

Changing this to the form  $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ , where  $\hat{\mathbf{n}}$  is a unit vector, we get

$$\mathbf{r} \cdot \begin{pmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ -\frac{5}{5\sqrt{2}} \end{pmatrix} = \frac{21}{5\sqrt{2}}$$

Therefore, the distance from the origin is  $\frac{21}{5\sqrt{2}}$ .

## Distance of a plane from a point

**Example 20** Find the distance from the point  $(3, -2, 6)$  to the plane  $3x + 4y - 5z = 21$ .

**SOLUTION**

**Method 1** First, we find the equation of the plane parallel to  $3x + 4y - 5z = 21$  which passes through the point  $(3, -2, 6)$ . Then we find the distance of each plane from the origin. The difference between these distances is equal to the distance of the plane  $3x + 4y - 5z = 21$  from  $(3, -2, 6)$ .

The equation of the plane parallel to  $3x + 4y - 5z = 21$  through  $(3, -2, 6)$  is

$$3x + 4y - 5z = (3 \times 3) - (2 \times 4) + (6 \times -5)$$

$$\Rightarrow 3x + 4y - 5z = -29$$

Changing this to the form  $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ , we get

$$\mathbf{r} \cdot \begin{pmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ -\frac{5}{5\sqrt{2}} \end{pmatrix} = -\frac{29}{5\sqrt{2}}$$

Therefore, the distance from the point  $(3, -2, 6)$  to the plane is

$$\frac{29}{5\sqrt{2}} + \frac{21}{5\sqrt{2}} = \frac{50}{5\sqrt{2}} = 5\sqrt{2}$$

**Method 2** Using the form  $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ , the equation of the line perpendicular to  $3x + 4y - 5z = 21$  through  $(3, -2, 6)$  is

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

This line meets the plane  $3x + 4y - 5z = 21$  when

$$3(3 + 3t) + 4(-2 + 4t) - 5(6 - 5t) = 21$$

$$\Rightarrow 50t = 50 \Rightarrow t = 1$$

Using  $\mathbf{r} = \mathbf{a} + t\mathbf{b}$  again, we see that the line meets the plane at (6, 2, 1).

The distance between the two points (3, -2, 6) and (6, 2, 1) is

$$\sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$

Therefore, the distance from the point (3, -2, 6) to the plane is  $5\sqrt{2}$ .

## Exercise 6B

1 Find  $\mathbf{a} \times \mathbf{b}$  when

a)  $\mathbf{a} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$       $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

b)  $\mathbf{a} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$       $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

c)  $\mathbf{a} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$       $\mathbf{b} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$

d)  $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$       $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

2 Find, in vector form, the equation of the plane through

a) A(4, 1, -5), B(2, -1, -6), C(-2, 3, 2)

b) P(2, 5, 3), Q(4, 1, -2), R(4, 3, 5)

c) D(4, 1, -3), E(2, 3, 2), F(-1, -3, 1)

3 Find, in cartesian form, the equation of the plane

a)  $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = 4$      b)  $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 8$      c)  $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + 7 = 0$

4 Find the angle between each pair of planes.

a)  $3x - y - 4z = 7$ ,  $2x + 3y - z = 11$

b)  $5x - 3y + z = 10$ ,  $2x - y - z = 8$

c)  $7x + 4y - 2z = 5$ ,  $6x + 7y + z = 4$

d)  $x - 2y - 9z = 1$ ,  $x + 3y + 2z = 0$

5 Find the angle between the line

$$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

and the plane  $2x + 4y - z = 7$ .

6 Find the angle between the line

$$\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$$

and the plane  $3x - y + 2z = 11$ .

7 Write the equation of the plane  $3x + 4y - 5z = 20$  in the form  $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ , where  $\hat{\mathbf{n}}$  is a unit vector. Hence write down the distance from the plane to the origin.

8 The points A, B and C have position vectors  $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$  respectively.

- a) Write down the vectors  $\mathbf{b} - \mathbf{a}$  and  $\mathbf{c} - \mathbf{a}$ , and hence determine
  - i)  $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})$
  - ii)  $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ .
- b) Using the results from part a, or otherwise, find
  - i) the cosine of the acute angle between the line AB and the line AC, giving your answer in an exact form
  - ii) the area of triangle ABC, giving your answer in an exact surd form
  - iii) a vector equation of the plane through A, B and C, giving your answer in the form  $\mathbf{r} \cdot \mathbf{n} = d$ . (AEB 98)

9 The plane  $\Pi_1$  has vector equation

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j}) + u(-4\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + v(\mathbf{j} + 2\mathbf{k})$$

where  $u$  and  $v$  are parameters.

- a) Find a vector  $\mathbf{n}_1$  normal to  $\Pi_1$ .
- The plane  $\Pi_2$  has equation  $3x + y - z = 3$ .
- b) Write down a vector  $\mathbf{n}_2$  normal to  $\Pi_2$ .
  - c) Show that  $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$  is normal to both  $\mathbf{n}_1$  and  $\mathbf{n}_2$ .

Given that the point (1, 1, 1) lies on both  $\Pi_1$  and  $\Pi_2$ ,

- d) write down an equation of the line of intersection of  $\Pi_1$  and  $\Pi_2$  in the form  $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ , where  $t$  is a parameter. (EDEXCEL)

10 The points A(24, 6, 0), B(30, 12, 12) and C(18, 6, 36) are referred to cartesian axes, origin O.

- a) Find a vector equation for the line passing through the points A and B.

The point P lies on the line passing through A and B.

- b) Show that  $\overrightarrow{CP}$  can be expressed as

$$(6 + t)\mathbf{i} + t\mathbf{j} + (2t - 36)\mathbf{k}$$

where  $t$  is a parameter.

