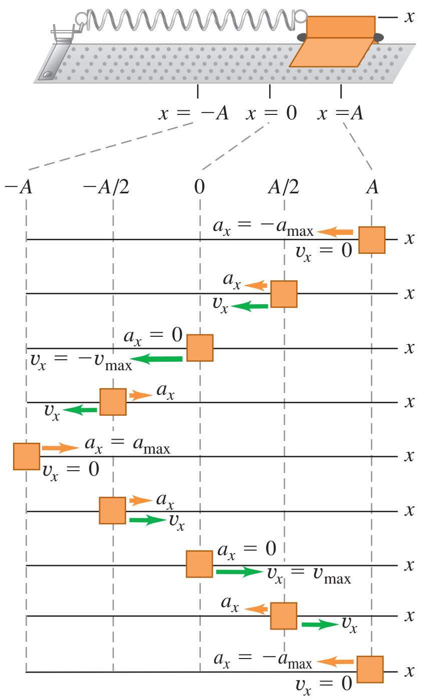
$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) \quad \text{(phase angle in SHM)} \qquad (14.18)$$

It is also easy to find the amplitude A if we are given x_0 and v_{0x}. We'll sketch the derivation, and you can fill in the details. Square Eq. (14.14); then divide Eq. (14.17) by ω, square it, and add to the square of Eq. (14.14). The right side will be $A^2(\sin^2\phi + \cos^2\phi)$, which is equal to A^2. The final result is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} \quad \text{(amplitude in SHM)} \qquad (14.19)$$

Note that when the body has both an initial displacement x_0 and a nonzero initial velocity v_{0x}, the amplitude A is *not* equal to the initial displacement. That's reasonable; if you start the body at a positive x_0 but give it a positive velocity v_{0x}, it will go *farther* than x_0 before it turns and comes back.

14.13 How x-velocity v_x and x-acceleration a_x vary during one cycle of SHM.



Problem-Solving Strategy 14.1 Simple Harmonic Motion I: Describing Motion

IDENTIFY *the relevant concepts:* An oscillating system undergoes simple harmonic motion (SHM) *only* if the restoring force is directly proportional to the displacement.

SET UP *the problem* using the following steps:
1. Identify the known and unknown quantities, and determine which are the target variables.
2. Distinguish between two kinds of quantities. *Properties of the system* include the mass m, the force constant k, and quantities derived from m and k, such as the period T, frequency f, and angular frequency ω. These are independent of *properties of the motion*, which describe how the system behaves when it is set into motion in a particular way; they include the amplitude A, maximum velocity v_{max}, and phase angle ϕ, and values of x, v_x, and a_x at particular times.
3. If necessary, define an x-axis as in Fig. 14.13, with the equilibrium position at $x = 0$.

EXECUTE *the solution* as follows:
1. Use the equations given in Sections 14.1 and 14.2 to solve for the target variables.
2. To find the values of x, v_x, and a_x at particular times, use Eqs. (14.13), (14.15), and (14.16), respectively. If the initial position x_0 and initial velocity v_{0x} are both given, determine ϕ and A from Eqs. (14.18) and (14.19). If the body has an initial positive displacement x_0 but zero initial velocity ($v_{0x} = 0$), then the amplitude is $A = x_0$ and the phase angle is $\phi = 0$. If it has an initial positive velocity v_{0x} but no initial displacement ($x_0 = 0$), the amplitude is $A = v_{0x}/\omega$ and the phase angle is $\phi = -\pi/2$. Express all phase angles in *radians*.

EVALUATE *your answer:* Make sure that your results are consistent. For example, suppose you used x_0 and v_{0x} to find general expressions for x and v_x at time t. If you substitute $t = 0$ into these expressions, you should get back the given values of x_0 and v_{0x}.

Example 14.3 Describing SHM

We give the glider of Example 14.2 an initial displacement $x_0 = +0.015$ m and an initial velocity $v_{0x} = +0.40$ m/s. (a) Find the period, amplitude, and phase angle of the resulting motion. (b) Write equations for the displacement, velocity, and acceleration as functions of time.

SOLUTION

IDENTIFY and SET UP: As in Example 14.2, the oscillations are SHM. We use equations from this section and the given values $k = 200$ N/m, $m = 0.50$ kg, x_0, and v_{0x} to calculate the target variables A and ϕ and to obtain expressions for x, v_x, and a_x.

Continued

EXECUTE: (a) In SHM the period and angular frequency are *properties of the system* that depend only on k and m, not on the amplitude, and so are the same as in Example 14.2 ($T = 0.31$ s and $\omega = 20$ rad/s). From Eq. (14.19), the amplitude is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} = \sqrt{(0.015\text{ m})^2 + \frac{(0.40\text{ m/s})^2}{(20\text{ rad/s})^2}} = 0.025\text{ m}$$

We use Eq. (14.18) to find the phase angle:

$$\begin{aligned}\phi &= \arctan\left(-\frac{v_{0x}}{\omega x_0}\right)\\ &= \arctan\left(-\frac{0.40\text{ m/s}}{(20\text{ rad/s})(0.015\text{ m})}\right) = -53° = -0.93\text{ rad}\end{aligned}$$

(b) The displacement, velocity, and acceleration at any time are given by Eqs. (14.13), (14.15), and (14.16), respectively. We substitute the values of A, ω, and ϕ into these equations:

$$\begin{aligned}x &= (0.025\text{ m})\cos[(20\text{ rad/s})t - 0.93\text{ rad}]\\ v_x &= -(0.50\text{ m/s})\sin[(20\text{ rad/s})t - 0.93\text{ rad}]\\ a_x &= -(10\text{ m/s}^2)\cos[(20\text{ rad/s})t - 0.93\text{ rad}]\end{aligned}$$

EVALUATE: You can check the expressions for x and v_x by confirming that if you substitute $t = 0$, they yield $x = x_0 = 0.015$ m and $v_x = v_{0x} = 0.40$ m/s.

Test Your Understanding of Section 14.2 A glider is attached to a spring as shown in Fig. 14.13. If the glider is moved to $x = 0.10$ m and released from rest at time $t = 0$, it will oscillate with amplitude $A = 0.10$ m and phase angle $\phi = 0$. (a) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the right in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero? (b) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the left in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero?

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PhET: Masses & Springs
ActivPhysics 9.3: Vibrational Energy
ActivPhysics 9.4: Two Ways to Weigh Young Tarzan
ActivPhysics 9.6: Releasing a Vibrating Skier I
ActivPhysics 9.7: Releasing a Vibrating Skier II
ActivPhysics 9.8: One- and Two-Spring Vibrating Systems
ActivPhysics 9.9: Vibro-Ride

14.3 Energy in Simple Harmonic Motion

We can learn even more about simple harmonic motion by using energy considerations. Take another look at the body oscillating on the end of a spring in Figs. 14.2 and 14.13. We've already noted that the spring force is the only horizontal force on the body. The force exerted by an ideal spring is a conservative force, and the vertical forces do no work, so the total mechanical energy of the system is *conserved*. We also assume that the mass of the spring itself is negligible.

The kinetic energy of the body is $K = \frac{1}{2}mv^2$ and the potential energy of the spring is $U = \frac{1}{2}kx^2$, just as in Section 7.2. (You'll find it helpful to review that section.) There are no nonconservative forces that do work, so the total mechanical energy $E = K + U$ is conserved:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \text{constant} \tag{14.20}$$

(Since the motion is one-dimensional, $v^2 = v_x^2$.)

