EXAMPLE 3.18

The first three terms in the expansion of $\left(ax + \frac{b}{x}\right)^6$ where $a > 0$, in descending powers of x, are $64x^6 - 576x^4 + cx^2$. Find the values of a, b and c.

SOLUTION

Find the first three terms in the expansion in terms of a and b:

$$\left(ax + \frac{b}{x}\right)^6 = \binom{6}{0}(ax)^6 + \binom{6}{1}(ax)^5\left(\frac{b}{x}\right) + \binom{6}{2}(ax)^4\left(\frac{b}{x}\right)^2$$

$x^4 \times \frac{1}{x^2} = x^2$

$$= a^6x^6 + 6a^5bx^4 + 15a^4b^2x^2$$

So $a^6x^6 + 6a^5bx^4 + 15a^4b^2x^2 = 64x^6 - 576x^4 + cx^2$

Compare the coefficients of x^6: $a^6 = 64 \Rightarrow a = 2$

Remember both $2^6 = 64$ and $(-2)^6 = 64$, but as $a > 0$ then $a = 2$.

Compare the coefficients of x^4: $6a^5b = -576$

Since $a = 2$ then $192b = -576 \Rightarrow b = -3$

Compare the coefficients of x^2: $15a^4b^2 = c$

Since $a = 2$ and $b = -3$ then $c = 15 \times 2^4 \times (-3)^2 \Rightarrow c = 2160$

A Pascal puzzle

$1.1^2 = 1.21 \qquad 1.1^3 = 1.331 \qquad 1.1^4 = 1.4641$

What is 1.1^5?

What is the connection between your results and the coefficients in Pascal's triangle?

---

e Relationships between binomial coefficients

There are several useful relationships between binomial coefficients.

Symmetry

Because Pascal's triangle is symmetrical about its middle, it follows that

$$\binom{n}{r} = \binom{n}{n-r}.$$

Adding terms

You have seen that each term in Pascal's triangle is formed by adding the two above it. This is written formally as

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}.$$

Sum of terms

You have seen that

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\ldots+\binom{n}{n}y^n$$

Substituting $x=y=1$ gives

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}.$$

Thus the sum of the binomial coefficients for power n is 2^n.

The binomial theorem and its applications

The binomial expansions covered in the last few pages can be stated formally as the binomial theorem for positive integer powers:

$$(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{for } n\in\mathbb{Z}^+, \quad \text{where}\binom{n}{r}=\frac{n!}{r!(n-r)!} \quad \text{and } 0!=1.$$

Note

Notice the use of the summation symbol, $\sum$. The right-hand side of the statement reads 'the sum of $\binom{n}{r}a^{n-r}b^r$ for values of r from 0 to n'.

It therefore means

$$\underset{r=0}{\binom{n}{0}a^n}+\underset{r=1}{\binom{n}{1}a^{n-1}b}+\underset{r=2}{\binom{n}{2}a^{n-2}b^2}+\ldots+\underset{r=k}{\binom{n}{k}a^{n-k}b^k}+\ldots+\underset{r=n}{\binom{n}{n}b^n}.$$

The binomial theorem is used on other types of expansion and it has applications in many areas of mathematics.

The binomial distribution

In some situations involving repetitions of trials with two possible outcomes, the probabilities of the various possible results are given by the terms of a binomial expansion. This is covered in *Probability and Statistics 1*.

Selections

The number of ways of selecting r objects from n (all different) is given by $\binom{n}{r}$. This is also covered in *Probability and Statistics 1*.

EXERCISE 3C

1 Write out the following binomial expansions.

(i) $(x+1)^4$ **(ii)** $(1+x)^7$ **(iii)** $(x+2)^5$

(iv) $(2x+1)^6$ **(v)** $(2x-3)^4$ **(vi)** $(2x+3y)^3$

(vii) $\left(x-\dfrac{2}{x}\right)^3$ **(viii)** $\left(x+\dfrac{2}{x^2}\right)^4$ **(ix)** $\left(3x^2-\dfrac{2}{x}\right)^5$

2 Use a non-calculator method to calculate the following binomial coefficients. Check your answers using your calculator's shortest method.

(i) $\binom{4}{2}$ **(ii)** $\binom{6}{2}$ **(iii)** $\binom{6}{3}$

(iv) $\binom{6}{4}$ **(v)** $\binom{6}{0}$ **(vi)** $\binom{12}{9}$

3 In these expansions, find the coefficient of these terms.

(i) x^5 in $(1+x)^8$ **(ii)** x^4 in $(1-x)^{10}$ **(iii)** x^6 in $(1+3x)^{12}$

(iv) x^7 in $(1-2x)^{15}$ **(v)** x^2 in $\left(x^2+\dfrac{2}{x}\right)^{10}$

4 (i) Simplify $(1+x)^3-(1-x)^3$.

(ii) Show that $a^3-b^3=(a-b)(a^2+ab+b^2)$.

(iii) Substitute $a=1+x$ and $b=1-x$ in the result in part **(ii)** and show that your answer is the same as that for part **(i)**.

5 Find the first three terms, in descending powers of x, in the expansion of $\left(2x-\dfrac{2}{x}\right)^4$.

6 Find the first three terms, in ascending powers of x, in the expansion $(2+kx)^6$.

7 (i) Find the first three terms, in ascending powers of x, in the expansion $(1-2x)^6$.

(ii) Hence find the coefficients of x and x^2 in the expansion of $(4-x)(2-4x)^6$.

8 (i) Find the first three terms, in descending powers of x, in the expansion $\left(4x-\dfrac{k}{x^2}\right)^6$.

(ii) Given that the value of the term in the expansion which is independent of x is 240, find possible values of k.

9 (i) Find the first three terms, in descending powers of x, in the expansion of $\left(x^2-\dfrac{1}{x}\right)^6$.

(ii) Find the coefficient of x^3 in the expansion of $\left(x^2-\dfrac{1}{x}\right)^6$.

10 **(i)** Find the first three terms, in descending powers of x, in the expansion of $\left(x - \dfrac{2}{x}\right)^5$.

(ii) Hence find the coefficient of x in the expansion of $\left(4 + \dfrac{1}{x^2}\right)\left(x - \dfrac{2}{x}\right)^5$.

11 **(i)** Show that $(2 + x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$ for all x.

(ii) Find the values of x for which $(2 + x)^4 = 16 + 16x + x^4$.

[MEI]

12 The first three terms in the expansion of $(2 + ax)^n$, in ascending powers of x, are $32 - 40x + bx^2$. Find the values of the constants n, a and b.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2006]

13 **(i)** Find the first three terms in the expansion of $(2 - x)^6$ in ascending powers of x.

(ii) Find the value of k for which there is no term in x^2 in the expansion of $(1 + kx)(2 - x)^6$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2005]

14 **(i)** Find the first three terms in the expansion of $(1 + ax)^5$ in ascending powers of x.

(ii) Given that there is no term in x in the expansion of $(1 - 2x)(1 + ax)^5$, find the value of the constant a.

(iii) For this value of a, find the coefficient of x^2 in the expansion of $(1 - 2x)(1 + ax)^5$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q6 June 2010]

INVESTIGATIONS

Routes to victory

In a recent soccer match, Juventus beat Manchester United 2–1.
What could the half-time score have been?

(i) How many different possible half-time scores are there if the final score is 2–1? How many if the final score is 4–3?

(ii) How many different 'routes' are there to any final score? For example, for the above match, putting Juventus' score first, the sequence could be:

0–0 → 0–1 → 1–1 → 2–1
or 0–0 → 1–0 → 1–1 → 2–1
or 0–0 → 1–0 → 2–0 → 2–1.

So in this case there are three routes.

Investigate the number of routes that exist to any final score (up to a maximum of five goals for either team).
Draw up a table of your results. Is there a pattern?

Cubes

A cube is painted red. It is then cut up into a number of identical cubes, as in figure 3.5.

How many of the cubes have the following numbers of faces painted red?

(i) 3 **(ii)** 2 **(iii)** 1 **(iv)** 0

In figure 3.5 there are 125 cubes but your answer should cover all possible cases.

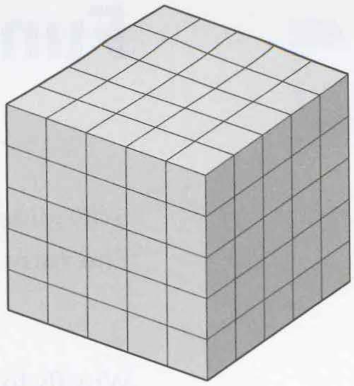


Figure 3.5

KEY POINTS

1. A sequence is an ordered set of numbers, $u_1, u_2, u_3, \ldots, u_k, \ldots u_n$, where u_k is the general term.
2. In an arithmetic sequence, $u_{k+1} = u_k + d$ where d is a fixed number called the common difference.
3. In a geometric sequence, $u_{k+1} = ru_k$ where r is a fixed number called the common ratio.
4. For an arithmetic progression with first term a, common difference d and n terms:
   - the kth term $u_k = a + (k-1)d$
   - the last term $l = a + (n-1)d$
   - the sum of the terms $= \frac{1}{2}n(a+l) = \frac{1}{2}n\left[2a + (n-1)d\right]$.
5. For a geometric progression with first term a, common ratio r and n terms:
   - the kth term $u_k = ar^{k-1}$
   - the last term $a_n = ar^{n-1}$
   - the sum of the terms $= \dfrac{a(r^n-1)}{(r-1)} = \dfrac{a(1-r^n)}{(1-r)}$.
6. For an infinite geometric series to converge, $-1 < r < 1$.

