Inequalities

In *Introducing Pure Mathematics* (pages 6 and 36), we found how to solve simple inequalities such as

$$4x+7>3(x-4) \quad \text{and} \quad x^2-7x+10 \geqslant 0$$

We established that we can add and subtract as usual with an inequality symbol, as if it were an equals symbol. **But** to multiply or divide by a negative number, we must **change the sign of the inequality**. For example, we have

$$3>2 \quad \text{but} \quad -3<-2$$
$$-2x>4 \quad \Rightarrow \quad x<-2$$

Hence, an inequality such as $\dfrac{ax+b}{cx+d}>2$ **cannot** be solved simply by multiplying both sides of the inequality by $cx+d$, since we do not know whether $cx+d$ is positive, giving

$$ax+b>2(cx+d)$$

or negative, giving

$$ax+b<2(cx+d)$$

To solve inequalities such as $\dfrac{ax+b}{cx+d}>k$, we can use either of the following two methods.

1 Multiply both sides of the inequality by $(cx+d)^2$, which we know must be positive.

2 Sketch $y=\dfrac{ax+b}{cx+d}$, solve $\dfrac{ax+b}{cx+d}=k$ and then, by comparing these two results, write down the solution to the inequality.

You should be able to use both methods, but the one which you prefer will probably depend on whichever is better, your algebraic skill or your graphical skill.

Example 6 Solve the inequality $\dfrac{5x-9}{x+3}>2$.

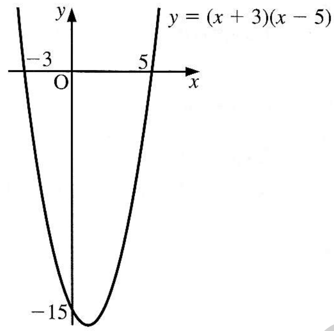
SOLUTION

Method 1
Multiplying by $(x+3)^2$, we obtain

$$\frac{5x-9}{x+3}(x+3)^2>2(x+3)^2$$
$$\Rightarrow \quad (5x-9)(x+3)>2(x+3)^2$$
$$\Rightarrow \quad (5x-9)(x+3)-2(x+3)^2>0$$

Noting that $(x + 3)$ is a factor, we factorise to obtain

$$(x+3)[5x - 9 - 2(x+3)] > 0$$
$$\Rightarrow \quad (x+3)(3x-15) > 0$$
$$\Rightarrow \quad (x+3)(x-5) > 0$$
$$\Rightarrow \quad x > 5 \quad \text{or} \quad x < -3$$



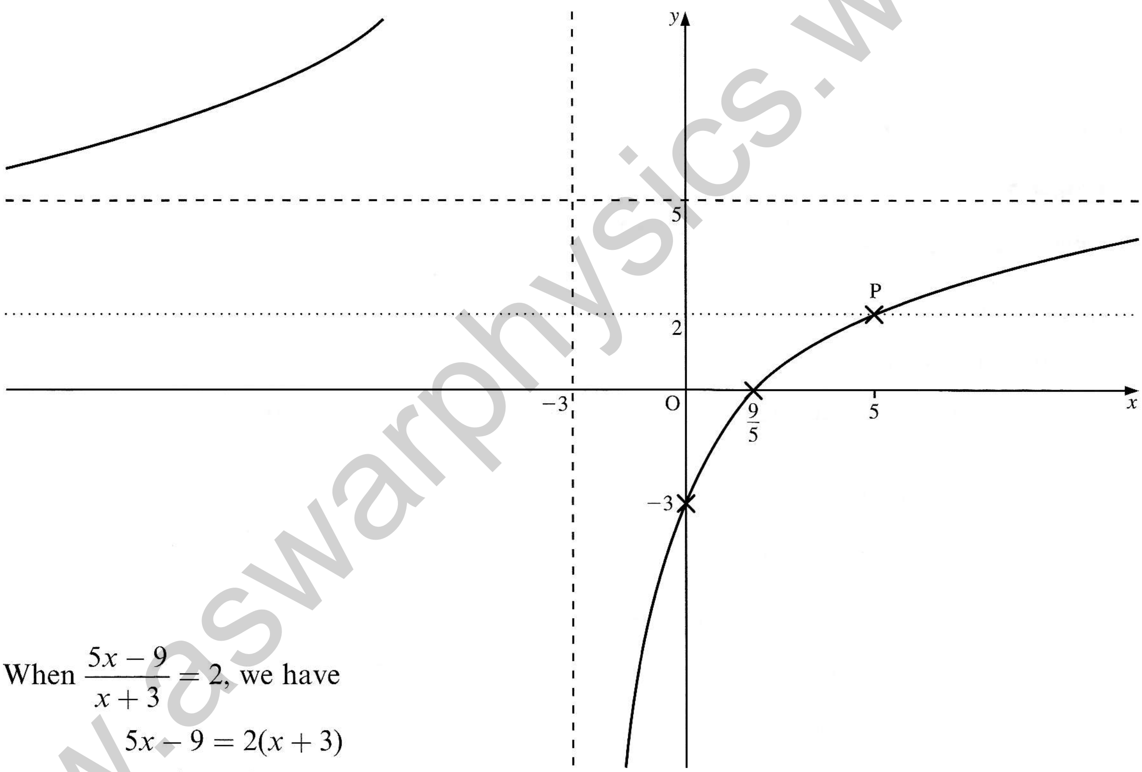
Method 2

Consider the curve $y = \dfrac{5x-9}{x+3}$.

The asymptotes are $x = -3$ and $y = 5$.

The curve cuts the axes at $\left(\dfrac{9}{5}, 0\right)$ and $(0, -3)$.

We can now sketch the curve of $y = \dfrac{5x-9}{x+3}$.



When $\dfrac{5x-9}{x+3} = 2$, we have

$$5x - 9 = 2(x+3)$$
$$\Rightarrow \quad 3x = 15$$
$$\Rightarrow \quad x = 5$$

We insert the point P (5, 2) on the curve. Hence, we can see that

$$\frac{5x-9}{x+3} > 2$$

is satisfied by the part of the graph above the dotted line $y = 2$. That is, where $x > 5$ or $x < -3$.

Example 7 Solve the inequality $\dfrac{(x+1)(x+4)}{(x-1)(x-2)} < 2$.

SOLUTION

Method 1

Multiplying by $(x-1)^2(x-2)^2$, we obtain

$$\frac{(x+1)(x+4)}{(x-1)(x-2)}(x-1)^2(x-2)^2 < 2(x-1)^2(x-2)^2$$
$$\Rightarrow \quad (x+1)(x+4)(x-1)(x-2) < 2(x-1)^2(x-2)^2$$
$$\Rightarrow \quad (x+1)(x+4)(x-1)(x-2) - 2(x-1)^2(x-2)^2 < 0$$

Noting that $(x-1)$ and $(x-2)$ are factors, we factorise to obtain

$$(x-1)(x-2)[(x+1)(x+4) - 2(x-1)(x-2)] < 0$$
$$\Rightarrow \quad (x-1)(x-2)[(x^2+5x+4-2x^2+6x-4] < 0$$
$$\Rightarrow \quad (x-1)(x-2)(-x^2+11x) < 0$$
$$\Rightarrow \quad (x-1)(x-2)(x^2-11x) > 0$$
$$\Rightarrow \quad (x-1)(x-2)x(x-11) > 0$$

Therefore, we have

$$\frac{(x+1)(x+4)}{(x-1)(x-2)} < 2$$

when $x > 11$, $1 < x < 2$, $x < 0$.

Method 2

Consider the curve of $y = \dfrac{(x+1)(x+4)}{(x-1)(x-2)}$.

The horizontal asymptote is $y = 1$.

The vertical asymptotes are $x = 1$ and $x = 2$.

The curve crosses the axes when $y = 0$, $x = -1, -4$, and when $x = 0$, $y = 2$.

When $y = 1$, we obtain

$$\frac{(x+1)(x+4)}{(x-1)(x-2)} = 1$$
$$\Rightarrow \quad x^2+5x+4 = x^2-3x+2$$
$$\Rightarrow \quad 8x = -2$$
$$\Rightarrow \quad x = -\frac{1}{4}$$

