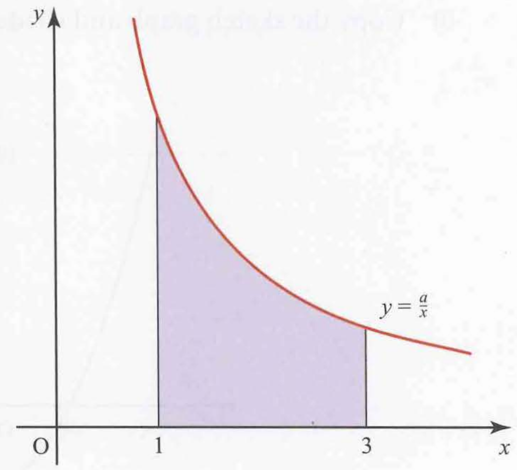
8 The diagram shows part of the curve $y = \dfrac{a}{x}$, where a is a positive constant.



Given that the volume obtained when the shaded region is rotated through 360° about the x axis is 24π, find the value of a.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q2 June 2010]

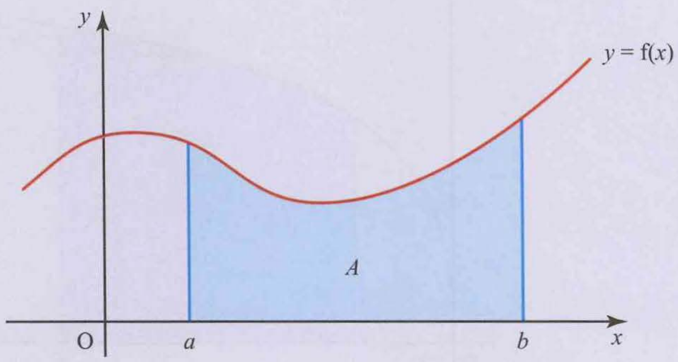
KEY POINTS

1 $\dfrac{dy}{dx} = x^n \quad \Rightarrow \quad y = \dfrac{x^{n+1}}{n+1} + c \qquad n \neq -1$

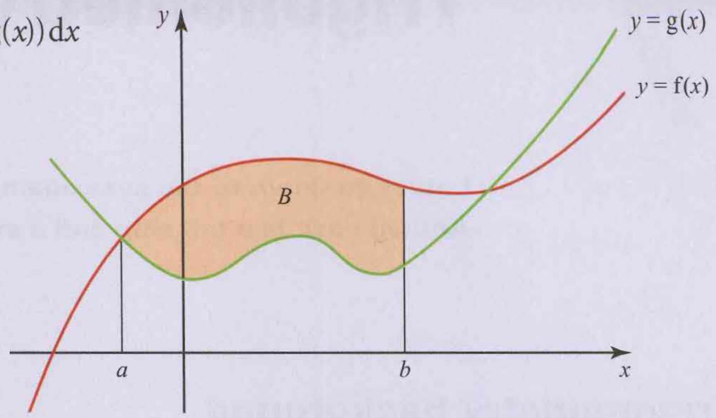
2 $\displaystyle\int_a^b x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1} \qquad n \neq -1$

3 Area $A = \displaystyle\int_a^b y \, dx$

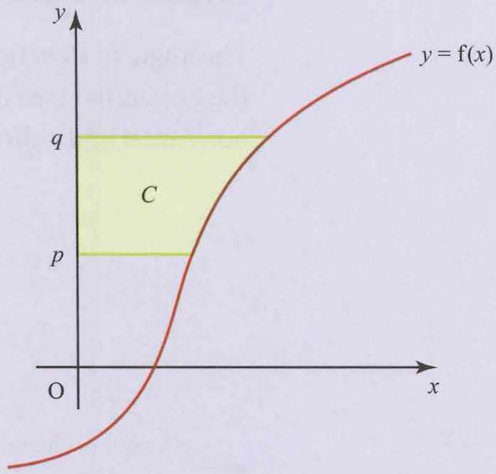
$= \displaystyle\int_a^b f(x) \, dx$



4 Area $B = \int_a^b (f(x) - g(x))\,dx$

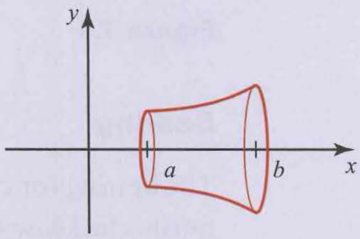


5 Area $C = \int_p^q x\,dy$

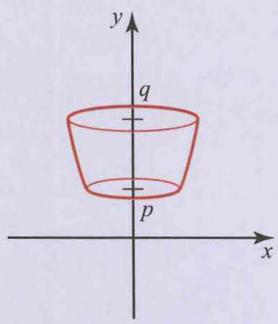


6 Volumes of revolution

About the x axis $V = \int_a^b \pi y^2\,dx$



About the y axis $V = \int_p^q \pi x^2\,dy$



7 Trigonometry

I must go down to the seas again, to the lonely sea and the sky,
And all I ask is a tall ship and a star to steer her by.

John Masefield

Trigonometry background

Angles of elevation and depression

The angle of elevation is the angle between the horizontal and a direction above the horizontal (see figure 7.1). The angle of depression is the angle between the horizontal and a direction below the horizontal (see figure 7.2).

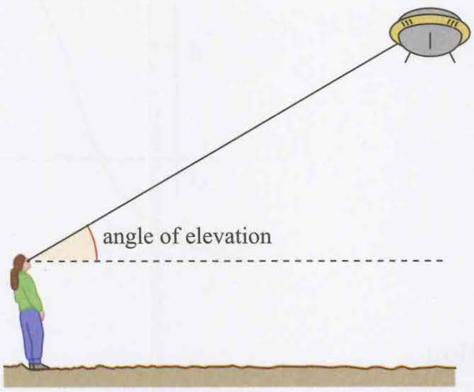


Figure 7.1

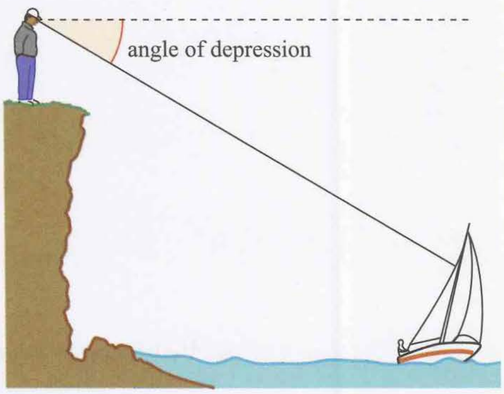


Figure 7.2

Bearing

The bearing (or compass bearing) is the direction measured as an angle from north, clockwise (see figure 7.3).

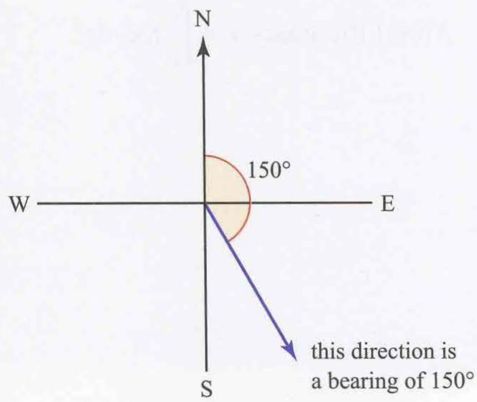


Figure 7.3

Trigonometrical functions

The simplest definitions of the trigonometrical functions are given in terms of the ratios of the sides of a right-angled triangle, for values of the angle θ between 0° and 90°.

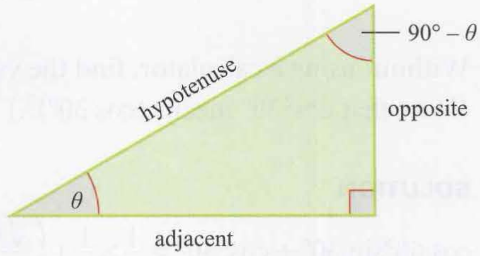


Figure 7.4

In figure 7.4

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Sin is an abbreviation of sine, cos of cosine and tan of tangent. You will see from the triangle in figure 7.4 that

$$\sin\theta = \cos(90° - \theta) \text{ and } \cos\theta = \sin(90° - \theta).$$

Special cases

Certain angles occur frequently in mathematics and you will find it helpful to know the value of their trigonometrical functions.

(i) *The angles 30° and 60°*

In figure 7.5, triangle ABC is an equilateral triangle with side 2 units, and AD is a line of symmetry.

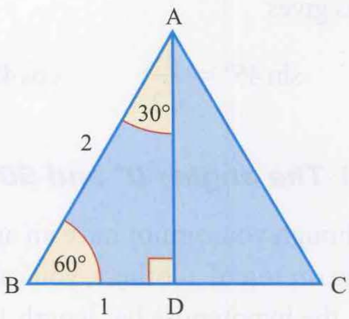


Figure 7.5

Using Pythagoras' theorem

$$AD^2 + 1^2 = 2^2 \Rightarrow AD = \sqrt{3}.$$

From triangle ABD,

$$\sin 60° = \frac{\sqrt{3}}{2}; \qquad \cos 60° = \frac{1}{2}; \qquad \tan 60° = \sqrt{3};$$

$$\sin 30° = \frac{1}{2}; \qquad \cos 30° = \frac{\sqrt{3}}{2}; \qquad \tan 30° = \frac{1}{\sqrt{3}}.$$

EXAMPLE 7.1

Without using a calculator, find the value of cos 60°sin 30° + cos²30°.
(Note that cos²30° means (cos 30°)².)

SOLUTION

$$\begin{aligned}\cos 60°\sin 30° + \cos^2 30° &= \frac{1}{2} \times \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1.\end{aligned}$$

(ii) *The angle 45°*

In figure 7.6, triangle PQR is a right-angled isosceles triangle with equal sides of length 1 unit.

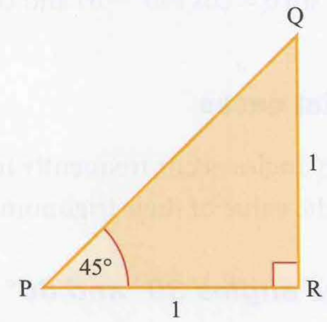


Figure 7.6

Using Pythagoras' theorem, $PQ = \sqrt{2}$.

This gives

$$\sin 45° = \frac{1}{\sqrt{2}}; \qquad \cos 45° = \frac{1}{\sqrt{2}}; \qquad \tan 45° = 1.$$

(iii) *The angles 0° and 90°*

Although you cannot have an angle of 0° in a triangle (because one side would be lying on top of another), you can still imagine what it might look like. In figure 7.7, the hypotenuse has length 1 unit and the angle at X is very small.

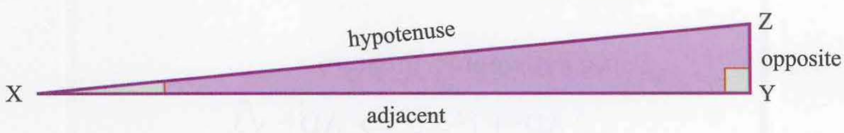


Figure 7.7

If you imagine the angle at X becoming smaller and smaller until it is zero, you can deduce that

$$\sin 0° = \tfrac{0}{1} = 0; \qquad \cos 0° = \tfrac{1}{1} = 1; \qquad \tan 0° = \tfrac{0}{1} = 0.$$

If the angle at X is 0°, then the angle at Z is 90°, and so you can also deduce that

$$\sin 90° = \tfrac{1}{1} = 1; \qquad \cos 90° = \tfrac{0}{1} = 0.$$

However when you come to find tan 90°, there is a problem. The triangle suggests this has value $\tfrac{1}{0}$, but you cannot divide by zero.

If you look at the triangle XYZ, you will see that what we actually did was to draw it with angle X not zero but just very small, and to argue:

'We can see from this what will happen if the angle becomes smaller and smaller so that it is effectively zero.'

? Compare this argument with the ideas about limits which you met in Chapters 5 and 6 on differentiation and integration.

In this case we are looking at the limits of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ as the angle θ approaches zero. The same approach can be used to look again at the problem of tan 90°.

