b Linear simultaneous equations

EXAMPLE 1.34 At a poultry farm, six hens and one duck cost \$40, while four hens and three ducks cost \$36. What is the cost of each type of bird?

SOLUTION

Let the cost of one hen be \$$h$ and the cost of one duck be \$$d$.

Then the information given can be written as:

$$6h + d = 40 \quad ①$$
$$4h + 3d = 36. \quad ②$$

There are several methods of solving this pair of equations.

Method 1: Elimination

Multiplying equation ① by 3	$\Rightarrow$	$18h + 3d = 120$
Leaving equation ②	$\Rightarrow$	$4h + 3d = 36$
Subtracting	$\Rightarrow$	$14h = 84$
Dividing both sides by 14	$\Rightarrow$	$h = 6$
Substituting $h = 6$ in equation ① gives		$36 + d = 40$
	$\Rightarrow$	$d = 4$

Therefore a hen costs \$6 and a duck \$4.

Note

1. The first step was to multiply equation ① by 3 so that there would be a term $3d$ in both equations. This meant that when equation ② was subtracted, the variable d was eliminated and so it was possible to find the value of h.
2. The value $h = 6$ was substituted in equation ① but it could equally well have been substituted in the other equation. Check for yourself that this too gives the answer $d = 4$.

Before looking at other methods for solving this pair of equations, here is another example.

EXAMPLE 1.35 Solve
$$3x + 5y = 12 \quad ①$$
$$2x - 6y = -20 \quad ②$$

SOLUTION

① × 6	$\Rightarrow$	$18x + 30y = 72$
② × 5	$\Rightarrow$	$10x - 30y = -100$
Adding	$\Rightarrow$	$28x = -28$
Giving		$x = -1$
Substituting $x = -1$ in equation ①	$\Rightarrow$	$-3 + 5y = 12$
Adding 3 to each side	$\Rightarrow$	$5y = 15$
Dividing by 5	$\Rightarrow$	$y = 3$

Therefore $x = -1$, $y = 3$.

Note

In this example, both equations were multiplied, the first by 6 to give $+30y$ and the second by 5 to give $-30y$. Because one of these terms was positive and the other negative, it was necessary to add rather than subtract in order to eliminate y.

Returning now to the pair of equations giving the prices of hens and ducks,

$$\begin{aligned} 6h + d &= 40 \quad ① \\ 4h + 3d &= 36 \quad ② \end{aligned}$$

here are two alternative methods of solving them.

Method 2: Substitution

The equation $6h + d = 40$ is rearranged to make d its subject:

$$d = 40 - 6h.$$

This expression for d is now substituted in the other equation, $4h + 3d = 36$, giving

$$\begin{aligned} 4h + 3(40 - 6h) &= 36 \\ \Rightarrow \quad 4h + 120 - 18h &= 36 \\ \Rightarrow \quad -14h &= -84 \\ \Rightarrow \quad h &= 6 \end{aligned}$$

Substituting for h in $d = 40 - 6h$ gives $d = 40 - 36 = 4$.

Therefore a hen costs \$6 and a duck \$4 (the same answer as before, of course).

Method 3: Intersection of the graphs of the equations

Figure 1.13 shows the graphs of the two equations, $6h + d = 40$ and $4h + 3d = 36$. As you can see, they intersect at the solution, $h = 6$ and $d = 4$.

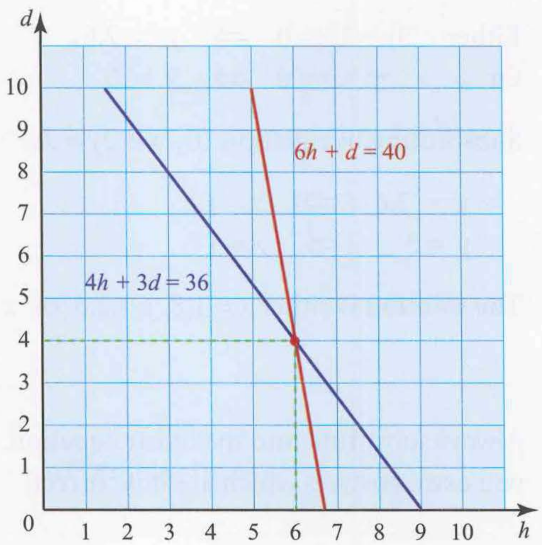


Figure 1.13

Non-linear simultaneous equations

The simultaneous equations in the examples so far have all been *linear*, that is their graphs have been straight lines. A linear equation in, say, x and y contains only terms in x and y and a constant term. So $7x + 2y = 11$ is linear but $7x^2 + 2y = 11$ is not linear, since it contains a term in x^2.

You can solve a pair of simultaneous equations, one of which is linear and the other not, using the substitution method. This is shown in the next example.

EXAMPLE 1.36

Solve
$$x + 2y = 7 \quad ①$$
$$x^2 + y^2 = 10 \quad ②$$

SOLUTION

Rearranging equation ① gives $x = 7 - 2y$.
Substituting for x in equation ②:

$$(7 - 2y)^2 + y^2 = 10$$

Multiplying out the $(7 - 2y) \times (7 - 2y)$

gives $49 - 14y - 14y + 4y^2 = 49 - 28y + 4y^2$,

so the equation is

$$49 - 28y + 4y^2 + y^2 = 10.$$

This is rearranged to give

$$\begin{aligned} 5y^2 - 28y + 39 &= 0 \\ \Rightarrow \quad 5y^2 - 15y - 13y + 39 &= 0 \\ \Rightarrow \quad 5y(y - 3) - 13(y - 3) &= 0 \\ \Rightarrow \quad (5y - 13)(y - 3) &= 0 \end{aligned}$$

A quadratic in y which you can now solve using factorisation or the formula.

Either $5y - 13 = 0 \Rightarrow y = 2.6$
Or $y - 3 = 0 \Rightarrow y = 3$

Substituting in equation ①, $x + 2y = 7$:

$y = 2.6 \Rightarrow x = 1.8$
$y = 3 \Rightarrow x = 1$

The solution is either $x = 1.8$, $y = 2.6$ or $x = 1$, $y = 3$.

Always substitute into the linear equation. Substituting in the quadratic will give you extra answers which are not correct.

EXERCISE 1G

1 Solve the following pairs of simultaneous equations.

(i) $2x+3y=8$
$3x+2y=7$

(ii) $x+4y=16$
$3x+5y=20$

(iii) $7x+y=15$
$4x+3y=11$

(iv) $5x-2y=3$
$x+4y=5$

(v) $8x-3y=21$
$5x+y=16$

(vi) $8x+y=32$
$7x-9y=28$

(vii) $4x+3y=5$
$2x-6y=-5$

(viii) $3u-2v=17$
$5u-3v=28$

(ix) $4l-3m=2$
$5l-7m=9$

2 A student wishes to spend exactly \$10 at a second-hand bookshop. All the paperbacks are one price, all the hardbacks another. She can buy five paperbacks and eight hardbacks. Alternatively she can buy ten paperbacks and six hardbacks.

(i) Write this information as a pair of simultaneous equations.

(ii) Solve your equations to find the cost of each type of book.

3 The cost of a pear is 5c greater than that of an apple. Eight apples and nine pears cost \$1.64.

(i) Write this information as a pair of simultaneous equations.

(ii) Solve your equations to find the cost of each type of fruit.

4 A car journey of 380 km lasts 4 hours. Part of this is on a motorway at an average speed of 110 $\mathrm{km\,h^{-1}}$, the rest on country roads at an average speed of 70 $\mathrm{km\,h^{-1}}$.

(i) Write this information as a pair of simultaneous equations.

(ii) Solve your equations to find how many kilometres of the journey is spent on each type of road.

5 Solve the following pairs of simultaneous equations.

(i) $x^2+y^2=10$
$x+y=4$

(ii) $x^2-2y^2=8$
$x+2y=8$

(iii) $2x^2+3y=12$
$x-y=-1$

(iv) $k^2+km=8$
$m=k-6$

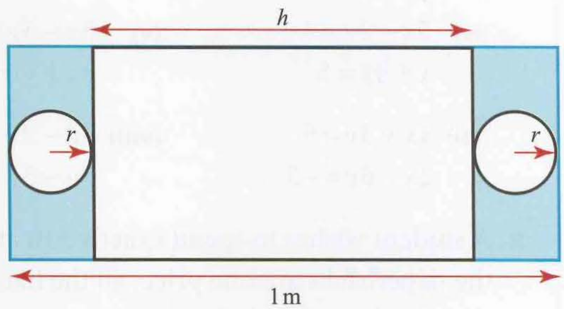
(v) $t_1{}^2-t_2{}^2=75$
$t_1=2t_2$

(vi) $p+q+5=0$
$p^2=q^2+5$

(vii) $k(k-m)=12$
$k(k+m)=60$

(viii) $p_1{}^2-p_2{}^2=0$
$p_1+p_2=2$

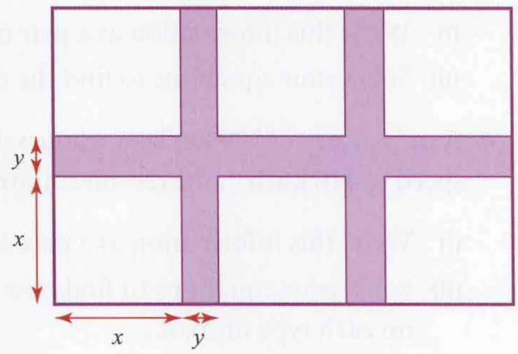
6 The diagram shows the net of a cylindrical container of radius r cm and height h cm. The full width of the metal sheet from which the container is made is 1 m, and the shaded area is waste. The surface area of the container is 1400π cm^2.



(i) Write down a pair of simultaneous equations for r and h.

(ii) Find the volume of the container, giving your answers in terms of π. (There are two possible answers.)

7 A large window consists of six square panes of glass as shown. Each pane is x m by x m, and all the dividing wood is y m wide.



(i) Write down the total area of the whole window in terms of x and y.

(ii) Show that the total area of the dividing wood is $7xy + 2y^2$.

(iii) The total area of glass is 1.5 m^2, and the total area of dividing wood is 1 m^2. Find x, and hence find an equation for y and solve it.

[MEI]

Inequalities

Not all algebraic statements involve the equals sign and it is just as important to be able to handle algebraic inequalities as it is to solve algebraic equations. The solution to an inequality is a range of possible values, not specific value(s) as in the case of an equation.

Linear inequalities

The methods for linear inequalities are much the same as those for equations but you must be careful when multiplying or dividing through an inequality by a negative number.

Take for example the following statement:

$$5 > 3 \text{ is true}$$

Multiply both sides by -1 $\quad -5 > -3$ is false.

It is actually the case that multiplying or dividing by a negative number reverses the inequality, but you may prefer to avoid the difficulty, as shown in the examples below.

EXAMPLE 1.37

Solve $5x - 3 \leqslant 2x - 15$.

SOLUTION

Add 3 to, and subtract $2x$ from, both sides $\Rightarrow 5x - 2x \leqslant -15 + 3$

Tidy up $\Rightarrow 3x \leqslant -12$

Divide both sides by 3 $\Rightarrow x \leqslant -4$

Note

Since there was no need to multiply or divide both sides by a negative number, no problems arose in this example.

EXAMPLE 1.38

Solve $2y + 6 > 7y + 11$.

SOLUTION

Subtract 6 and $7y$ from both sides $\Rightarrow 2y - 7y > 11 - 6$

Tidy up $\Rightarrow -5y > +5$

Beware: do not divide both sides by -5.

Add $5y$ to both sides and subtract 5 $\Rightarrow -5 > +5y$

This now allows you to divide both sides by +5.

Divide both sides by +5 $\Rightarrow -1 > y$

Note that logically $-1 > y$ is the same as $y < -1$, so the solution is $y < -1$.

Quadratic inequalities

EXAMPLE 1.39

Solve **(i)** $x^2 - 4x + 3 > 0$ **(ii)** $x^2 - 4x + 3 \leqslant 0$.