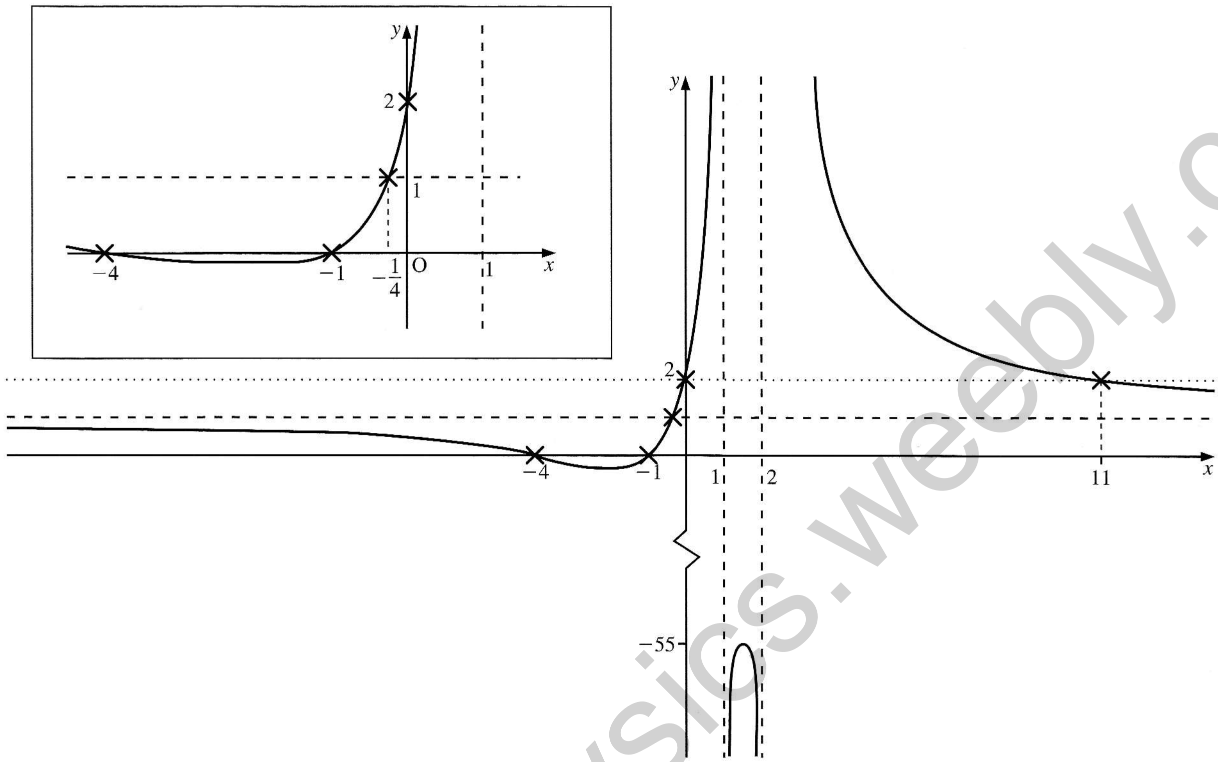
When $y = 2$, we obtain

$$\frac{(x+1)(x+4)}{(x-1)(x-2)} = 2$$
$$\Rightarrow \quad x^2+5x+4 = 2(x^2-3x+2)$$
$$\Rightarrow \quad 0 = x^2-11x$$
$$\Rightarrow \quad x = 0 \quad \text{and} \quad 11$$

Therefore, we have

$$\frac{(x+1)(x+4)}{(x-1)(x-2)} < 2$$

when $x > 11$, $1 < x < 2$, $x < 0$.



Inequalities involving modulus curves

In *Introducing Pure Mathematics* (page 95), we found how to solve simple modulus inequalities. Here, we consider modulus inequalities involving algebraic fractions.

Take, for example, the modulus inequality

$$\left|\frac{5x-9}{x+3}\right| > 2$$

We solve this by first solving

$$\frac{5x-9}{x+3} = +2 \quad \text{and} \quad \frac{5x-9}{x+3} = -2$$

and then deducing the required values of x from the sketch of the curve

$$y = \left|\frac{5x-9}{x+3}\right|$$

The sketch of $y = \left|\frac{5x-9}{x+3}\right|$ is obtained by sketching $y = \frac{5x-9}{x+3}$ and reflecting in the x-axis that part of the curve below the x-axis.

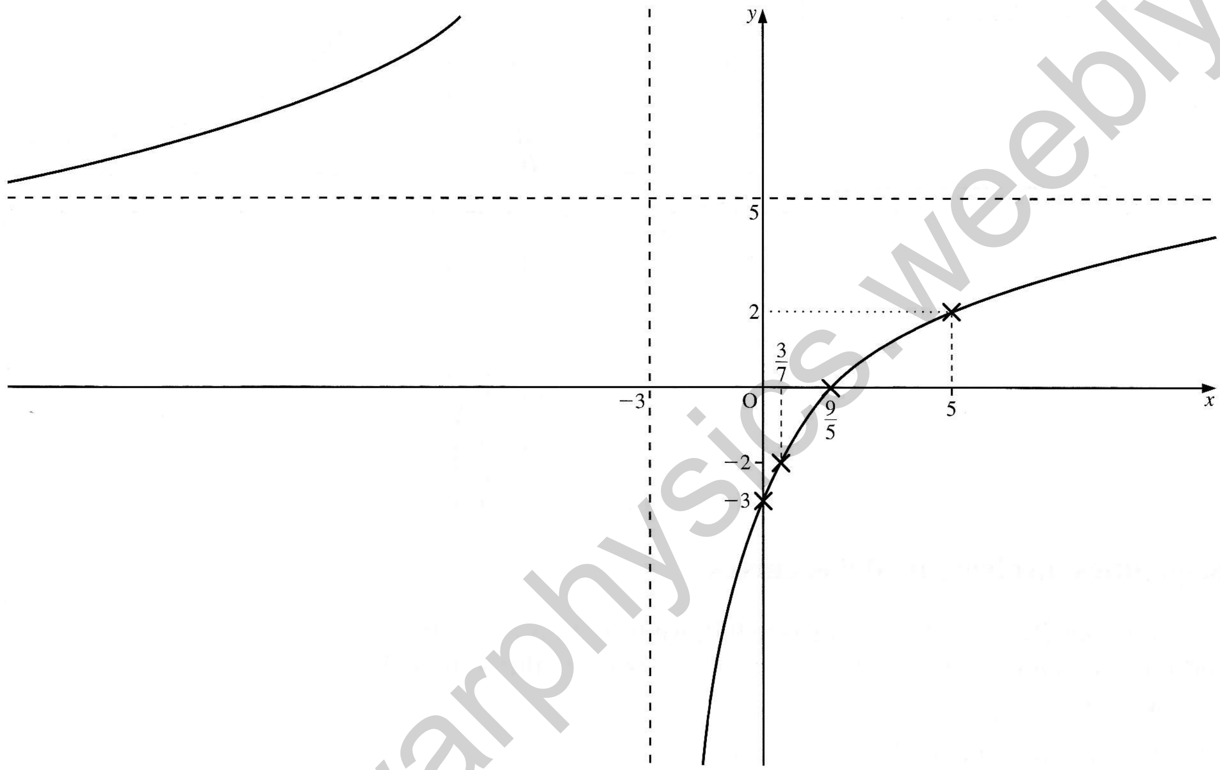
Thus, to solve $\left|\dfrac{5x-9}{x+3}\right| > 2$, we proceed as follows.

First, we solve $\dfrac{5x-9}{x+3} = 2$, which gives $x = 5$ (as in Example 6, pages 140–1).

Next, we solve $\dfrac{5x-9}{x+3} = -2$, which gives

$$5x - 9 = -2(x+3) \quad \Rightarrow \quad x = \tfrac{3}{7}$$

Then, we sketch $y = \dfrac{5x-9}{x+3}$.



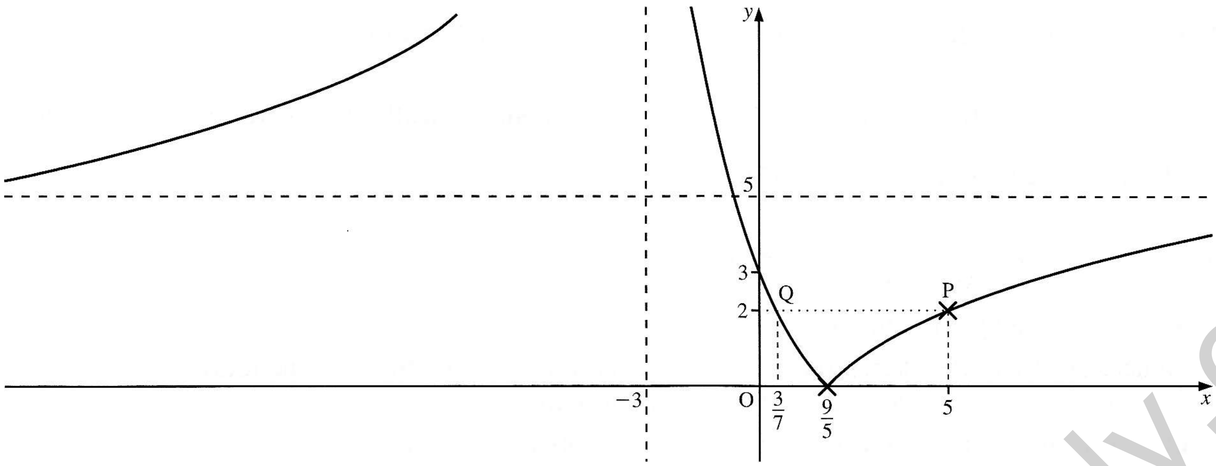
Finally, we sketch $y = \left|\dfrac{5x-9}{x+3}\right|$. (See top of page 145.)

Insert the point P where $\dfrac{5x-9}{x+3} = +2$, and the point Q where $\dfrac{5x-9}{x+3} = -2$.

Hence, we have

$$\left|\frac{5x-9}{x+3}\right| > 2$$

when $x > 5$ and $x < \frac{3}{7}$, **excluding** $x = -3$, where the curve is not defined.



Exercise 7B

In Questions **1** to **4**, solve each of the inequalities for x.