If the angle X is not quite zero, then the side ZY is also not quite zero, and tan Z is 1 (XY is almost 1) divided by a very small number and so is large. The smaller the angle X, the smaller the side ZY and so the larger the value of tan Z. We conclude that in the limit when angle X becomes zero and angle Z becomes 90°, tan Z is infinitely large, and so we say

Read these arrows as 'tends to'.

as $Z \to 90°$, $\tan Z \to \infty$ (infinity).

You can see this happening in the table of values below.

Z	$\tan Z$
80°	5.67
89°	57.29
89.9°	572.96
89.99°	5729.6
89.999°	57 296

When Z actually equals 90°, we say that tan Z is *undefined*.

Positive and negative angles

Unless given in the form of bearings, angles are measured from the x axis (see figure 7.8). Anticlockwise is taken to be positive and clockwise to be negative.

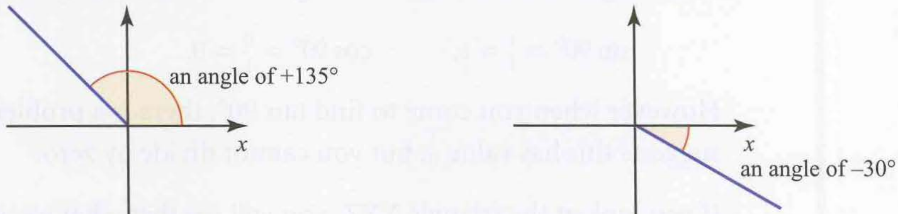


Figure 7.8

EXAMPLE 7.2

In the diagram, angles ADB and CBD are right angles, angle BAD = 60°, AB = $2l$ and BC = $3l$.

Find the angle θ.

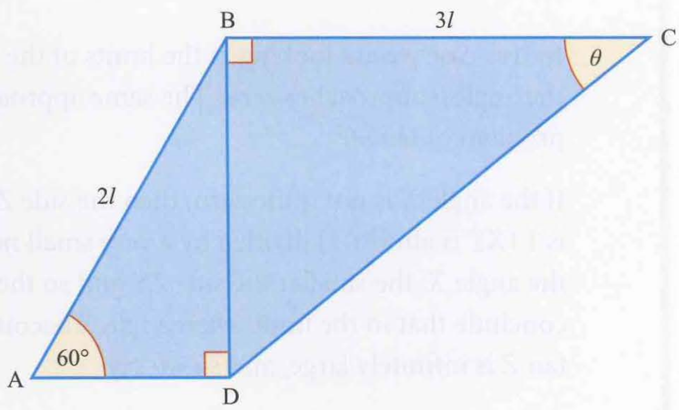


Figure 7.9

SOLUTION

First, find an expression for BD.

In triangle ABD, $\dfrac{BD}{AB} = \sin 60°$ AB = $2l$

$$\Rightarrow \quad BD = 2l\sin 60°$$

$$= 2l \times \frac{\sqrt{3}}{2}$$

$$= \sqrt{3}l$$

In triangle BCD, $\tan\theta = \frac{BD}{BC}$

$$= \frac{\sqrt{3}l}{3l}$$

$$= \frac{1}{\sqrt{3}}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 30°$$

EXERCISE 7A

1 In the triangle PQR, PQ = 17 cm, QR = 15 cm and PR = 8 cm.

(i) Show that the triangle is right-angled.

(ii) Write down the values of $\sin Q$, $\cos Q$ and $\tan Q$, leaving your answers as fractions.

(iii) Use your answers to part **(ii)** to show that

(a) $\sin^2 Q + \cos^2 Q = 1$

(b) $\tan Q = \frac{\sin Q}{\cos Q}$

2 Without using a calculator, show that:

(i) $\sin 60°\cos 30° + \cos 60°\sin 30° = 1$

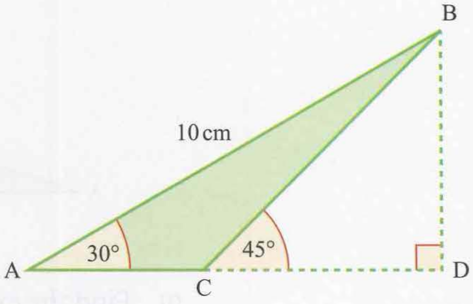
(ii) $\sin^2 30° + \sin^2 45° = \sin^2 60°$

(iii) $3\sin^2 30° = \cos^2 30°$.

3 In the diagram, AB = 10 cm, angle BAC = 30°, angle BCD = 45° and angle BDC = 90°.

(i) Find the length of BD.

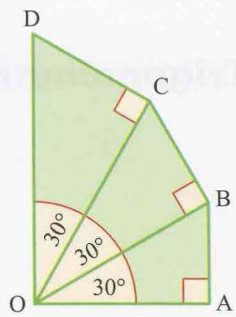
(ii) Show that AC $= 5\left(\sqrt{3} - 1\right)$ cm.



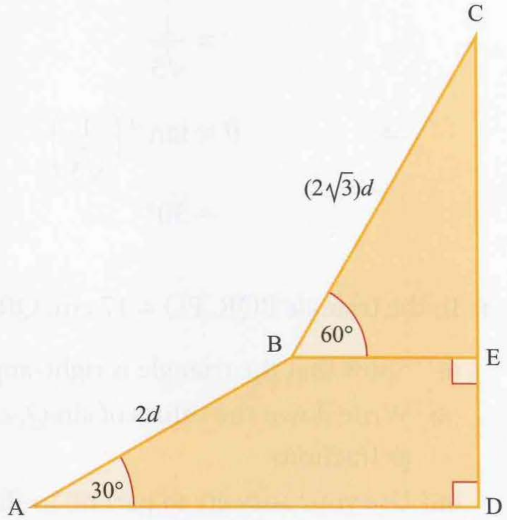
4 In the diagram, OA = 1 cm, angle AOB = angle BOC = angle COD = 30° and angle OAB = angle OBC = angle OCD = 90°.

(i) Find the length of OD giving your answer in the form $a\sqrt{3}$.

(ii) Show that the perimeter of OABCD is $\frac{5}{3}\left(1 + \sqrt{3}\right)$ cm.



5 In the diagram, ABED is a trapezium with right angles at E and D, and CED is a straight line. The lengths of AB and BC are $2d$ and $\left(2\sqrt{3}\right)d$ respectively, and angles BAD and CBE are 30° and 60° respectively.

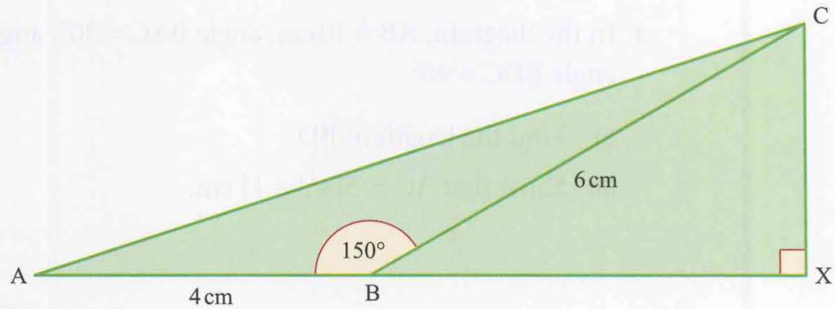


(i) Find the length of CD in terms of d.

(ii) Show that angle CAD $= \tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 November 2005]

6 In the diagram, ABC is a triangle in which AB = 4 cm, BC = 6 cm and angle ABC = 150°. The line CX is perpendicular to the line ABX.



(i) Find the exact length of BX and show that angle CAB $= \tan^{-1}\left(\dfrac{3}{4+3\sqrt{3}}\right)$

(ii) Show that the exact length of AC is $\sqrt{(52+24\sqrt{3})}$ cm.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 June 2006]

Trigonometrical functions for angles of any size

Is it possible to extend the use of the trigonometrical functions to angles greater than 90°, like sin 120°, cos 275° or tan 692°? The answer is yes – provided you change the definition of sine, cosine and tangent to one that does not require the angle to be in a right-angled triangle. It is not difficult to extend the definitions, as follows.

First look at the right-angled triangle in figure 7.10 which has hypotenuse of unit length.

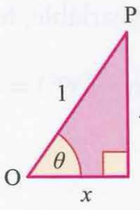


Figure 7.10

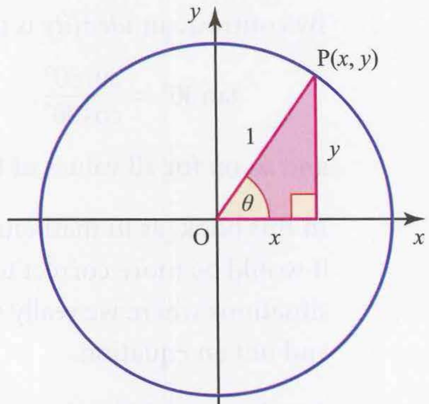


Figure 7.11

This gives rise to the definitions:

$$\sin\theta = \frac{y}{1} = y; \qquad \cos\theta = \frac{x}{1} = x; \qquad \tan\theta = \frac{y}{x}.$$

Now think of the angle θ being situated at the origin, as in figure 7.11, and allow θ to take any value. The vertex marked P has co-ordinates (x, y) and can now be anywhere on the unit circle.

You can now see that the definitions above can be applied to *any* angle θ, whether it is positive or negative, and whether it is less than or greater than 90°

$$\sin\theta = y, \qquad \cos\theta = x, \qquad \tan\theta = \frac{y}{x}.$$

For some angles, x or y (or both) will take a negative value, so the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$ will vary accordingly.

ACTIVITY 7.1 Draw x and y axes. For each of the four quadrants formed, work out the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$, from the definitions above.

Identities involving sin θ, cos θ and tan θ

Since $\tan\theta = \frac{y}{x}$ and $y = \sin\theta$ and $x = \cos\theta$ it follows that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

It would be more accurate here to use the identity sign, ≡, since the relationship is true for all values of θ

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}.$$

An *identity* is different from an equation since an equation is only true for certain values of the variable, called the *solution* of the equation. For example, $\tan\theta = 1$ is an equation: it is true when $\theta = 45°$ or $225°$, but not when it takes any other value in the range $0° \leqslant \theta \leqslant 360°$.

By contrast, an *identity* is true for all values of the variable, for example

$$\tan 30° = \frac{\sin 30°}{\cos 30°}, \quad \tan 72° = \frac{\sin 72°}{\cos 72°}, \quad \tan(-339°) = \frac{\sin(-399°)}{\cos(-399°)},$$

and so on for all values of the angle.

In this book, as in mathematics generally, we often use an equals sign where it would be more correct to use an identity sign. The identity sign is kept for situations where we really want to emphasise that the relationship is an identity and not an equation.

Another useful identity can be found by applying Pythagoras' theorem to any point $P(x, y)$ on the unit circle

$$y^2 + x^2 \equiv OP^2$$

$$(\sin\theta)^2 + (\cos\theta)^2 \equiv 1.$$

This is written as

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

You can use the identities $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ to prove other identities are true.

There are two methods you can use to prove an identity; you can use either method or a mixture of both.

Method 1

When both sides of the identity look equally complicated you can work with both the left-hand side (LHS) and the right-hand side (RHS) and show that LHS – RHS = 0.

EXAMPLE 7.3

Prove the identity $\cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1$.

SOLUTION

Both sides look equally complicated, so show LHS – RHS = 0.

So you need to show $\cos^2\theta - \sin^2\theta - 2\cos^2\theta + 1 \equiv 0$.

Simplifying:

$$\begin{aligned}\cos^2\theta - \sin^2\theta - 2\cos^2\theta + 1 &\equiv -\cos^2\theta - \sin^2\theta + 1\\ &\equiv -(\cos^2\theta + \sin^2\theta) + 1\\ &\equiv -1 + 1 \quad \text{Using } \sin^2\theta + \cos^2\theta = 1.\\ &\equiv 0 \text{ as required}\end{aligned}$$

Method 2

When one side of the identity looks more complicated than the other side, you can work with this side until you end up with the same as the simpler side.

EXAMPLE 7.4

Prove the identity $\dfrac{\cos\theta}{1-\sin\theta} - \dfrac{1}{\cos\theta} \equiv \tan\theta$.