SOLUTION

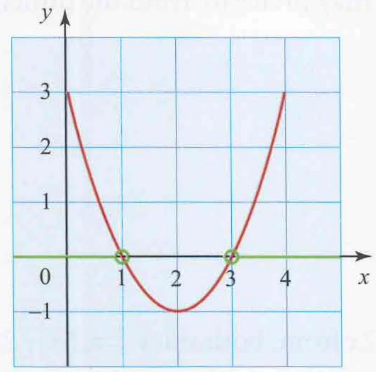
The graph of $y = x^2 - 4x + 3$ is shown in figure 1.14 with the green parts of the x axis corresponding to the solutions to the two parts of the question.

(i) You want the values of x for which $y > 0$, which that is where the curve is above the x axis.

(ii) You want the values of x for $y \leqslant 0$, that is where the curve crosses or is below the x axis.

Here the end points *are not* included in the inequality so you draw open circles: ○

Here the end points *are* included in the inequality so you draw solid circles: ●



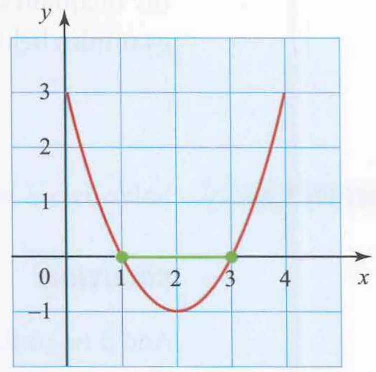


Figure 1.14

The solution is $x < 1$ or $x > 3$.

The solution is $x \geqslant 1$ and $x \leqslant 3$, usually witten $1 \leqslant x \leqslant 3$.

EXAMPLE 1.40

Find the set of values of k for which $x^2 + kx + 4 = 0$ has real roots.

SOLUTION

A quadratic equation, $ax^2 + bx + c = 0$, has real roots if $b^2 - 4ac \geqslant 0$.

So $x^2 + kx + 4 = 0$ has real roots if $k^2 - 4 \times 4 \times 1 \geqslant 0$.

$\Rightarrow k^2 - 16 \geqslant 0$

$\Rightarrow k^2 \geqslant 16$

Take the square root of both sides.

So the set of values is $k \geqslant 4$ and $k \leqslant -4$.

Take care: $(-5)^2 = 25$ and $(-3)^2 = 9$, so k must be less than or equal to -4.

EXERCISE 1H

1 Solve the following inequalities.

(i) $5a + 6 > 2a + 24$ **(ii)** $3b - 5 \leqslant b - 1$

(iii) $4(c - 1) > 3(c - 2)$ **(iv)** $d - 3(d + 2) \geqslant 2(1 + 2d)$

(v) $\frac{1}{2}e + 3\frac{1}{2} < e$ **(vi)** $-f - 2f - 3 < 4(1 + f)$

(vii) $5(2 - 3g) + g \geqslant 8(2g - 4)$ **(viii)** $3(h + 2) - 2(h - 4) > 7(h + 2)$

2 Solve the following inequalities by sketching the curves of the functions involved.

(i) $p^2 - 5p + 4 < 0$ **(ii)** $p^2 - 5p + 4 \geqslant 0$

(iii) $x^2 + 3x + 2 \leqslant 0$ **(iv)** $x^2 + 3x > -2$

(v) $y^2 - 2y - 3 > 0$ **(vi)** $z(z - 1) \leqslant 20$

(vii) $q^2 - 4q + 4 > 0$ **(viii)** $y(y - 2) > 8$

(ix) $3x^2 + 5x - 2 < 0$ **(x)** $2y^2 - 11y - 6 \geqslant 0$

(xi) $4x - 3 \geqslant x^2$ **(xii)** $10y^2 > y + 3$

3 Find the set of values of k for which each of these equations has two real roots.

(i) $2x^2 - 3x + k = 0$ **(ii)** $kx^2 + 4x - 1 = 0$

(iii) $5x^2 + kx + 5 = 0$ **(iv)** $3x^2 + 2kx + k = 0$

4 Find the set of values of k for which each of these equations has no real roots.

(i) $x^2 - 6x + k = 0$ **(ii)** $kx^2 + x - 2 = 0$

(iii) $4x^2 - kx + 4 = 0$ **(iv)** $2kx^2 - kx + 1 = 0$

KEY POINTS

1 The quadratic formula for solving $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b^2 - 4ac$ is called the discriminant.

If $b^2 - 4ac > 0$, the equation has two real roots.
If $b^2 - 4ac = 0$, the equation has one repeated root.
If $b^2 - 4ac < 0$, the equation has no real roots.

2 To solve a pair of simultaneous equations where one equation is non-linear:

- first make x or y the subject of the *linear* equation
- then substitute this rearranged equation for x or y in the *non-linear* equation
- solve to find y or x
- substitute back into the linear equation to find pairs of solutions.

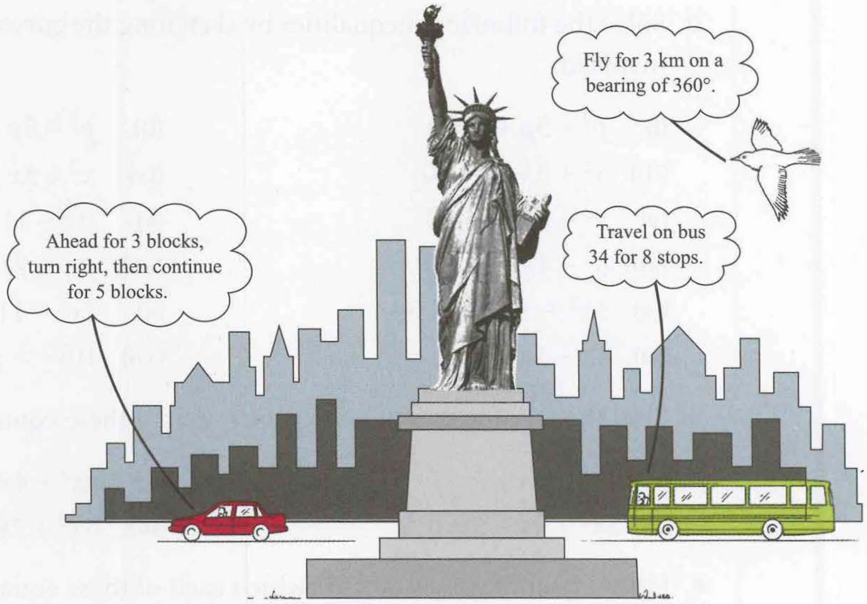
3 Linear inequalities are dealt with like equations *but* if you multiply or divide by a negative number you must reverse the inequality sign.

4 When solving a quadratic inequality it is advisable to sketch the graph.

2 Co-ordinate geometry

A place for everything, and everything in its place

Samuel Smiles



Co-ordinates

Co-ordinates are a means of describing a position relative to some fixed point, or origin. In two dimensions you need two pieces of information; in three dimensions, you need three pieces of information.

In the Cartesian system (named after René Descartes), position is given in perpendicular directions: x, y in two dimensions; x, y, z in three dimensions (see figure 2.1). This chapter concentrates exclusively on two dimensions.

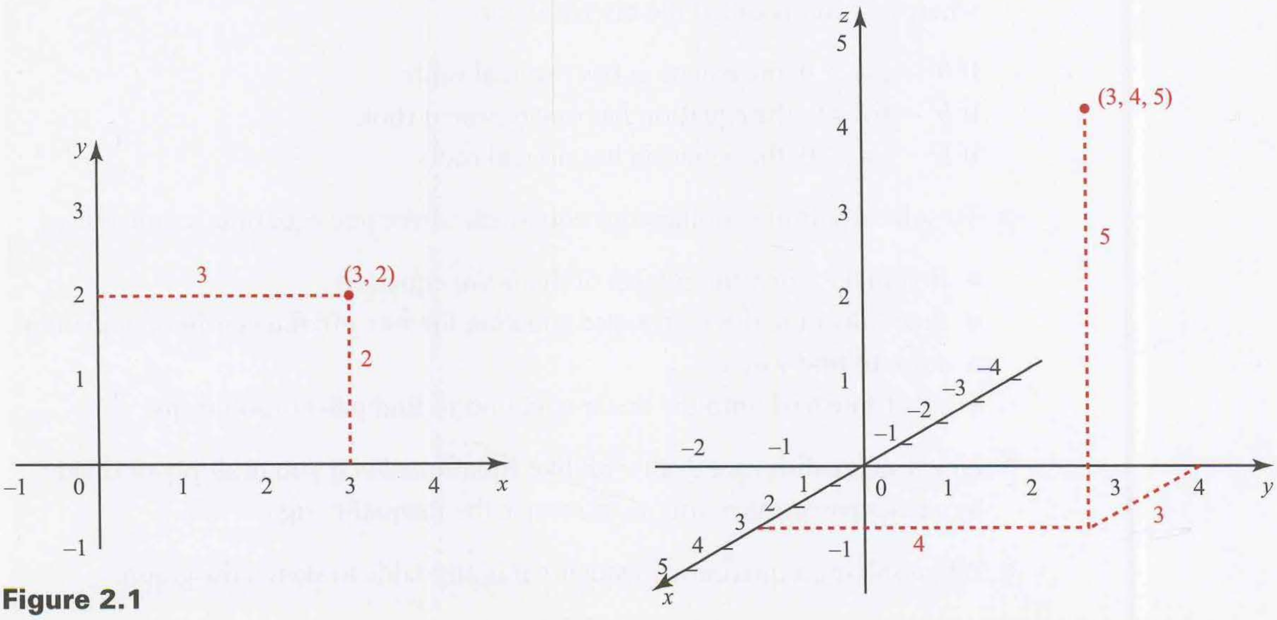


Figure 2.1

Plotting, sketching and drawing

In two dimensions, the co-ordinates of points are often marked on paper and joined up to form lines or curves. A number of words are used to describe this process.

Plot (a line or curve) means mark the points and join them up as accurately as you can. You would expect to do this on graph paper and be prepared to read information from the graph.

Sketch means mark points in approximately the right positions and join them up in the right general shape. You would not expect to use graph paper for a sketch and would not read precise information from one. You would however mark on the co-ordinates of important points, like intersections with the x and y axes and points at which the curve changes direction.

Draw means that you are to use a level of accuracy appropriate to the circumstances, and this could be anything between a rough sketch and a very accurately plotted graph.

The gradient of a line

In everyday English, the word *line* is used to mean a straight line or a curve. In mathematics, it is usually understood to mean a straight line. If you know the co-ordinates of any two points on a line, then you can draw the line.

The slope of a line is measured by its *gradient*. It is often denoted by the letter m.

In figure 2.2, A and B are two points on the line. The gradient of the line AB is given by the increase in the y co-ordinate from A to B divided by the increase in the x co-ordinate from A to B.

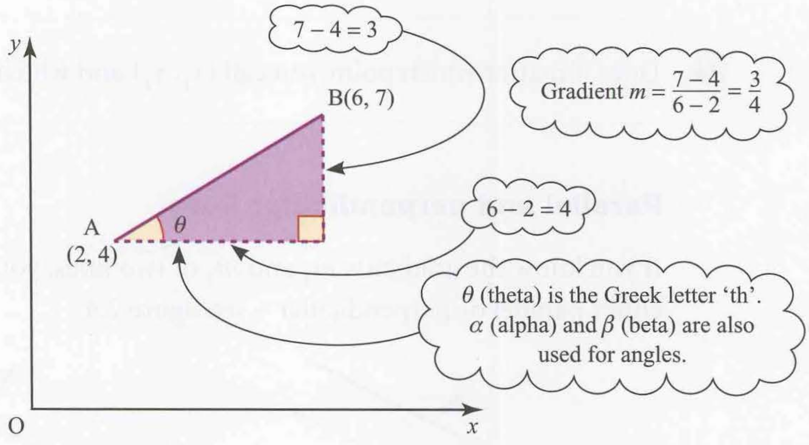


Figure 2.2

In general, when A is the point (x_1, y_1) and B is the point (x_2, y_2), the gradient is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

When the same scale is used on both axes, $m = \tan\theta$ (see figure 2.2). Figure 2.3 shows four lines. Looking at each one from left to right: line A goes uphill and its gradient is positive; line B goes downhill and its gradient is negative. Line C is horizontal and its gradient is 0; the vertical line D has an infinite gradient.