1 **a)** $\dfrac{x+3}{x+2} < 2$ **b)** $\dfrac{x+5}{x-3} > 1$ **c)** $\dfrac{2x-1}{x+3} > 3$

d) $\dfrac{3x+4}{x-5} > 2$ **e)** $\dfrac{1-2x}{4x+2} > 2$ **f)** $\dfrac{3+4x}{5x-1} > 3$

2 **a)** $\dfrac{(x-1)(x-2)}{(x+1)(x+2)} > 1$ **b)** $\dfrac{(x+2)(x-5)}{(x-3)(x-2)} > 1$ **c)** $\dfrac{(x-1)(x-4)}{(x+1)(x-5)} > 2$

d) $\dfrac{(2x-1)(x-2)}{(x-3)(x+7)} > 2$ **e)** $\dfrac{(x+1)(x+5)}{(x+2)(2x+3)} > 3$

3 **a)** $\left|\dfrac{x+3}{x+2}\right| > 1$ **b)** $\left|\dfrac{x-1}{x+2}\right| > 2$ **c)** $\left|\dfrac{x+3}{x-4}\right| > 2$

d) $\left|\dfrac{2x-1}{x+5}\right| > 1$ **e)** $\left|\dfrac{3x-1}{x+2}\right| > 2$ **f)** $\left|\dfrac{x+2}{x+3}\right| \leqslant 1$

4 **a)** $\dfrac{x^2+x-3}{x^2+x-2} > 1$ **b)** $\dfrac{2x^2+x-5}{2x^2+x-3} < 1$ **c)** $\dfrac{x^2-x-2}{x^2+3x+2} > 1$

5 Find the complete set of values of x for which $\dfrac{x^2-12}{x} > 1$. (EDEXCEL)

6 Given that $|x| \neq 1$, find the complete set of values of x for which $\dfrac{x}{x-1} > \dfrac{1}{x+1}$. (EDEXCEL)

7 Find the set of values of x for which $x+2 > \dfrac{3}{2-x}$. (EDEXCEL)

8 Find the set of values of x for which $\dfrac{x}{x+2} < \dfrac{2}{x-1}$. (EDEXCEL)

9 Find the set of values of x for which $x < \dfrac{2x+5}{x-2}$. (EDEXCEL)

10 For the curve with equation $y = \dfrac{2x^2 - x - 7}{x - 3}$, prove **algebraically** that there are no real values of x for which $3 < y < 19$. (AEB 98)

11 $$f(x) = \frac{3x-6}{x(x+6)} \qquad x \in \mathbb{R}, x \neq 0, x \neq -6$$

a) Find the range of values of $f(x)$.

Hence, or otherwise, sketch the curve with equation $y = f(x)$. State the equations of any asymptotes and the coordinates of any turning points.

b) Use your graph to find the number of real roots of the equation

$$x^3 + 6x^2 - 3x + 6 = 0$$

c) On a separate diagram, sketch the curve with equation $y = |f(x)|$. (EDEXCEL)

12 On the same diagram, sketch the graphs of

$$y = |x-5| \quad \text{and} \quad y = |3x-2|$$

distinguishing between them clearly.

Find the set of values of x for which $|x-5| < |3x-2|$. (EDEXCEL)

13 Find the complete set of values of x for which $\dfrac{x-2}{x+1} < 3$. (EDEXCEL)

14 Find the constants P, Q and R in the identity

$$\frac{x^2+x+2}{x-1} \equiv Px + Q + \frac{R}{x-1}$$

Hence write down the equation of the oblique asymptote of the curve C whose equation is

$$y = \frac{x^2+x+2}{x-1}$$

Show that C does not intersect this asymptote.

The points $(-1, -1)$ and $(3, 7)$ are stationary points of C. Sketch C, indicating the asymptotes. (NEAB)

15 **a)** Sketch the graph of $y = |2x+3|$, giving the coordinates of the points where the graph meets the coordinate axes.

b) Hence, or otherwise, find the set of values of x for which $4x + 10 > |2x+3|$. (EDEXCEL)

8 Roots of polynomial equations

And the equation will come at last.
LOUIS MACNEICE

Roots of a quadratic equation

If α and β are the roots of a quadratic equation, $f(x) \equiv ax^2 + bx + c = 0$, then the equation must be of the form

$$f(x) = k(x - \alpha)(x - \beta) \quad \text{for some constant } k$$

Therefore, we have

$$k(x - \alpha)(x - \beta) \equiv ax^2 + bx + c$$
$$\Rightarrow \quad k(x^2 - [\alpha + \beta]x + \alpha\beta) \equiv ax^2 + bx + c$$

Equating the coefficients of x^2 gives: $k = a$

Equating the coefficients of x gives: $-k(\alpha + \beta) = b$

And equating the constants gives: $k\alpha\beta = c$

Therefore, we obtain

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Or

The **sum** of the roots is $-\dfrac{b}{a}$ and the **product** of the roots is $\dfrac{c}{a}$.

Example 1 In the equation $3x^2 - 7x + 11 = 0$, find

a) the sum of the roots
b) the product of the roots.

SOLUTION

a) Using $\alpha + \beta = -\dfrac{b}{a}$, we have

$$\text{Sum of the roots, } \alpha + \beta = -\frac{-7}{3} = +\frac{7}{3}$$

b) Using $\alpha\beta = \dfrac{c}{a}$, we have

$$\text{Product of the roots, } \alpha\beta = \frac{11}{3}$$

Conversely, we may write the quadratic equation as

$$\mathbf{x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0}$$

Example 2 Find the equation whose roots have a sum of $\frac{1}{2}$ and a product of $-\frac{5}{2}$.

SOLUTION

Using $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$, we have

$$x^2 - \tfrac{1}{2}x - \tfrac{5}{2} = 0 \quad \text{or} \quad 2x^2 - x - 5 = 0$$

Example 3 The equation $3x^2 + 9x - 11 = 0$ has roots α and β. Find the equation whose roots are $\alpha + \beta$ and $\alpha\beta$.

SOLUTION

From $3x^2 + 9x - 11 = 0$, we have

$$\alpha + \beta = -3 \quad \text{and} \quad \alpha\beta = -\frac{11}{3}$$

The sum of the **new roots** is: $\alpha + \beta + \alpha\beta = -3 - \dfrac{11}{3} = -\dfrac{20}{3}$

The product of the **new roots** is: $(\alpha + \beta) \times \alpha\beta = -3 \times -\dfrac{11}{3} = 11$

Therefore, the new equation is

$$x^2 + \frac{20}{3}x + 11 = 0 \quad \text{or} \quad 3x^2 + 20x + 33 = 0$$

Example 4 The equation $4x^2 + 7x - 5 = 0$ has roots α and β. Find the equation whose roots are α^2 and β^2.

SOLUTION

From $4x^2 + 7x - 5 = 0$, we have

$$\alpha + \beta = -\frac{7}{4} \quad \text{and} \quad \alpha\beta = -\frac{5}{4}$$

The sum of the new roots is

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Substituting the above values in the RHS, we obtain

$$\alpha^2 + \beta^2 = \left(-\frac{7}{4}\right)^2 - 2 \times -\frac{5}{4} = \frac{89}{16}$$

The product of the new roots is $\alpha^2\beta^2 = (\alpha\beta)^2$. Substituting the value for $\alpha\beta$, we obtain

$$(\alpha\beta)^2 = \left(-\frac{5}{4}\right)^2 = \frac{25}{16}$$

Therefore, the new equation is

$$x^2 - \frac{89}{16}x + \frac{25}{16} = 0 \quad \text{or} \quad 16x^2 - 89x + 25 = 0$$

Roots of a cubic equation

In a similar manner, if α, β and γ are the roots of a cubic equation, $ax^3 + bx^2 + cx + d = 0$, then we have

$$ax^3 + bx^2 + cx + d \equiv k(x - \alpha)(x - \beta)(x - y)$$

$$\Rightarrow \quad ax^3 + bx^2 + cx + d \equiv k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$$

Equating coefficients of x^2 gives: $\alpha + \beta + \gamma = -\dfrac{b}{a}$

Equating coefficients of x gives: $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

And equating the constants gives: $\alpha\beta\gamma = -\dfrac{d}{a}$

Example 5 Find the cubic equation in x which has roots 4, 3 and -2.

SOLUTION

The sum of the roots is

$$\alpha + \beta + \gamma = 4 + 3 + (-2) = 5$$

The sum of the roots taken two at a time is

$$\alpha\beta + \beta\gamma + \gamma\alpha = 4 \times 3 + 3 \times -2 + (-2 \times 4) = -2$$

The product of the roots is

$$\alpha\beta\gamma = 4 \times 3 \times -2 = -24$$

Therefore, the equation is

$$x^3 - 5x^2 - 2x + 24 = 0$$

Example 6 The cubic equation $x^3 + 3x^2 - 7x + 2 = 0$ has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

SOLUTION

From the cubic equation, we have

$$\alpha + \beta + \gamma = -3$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = -7$$
$$\alpha\beta\gamma = -2$$

We now expand $(\alpha + \beta + \gamma)^2$ to obtain

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Substituting the values, we obtain

$$\alpha^2 + \beta^2 + \gamma^2 = (-3)^2 - 2 \times -7 = 23$$

Therefore, we have

$$\alpha^2 + \beta^2 + \gamma^2 = 23$$

Roots of a polynomial equation of degree n

From the properties of the roots of a quadratic equation and of a cubic equation, we see that in a polynomial equation of degree n, $ax^n + bx^{n-1} + cx^{n-2} + \ldots = 0$, the sum of the roots is $-\dfrac{b}{a}$ and the product of the roots is given by

$$(-1)^n \frac{\text{Last term}}{\text{First term}}$$

since the last term is the product of $-\alpha, -\beta, -\gamma, -\delta, \ldots$.