   In this case the sum of all the terms is given by $\dfrac{a}{(1-r)}$.
7. Binomial coefficients, denoted by $\dbinom{n}{r}$ or ${}^n\mathrm{C}_r$, can be found
   - using Pascal's triangle
   - using tables
   - using the formula $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$.
8. The binomial expansion of $(1+x)^n$ may also be written

   $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n.$$

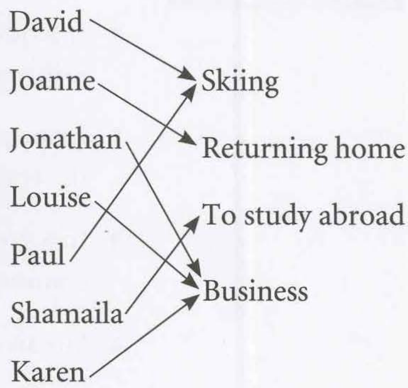
4 Functions

Still glides the stream and shall forever glide;
The form remains, the function never dies.

William Wordsworth

Why fly to Geneva in January?

Several people arriving at Geneva airport from London were asked the main purpose of their visit. Their answers were recorded.



This is an example of a *mapping*.

The language of functions

A mapping is any rule which associates two sets of items. In this example, each of the names on the left is an *object*, or *input*, and each of the reasons on the right is an *image*, or *output*.

For a mapping to make sense or to have any practical application, the inputs and outputs must each form a natural collection or set. The set of possible inputs (in this case, all of the people who flew to Geneva from London in January) is called the *domain* of the mapping.

The seven people questioned in this example gave a set of four reasons, or outputs. These form the *range* of the mapping for this particular set of inputs.

Notice that Jonathan, Louise and Karen are all visiting Geneva on business: each person gave only one reason for the trip, but the same reason was given by several people. This mapping is said to be *many-to-one*. A mapping can also be *one-to-one*, *one-to-many* or *many-to-many*. The relationship between the people from any country and their passport numbers will be one-to-one. The relationship between the people and their items of luggage is likely to be one-to-many, and that between the people and the countries they have visited in the last 10 years will be many-to-many.

Mappings

In mathematics, many (but not all) mappings can be expressed using algebra. Here are some examples of mathematical mappings.

(a)

Domain: integers		**Range**
Objects		Images
-1	$\longrightarrow$	3
0	$\longrightarrow$	5
1	$\longrightarrow$	7
2	$\longrightarrow$	9
3	$\longrightarrow$	11
General rule: x	$\longrightarrow$	$2x+5$

(b)

Domain: integers		**Range**
Objects		Images
2	$\longrightarrow$	1.9
2	$\longrightarrow$	2.1
2	$\longrightarrow$	2.33
3	$\longrightarrow$	2.52
3	$\longrightarrow$	2.99
3	$\longrightarrow$	π
General rule: Rounded whole numbers	$\longrightarrow$	Unrounded numbers

(c)

Domain: real numbers		**Range**
Objects		Images
0	$\longrightarrow$	0
45	$\longrightarrow$	0.707
90	$\longrightarrow$	1
135	$\longrightarrow$	0.707
180	$\longrightarrow$	0
General rule: $x°$	$\longrightarrow$	$\sin x°$

(d)

Domain: quadratic equations with real roots		**Range**
Objects		Images
$x^2-4x+3=0$	$\longrightarrow$	1
$x^2-4x+3=0$	$\longrightarrow$	3
$x^2-x=0$	$\longrightarrow$	0
$x^2-x=0$	$\longrightarrow$	1
$x^2-3x+2=0$	$\longrightarrow$	1
$x^2-3x+2=0$	$\longrightarrow$	2
General rule: $ax^2+bx+c=0$	$\longrightarrow$	$x=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$
General rule: $ax^2+bx+c=0$	$\longrightarrow$	$x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$

For each of the examples above:

(i) decide whether the mapping is one-to-one, many-to-many, one-to-many or many-to-one

(ii) take a different set of inputs and identify the corresponding range.

Functions

Mappings which are one-to-one or many-to-one are of particular importance, since in these cases there is only one possible image for any object. Mappings of these types are called *functions*. For example, $x \mapsto x^2$ and $x \mapsto \cos x$ are both functions, because in each case for any value of x there is only one possible answer. By contrast, the mapping of rounded whole numbers (objects) on to unrounded numbers (images) is not a function, since, for example, the rounded number 5 could map on to any unrounded number between 4.5 and 5.5.

There are several different but equivalent ways of writing a function. For example, the function which maps the real numbers, x, on to x^2 can be written in any of the following ways.

- $y = x^2 \qquad x \in \mathbb{R}$

  This is a short way of writing 'x is a real number'.
- $\mathrm{f}(x) = x^2 \qquad x \in \mathbb{R}$
- $\mathrm{f}: x \mapsto x^2 \qquad x \in \mathbb{R}$

  Read this as 'f maps x on to x^2'.

To define a function you need to specify a suitable domain. For example, you cannot choose a domain of $x \in \mathbb{R}$ (all the real numbers) for the function $\mathrm{f}: x \mapsto \sqrt{x-5}$ because when, say, $x = 3$, you would be trying to take the square root of a negative number; so you need to define the function as $\mathrm{f}: x \mapsto \sqrt{x-5}$ for $x \geqslant 5$, so that the function is valid for all values in its domain.

Likewise, when choosing a suitable domain for the function $\mathrm{g}: x \mapsto \dfrac{1}{x-5}$, you need to remember that division by 0 is undefined and therefore you cannot input $x = 5$. So the function g is defined as $\mathrm{g}: x \mapsto \dfrac{1}{x-5}$, $x \neq 5$.

It is often helpful to represent a function graphically, as in the following example, which also illustrates the importance of knowing the domain.

EXAMPLE 4.1 Sketch the graph of $y = 3x + 2$ when the domain of x is

(i) $x \in \mathbb{R}$

(ii) $x \in \mathbb{R}^+$ — This means x is a positive real number.

(iii) $x \in \mathbb{N}$. — This means x is a natural number, i.e. a positive integer or zero.

SOLUTION

(i) When the domain is $\mathbb{R}$, all values of y are possible. The range is therefore $\mathbb{R}$, also.

(ii) When x is restricted to positive values, all the values of y are greater than 2, so the range is $y > 2$.

(iii) In this case the range is the set of points $\{2, 5, 8, \ldots\}$. These are clearly all of the form $3x + 2$ where x is a natural number $(0, 1, 2, \ldots)$. This set can be written neatly as $\{3x + 2 : x \in \mathbb{N}\}$.

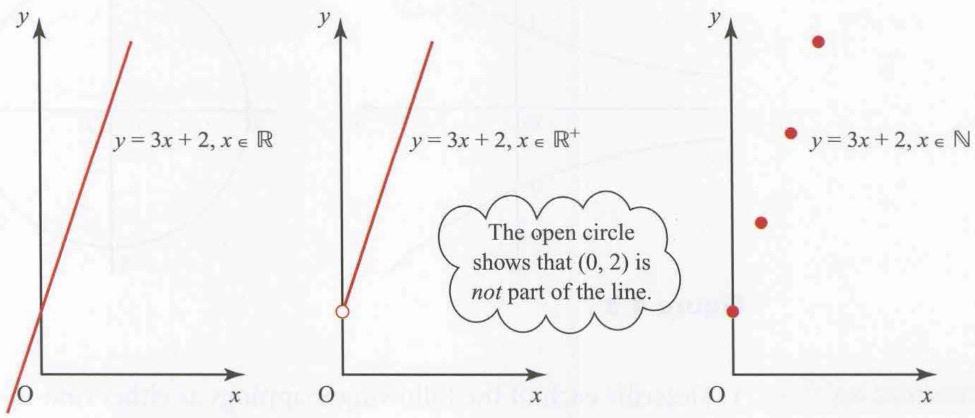


Figure 4.1

When you draw the graph of a mapping, the x co-ordinate of each point is an input value, the y co-ordinate is the corresponding output value. The table below shows this for the mapping $x \mapsto x^2$, or $y = x^2$, and figure 4.2 shows the resulting points on a graph.

Input (x)	Output (y)	Point plotted
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)

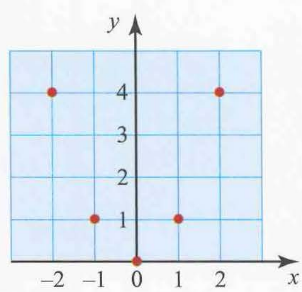


Figure 4.2

If the mapping is a function, there is one and only one value of y for every value of x in the domain. Consequently the graph of a function is a simple curve or line going from left to right, with no doubling back.

Figure 4.3 illustrates some different types of mapping. The graphs in **(a)** and **(b)** illustrate functions, those in **(c)** and **(d)** do not.