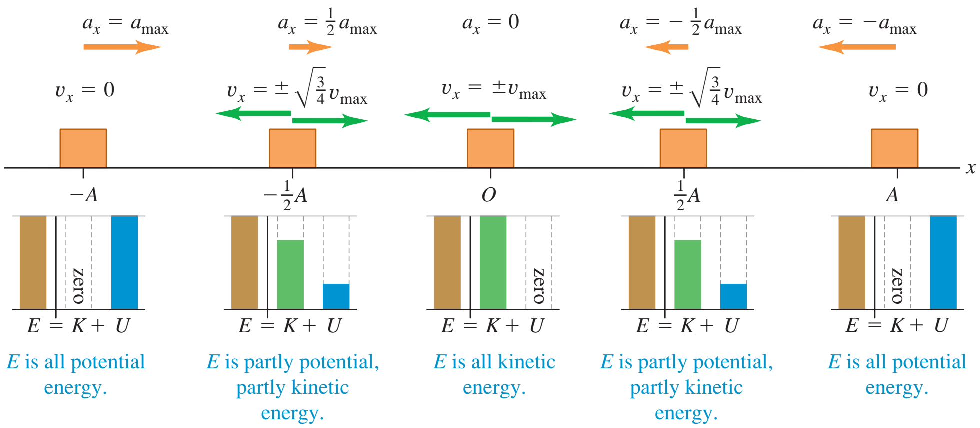
The total mechanical energy E is also directly related to the amplitude A of the motion. When the body reaches the point $x = A$, its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A$ (or $-A$), $v_x = 0$. At this point the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $\frac{1}{2}kA^2$ at any other point. Combining this expression with Eq. (14.20), we get

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \text{constant} \quad \text{(total mechanical energy in SHM)} \tag{14.21}$$

We can verify this equation by substituting x and v_x from Eqs. (14.13) and (14.15) and using $\omega^2 = k/m$ from Eq. (14.9):

$$\begin{aligned}E &= \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \tfrac{1}{2}k[A\cos(\omega t + \phi)]^2\\ &= \tfrac{1}{2}kA^2\sin^2(\omega t + \phi) + \tfrac{1}{2}kA^2\cos^2(\omega t + \phi)\\ &= \tfrac{1}{2}kA^2\end{aligned}$$

14.14 Graphs of E, K, and U versus displacement in SHM. The velocity of the body is *not* constant, so these images of the body at equally spaced positions are *not* equally spaced in time.



(Recall that $\sin^2\alpha + \cos^2\alpha = 1$.) Hence our expressions for displacement and velocity in SHM are consistent with energy conservation, as they must be.

We can use Eq. (14.21) to solve for the velocity v_x of the body at a given displacement x:

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \tag{14.22}$$

The $\pm$ sign means that at a given value of x the body can be moving in either direction. For example, when $x = \pm A/2$,

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\pm\frac{A}{2}\right)^2} = \pm\sqrt{\frac{3}{4}}\sqrt{\frac{k}{m}}A$$

Equation (14.22) also shows that the *maximum* speed v_{max} occurs at $x = 0$. Using Eq. (14.10), $\omega = \sqrt{k/m}$, we find that

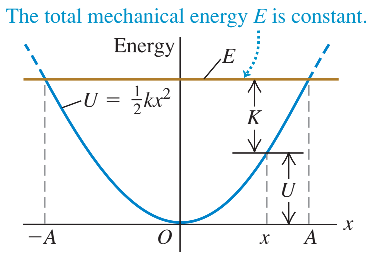
$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \tag{14.23}$$

This agrees with Eq. (14.15): v_x oscillates between $-\omega A$ and $+\omega A$.

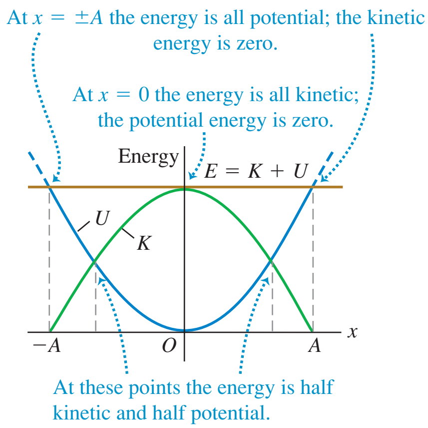
Interpreting E, K, and U in SHM

Figure 14.14 shows the energy quantities E, K, and U at $x = 0$, $x = \pm A/2$, and $x = \pm A$. Figure 14.15 is a graphical display of Eq. (14.21); energy (kinetic, potential, and total) is plotted vertically and the coordinate x is plotted horizontally.

(a) The potential energy U and total mechanical energy E for a body in SHM as a function of displacement x



(b) The same graph as in **(a)**, showing kinetic energy K as well



14.15 Kinetic energy K, potential energy U, and total mechanical energy E as functions of position for SHM. At each value of x the sum of the values of K and U equals the constant value of E. Can you show that the energy is half kinetic and half potential at $x = \pm\sqrt{\frac{1}{2}}A$?

The parabolic curve in Fig. 14.15a represents the potential energy $U = \frac{1}{2}kx^2$. The horizontal line represents the total mechanical energy E, which is constant and does not vary with x. At any value of x between $-A$ and A, the vertical distance from the x-axis to the parabola is U; since $E = K + U$, the remaining vertical distance up to the horizontal line is K. Figure 14.15b shows both K and U as functions of x. The horizontal line for E intersects the potential-energy curve at $x = -A$ and $x = A$, so at these points the energy is entirely potential, the kinetic energy is zero, and the body comes momentarily to rest before reversing direction. As the body oscillates between $-A$ and A, the energy is continuously transformed from potential to kinetic and back again.

Figure 14.15a shows the connection between the amplitude A and the corresponding total mechanical energy $E = \frac{1}{2}kA^2$. If we tried to make x greater than A (or less than $-A$), U would be greater than E, and K would have to be negative. But K can never be negative, so x can't be greater than A or less than $-A$.

Problem-Solving Strategy 14.2 Simple Harmonic Motion II: Energy

The SHM energy equation, Eq. (14.21), is a useful relationship among velocity, position, and total mechanical energy. If the problem requires you to relate position, velocity, and acceleration without reference to time, consider using Eq. (14.4) (from Newton's second law) or Eq. (14.21) (from energy conservation). Because Eq. (14.21) involves x^2 and v_x^2, you must infer the *signs* of x and v_x from the situation. For instance, if the body is moving from the equilibrium position toward the point of greatest positive displacement, then x is positive and v_x is positive.

Example 14.4 Velocity, acceleration, and energy in SHM

(a) Find the maximum and minimum velocities attained by the oscillating glider of Example 14.2. (b) Find the maximum and minimum accelerations. (c) Find the velocity v_x and acceleration a_x when the glider is halfway from its initial position to the equilibrium position $x = 0$. (d) Find the total energy, potential energy, and kinetic energy at this position.

SOLUTION

IDENTIFY and SET UP: The problem concerns properties of the motion at specified *positions,* not at specified *times,* so we can use the energy relationships of this section. Figure 14.13 shows our choice of x-axis. The maximum displacement from equilibrium is $A = 0.020$ m. We use Eqs. (14.22) and (14.4) to find v_x and a_x for a given x. We then use Eq. (14.21) for given x and v_x to find the total, potential, and kinetic energies E, U, and K.

EXECUTE: (a) From Eq. (14.22), the velocity v_x at any displacement x is

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

The glider's maximum *speed* occurs when it is moving through $x = 0$:

$$v_{\text{max}} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}(0.020\ \text{m}) = 0.40\ \text{m/s}$$

Its maximum and minimum (most negative) *velocities* are $+0.40$ m/s and -0.40 m/s, which occur when it is moving through $x = 0$ to the right and left, respectively.