- c) Given that  $\overrightarrow{CP}$  is perpendicular to  $\overrightarrow{AB}$ , find the coordinates of P.
- d) Hence, or otherwise, find the area of the triangle ABC, giving your answer to three significant figures. (EDEXCEL)

11 The plane  $\Pi$  passes through the points A(-2, 3, 5), B(1, -3, 1) and C(4, -6, -7).

- a) Find  $\overrightarrow{AC} \times \overrightarrow{BC}$ .
- b) Hence, or otherwise, find the equation of the plane  $\Pi$  in the form  $\mathbf{r} \cdot \mathbf{n} = p$ .

The perpendicular from the point (25, 5, 7) to  $\Pi$  meets the plane at the point F.

- c) Find the coordinates of F. (EDEXCEL)

- 12** The plane  $p$  has equation

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + s(\mathbf{i} + \mathbf{k}) + t(\mathbf{j} - \mathbf{k})$$

and the line  $l$  has equation

$$\mathbf{r} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j})$$

- Find a vector which is normal to  $p$ .
- Show that the acute angle between  $p$  and  $l$  is  $\sin^{-1}(\frac{1}{5}\sqrt{15})$ . (OCR)

- 13** The planes  $P_1$  and  $P_2$  have equations

$$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 4 \quad \text{and} \quad \mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5$$

respectively. Find, in the form  $\mathbf{r} \cdot \mathbf{n} = d$ , the equation of the plane which is perpendicular to both  $P_1$  and  $P_2$  and which passes through the point with position vector  $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ . (OCR)

- 14** The line  $l$  passes through the points with position vectors  $\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$  and  $7\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ . Find an equation of  $l$  in vector form.

The points A, B, C have position vectors  $3\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}$ ,  $5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$  and  $4\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$  respectively.

- Find the vector product  $\overrightarrow{AB} \times \overrightarrow{AC}$ . Hence or otherwise find the equation of the plane ABC.
- Show that the angle between  $l$  and the plane ABC is  $24.5^\circ$ , correct to the nearest  $0.1^\circ$ .
- Find the position vector of the point of intersection of  $l$  and the plane ABC. (OCR)

- 15** The point A has position vector  $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$  and the line  $l$  has equation

$$\mathbf{r} = -5\mathbf{i} + 6\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})$$

- Find the position vector of the point N on  $l$  such that AN is perpendicular to  $l$ .
- Show that the perpendicular distance from A to  $l$  is  $\sqrt{26}$ .

The points B and C have position vectors  $-5\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$  and  $6\mathbf{i} + 13\mathbf{j} - 7\mathbf{k}$  respectively, and the point D is the mid-point of BN.

- Show that the plane ANC is perpendicular to  $l$ .
- Find the acute angle between the planes ANC and ACD. (OCR)

- 16** The line  $l$  has equation

$$\mathbf{r} = 5\mathbf{i} + 8\mathbf{j} + \mathbf{k} + t(\mathbf{i} + 8\mathbf{k})$$

and the plane  $P$  has equation

$$2x - 2y - z - 5 = 0$$

Find the position vector of the point at which  $l$  and  $P$  intersect.

Find also the acute angle between  $l$  and  $P$ , giving your answer to the nearest degree. (OCR)

- 17** Consider the plane  $P_1$ , the line  $L_1$  and the line  $L_2$  given by the equations,

$$P_1: x + 2y - z = 5$$

$$L_1: \frac{x-11}{-4} = \frac{y+2}{2} = \frac{z+8}{5}$$

$$L_2: \frac{x-1}{1} = \frac{y+2}{-3}, z = 7$$

- Show that  $L_1$  and  $L_2$  are coplanar.
- Find the equation of the plane,  $P_2$ , which contains  $L_1$  and  $L_2$ .
- Find the equation of the line of intersection of the planes  $P_1$  and  $P_2$ . (NICCEA)



- 18 The points A, B and C have position vectors  $(\mathbf{j} + 2\mathbf{k})$ ,  $(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$  and  $(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ , respectively, relative to the origin O. The plane  $\Pi$  contains the points A, B and C.

- Find a vector which is perpendicular to  $\Pi$ .
- Find the area of  $\triangle ABC$ .
- Find a vector equation of  $\Pi$  in the form  $\mathbf{r} \cdot \mathbf{n} = p$ .
- Hence, or otherwise, obtain a cartesian equation of  $\Pi$ .
- Find the distance of the origin O from  $\Pi$ .

The point D has position vector  $(3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$ . The distance of D from  $\Pi$  is  $\frac{1}{\sqrt{17}}$ .

- Using this distance, or otherwise, calculate the acute angle between the line AD and  $\Pi$ , giving your answer in degrees to one decimal place. (EDEXCEL)

- 19 Given that

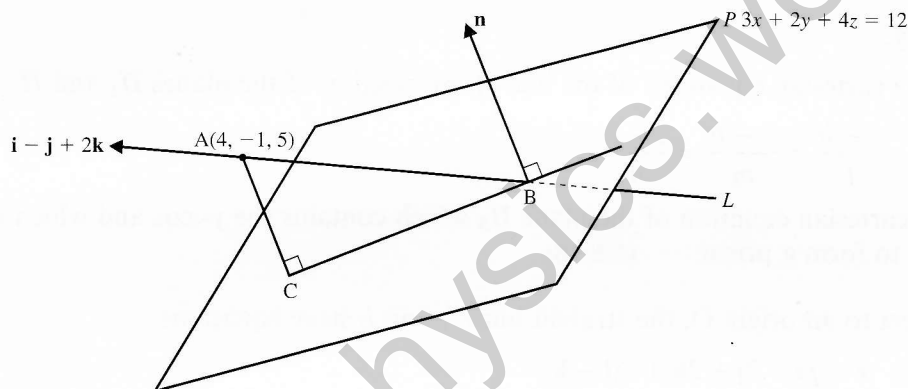
$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \quad \mathbf{b} \times \mathbf{c} = \mathbf{j} \quad \mathbf{c} \times \mathbf{a} = \mathbf{k}$$

express

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b} + 3\mathbf{c})$$

in terms of  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . (NEAB)

20



The figure above represents the line  $L$  and the plane  $P$  given by

$$L: \mathbf{r} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$P: 3x + 2y + 4z = 12$$

- Find the coordinates of B, the point of intersection of the line  $L$  and the plane  $P$ .
- Write down a vector  $\mathbf{n}$  which is perpendicular to the plane  $P$ .
- Calculate the vector  $\mathbf{q}$  given by

$$\mathbf{q} = \mathbf{n} \times (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

and mark it on a copy of the figure starting at B.