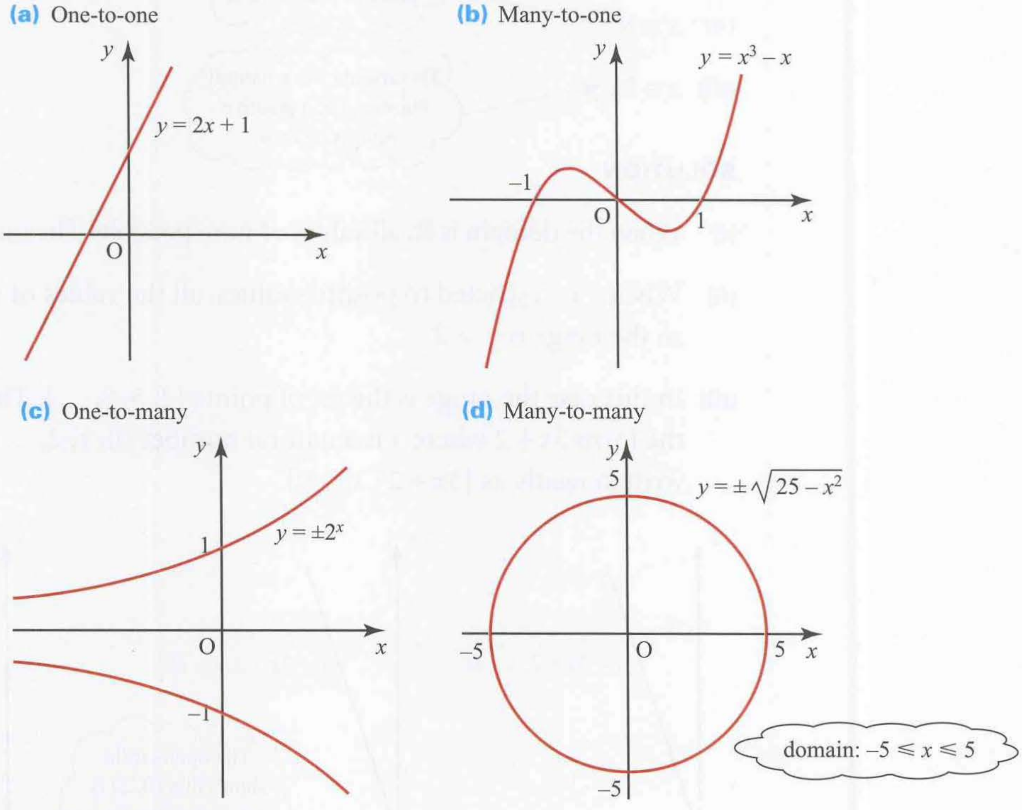
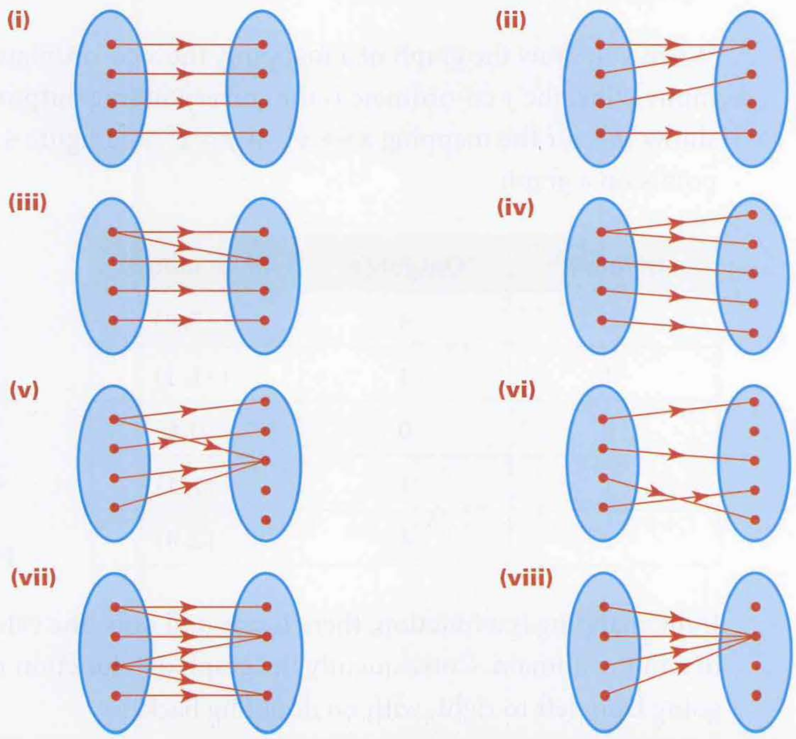


Figure 4.3

EXERCISE 4A

1 Describe each of the following mappings as either one-to-one, many-to-one, one-to-many or many-to-many, and say whether it represents a function.



2 For each of the following mappings:

(a) write down a few examples of inputs and corresponding outputs
(b) state the type of mapping (one-to-one, many-to-one, etc.)
(c) suggest a suitable domain.

(i) Words $\mapsto$ number of letters they contain
(ii) Side of a square in cm $\mapsto$ its perimeter in cm
(iii) Natural numbers $\mapsto$ the number of factors (including 1 and the number itself)
(iv) $x \mapsto 2x - 5$
(v) $x \mapsto \sqrt{x}$
(vi) The volume of a sphere in cm^3 $\mapsto$ its radius in cm
(vii) The volume of a cylinder in cm^3 $\mapsto$ its height in cm
(viii) The length of a side of a regular hexagon in cm $\mapsto$ its area in cm^2
(ix) $x \mapsto x^2$

3 **(i)** A function is defined by $f(x) = 2x - 5$, $x \in \mathbb{R}$. Write down the values of
(a) $f(0)$ **(b)** $f(7)$ **(c)** $f(-3)$.

(ii) A function is defined by g:(polygons) $\mapsto$ (number of sides). What are
(a) g(triangle) **(b)** g(pentagon) **(c)** g(decagon)?

(iii) The function t maps Celsius temperatures on to Fahrenheit temperatures. It is defined by t: $C \mapsto \frac{9C}{5} + 32$, $C \in \mathbb{R}$. Find
(a) $t(0)$ **(b)** $t(28)$ **(c)** $t(-10)$
(d) the value of C when $t(C) = C$.

4 Find the range of each of the following functions.
(You may find it helpful to draw the graph first.)

(i)	$f(x) = 2 - 3x$	$x \geqslant 0$
(ii)	$f(\theta) = \sin\theta$	$0° \leqslant \theta \leqslant 180°$
(iii)	$y = x^2 + 2$	$x \in \{0, 1, 2, 3, 4\}$
(iv)	$y = \tan\theta$	$0° < \theta < 90°$
(v)	$f: x \mapsto 3x - 5$	$x \in \mathbb{R}$
(vi)	$f: x \mapsto 2^x$	$x \in \{-1, 0, 1, 2\}$
(vii)	$y = \cos x$	$-90° \leqslant x \leqslant 90°$
(viii)	$f: x \mapsto x^3 - 4$	$x \in \mathbb{R}$
(ix)	$f(x) = \dfrac{1}{1 + x^2}$	$x \in \mathbb{R}$
(x)	$f(x) = \sqrt{x - 3} + 3$	$x \geqslant 3$

5 The mapping f is defined by $f(x) = x^2 \quad 0 \leqslant x \leqslant 3$
$f(x) = 3x \quad 3 \leqslant x \leqslant 10$.

The mapping g is defined by $g(x) = x^2 \quad 0 \leqslant x \leqslant 2$
$g(x) = 3x \quad 2 \leqslant x \leqslant 10$.

Explain why f is a function and g is not.

Composite functions

It is possible to combine functions in several different ways, and you have already met some of these. For example, if $f(x) = x^2$ and $g(x) = 2x$, then you could write

$$f(x) + g(x) = x^2 + 2x.$$

In this example, two functions are added.

Similarly if $f(x) = x$ and $g(x) = \sin x$, then

$$f(x).g(x) = x \sin x.$$

In this example, two functions are multiplied.

Sometimes you need to apply one function and then apply another to the answer. You are then creating a *composite function* or a *function of a function.*

EXAMPLE 4.2

A new mother is bathing her baby for the first time. She takes the temperature of the bath water with a thermometer which reads in Celsius, but then has to convert the temperature to degrees Fahrenheit to apply the rule that her own mother taught her:

At one o five
He'll cook alive
But ninety four
is rather raw.

Write down the two functions that are involved, and apply them to readings of

(i) 30°C **(ii)** 38°C **(iii)** 45°C.

SOLUTION

The first function converts the Celsius temperature C into a Fahrenheit temperature, F.

$$F = \frac{9C}{5} + 32$$

The second function maps Fahrenheit temperatures on to the state of the bath.

$F \leqslant 94$	too cold
$94 < F < 105$	all right
$F \geqslant 105$	too hot

This gives

(i) 30°C $\mapsto$ 86°F $\mapsto$ too cold
(ii) 38°C $\mapsto$ 100.4°F $\mapsto$ all right
(iii) 45°C $\mapsto$ 113°C $\mapsto$ too hot.

In this case the composite function would be (to the nearest degree)

$$C \leqslant 34°\text{C} \qquad \text{too cold}$$
$$35°\text{C} \leqslant C \leqslant 40°\text{C} \qquad \text{all right}$$
$$C \geqslant 41°\text{C} \qquad \text{too hot.}$$

In algebraic terms, a composite function is constructed as

$$\text{Input } x \overset{\text{f}}{\mapsto} \text{Output f}(x)$$

$$\text{Input f}(x) \overset{\text{g}}{\mapsto} \text{Output g}[\text{f}(x)] \qquad (\text{or gf}(x)).$$

Read this as 'g of f of x'.

Thus the composite function gf(x) should be performed from right to left: start with x then apply f and then g.

Notation

To indicate that f is being applied twice in succession, you could write ff(x) but you would usually use $\text{f}^2(x)$ instead. Similarly $\text{g}^3(x)$ means three applications of g.

In order to apply a function repeatedly its range must be completely contained within its domain.

Order of functions

If f is the rule 'square the input value' and g is the rule 'add 1', then

$$x \underset{\text{square}}{\overset{\text{f}}{\mapsto}} x^2 \underset{\text{add 1}}{\overset{\text{g}}{\mapsto}} x^2 + 1.$$

So $\text{gf}(x) = x^2 + 1$.

Notice that gf(x) is not the same as fg(x), since for fg(x) you must apply g first. In the example above, this would give:

$$x \underset{\text{add 1}}{\overset{\text{g}}{\mapsto}} (x+1) \underset{\text{square}}{\overset{\text{f}}{\mapsto}} (x+1)^2$$

and so $\text{fg}(x) = (x+1)^2$.

Clearly this is *not* the same result.

Figure 4.4 illustrates the relationship between the domains and ranges of the functions f and g, and the range of the composite function gf.