Example 7 The roots of $\mathrm{f}(x) \equiv 4x^5 + 6x^4 - 3x^3 + 7x^2 - 11x - 3 = 0$ are $\alpha, \beta, \gamma, \delta$ and ε.

a) Find the product of the five roots.

b) i) Show that $x = 1$ is a root of the equation.

ii) Hence show that the sum of the roots other than 1 is $-\dfrac{5}{2}$.

SOLUTION

a) The sum of all five roots, $\alpha, \beta, \gamma, \delta$ and ε, is $-\dfrac{b}{a} = -\dfrac{6}{4} = -\dfrac{3}{2}$.

b) i) When $x = 1$, we have

$$\mathrm{f}(1) = 4 + 6 - 3 + 7 - 11 - 3 = 0$$

Therefore, from the factor theorem, $x = 1$ is one root of the equation.

ii) The sum of all five roots is $-\dfrac{3}{2}$ (from part **a**). That is,

$$\alpha + \beta + \gamma + \delta + \varepsilon = -\frac{3}{2}$$

Putting $\varepsilon = 1$, we have

$$\alpha + \beta + \gamma + \delta + 1 = -\frac{3}{2} \quad \Rightarrow \quad \alpha + \beta + \gamma + \delta = -\frac{5}{2}$$

Therefore, the sum of the other four roots is $-\dfrac{5}{2}$.

Example 8 The equation $z^2 + (3 + \mathrm{i})z + p = 0$ has a root of $2 - \mathrm{i}$. Find the value of p and the other root of the equation.

SOLUTION

Since $2 - \mathrm{i}$ is a root, $z = 2 - \mathrm{i}$ satisfies the equation. Therefore, we have

$$(2 - \mathrm{i})^2 + (3 + \mathrm{i})(2 - \mathrm{i}) + p = 0$$
$$\Rightarrow \quad p = -10 + 5\mathrm{i}$$

The sum of the roots, $\alpha + \beta = -\dfrac{b}{a}$, is $-(3 + \mathrm{i})$. Therefore, the other root is

$$-(3 + \mathrm{i}) - (2 - \mathrm{i}) = -5$$

Exercise 8A

1 Write down the sum and the product of the roots of each of the following equations.

a) $x^2 + 3x - 7 = 0$ **b)** $x^2 - 11x + 5 = 0$ **c)** $x^2 + 5x - 4 = 0$

d) $3x^2 + 11x + 2 = 0$ **e)** $x + 2 = \dfrac{5}{x}$ **f)** $2x^2 = 7 - 4x$

2 Write down the equation whose roots have the sum and the product given below.

a) Sum 7; product 15 **b)** Sum -3; product $+5$

c) Sum -2; product -4 **d)** Sum -5; product -11

3 If α, β, γ are the roots of the equation $x^3 - 5x + 3 = 0$, find the values of

a) $\alpha + \beta + \gamma$ **b)** $\alpha^2 + \beta^2 + \gamma^2$ **c)** $\alpha^3 + \beta^3 + \gamma^3$

4 The equation $2z^2 - (7 - 2\mathrm{i})z + q = 0$ has a root of $1 + \mathrm{i}$. Find **i)** the value of q and **ii)** the other root of the equation.

5 The equation $3z^2 - (1 - \mathrm{i})z + t = 0$ has a root of $3 + 2\mathrm{i}$. Find **i)** the value of t and **ii)** the other root of the equation.

6 Given that α, β, γ are the roots of the equation $x^3 + x^2 + 4x - 5 = 0$, find the cubic equation whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. (WJEC)

7 Given the cubic equation $x^3 - 7x + q = 0$ has roots α, 2α and β, find the possible values of q. (WJEC)

8 The equation $3x^2 - 5x + 6 = 0$ has roots α and β. Without solving the given equation, find an equation with integer coefficients whose roots are $(\alpha + \beta)$ and $\alpha\beta$. (EDEXCEL)

9 The roots of the equation $x^3 - 3x^2 - 3x - 7 = 0$ are α, β and γ.

a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

b) Show that

$$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 0 \qquad \text{(NEAB)}$$

Equations with related roots

If α and β are the roots of $ax^2 + bx + c = 0$, then we can obtain the equation whose roots are 2α and 2β by making a substitution for x.

First, we express $ax^2 + bx + c = 0$ as

$$a(x - \alpha)(x - \beta) = 0$$

which gives

$$a(2x - 2\alpha)(2x - 2\beta) = 0$$

We obtain the required equation, whose roots are 2α and 2β, by putting $y = 2x$, which gives

$$a(y - 2\alpha)(y - 2\beta) = 0$$

Hence, replacing x by $\frac{y}{2}$ gives an equation whose roots are twice those of the original equation.

Example 9 Find the equation whose roots are 3α and 3β, where α and β are the roots of the equation $2x^2 - 5x + 3 = 0$.

SOLUTION

Replacing x by $\frac{y}{3}$ in $2x^2 - 5x + 3 = 0$, we obtain an equation in y whose roots for $\frac{y}{3}$ are the same as those for x: that is, α and β. Hence, the roots for y will be 3α and 3β.

Therefore, the required equation is

$$2\left(\frac{y}{3}\right)^2 - 5\left(\frac{y}{3}\right) + 3 = 0$$

$$\Rightarrow \quad 2y^2 - 15y + 27 = 0$$

If the equation is to be expressed in terms of x, it would be

$$2x^2 - 15x + 27 = 0$$

Example 10 Find the equation whose roots are α^2, β^2, γ^2, where α, β, γ are the roots of $3x^3 - 7x^2 + 11x - 5 = 0$.

SOLUTION

Replacing x by $\sqrt{y}$ in $3x^3 - 7x^2 + 11x - 5 = 0$, we obtain α, β, γ as the roots for $\sqrt{y}$. Hence, the roots for y are α^2, β^2, γ^2

Therefore, the equation in $\sqrt{y}$ is

$$3(\sqrt{y})^3 - 7(\sqrt{y})^2 + 11(\sqrt{y}) - 5 = 0$$

$$\Rightarrow \quad 3y\sqrt{y} + 11\sqrt{y} = 7y + 5$$

Squaring both sides, we have

$$9y^3 + 66y^2 + 121y = 49y^2 + 70y + 25$$

Therefore, the required equation is

$$9y^3 + 17y^2 + 51y - 25 = 0$$

Exercise 8B

1 The roots of the equation $x^2 + 7x + 11 = 0$ are α and β. Find the equation whose roots are 2α and 2β.

2 The roots of the equation $x^2 - 15x + 7 = 0$ are α and β. Find the equation whose roots are 3α and 3β.

3 The roots of the equation $3x^3 - 4x^2 + 8x - 7 = 0$ are α, β and γ. Find the equation whose roots are 2α, 2β and 2γ.

4 The roots of the equation $x^3 - 3x^2 - 11x + 5 = 0$ are α, β and γ. Find the equation whose roots are $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$.

5 The roots of the equation $2x^2 + 3x + 17 = 0$ are α and β. Find the equation whose roots are α^2 and β^2.

6 The roots of the equation $3x^2 - 7x + 15 = 0$ are α and β. Find the equation whose roots are α^2 and β^2.

7 The equation $2x^2 + 7x + 3 = 0$ has roots α and β. Find the equation whose roots are

a) $2\alpha, 2\beta$ **b)** $\frac{\alpha}{3}, \frac{\beta}{3}$ **c)** α^2, β^2 **d)** $\alpha + 2, \beta + 2$

8 The equation $3x^2 + 9x - 2 = 0$ has roots α and β. Find the equation whose roots are

a) $4\alpha, 4\beta$ **b)** $\frac{\alpha}{2}, \frac{\beta}{2}$ **c)** α^2, β^2 **d)** $\alpha - 3, \beta - 3$

9 The roots of the equation $x^3 + 3x^2 + 5x + 7 = 0$ are α, β and γ. Find the equation whose roots are

a) $3\alpha, 3\beta, 3\gamma$ **b)** $\alpha^2, \beta^2, \gamma^2$ **c)** $\alpha + 3, \beta + 3, \gamma + 3$

10 The roots of the equation $x^4 + 3x^3 + 7x^2 - 11x + 1 = 0$ are α, β, γ and δ. Find the equation whose roots are 3α, 3β, 3γ and 3δ.

11 The equation $x + 2 + \frac{3}{x} = 0$ has roots α and β. Find the equation whose roots are 5α and 5β.

12 The roots of the quadratic equation $x^2 - 3x + 4 = 0$ are α and β. Without solving the equation, find a quadratic equation, with integer coefficients, whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. (EDEXCEL)

Complex roots of a polynomial equation

If $z \equiv x + iy$ is a root of a polynomial equation with **real coefficients**, then $\bar{z} \equiv x - iy$ is also a root of the polynomial equation, where $\bar{z}$ is the conjugate of z (see page 3).

Proof

Suppose z is a root of the polynomial

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0 = 0$$

Then, taking the conjugate of both sides, we have

$$\overline{a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0} = 0$$

Using $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, we obtain

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \ldots + \overline{a_0} = 0$$

And using $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, we obtain

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \overline{a_{n-2}}\,\overline{z^{n-2}} + \ldots + \overline{a_0} = 0$$

which gives

$$\overline{a_n}\,(\bar{z})^n + \overline{a_{n-1}}\,(\bar{z})^{n-1} + \overline{a_{n-2}}\,(\bar{z})^{n-2} + \ldots + \overline{a_0} = 0$$

Since all the a_i are real, $\overline{a_i} = a_i$. Therefore, we have

$$a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + a_{n-2}(\bar{z})^{n-2} + \ldots + a_0 = 0$$

Hence, $\bar{z}$ is also a root of the polynomial.