SOLUTION

The LHS of this identity is more complicated, so manipulate the LHS until you end up with $\tan\theta$.

Write the LHS as a single fraction:

$$\frac{\cos\theta}{1-\sin\theta} - \frac{1}{\cos\theta} \equiv \frac{\cos^2\theta - (1-\sin\theta)}{\cos\theta(1-\sin\theta)}$$

$$\equiv \frac{\cos^2\theta + \sin\theta - 1}{\cos\theta(1-\sin\theta)}$$

Since $\sin^2\theta + \cos^2\theta \equiv 1$, $\cos^2\theta \equiv 1 - \sin^2\theta$

$$\equiv \frac{1-\sin^2\theta + \sin\theta - 1}{\cos\theta(1-\sin\theta)}$$

$$\equiv \frac{\sin\theta - \sin^2\theta}{\cos\theta(1-\sin\theta)} \equiv \frac{\sin\theta(1-\sin\theta)}{\cos\theta(1-\sin\theta)}$$

$$\equiv \frac{\sin\theta}{\cos\theta}$$

$\equiv \tan\theta$ as required

EXERCISE 7B

Prove each of the following identities.

1 $1-\cos^2\theta \equiv \sin^2\theta$

2 $(1-\sin^2\theta)\tan\theta \equiv \cos\theta\sin\theta$

3 $\dfrac{1}{\sin^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta} \equiv 1$

4 $\tan^2\theta \equiv \dfrac{1}{\cos^2\theta} - 1$

5 $\dfrac{\sin^2\theta - 3\cos^2\theta + 1}{\sin^2\theta - \cos^2\theta} \equiv 2$

6 $\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} \equiv \dfrac{1}{\cos^2\theta\sin^2\theta}$

7 $\tan\theta + \dfrac{\cos\theta}{\sin\theta} \equiv \dfrac{1}{\sin\theta\cos\theta}$

8 $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} \equiv \dfrac{2}{\cos^2\theta}$

9 Prove the identity $\dfrac{1-\tan^2 x}{1+\tan^2 x} \equiv 1 - 2\sin^2 x$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 June 2007]

10 Prove the identity $\dfrac{1+\sin x}{\cos x}+\dfrac{\cos x}{1+\sin x}\equiv\dfrac{2}{\cos x}$

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 November 2008]

11 Prove the identity $\dfrac{\sin x}{1-\sin x}-\dfrac{\sin x}{1+\sin x}\equiv 2\tan^2 x$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 June 2009]

The sine and cosine graphs

In figure 7.12, angles have been drawn at intervals of 30° in the unit circle, and the resulting y co-ordinates plotted relative to the axes on the right. They have been joined with a continuous curve to give the graph of $\sin\theta$ for $0° \leqslant \theta \leqslant 360°$.

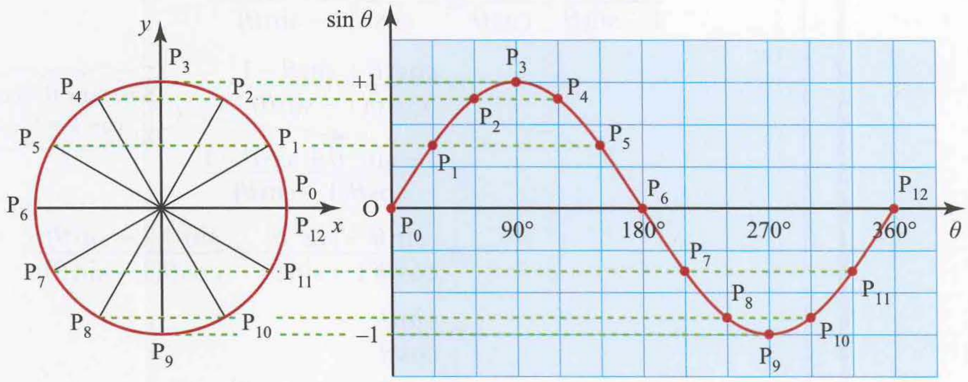


Figure 7.12

The angle 390° gives the same point P_1 on the circle as the angle 30°, the angle 420° gives point P_2 and so on. You can see that for angles from 360° to 720° the sine wave will simply repeat itself, as shown in figure 7.13. This is true also for angles from 720° to 1080° and so on.

Since the curve repeats itself every 360° the sine function is described as *periodic*, with *period* 360°.

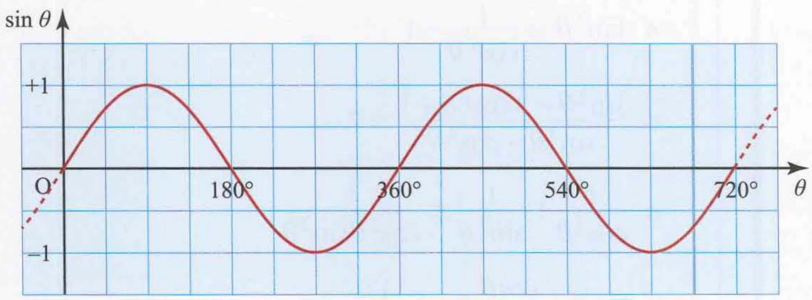


Figure 7.13

In a similar way you can transfer the x co-ordinates on to a set of axes to obtain the graph of $\cos\theta$. This is most easily illustrated if you first rotate the circle through 90° anticlockwise.

Figure 7.14 shows the circle in this new orientation, together with the resulting graph.

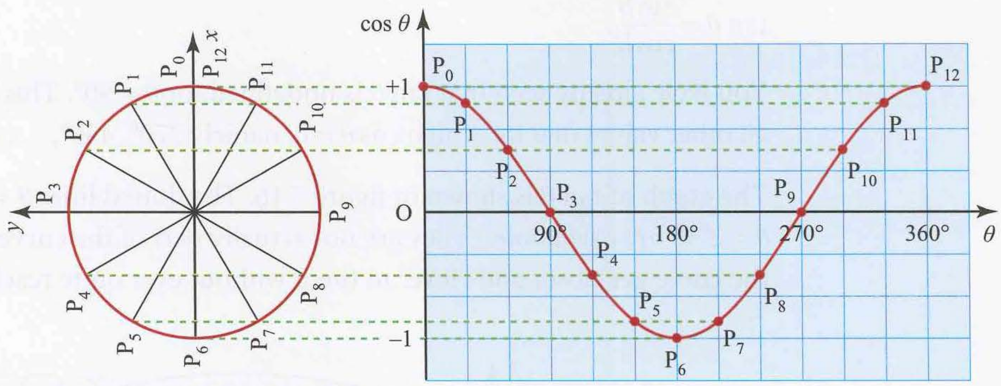


Figure 7.14

For angles in the interval $360° < \theta < 720°$, the cosine curve will repeat itself. You can see that the cosine function is also periodic with a period of 360°.

Notice that the graphs of $\sin\theta$ and $\cos\theta$ have exactly the same shape. The cosine graph can be obtained by translating the sine graph 90° to the left, as shown in figure 7.15.

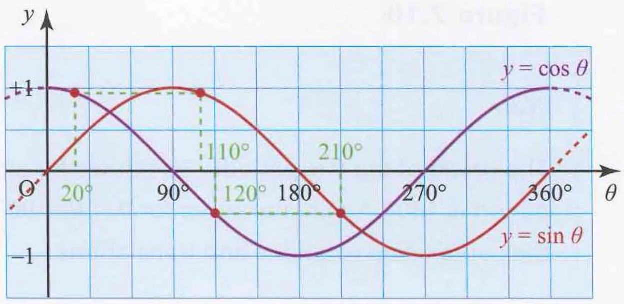


Figure 7.15

From the graphs it can be seen that, for example

$$\cos 20° = \sin 110°,\ \cos 90° = \sin 180°,\ \cos 120° = \sin 210°, \text{ etc.}$$

In general

$$\cos\theta \equiv \sin(\theta + 90°).$$

? **1** What do the graphs of $\sin\theta$ and $\cos\theta$ look like for negative angles?

2 Draw the curve of $\sin\theta$ for $0° \leqslant \theta \leqslant 90°$.

Using only reflections, rotations and translations of this curve, how can you generate the curves of $\sin\theta$ and $\cos\theta$ for $0° \leqslant \theta \leqslant 360°$?

The tangent graph

The value of $\tan\theta$ can be worked out from the definition $\tan\theta = \dfrac{y}{x}$ or by using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

You have already seen that $\tan\theta$ is undefined for $\theta = 90°$. This is also the case for all other values of θ for which $\cos\theta = 0$, namely 270°, 450°, …, and −90°, −270°, …

The graph of $\tan\theta$ is shown in figure 7.16. The dotted lines $\theta = \pm 90°$ and $\theta = 270°$ are *asymptotes.* They are not actually part of the curve. The branches of the curve get closer and closer to them without ever quite reaching them.

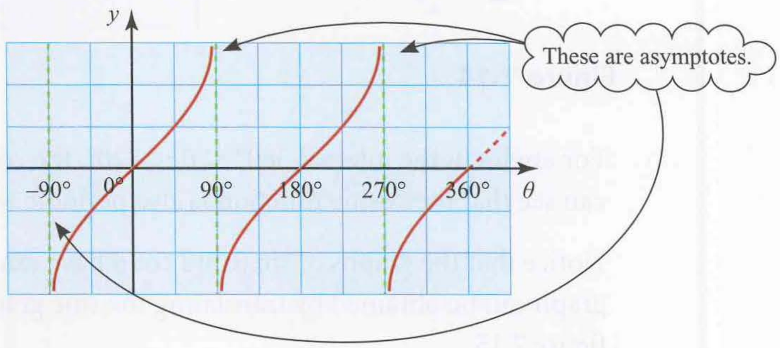


Figure 7.16

> *Note*
>
> The graph of $\tan\theta$ is periodic, like those for $\sin\theta$ and $\cos\theta$, but in this case the period is 180°. Again, the curve for $0 \leqslant \theta < 90°$ can be used to generate the rest of the curve using rotations and translations.

ACTIVITY 7.2 Draw the graphs of $y = \sin\theta$, $y = \cos\theta$, and $y = \tan\theta$ for values of θ between −90° and 450°.

These graphs are very important. Keep them handy because they will be useful for solving trigonometrical equations.

> *Note*
>
> Some people use this diagram to help them remember when sin, cos and tan are positive, and when they are negative. A means all positive in this quadrant, S means sin positive, cos and tan negative, etc.

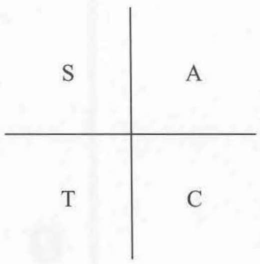


Figure 7.17

Solving equations using graphs of trigonometrical functions

Suppose that you want to solve the equation $\cos\theta = 0.5$.

You press the calculator keys for $\cos^{-1} 0.5$ (or arccos 0.5 or invcos 0.5), and the answer comes up as 60°.

However, by looking at the graph of $y = \cos\theta$ (your own or figure 7.18) you can see that there are in fact infinitely many roots to this equation.