(b) From Eq. (14.4), $a_x = -(k/m)x$. The glider's maximum (most positive) acceleration occurs at the most negative value of x, $x = -A$:

$$a_{\text{max}} = -\frac{k}{m}(-A) = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(-0.020\ \text{m}) = 8.0\ \text{m/s}^2$$

The minimum (most negative) acceleration is $a_{\text{min}} = -8.0\ \text{m/s}^2$, which occurs at $x = +A = +0.020$ m.

(c) The point halfway from $x = x_0 = A$ to $x = 0$ is $x = A/2 = 0.010$ m. From Eq. (14.22), at this point

$$v_x = -\sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}\sqrt{(0.020\ \text{m})^2 - (0.010\ \text{m})^2} = -0.35\ \text{m/s}$$

We choose the negative square root because the glider is moving from $x = A$ toward $x = 0$. From Eq. (14.4),

$$a_x = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(0.010\ \text{m}) = -4.0\ \text{m/s}^2$$

Figure 14.14 shows the conditions at $x = 0$, $\pm A/2$, and $\pm A$.

(d) The energies are

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(200\ \text{N/m})(0.020\ \text{m})^2 = 0.040\ \text{J}$$
$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(200\ \text{N/m})(0.010\ \text{m})^2 = 0.010\ \text{J}$$
$$K = \tfrac{1}{2}mv_x^2 = \tfrac{1}{2}(0.50\ \text{kg})(-0.35\ \text{m/s})^2 = 0.030\ \text{J}$$

EVALUATE: At $x = A/2$, the total energy is one-fourth potential energy and three-fourths kinetic energy. You can confirm this by inspecting Fig. 14.15b.

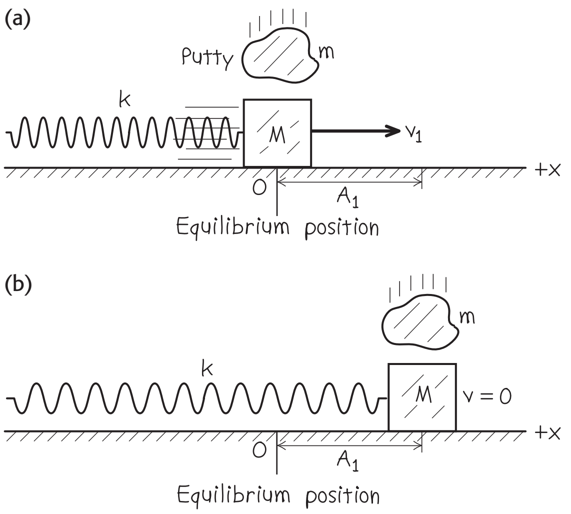
Example 14.5 Energy and momentum in SHM

A block of mass M attached to a horizontal spring with force constant k is moving in SHM with amplitude A_1. As the block passes through its equilibrium position, a lump of putty of mass m is dropped from a small height and sticks to it. (a) Find the new amplitude and period of the motion. (b) Repeat part (a) if the putty is dropped onto the block when it is at one end of its path.

SOLUTION

IDENTIFY and SET UP: The problem involves the motion at a given position, not a given time, so we can use energy methods. Figure 14.16 shows our sketches. Before the putty falls, the mechanical energy of the block–spring system is constant. In part (a), the putty–block collision is completely inelastic: The horizontal component of momentum is conserved, kinetic energy decreases, and the amount of mass that's oscillating increases. After the collision, the mechanical energy remains constant at its new value. In part (b) the oscillating mass also increases, but the block isn't moving when the putty is added; there is effectively no collision at all, and no mechanical energy is lost. We find the amplitude A_2 after each collision from the final energy of the system using Eq. (14.21) and conservation of momentum. The period T_2 after the collision is a *property of the system*, so it is the same in both parts (a) and (b); we find it using Eq. (14.12).

14.16 Our sketches for this problem.



EXECUTE: (a) Before the collision the total mechanical energy of the block and spring is $E_1 = \frac{1}{2}kA_1^2$. The block is at $x = 0$, so $U = 0$ and the energy is purely kinetic (Fig. 14.16a). If we let v_1 be the speed of the block at this point, then $E_1 = \frac{1}{2}kA_1^2 = \frac{1}{2}Mv_1^2$ and

$$v_1 = \sqrt{\frac{k}{M}}A_1$$

During the collision the x-component of momentum of the block–putty system is conserved. (Why?) Just before the collision this component is the sum of Mv_1 (for the block) and zero (for the putty). Just after the collision the block and putty move together with speed v_2, so their combined x-component of momentum is $(M + m)v_2$. From conservation of momentum,

$$Mv_1 + 0 = (M + m)v_2 \qquad \text{so} \qquad v_2 = \frac{M}{M + m}v_1$$

We assume that the collision lasts a very short time, so that the block and putty are still at the equilibrium position just after the collision. The energy is still purely kinetic but is *less* than before the collision:

$$E_2 = \tfrac{1}{2}(M + m)v_2^2 = \tfrac{1}{2}\frac{M^2}{M + m}v_1^2$$

$$= \frac{M}{M + m}\left(\tfrac{1}{2}Mv_1^2\right) = \left(\frac{M}{M + m}\right)E_1$$

Since $E_2 = \frac{1}{2}kA_2^2$, where A_2 is the amplitude after the collision, we have

$$\tfrac{1}{2}kA_2^2 = \left(\frac{M}{M + m}\right)\tfrac{1}{2}kA_1^2$$

$$A_2 = A_1\sqrt{\frac{M}{M + m}}$$

From Eq. (14.12), the period of oscillation after the collision is

$$T_2 = 2\pi\sqrt{\frac{M + m}{k}}$$

(b) When the putty falls, the block is instantaneously at rest (Fig. 14.16b). The x-component of momentum is zero both before and after the collision. The block and putty have zero kinetic energy just before and just after the collision. The energy is all potential energy stored in the spring, so adding the putty has *no effect* on the mechanical energy. That is, $E_2 = E_1 = \frac{1}{2}kA_1^2$, and the amplitude is unchanged: $A_2 = A_1$. The period is again $T_2 = 2\pi\sqrt{(M + m)/k}$.

EVALUATE: Energy is lost in part (a) because the putty slides against the moving block during the collision, and energy is dissipated by kinetic friction. No energy is lost in part (b), because there is no sliding during the collision.

Test Your Understanding of Section 14.3 (a) To double the total energy for a mass-spring system oscillating in SHM, by what factor must the amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$. (b) By what factor will the frequency change due to this amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$; (v) it does not change.

MP

14.4 Applications of Simple Harmonic Motion

So far, we've looked at a grand total of *one* situation in which simple harmonic motion (SHM) occurs: a body attached to an ideal horizontal spring. But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), $F_x = -kx$. The restoring force will originate in different ways in different situations, so the force constant k has to be found for each case by examining the net force on the system. Once this is done, it's straightforward to find the angular frequency ω, frequency f, and period T; we just substitute the value of k into Eqs. (14.10), (14.11), and (14.12), respectively. Let's use these ideas to examine several examples of simple harmonic motion.