The complex roots of a polynomial with **real coefficients** always occur in **conjugate complex pairs**.

Note We found in Example 8 (page 150) that when a quadratic equation does **not have real coefficients**, the roots are **not conjugate complex pairs**. (In Example 8, they are $2 - i$ and -5.)

Example 11 Show that $4 - i$ is a root of the polynomial equation

$$f(z) \equiv z^3 - 6z^2 + z + 34 = 0$$

Hence find the other roots.

SOLUTION

To prove that $z = 4 - i$ is a root, we prove that $f(4 - i) = 0$. If $z = 4 - i$ is a root, then $z = 4 + i$ is also a root, since the roots occur as conjugate complex pairs.

Next, we find the quadratic with **real** coefficients which is a factor. We then divide $f(z)$ by this quadratic to find the other factor.

Substituting $z = 4 - \mathrm{i}$ in $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$, we have

$$\begin{aligned} \mathrm{f}(4-\mathrm{i}) &= (4-\mathrm{i})^3 - 6(4-\mathrm{i})^2 + (4-\mathrm{i}) + 34 \\ &= 52 - 47\mathrm{i} - 90 + 48\mathrm{i} + 4 - \mathrm{i} + 34 \\ &= 0 \end{aligned}$$

Therefore, $4 - \mathrm{i}$ is a root of $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$. Hence, $4 + \mathrm{i}$ is also a root.

If $z - (4+\mathrm{i})$ and $z - (4-\mathrm{i})$ are factors of the polynomial, so is

$$[z-(4+\mathrm{i})][z-(4-\mathrm{i})] = z^2 - 8z + 17$$

Dividing $z^3 - 6z^2 + z + 34 = 0$ by $z^2 - 8z + 17$, we obtain

$$\mathrm{f}(z) = (z^2 - 8z + 17)(z + 2)$$

Therefore, the three roots of $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$ are $4 + \mathrm{i}$, $4 - \mathrm{i}$ and -2.

Example 12 Show that $2 + \mathrm{i}$ is a root of the polynomial equation

$$\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$$

Hence find the other roots.

SOLUTION

As in Example 11, to prove that $z = 2 + \mathrm{i}$ is a root, we prove that $\mathrm{f}(2+\mathrm{i}) = 0$. If $z = 2 + \mathrm{i}$ is a root, then $z = 2 - \mathrm{i}$ is also a root.

Next, we find the quadratic with **real** coefficients which is a factor. We then divide $\mathrm{f}(z)$ by this quadratic to find the other factors.

Substituting $z = 2 + \mathrm{i}$ in $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$, we have

$$\begin{aligned} \mathrm{f}(2+\mathrm{i}) &= (2+\mathrm{i})^4 - 12(2+\mathrm{i})^3 + 62(2+\mathrm{i})^2 - 140(2+\mathrm{i}) + 125 \\ &= -7 + 24\mathrm{i} - 24 - 132\mathrm{i} + 186 + 248\mathrm{i} - 280 - 140\mathrm{i} + 125 \\ &= 0 \end{aligned}$$

Therefore, $(2+\mathrm{i})$ is a root of $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$. Hence, $(2-\mathrm{i})$ is also a root.

If $z - (2+\mathrm{i})$ and $z - (2-\mathrm{i})$ are factors of the polynomial, so is

$$[z-(2+\mathrm{i})][z-(2-\mathrm{i})] = z^2 - 4z + 5$$

Dividing $z^4 - 12z^3 + 62z^2 - 140z + 125$ by $z^2 - 4z + 5$, we obtain

$$\mathrm{f}(z) = (z^2 - 4z + 5)(z^2 - 8z + 25)$$

Using the quadratic formula, we find that the roots of $z^2 - 8z + 25 = 0$ are $4 \pm 3\mathrm{i}$.

Therefore, the four roots of $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$ are $2 + \mathrm{i}$, $2 - \mathrm{i}$, $4 + 3\mathrm{i}$ and $4 - 3\mathrm{i}$.

Example 13 The roots of the equation $f(x) \equiv 2x^3 - 3x^2 + 7x - 19 = 0$ are α, β and γ. Show that

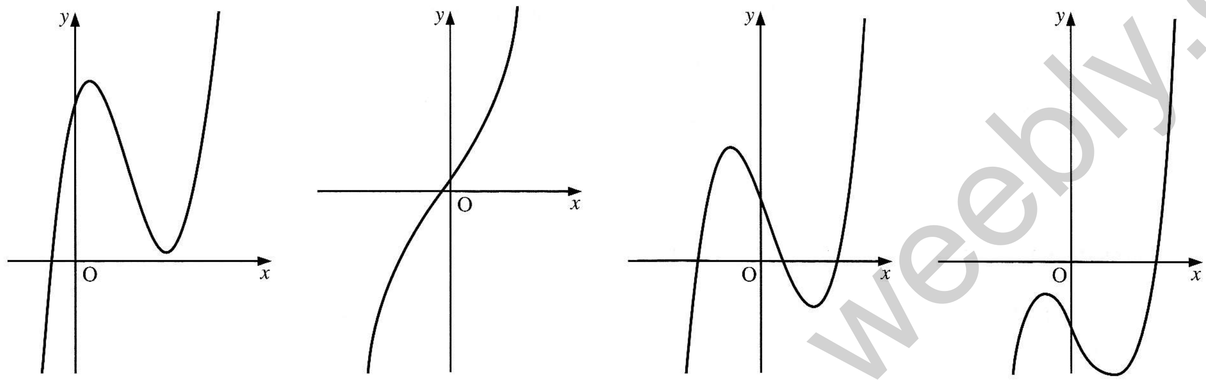
a) there is only one real root

b) the real root lies between $x = 2$ and $x = 3$

c) the real part of the two complex roots lies between $-\frac{1}{4}$ and $-\frac{3}{4}$.

SOLUTION

To show that a cubic equation has only one real root, we find the values of $f(x)$ at its turning points. Hence, we will be able to see which of the following curves is $f(x)$.



Note When the values of $f(x)$ at its turning points are of opposite sign, $f(x) = 0$ has three real roots.

a) To find the values of $f(x)$ at its turning points, we differentiate $f(x)$:

$$f(x) \equiv 2x^3 - 3x^2 + 7x - 19$$
$$f'(x) = 6x^2 - 6x + 7$$

Hence, we have

$$6x^2 - 6x + 7 = 0$$
$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 - 168}}{12}$$

That is, $f'(x) = 0$ has no real roots. Hence, the cubic $f(x)$ has no turning points, which means that $f(x) = 0$ has only one real root.

b) We find that

$$f(2) = -1 \quad \text{and} \quad f(3) = +29$$

So, $f(x)$ has opposite signs at $x = 2$ and $x = 3$ and is continuous for $2 \leqslant x \leqslant 3$. Therefore, the real root of $f(x) = 0$ lies between $x = 2$ and $x = 3$.

c) Let the three roots of the equation be α, β, γ, where α is a real number between 2 and 3, and β and γ are complex numbers.

Since the roots of a polynomial with real coefficients occur in conjugate complex pairs, β and γ are conjugate complex numbers, which we will represent by $p + \mathrm{i}q$ and $p - \mathrm{i}q$.

Using $\alpha + \beta + \gamma = -\frac{b}{a}$, we find

$$\alpha + \beta + \gamma = \frac{3}{2}$$

which gives

$$\alpha + p + iq + p - iq = \frac{3}{2}$$

$$\Rightarrow \quad 2p = \frac{3}{2} - \alpha$$

Since $2 < \alpha < 3$, we therefore have

$$\frac{3}{2} - 3 < 2p < \frac{3}{2} - 2$$

$$\Rightarrow \quad -\frac{3}{2} < 2p < -\frac{1}{2}$$

$$\Rightarrow \quad -\frac{3}{4} < p < -\frac{1}{4}$$

Hence, the real part of each complex root lies between $-\frac{1}{4}$ and $-\frac{3}{4}$.

Exercise 8C

1 Solve the equation $x^4 - 5x^3 + 2x^2 - 5x + 1 = 0$, given that i is a root.

2 Solve the equation $3x^4 - x^3 + 2x^2 - 4x - 40 = 0$, given that 2i is a root.

3 Determine the number of real roots of the equation $2x^3 + x^2 = 3$.

4 Determine the number of real roots of the equation $2x^3 - 7x + 2 = 0$.

5 Determine the range of possible values of k if the equation $x^3 + 3x^2 = k$ has three real roots.

6 One root of the equation $z^4 - 5z^3 + 13z^2 - 16z + 10 = 0$ is $1 + i$. Find the other roots.

7 **a)** Show that one root of the equation $z^3 + 5z^2 - 56z + 110 = 0$ is $3 + i$.
b) Find the other roots of the equation.

8 **a)** Show that one root of the equation $z^4 - 2z^3 + 6z^2 + 22z + 13 = 0$ is $2 - 3i$.
b) **i)** Find the other roots of the equation.
ii) Hence factorise $z^4 - 2z^3 + 6z^2 + 22z + 13$ into two quadratics, each of which has real coefficients.