- Using  $\mathbf{q}$ , or otherwise, find the vector equation of the line BC which is the projection of the line  $L$  on the plane  $P$ . (NICCEA)

- 21 Simplify

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$$

Given that  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors and that

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{0}$$

write down the possible values of the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . (NEAB)

- 22** The points A and B have position vectors  $\mathbf{a} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$  respectively, and the

plane  $\Pi$  has equation  $\mathbf{r} \cdot \mathbf{n} = 1$ , where  $\mathbf{n} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ .

- Show that B lies in  $\Pi$  and that A does not lie in  $\Pi$ .
- Write down the vector  $\overrightarrow{AB}$ .
- The angle between  $\overrightarrow{AB}$  and  $\mathbf{n}$  is  $\theta$ . Find the value of  $\theta$ , giving your answer correct to the nearest  $0.1^\circ$ .
- The point C lies in the plane  $\Pi$  and is such that  $\overrightarrow{AC}$  is perpendicular to  $\Pi$ . Explain why  $\overrightarrow{AC} = \lambda \mathbf{n}$  for some scalar parameter  $\lambda$ . By finding the value of  $\lambda$ , or otherwise, determine the position vector of C.
- Find the shortest distance of the point A from  $\Pi$ . (AEB 96)

- 23** The planes  $\Pi_1$  and  $\Pi_2$  have cartesian equations

$$x + 2y - z = 7 \quad \text{and} \quad 2x + y + z = -1$$

respectively.

- Find the cartesian equations of the line of intersection of the planes  $\Pi_1$  and  $\Pi_2$  in the form

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

- Find a cartesian equation of the plane  $\Pi_3$  which contains the  $y$ -axis and which intersects  $\Pi_1$  and  $\Pi_2$  to form a prism. (NEAB)

- 24** With respect to an origin O, the straight lines  $l_1$  and  $l_2$  have equations

$$l_1: \mathbf{r} = p\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{k})$$

$$l_2: \mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters and  $p$  is a scalar constant. The lines intersect at the point A.

- Find the coordinates of A and show that  $p = 2$ .  
The plane  $\Pi$  passes through A and is perpendicular to  $l_2$ .
- Find a cartesian equation of  $\Pi$ .
- Find the acute angle between the plane  $\Pi$  and the line  $l_1$ , giving your answer in degrees to one decimal place. (EDEXCEL)

- 25** The point P has coordinates  $(4, k, 5)$ , where  $k$  is a constant. The line  $L$  has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad \text{The line } M \text{ has equation } \mathbf{r} = \begin{pmatrix} 4 \\ k \\ 5 \end{pmatrix} + t \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix}.$$

- Show that the shortest distance from the point P to the line  $L$  is  $\frac{1}{3}\sqrt{5(k^2 + 12k + 117)}$ .
- Find (in terms of  $k$ ) the shortest distance between lines  $L$  and  $M$ .
- Find the value of  $k$  for which the lines  $L$  and  $M$  intersect.
- When  $k = 12$ , show that the distances in parts i and ii are equal. In this case, find the equation of the line which is perpendicular to, and intersects, both  $L$  and  $M$ . (MEI)

- 26 The planes  $\Pi_1$  and  $\Pi_2$  have equations

$$x + 2y - z = 3 \quad \text{and} \quad 3x + 4y - z = 1$$

respectively. Find

- a vector which is parallel to both  $\Pi_1$  and  $\Pi_2$
- the equation of the plane which is perpendicular to both  $\Pi_1$  and  $\Pi_2$  and passes through the point  $(3, -4, -5)$ . (NEAB)

- 27 The lines  $l_1$  and  $l_2$  have vector equations

$$\mathbf{r} = (2\lambda - 3)\mathbf{i} + \lambda\mathbf{j} + (1 - \lambda)\mathbf{k} \quad \text{and} \quad \mathbf{r} = (2 + 5\mu)\mathbf{i} + (1 + \mu)\mathbf{j} + (3 + 2\mu)\mathbf{k}$$

respectively, where  $\lambda$  and  $\mu$  are scalar parameters.

- Show that  $l_1$  and  $l_2$  intersect, stating the position vector of the point of intersection.
- The vector  $\mathbf{i} + a\mathbf{j} + b\mathbf{k}$  is perpendicular to both lines. Determine the value of the constants  $a$  and  $b$ .
- Find a cartesian equation of the plane which contains  $l_1$  and  $l_2$ . (AEB 98)

- 28 The lines  $L_1$  and  $L_2$  have equations

$$\mathbf{r} = \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

and 
$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

respectively. Find direction ratios of a line which is perpendicular to both  $L_1$  and  $L_2$ .