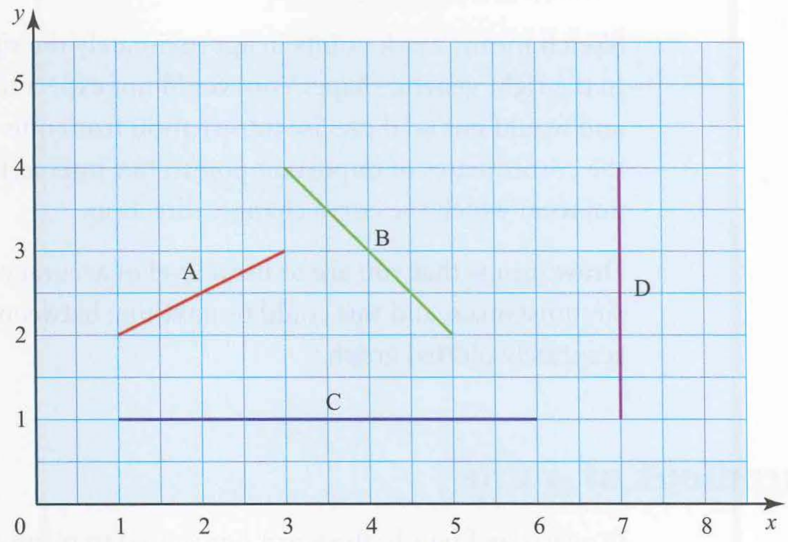


Figure 2.3

ACTIVITY 2.1

On each line in figure 2.3, take any two points and call them (x_1, y_1) and (x_2, y_2). Substitute the values of x_1, y_1, x_2 and y_2 in the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and so find the gradient.

? Does it matter which point you call (x_1, y_1) and which (x_2, y_2)?

Parallel and perpendicular lines

If you know the gradients m_1 and m_2 of two lines, you can tell at once if they are either parallel or perpendicular – see figure 2.4.

Figure 2.4 parallel lines: $m_1 = m_2$ perpendicular lines: $m_1 m_2 = -1$

Lines which are parallel have the same slope and so $m_1 = m_2$. If the lines are perpendicular, $m_1 m_2 = -1$. You can see why this is so in the activities below.

ACTIVITY 2.2 Draw the line L_1 joining (0, 2) to (4, 4), and draw another line L_2 perpendicular to L_1. Find the gradients m_1 and m_2 of these two lines and show that $m_1 m_2 = -1$.

ACTIVITY 2.3 The lines AB and BC in figure 2.5 are equal in length and perpendicular. By showing that triangles ABE and BCD are congruent prove that the gradients m_1 and m_2 must satisfy $m_1 m_2 = -1$.

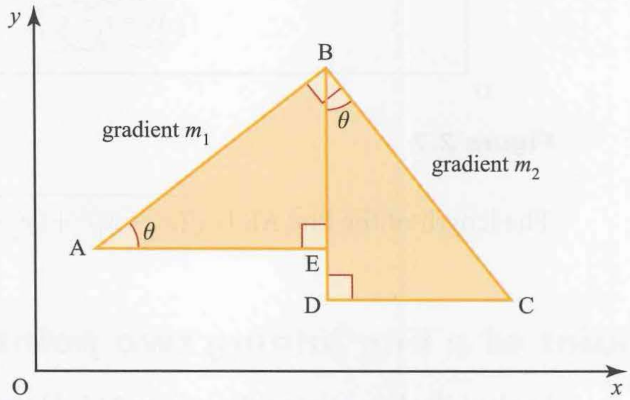


Figure 2.5

Lines for which $m_1 m_2 = -1$ will only look perpendicular if the same scale has been used for both axes.

The distance between two points

When the co-ordinates of two points are known, the distance between them can be calculated using Pythagoras' theorem, as shown in figure 2.6.

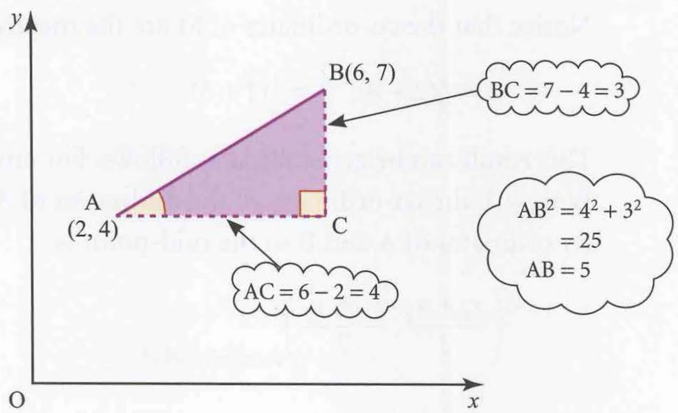


Figure 2.6

This method can be generalised to find the distance between any two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, as in figure 2.7.

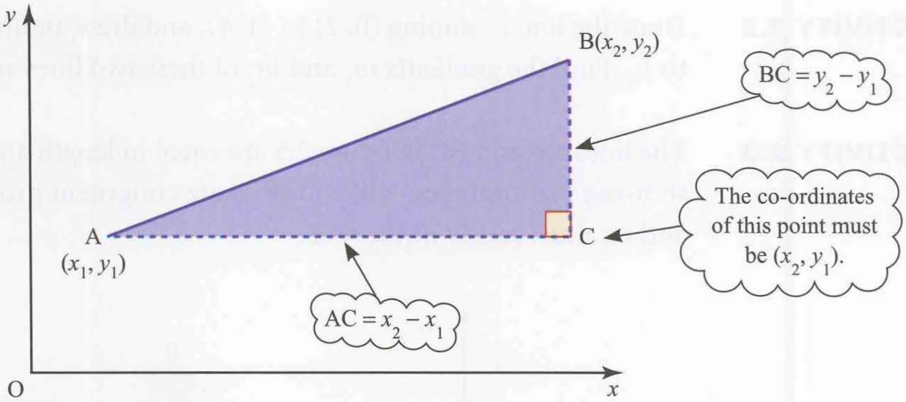


Figure 2.7

The length of the line AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The mid-point of a line joining two points

Look at the line joining the points A(2, 1) and B(8, 5) in figure 2.8. The point M(5, 3) is the mid-point of AB.

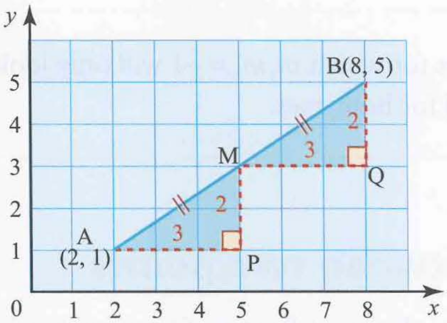


Figure 2.8

Notice that the co-ordinates of M are the means of the co-ordinates of A and B.

$$5 = \tfrac{1}{2}(2 + 8); \quad 3 = \tfrac{1}{2}(1 + 5).$$

This result can be generalised as follows. For any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the co-ordinates of the mid-point of AB are the means of the co-ordinates of A and B so the mid-point is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

EXAMPLE 2.1

A and B are the points (2, 5) and (6, 3) respectively (see figure 2.9). Find:

(i) the gradient of AB
(ii) the length of AB
(iii) the mid-point of AB
(iv) the gradient of a line perpendicular to AB.

SOLUTION

Taking A(2, 5) as the point (x_1, y_1), and B(6, 3) as the point (x_2, y_2) gives $x_1 = 2$, $y_1 = 5$, $x_2 = 6$, $y_2 = 3$.

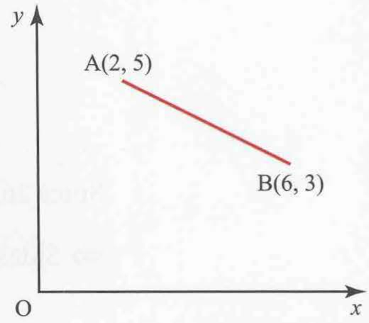


Figure 2.9

(i) $$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3-5}{6-2} = -\tfrac{1}{2}$$

(ii) $$\text{Length AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(6-2)^2 + (3-5)^2}$$
$$= \sqrt{16 + 4} = \sqrt{20}$$

(iii) $$\text{Mid-point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2+6}{2}, \frac{5+3}{2}\right) = (4, 4)$$

(iv) Gradient of AB $= m_1 = -\tfrac{1}{2}$.

If m_2 is the gradient of a line perpendicular to AB, then $m_1 m_2 = -1$

$$\Rightarrow -\tfrac{1}{2} m_2 = -1$$
$$m_2 = 2.$$

EXAMPLE 2.2

Using two different methods, show that the lines joining P(2, 7), Q(3, 2) and R(0, 5) form a right-angled triangle (see figure 2.10).

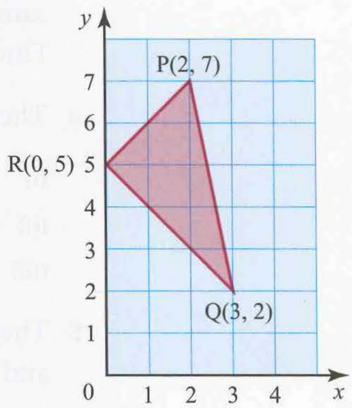


Figure 2.10

SOLUTION

Method 1

$$\text{Gradient of RP} = \frac{7-5}{2-0} = 1$$

$$\text{Gradient of RQ} = \frac{2-5}{3-0} = -1$$

$\Rightarrow$ Product of gradients $= 1 \times (-1) = -1$

$\Rightarrow$ Sides RP and RQ are at right angles.

Method 2

Pythagoras' theorem states that for a right-angled triangle whose hypotenuse has length a and whose other sides have lengths b and c, $a^2 = b^2 + c^2$.

Conversely, if you can show that $a^2 = b^2 + c^2$ for a triangle with sides of lengths a, b, and c, then the triangle has a right angle and the side of length a is the hypotenuse.

This is the basis for the alternative proof, in which you use

$$\text{length}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

$$PQ^2 = (3-2)^2 + (2-7)^2 = 1 + 25 = 26$$

$$RP^2 = (2-0)^2 + (7-5)^2 = 4 + 4 = 8$$

$$RQ^2 = (3-0)^2 + (2-5)^2 = 9 + 9 = 18$$

Since $26 = 8 + 18$, $\quad PQ^2 = RP^2 + RQ^2$

$\Rightarrow$ Sides RP and RQ are at right angles.

EXERCISE 2A

1 For the following pairs of points A and B, calculate:

(a) the gradient of the line AB

(b) the mid-point of the line joining A to B

(c) the distance AB

(d) the gradient of the line perpendicular to AB.

(i)	A(0, 1)	B(2, –3)	**(ii)**	A(3, 2)	B(4, –1)
(iii)	A(–6, 3)	B(6, 3)	**(iv)**	A(5, 2)	B(2, –8)
(v)	A(4, 3)	B(2, 0)	**(vi)**	A(1, 4)	B(1, –2)

2 The line joining the point P(3, –4) to Q(q, 0) has a gradient of 2.
Find the value of q.

3 The three points X(2, –1), Y(8, y) and Z(11, 2) are collinear (i.e. they lie on the same straight line).
Find the value of y.

4 The points A, B, C and D have co-ordinates (1, 2), (7, 5), (9, 8) and (3, 5).

(i) Find the gradients of the lines AB, BC, CD and DA.

(ii) What do these gradients tell you about the quadrilateral ABCD?

(iii) Draw a diagram to check your answer to part **(ii)**.

5 The points A, B and C have co-ordinates (2, 1), (b, 3) and (5, 5), where $b > 3$ and $\angle ABC = 90°$. Find:

(i) the value of b

(ii) the lengths of AB and BC

(iii) the area of triangle ABC.

6 The triangle PQR has vertices P(8, 6), Q(0, 2) and R(2, r). Find the values of r when the triangle:

(i) has a right angle at P
(ii) has a right angle at Q
(iii) has a right angle at R
(iv) is isosceles with RQ = RP.