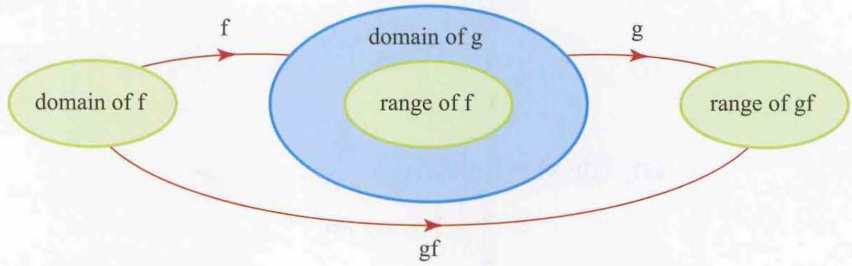


Figure 4.4

Notice the range of f must be completely contained within the domain of g. If this wasn't the case you wouldn't be able to form the composite function gf because you would be trying to input values into g that weren't in its domain.

For example, consider these functions f and g.

$$f: x \mapsto 2x, \ x > 0$$
$$g: x \mapsto \sqrt{x}, \ x > 0$$

You need this restriction so you are not taking the square root of a negative number.

The composite function gf can be formed:

$$x \ \underset{\times 2}{\overset{f}{\mapsto}} \ 2x \ \underset{\text{square root}}{\overset{g}{\mapsto}} \ \sqrt{2x}$$

and so $gf: x \mapsto \sqrt{2x}, \ x > 0$

Now think about a different function h.

$$h: x \mapsto 2x, \ x \in \mathbb{R}$$

This function looks like f but h has a different domain; it is all the real numbers whereas f was restricted to positive numbers. The range of h is also all real numbers and so it includes negative numbers, which are not in the domain of g.

So you cannot form the composite function gh. If you tried, h would input negative numbers into g and you cannot take the square root of a negative number.

EXAMPLE 4.3

The functions f, g and h are defined by:

$f(x) = 2x$ for $x \in \mathbb{R}$, $g(x) = x^2$ for $x \in \mathbb{R}$, $h(x) = \dfrac{1}{x}$ for $x \in \mathbb{R}, x \neq 0$.

Find the following.

(i) $fg(x)$ **(ii)** $gf(x)$ **(iii)** $gh(x)$

(iv) $f^2(x)$ **(v)** $fgh(x)$

SOLUTION

(i)
$$\begin{aligned} fg(x) &= f[g(x)] \\ &= f(x^2) \\ &= 2x^2 \end{aligned}$$

(ii)
$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g(2x) \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

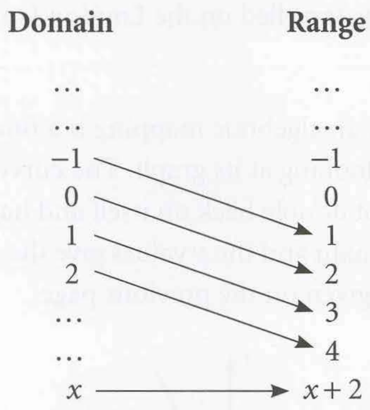
(iii)
$$\begin{aligned} gh(x) &= g[h(x)] \\ &= g\left(\frac{1}{x}\right) \\ &= \frac{1}{x^2} \end{aligned}$$

(iv)
$$\begin{aligned} f^2(x) &= f[f(x)] \\ &= f(2x) \\ &= 2(2x) \\ &= 4x \end{aligned}$$

(v)
$$\begin{aligned} fgh(x) &= f[gh(x)] \\ &= f\left(\frac{1}{x^2}\right) \text{ using (iii)} \\ &= \frac{2}{x^2} \end{aligned}$$

Inverse functions

Look at the mapping $x \mapsto x+2$ with domain the set of integers.



The mapping is clearly a function, since for every input there is one and only one output, the number that is two greater than that input.

This mapping can also be seen in reverse. In that case, each number maps on to the number two less than itself: $x \mapsto x-2$. The reverse mapping is also a function because for any input there is one and only one output. The reverse mapping is called the *inverse function*, f^{-1}.

Function: $\quad f \;: x \mapsto x+2 \qquad x \in \mathbb{Z}.$ ← This is a short way of writing x is an integer.

Inverse function: $f^{-1} : x \mapsto x-2 \qquad x \in \mathbb{Z}.$

For a mapping to be a function which also has an inverse function, every object in the domain must have one and only one image in the range, and vice versa. This can only be the case if the mapping is one-to-one.

So the condition for a function f to have an inverse function is that, over the given domain, f represents a one-to-one mapping. This is a common situation, and many inverse functions are self-evident as in the following examples, for all of which the domain is the real numbers.

$f : x \mapsto x-1; \qquad f^{-1} : x \mapsto x+1$

$g : x \mapsto 2x; \qquad g^{-1} : x \mapsto \frac{1}{2}x$

$h : x \mapsto x^3; \qquad h^{-1} : x \mapsto \sqrt[3]{x}$

? Some of the following mappings are functions which have inverse functions, and others are not.

(a) Decide which mappings fall into each category, and for those which do not have inverse functions, explain why.

(b) For those which have inverse functions, how can the functions and their inverses be written down algebraically?

(i) Temperature measured in Celsius $\mapsto$ temperature measured in Fahrenheit.
(ii) Marks in an examination $\mapsto$ grade awarded.
(iii) Distance measured in light years $\mapsto$ distance measured in metres.
(iv) Number of stops travelled on the London Underground $\mapsto$ fare.

You can decide whether an algebraic mapping is a function, and whether it has an inverse function, by looking at its graph. The curve or line representing a one-to-one function does not double back on itself and has no turning points. The x values cover the full domain and the y values give the range. Figure 4.5 illustrates the functions f, g and h given on the previous page.

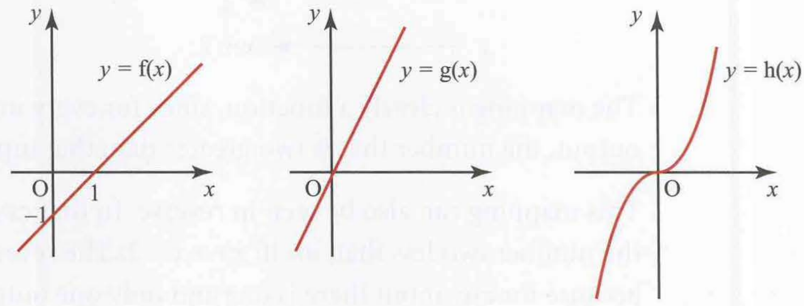


Figure 4.5

Now look at $f(x) = x^2$ for $x \in \mathbb{R}$ (figure 4.6). You can see that there are two distinct input values giving the same output: for example $f(2) = f(-2) = 4$. When you want to reverse the effect of the function, you have a mapping which for a single input of 4 gives two outputs, -2 and $+2$. Such a mapping is not a function.

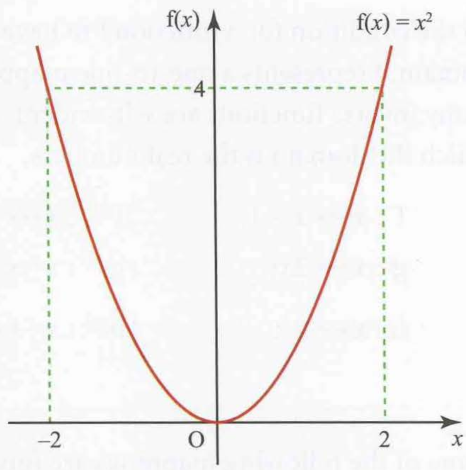


Figure 4.6

You can make a new function, $g(x) = x^2$ by restricting the domain to $\mathbb{R}^+$ (the set of positive real numbers). This is shown in figure 4.7. The function $g(x)$ is a one-to-one function and its inverse is given by $g^{-1}(x) = \sqrt{x}$ since the sign $\sqrt{}$ means 'the positive square root of'.

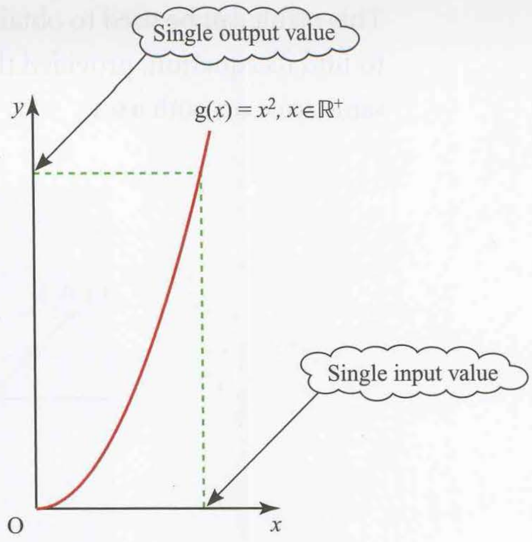


Figure 4.7

It is often helpful to define a function with a restricted domain so that its inverse is also a function. When you use the inv sin (i.e. $\sin^{-1}$ or arcsin) key on your calculator the answer is restricted to the range –90° to 90°, and is described as the *principal value.* Although there are infinitely many roots of the equation $\sin x = 0.5$ (…, –330°, –210°, 30°, 150°, …), only one of these, 30°, lies in the restricted range and this is the value your calculator will give you.

The graph of a function and its inverse

ACTIVITY 4.1

For each of the following functions, work out the inverse function, and draw the graphs of both the original and the inverse on the same axes, using the same scale on both axes.

(i) $f(x) = x^2, x \in \mathbb{R}^+$ **(ii)** $f(x) = 2x, x \in \mathbb{R}$

(iii) $f(x) = x + 2, x \in \mathbb{R}$ **(iv)** $f(x) = x^3 + 2, x \in \mathbb{R}$

Look at your graphs and see if there is any pattern emerging.