9 The polynomial f(z) is defined by

$$f(z) \equiv z^4 - 2z^3 + 3z^2 - 2z + 2$$

a) Verify that i is a root of the equation $f(z) = 0$.
b) Find all the other roots of the equation $f(z) = 0$. (EDEXCEL)

10 Given that $2 + i$ is a root of the equation $3x^3 - 14x^2 + 23x - 10 = 0$, find the other roots of the equation. (WJEC)

11 One of the complex roots of $2z^4 - 13z^3 + 33z^2 - 80z - 50 = 0$ is $(1 - 3\mathrm{i})$, where $\mathrm{i}^2 = -1$.

i) State one other complex root.
ii) Find the other two roots and plot all four on an Argand diagram. (NICCEA)

12 Given that $3\mathrm{i}$ is a root of the equation $3z^3 - 5z^2 + 27z - 45 = 0$, find the other two roots. (OCR)

13 **a)** Verify that $z = 2$ is a solution of the equation $z^3 - 8z^2 + 22z - 20 = 0$.
b) Express $z^3 - 8z^2 + 22z - 20$ as a product of a linear factor and a quadratic factor with real coefficients. Hence find **all** the solutions of $z^3 - 8z^2 + 22z - 20 = 0$. (SQA/CSYS)

14 Two of the roots of a cubic equation, in which all the coefficients are real, are 2 and $1 + 3\mathrm{i}$.

i) State the third root.
ii) Find the cubic equation, giving it in the form $z^3 + az^2 + bz + c = 0$. (OCR)

15 Verify that $z = 1 + \mathrm{i}$ is a solution of the equation $z^3 + 16z^2 - 34z + 36 = 0$.

Write down a second solution of the equation.

Hence find constants α and β such that

$$z^3 + 16z^2 - 34z + 36 = (z^2 - \alpha z + \alpha)(z + \beta) \qquad \text{(SQA/CSYS)}$$

16 The roots of the equation $7x^3 - 8x^2 + 23x + 30 = 0$ are α, β, γ.

a) Write down the value of $\alpha + \beta + \gamma$.
b) Given that $1 + 2\mathrm{i}$ is a root of the equation, find the other two roots. (NEAB)

17 Derive expressions for the three cube roots of unity in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Represent the roots on an Argand diagram.

Let ω denote one of the non-real roots. Show that the other non-real root is ω^2. Show also that

$$1 + \omega + \omega^2 = 0$$

Given that

$$\alpha = p + q \qquad \beta = p + q\omega \qquad \gamma = p + q\omega^2$$

where p and q are real,

i) find, in terms of p, $\alpha\beta + \beta\gamma + \gamma\alpha$
ii) show that $\alpha\beta\gamma = p^3 + q^3$
iii) find a cubic equation, with coefficients in terms of p and q, whose roots are α, β, γ.

(NEAB)

18 The polynomial $\mathrm{f}(z)$ has real coefficients and one root of the equation $\mathrm{f}(z) = 0$ is $5 + 4\mathrm{i}$. Show that $z^2 - 10z + 41$ is a factor of $\mathrm{f}(z)$.

Given now that

$$\mathrm{f}(z) = z^6 - 10z^5 + 41z^4 + 16z^2 - 160z + 656,$$

solve the equation $\mathrm{f}(z) = 0$, giving each root exactly in the form $a + \mathrm{i}b$. (OCR)

9 Proof, sequences and series

We must never assume that which is incapable of proof.
G. H. LEWES

We studied some aspects of proof in *Introducing Pure Mathematics* (pages 515–22). Here, we will examine **proof by induction**, including its application to divisibility, and will revisit **proof by contradiction**.

Proof by induction

Proof by induction is used when we are given a statement which applies to any natural number, n.

To prove a statement by induction, we proceed in two steps:

1 We assume that the statement is true for $n = k$, and then use this assumption to prove that it is true for $n = k + 1$.

2 We then prove the statement for $n = 1$.

Step 2 tells us that the statement is true for $n = 1$.

Step 1 then tells us that, when $k = 1$, the statement is true for $n = 2$.

Using step 1 again, when $k = 2$, the statement must be true for $n = 3$.

Using step 1 yet again, the statement is true for $n = 4$.

Similarly, step 1 can be repeated for $n = 5$, $n = 6$, and so on.

Therefore, the statement is true for all integer $n\,(\geqslant 1)$.

Example 1 Prove that $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

SOLUTION

We assume that the formula is true for $n = k$. Therefore, we have

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

We are trying to prove that $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ is true for $n = k + 1$.

That is, we are trying to prove that $\displaystyle\sum_{r=1}^{k+1} r = \frac{1}{2}(k+1)(k+2)$.

We have

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)\text{th term}$$

which gives

$$\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + k + 1$$

$$= \frac{1}{2}[k(k+1) + 2(k+1)]$$

$$= \frac{1}{2}(k+1)(k+2)$$

Therefore, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ is true for $n = k+1$.

When $n = 1$: LHS of the formula $= 1$

RHS of the formula $= \frac{1}{2} \times 1 \times 2 = 1$

Therefore, the formula is true for $n = 1$.

Therefore, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ is true for all $n \geqslant 1$.

Note In a mathematical proof by induction, it is vital that we **write these last four lines of the proof in full**.

Example 2 Prove that $\sum_{r=1}^{n} r.r! = (n+1)! - 1$.

SOLUTION

We assume that the formula is true for $n = k$, which gives

$$\sum_{r=1}^{k} r.r! = (k+1)! - 1$$

Therefore, we have

$$\sum_{r=1}^{k+1} r.r! = (k+1)! - 1 + (k+1)\text{th term}$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!(1 + k + 1) - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

Therefore, the formula is true for $n = k+1$.

When $n = 1$: LHS of $\sum_{r=1}^{n} r.r! = 1$

RHS of $\sum_{r=1}^{n} r.r! = (n+1)! - 1 = 2! - 1 = 1$

Therefore, the formula is true for $n = 1$.

Therefore, $\sum_{r=1}^{n} r.r! = (n+1)! - 1$ is true for all $n \geqslant 1$.

Example 3 Prove that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{n}{2}}\mathrm{e}^x \sin(x + \frac{1}{4}n\pi)$.

SOLUTION

We assume that the formula is true for $n = k$, which gives

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathrm{e}^x \sin x) = 2^{\frac{k}{2}}\mathrm{e}^x \sin(x + \tfrac{1}{4}k\pi)$$

Therefore, we have

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\mathrm{e}^x \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathrm{e}^x \sin x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}[2^{\frac{k}{2}}\mathrm{e}^x \sin(x + \tfrac{1}{4}k\pi)]$$

$$= 22^{\frac{k}{2}}\mathrm{e}^x \sin(x + \tfrac{1}{4}k\pi) + 2^{\frac{k}{2}}\mathrm{e}^x \cos(x + \tfrac{1}{4}k\pi)$$

$$= 2^{\frac{k}{2}}\mathrm{e}^x [\sin(x + \tfrac{1}{4}k\pi) + \cos(x + \tfrac{1}{4}k\pi)]$$

Using $a \sin\theta + b\cos\theta = R\sin(\theta + \alpha)$, we obtain

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\mathrm{e}^x \sin x) = 2^{\frac{k}{2}}\mathrm{e}^x \sqrt{2}\sin[(x + \tfrac{1}{4}k\pi) + \tfrac{1}{4}\pi]$$

$$= 2^{\frac{1}{2}(k+1)}\mathrm{e}^x \sin[x + \tfrac{1}{4}(k+1)\pi]$$

Therefore, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{n}{2}}\mathrm{e}^x \sin(x + \frac{1}{4}n\pi)$ is true for $n = k + 1$.

When $n = 1$: $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x \sin x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$

$$= \sqrt{2}\mathrm{e}^x \sin(x + \tfrac{1}{4}\pi)$$

Therefore, the formula is true for $n = 1$.

Therefore, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = 2^{\frac{n}{2}}\mathrm{e}^x \sin(x + \frac{1}{4}n\pi)$ is true for all $n \geqslant 1$.

Example 4 If $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$.

SOLUTION

We assume that the statement is true for $n = k$, which gives

$$A^k = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

Therefore, we have

$$A^{k+1} = A^k \times A = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2^k + 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad A^{k+1} = \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$$

Therefore, the statement is true for $n = k + 1$.

When $n = 1$, the statement is true.

Therefore, if $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $A^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for all $n \geqslant 1$.

Divisibility

Proof by induction can also be used to prove that a term is divisible by a certain integer.

Example 5 Prove that $5^{2n} + 2^{2n-2}3^{n-1}$ is divisible by 13.

SOLUTION

Let $u_n = 5^{2n} + 2^{2n-2}3^{n-1}$. Therefore, we have

$$u_{n+1} = 5^{2(n+1)} + 2^{2(n+1)-2}3^{(n+1)-1}$$

Expressing u_{n+1} in the powers of u_n, we obtain

$$u_{n+1} = 5^2 5^{2n} + 2^2 2^{2n-2} 3^1 3^{n-1}$$
$$= 25 \times 5^{2n} + 12 \times 2^{2n-2}3^{n-1}$$

Adding u_n and u_{n+1}, we obtain

$$u_n + u_{n+1} = 26 \times 5^{2n} + 13 \times 2^{2n-2}3^{n-1}$$

Both 26 and 13 are divisible by 13. Therefore, since the sum of u_n and u_{n+1} is divisible by 13, either

both u_n and u_{n+1} **are** divisible by 13, or
both u_n and u_{n+1} **are not** divisible by 13.