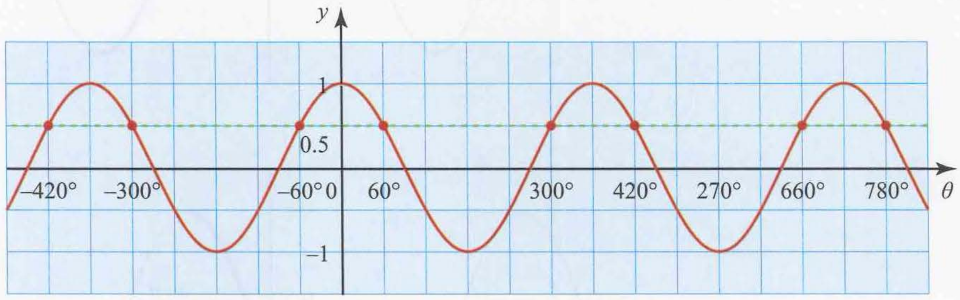


Figure 7.18

You can see from the graph of $y = \cos\theta$ that the roots for $\cos\theta = 0.5$ are:

$$\theta = \ldots, -420°, -300°, -60°, 60°, 300°, 420°, 660°, 780°, \ldots .$$

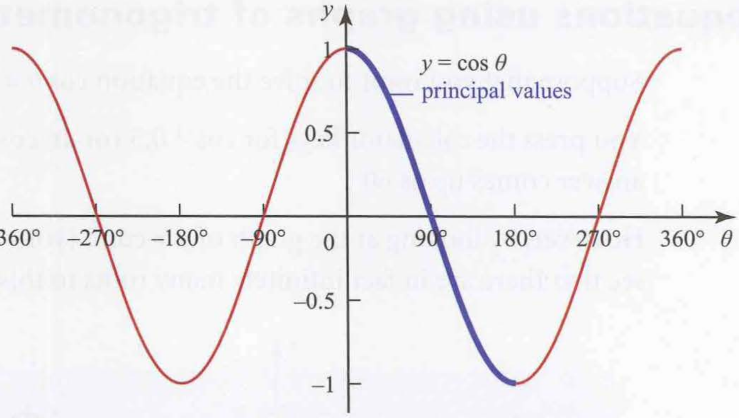
The functions cosine, sine and tangent are all many-to-one mappings, so their inverse mappings are one-to-many. Thus the problem 'find cos 60°' has only one solution, 0.5, whilst 'find θ such that $\cos\theta = 0.5$' has infinitely many solutions.

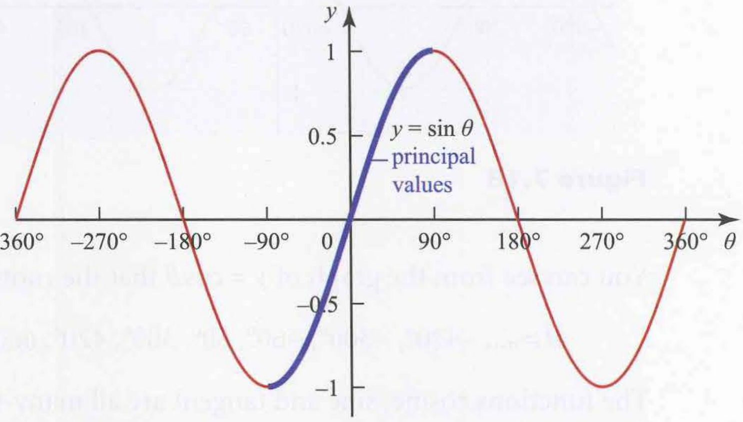
Remember, that a function has to be either one-to-one or many-to-one; so in order to define inverse functions for cosine, sine and tangent, a restriction has to be placed on the domain of each so that it becomes a one-to-one mapping. This means your calculator only gives one of the infinitely many solutions to the equation $\cos\theta = 0.5$. In fact, your calculator will always give the value of the solution between:

$$0° \leqslant \theta \leqslant 180° \quad \text{(cos)}$$
$$-90° \leqslant \theta \leqslant 90° \quad \text{(sin)}$$
$$-90° < \theta < 90° \quad \text{(tan).}$$

The solution that your calculator gives you is called *principal value.*

Figure 7.19 shows the graphs of cosine, sine and tangent together with their principal values. You can see from the graph that the principal values cover the whole of the range (y values) for each function.


y
1
0.5
−0.5
−1
y = cos θ
principal values
−360°
−270°
−180°
−90°
0
90°
180°
270°
360°
θ


y
1
0.5
−0.5
−1
y = sin θ
principal values
−360°
−270°
−180°
−90°
0
90°
180°
270°
360°
θ

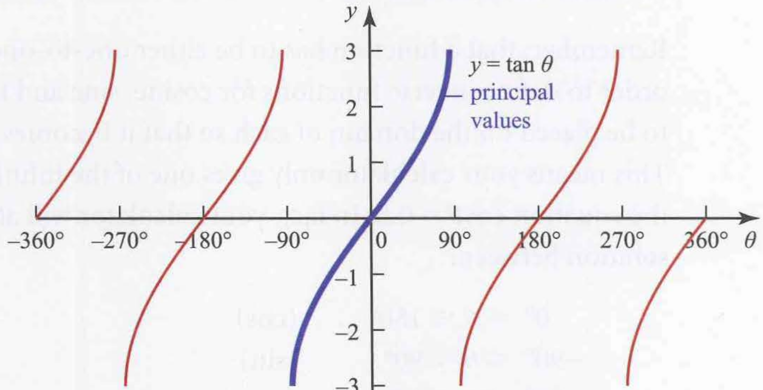

y
3
2
1
−1
−2
−3
y = tan θ
principal values
−360°
−270°
−180°
−90°
0
90°
180°
270°
360°
θ

Figure 7.19

EXAMPLE 7.5

Find values of θ in the interval $-360° \leqslant \theta \leqslant 360°$ for which $\sin\theta = 0.5$.

SOLUTION

$\sin\theta = 0.5 \Rightarrow \sin^{-1}0.5 = 30° \Rightarrow \theta = 30°$. Figure 7.20 shows the graph of $\sin\theta$.

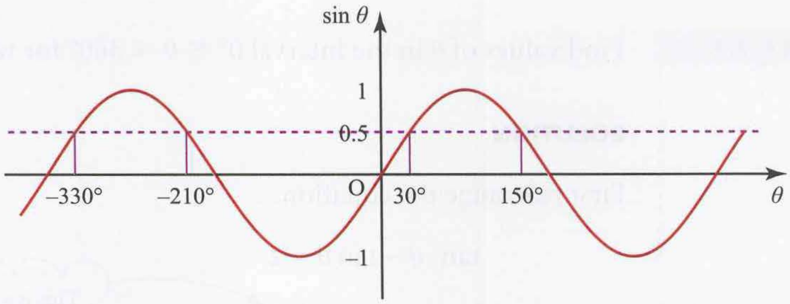


Figure 7.20

The values of θ for which $\sin\theta = 0.5$ are $-330°, -210°, 30°, 150°$.

EXAMPLE 7.6

Solve the equation $3\tan\theta = -1$ for $-180° \leqslant \theta \leqslant 180°$.

SOLUTION

$$\begin{aligned}
& 3\tan\theta = -1 \\
\Rightarrow\quad & \tan\theta = -\tfrac{1}{3} \\
\Rightarrow\quad & \theta = \tan^{-1}\left(-\tfrac{1}{3}\right) \\
\Rightarrow\quad & \theta = -18.4° \text{ to 1 d.p. (calculator).}
\end{aligned}$$

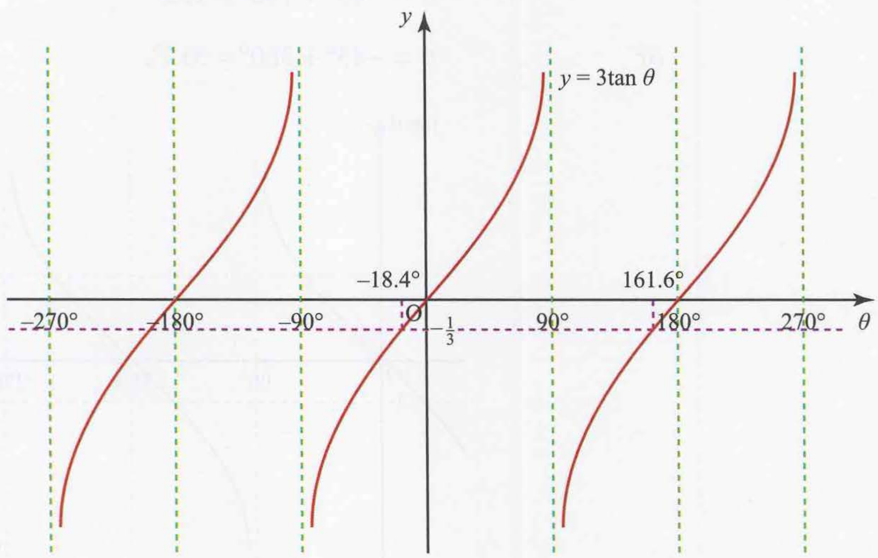


Figure 7.21

From figure 7.21, the other answer in the range is

$$\begin{aligned}
\theta &= -18.4° + 180° \\
&= 161.6°
\end{aligned}$$

The values of θ are $-18.4°$ or $161.6°$ to 1 d.p.

How can you find further roots of the equation $3\tan\theta = -1$, outside the range $-180° \leqslant \theta \leqslant 180°$?

EXAMPLE 7.7 Find values of θ in the interval $0° \leqslant \theta \leqslant 360°$ for which $\tan^2\theta - \tan\theta = 2$.

SOLUTION

First rearrange the equation.

$$\tan^2\theta - \tan\theta = 2$$

$$\Rightarrow \quad \tan^2\theta - \tan\theta - 2 = 0$$

This is a quadratic equation like $x^2 - x - 2 = 0$.

$$\Rightarrow \quad (\tan\theta - 2)(\tan\theta + 1) = 0$$

$$\Rightarrow \quad \tan\theta = 2 \text{ or } \tan\theta = -1.$$

$\tan\theta = 2 \;\Rightarrow\; \theta = 63.4°$ (calculator)

or $\theta = 63.4° + 180°$ (see figure 7.22)

$= 243.4°$.

$\tan\theta = -1 \;\Rightarrow\; \theta = -45°$ (calculator).

This is not in the range $0° \leqslant \theta \leqslant 360°$ so figure 7.22 is used to give

$\theta = -45° + 180° = 135°$

or $\theta = -45° + 360° = 315°$.

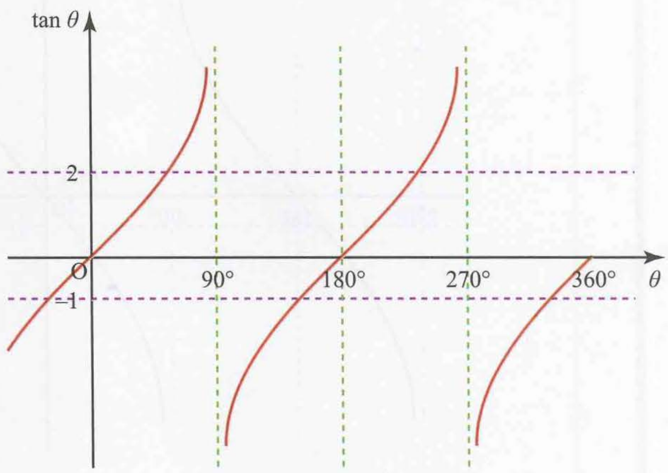


Figure 7.22

The values of θ are 63.4°, 135°, 243.4°, 315°.

EXAMPLE 7.8

Solve the equation $2\sin^2\theta = \cos\theta + 1$ for $0° \leqslant \theta \leqslant 360°$.

SOLUTION

First use the identity $\sin^2\theta + \cos^2\theta = 1$ to obtain an equation containing only one trigonometrical function.

$$2\sin^2\theta = \cos\theta + 1$$

$$\Rightarrow \quad 2(1 - \cos^2\theta) = \cos\theta + 1$$

$$\Rightarrow \quad 2 - 2\cos^2\theta = \cos\theta + 1$$

This is a quadratic equation in $\cos\theta$. Rearrange it to equal zero and factorise it to solve the equation.

$$\Rightarrow \quad 0 = 2\cos^2\theta + \cos\theta - 1$$

$$\Rightarrow \quad 0 = (2\cos\theta - 1)(\cos\theta + 1)$$

$$\Rightarrow \quad 2\cos\theta - 1 = 0 \text{ or } \cos\theta + 1 = 0$$

$$\Rightarrow \quad \cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = -1.$$

$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 60°$

or $\theta = 360° - 60° = 300°$ (see figure 7.23).

$\cos\theta = -1 \quad \Rightarrow \quad \theta = 180°.$

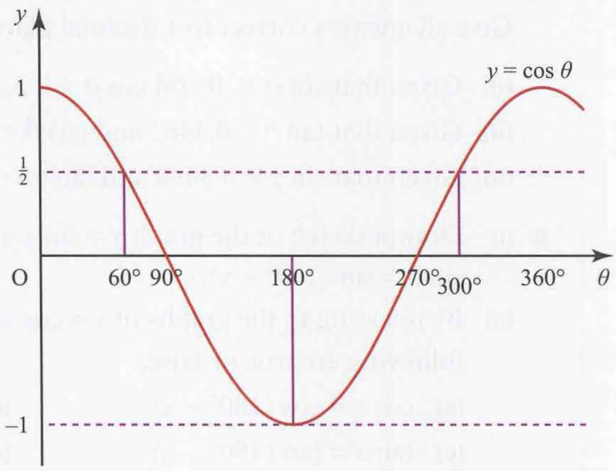


Figure 7.23

The values of θ are 60°, 180° or 300°.