Try out a few more functions of your own to check your ideas.

Make a conjecture about the relationship between the graph of a function and its inverse.

You have probably realised by now that the graph of the inverse function is the same shape as that of the function, but reflected in the line $y = x$. To see why this is so, think of a function $f(x)$ mapping a on to b; (a, b) is clearly a point on the graph of $f(x)$. The inverse function $f^{-1}(x)$, maps b on to a and so (b, a) is a point on the graph of $f^{-1}(x)$.

The point (b, a) is the reflection of the point (a, b) in the line $y = x$. This is shown for a number of points in figure 4.8.

This result can be used to obtain a sketch of the inverse function without having to find its equation, provided that the sketch of the original function uses the same scale on both axes.

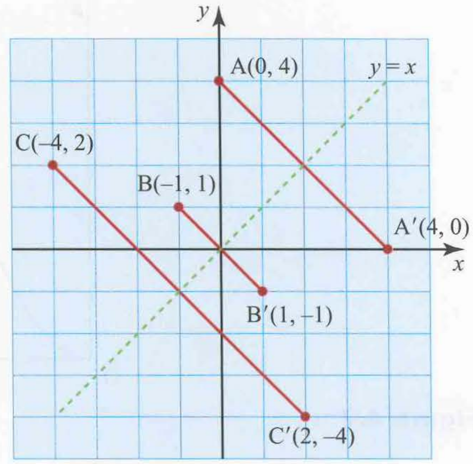


Figure 4.8

Finding the algebraic form of the inverse function

To find the algebraic form of the inverse of a function $f(x)$, you should start by changing notation and writing it in the form $y = \dots$.

Since the graph of the inverse function is the reflection of the graph of the original function in the line $y = x$, it follows that you may find its equation by interchanging y and x in the equation of the original function. You will then need to make y the subject of your new equation. This procedure is illustrated in Example 4.4.

EXAMPLE 4.4

Find $f^{-1}(x)$ when $f(x) = 2x + 1$, $x \in \mathbb{R}$.

SOLUTION

The function $f(x)$ is given by $\quad y = 2x + 1$

Interchanging x and y gives $\quad x = 2y + 1$

Rearranging to make y the subject: $\quad y = \dfrac{x-1}{2}$

So $f^{-1}(x) = \dfrac{x-1}{2}$, $x \in \mathbb{R}$

Sometimes the domain of the function f will not include the whole of $\mathbb{R}$. When any real numbers are excluded from the domain of f, it follows that they will be excluded from the range of f^{-1}, and vice versa.

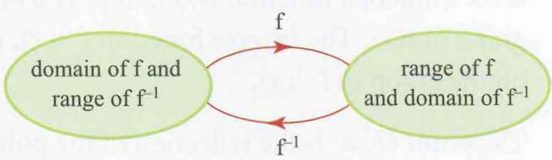


Figure 4.9

EXAMPLE 4.5

Find $f^{-1}(x)$ when $f(x) = 2x - 3$ and the domain of f is $x \geqslant 4$.

SOLUTION

	Domain	Range
Function: $y = 2x - 3$	$x \geqslant 4$	$y \geqslant 5$
Inverse function: $x = 2y - 3$	$x \geqslant 5$	$y \geqslant 4$

Rearranging the inverse function to make y the subject: $y = \dfrac{x+3}{2}$.

The full definition of the inverse function is therefore:

$$f^{-1}(x) = \frac{x+3}{2} \text{ for } x \geqslant 5.$$

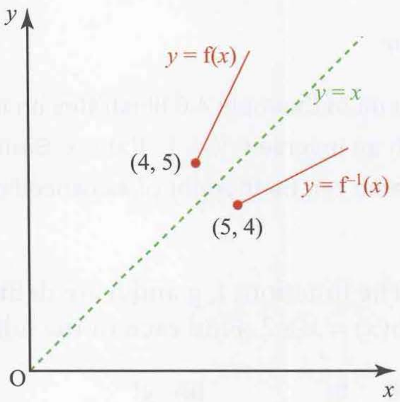


Figure 4.10

You can see in figure 4.10 that the inverse function is the reflection of a restricted part of the line $y = 2x - 3$.

EXAMPLE 4.6

(i) Find $f^{-1}(x)$ when $f(x) = x^2 + 2, \ x \geqslant 0$.

(ii) Find $f(7)$ and $f^{-1}f(7)$. What do you notice?

SOLUTION

(i)

	Domain	Range
Function: $y = x^2 + 2$	$x \geqslant 0$	$y \geqslant 2$
Inverse function: $x = y^2 + 2$	$x \geqslant 2$	$y \geqslant 0$

Rearranging the inverse function to make y its subject: $y^2 = x - 2$.

This gives $y = \pm\sqrt{x-2}$, but since you know the range of the inverse function to be $y \geqslant 0$ you can write:

$$y = +\sqrt{x-2} \text{ or just } y = \sqrt{x-2}.$$

The full definition of the inverse function is therefore:

$$f^{-1}(x) = \sqrt{x-2} \text{ for } x \geqslant 2.$$

The function and its inverse function are shown in figure 4.11.

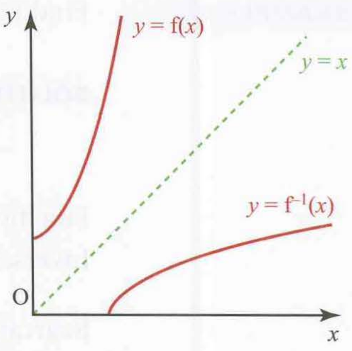


Figure 4.11

(ii) $f(7) = 7^2 + 2 = 51$

$f^{-1}f(7) = f^{-1}(51) = \sqrt{51-2} = 7$

Applying the function followed by its inverse brings you back to the original input value.

Note

Part **(ii)** of Example 4.6 illustrates an important general result. For any function $f(x)$ with an inverse $f^{-1}(x)$, $f^{-1}f(x) = x$. Similarly $ff^{-1}(x) = x$. The effects of a function and its inverse can be thought of as cancelling each other out.

EXERCISE 4B

1 The functions f, g and h are defined for $x \in \mathbb{R}$ by $f(x) = x^3$, $g(x) = 2x$ and $h(x) = x + 2$. Find each of the following, in terms of x.

(i) fg **(ii)** gf **(iii)** fh **(iv)** hf **(v)** fgh

(vi) ghf **(vii)** g^2 **(viii)** $(fh)^2$ **(ix)** h^2

2 Find the inverses of the following functions.

(i) $f(x) = 2x + 7, x \in \mathbb{R}$ **(ii)** $f(x) = 4 - x, x \in \mathbb{R}$

(iii) $f(x) = \dfrac{4}{2-x}, x \neq 2$ **(iv)** $f(x) = x^2 - 3, x \geqslant 0$

3 The function f is defined by $f(x) = (x-2)^2 + 3$ for $x \geqslant 2$.

(i) Sketch the graph of $f(x)$.

(ii) On the same axes, sketch the graph of $f^{-1}(x)$ without finding its equation.

4 Express the following in terms of the functions $f: x \mapsto \sqrt{x}$ and $g: x \mapsto x + 4$ for $x > 0$.

(i) $x \mapsto \sqrt{x+4}$ **(ii)** $x \mapsto x + 8$

(iii) $x \mapsto \sqrt{x+8}$ **(iv)** $x \mapsto \sqrt{x} + 4$

5 A function f is defined by:

$$f: x \mapsto \frac{1}{x} \qquad x \in \mathbb{R}, x \neq 0.$$

Find **(i)** $f^2(x)$ **(ii)** $f^3(x)$ **(iii)** $f^{-1}(x)$ **(iv)** $f^{999}(x)$.

6 **(i)** Show that $x^2 + 4x + 7 = (x + 2)^2 + a$, where a is to be determined.

(ii) Sketch the graph of $y = x^2 + 4x + 7$, giving the equation of its axis of symmetry and the co-ordinates of its vertex.

The function f is defined by f: $x \mapsto x^2 + 4x + 7$ with domain the set of all real numbers.

(iii) Find the range of f.

(iv) Explain, with reference to your sketch, why f has no inverse with its given domain. Suggest a domain for f for which it has an inverse.

[MEI]

7 The function f is defined by f: $x \mapsto 4x^3 + 3$, $x \in \mathbb{R}$.
Give the corresponding definition of f^{-1}.
State the relationship between the graphs of f and f^{-1}.

[UCLES]

8 Two functions are defined for $x \in \mathbb{R}$ as $f(x) = x^2$ and $g(x) = x^2 + 4x - 1$.

(i) Find a and b so that $g(x) = f(x + a) + b$.

(ii) Show how the graph of $y = g(x)$ is related to the graph of $y = f(x)$ and sketch the graph of $y = g(x)$.

(iii) State the range of the function $g(x)$.

(iv) State the least value of c so that $g(x)$ is one-to-one for $x \geqslant c$.

(v) With this restriction, sketch $g(x)$ and $g^{-1}(x)$ on the same axes.

9 The functions f and g are defined for $x \in \mathbb{R}$ by

$$f: x \mapsto 4x - 2x^2;$$

$$g: x \mapsto 5x + 3.$$

(i) Find the range of f.

(ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q3 June 2010]

10 Functions f and g are defined by

$$f: x \mapsto k - x \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$

$$g: x \mapsto \frac{9}{x + 2} \quad \text{for } x \in \mathbb{R}, x \neq -2.$$

(i) Find the values of k for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases.

(ii) Solve the equation $fg(x) = 5$ when $k = 6$.

(iii) Express $g^{-1}(x)$ in terms of x.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2006]

11 The function f is defined by $f: x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants.

(ii) State the range of f.