Vertical SHM

Suppose we hang a spring with force constant k (Fig. 14.17a) and suspend from it a body with mass m. Oscillations will now be vertical; will they still be SHM? In Fig. 14.17b the body hangs at rest, in equilibrium. In this position the spring is stretched an amount Δl just great enough that the spring's upward vertical force $k\,\Delta l$ on the body balances its weight mg:

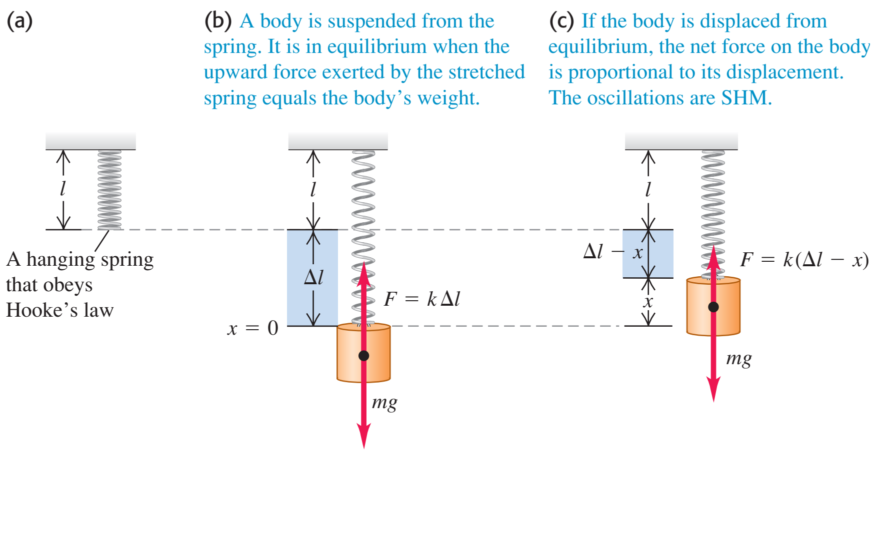
$$k\,\Delta l = mg$$

Take $x = 0$ to be this equilibrium position and take the positive x-direction to be upward. When the body is a distance x *above* its equilibrium position (Fig. 14.17c), the extension of the spring is $\Delta l - x$. The upward force it exerts on the body is then $k(\Delta l - x)$, and the net x-component of force on the body is

$$F_{\text{net}} = k(\Delta l - x) + (-mg) = -kx$$

that is, a net downward force of magnitude kx. Similarly, when the body is *below* the equilibrium position, there is a net upward force with magnitude kx. In either case there is a restoring force with magnitude kx. If the body is set in vertical motion, it oscillates in SHM with the same angular frequency as though it were horizontal, $\omega = \sqrt{k/m}$. So vertical SHM doesn't differ in any essential way from horizontal SHM. The only real change is that the equilibrium position $x = 0$ no longer corresponds to the point at which the spring is unstretched. The same ideas hold if a body with weight mg is placed atop a compressible spring (Fig. 14.18) and compresses it a distance Δl.

14.17 A body attached to a hanging spring.



Example 14.6 Vertical SHM in an old car

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980-N person climbs slowly into the car at its center of gravity, the car sinks 2.8 cm. The car (with the person aboard) hits a bump, and the car starts oscillating up and down in SHM. Model the car and person as a single body on a single spring, and find the period and frequency of the oscillation.

SOLUTION

IDENTIFY and SET UP: The situation is like that shown in Fig. 14.18. The compression of the spring when the person's weight is added tells us the force constant, which we can use to find the period and frequency (the target variables).

EXECUTE: When the force increases by 980 N, the spring compresses an additional 0.028 m, and the *x*-coordinate of the car changes by −0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}} = 3.5 \times 10^4\ \text{kg/s}^2$$

The person's mass is $w/g = (980\ \text{N})/(9.8\ \text{m/s}^2) = 100$ kg. The *total* oscillating mass is $m = 1000\ \text{kg} + 100\ \text{kg} = 1100$ kg. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

The frequency is $f = 1/T = 1/(1.11\ \text{s}) = 0.90$ Hz.

EVALUATE: A persistent oscillation with a period of about 1 second makes for a very unpleasant ride. The purpose of shock absorbers is to make such oscillations die out (see Section 14.7).

Angular SHM

A mechanical watch keeps time based on the oscillations of a balance wheel (Fig. 14.19). The wheel has a moment of inertia I about its axis. A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position. We write $\tau_z = -\kappa\theta$, where κ (the Greek letter kappa) is a constant called the *torsion constant*. Using the rotational analog of Newton's second law for a rigid body, $\Sigma\tau_z = I\alpha_z = I\,d^2\theta/dt^2$, we can find the equation of motion:

$$-\kappa\theta = I\alpha \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

The form of this equation is exactly the same as Eq. (14.4) for the acceleration in simple harmonic motion, with x replaced by θ and k/m replaced by κ/I. So we are dealing with a form of *angular* simple harmonic motion. The angular frequency ω and frequency f are given by Eqs. (14.10) and (14.11), respectively, with the same replacement:

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \quad \text{(angular SHM)} \qquad (14.24)$$

The motion is described by the function

$$\theta = \Theta\cos(\omega t + \phi)$$

where Θ (the Greek letter theta) plays the role of an angular amplitude.

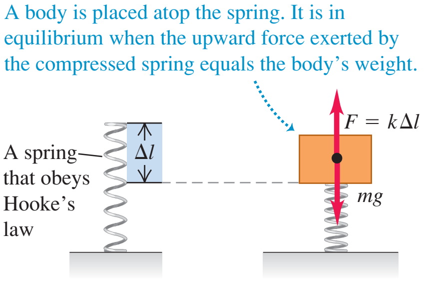
It's a good thing that the motion of a balance wheel *is* simple harmonic. If it weren't, the frequency might depend on the amplitude, and the watch would run too fast or too slow as the spring ran down.

Vibrations of Molecules

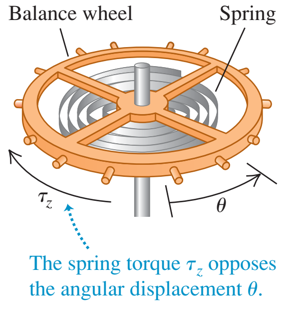
The following discussion of the vibrations of molecules uses the binomial theorem. If you aren't familiar with this theorem, you should read about it in the appropriate section of a math textbook.

When two atoms are separated from each other by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close to each other that their electron shells overlap, the forces between the atoms are repulsive. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule*. If these atoms are displaced slightly from equilibrium, they will oscillate.

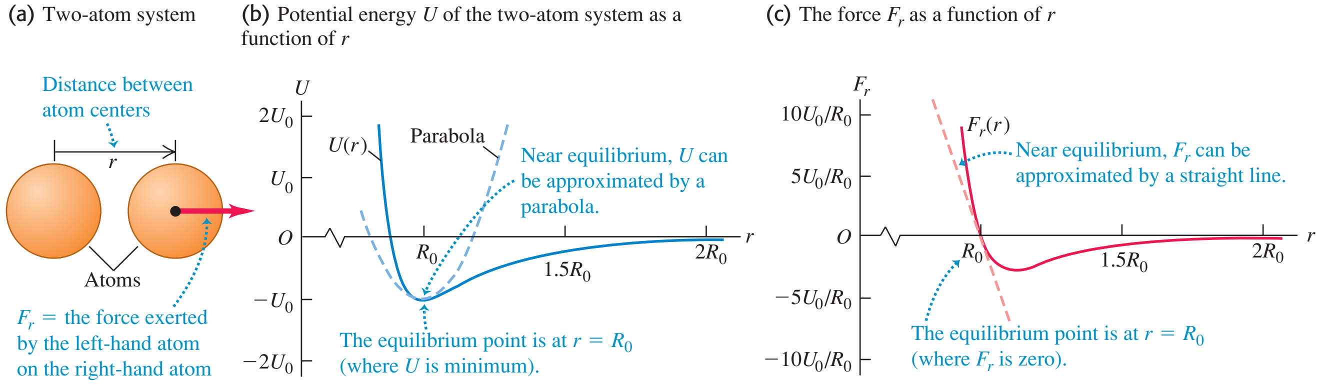
14.18 If the weight mg compresses the spring a distance Δl, the force constant is $k = mg/\Delta l$ and the angular frequency for vertical SHM is $\omega = \sqrt{k/m}$—the same as if the body were suspended from the spring (see Fig. 14.17).