7 The points A, B, and C have co-ordinates (−4, 2), (7, 4) and (−3, −1).

(i) Draw the triangle ABC.
(ii) Show by calculation that the triangle ABC is isosceles and name the two equal sides.
(iii) Find the mid-point of the third side.
(iv) By calculating appropriate lengths, calculate the area of the triangle ABC.

8 For the points P(x, y), and Q($3x$, $5y$), find in terms of x and y:

(i) the gradient of the line PQ
(ii) the mid-point of the line PQ
(iii) the length of the line PQ.

9 A quadrilateral has vertices A(0, 0), B(0, 3), C(6, 6) and D(12, 6).

(i) Draw the quadrilateral.
(ii) Show by calculation that it is a trapezium.
(iii) Find the co-ordinates of E when EBCD is a parallelogram.

10 Three points A, B and C have co-ordinates (1, 3), (3, 5) and (−1, y). Find the values of y when:

(i) AB = AC
(ii) AC = BC
(iii) AB is perpendicular to BC
(iv) A, B and C are collinear.

11 The diagonals of a rhombus bisect each other at 90°, and conversely, when two lines bisect each other at 90°, the quadrilateral formed by joining the end points of the lines is a rhombus.

Use the converse result to show that the points with co-ordinates (1, 2), (8, −2), (7, 6) and (0, 10) are the vertices of a rhombus, and find its area.

The equation of a straight line

The word *straight* means going in a constant direction, that is with fixed gradient. This fact allows you to find the equation of a straight line from first principles.

EXAMPLE 2.3

Find the equation of the straight line with gradient 2 through the point $(0, -5)$.

SOLUTION

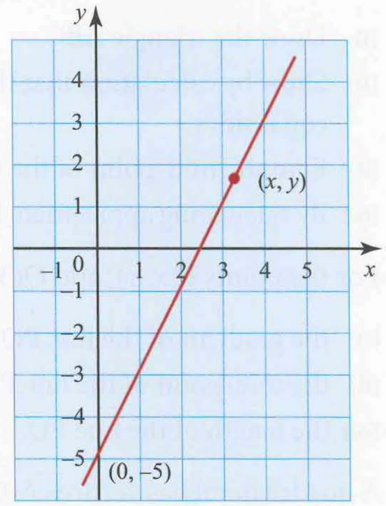


Figure 2.11

Take a general point (x, y) on the line, as shown in figure 2.11. The gradient of the line joining $(0, -5)$ to (x, y) is given by

$$\text{gradient} = \frac{y-(-5)}{x-0} = \frac{y+5}{x}.$$

Since we are told that the gradient of the line is 2, this gives

$$\frac{y+5}{x} = 2$$

$$\Rightarrow \qquad y = 2x - 5.$$

Since (x, y) is a general point on the line, this holds for any point on the line and is therefore the equation of the line.

The example above can easily be generalised (see page 50) to give the result that the equation of the line with gradient m cutting the y axis at the point $(0, c)$ is

$$y = mx + c.$$

(In the example above, m is 2 and c is -5.)

This is a well-known standard form for the equation of a straight line.

Drawing a line, given its equation

There are several standard forms for the equation of a straight line, as shown in figure 2.12.

When you need to draw the graph of a straight line, given its equation, the first thing to do is to look carefully at the form of the equation and see if you can recognise it.

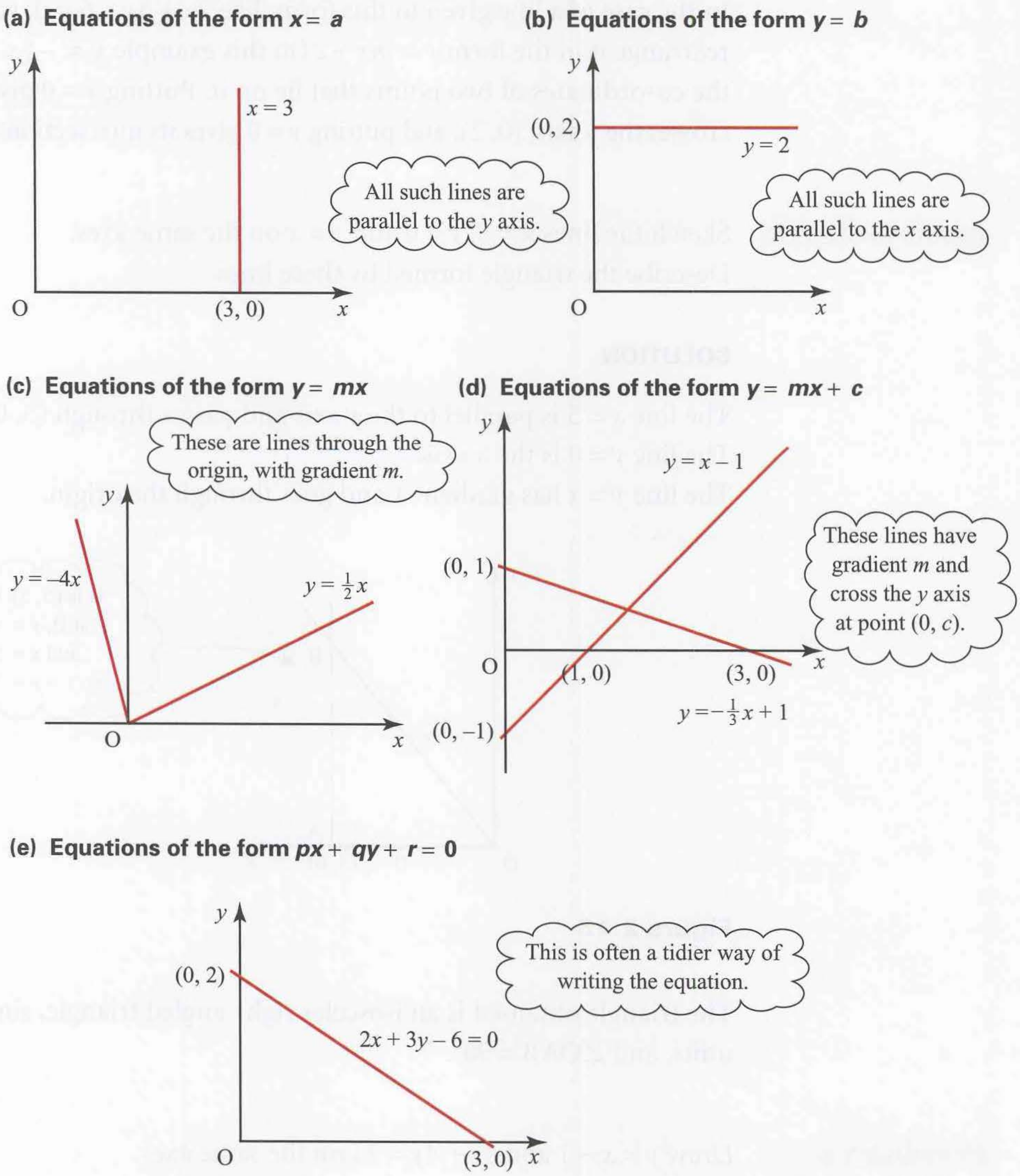


Figure 2.12

(a), (b): *Lines parallel to the axes*

Lines parallel to the x axis have the form $y =$ constant, those parallel to the y axis the form $x =$ constant. Such lines are easily recognised and drawn.

(c), (d): *Equations of the form* $y = mx + c$

The line $y = mx + c$ crosses the y axis at the point $(0, c)$ and has gradient m. If $c = 0$, it goes through the origin. In either case you know one point and can complete the line either by finding one more point, for example by substituting $x = 1$, or by following the gradient (e.g. 1 along and 2 up for gradient 2).

(e): *Equations of the form* $px + qy + r = 0$

In the case of a line given in this form, like $2x + 3y - 6 = 0$, you can either rearrange it in the form $y = mx + c$ (in this example $y = -\frac{2}{3}x + 2$), or you can find the co-ordinates of two points that lie on it. Putting $x = 0$ gives the point where it crosses the y axis, (0, 2), and putting $y = 0$ gives its intersection with the x axis, (3, 0).

EXAMPLE 2.4

Sketch the lines $x = 5$, $y = 0$ and $y = x$ on the same axes.
Describe the triangle formed by these lines.

SOLUTION

The line $x = 5$ is parallel to the y axis and passes through (5, 0).
The line $y = 0$ is the x axis.
The line $y = x$ has gradient 1 and goes through the origin.

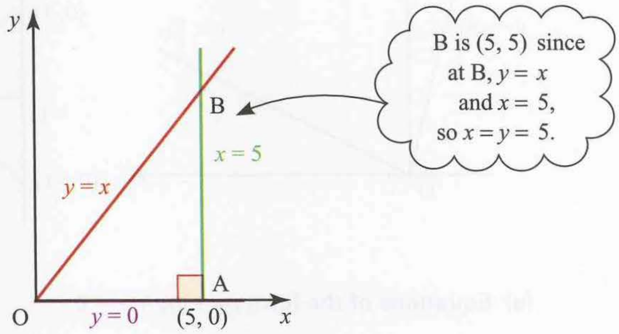


Figure 2.13

The triangle obtained is an isosceles right-angled triangle, since OA = AB = 5 units, and ∠OAB = 90°.

EXAMPLE 2.5

Draw $y = x - 1$ and $3x + 4y = 24$ on the same axes.

SOLUTION

The line $y = x - 1$ has gradient 1 and passes through the point (0, −1).
Substituting $y = 0$ gives $x = 1$, so the line also passes through (1, 0).

Find two points on the line $3x + 4y = 24$.

Substituting $x = 0$	gives	$4y = 24$	so	$y = 6$.
Substituting $y = 0$	gives	$3x = 24$	so	$x = 8$.

The line passes through (0, 6) and (8, 0).

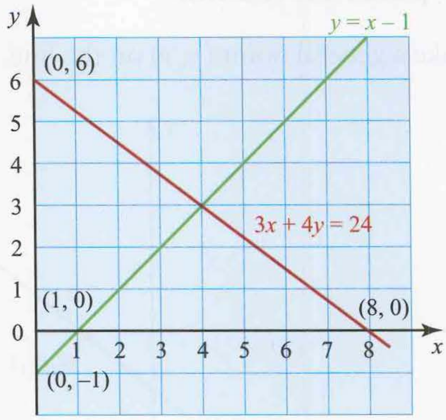


Figure 2.14

EXERCISE 2B

1 Sketch the following lines.

(i) $y = -2$	**(ii)** $x = 5$	**(iii)** $y = 2x$
(iv) $y = -3x$	**(v)** $y = 3x + 5$	**(vi)** $y = x - 4$
(vii) $y = x + 4$	**(viii)** $y = \frac{1}{2}x + 2$	**(ix)** $y = 2x + \frac{1}{2}$
(x) $y = -4x + 8$	**(xi)** $y = 4x - 8$	**(xii)** $y = -x + 1$
(xiii) $y = -\frac{1}{2}x - 2$	**(xiv)** $y = 1 - 2x$	**(xv)** $3x - 2y = 6$
(xvi) $2x + 5y = 10$	**(xvii)** $2x + y - 3 = 0$	**(xviii)** $2y = 5x - 4$
(xix) $x + 3y - 6 = 0$	**(xx)** $y = 2 - x$	

2 By calculating the gradients of the following pairs of lines, state whether they are parallel, perpendicular or neither.

(i)	$y = -4$	$x = 2$	**(ii)**	$y = 3x$	$x = 3y$
(iii)	$2x + y = 1$	$x - 2y = 1$	**(iv)**	$y = 2x + 3$	$4x - y + 1 = 0$
(v)	$3x - y + 2 = 0$	$3x + y = 0$	**(vi)**	$2x + 3y = 4$	$2y = 3x - 2$
(vii)	$x + 2y - 1 = 0$	$x + 2y + 1 = 0$	**(viii)**	$y = 2x - 1$	$2x - y + 3 = 0$
(ix)	$y = x - 2$	$x + y = 6$	**(x)**	$y = 4 - 2x$	$x + 2y = 8$
(xi)	$x + 3y - 2 = 0$	$y = 3x + 2$	**(xii)**	$y = 2x$	$4x + 2y = 5$

Finding the equation of a line

The simplest way to find the equation of a straight line depends on what information you have been given.