When $n = 1$, $u_1 = 5^2 + 2^0 3^0 = 26$, which is divisible by 13.

Therefore, u_n is divisible by 13 for all integer $n \geqslant 1$.

It is not necessary to use simply $u_{n+1} + u_n$ as the term to be divisible by the required integer divisor. We can add, or subtract, any multiple of u_{n+1} and u_n, as long as that multiple is not the divisor, or a factor of the divisor.

In Example 5, we could have used $u_{n+1} - 12u_n = 13 \times 5^{2n}$. But obviously we could not use $13u_{n+1} - 13u_n$, which is divisible by 13, to prove anything about the divisibility of u_{n+1} or u_n.

Example 6 Prove that $3^{4n+2} + 5^{2n+1}$ is divisible by 14.

SOLUTION

Let $u_n = 3^{4n+2} + 5^{2n+1}$. Therefore, we have

$$\begin{aligned} u_{n+1} &= 3^{4(n+1)+2} + 5^{2(n+1)+1} \\ &= 3^4 3^{4n+2} + 5^2 5^{2n+1} \\ &= 81 \times 3^{4n+2} + 25 \times 5^{2n+1} \end{aligned}$$

Note We are trying to prove divisibility by 14. But for the term in 5^{2n+1},

$$u_{n+1} + u_n \text{ gives } (25+1)5^{2n+1} \quad \text{and} \quad u_{n+1} - u_n \text{ gives } (25-1)5^{2n+1}$$

Neither $25 + 1 = 26$, nor $25 - 1 = 24$ are divisible by 14, and so are unhelpful. However, we can see that both $u_{n+1} + 3u_n$ **and** $u_{n+1} - 11u_n$ make the term in 5^{2n+1} divisible by 14, giving respectively $(25 + 3) = 28$ and $(25 - 11) = 14$.

We need to check that the term in 3^{4n+2} also satisfies this divisibility:

$$\begin{aligned} u_{n+1} - 11u_n &= 81 \times 3^{4n+2} + 25 \times 5^{2n+1} - 11(3^{4n+2} + 5^{2n+1}) \\ &= 81 \times 3^{4n+2} + 25 \times 5^{2n+1} - 11 \times 3^{4n+2} - 11 \times 5^{2n+1} \\ &= 70 \times 3^{4n+2} + 14 \times 5^{2n+1} \end{aligned}$$

which is divisible by 14.

Therefore, either both u_{n+1} and u_n **are** divisible by 14, or both u_{n+1} and u_n **are not** divisible by 14.

When $n = 1$, $3^{4n+2} + 5^{2n+1} = 3^6 + 5^3 = 854$, which **is** divisible by 14.

Therefore, all u_n are divisible by 14.

Therefore, $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all $n \geqslant 1$.

Example 7 Prove that $7^n + 4^n + 1$ is divisible by 6.

SOLUTION

Let $u_n = 7^n + 4^n + 1$. Therefore, we have

$$\begin{aligned} u_{n+1} &= 7^{n+1} + 4^{n+1} + 1 \\ &= 7 \times 7^n + 4 \times 4^n + 1 \end{aligned}$$

To eliminate the +1, we need to subtract u_n from u_{n+1}, giving

$$u_{n+1} - u_n = 6 \times 7^n + 3 \times 4^n$$

We cannot use $2u_{n+1} - 2u_n$, as this would involve multiplying by 2, which is a factor of 6, which we are trying to prove is a factor of the given expression. Hence, we need to show that 3×4^n, as well as 6×7^n, is divisible by 6:

$$\begin{aligned} u_{n+1} - u_n &= 6 \times 7^n + 3 \times 4^n \\ &= 6 \times 7^n + 3 \times 2^{2n} \\ &= 6 \times 7^n + 6 \times 2^{2n-1} \end{aligned}$$

which is divisible by 6.

Therefore, either both u_{n+1} and u_n **are** divisible by 6, or both u_{n+1} and u_n **are not** divisible by 6.

When $n = 1$, $7^n + 4^n + 1 = 7 + 4 + 1 = 12$, which is divisible by 6.

Therefore, all u_n are divisible by 6.

Therefore, $7^n + 4^n + 1$ is divisible by 6 for all $n \geqslant 1$.

Proof by contradiction

Another way to prove that something is true is to assume that it is false, and then to arrive at a contradiction. (See also *Introducing Pure Mathematics*, pages 521–3.)

Suppose, for example, that we want to prove the statement

There is no biggest integer.

It seems obvious that there is no biggest whole number, but 'it seems obvious' is not a proper mathematical proof. One way to prove this statement is to assume that there **is** a biggest integer.

Call the biggest integer M. Then $M + 1$ must also be an integer. Now, $M + 1 > M$. But M was supposed to be the **biggest** integer. Therefore, we have a contradiction.

So our original assumption is false: there is no biggest integer.

One of the most beautiful proofs in all of mathematics concerns the statement

There are an infinite number of prime numbers

We suppose there are **not** an infinite number of prime numbers, and prove that this is nonsense.

Assume that there are a finite number of prime numbers. Then we can write them down as $\{p_1, p_2, \ldots, p_n\}$. The number $p_1 \times p_2 \times \ldots \times p_n + 1$ is not divisible by any of the prime numbers $\{p_1, p_2, \ldots, p_n\}$. This is nonsense because $\{p_1, p_2, \ldots, p_n\}$ was supposed to be a list of all the prime numbers. This contradiction tells us that our original assumption is wrong. Hence, there are infinitely many prime numbers.

Exercise 9A

1 Use proof by induction to prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

2 Use proof by induction to prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

3 Prove that $13^n - 6^{n-2}$ is divisible by 7.

4 Prove that $2^{6n} + 3^{2n-2}$ is divisible by 5.

5 It is given that $\phi(n) = 7^n(6n+1) - 1$, for $n = 1, 2, 3, \ldots$

i) Show that

$$\phi(n+1) - \phi(n) = 7^n(36n+48)$$

ii) Hence prove by induction that $\phi(n)$ is divisible by 12 for every positive integer n. (OCR)

6 Verify that $5^5 \equiv 1 \pmod{11}$. Hence find the remainder obtained on dividing 5^{1998} by 11. (OCR)

7 Use mathematical induction to prove that

$$\sum_{r=1}^{n} (r-1)(3r-2) = n^2(n-1)$$

for all positive integers n. (AEB 97)

8 $f(n) \equiv 24 \times 2^{4n} + 3^{4n}$, where n is a non-negative integer.

a) Write down $f(n+1) - f(n)$.

b) Prove, by induction, that $f(n)$ is divisible by 5. (EDEXCEL)

9 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 24 for all positive integers n. (WJEC)

10 Prove, by induction, that $\displaystyle\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$, $n \in \mathbb{N}$. (EDEXCEL)

11 Prove by induction that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Find the sum of the squares of the first n positive odd integers. (OCR)

12 Use induction to prove that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n. (SQA/CSYS)

13 Prove by induction that

$$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3) \qquad \text{(NICCEA)}$$

14 a) Write down an expression for the nth term of the series

$$\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \frac{4^2}{7 \times 9} + \ldots$$

b) Prove by induction, or otherwise, that the sum, S_n, of the first n terms of the above series is given by

$$S_n = \frac{n(n+1)}{2(2n+1)} \qquad \text{(NEAB)}$$

15 Show by mathematical induction that

$$1 + 2.2 + 3.2^2 + \ldots + n2^{n-1} = (n-1)2^n + 1$$

for all positive integer values of n. (WJEC)

16 **a)** Use the results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ to find an expression, in terms of n, for $\sum_{r=1}^{n} r(r-1)(r+1)$, factorising your answer as fully as possible.

b) Use mathematical induction to prove that

$$\sum_{r=2}^{n} \frac{1}{r(r-1)(r+1)} = \frac{1}{4} - \frac{1}{2n(n+1)}$$

for all positive integers $n \geqslant 2$. (AEB 97)

17 Use mathematical induction to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n. (AEB 98)

18 Show, by means of a counter-example, that the statement

$\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$

is false.

Find a unit vector $\mathbf{n}$ such that $\mathbf{n} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0}$. (NEAB)

19 Prove by induction that

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} \qquad \text{(OCR)}$$

20 Prove that there is no smallest positive rational number. [**Hint** Prove this by contradiction.]

21 Prove, by induction, that

$$\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2) \qquad \text{(EDEXCEL)}$$

22 **i)** Show that, if $n = k+1$, then

$$\frac{(3n+2)(n-1)}{n(n+1)} = \frac{3k^3+5k^2}{k(k+1)(k+2)}$$

provided $k > 0$.

ii) Prove by induction

$$\sum_{r=2}^{n} \frac{4}{r^2-1} \equiv \frac{(3n+2)(n-1)}{n(n+1)} \qquad \text{(NICCEA)}$$

23 Show that $\sum_{r=1}^{n} r(r+2) = \frac{n}{6}(n+1)(2n+7)$.

Using this result, or otherwise, find, in terms of n, the sum of the series

$$3\ln 2 + 4\ln 2^2 + 5\ln 2^3 + \ldots + (n+2)\ln 2^n$$

Express your answer in its simplest form. (EDEXCEL)

24 Consider the sequence defined by the relationship $u_{n+1} = 5u_n + 2$ whose first term is $u_1 = 1$.

i) Show that the first four terms are 1, 7, 37, 187, ...

ii) Use the method of induction to prove that $u_n = \frac{1}{2}[3(5^{n-1}) - 1]$. (NICCEA)

25 A sequence $u_0, u_1, u_2, \ldots$ is defined by

$$u_0 = 2 \quad \text{and} \quad u_{n+1} = 1 - 2u_n \quad (n \geqslant 0)$$

a) Prove by induction that, for all $n \geqslant 0$,

$$u_n = \frac{1}{3}\{1 + 5(-2)^n\}$$

b) State, briefly giving a reason for your answer, whether the sequence is convergent.