14.19 The balance wheel of a mechanical watch. The spring exerts a restoring torque that is proportional to the angular displacement θ, so the motion is angular SHM.



14.20 (a) Two atoms with centers separated by r. (b) Potential energy U in the van der Waals interaction as a function of r. (c) Force F_r on the right-hand atom as a function of r.



As an example, we'll consider one type of interaction between atoms called the *van der Waals interaction.* Our immediate task here is to study oscillations, so we won't go into the details of how this interaction arises. Let the center of one atom be at the origin and let the center of the other atom be a distance r away (Fig. 14.20a); the equilibrium distance between centers is $r = R_0$. Experiment shows that the van der Waals interaction can be described by the potential-energy function

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right] \tag{14.25}$$

where U_0 is a positive constant with units of joules. When the two atoms are very far apart, $U = 0$; when they are separated by the equilibrium distance $r = R_0$, $U = -U_0$. The force on the second atom is the negative derivative of Eq. (14.25):

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{\,12}}{r^{13}} - 2\frac{6R_0^{\,6}}{r^{7}}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right] \tag{14.26}$$

Figures 14.20b and 14.20c plot the potential energy and force, respectively. The force is positive for $r < R_0$ and negative for $r > R_0$, so it is a *restoring* force.

Let's examine the restoring force F_r in Eq. (14.26). We let x represent the displacement from equilibrium:

$$x = r - R_0 \qquad \text{so} \qquad r = R_0 + x$$

In terms of x, the force F_r in Eq. (14.26) becomes

$$\begin{aligned} F_r &= 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^{7}\right] \\ &= 12\frac{U_0}{R_0}\left[\frac{1}{(1+x/R_0)^{13}} - \frac{1}{(1+x/R_0)^{7}}\right] \end{aligned} \tag{14.27}$$

This looks nothing like Hooke's law, $F_x = -kx$, so we might be tempted to conclude that molecular oscillations cannot be SHM. But let us restrict ourselves to *small-amplitude* oscillations so that the absolute value of the displacement x is small in comparison to R_0 and the absolute value of the ratio x/R_0 is much less than 1. We can then simplify Eq. (14.27) by using the *binomial theorem:*

$$(1 + u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots \tag{14.28}$$

If $|u|$ is much less than 1, each successive term in Eq. (14.28) is much smaller than the one it follows, and we can safely approximate $(1 + u)^n$ by just the first two terms. In Eq. (14.27), u is replaced by x/R_0 and n equals -13 or -7, so

$$\frac{1}{(1 + x/R_0)^{13}} = (1 + x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0}$$

$$\frac{1}{(1 + x/R_0)^{7}} = (1 + x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0}$$

$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1 + (-13)\frac{x}{R_0}\right) - \left(1 + (-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x \qquad (14.29)$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$. (Note that k has the correct units, J/m^2 or N/m.) So oscillations of molecules bound by the van der Waals interaction can be simple harmonic motion, provided that the amplitude is small in comparison to R_0 so that the approximation $|x/R_0| \ll 1$ used in the derivation of Eq. (14.29) is valid.

You can also use the binomial theorem to show that the potential energy U in Eq. (14.25) can be written as $U \approx \frac{1}{2}kx^2 + C$, where $C = -U_0$ and k is again equal to $72U_0/R_0^2$. Adding a constant to the potential energy has no effect on the physics, so the system of two atoms is fundamentally no different from a mass attached to a horizontal spring for which $U = \frac{1}{2}kx^2$.

Example 14.7 Molecular vibration

Two argon atoms form the molecule Ar_2 as a result of a van der Waals interaction with $U_0 = 1.68 \times 10^{-21}$ J and $R_0 = 3.82 \times 10^{-10}$ m. Find the frequency of small oscillations of one Ar atom about its equilibrium position.

SOLUTION

IDENTIFY and SET UP This is just the situation shown in Fig. 14.20. Because the oscillations are small, we can use Eq. (14.29) to find the force constant k and Eq. (14.11) to find the frequency f of SHM.

EXECUTE: From Eq. (14.29),

$$k = \frac{72U_0}{R_0^2} = \frac{72(1.68 \times 10^{-21}\ \text{J})}{(3.82 \times 10^{-10}\ \text{m})^2} = 0.829\ \text{J/m}^2 = 0.829\ \text{N/m}$$

(This force constant is comparable to that of a loose toy spring like a Slinky™.) From Appendix D, the average atomic mass of argon is $(39.948\ \text{u})(1.66 \times 10^{-27}\ \text{kg/1 u}) = 6.63 \times 10^{-26}$ kg.

From Eq. (14.11), if one atom is fixed and the other oscillates,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.829\ \text{N/m}}{6.63 \times 10^{-26}\ \text{kg}}} = 5.63 \times 10^{11}\ \text{Hz}$$

EVALUATE: Our answer for f isn't quite right. If no net external force acts on the molecule, its center of mass (halfway between the atoms) doesn't accelerate, so *both* atoms must oscillate with the same amplitude in opposite directions. It turns out that we can account for this by replacing m with $m/2$ in our expression for f. This makes f larger by a factor of $\sqrt{2}$, so the correct frequency is $f = \sqrt{2}(5.63 \times 10^{11}\ \text{Hz}) = 7.96 \times 10^{11}$ Hz. What's more, on the atomic scale we must use *quantum mechanics* rather than Newtonian mechanics to describe motion; happily, quantum mechanics also yields $f = 7.96 \times 10^{11}$ Hz.

Test Your Understanding of Section 14.4 A block attached to a hanging ideal spring oscillates up and down with a period of 10 s on earth. If you take the block and spring to Mars, where the acceleration due to gravity is only about 40% as large as on earth, what will be the new period of oscillation? (i) 10 s; (ii) more than 10 s; (iii) less than 10 s.

14.5 The Simple Pendulum

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing (Fig. 14.21a) can be modeled as simple pendulums.

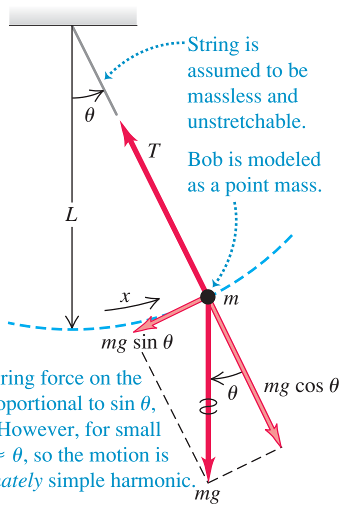
MasteringPHYSICS

PhET: Pendulum Lab
ActivPhysics 9.10: Pendulum Frequency
ActivPhysics 9.11: Risky Pendulum Walk
ActivPhysics 9.12: Physical Pendulum

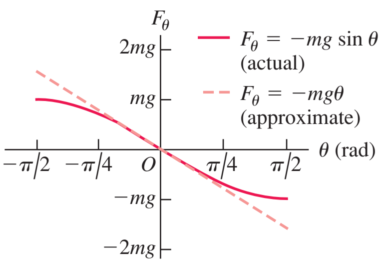
14.21 The dynamics of a simple pendulum.