(iii) Explain why f does not have an inverse.

The function g is defined by $g: x \mapsto 2x^2 - 8x + 11$ for $x \leqslant A$, where A is a constant.

(iv) State the largest value of A for which g has an inverse.

(v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 November 2007]

12 The function f is defined by $f: x \mapsto 3x - 2$ for $x \in \mathbb{R}$.

(i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs.

The function g is defined by $g: x \mapsto 6x - x^2$ for $x \in \mathbb{R}$.

(ii) Express $gf(x)$ in terms of x, and hence show that the maximum value of $gf(x)$ is 9.

The function h is defined by $h: x \mapsto 6x - x^2$ for $x \geqslant 3$.

(iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where a and b are positive constants.

(iv) Express $h^{-1}(x)$ in terms of x.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2008]

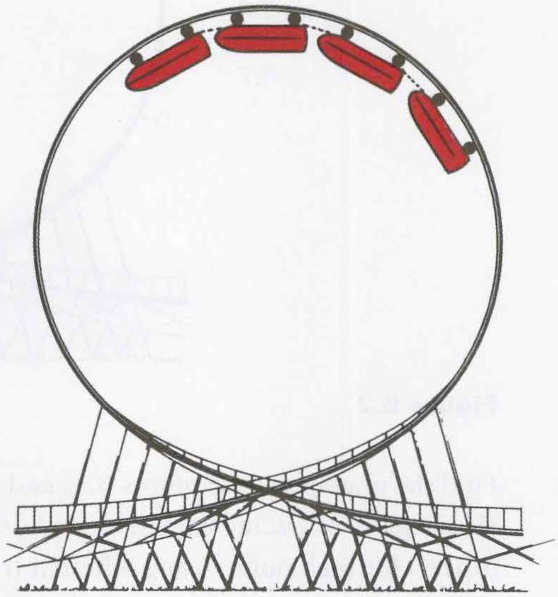
KEY POINTS

1. A mapping is any rule connecting input values (objects) and output values (images). It can be many-to-one, one-to-many, one-to-one or many-to-many.
2. A many-to-one or one-to-one mapping is called a function. It is a mapping for which each input value gives exactly one output value.
3. The domain of a mapping or function is the set of possible input values (values of x).
4. The range of a mapping or function is the set of output values.
5. A composite function is obtained when one function (say g) is applied after another (say f). The notation used is $g[f(x)]$ or $gf(x)$.
6. For any one-to-one function $f(x)$, there is an inverse function $f^{-1}(x)$.
7. The curves of a function and its inverse are reflections of each other in the line $y = x$.

5 Differentiation

Hold infinity in the palm of your hand.

William Blake



This picture illustrates one of the more frightening rides at an amusement park. To ensure that the ride is absolutely safe, its designers need to know the gradient of the curve at any point. What do we mean by the gradient of a curve?

The gradient of a curve

To understand what this means, think of a log on a log-flume, as in figure 5.1. If you draw the straight line $y = mx + c$ passing along the bottom of the log, then this line is a tangent to the curve at the point of contact. The gradient m of the tangent is the gradient of the curve at the point of contact.

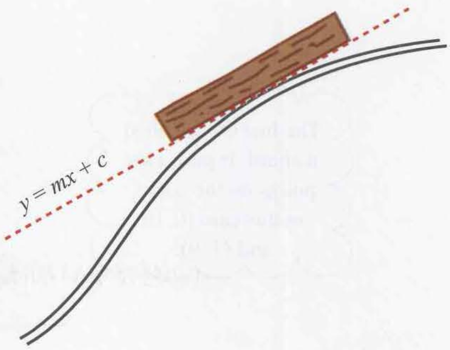


Figure 5.1

One method of finding the gradient of a curve is shown for point A in figure 5.2.

$$\text{Gradient} = \frac{y\text{ step}}{x\text{ step}} = \frac{5.5}{1.5} = 3.7$$

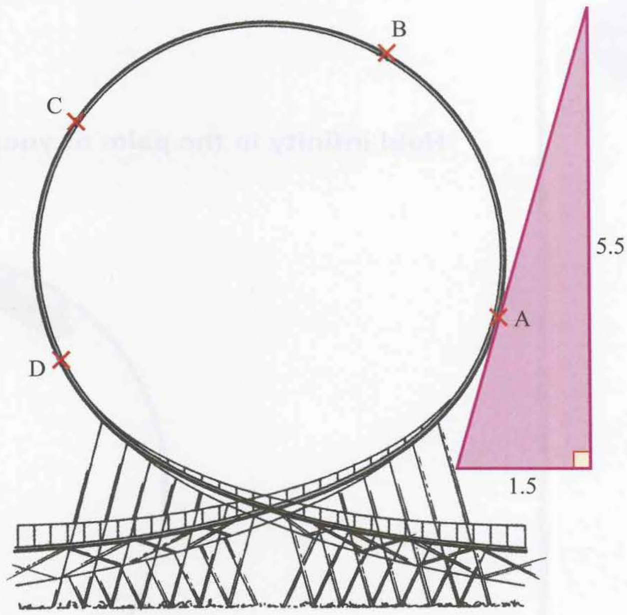


Figure 5.2

ACTIVITY 5.1 Find the gradient at the points B, C and D using the method shown in figure 5.2. (Use a piece of tracing paper to avoid drawing directly on the book!) Repeat the process for each point, using different triangles, and see whether you get the same answers.

You probably found that your answers were slightly different each time, because they depended on the accuracy of your drawing and measuring. Clearly you need a more accurate method of finding the gradient at a point. As you will see in this chapter, a method is available which can be used on many types of curve, and which does not involve any drawing at all.

Finding the gradient of a curve

Figure 5.3 shows the part of the graph $y = x^2$ which lies between $x = -1$ and $x = 3$. What is the value of the gradient at the point P(3, 9)?

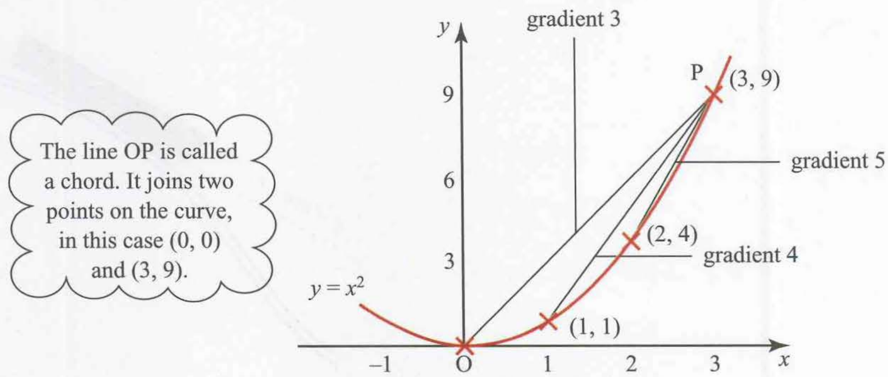


Figure 5.3

You have already seen that drawing the tangent at the point by hand provides only an approximate answer. A different approach is to calculate the gradients of chords to the curve. These will also give only approximate answers for the gradient of the curve, but they will be based entirely on calculation and not depend on your drawing skill. Three chords are marked on figure 5.3.

Chord (0, 0) to (3, 9): gradient $= \frac{9-0}{3-0} = 3$

Chord (1, 1) to (3, 9): gradient $= \frac{9-1}{3-1} = 4$

Chord (2, 4) to (3, 9): gradient $= \frac{9-4}{3-2} = 5$

Clearly none of these three answers is exact, but which of them is the most accurate?

Of the three chords, the one closest to being a tangent is that joining (2, 4) to (3, 9), the two points that are closest together.

You can take this process further by 'zooming in' on the point (3, 9) and using points which are much closer to it, as in figure 5.4.

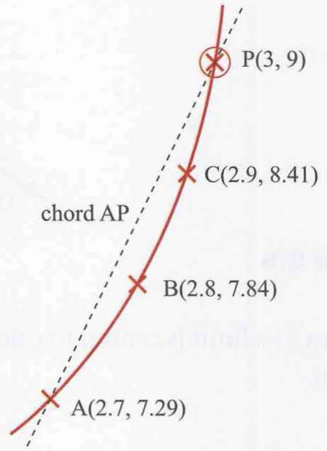


Figure 5.4

The x co-ordinate of point A is 2.7, the y co-ordinate 2.7^2, or 7.29 (since the point lies on the curve $y = x^2$). Similarly B and C are (2.8, 7.84) and (2.9, 8.41). The gradients of the chords joining each point to (3, 9) are as follows.

Chord (2.7, 7.29) to (3, 9): gradient $= \frac{9-7.29}{3-2.7} = 5.7$

Chord (2.8, 7.84) to (3, 9): gradient $= \frac{9-7.84}{3-2.8} = 5.8$

Chord (2.9, 8.41) to (3, 9): gradient $= \frac{9-8.41}{3-2.9} = 5.9$

These results are getting closer to the gradient of the tangent. What happens if you take points much closer to (3, 9), for example (2.99, 8.9401) and (2.999, 8.994001)?

The gradients of the chords joining these to (3, 9) work out to be 5.99 and 5.999 respectively.

ACTIVITY 5.2 Take points X, Y, Z on the curve $y = x^2$ with x co-ordinates 3.1, 3.01 and 3.001 respectively, and find the gradients of the chords joining each of these points to (3, 9).

It looks as if the gradients are approaching the value 6, and if so this is the gradient of the tangent at (3, 9).

Taking this method to its logical conclusion, you might try to calculate the gradient of the 'chord' from (3, 9) to (3, 9), but this is undefined because there is a zero in the denominator. So although you can find the gradient of a chord which is as close as you like to the tangent, it can never be exactly that of the tangent. What you need is a way of making that final step from a chord to a tangent.