(i) *Given the gradient, m, and the co-ordinates (x_1, y_1) of one point on the line*

Take a general point (x, y) on the line, as shown in figure 2.15.

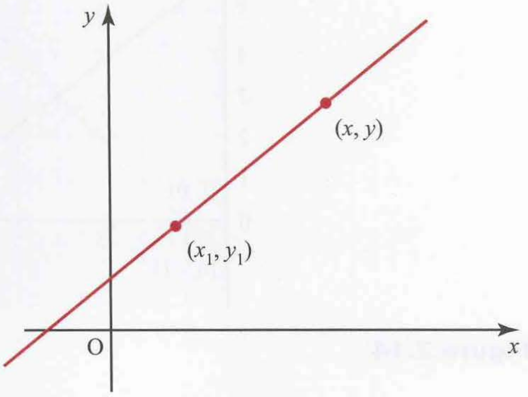


Figure 2.15

The gradient, m, of the line joining (x_1, y_1) to (x, y) is given by

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \quad y - y_1 = m(x - x_1).$$

This is a very useful form of the equation of a straight line. Two positions of the point (x_1, y_1) lead to particularly important forms of the equation.

(a) When the given point (x_1, y_1) is the point $(0, c)$, where the line crosses the y axis, the equation takes the familiar form

$$y = mx + c$$

as shown in figure 2.16.

(b) When the given point (x_1, y_1) is the origin, the equation takes the form

$$y = mx$$

as shown in figure 2.17.

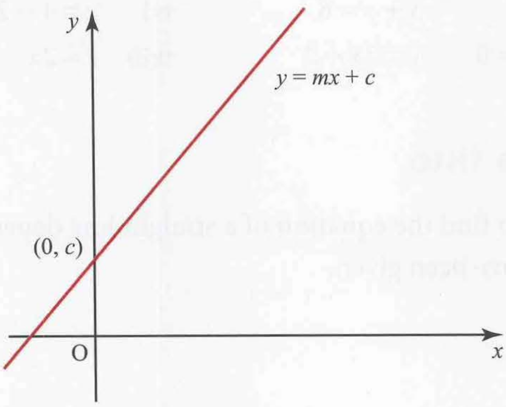


Figure 2.16

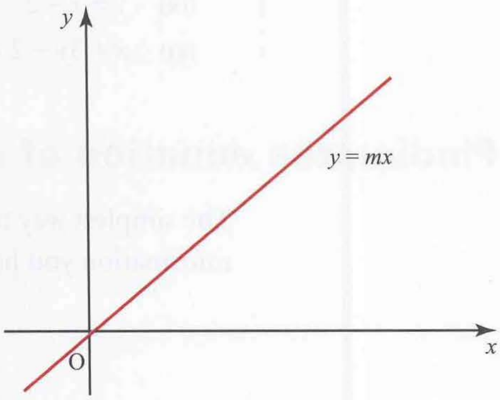


Figure 2.17

EXAMPLE 2.6

Find the equation of the line with gradient 3 which passes through the point $(2, -4)$.

SOLUTION

Using $y - y_1 = m(x - x_1)$

$$\Rightarrow y - (-4) = 3(x - 2)$$
$$\Rightarrow y + 4 = 3x - 6$$
$$\Rightarrow y = 3x - 10.$$

(ii) *Given two points, (x_1, y_1) and (x_2, y_2)*

The two points are used to find the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

This value of m is then substituted in the equation

$$y - y_1 = m(x - x_1).$$

This gives

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

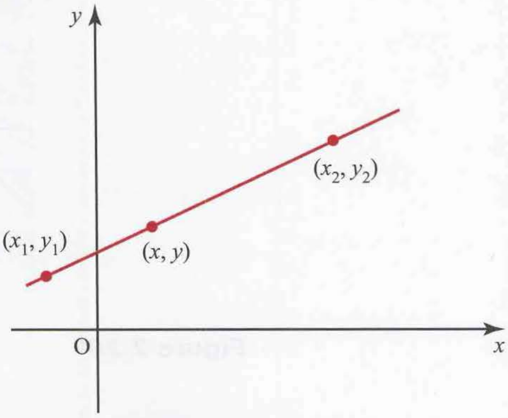


Figure 2.18

Rearranging the equation gives

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \text{or} \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 2.7

Find the equation of the line joining $(2, 4)$ to $(5, 3)$.

SOLUTION

Taking (x_1, y_1) to be $(2, 4)$ and (x_2, y_2) to be $(5, 3)$, and substituting the values in

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

gives $\dfrac{y - 4}{3 - 4} = \dfrac{x - 2}{5 - 2}$.

This can be simplified to $x + 3y - 14 = 0$.

? Show that the equation of the line in figure 2.19 can be written

$$\frac{x}{a} + \frac{y}{b} = 1.$$

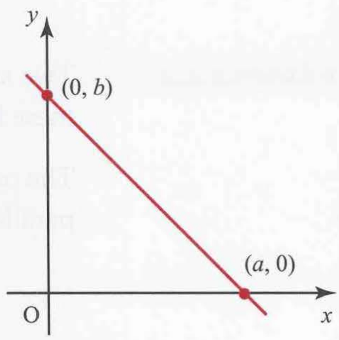


Figure 2.19

Different techniques to solve problems

The following examples illustrate the different techniques and show how these can be used to solve a problem.

EXAMPLE 2.8

Find the equations of the lines (a) – (e) in figure 2.20.

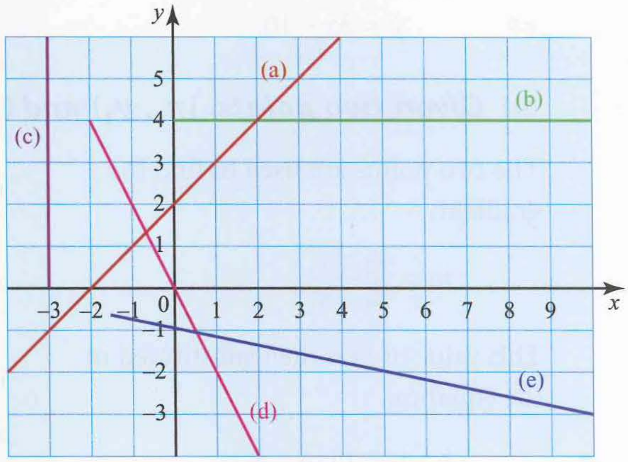


Figure 2.20

SOLUTION

Line (a) passes through (0, 2) and has gradient 1
$\Rightarrow$ equation of (a) is $y = x + 2$.

Line (b) is parallel to the x axis and passes through (0, 4)
$\Rightarrow$ equation of (b) is $y = 4$.

Line (c) is parallel to the y axis and passes through (–3, 0)
$\Rightarrow$ equation of (c) is $x = -3$.

Line (d) passes through (0, 0) and has gradient –2
$\Rightarrow$ equation of (d) is $y = -2x$.

Line (e) passes through (0, –1) and has gradient $-\frac{1}{5}$
$\Rightarrow$ equation of (e) is $y = -\frac{1}{5}x - 1$.

This can be rearranged to give $x + 5y + 5 = 0$.

EXAMPLE 2.9

Two sides of a parallelogram are the lines $2y = x + 12$ and $y = 4x - 10$. Sketch these lines on the same diagram.

The origin is a vertex of the parallelogram. Complete the sketch of the parallelogram and find the equations of the other two sides.

SOLUTION

The line $2y = x + 12$ has gradient $\frac{1}{2}$ and passes through the point (0, 6) (since dividing by 2 gives $y = \frac{1}{2}x + 6$).

The line $y = 4x - 10$ has gradient 4 and passes through the point (0, –10).

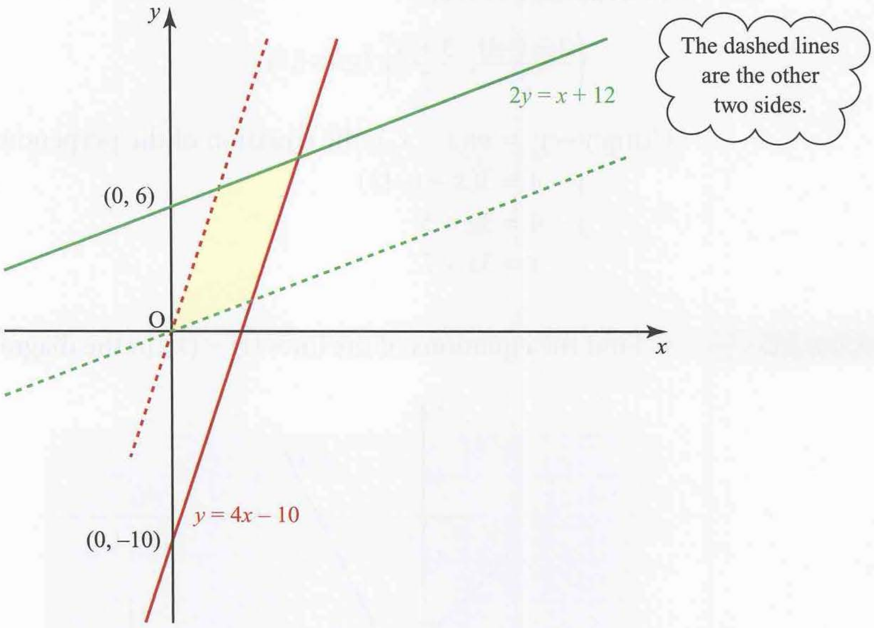


Figure 2.21

The other two sides are lines with gradients $\frac{1}{2}$ and 4 which pass through (0, 0), i.e. $y = \frac{1}{2}x$ and $y = 4x$.

EXAMPLE 2.10

Find the equation of the perpendicular bisector of the line joining P(–4, 5) to Q(2, 3).

SOLUTION

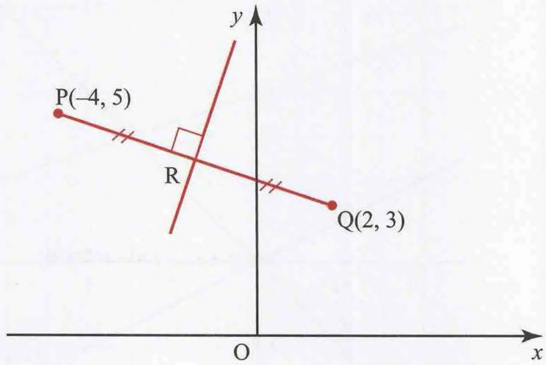


Figure 2.22

The gradient of the line PQ is

$$\frac{3-5}{2-(-4)} = \frac{-2}{6} = -\frac{1}{3}$$

and so the gradient of the perpendicular bisector is +3.