(a) A real pendulum

(b) An idealized simple pendulum



14.22 For small angular displacements θ, the restoring force $F_\theta = -mg\sin\theta$ on a simple pendulum is approximately equal to $-mg\theta$; that is, it is approximately proportional to the displacement θ. Hence for small angles the oscillations are simple harmonic.



The path of the point mass (sometimes called a pendulum bob) is not a straight line but the arc of a circle with radius L equal to the length of the string (Fig. 14.21b). We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ. Is it?

In Fig. 14.21b we represent the forces on the mass in terms of tangential and radial components. The restoring force F_θ is the tangential component of the net force:

$$F_\theta = -mg\sin\theta \tag{14.30}$$

The restoring force is provided by gravity; the tension T merely acts to make the point mass move in an arc. The restoring force is proportional *not* to θ but to $\sin\theta$, so the motion is *not* simple harmonic. However, if the angle θ is *small,* $\sin\theta$ is very nearly equal to θ in radians (Fig. 14.22). For example, when $\theta = 0.1$ rad (about 6°), $\sin\theta = 0.0998$, a difference of only 0.2%. With this approximation, Eq. (14.30) becomes

$$F_\theta = -mg\theta = -mg\frac{x}{L} \qquad \text{or}$$

$$F_\theta = -\frac{mg}{L}x \tag{14.31}$$

The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg/L$. From Eq. (14.10) the angular frequency ω of a simple pendulum with small amplitude is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \tag{14.32}$$

The corresponding frequency and period relationships are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \tag{14.33}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \quad \text{(simple pendulum, small amplitude)} \tag{14.34}$$

Note that these expressions do not involve the *mass* of the particle. This is because the restoring force, a component of the particle's weight, is proportional to m. Thus the mass appears on *both* sides of $\Sigma\vec{F} = m\vec{a}$ and cancels out. (This is the same physics that explains why bodies of different masses fall with the same acceleration in a vacuum.) For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length.

The dependence on L and g in Eqs. (14.32) through (14.34) is just what we should expect. A long pendulum has a longer period than a shorter one. Increasing g increases the restoring force, causing the frequency to increase and the period to decrease.

We emphasize again that the motion of a pendulum is only *approximately* simple harmonic. When the amplitude is not small, the departures from simple harmonic motion can be substantial. But how small is "small"? The period can be expressed by an infinite series; when the maximum angular displacement is Θ, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right) \tag{14.35}$$

We can compute the period to any desired degree of precision by taking enough terms in the series. We invite you to check that when $\Theta = 15°$ (on either side of the central position), the true period is longer than that given by the approximate Eq. (14.34) by less than 0.5%.

The usefulness of the pendulum as a timekeeper depends on the period being *very nearly* independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

Example 14.8 A simple pendulum

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \text{m/s}^2$.

SOLUTION

IDENTIFY and SET UP: This is a simple pendulum, so we can use the ideas of this section. We use Eq. (14.34) to determine the pendulum's period T from its length, and Eq. (14.1) to find the frequency f from T.

EXECUTE: From Eqs. (14.34) and (14.1),

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \text{m}}{9.800\ \text{m/s}^2}} = 2.007\ \text{s}$$

$$f = \frac{1}{T} = \frac{1}{2.007\ \text{s}} = 0.4983\ \text{Hz}$$

EVALUATE: The period is almost exactly 2 s. When the metric system was established, the second was *defined* as half the period of a 1-m simple pendulum. This was a poor standard, however, because the value of g varies from place to place. We discussed more modern time standards in Section 1.3.

Test Your Understanding of Section 14.5 When a body oscillating on a horizontal spring passes through its equilibrium position, its acceleration is zero (see Fig. 14.2b). When the bob of an oscillating simple pendulum passes through its equilibrium position, is its acceleration zero?

14.6 The Physical Pendulum

A **physical pendulum** is any *real* pendulum that uses an extended body, as contrasted to the idealized model of the *simple* pendulum with all the mass concentrated at a single point. For small oscillations, analyzing the motion of a real, physical pendulum is almost as easy as for a simple pendulum. Figure 14.23 shows a body of irregular shape pivoted so that it can turn without friction about an axis through point O. In the equilibrium position the center of gravity is directly below the pivot; in the position shown in the figure, the body is displaced from equilibrium by an angle θ, which we use as a coordinate for the system. The distance from O to the center of gravity is d, the moment of inertia of the body about the axis of rotation through O is I, and the total mass is m. When the body is displaced as shown, the weight mg causes a restoring torque

$$\tau_z = -(mg)(d\sin\theta) \tag{14.36}$$

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise, and vice versa.

When the body is released, it oscillates about its equilibrium position. The motion is not simple harmonic because the torque τ_z is proportional to $\sin\theta$ rather than to θ itself. However, if θ is small, we can approximate $\sin\theta$ by θ in radians, just as we did in analyzing the simple pendulum. Then the motion is *approximately* simple harmonic. With this approximation,

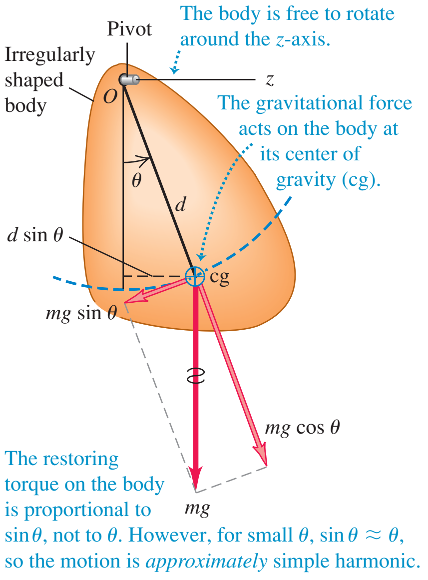
$$\tau_z = -(mgd)\theta$$

The equation of motion is $\sum\tau_z = I\alpha_z$, so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta \tag{14.37}$$

14.23 Dynamics of a physical pendulum.



Comparing this with Eq. (14.4), we see that the role of (k/m) for the spring-mass system is played here by the quantity (mgd/I). Thus the angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}} \quad \text{(physical pendulum, small amplitude)} \tag{14.38}$$

The frequency f is $1/2\pi$ times this, and the period T is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad \text{(physical pendulum, small amplitude)} \tag{14.39}$$

Equation (14.39) is the basis of a common method for experimentally determining the moment of inertia of a body with a complicated shape. First locate the center of gravity of the body by balancing. Then suspend the body so that it is free to oscillate about an axis, and measure the period T of small-amplitude oscillations. Finally, use Eq. (14.39) to calculate the moment of inertia I of the body about this axis from T, the body's mass m, and the distance d from the axis to the center of gravity (see Exercise 14.53). Biomechanics researchers use this method to find the moments of inertia of an animal's limbs. This information is important for analyzing how an animal walks, as we'll see in the second of the two following examples.

Example 14.9 Physical pendulum versus simple pendulum

If the body in Fig. 14.23 is a uniform rod with length L, pivoted at one end, what is the period of its motion as a pendulum?

SOLUTION

IDENTIFY and SET UP: Our target variable is the oscillation period T of a rod that acts as a physical pendulum. We find the rod's moment of inertia in Table 9.2, and then determine T using Eq. (14.39).