Verify that the plane  $\Pi_1$ , through the origin O, whose equation is

$$10x + y - 2z = 0$$

contains  $L_1$ . Find the equation of the plane  $\Pi_2$  containing O and  $L_2$ . Show that the cartesian equations of the line  $L$  in which  $\Pi_1$  and  $\Pi_2$  intersect are

$$x = -\frac{y}{2} = \frac{z}{4}$$

Explain why  $L$  must be the common perpendicular of  $L_1$  and  $L_2$ . (NEAB)

- 29 A plane  $\Pi$  contains the points A(1, -2, 1), B(4, 0, 1) and C(1, 0, 2).

- Calculate the vector  $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ .
- Explain why  $\mathbf{n}$  is perpendicular to  $\Pi$ .
- Express the equation of  $\Pi$  in the form

$$\mathbf{r} \cdot \mathbf{n} = p$$

where  $p$  is a constant.

- The plane  $\Pi$  divides three-dimensional space into two regions. Show, with the aid of a diagram, that the region into which  $\mathbf{n}$  is directed does not contain the origin.
- b) A straight line  $L$  passes through the point D(3, -1, 2) and has direction ratios 2 : 1 : 1.
- Write down a vector equation for  $L$  and verify that  $L$  passes through A.
  - Show that the resolved part of the vector  $\overrightarrow{DA}$  in the direction of  $\mathbf{n}$  is -1.
  - Write down two conclusions that can be drawn from this result about the position of D with respect to the plane  $\Pi$ . (NEAB)

- 30** The planes  $\Pi_1$  and  $\Pi_2$  with equations

$$x + 2y + z + 2 = 0 \quad \text{and} \quad 2x + 3y + 2z - 1 = 0$$

respectively, meet in a line  $L$ . The point  $A$  has coordinates  $(2, -2, 1)$ .

- a) i)** Explain why the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

is in the direction of  $L$ .

- ii)** Hence find in the form  $\mathbf{r} \cdot \mathbf{n} = a$  the equation of the plane which is perpendicular to  $L$  and contains  $A$ .

- b) i)** Explain why, for any constant  $\lambda$ , the plane  $\Pi_3$  with equation

$$(x + 2y + z + 2) + \lambda(2x + 3y + 2z - 1) = 0$$

contains  $L$ .

- ii)** Hence, or otherwise, find the cartesian equation of the plane which contains  $L$  and the point  $A$ . (NEAB)

- 31** The plane  $\Pi$  has equation  $2x + y + 3z = 21$  and the origin is  $O$ . The line  $l$  passes through the point  $P(1, 2, 1)$  and is perpendicular to  $\Pi$ .

- a)** Find a vector equation of  $l$ .

The line  $l$  meets the plane  $\Pi$  at the point  $M$ .

- b)** Find the coordinates of  $M$ .

- c)** Find  $\overrightarrow{OP} \times \overrightarrow{OM}$ .

- d)** Hence, or otherwise, find the distance from  $P$  to the line  $OM$ , giving your answer in surd form.

The point  $Q$  is in the reflection of  $P$  in  $\Pi$ .

- e)** Find the coordinates of  $Q$ . (EDEXCEL)

- 32** With respect to an origin  $O$ , the points  $A, B, C$  have position vectors  $2\mathbf{i}, 4\mathbf{j}, 6\mathbf{k}$  respectively. The points  $P$  and  $Q$  are the mid-points of  $AB$  and  $BC$  respectively, and the point  $N$  has position vector  $5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ . The line  $l$  passes through  $P$  and  $N$ .

- i)** Find a vector equation of  $l$  and find the perpendicular distance from the point  $Q$  to  $l$ .  
**ii)** Find a vector equation of the line of intersection of the planes  $ABC$  and  $OPQ$ , and find the acute angle between these two planes.  
**iii)** Find the shortest distance between the lines  $OB$  and  $PQ$ . (OCR)

- 33** The line  $l_1$  passes through the point  $A$ , whose position vector is  $\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ , and is parallel to the vector  $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ . The line  $l_2$  passes through the point  $B$ , whose position vector is  $2\mathbf{i} - 9\mathbf{j} - 14\mathbf{k}$ , and is parallel to the vector  $2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ . The point  $P$  on  $l_1$  and the point  $Q$  on  $l_2$  are such that  $PQ$  is perpendicular to both  $l_1$  and  $l_2$ .

- i)** Find the length of  $PQ$ .  
**ii)** Find a vector perpendicular to the plane  $\Pi$  which contains  $PQ$  and  $l_2$ .  
**iii)** Find the perpendicular distance from  $A$  to  $\Pi$ . (OCR)

- 34** Let  $A, B, C$ , be the points  $(2, 1, 0), (3, 3, -1), (5, 0, 2)$  respectively. Find  $\overrightarrow{AB} \times \overrightarrow{AC}$ .

Hence or otherwise obtain an equation for the plane containing  $A, B$  and  $C$ . (SQA/CSYS)

- 35 The plane  $\pi$  has equation  $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 0$ , and P and Q are the points with position vectors  $7\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$  and  $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  respectively. Find the position vector of the point in which the line passing through P and Q meets the plane  $\pi$ .

Find, in the form  $ax + by + cz = d$ , the equation of the plane which contains the line PQ and which is perpendicular to  $\pi$ . (OCR)

- 36 a) With the help of Fig. 1 below and using, where appropriate, the notation in the figure, show that the volume of the tetrahedron OABC is

$$\frac{1}{6} |\mathbf{n} \cdot \mathbf{c}|$$

where  $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ .

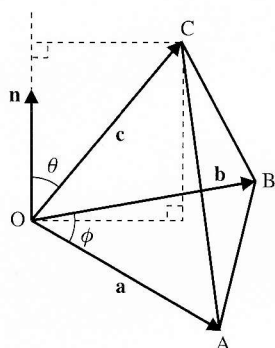


Fig. 1

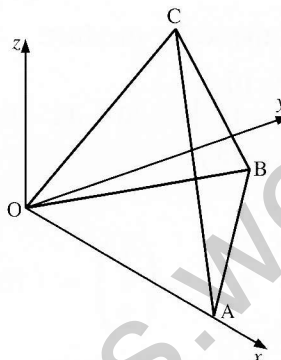


Fig. 2

- b) In the tetrahedron OABC, shown in Fig. 2 above, the equation of the plane ABC is

$$12x + 4y + 5z = 48$$

- i) Given that A is on the  $x$ -axis, find its coordinates.