The perpendicular bisector passes throught the mid-point, R, of the line PQ. The co-ordinates of R are

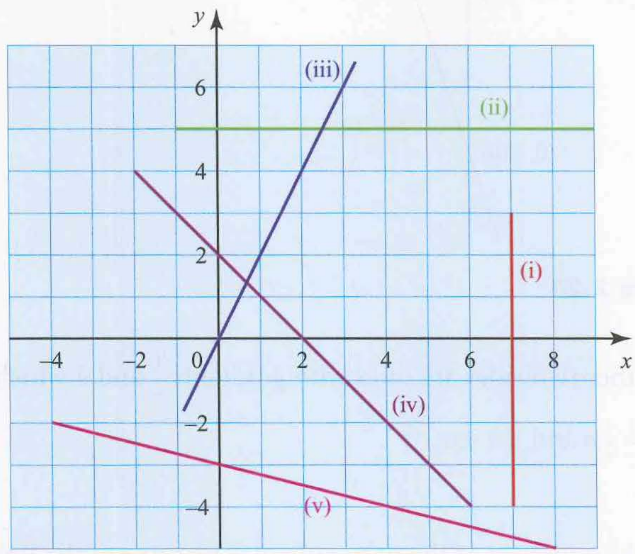
$$\left(\frac{2+(-4)}{2}, \frac{3+5}{2}\right) \text{ i.e. } (-1, 4).$$

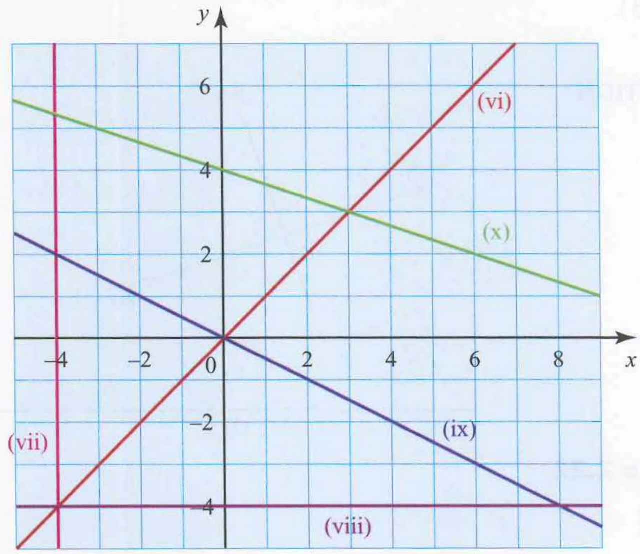
Using $y - y_1 = m(x - x_1)$, the equation of the perpendicular bisector is

$$y - 4 = 3(x - (-1))$$
$$y - 4 = 3x + 3$$
$$y = 3x + 7.$$

EXERCISE 2C

1 Find the equations of the lines (i) – (x) in the diagrams below.





2 Find the equations of the following lines.

(i) parallel to $y = 2x$ and passing through (1, 5)
(ii) parallel to $y = 3x - 1$ and passing through (0, 0)
(iii) parallel to $2x + y - 3 = 0$ and passing through (–4, 5)
(iv) parallel to $3x - y - 1 = 0$ and passing through (4, –2)
(v) parallel to $2x + 3y = 4$ and passing through (2, 2)
(vi) parallel to $2x - y - 8 = 0$ and passing through (–1, –5)

3 Find the equations of the following lines.

(i) perpendicular to $y = 3x$ and passing through (0, 0)
(ii) perpendicular to $y = 2x + 3$ and passing through (2, –1)
(iii) perpendicular to $2x + y = 4$ and passing through (3, 1)
(iv) perpendicular to $2y = x + 5$ and passing through (–1, 4)
(v) perpendicular to $2x + 3y = 4$ and passing through (5, –1)
(vi) perpendicular to $4x - y + 1 = 0$ and passing through (0, 6)

4 Find the equations of the line AB in each of the following cases.

(i) A(0, 0) B(4, 3) **(ii)** A(2, –1) B(3, 0)
(iii) A(2, 7) B(2, –3) **(iv)** A(3, 5) B(5, –1)
(v) A(–2, 4) B(5, 3) **(vi)** A(–4, –2) B(3, –2)

5 Triangle ABC has an angle of 90° at B. Point A is on the y axis, AB is part of the line $x - 2y + 8 = 0$ and C is the point (6, 2).

(i) Sketch the triangle.
(ii) Find the equations of AC and BC.
(iii) Find the lengths of AB and BC and hence find the area of the triangle.
(iv) Using your answer to part **(iii)**, find the length of the perpendicular from B to AC.

6 A median of a triangle is a line joining one of the vertices to the mid-point of the opposite side.

In a triangle OAB, O is at the origin, A is the point (0, 6) and B is the point (6, 0).

(i) Sketch the triangle.
(ii) Find the equations of the three medians of the triangle.
(iii) Show that the point (2, 2) lies on all three medians. (This shows that the medians of this triangle are concurrent.)

7 A quadrilateral ABCD has its vertices at the points (0, 0), (12, 5), (0, 10) and (–6, 8) respectively.

(i) Sketch the quadrilateral.
(ii) Find the gradient of each side.
(iii) Find the length of each side.
(iv) Find the equation of each side.
(v) Find the area of the quadrilateral.

The intersection of two lines

The intersection of any two curves (or lines) can be found by solving their equations simultaneously. In the case of two distinct lines, there are two possibilities:

(i) they are parallel

(ii) they intersect at a single point.

EXAMPLE 2.11

Sketch the lines $x + 2y = 1$ and $2x + 3y = 4$ on the same axes, and find the co-ordinates of the point where they intersect.

SOLUTION

The line $x + 2y = 1$ passes through $\left(0, \frac{1}{2}\right)$ and $(1, 0)$.

The line $2x + 3y = 4$ passes through $\left(0, \frac{4}{3}\right)$ and $(2, 0)$.

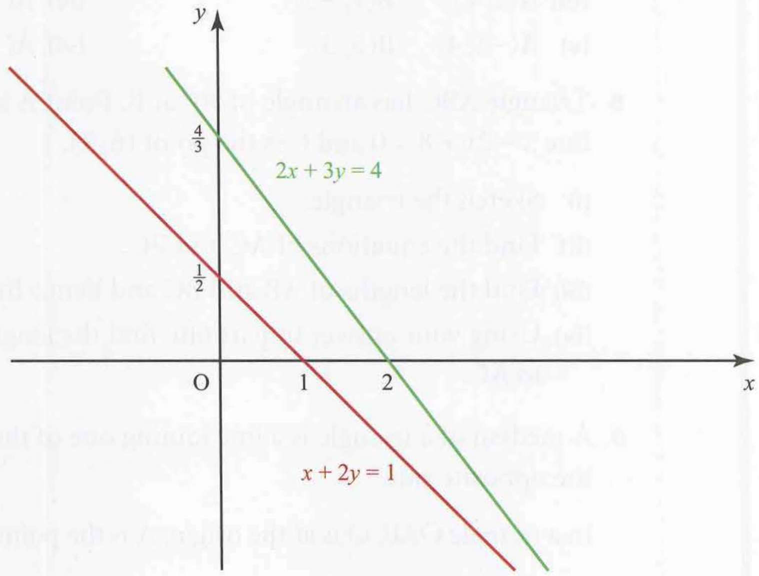


Figure 2.23

$$\begin{array}{llll} ①: & x + 2y = 1 & ①\times 2: & 2x + 4y = 2 \\ ②: & 2x + 3y = 4 & ②: & \underline{2x + 3y = 4} \\ & & \text{Subtract:} & y = -2. \end{array}$$

$$\begin{array}{ll} \text{Substituting } y = -2 \text{ in } ①: & x - 4 = 1 \\ & \Rightarrow x = 5. \end{array}$$

The co-ordinates of the point of intersection are $(5, -2)$.

EXAMPLE 2.12

Find the co-ordinates of the vertices of the triangle whose sides have the equations $x+y=4$, $2x-y=8$ and $x+2y=-1$.

SOLUTION

A sketch will be helpful, so first find where each line crosses the axes.

① $x+y=4$ crosses the axes at $(0, 4)$ and $(4, 0)$.

② $2x-y=8$ crosses the axes at $(0, -8)$ and $(4, 0)$.

③ $x+2y=-1$ crosses the axes at $\left(0, -\frac{1}{2}\right)$ and $(-1, 0)$.

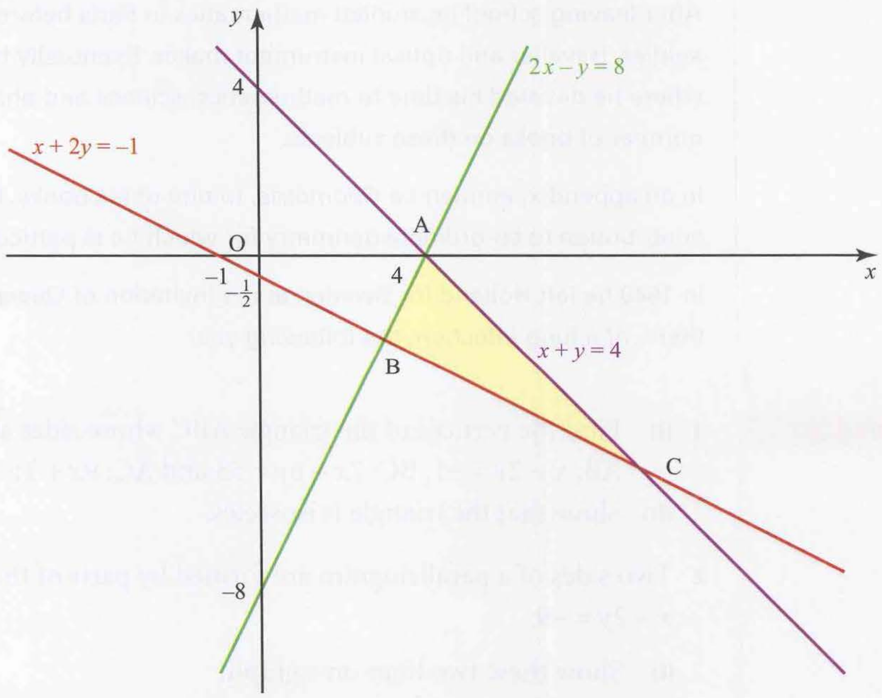


Figure 2.24

Since two lines pass through the point $(4, 0)$ this is clearly one of the vertices. It has been labelled A on figure 2.24.

Point B is found by solving ② and ③ simultaneously:

$$\begin{array}{lrl} ②\times 2: & 4x-2y & =16 \\ ③: & x+2y & =-1 \\ \hline \text{Add} & 5x \quad & =15 \quad \text{so} \quad x=3. \end{array}$$

Substituting $x=3$ in ② gives $y=-2$, so B is the point $(3, -2)$.

Point C is found by solving ① and ③ simultaneously:

$$\begin{array}{lrl} ①: & x+\ \ y & =4 \\ ③: & x+2y & =-1 \\ \hline \text{Subtract} & -y & =5 \quad \text{so} \quad y=-5. \end{array}$$

Substituting $y=-5$ in ① gives $x=9$, so C is the point $(9, -5)$.

? The line l has equation $2x - y = 4$ and the line m has equation $y = 2x - 3$.

What can you say about the intersection of these two lines?

Historical note

René Descartes was born near Tours in France in 1596. At the age of eight he was sent to a Jesuit boarding school where, because of his frail health, he was allowed to stay in bed until late in the morning. This habit stayed with him for the rest of his life and he claimed that he was at his most productive before getting up.

After leaving school he studied mathematics in Paris before becoming in turn a soldier, traveller and optical instrument maker. Eventually he settled in Holland where he devoted his time to mathematics, science and philosophy, and wrote a number of books on these subjects.

In an appendix, entitled *La Géométrie*, to one of his books, Descartes made the contribution to co-ordinate geometry for which he is particularly remembered.

In 1649 he left Holland for Sweden at the invitation of Queen Christina but died there, of a lung infection, the following year.

EXERCISE 2D

1 **(i)** Find the vertices of the triangle ABC whose sides are given by the lines AB: $x - 2y = -1$, BC: $7x + 6y = 53$ and AC: $9x + 2y = 11$.

(ii) Show that the triangle is isosceles.

2 Two sides of a parallelogram are formed by parts of the lines $2x - y = -9$ and $x - 2y = -9$.

(i) Show these two lines on a graph.

(ii) Find the co-ordinates of the vertex where they intersect.

Another vertex of the parallelogram is the point (2, 1).

(iii) Find the equations of the other two sides of the parallelogram.

(iv) Find the co-ordinates of the other two vertices.

3 A(0, 1), B(1, 4), C(4, 3) and D(3, 0) are the vertices of a quadrilateral ABCD.

(i) Find the equations of the diagonals AC and BD.

(ii) Show that the diagonals AC and BD bisect each other at right angles.

(iii) Find the lengths of AC and BD.

(iv) What type of quadrilateral is ABCD?

4 The line with equation $5x + y = 20$ meets the x axis at A and the line with equation $x + 2y = 22$ meets the y axis at B. The two lines intersect at a point C.