EXECUTE: The moment of inertia of a uniform rod about an axis through one end is $I = \frac{1}{3}ML^2$. The distance from the pivot to the rod's center of gravity is $d = L/2$. Then from Eq. (14.39),

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

EVALUATE: If the rod is a meter stick ($L = 1.00$ m) and $g = 9.80\ \text{m/s}^2$, then

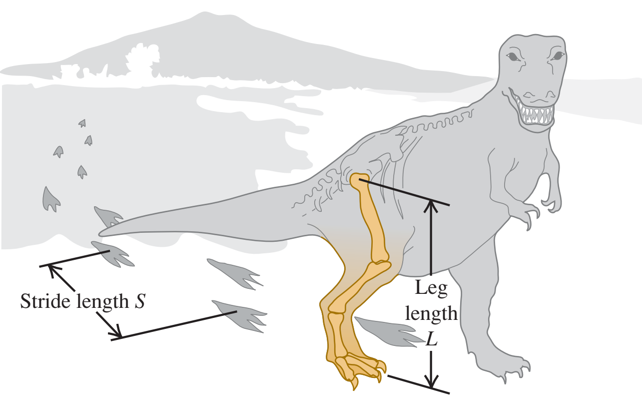
$$T = 2\pi\sqrt{\frac{2(1.00\ \text{m})}{3(9.80\ \text{m/s}^2)}} = 1.64\ \text{s}$$

The period is smaller by a factor of $\sqrt{\frac{2}{3}} = 0.816$ than that of a simple pendulum of the same length (see Example 14.8). The rod's moment of inertia around one end, $I = \frac{1}{3}ML^2$, is one-third that of the simple pendulum, and the rod's cg is half as far from the pivot as that of the simple pendulum. You can show that, taken together in Eq. (14.39), these two differences account for the factor $\sqrt{\frac{2}{3}}$ by which the periods differ.

Example 14.10 *Tyrannosaurus rex* and the physical pendulum

All walking animals, including humans, have a natural walking pace—a number of steps per minute that is more comfortable than a faster or slower pace. Suppose that this pace corresponds to the oscillation of the leg as a physical pendulum. (a) How does this pace depend on the length L of the leg from hip to foot? Treat the leg as a uniform rod pivoted at the hip joint. (b) Fossil evidence shows that *T. rex,* a two-legged dinosaur that lived about 65 million years ago, had a leg length $L = 3.1$ m and a stride length $S = 4.0$ m (the distance from one footprint to the next print of the same foot; see Fig. 14.24). Estimate the walking speed of *T. rex.*

14.24 The walking speed of *Tyrannosaurus rex* can be estimated from leg length L and stride length S.



SOLUTION

IDENTIFY and SET UP: Our target variables are (a) the relationship between walking pace and leg length L and (b) the walking speed of *T. rex.* We treat the leg as a physical pendulum, with a period of oscillation as found in Example 14.9. We can find the walking speed from the period and the stride length.

EXECUTE: (a) From Example 14.9 the period of oscillation of the leg is $T = 2\pi\sqrt{2L/3g}$, which is proportional to $\sqrt{L}$. Each step takes one-half a period, so the walking pace (in steps per second) is twice the oscillation frequency $f = 1/T$, which is proportional to $1/\sqrt{L}$. The greater the leg length L, the slower the walking pace.

(b) According to our model, *T. rex* traveled one stride length S in a time

$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(3.1\text{ m})}{3(9.8\text{ m/s}^2)}} = 2.9\text{ s}$$

so its walking speed was

$$v = \frac{S}{T} = \frac{4.0\text{ m}}{2.9\text{ s}} = 1.4\text{ m/s} = 5.0\text{ km/h} = 3.1\text{ mi/h}$$

This is roughly the walking speed of an adult human.

EVALUATE: A uniform rod isn't a very good model for a leg. The legs of many animals, including both *T. rex* and humans, are tapered; there is more mass between hip and knee than between knee and foot. The center of mass is therefore less than $L/2$ from the hip; a reasonable guess would be about $L/4$. The moment of inertia is therefore *considerably* less than $ML^2/3$—say, $ML^2/15$. Use the analysis of Example 14.9 with these corrections; you'll get a shorter oscillation period and an even greater walking speed for *T. rex*.

Test Your Understanding of Section 14.6 The center of gravity of a simple pendulum of mass m and length L is located at the position of the pendulum bob, a distance L from the pivot point. The center of gravity of a uniform rod of the same mass m and length $2L$ pivoted at one end is also a distance L from the pivot point. How does the period of this uniform rod compare to the period of the simple pendulum? (i) The rod has a longer period; (ii) the rod has a shorter period; (iii) the rod has the same period.

14.7 Damped Oscillations

The idealized oscillating systems we have discussed so far are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces, however, and oscillations die out with time unless we replace the dissipated mechanical energy (Fig. 14.25). A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears. But eventually the spring runs down or the weights reach the bottom of their travel. Then no more energy is available, and the pendulum swings decrease in amplitude and stop.

14.25 A swinging bell left to itself will eventually stop oscillating due to damping forces (air resistance and friction at the point of suspension).

The decrease in amplitude caused by dissipative forces is called **damping,** and the corresponding motion is called **damped oscillation.** The simplest case to analyze in detail is a simple harmonic oscillator with a frictional damping force that is directly proportional to the *velocity* of the oscillating body. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces. We then have an additional force on the body due to friction, $F_x = -bv_x$, where $v_x = dx/dt$ is the velocity and b is a constant that describes the strength of the damping force. The negative sign shows that the force is always opposite in direction to the velocity. The *net* force on the body is then

$$\sum F_x = -kx - bv_x \tag{14.40}$$

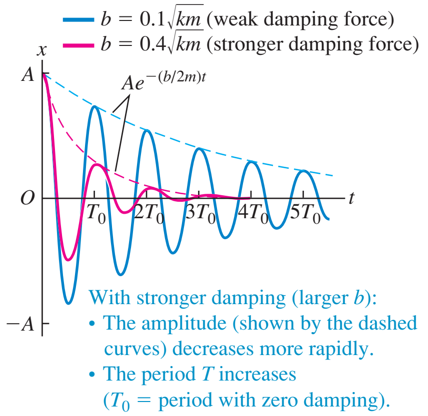
and Newton's second law for the system is

$$-kx - bv_x = ma_x \qquad \text{or} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{14.41}$$

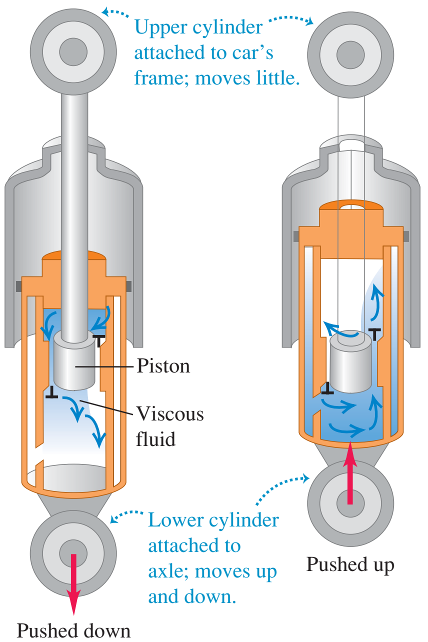
Equation (14.41) is a differential equation for x; it would be the same as Eq. (14.4), the equation for the acceleration in SHM, except for the added term $-b\,dx/dt$. Solving this equation is a straightforward problem in differential equations, but we won't go into the details here. If the damping force is relatively small, the motion is described by

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \quad \text{(oscillator with little damping)} \tag{14.42}$$

14.26 Graph of displacement versus time for an oscillator with little damping [see Eq. (14.42)] and with phase angle $\phi = 0$. The curves are for two values of the damping constant b.



14.27 An automobile shock absorber. The viscous fluid causes a damping force that depends on the relative velocity of the two ends of the unit.