The equation of the plane OBC is  $-4x + 4y + z = 0$ .

- ii) Show that the equation of BC is

$$\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

- iii) Given that B is in the  $xy$ -plane show that it is the point (3, 3, 0).

The cartesian equation of AC is  $\frac{x}{2} = \frac{2-y}{1} = \frac{8-z}{4}$ .

- iv) Find the coordinates of C.

- v) Find the volume of this tetrahedron. (NICCEA)

## Scalar triple product and its applications

The scalar triple product of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  is defined as  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ .

**Note** We must calculate  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$  as  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ . If we tried to calculate it as  $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ , we would have the vector product of a scalar and a vector, which, by definition, cannot exist.

**Example 21** Calculate  $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix}$ .

**SOLUTION**

We must calculate the **vector product** first:

$$\begin{aligned} \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} &= \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \left( \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -11 \\ -13 \end{pmatrix} \end{aligned}$$

Then we calculate the scalar product:

$$\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -11 \\ -13 \end{pmatrix} = 30 - 44 - 91 = -105$$

Therefore, we have

$$\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} = -105$$

A quicker way to find  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$  is as follows.

The vector product  $\mathbf{b} \times \mathbf{c}$  is given by (see page 104)

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \mathbf{i}(b_2c_3 - b_3c_2) - \mathbf{j}(b_1c_3 - b_3c_1) + \mathbf{k}(b_1c_2 - b_2c_1) \end{aligned}$$

Therefore, the scalar triple product  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$  is given by (see *Introducing Pure Mathematics*, page 503)

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

That is,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Applying this result to Example 21, we would have

$$\begin{aligned} \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} &= \begin{vmatrix} 3 & 4 & 7 \\ 2 & 3 & -1 \\ 7 & 4 & 2 \end{vmatrix} \\ &= 3 \times 10 - 4 \times 11 + 7 \times -13 \\ &= -105 \end{aligned}$$

## Coplanar vectors

We have

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} &= \mathbf{a} \cdot (bc \sin \theta \hat{\mathbf{n}}) \\ &= abc \sin \theta \cos \phi\end{aligned}$$

where  $\theta$  is the angle between  $\mathbf{b}$  and  $\mathbf{c}$ , and  $\phi$  is the angle between  $\mathbf{a}$  and  $\hat{\mathbf{n}}$ , which is perpendicular to the plane containing  $\mathbf{b}$  and  $\mathbf{c}$ . Therefore, we get

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = abc \sin \theta \sin \psi$$

where  $\psi = (90^\circ - \phi)$  is the angle between  $\mathbf{a}$  and the plane containing  $\mathbf{b}$  and  $\mathbf{c}$ .

Hence, when  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are **coplanar** ( $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  lie in the same plane), we have

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = 0$$

## Volume of a cuboid

Consider cuboid OBDCAQRS, which has adjacent edges  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$  and  $\overrightarrow{OC} = \mathbf{c}$ .

The volume of a cuboid is given by

$$\text{Volume} = \text{Area of base} \times \text{Perpendicular height}$$

Therefore, the volume,  $V$ , of OBDCAQRS is

$$V = (b \times c) \times a = abc$$

Now,  $\mathbf{b}$  and  $\mathbf{c}$  are perpendicular to each other, and  $\mathbf{a}$  is perpendicular to the plane containing them. Hence, we have

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = abc \sin 90^\circ \sin 90^\circ = abc$$

Therefore, the volume of a cuboid is given by

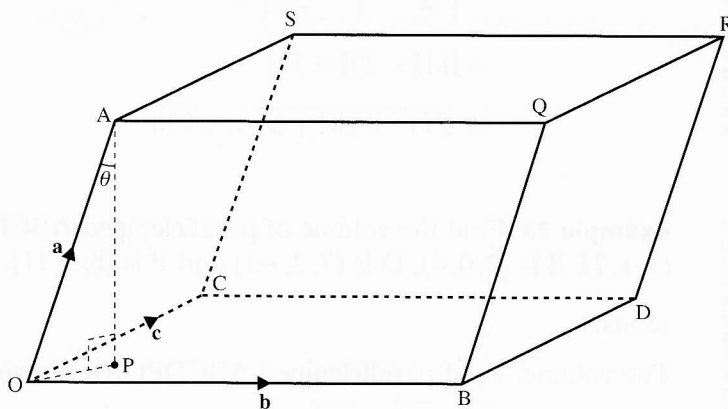
$$V = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  represent three adjacent edges of the cuboid.

**Note** Since the volume of any shape must be **positive**, we always use the **magnitude** of  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$  in volume calculations.

## Volume of a parallelepiped

A parallelepiped is a polyhedron with six faces, each of which is a parallelogram.





Consider the parallelepiped OBDCAQRS, which has adjacent edges  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$  and  $\overrightarrow{OC} = \mathbf{c}$ .

The volume of a parallelepiped is given by

$$\text{Volume} = \text{Area of base} \times \text{Perpendicular height}$$

Therefore, the volume,  $V$ , of OBDCAQRS is

$$V = |\mathbf{b} \times \mathbf{c}| \times \text{Perpendicular height}$$

Now, the perpendicular height, AP, is  $|\mathbf{a}| \cos \theta$ . Therefore, we have

$$\begin{aligned} V &= |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \cos \theta \\ &= |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \theta \end{aligned}$$

We note that this is identical to the scalar product  $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta$ , with  $|\mathbf{x}| = |\mathbf{a}|$ ,  $|\mathbf{y}| = |\mathbf{b} \times \mathbf{c}|$  and  $\mathbf{b} \times \mathbf{c}$  having the same sense as PA. Therefore, the volume of a parallelepiped is given by

$$V = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  represent three adjacent edges of the parallelepiped.