EXERCISE 7C

1 **(i)** Sketch the curve $y = \sin x$ for $0° \leqslant x \leqslant 360°$.

(ii) Solve the equation $\sin x = 0.5$ for $0° \leqslant x \leqslant 360°$, and illustrate the two roots on your sketch.

(iii) State the other roots for $\sin x = 0.5$, given that x is no longer restricted to values between 0° and 360°.

(iv) Write down, without using your calculator, the value of $\sin 330°$.

2 **(i)** Sketch the curve $y = \cos x$ for $-90° \leqslant x \leqslant 450°$.

(ii) Solve the equation $\cos x = 0.6$ for $-90° \leqslant x \leqslant 450°$, and illustrate all the roots on your sketch.

(iii) Sketch the curve $y = \sin x$ for $-90° \leqslant x \leqslant 450°$.

(iv) Solve the equation $\sin x = 0.8$ for $-90° \leqslant x \leqslant 450°$, and illustrate all the roots on your sketch.

(v) Explain why some of the roots of $\cos x = 0.6$ are the same as those for $\sin x = 0.8$, and why some are different.

3 Solve the following equations for $0° \leqslant x \leqslant 360°$.

(i) $\tan x = 1$ **(ii)** $\cos x = 0.5$ **(iii)** $\sin x = -\frac{\sqrt{3}}{2}$

(iv) $\tan x = -1$ **(v)** $\cos x = -0.9$ **(vi)** $\cos x = 0.2$

(vii) $\sin x = -0.25$ **(viii)** $\cos x = -1$

4 Write the following as integers, fractions, or using square roots. You should not need your calculator.

(i) $\sin 60°$ **(ii)** $\cos 45°$ **(iii)** $\tan 45°$

(iv) $\sin 150°$ **(v)** $\cos 120°$ **(vi)** $\tan 180°$

(vii) $\sin 390°$ **(viii)** $\cos(-30°)$ **(ix)** $\tan 315°$

5 In this question all the angles are in the interval $-180°$ to $180°$. Give all answers correct to 1 decimal place.

(i) Given that $\sin\alpha < 0$ and $\cos\alpha = 0.5$, find α.

(ii) Given that $\tan\beta = 0.4463$ and $\cos\beta < 0$, find β.

(iii) Given that $\sin\gamma = 0.8090$ and $\tan\gamma > 0$, find γ.

6 **(i)** Draw a sketch of the graph $y = \sin x$ and use it to demonstrate why $\sin x = \sin(180° - x)$.

(ii) By referring to the graphs of $y = \cos x$ and $y = \tan x$, state whether the following are true or false.

(a) $\cos x = \cos(180° - x)$ **(b)** $\cos x = -\cos(180° - x)$

(c) $\tan x = \tan(180° - x)$ **(d)** $\tan x = -\tan(180° - x)$

7 **(i)** For what values of α are $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ all positive?

(ii) Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all negative? Explain your answer.

(iii) Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all equal? Explain your answer.

8 Solve the following equations for $0° \leqslant x \leqslant 360°$.

(i) $\sin x = 0.1$ **(ii)** $\cos x = 0.5$

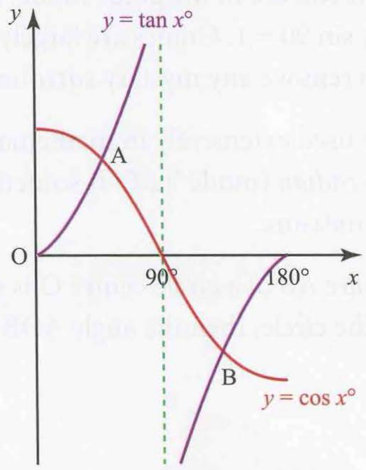
(iii) $\tan x = -2$ **(iv)** $\sin x = -0.4$

(v) $\sin^2 x = 1 - \cos x$ **(vi)** $\sin^2 x = 1$

(vii) $1 - \cos^2 x = 2\sin x$ **(viii)** $\sin^2 x = 2\cos^2 x$

(ix) $2\sin^2 x = 3\cos x$ **(x)** $3\tan^2 x - 10\tan x + 3 = 0$

9 The diagram shows part of the curves $y = \cos x°$ and $y = \tan x°$ which intersect at the points A and B. Find the co-ordinates of A and B.



10 (i) Show that the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$ can be written in the form $\tan x = -\frac{3}{4}$.

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \leqslant x \leqslant 360°$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q1 June 2010]

11 (i) Prove the identity $(\sin x + \cos x)(1 - \sin x \cos x) \equiv \sin^3 x + \cos^3 x$.

(ii) Solve the equation $(\sin x + \cos x)(1 - \sin x \cos x) = 9\sin^3 x$ for $0° \leqslant x \leqslant 360°$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q5 November 2009]

12 (i) Show that the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$ can be expressed as $\tan\theta = 3$.

(ii) Hence solve the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$, for $0° \leqslant \theta \leqslant 360°$

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 June 2005]

13 Solve the equation $3\sin^2\theta - 2\cos\theta - 3 = 0$, for $0° \leqslant x \leqslant 180°$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 November 2005]

Circular measure

Have you ever wondered why angles are measured in degrees, and why there are 360° in one revolution?

There are various legends to support the choice of 360, most of them based in astronomy. One of these is that since the shepherd-astronomers of Sumeria thought that the solar year was 360 days long, this number was then used by the ancient Babylonian mathematicians to divide one revolution into 360 equal parts.

Degrees are not the only way in which you can measure angles. Some calculators have modes which are called 'rad' and 'gra' (or 'grad'); if yours is one of these, you have probably noticed that these give different answers when you are using the sin, cos or tan keys. These answers are only wrong when the calculator mode is different from the units being used in the calculation.

The *grade* (mode 'gra') is a unit which was introduced to give a means of angle measurement which was compatible with the metric system. There are 100 grades in a right angle, so when you are in the grade mode, sin 100 = 1, just as when you are in the degree mode, sin 90 = 1. Grades are largely of historical interest and are only mentioned here to remove any mystery surrounding this calculator mode.

By contrast, radians are used extensively in mathematics because they simplify many calculations. The *radian* (mode 'rad') is sometimes referred to as the natural unit of angular measure.

If, as in figure 7.24, the arc AB of a circle centre O is drawn so that it is equal in length to the radius of the circle, then the angle AOB is 1 radian, about 57.3°.

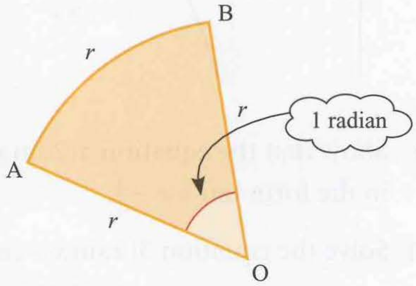


Figure 7.24

You will sometimes see 1 radian written as 1^c, just as 1 degree is written 1°.

Since the circumference of a circle is given by $2\pi r$, it follows that the angle of a complete turn is 2π radians.

$$360° = 2\pi \text{ radians}$$

Consequently

$$180° = \pi \text{ radians}$$

$$90° = \frac{\pi}{2} \text{ radians}$$

$$60° = \frac{\pi}{3} \text{ radians}$$

$$45° = \frac{\pi}{4} \text{ radians}$$

$$30° = \frac{\pi}{6} \text{ radians}$$

To convert degrees into radians you multiply by $\frac{\pi}{180}$.

To convert radians into degrees multipy by $\frac{180}{\pi}$.

Note

1 If an angle is a simple fraction or multiple of 180° and you wish to give its value in radians, it is usual to leave the answer as a fraction of π.

2 When an angle is given as a multiple of π it is assumed to be in radians.

EXAMPLE 7.9

(i) Express in radians **(a)** 30° **(b)** 315° **(c)** 29°.

(ii) Express in degrees **(a)** $\frac{\pi}{12}$ **(b)** $\frac{8\pi}{3}$ **(c)** 1.2 radians.

SOLUTION

(i) (a) $30° = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

(b) $315° = 315 \times \frac{\pi}{180} = \frac{7\pi}{4}$

(c) $29° = 29 \times \frac{\pi}{180} = 0.506$ radians (to 3 s.f.).

(ii) (a) $\frac{\pi}{12} = \frac{\pi}{12} \times \frac{180}{\pi} = 15°$

(b) $\frac{8\pi}{3} = \frac{8\pi}{3} \times \frac{180}{\pi} = 480°$

(c) $1.2 \text{ radians} = 1.2 \times \frac{180}{\pi} = 68.8°$ (to 3 s.f.).

Using your calculator in radian mode

If you wish to find the value of, say, $\sin 1.4^c$ or $\cos \frac{\pi}{12}$, use the 'rad' mode on your calculator. This will give the answers directly – in these examples 0.9854… and 0.9659… .

You could alternatively convert the angles into degrees $\left(\text{by multiplying by } \frac{180}{\pi}\right)$ but this would usually be a clumsy method. It is much better to get into the habit of working in radians.

EXAMPLE 7.10

Solve $\sin\theta = \frac{1}{2}$ for $0 < \theta < 2\pi$ giving your answers as multiples of π.

SOLUTION

Since the answers are required as multiples of π it is easier to work in degrees first.

$$\sin\theta = \tfrac{1}{2} \Rightarrow \theta = 30°$$

$$\theta = 30 \times \frac{\pi}{180} = \frac{\pi}{6}.$$

From figure 7.25 there is a second value

$$\theta = 150° = \frac{5\pi}{6}.$$

The values of θ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

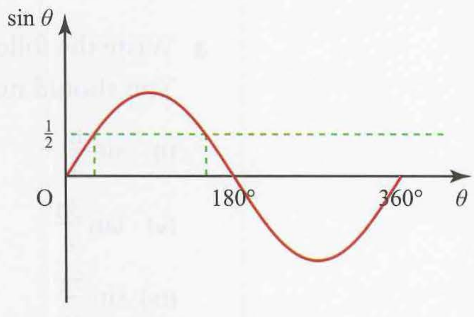


Figure 7.25

P1 7 Trigonometry

EXAMPLE 7.11

Solve $\tan^2\theta = 2$ for $0 < \theta < \pi$.

SOLUTION

Here the range $0 < \theta < \pi$ indicates that radians are required.

Since there is no request for multiples of π, set your calculator to radians.

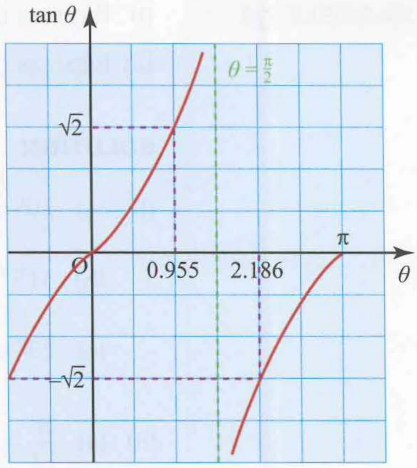


Figure 7.26

$\tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$ or $\tan\theta = -\sqrt{2}$.

$\tan\theta = \sqrt{2} \quad \Rightarrow \theta = 0.955$ radians

$\tan\theta = -\sqrt{2} \quad \Rightarrow \theta = -0.955$ (not in range)

or $\theta = -0.955 + \pi = 2.186$ radians.

The values of θ are 0.955 radians and 2.186 radians.