The concept of a *limit* enables us to do this, as you will see in the next section. It allows us to confirm that in the limit as point Q tends to point P(3, 9), the chord QP tends to the tangent of the curve at P, and the gradient of QP tends to 6 (see figure 5.5).

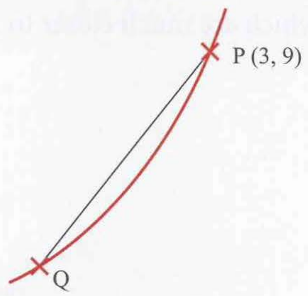


Figure 5.5

The idea of a limit is central to calculus, which is sometimes described as the study of limits.

Historical note This method of using chords approaching the tangent at P to *calculate* the gradient of the tangent was first described clearly by Pierre de Fermat (*c.*1608–65). He spent his working life as a civil servant in Toulouse and produced an astonishing amount of original mathematics in his spare time.

e Finding the gradient from first principles

Although the work in the previous section was more formal than the method of drawing a tangent and measuring its gradient, it was still somewhat experimental. The result that the gradient of $y = x^2$ at (3, 9) is 6 was a sensible conclusion, rather than a proved fact.

In this section the method is formalised and extended.

Take the point P(3, 9) and another point Q close to (3, 9) on the curve $y = x^2$. Let the x co-ordinate of Q be $3 + h$ where h is small. Since $y = x^2$ at Q, the y co-ordinate of Q will be $(3 + h)^2$.

Figure 5.6 shows Q in a position where h is positive, but negative values of h would put Q to the left of P.

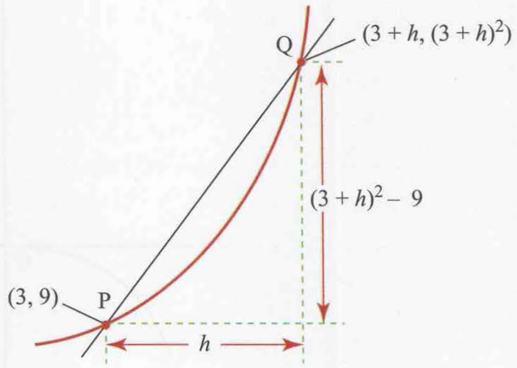


Figure 5.6

From figure 5.6, the gradient of PQ is $\dfrac{(3+h)^2-9}{h}$

$$= \frac{9+6h+h^2-9}{h}$$

$$= \frac{6h+h^2}{h}$$

$$= \frac{h(6+h)}{h}$$

$$= 6+h.$$

For example, when $h = 0.001$, the gradient of PQ is 6.001, and when $h = -0.001$, the gradient of PQ is 5.999. The gradient of the tangent at P is between these two values. Similarly the gradient of the tangent would be between $6 - h$ and $6 + h$ for all small non-zero values of h.

For this to be true the gradient of the tangent at (3, 9) must be *exactly* 6.

ACTIVITY 5.3

Using a similar method, find the gradient of the tangent to the curve at

(i) (1, 1)
(ii) (−2, 4)
(iii) (4, 16).

What do you notice?

The gradient function

The work so far has involved finding the gradient of the curve $y = x^2$ at a particular point (3, 9), but this is not the way in which you would normally find the gradient at a point. Rather you would consider the general point, (x, y), and then substitute the value(s) of x (and/or y) corresponding to the point of interest.

EXAMPLE 5.1

Find the gradient of the curve $y = x^3$ at the general point (x, y).

SOLUTION

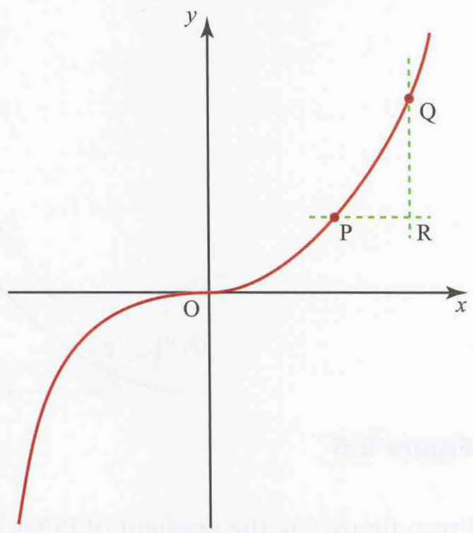


Figure 5.7

Let P have the general value x as its x co-ordinate, so P is the point (x, x^3) (since it is on the curve $y = x^3$). Let the x co-ordinate of Q be $(x + h)$ so Q is $((x+h), (x+h)^3)$. The gradient of the chord PQ is given by

$$\begin{aligned}\frac{\text{QR}}{\text{PR}} &= \frac{(x+h)^3 - x^3}{(x+h) - x} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= 3x^2 + 3xh + h^2\end{aligned}$$

As Q takes values closer to P, h takes smaller and smaller values and the gradient approaches the value of $3x^2$ which is the gradient of the tangent at P. The gradient of the curve $y = x^3$ at the point (x, y) is equal to $3x^2$.

Note

If the equation of the curve is written as $y = \mathrm{f}(x)$, then the *gradient function* (i.e. the gradient at the general point (x, y)) is written as $\mathrm{f}'(x)$. Using this notation the result above can be written as $\mathrm{f}(x) = x^3 \Rightarrow \mathrm{f}'(x) = 3x^2$.

EXERCISE 5A

1 Use the method in Example 5.1 to prove that the gradient of the curve $y = x^2$ at the point (x, y) is equal to $2x$.

2 Use the binomial theorem to expand $(x + h)^4$ and hence find the gradient of the curve $y = x^4$ at the point (x, y).

3 Copy the table below, enter your answer to question **2**, and suggest how the gradient pattern should continue when $f(x) = x^5$, $f(x) = x^6$ and $f(x) = x^n$ (where n is a positive whole number).

$f(x)$	$f'(x)$ (gradient at (x, y))
x^2	$2x$
x^3	$3x^2$
x^4	
x^5	
x^6	
$\vdots$	
x^n	

4 Prove the result when $f(x) = x^5$.

Note

The result you should have obtained from question 3 is known as *Wallis's rule* and can be used as a formula.

? How can you use the binomial theorem to prove this general result for integer values of n?

An alternative notation

So far h has been used to denote the difference between the x co-ordinates of our points P and Q, where Q is close to P.

h is sometimes replaced by δx. The Greek letter δ (delta) is shorthand for 'a small change in' and so δx represents a small change in x and δy a corresponding small change in y.

In figure 5.8 the gradient of the chord PQ is $\frac{\delta y}{\delta x}$.

In the limit as $\delta x \to 0$, δx and δy both become infinitesimally small and the value obtained for $\frac{\delta y}{\delta x}$ approaches the gradient of the tangent at P.

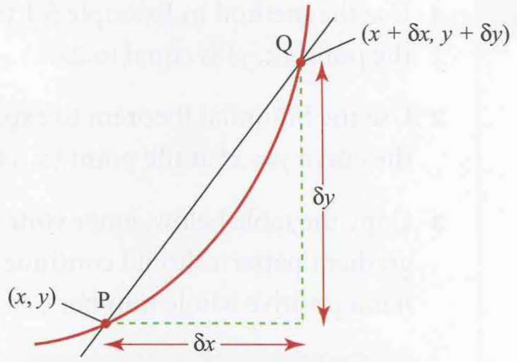


Figure 5.8

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \text{ is written as } \frac{dy}{dx}.$$

Read this as 'the limit as δx tends towards zero'.

Using this notation, Wallis's rule becomes

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}.$$

The gradient function, $\frac{dy}{dx}$ or $f'(x)$ is sometimes called the *derivative* of y with respect to x, and when you find it you have *differentiated* y with respect to x.

Note

There is nothing special about the letters x, y and f.

If, for example, your curve represented time (t) on the horizontal axis and velocity (v) on the vertical axis, then the relationship may be referred to as $v = g(t)$, i.e. v is a function of t, and the gradient function is given by $\frac{dv}{dt} = g'(t)$.

ACTIVITY 5.4 Plot the curve with equation $y = x^3 + 2$, for values of x from -2 to $+2$. On the same axes and for the same range of values of x, plot the curves $y = x^3 - 1$, $y = x^3$ and $y = x^3 + 1$.

What do you notice about the gradients of this family of curves when $x = 0$?

What about when $x = 1$ or $x = -1$?

ACTIVITY 5.5 Differentiate the equation $y = x^3 + c$, where c is a constant. How does this result help you to explain your findings in Activity 5.4?

Historical note

The notation $\dfrac{dy}{dx}$ was first used by the German mathematician and philosopher Gottfried Leibniz (1646–1716) in 1675. Leibniz was a child prodigy and a self-taught mathematician. The terms 'function' and 'co-ordinates' are due to him and, because of his influence, the sign '=' is used for equality and '×' for multiplication. In 1684 he published his work on calculus (which deals with the way in which quantities change) in a six-page article in the periodical *Acta Eruditorum*.

Sir Isaac Newton (1642–1727) worked independently on calculus but Leibniz published his work first. Newton always hesitated to publish his discoveries. Newton used different notation (introducing 'fluxions' and 'moments of fluxions') and his expressions were thought to be rather vague. Over the years the best aspects of the two approaches have been combined, but at the time the dispute as to who 'discovered' calculus first was the subject of many articles and reports, and indeed nearly caused a war between England and Germany.

Differentiating by using standard results

The method of differentiation from first principles will always give the gradient function, but it is rather tedious and, in practice, it is hardly ever used. Its value is in establishing a formal basis for differentiation rather than as a working tool.