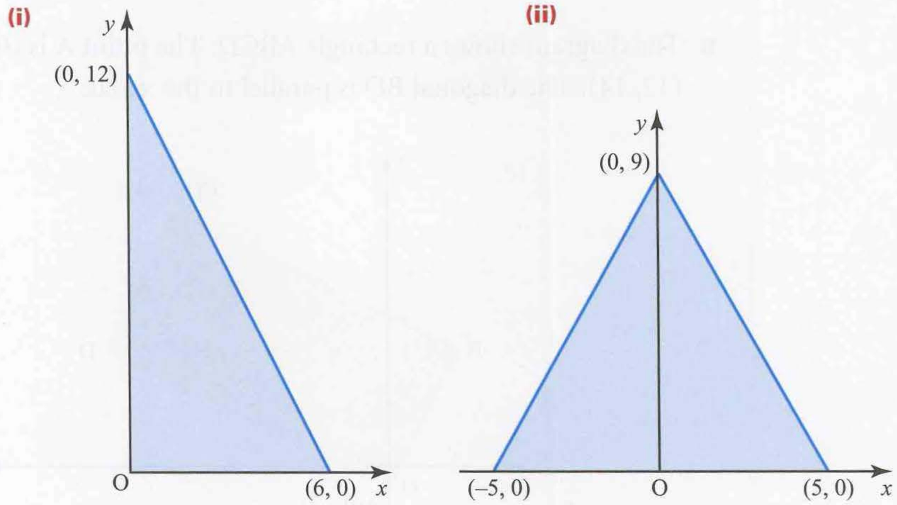
(i) Sketch the two lines on the same diagram.

(ii) Calculate the co-ordinates of A, B and C.

(iii) Calculate the area of triangle OBC where O is the origin.

(iv) Find the co-ordinates of the point E such that ABEC is a parallelogram.

5 A median of a triangle is a line joining a vertex to the mid-point of the opposite side. In any triangle, the three medians meet at a point.
The centroid of a triangle is at the point of intersection of the medians.
Find the co-ordinates of the centroid for each triangle shown.



6 You are given the co-ordinates of the four points A(6, 2), B(2, 4), C(–6, –2) and D(–2, –4).

(i) Calculate the gradients of the lines AB, CB, DC and DA.
Hence describe the shape of the figure ABCD.

(ii) Show that the equation of the line DA is $4y - 3x = -10$ and find the length DA.

(iii) Calculate the gradient of a line which is perpendicular to DA and hence find the equation of the line l through B which is perpendicular to DA.

(iv) Calculate the co-ordinates of the point P where l meets DA.

(v) Calculate the area of the figure ABCD.

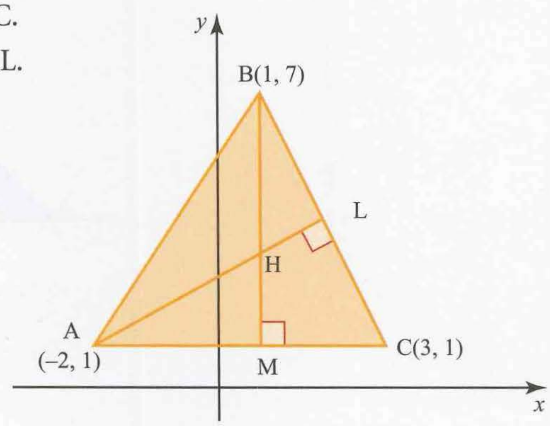
[MEI]

7 The diagram shows a triangle whose vertices are A(–2, 1), B(1, 7) and C(3, 1). The point L is the foot of the perpendicular from A to BC, and M is the foot of the perpendicular from B to AC.

(i) Find the gradient of the line BC.

(ii) Find the equation of the line AL.

(iii) Write down the equation of the line BM.



The lines AL and BM meet at H.

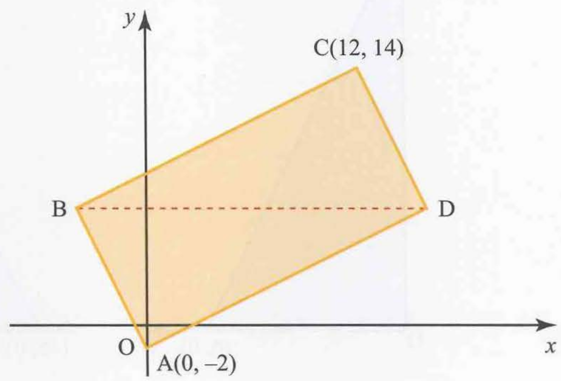
(iv) Find the co-ordinates of H.

(v) Show that CH is perpendicular to AB.

(vi) Find the area of the triangle BLH.

[MEI]

8 The diagram shows a rectangle ABCD. The point A is (0, −2) and C is (12, 14). The diagonal BD is parallel to the x axis.



(i) Explain why the y co-ordinate of D is 6.

The x co-ordinate of D is h.

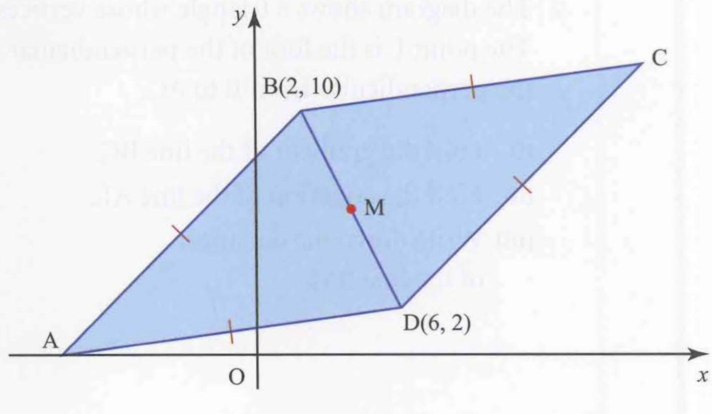
(ii) Express the gradients of AD and CD in terms of h.

(iii) Calculate the x co-ordinates of D and B.

(iv) Calculate the area of the rectangle ABCD.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q9 November 2009]

9 The diagram shows a rhombus ABCD. The points B and D have co-ordinates (2, 10) and (6, 2) respectively, and A lies on the x axis. The mid-point of BD is M. Find, by calculation, the co-ordinates of each of M, A and C.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 June 2005]

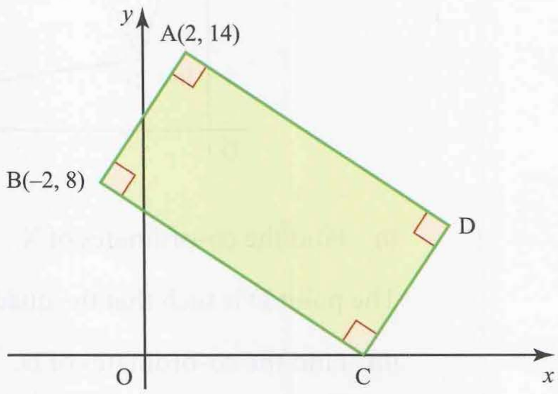
10 Three points have co-ordinates A(2, 6), B(8, 10) and C(6, 0). The perpendicular bisector of AB meets the line BC at D. Find

(i) the equation of the perpendicular bisector of AB in the form $ax + by = c$

(ii) the co-ordinates of D.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 November 2005]

11 The diagram shows a rectangle ABCD. The point A is (2, 14), B is (–2, 8) and C lies on the x axis.



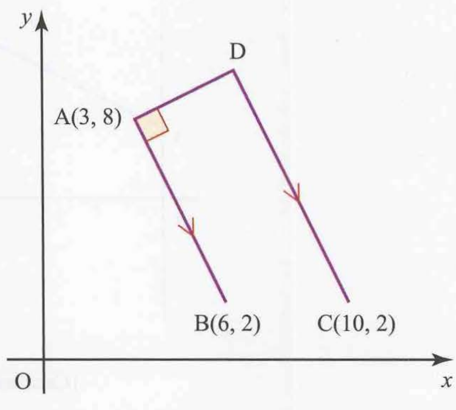
Find

(i) the equation of BC.

(ii) the co-ordinates of C and D.

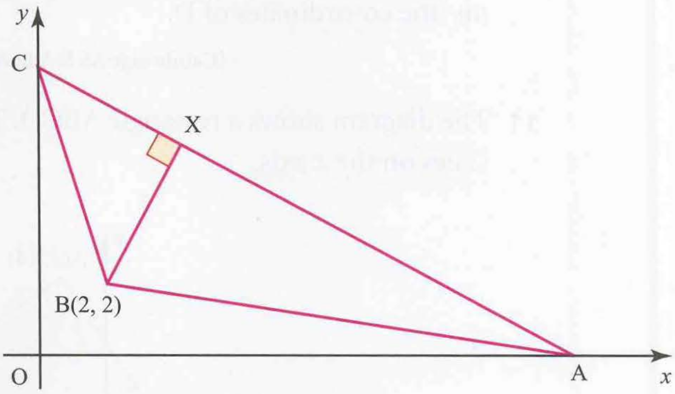
[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 June 2007]

12 The three points A(3, 8), B(6, 2) and C(10, 2) are shown in the diagram. The point D is such that the line DA is perpendicular to AB and DC is parallel to AB. Calculate the co-ordinates of D.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 November 2007]

13 In the diagram, the points A and C lie on the x and y axes respectively and the equation of AC is $2y + x = 16$. The point B has co-ordinates (2, 2). The perpendicular from B to AC meets AC at the point X.



(i) Find the co-ordinates of X.

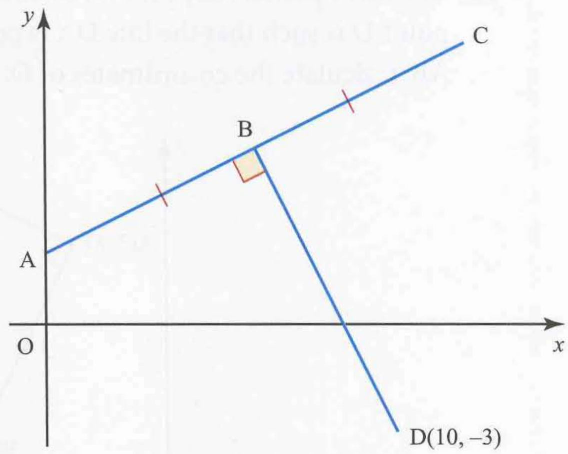
The point D is such that the quadrilateral ABCD has AC as a line of symmetry.

(ii) Find the co-ordinates of D.

(iii) Find, correct to 1 decimal place, the perimeter of ABCD.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2008]

14 The diagram shows points A, B and C lying on the line $2y = x + 4$. The point A lies on the y axis and AB = BC. The line from D(10, –3) to B is perpendicular to AC. Calculate the co-ordinates of B and C.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2009]

e Drawing curves

You can always plot a curve, point by point, if you know its equation. Often, however, all you need is a general idea of its shape and a sketch is quite sufficient.

Figures 2.25 and 2.26 show some common curves of the form $y = x^n$ for $n = 1, 2, 3$ and 4 and $y = \frac{1}{x^n}$ for $n = 1$ and 2.