EXERCISE 7D

1 Express the following angles in radians, leaving your answers in terms of π where appropriate.

(i) 45° **(ii)** 90° **(iii)** 120° **(iv)** 75°
(v) 300° **(vi)** 23° **(vii)** 450° **(viii)** 209°
(ix) 150° **(x)** 7.2°

2 Express the following angles in degrees, using a suitable approximation where necessary.

(i) $\dfrac{\pi}{10}$ **(ii)** $\dfrac{3\pi}{5}$ **(iii)** 2 radians **(iv)** $\dfrac{4\pi}{9}$
(v) 3π **(vi)** $\dfrac{5\pi}{3}$ **(vii)** 0.4 radians **(viii)** $\dfrac{3\pi}{4}$
(ix) $\dfrac{7\pi}{3}$ **(x)** $\dfrac{3\pi}{7}$

3 Write the following as fractions, or using square roots.
You should not need your calculator.

(i) $\sin\dfrac{\pi}{4}$ **(ii)** $\tan\dfrac{\pi}{3}$ **(iii)** $\cos\dfrac{\pi}{6}$ **(iv)** $\cos\pi$
(v) $\tan\dfrac{3\pi}{4}$ **(vi)** $\sin\dfrac{2\pi}{3}$ **(vii)** $\tan\dfrac{4\pi}{3}$ **(viii)** $\cos\dfrac{3\pi}{4}$
(ix) $\sin\dfrac{5\pi}{6}$ **(x)** $\cos\dfrac{5\pi}{3}$

4 Solve the following equation for $0 \leqslant \theta \leqslant 2\pi$, giving your answers as multiples of π.

(i) $\cos\theta = \dfrac{\sqrt{3}}{2}$ **(ii)** $\tan\theta = 1$ **(iii)** $\sin\theta = \dfrac{1}{\sqrt{2}}$
(iv) $\sin\theta = -\dfrac{1}{2}$ **(v)** $\cos\theta = -\dfrac{1}{\sqrt{2}}$ **(vi)** $\tan\theta = \sqrt{3}$

5 Solve the following equations for $-\pi < \theta < \pi$.

(i) $\sin\theta = 0.2$ **(ii)** $\cos\theta = 0.74$ **(iii)** $\tan\theta = 3$

(iv) $4\sin\theta = -1$ **(v)** $\cos\theta = -0.4$ **(vi)** $2\tan\theta = -1$

6 Solve $3\cos^2\theta + 2\sin\theta - 3 = 0$ for $0 \leqslant \theta \leqslant \pi$.

The length of an arc of a circle

From the definition of a radian, an angle of 1 radian at the centre of a circle corresponds to an arc of length r (the radius of the circle). Similarly, an angle of 2 radians corresponds to an arc length of $2r$ and, in general, an angle of θ radians corresponds to an arc length of θr, which is usually written $r\theta$ (figure 7.27).

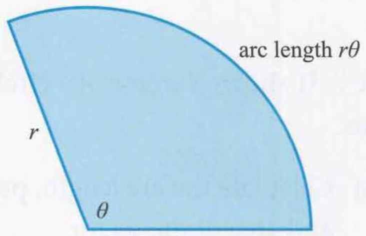


Figure 7.27

The area of a sector of a circle

A *sector* of a circle is the shape enclosed by an arc of the circle and two radii. It is the shape of a piece of cake. If the sector is smaller than a semi-circle it is called a *minor sector*; if it is larger than a semi-circle it is a *major sector*, see figure 7.28.

The area of a sector is a fraction of the area of the whole circle. The fraction is found by writing the angle θ as a fraction of one revolution, i.e. 2π (figure 7.29).

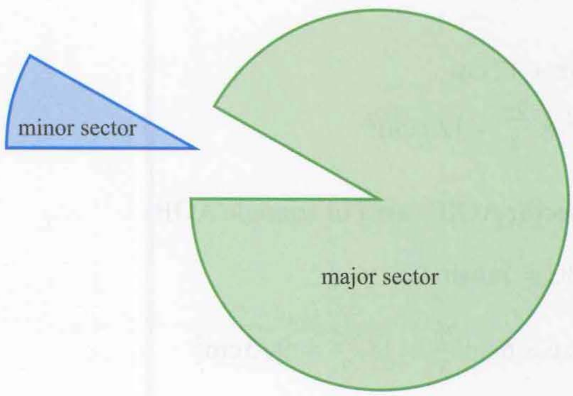


Figure 7.28

$$\text{Area} = \frac{\theta}{2\pi} \times \pi r^2 = \tfrac{1}{2}r^2\theta.$$

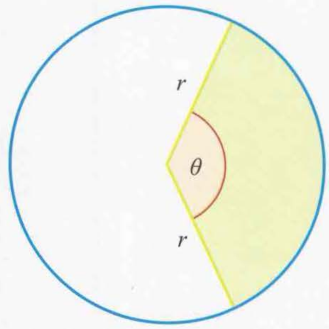


Figure 7.29

The following formulae often come in useful when solving problems involving sectors of circles.

For any triangle ABC:

The sine rule: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The cosine rule: $$a^2 = b^2 + c^2 - 2bc\cos A$$

or $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The area of any triangle ABC $= \frac{1}{2}ab\sin C$.

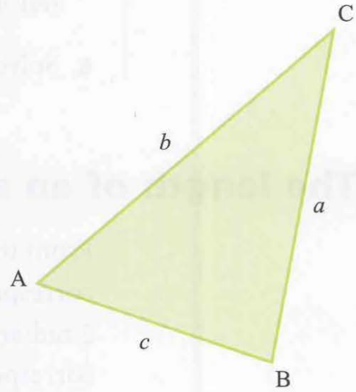


Figure 7.30

EXAMPLE 7.12

Figure 7.31 shows a sector of a circle, centre O, radius 6 cm. Angle AOB $= \frac{2\pi}{3}$ radians.

(i) **(a)** Calculate the arc length, perimeter and area of the sector.

(b) Find the area of the blue region.

This is called a segment of the circle.

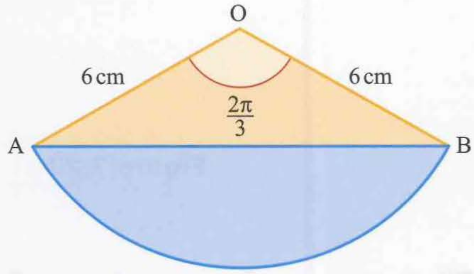


Figure 7.31

(ii) Find the exact length of the chord AB.

SOLUTION

(i) **(a)** Arc length $= r\theta$

$$= 6 \times \frac{2\pi}{3}$$

$$= 4\pi \text{ cm}$$

Perimeter $= 4\pi + 6 + 6 = 4\pi + 12$ cm

Area $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 6^2 \times \frac{2\pi}{3} = 12\pi \text{ cm}^2$

(b) Area of segment = area of sector AOB – area of triangle AOB

The area of any triangle ABC $= \frac{1}{2}ab\sin C$.

Area of triangle AOB $= \frac{1}{2} \times 6 \times 6 \sin\frac{2\pi}{3} = 18\frac{\sqrt{3}}{2} = 9\sqrt{3}\text{ cm}^2$

So area of segment $= 12\pi - 9\sqrt{3}$

$$= 22.1 \text{ cm}^2$$

(ii) Use the cosine rule to find the length of the chord AB

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute in $b = 6$, $c = 6$ and $A = \frac{2\pi}{3}$

So $a^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \cos\frac{2\pi}{3}$

$$= 72 - 72 \times \left(-\tfrac{1}{2}\right) = 108$$

$$a = \sqrt{108} = 6\sqrt{3}\,\text{cm}$$

? How else could you find the area of triangle AOB and the length of AB?

EXERCISE 7E

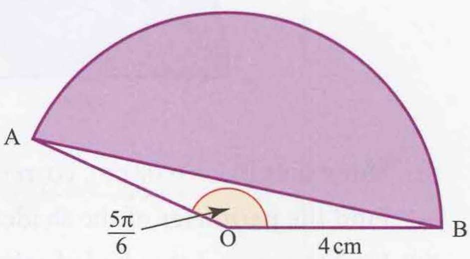
1 Each row of the table gives dimensions of a sector of a circle of radius r cm. The angle subtended at the centre of the circle is θ radians, the arc length of the sector is s cm and its area is A cm^2. Copy and complete the table.

r (cm)	θ (rad)	s (cm)	A (cm^2)
5	$\frac{\pi}{4}$		
8	1		
4		2	
	$\frac{\pi}{3}$	$\frac{\pi}{2}$	
5			10
	0.8	1.5	
	$\frac{2\pi}{3}$		4π

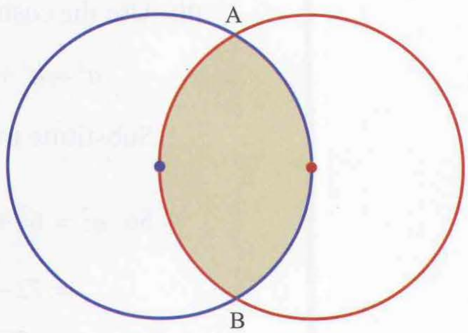
2 **(i)** **(a)** Find the area of the sector OAB in the diagram.

(b) Show that the area of triangle OAB is $16\sin\frac{5\pi}{12}\cos\frac{5\pi}{12}$.

(c) Find the shaded area.



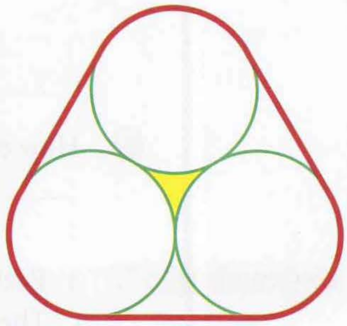
(ii) The diagram shows two circles, each of radius 4 cm, with each one passing through the centre of the other. Calculate the shaded area. (Hint: Add the common chord AB to the sketch.)



3 The diagram shows the cross-section of three pencils, each of radius 3.5 mm, held together by a stretched elastic band. Find

(i) the shaded area

(ii) the stretched length of the band.



4 A circle, centre O, has two radii OA and OB. The line AB divides the circle into two regions with areas in the ratio 3:1.
If the angle AOB is θ (radians), show that

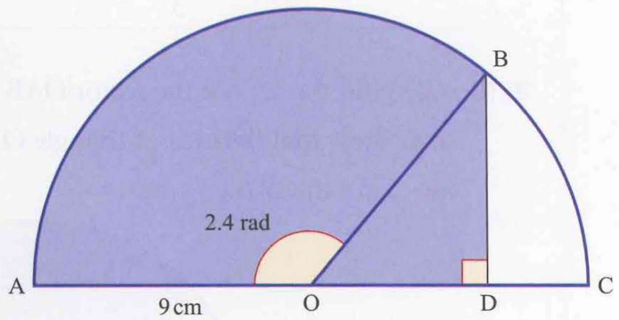
$$\theta - \sin\theta = \frac{\pi}{2}.$$

5 In a cricket match, a particular cricketer generally hits the ball anywhere in a sector of angle 100°. If the boundary (assumed circular) is 80 yards away, find

(i) the length of boundary which the fielders should patrol

(ii) the area of the ground which the fielders need to cover.

6 In the diagram, ABC is a semi-circle, centre O and radius 9 cm. The line BD is perpendicular to the diameter AC and angle AOB = 2.4 radians.



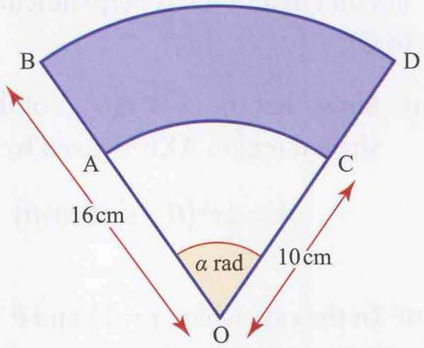
(i) Show that BD = 6.08 cm, correct to 3 significant figures.

(ii) Find the perimeter of the shaded region.

(iii) Find the area of the shaded region.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2005]

7 In the diagram, OAB and OCD are radii of a circle, centre O and radius 16 cm. Angle AOC = α radians. AC and BD are arcs of circles, centre O and radii 10 cm and 16 cm respectively.

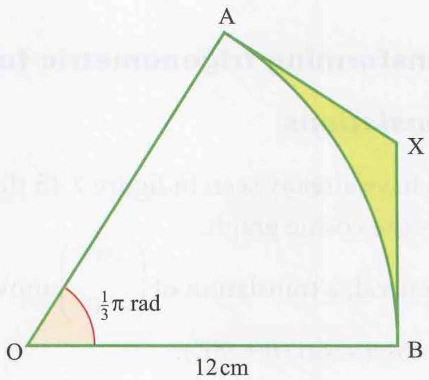


(i) In the case where $\alpha = 0.8$, find the area of the shaded region.