If you look at the results of differentiating $y = x^n$ for different values of n a pattern is immediately apparent, particularly when you include the result that the line $y = x$ has constant gradient 1.

y	$\dfrac{dy}{dx}$
x^1	1
x^2	$2x^1$
x^3	$3x^2$

This pattern continues and, in general

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}.$$

This can be extended to functions of the type $y = kx^n$ for any constant k, to give

$$y = kx^n \Rightarrow \frac{dy}{dx} = knx^{n-1}.$$

The power n can be any real number and this includes positive and negative integers and fractions, i.e. all rational numbers.

Another important result is that

$$y = c \Rightarrow \frac{dy}{dx} = 0 \quad \text{where } c \text{ is any constant.}$$

This follows from the fact that the graph of $y = c$ is a horizontal line with gradient zero (see figure 5.9).

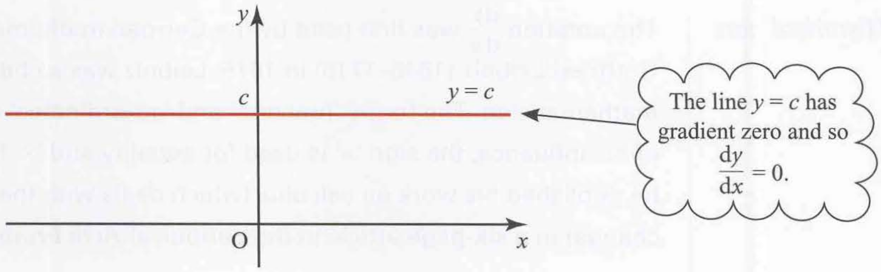


Figure 5.9

EXAMPLE 5.2

For each of these functions of x, find the gradient function.

(i) $y = x^5$ **(ii)** $z = 7x^6$ **(iii)** $p = 11$ **(iv)** $f(x) = \dfrac{3}{x}$

SOLUTION

(i) $\dfrac{dy}{dx} = 5x^4$

(ii) $\dfrac{dz}{dx} = 6 \times 7x^5 = 42x^5$

(iii) $\dfrac{dp}{dx} = 0$

(iv) $f(x) = 3x^{-1}$ ← You many find it easier to write $\frac{1}{x}$ as x^{-1}.

$$\Rightarrow f'(x) = (-1) \times 3x^{-2}$$

$$= -\frac{3}{x^2}$$

Sums and differences of functions

Many of the functions you will meet are sums or differences of simpler ones. For example, the function $(3x^2 + 4x^3)$ is the sum of the functions $3x^2$ and $4x^3$.

To differentiate a function such as this you differentiate each part separately and then add the results together.

EXAMPLE 5.3

Differentiate $y = 3x^2 + 4x^3$.

SOLUTION

$$\frac{dy}{dx} = 6x + 12x^2$$

Note

This may be written in general form as:

$$y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$$

EXAMPLE 5.4

Differentiate $f(x) = \dfrac{(x^2+1)(x-5)}{x}$

SOLUTION

You cannot differentiate f(x) as it stands, so you need to start by rewriting it.

Expanding the brackets: $f(x) = \dfrac{x^3 - 5x^2 + x - 5}{x}$

$$= \frac{x^3}{x} - \frac{5x^2}{x} + \frac{x}{x} - \frac{5}{x}$$

$$= x^2 - 5x + 1 - 5x^{-1}$$

Now you can differentiate f(x) to give $f'(x) = 2x - 5 + 5x^{-2}$

$$= 2x + \frac{5}{x^2} - 5$$

EXERCISE 5B

Differentiate the following functions using the rules

$$y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = knx^{n-1}$$

and $y = f(x) + g(x) \quad \Rightarrow \quad \dfrac{dy}{dx} = f'(x) + g'(x)$.

1 $y = x^5$

2 $y = 4x^2$

3 $y = 2x^3$

4 $y = x^{11}$

5 $y = 4x^{10}$

6 $y = 3x^5$

7 $y = 7$

8 $y = 7x$

9 $y = 2x^3 + 3x^5$

10 $y = x^7 - x^4$

11 $y = x^2 + 1$

12 $y = x^3 + 3x^2 + 3x + 1$

13 $y = x^3 - 9$

14 $y = \frac{1}{2}x^2 + x + 1$

15 $y = 3x^2 + 6x + 6$

16 $A = 4\pi r^2$

17 $A = \frac{4}{3}\pi r^3$

18 $d = \frac{1}{4}t^2$

19 $C = 2\pi r$

20 $V = l^3$

21 $f(x) = x^{\frac{3}{2}}$

22 $y = \dfrac{1}{x}$

23 $y = \sqrt{x}$

24 $y = \frac{1}{5}x^{\frac{5}{2}}$

25 $f(x) = \dfrac{1}{x^2}$

26 $f(x) = \dfrac{5}{x^3}$

27 $y = \dfrac{2}{\sqrt{x}}$

28 $f(x) = 4\sqrt{x} - \dfrac{8}{\sqrt{x}}$

29 $f(x) = x^{\frac{3}{2}} + x^{-\frac{3}{2}}$

30 $f(x) = x^{\frac{5}{3}} - x^{-\frac{2}{3}}$

31 $y = x(4x - 1)$

32 $f(x) = (2x - 1)(x + 3)$

33 $y = \dfrac{x^2 + 6x}{x}$

34 $y = \dfrac{4x^6 - 5x^4}{x^2}$

35 $y = x\sqrt{x}$

36 $f(x) = \dfrac{2x}{\sqrt{x}}$

37 $g(x) = \dfrac{3x^2 - 2x}{\sqrt{x}}$

38 $y = \left(\dfrac{x}{4} + \dfrac{4}{x}\right)(x^2 - x)$

39 $h(x) = \left(\sqrt{x}\right)^3$

40 $y = \dfrac{(x^2 + 2x)(x - 4)}{2\sqrt{x}}$

Using differentiation

EXAMPLE 5.5

Given that $y = \sqrt{x} - \frac{8}{x^2}$, find

(i) $\frac{dy}{dx}$

(ii) the gradient of the curve at the point $\left(4, 1\frac{1}{2}\right)$.

SOLUTION

(i) Rewrite $y = \sqrt{x} - \frac{8}{x^2}$ as $y = x^{\frac{1}{2}} - 8x^{-2}$.

Now you can differentiate using the rule $y = kx^n \Rightarrow \frac{dy}{dx} = knx^{n-1}$.

$$\frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} + 16x^{-3}$$
$$= \frac{1}{2\sqrt{x}} + \frac{16}{x^3}$$

(ii) At $\left(4, 1\frac{1}{2}\right)$, $x = 4$

Substituting $x = 4$ into the expression for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4}} + \frac{16}{4^3}$$
$$= \tfrac{1}{4} + \tfrac{16}{64}$$
$$= \tfrac{1}{2}$$

EXAMPLE 5.6

Figure 5.10 shows the graph of

$y = x^2(x - 6) = x^3 - 6x^2$.

Find the gradient of the curve at the points A and B where it meets the x axis.

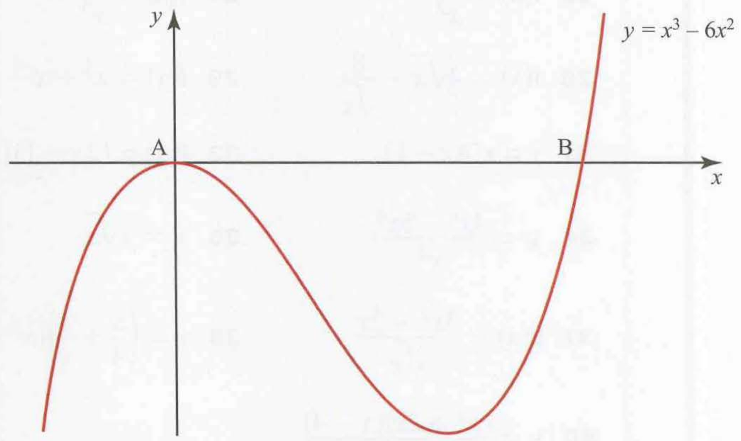


Figure 5.10

SOLUTION

The curve cuts the x axis when $y = 0$, and so at these points

$$x^2(x-6) = 0$$
$$\Rightarrow \quad x = 0 \text{ (twice) or } x = 6.$$

Differentiating $y = x^3 - 6x^2$ gives

$$\frac{dy}{dx} = 3x^2 - 12x.$$

At the point (0, 0), $\frac{dy}{dx} = 0$

and at (6, 0), $\frac{dy}{dx} = 3 \times 6^2 - 12 \times 6 = 36.$

At A(0, 0) the gradient of the curve is 0 and at B(6, 0) the gradient of the curve is 36.

Note

This curve goes through the origin. You can see from the graph and from the value of $\frac{dy}{dx}$ that the x axis is a tangent to the curve at this point. You could also have deduced this from the fact that $x = 0$ is a repeated root of the equation $x^3 - 6x^2 = 0$.

EXAMPLE 5.7

Find the points on the curve with equation $y = x^3 + 6x^2 + 5$ where the value of the gradient is −9.

SOLUTION

The gradient at any point on the curve is given by

$$\frac{dy}{dx} = 3x^2 + 12x.$$

Therefore you need to find points at which $\frac{dy}{dx} = -9$, i.e.

$$\begin{aligned} 3x^2 + 12x &= -9 \\ 3x^2 + 12x + 9 &= 0 \\ 3(x^2 + 4x + 3) &= 0 \\ 3(x+1)(x+3) &= 0 \\ \Rightarrow \quad x &= -1 \text{ or } x = -3. \end{aligned}$$

When $x = -1$, $y = (-1)^3 + 6(-1)^2 + 5 = 10$.

When $x = -3$, $y = (-3)^3 + 6(-3)^2 + 5 = 32$.

Therefore the gradient is −9 at the points (−1, 10) and (−3, 32)(see figure 5.11).