(NEAB)

26 Prove by contradiction that if the sum of two numbers is greater than 50, then at least one of the original numbers must have been greater than 25.

27 Let

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

Use induction to prove that, for all positive integers n,

$$A^n = \begin{pmatrix} 1 & 0 \\ 1 - 2^n & 2^n \end{pmatrix}$$

Determine whether or not this formula for A^n is also valid when $n = -1$. (SQA/CSYS)

28 Prove by induction that, for every positive integer N,

$$\sum_{n=1}^{N} \frac{n4^n}{(n+4)!} = \frac{1}{6} - \frac{4^{N+1}}{(N+4)!}$$

Given that, for every positive integer N,

$$\frac{4^{N+1}}{(N+4)!} \leqslant \frac{1}{6}\left(\frac{4}{5}\right)^N$$

show that the infinite series

$$\frac{1 \times 4^1}{5!} + \frac{2 \times 4^2}{6!} + \frac{3 \times 4^3}{7!} + \ldots$$

is convergent, and give the sum to infinity. (OCR)

29 Let u, v, w be positive integers. For each of the following, decide whether the statement is true or false. Where false, give a counter-example; where true, give a proof.

i) If u and v both divide w then $u + v$ divides w.
ii) If u divides both v and w then u divides $v + w$.
iii) If u divides v and v divides w then u divides $v + w$.

Write down the converse of statement **ii**, and determine whether or not this converse is true.

(SQA/CSYS)

Summation of series

As we have already seen on pages 159–61, proof by induction can be used to prove that a series has a known sum. Unfortunately, it is of no use when we do not know the sum in advance. Therefore, we will now introduce two other methods of summing a series: **applying standard formulae** and **differencing**.

Applying standard formulae

On pages 159 and 164, we found that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

We also have

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$$

These four formulae can be used to find the sums of many series.

Note $\sum_{r=1}^{n} r$ is often expressed as $\sum_{1}^{n} r$.

Example 8 Find the sum of $\sum_{r=1}^{n}(4r^2+1)$.

SOLUTION

First, we split the given term into its parts, and then use the formulae above, as appropriate.

Note $\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \ldots + 1 = n$ (total of n terms of 1)

Splitting the given term, we have

$$\sum_{r=1}^{n}(4r^2+1) = 4\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 1$$

which gives

$$\begin{aligned}\sum_{r=1}^{n}(4r^2+1) &= 4 \times \frac{1}{6}n(n+1)(2n+1) + n \\ &= \frac{2}{3}n(n+1)(2n+1) + n \\ &= \frac{1}{3}[2n(n+1)(2n+1) + 3n]\end{aligned}$$

$$\Rightarrow \quad \sum_{r=1}^{n}(4r^2+1)=\frac{1}{3}n[2(n+1)(2n+1)+3]$$

Therefore, we have

$$\sum_{r=1}^{n}(4r^2+1)=\frac{1}{3}n(4n^2+6n+5)$$

Example 9 Find the sum of $\sum_{r=1}^{n}(2r^3+3r^2+1)$.

SOLUTION

Splitting the given term, we have

$$\begin{aligned}\sum_{r=1}^{n}(2r^3+3r^2+1)&=\sum_{r=1}^{n}2r^3+\sum_{r=1}^{n}3r^2+\sum_{r=1}^{n}1\\&=2\sum_{r=1}^{n}r^3+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}1\end{aligned}$$

which gives

$$\begin{aligned}\sum_{r=1}^{n}(2r^3+3r^2+1)&=2\times\frac{1}{4}n^2(n+1)^2+3\times\frac{1}{6}n(n+1)(2n+1)+n\\&=\frac{n}{2}[n(n+1)^2+(n+1)(2n+1)+2]\end{aligned}$$

Therefore, we have

$$\sum_{r=1}^{n}(2r^3+3r^2+1)=\frac{n}{2}(n^3+4n^2+4n+3)$$

Example 10 Find the sum of $\sum_{r=n+1}^{2n}(4r^3-3)$.

SOLUTION

Splitting the given term, we have

$$\sum_{r=n+1}^{2n}(4r^3-3)=\sum_{r=1}^{2n}(4r^3-3)-\sum_{r=1}^{n}(4r^3-3)$$

which gives

$$\begin{aligned}\sum_{r=n+1}^{2n}(4r^3-3)&=4\sum_{1}^{2n}r^3-3\sum_{1}^{2n}1-\left(4\sum_{1}^{n}r^3-3\sum_{1}^{n}1\right)\\&=4\times\frac{1}{4}(2n)^2(2n+1)^2-3\times 2n-\left[4\times\frac{1}{4}n^2(n+1)^2-3n\right]\\&=4n^2(2n+1)^2-6n-n^2(n+1)^2+3n\\&=4n^2(4n^2+4n+1)-n^2(n^2+2n+1)-3n\\&=n^2(15n^2+14n+3)-3n\end{aligned}$$

Therefore, we have

$$\sum_{r=n+1}^{2n} (4r^3 - 3) = 15n^4 + 14n^3 + 3n^2 - 3n$$

Example 11 Find $\sum_{r=1}^{8} (r^2 + 2)$.

SOLUTION

Splitting the given term, we have

$$\sum_{r=1}^{8} (r^2 + 2) = \sum_{1}^{8} r^2 + \sum_{1}^{8} 2$$

Using

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and remembering that $\sum_{r=1}^{n} 1 = n$, therefore $\sum_{r=1}^{n} 2 = 2n$, we obtain

$$\sum_{r=1}^{n} (r^2 + 2) = \frac{1}{6}n(n+1)(2n+1) + 2n$$

Now, $n = 8$, therefore,

$$\sum_{r=1}^{8} (r^2 + 2) = \frac{1}{6} \times 8 \times 9 \times 17 + 16 = 220$$

Hence, we have

$$\sum_{r=1}^{8} (r^2 + 2) = 220$$

Exercise 9B

1 Find $\sum_{r=1}^{n} (2r^2 + 2r)$.

2 Find $\sum_{r=1}^{n} (2r^3 + r)$.

3 Find $\sum_{r=1}^{n} (r + 1)(r - 2)$.

4 Find $\sum_{r=1}^{n} (2r - 1)(r + 5)$.

5 a) Show that $\sum_{r=1}^{n} (2r - 1)(2r + 3) = \frac{1}{3}n(4n^2 + 12n - 1)$.

b) Hence find $\sum_{r=5}^{35} (2r - 1)(2r + 3)$. (EDEXCEL)

6 Given that n is a positive integer, find $\sum_{r=1}^{n} (2r - 1)^3$, giving your answer in its simplest form. (EDEXCEL)

7 Show that $\sum_{r=1}^{n} r(2r + 1) = \frac{1}{6}n(n + 1)(4n + 5)$. Hence evaluate $\sum_{r=10}^{30} r(2r + 1)$. (EDEXCEL)

8 Write down the sum

$$\sum_{n=1}^{2N} n^3$$

in terms of N, and hence find

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots - (2N)^3$$

in terms of N, simplifying your answer. (OCR)

Differencing

Some series can be summed using partial fractions (see *Introducing Pure Mathematics*, pages 280–89). The basis of this method is that most of the terms cancel out.

Example 12 Find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)}$.

SOLUTION

First, we write $\dfrac{1}{r(r+1)}$ as the sum of partial fractions:

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

Hence, we have

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

We notice that all the terms except the first and the last cancel one another. Therefore, we have

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

Example 13 Find $\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$.

SOLUTION

First, we write $\dfrac{2}{r(r+1)(r+2)}$ as the sum of partial fractions:

$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

Hence, we have

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}\right)$$

$$= \left(1 - \frac{2}{2} + \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}\right) +$$

$$+ \left(\cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}\right) + \ldots + \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{2}{n-1}} + \cancel{\frac{1}{n}}\right) +$$

$$+ \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

Note **Do not reduce** fractions to their lowest terms, since this obscures the cancellation which should occur.

We notice that almost all the terms cancel one another. We are left with

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = 1 - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

Example 14 Use the identity $r \equiv \frac{1}{2}[r(r+1) - (r-1)r]$ to find the sum $\sum_{r=1}^{n} r$.

SOLUTION

Making the given substitution, we obtain

$$\sum_{r=1}^{n} r = \sum_{r=1}^{n} \frac{1}{2}[r(r+1) - (r-1)r]$$

$$= \frac{1}{2}(1 \times 2 - 0 \times 1) + \frac{1}{2}(2 \times 3 - 1 \times 2) + \frac{1}{2}(3 \times 4 - 2 \times 3) + \ldots$$

$$+ \frac{1}{2}[(n-1)n - (n-2)(n-1)] + \frac{1}{2}[n(n+1) - (n-1)n]$$

We notice that almost all the terms cancel one another. We are left with

$$\sum_{r=1}^{n} r = \frac{1}{2}[-0 \times 1 + n(n+1)]$$

$$= \frac{1}{2}n(n+1)$$

Note This result was also found on pages 159–60, using a different method.