(ii) Find the value of α for which the perimeter of the shaded region is 28.9 cm.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 November 2005]

8 In the diagram, OAB is a sector of a circle with centre O and radius 12 cm. The lines AX and BX are tangents to the circle at A and B respectively. Angle AOB = $\frac{1}{3}\pi$ radians.



(i) Find the exact length of AX, giving your answer in terms of $\sqrt{3}$.

(ii) Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$.

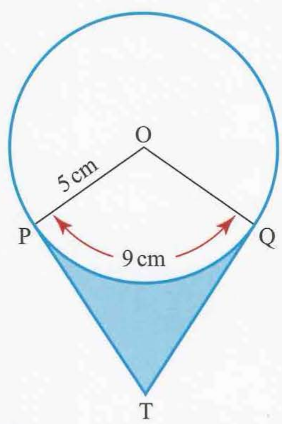
[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 June 2007]

9 In the diagram, the circle has centre O and radius 5 cm. The points P and Q lie on the circle, and the arc length PQ is 9 cm. The tangents to the circle at P and Q meet at the point T. Calculate

(i) angle POQ in radians

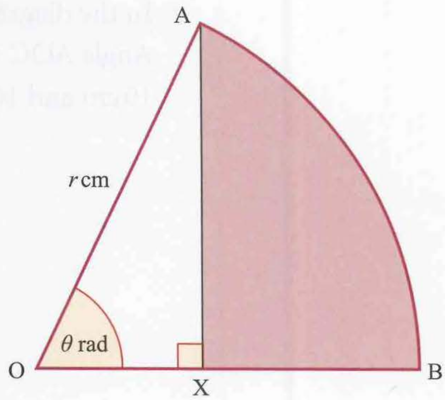
(ii) the length of PT

(iii) the area of the shaded region.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 November 2008]

10 In the diagram, AB is an arc of a circle, centre O and radius r cm, and angle AOB = θ radians. The point X lies on OB and AX is perpendicular to OB.



(i) Show that the area, A cm^2, of the shaded region AXB is given by

$$A = \tfrac{1}{2}r^2(\theta - \sin\theta\cos\theta)$$

(ii) In the case where $r = 12$ and $\theta = \frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 November 2007]

Other trigonometrical functions

You need to be able to sketch and work with other trigonometrical functions. Using transformations often helps you to do this.

e Transforming trigonometric functions

Translations

You have already seen in figure 7.15 that translating the sine graph 90° to the left gives the cosine graph.

In general, a translation of $\begin{pmatrix} -90° \\ 0 \end{pmatrix}$ moves the graph of $y = f(\theta)$ to $y = f(\theta + 90°)$.

So $\cos\theta = \sin(\theta + 90°)$.

Results from translations can also be used in plotting graphs such as $y = \sin\theta + 1$. This is the graph of $y = \sin\theta$ translated by 1 unit upwards, as shown in figure 7.32.

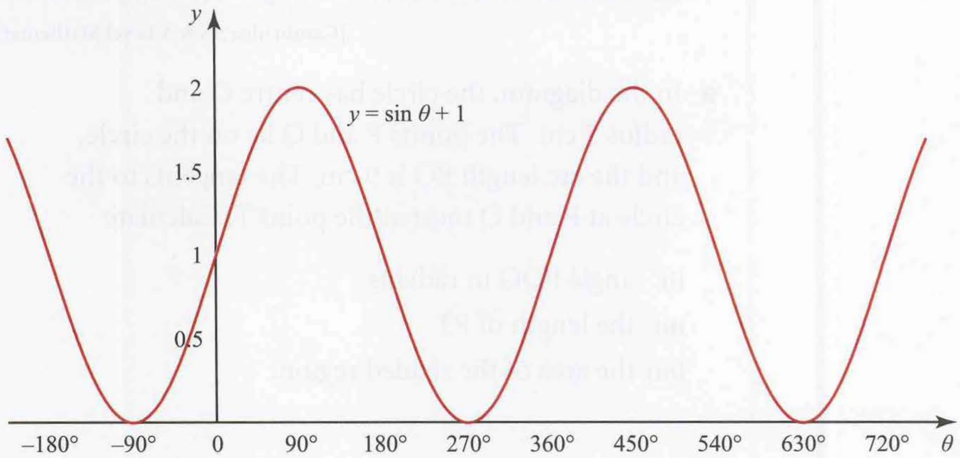
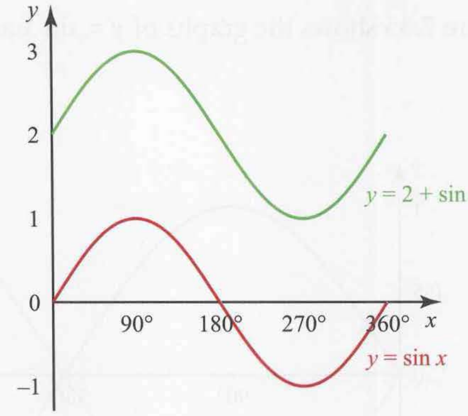


Figure 7.32

ACTIVITY 7.3 Figure 7.33 shows the graphs of $y = \sin x$ and $y = 2 + \sin x$ for $0° \leqslant x \leqslant 360°$.



If you have a graphics calculator, use it to experiment with other curves like these.

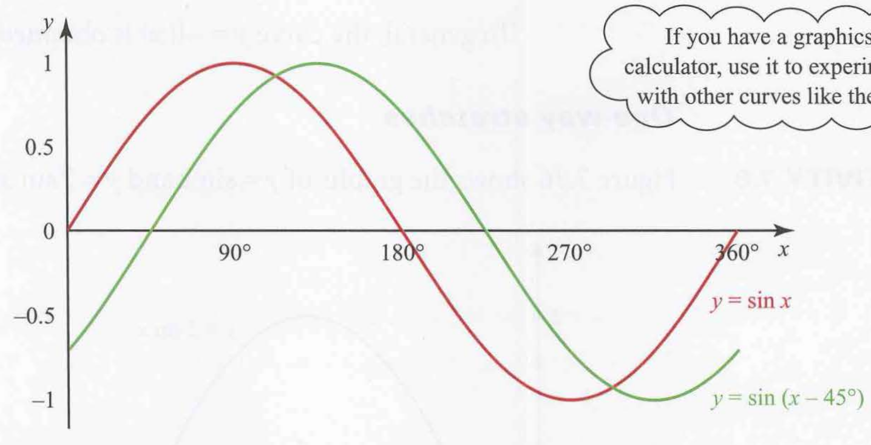
Figure 7.33

Describe the transformation that maps the curve $y = \sin x$ on to the curve $y = 2 + \sin x$.

Complete this statement.

'In general, the curve $y = f(x) + s$ is obtained from $y = f(x)$ by ... .'

ACTIVITY 7.4 Figure 7.34 shows the graphs of $y = \sin x$ and $y = \sin (x - 45°)$ for $0° \leqslant x \leqslant 360°$.



If you have a graphics calculator, use it to experiment with other curves like these.

Figure 7.34

Describe the transformation that maps the curve $y = \sin x$ on to the curve $y = \sin (x - 45°)$.

Complete this statement.

'In general, the curve $y = f(x - t)$ is obtained from $y = f(x)$ by ... .'

Reflections

ACTIVITY 7.5 Figure 7.35 shows the graphs of $y = \sin x$ and $y = -\sin x$ for $0° \leqslant x \leqslant 360°$.

If you have a graphics calculator, use it to experiment with other curves like these.

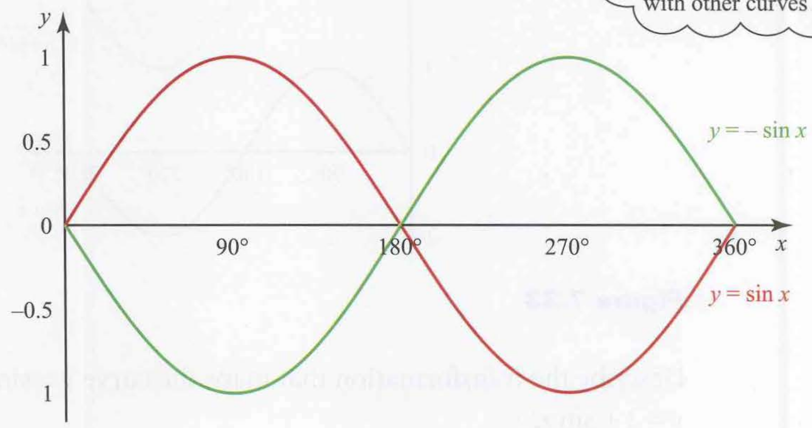


Figure 7.35

Describe the transformation that maps the curve $y = \sin x$ on to the curve $y = -\sin x$.

Complete this statement.

'In general, the curve $y = -f(x)$ is obtained from $y = f(x)$ by ... .'

One-way stretches

ACTIVITY 7.6 Figure 7.36 shows the graphs of $y = \sin x$ and $y = 2 \sin x$ for $0° \leqslant x \leqslant 180°$.

If you have a graphics calculator, use it to experiment with other curves like these.

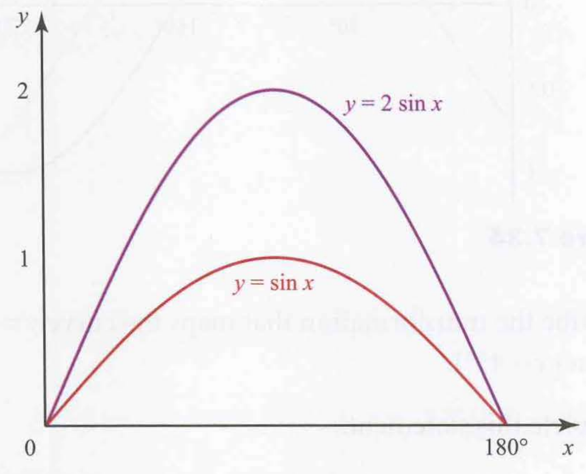


Figure 7.36

What do you notice about the value of the y co-ordinate of a point on the curve $y = \sin x$ and the y co-ordinate of a point on the curve $y = 2 \sin x$ for any value of x?

Can you describe the transformation that maps the curve $y = \sin x$ on to the curve $y = 2 \sin x$?

ACTIVITY 7.7 Figure 7.37 shows the graphs of $y = \sin x$ and $y = \sin 2x$ for $0° \leqslant x \leqslant 360°$.

If you have a graphics calculator, use it to experiment with other curves like these.

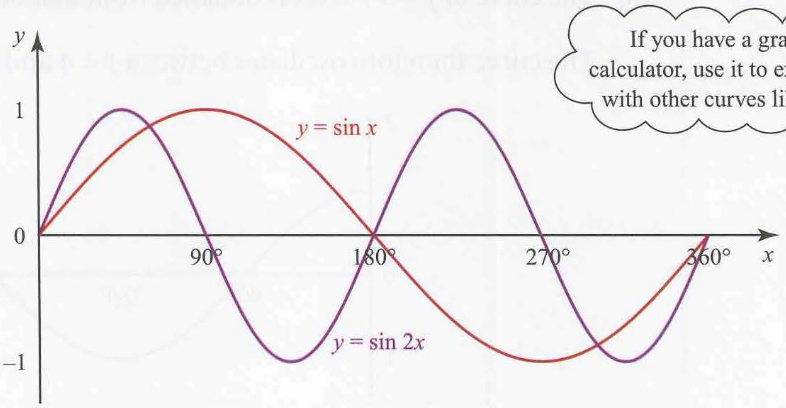


Figure 7.37

What do you notice about the value of the x co-ordinate of a point on the curve $y = \sin x$ and the x co-ordinate of a point on the curve $y = \sin 2x$ for any value of y?

Can you describe the transformation that maps the curve $y = \sin x$ on to the curve $y = \sin 2x$?

EXAMPLE 7.13

Starting with the curve $y = \cos x$, show how transformations can be used to sketch these curves.