The angular frequency of oscillation ω' is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad \text{(oscillator with little damping)} \tag{14.43}$$

You can verify that Eq. (14.42) is a solution of Eq. (14.41) by calculating the first and second derivatives of x, substituting them into Eq. (14.41), and checking whether the left and right sides are equal. This is a straightforward but slightly tedious procedure.

The motion described by Eq. (14.42) differs from the undamped case in two ways. First, the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the decreasing exponential factor $e^{-(b/2m)t}$. Figure 14.26 is a graph of Eq. (14.42) for the case $\phi = 0$; it shows that the larger the value of b, the more quickly the amplitude decreases.

Second, the angular frequency ω', given by Eq. (14.43), is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \qquad \text{or} \qquad b = 2\sqrt{km} \tag{14.44}$$

When Eq. (14.44) is satisfied, the condition is called **critical damping.** The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

If b is greater than $2\sqrt{km}$, the condition is called **overdamping.** Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping. For the overdamped case the solutions of Eq. (14.41) have the form

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}$$

where C_1 and C_2 are constants that depend on the initial conditions and a_1 and a_2 are constants determined by m, k, and b.

When b is less than the critical value, as in Eq. (14.42), the condition is called **underdamping.** The system oscillates with steadily decreasing amplitude.

In a vibrating tuning fork or guitar string, it is usually desirable to have as little damping as possible. By contrast, damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it doesn't continue bouncing forever (Fig. 14.27). For optimal passenger comfort, the system should be critically damped or slightly underdamped. Too much damping would be counterproductive; if the suspension is overdamped and the car hits a second bump just after the first one, the springs in the suspension will still be compressed somewhat from the first bump and will not be able to fully absorb the impact.

Energy in Damped Oscillations

In damped oscillations the damping force is nonconservative; the mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy E at any instant:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2$$

To find the rate of change of this quantity, we take its time derivative:

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

But $dv_x/dt = a_x$ and $dx/dt = v_x$, so

$$\frac{dE}{dt} = v_x(ma_x + kx)$$

From Eq. (14.41), $ma_x + kx = -bdx/dt = -bv_x$, so

$$\frac{dE}{dt} = v_x(-bv_x) = -bv_x^2 \quad \text{(damped oscillations)} \tag{14.45}$$

The right side of Eq. (14.45) is **negative** whenever the oscillating body is in motion, whether the x-velocity v_x is positive or negative. This shows that as the body moves, the energy decreases, though not at a uniform rate. The term $-bv_x^2 = (-bv_x)v_x$ (force times velocity) is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance, capacitance, and resistance. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant b. We will study these circuits in detail in Chapters 30 and 31.

Test Your Understanding of Section 14.7 An airplane is flying in a straight line at a constant altitude. If a wind gust strikes and raises the nose of the airplane, the nose will bob up and down until the airplane eventually returns to its original attitude. Are these oscillations (i) undamped, (ii) underdamped, (iii) critically damped, or (iv) overdamped?

14.8 Forced Oscillations and Resonance

A damped oscillator left to itself will eventually stop moving altogether. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic or cyclic way, with a definite period and frequency. As an example, consider your cousin Throckmorton on a playground swing. You can keep him swinging with constant amplitude by giving him a little push once each cycle. We call this additional force a **driving force.**

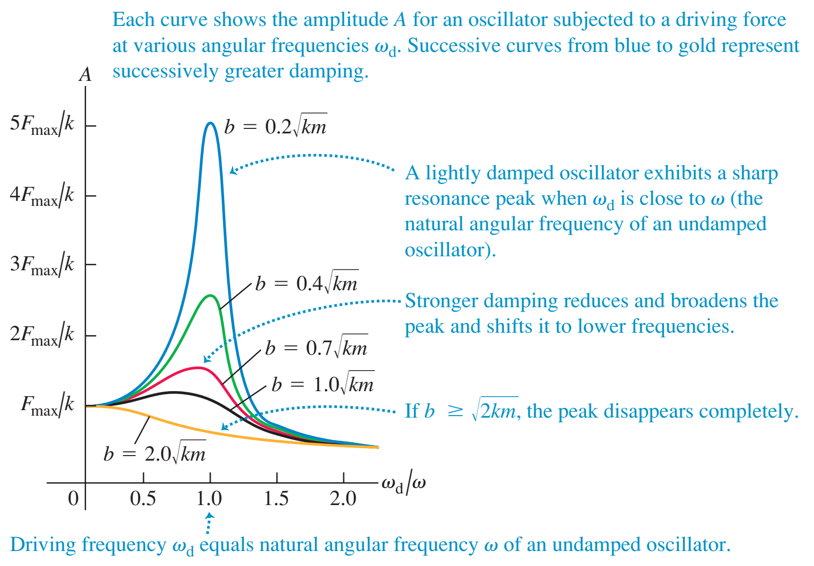
Damped Oscillation with a Periodic Driving Force

If we apply a periodically varying driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation.* It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency** ω' determined by m, k, and b, as in Eq. (14.43). In a forced oscillation, however, the angular frequency with which the mass oscillates is equal to the driving angular frequency ω_d. This does *not* have to be equal to the angular frequency ω' with which the system would oscillate without a driving force. If you grab the ropes of Throckmorton's swing, you can force the swing to oscillate with any frequency you like.

Suppose we force the oscillator to vibrate with an angular frequency ω_d that is nearly *equal* to the angular frequency ω' it would have with no driving force. What happens? The oscillator is naturally disposed to oscillate at $\omega = \omega'$, so we expect the amplitude of the resulting oscillation to be larger than when the two frequencies are very different. Detailed analysis and experiment show that this is just what happens. The easiest case to analyze is a *sinusoidally* varying force—say, $F(t) = F_{max} \cos \omega_d t$. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way (Fig. 14.28). When there is very little damping (small b), the amplitude goes through a sharp peak as the driving angular frequency ω_d nears the natural oscillation angular frequency ω'. When the damping is increased (larger b), the peak becomes broader and smaller in height and shifts toward lower frequencies.

We could work out an expression that shows how the amplitude A of the forced oscillation depends on the frequency of a sinusoidal driving force, with

14.28 Graph of the amplitude A of forced oscillation as a function of the angular frequency ω_d of the driving force. The horizontal axis shows the ratio of ω_d to the angular frequency $\omega = \sqrt{k/m}$ of an undamped oscillator. Each curve has a different value of the damping constant b.



maximum value F_{max}. That would involve more differential equations than we're ready for, but here is the result:

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad \text{(amplitude of a driven oscillator)} \tag{14.46}$$

When $k - m\omega_d^2 = 0$, the first term under the radical is zero, so A has a maximum near $\omega_d = \sqrt{k/m}$. The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low-frequency extreme, when $\omega_d = 0$, we get $A = F_{max}/k$. This corresponds to a *constant* force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium, as we might expect.

Resonance and Its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. A vibrating rattle in a car that occurs only at a certain engine speed or wheel-rotation speed is an all-too-familiar example. Inexpensive loudspeakers often have an annoying boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone or the speaker housing. In Chapter 16 we will study other examples of resonance that involve sound. Resonance also occurs in electric circuits, as we will see in Chapter 31; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.

Application Canine Resonance
Unlike humans, dogs have no sweat glands and so must pant in order to cool down. The frequency at which a dog pants is very close to the resonant frequency of its respiratory system. This causes the maximum amount of air to move in and out of the dog and so minimizes the effort that the dog must exert to cool itself.

Test Your Understanding of Section 14.8 When driven at a frequency near its natural frequency, an oscillator with very little damping has a much greater response than the same oscillator with more damping. When driven at a frequency that is much higher or lower than the natural frequency, which oscillator will have the greater response: (i) the one with very little damping or (ii) the one with more damping?