**Example 22** Find the area of parallelogram ABCD, where A is (3, 1, 7), B is (2, 0, 4) and D is (7, 2, -1).

**SOLUTION**

We have the adjacent sides

$$\begin{aligned} \overrightarrow{AB} &= \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix} \\ \overrightarrow{AD} &= \mathbf{d} - \mathbf{a} = \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -8 \end{pmatrix} \end{aligned}$$

The area of parallelogram ABCD is  $|\overrightarrow{AB} \times \overrightarrow{AD}|$ , which gives

$$\begin{aligned} \text{Area} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -3 \\ 4 & 1 & -8 \end{vmatrix} \\ &= |11\mathbf{i} - 20\mathbf{j} + 3\mathbf{k}| \\ &= \sqrt{11^2 + 20^2 + 3^2} = \sqrt{530} \end{aligned}$$

**Example 23** Find the volume of parallelepiped ABCDPQRS, where A is (3, 1, 7), B is (2, 0, 4), D is (7, 2, -1) and P is (8, 3, 11).

**SOLUTION**

The volume,  $V$ , of parallelepiped ABCDPQRS is given by

$$V = \overrightarrow{AP} \cdot \overrightarrow{AB} \times \overrightarrow{AD}$$



We have

$$\vec{AP} = \mathbf{p} - \mathbf{a} = \begin{pmatrix} 8 \\ 3 \\ 11 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$$

Using  $\vec{AB} \times \vec{AD} = \begin{pmatrix} 11 \\ -20 \\ 3 \end{pmatrix}$  from Example 22, we get

$$\begin{aligned} V &= \vec{AP} \cdot \vec{AB} \times \vec{AD} \\ &= \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -20 \\ 3 \end{pmatrix} = 55 - 40 + 12 = 27 \end{aligned}$$

Or we can use (see pages 122 and 124)

$$V = \begin{vmatrix} 5 & 2 & 4 \\ -1 & -1 & -3 \\ 4 & 1 & -8 \end{vmatrix} = 5(8 + 3) - 2(8 + 12) + 4(-1 + 4) = 27$$

**Note**  $\vec{AB} \cdot \vec{AD} \times \vec{AP}$  could be used, since the order in which we select the three adjacent edges is not relevant, but each vector must be **away from, or towards, the same point** of the parallelepiped.

### Volume of a tetrahedron

A tetrahedron is a polyhedron with four faces, each of which is a triangle. That is, it is a pyramid with a triangular base.

Consider the adjacent edges AD, AB and AC, represented by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  respectively.

The volume of a tetrahedron is given by

$$\text{Volume} = \frac{1}{3} \times \text{Area of base} \times \text{Perpendicular height}$$

Therefore, the volume,  $V$ , of tetrahedron ABCD is

$$V = \frac{1}{3} \times \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \times \text{Perpendicular height}$$

Now, the perpendicular height, DP, is  $|\mathbf{a}| \cos \theta$ . Therefore,

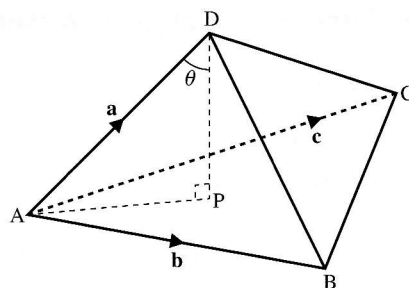
$$V = \frac{1}{6} |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \cos \theta = \frac{1}{6} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \theta$$

Because  $\mathbf{b} \times \mathbf{c}$  has the same sense as PD, this gives

$$V = \frac{1}{6} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are the vectors representing three adjacent edges of the tetrahedron.

Therefore, the volume of a tetrahedron is one sixth of the volume of a parallelepiped.



**Example 24** Find the volume of tetrahedron PQRS, where P is (3, 4, 7), Q is (-2, 1, 5), R is (1, 3, -1) and S is (-3, 6, 8).

**SOLUTION**

We have

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix} \quad \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{pmatrix} -2 \\ -1 \\ -8 \end{pmatrix} \quad \overrightarrow{PS} = \mathbf{s} - \mathbf{p} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$$

Therefore, the volume,  $V$ , of tetrahedron PQRS is given by

$$\begin{aligned} V &= \frac{1}{6} \times \overrightarrow{PQ} \cdot \overrightarrow{PR} \times \overrightarrow{PS} \\ &= \frac{1}{6} \begin{vmatrix} -5 & -3 & -2 \\ -2 & -1 & -8 \\ -6 & 2 & 1 \end{vmatrix} \\ &= \frac{1}{6} (-5 \times 15 + 3 \times -50 - 2 \times -10) \\ &= -\frac{205}{6} \end{aligned}$$

Therefore, the volume of tetrahedron PQRS is  $\frac{205}{6}$ .

### Volume of a triangular prism

The volume of a triangular prism is given by

$$\text{Volume} = \text{Area of base} \times \text{Perpendicular height}$$

By definition, the base is a triangle. So, we have

$$\text{Area of base} = \frac{1}{2} |\mathbf{b} \times \mathbf{c}|$$

Therefore, the volume,  $V$ , of the prism is

$$\begin{aligned} V &= \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \times \text{Perpendicular height} \\ &= \frac{1}{2} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \end{aligned}$$

which gives

$$V = \frac{1}{2} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where  $\mathbf{a}$  is the vector representing an edge of the prism, and  $\mathbf{b}$  and  $\mathbf{c}$  are the vectors representing two sides of its triangular base, adjacent to  $\mathbf{a}$ .