(i) $y = \cos 3x$ **(ii)** $y = 3 + \cos x$

(iii) $y = \cos(x - 60°)$ **(iv)** $y = 2\cos x$

SOLUTION

(i) The curve with equation $y = \cos 3x$ is obtained from the curve with equation $y = \cos x$ by a stretch of scale factor $\frac{1}{3}$ parallel to the x axis. There will therefore be one complete oscillation of the curve in 120° (instead of 360°). This is shown in figure 7.38.

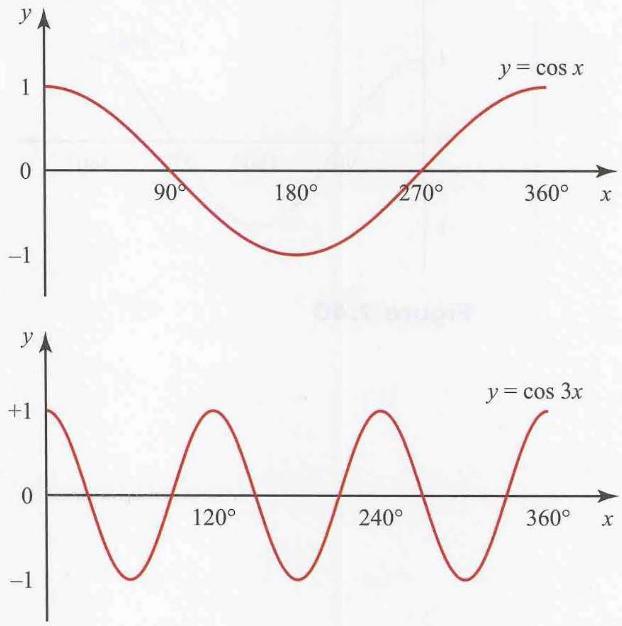


Figure 7.38

(ii) The curve of $y = 3 + \cos x$ is obtained from that of $y = \cos x$ by a translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

The curve therefore oscillates between $y = 4$ and $y = 2$ (see figure 7.39).

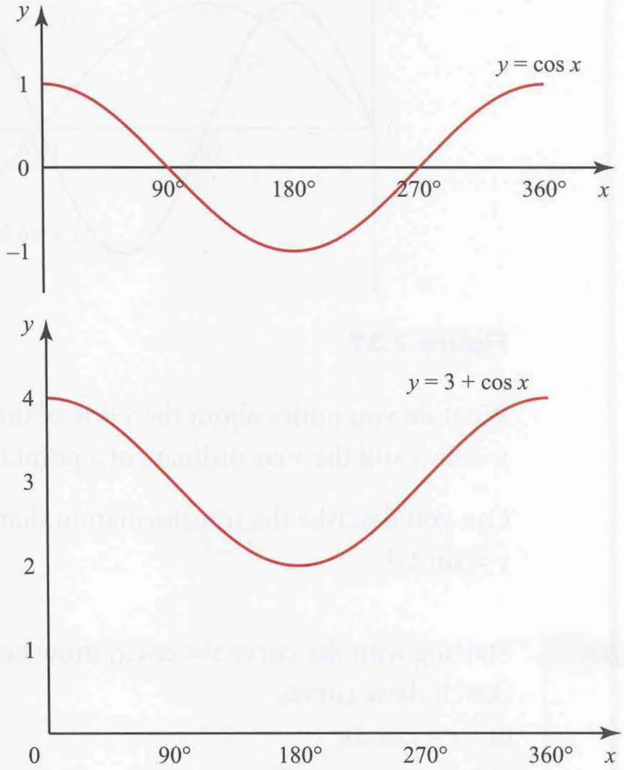


Figure 7.39

(iii) The curve of $y = \cos(x - 60°)$ is obtained from that of $y = \cos x$ by a translation of $\begin{pmatrix} 60° \\ 0 \end{pmatrix}$ (see figure 7.40).

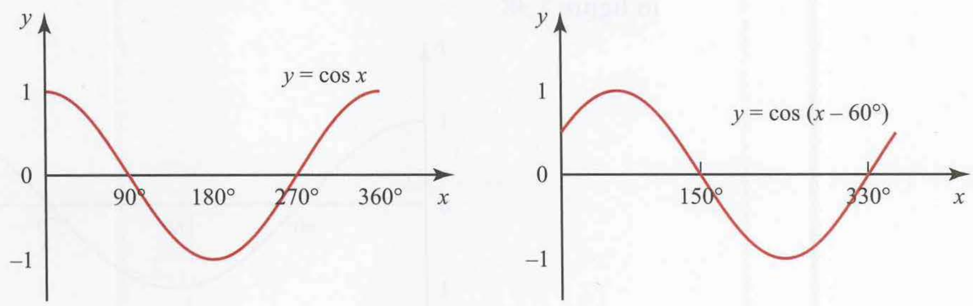


Figure 7.40

(iv) The curve of $y = 2\cos x$ is obtained from that of $y = \cos x$ by a stretch of scale factor 2 parallel to the y axis. The curve therefore oscillates between $y = 2$ and $y = -2$ (instead of between $y = 1$ and $y = -1$). This is shown in figure 7.41.

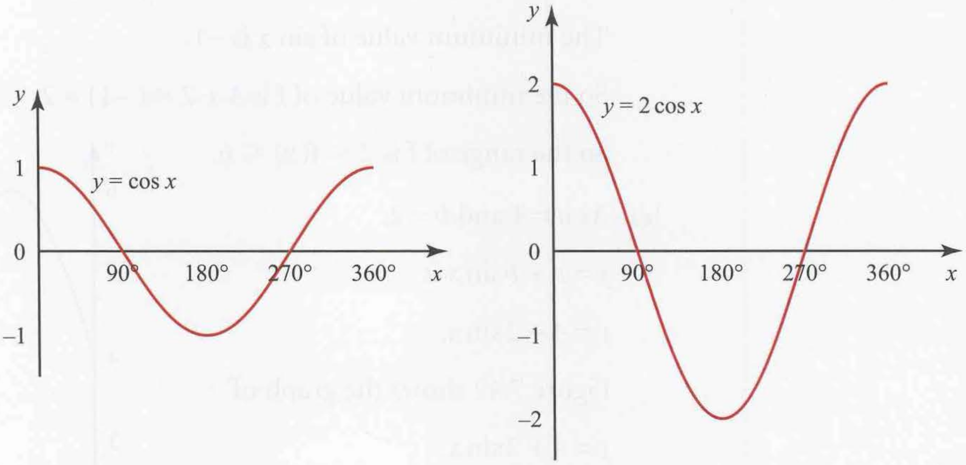


Figure 7.41

⚠ It is always a good idea to check your results using a graphic calculator whenever possible.

EXAMPLE 7.14

(i) The function $f : x \mapsto a + b\sin x$ is defined for $0 \leqslant x \leqslant 2\pi$.
Given that $f(0) = 4$ and $f\left(\frac{\pi}{6}\right) = 5$,

(a) find the values of a and b

(b) the range of f

(c) sketch the graph of $y = a + b\sin x$ for $0 \leqslant x \leqslant 2\pi$.

(ii) The function $g : x \mapsto a + b\sin x$, where a and b have the same value as found in part **(i)** is defined for the domain $\frac{\pi}{2} \leqslant x \leqslant k$. Find the largest value of k for which $g(x)$ has an inverse.

SOLUTION

(i) **(a)** $f(0) = 4 \Rightarrow a + b\sin 0 = 4$

$\Rightarrow a = 4$ since $\sin 0 = 0$

$$f\left(\frac{\pi}{6}\right) = 5 \Rightarrow 4 + b\sin\left(\frac{\pi}{6}\right) = 5$$

$$\Rightarrow 4 + \tfrac{1}{2}b = 5$$

$$\Rightarrow b = 2$$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

(b) $f : x \mapsto 4 + 2\sin x$

The maximum value of $\sin x$ is 1.

So the maximum value of f is $4 + 2 \times 1 = 6$.

The minimum value of $\sin x$ is -1.

So the minimum value of f is $4 + 2 \times (-1) = 2$.

So the range of f is $2 \leqslant f(x) \leqslant 6$.

(c) As $a = 4$ and $b = 2$,

$y = a + b\sin x$ is

$y = 4 + 2\sin x$.

Figure 7.42 shows the graph of

$y = 4 + 2\sin x$.

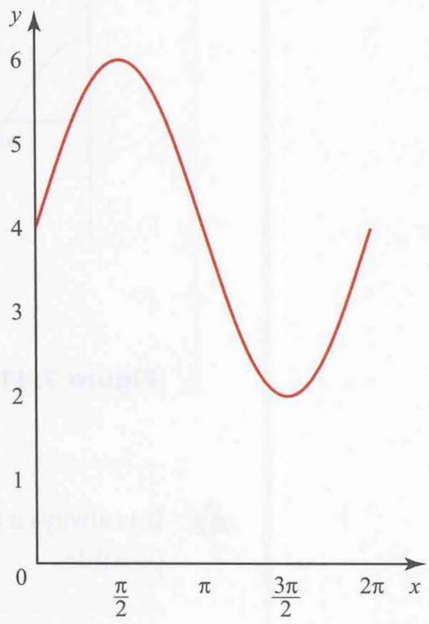


Figure 7.42

(ii) For a function to have an inverse it must be one-to-one.

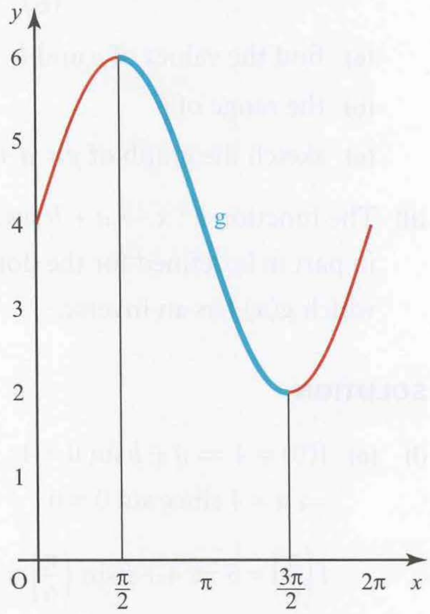


Figure 7.43

The domain of g starts at $\frac{\pi}{2}$ and must end at $\frac{3\pi}{2}$, as the curve turns here.

So $k = \frac{3\pi}{2}$.

EXERCISE 7F

1 Starting with the graph of $y = \sin x$, state the transformations which can be used to sketch each of the following curves.

(i) $y = \sin(x - 90°)$ **(ii)** $y = \sin 3x$

(iii) $2y = \sin x$ **(iv)** $y = \sin \dfrac{x}{2}$

(v) $y = 2 + \sin x$

2 Starting with the graph of $y = \cos x$, state the transformations which can be used to sketch each of the following curves.

(i) $y = \cos(x + 60°)$ **(ii)** $3y = \cos x$

(iii) $y = \cos x + 1$ **(iv)** $y = \cos 2x$

3 For each of the following curves

(a) sketch the curve

(b) identify the curve as being the same as one of the following:

$y = \pm \sin x$, $y = \pm \cos x$, or $y = \pm \tan x$.

(i) $y = \sin(x + 360°)$ **(ii)** $y = \sin(x + 90°)$

(iii) $y = \tan(x - 180°)$ **(iv)** $y = \cos(x - 90°)$

(v) $y = \cos(x + 180°)$

4 Starting with the graph of $y = \tan x$, find the equation of the graph and sketch the graph after the following transformations.

(i) Translation of $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

(ii) Translation of $\begin{pmatrix} -30° \\ 0 \end{pmatrix}$

(iii) One-way stretch with scale factor 2 parallel to the x axis

5 The graph of $y = \sin x$ is stretched with scale factor 4 parallel to the y axis.

(i) State the equation of the new graph.

(ii) Find the exact value of y on the new graph when $x = 240°$.

6 The function f is defined by $f(x) = a + b\cos 2x$, for $0 \leqslant x \leqslant \pi$. It is given that $f(0) = -1$ and $f\left(\frac{1}{2}\pi\right) = 7$.

(i) Find the values of a and b.

(ii) Find the x co-ordinates of the points where the curve $y = f(x)$ intersects the x axis.

(iii) Sketch the graph of $y = f(x)$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2007]