Curves of the form $y = x^n$ for $n = 1, 2, 3$ and 4

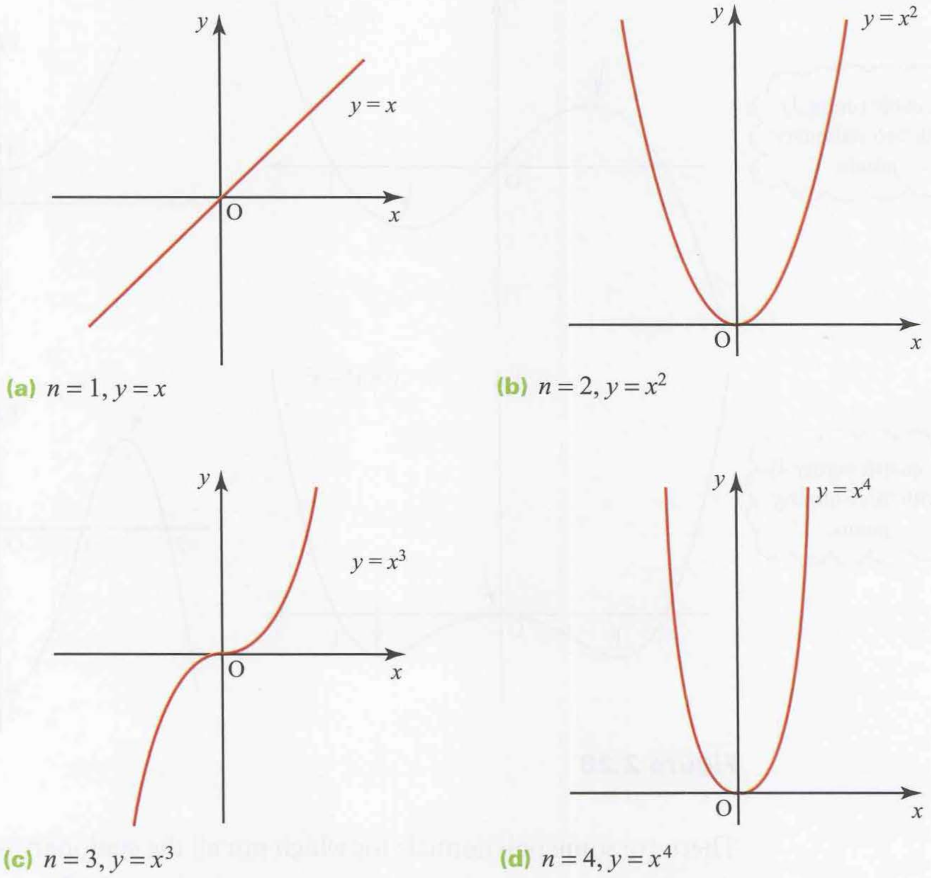


Figure 2.25

? How are the curves for even values of n different from those for odd values of n?

Stationary points

A turning point is a place where a curve changes from increasing (curve going up) to decreasing (curve going down), or vice versa. A *turning point* may be described as a *maximum* (change from increasing to decreasing) or a *minimum* (change from decreasing to increasing). Turning points are examples of *stationary points*, where the gradient is zero. In general, the curve of a polynomial of order n has up to $n - 1$ turning points as shown in figure 2.26.

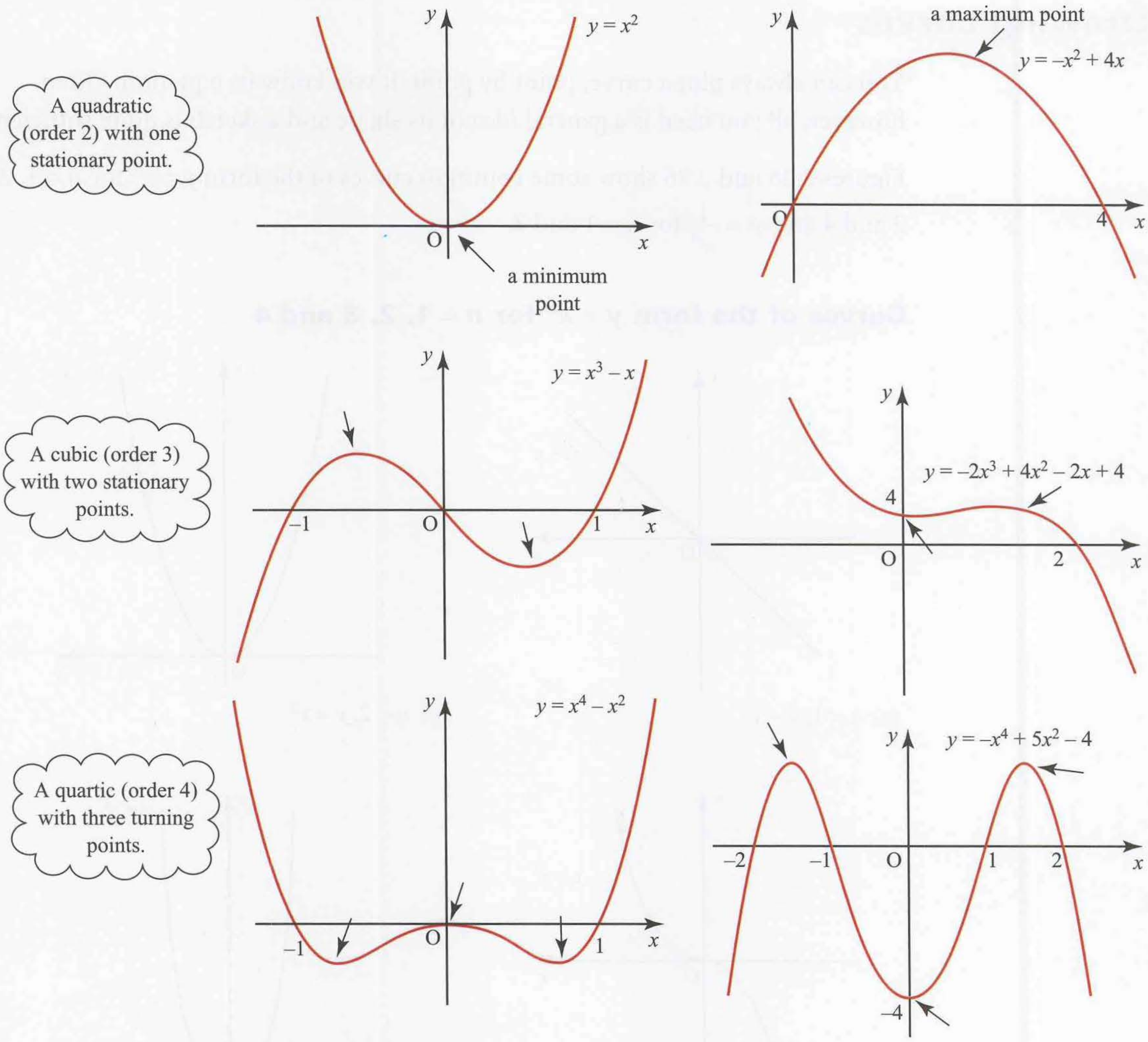


Figure 2.26

There are some polynomials for which not all the stationary points materialise, as in the case of $y = x^4 - 4x^3 + 5x^2$ (whose curve is shown in figure 2.27). To be accurate, you say that the curve of a polynomial of order n has *at most* $n - 1$ stationary points.

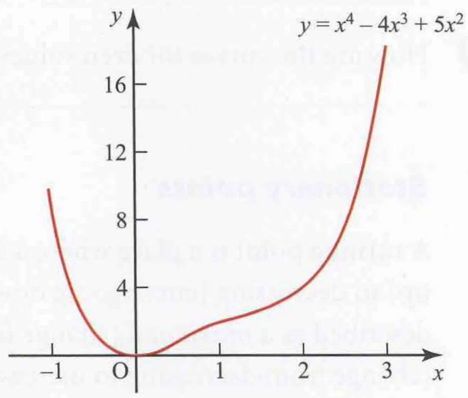


Figure 2.27

Behaviour for large x (positive and negative)

What can you say about the value of a polynomial for large positive values and large negative values of x? As an example, look at

$$f(x) = x^3 + 2x^2 + 3x + 9,$$

and take 1000 as a large number.

$$\begin{aligned} f(1000) &= 1\,000\,000\,000 + 2\,000\,000 + 3000 + 9 \\ &= 1\,002\,003\,009 \end{aligned}$$

Similarly,

$$\begin{aligned} f(-1000) &= -1\,000\,000\,000 + 2\,000\,000 - 3000 + 9 \\ &= -998\,002\,991. \end{aligned}$$

Note

1 The term x^3 makes by far the largest contribution to the answers. It is the *dominant* term.
For a polynomial of order n, the term in x^n is dominant as $x \to \pm\infty$.

2 In both cases the answers are extremely large numbers. You will probably have noticed already that away from their turning points, polynomial curves quickly disappear off the top or bottom of the page.
For all polynomials as $x \to \pm\infty$, either $f(x) \to +\infty$ or $f(x) \to -\infty$.

When investigating the behaviour of a polynomial of order n as $x \to \pm\infty$, you need to look at the term in x^n and ask two questions.

(i) Is n even or odd?
(ii) Is the coefficient of x^n positive or negative?

According to the answers, the curve will have one of the four types of shape illustrated in figure 2.28.

Intersections with the x and y axes

The constant term in the polynomial gives the value of y where the curve intersects the y axis. So $y = x^8 + 5x^6 + 17x^3 + 23$ crosses the y axis at the point (0, 23). Similarly, $y = x^3 + x$ crosses the y axis at (0, 0), the origin, since the constant term is zero.

When the polynomial is given, or known, in factorised form you can see at once where it crosses the x axis. The curve $y = (x-2)(x-8)(x-9)$, for example, crosses the x axis at $x = 2$, $x = 8$ and $x = 9$. Each of these values makes one of the brackets equal to zero, and so $y = 0$.

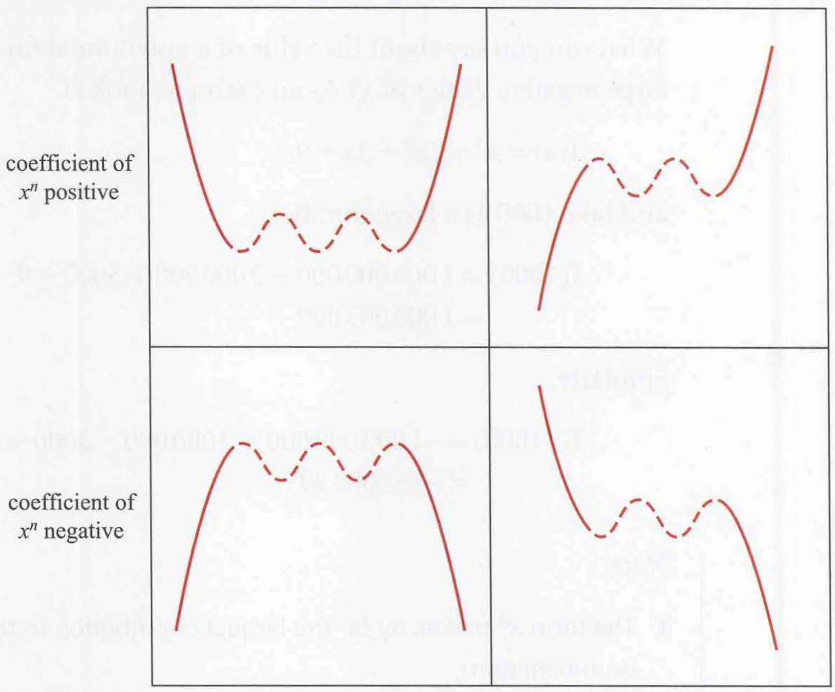


Figure 2.28

EXAMPLE 2.13

Sketch the curve $y = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3)$.

SOLUTION

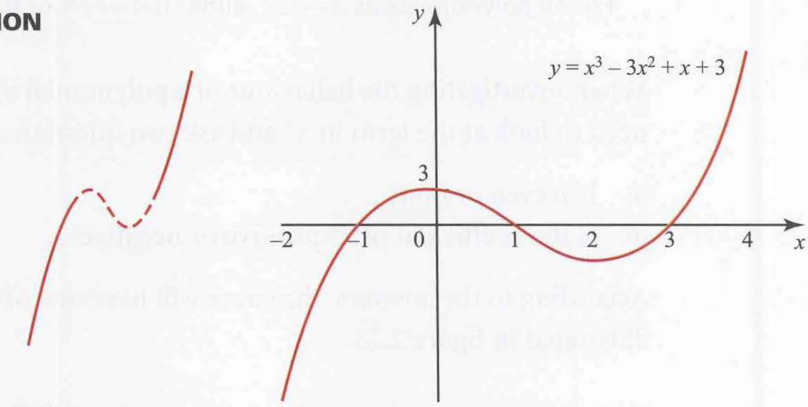


Figure 2.29

Since the polynomial is of order 3, the curve has up to two stationary points. The term in x^3 has a positive coefficient (+1) and 3 is an odd number, so the general shape is as shown on the left of figure 2.29.

The actual equation

$$y = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3)$$

tells you that the curve:

– crosses the y axis at (0, 3)
– crosses the x axis at (–1, 0), (1, 0) and (3, 0).

This is enough information to sketch the curve (see the right of figure 2.29).