## Volume of a pyramid

The volume of a pyramid is given by

$$V = \frac{1}{3} \times \text{Area of base} \times \text{Perpendicular height}$$

Taking the case of a rectangular (or parallelogram) base, we have

$$V = \frac{1}{3} |\mathbf{b} \times \mathbf{c}| \times \text{Perpendicular height}$$

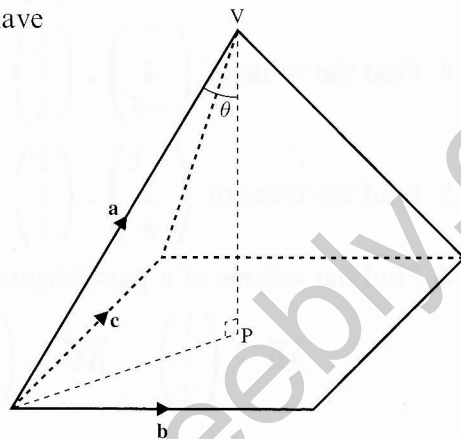
where  $\mathbf{b}$  and  $\mathbf{c}$  represent adjacent sides of the base, as shown in the diagram on the right.

From the diagram, we see that the perpendicular height is  $|\mathbf{a}| \cos \theta$ . Therefore, we have

$$V = \frac{1}{3} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \theta$$

Because  $\mathbf{b} \times \mathbf{c}$  has the same sense as  $\mathbf{PV}$ , this gives

$$V = \frac{1}{3} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$



Therefore, the volume of a pyramid with a rectangular (or parallelogram) base is one third of the volume of a parallelepiped.

**Example 25** Find the volume of pyramid ABCDV, where ABCD is a parallelogram, and V is the vertex. A is (2, 1, 5), B is (3, 4, -2), D is (5, 2, 3) and V is (0, 6, 4).

**SOLUTION**

We have

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} \quad \overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \quad \overrightarrow{AV} = \mathbf{v} - \mathbf{a} = \begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$$

Therefore, the volume of pyramid ABCDV is

$$\begin{aligned} \frac{1}{3} \overrightarrow{AV} \cdot \overrightarrow{AB} \times \overrightarrow{AD} &= \frac{1}{3} \begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix} \cdot \left( \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right) \\ &= \frac{1}{3} \begin{vmatrix} -2 & 5 & -1 \\ 1 & 3 & -7 \\ 3 & 1 & -2 \end{vmatrix} \\ &= \frac{1}{3} [-2(-6 + 7) - 5(-2 + 21) - 1(1 - 9)] \\ &= \frac{1}{3} (-2 \times 1 + -5 \times 19 - 1 \times -8) \\ &= \frac{1}{3} (-2 - 95 + 8) \end{aligned}$$

Therefore, the volume of pyramid ABCDV is  $\frac{89}{3}$ .

## Exercise 6C

1 Find the value of  $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ .

2 Find the value of  $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ .

3 Find the value of  $\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 8 \\ 4 \end{pmatrix}$ .

4 Find the volume of a parallelepiped ABCDEFGH,

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \quad \overrightarrow{AE} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

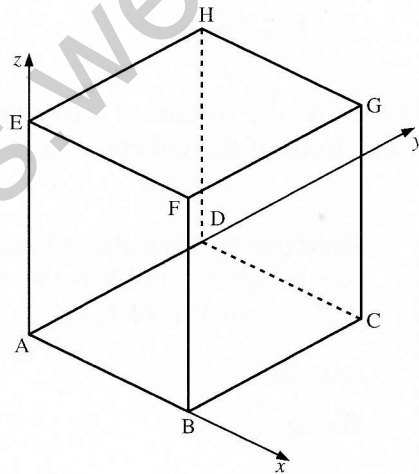
5 The figure on the right represents a cube with side of unit length.

i) Find  $\overrightarrow{AB} \cdot \overrightarrow{AC}$ .

ii) Find a vector, using the letters in the diagram, which is equal to  $\overrightarrow{EA} \times \overrightarrow{EH}$ .

iii) Find the value of  $\lambda$  in the following equation

$$\overrightarrow{EA} \times \overrightarrow{EC} = \lambda \overrightarrow{BD} \quad (\text{NICCEA})$$



6 The points A, B, C and D have coordinates (3, 1, 2), (5, 2, -1), (6, 4, 5) and (-7, 6, -3) respectively.

a) Find  $\overrightarrow{AC} \times \overrightarrow{AD}$ .

b) Find a vector equation of the line through A which is perpendicular to  $\overrightarrow{AC}$  and  $\overrightarrow{AD}$ .

c) Verify that B lies on this line.

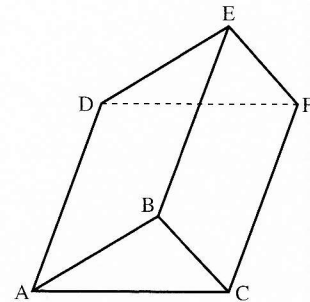
d) Find the volume of the tetrahedron ABCD. (EDEXCEL)

7 The figure on the right shows a right prism with triangular ends ABC and DEF, and parallel edges AD, BE, CF. Given that A is (2, 7, -1), B is (5, 8, 2), C is (6, 7, 4) and D is (12, 1, -9),

a) find  $\overrightarrow{AB} \times \overrightarrow{AC}$

b) find  $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$ .

c) Calculate the volume of the prism. (EDEXCEL)



- 8 The points A, B and C have position vectors  $\mathbf{a}$  m,  $\mathbf{b}$  m and  $\mathbf{c}$  m respectively, relative to an origin O, where

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \mathbf{b} = 4\mathbf{i} + 6\mathbf{j} + 7\mathbf{k} \quad \mathbf{c} = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$$

- Find  $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ .
- Hence, or otherwise, find the area of  $\triangle ABC$  and the volume of tetrahedron OABC.
- Find an equation of the plane ABC in the form  $\mathbf{r} \cdot \mathbf{n} = p$ .

Given that the point D has position vector  $(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$  m,

- find the coordinates of the point of intersection, E, of the OD with the plane ABC
- find the acute angle between ED and the plane ABC. (EDEXCEL)

## 7 Curve sketching and inequalities

*And of the curviship lend a myth to God.*

HART CRANE

### Curve sketching

On page 306 of *Introducing Pure Mathematics*, there is a brief introduction to the use of asymptotes in curve sketching. We are now going to extend the procedure to more complex curves.

**Remember** An asymptote is a line which becomes a tangent to a curve as  $x$  or  $y$  tends to infinity.

We need to be able to find asymptotes if we want to sketch functions which are not trigonometric or polynomial.

Consider, for example, the curve

$$y = \frac{4x - 8}{x + 3}$$

As  $y \rightarrow \pm\infty$ , the denominator of this function must tend to zero. That is, as  $x + 3 \rightarrow 0$ ,  $x \rightarrow -3$ . Hence,  $x = -3$  is an asymptote.

To find the asymptote as  $x \rightarrow \pm\infty$ , we express the function as

$$y = \frac{4 - \frac{8}{x}}{1 + \frac{3}{x}}$$

As  $x \rightarrow \pm\infty$ ,  $\frac{3}{x} \rightarrow 0$ , and  $\frac{8}{x} \rightarrow 0$ . Therefore,  $y \rightarrow \frac{4}{1} = 4$ . Hence,  $y = 4$  is also an asymptote.

Notice that, as  $x \rightarrow \pm\infty$ , the largest terms in the numerator and the denominator are  $4x$  and  $x$  respectively, and so  $y \approx 4x \div x = 4$ .

$x = -3$  is a **vertical asymptote**, as it is parallel to the  $y$ -axis, and  $y = 4$  is a **horizontal asymptote**, as it is parallel to the  $x$ -axis.

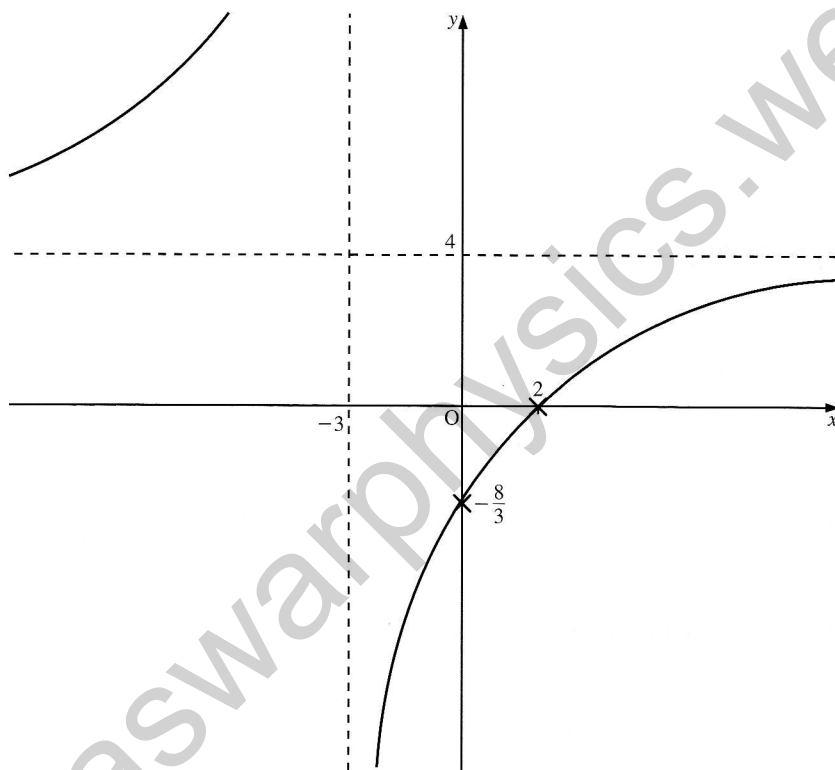
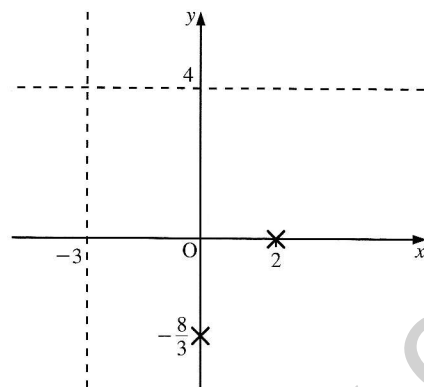
To be able to sketch  $y = \frac{4x - 8}{x + 3}$ , we also need to find where it crosses the  $x$ - and  $y$ -axes:

$$\text{When } x = 0: y = -\frac{8}{3}$$

$$\text{When } y = 0: 4x - 8 = 0 \Rightarrow x = 2$$

To sketch the curve, we proceed as follows (see the diagram on the right):

- First, draw the asymptotes, using dashed lines.
- Next, mark the points where the curve crosses the axes.
- As the numerator **and** the denominator of the function each contain only a linear term in  $x$ , the curve cannot cross either asymptote.
- Considering the curve for  $x > -3$ , we see that it tends to  $-\infty$  as  $x$  approaches  $-3$  from values of  $x$  greater than  $-3$ . Hence, the curve tends to  $+\infty$  as  $x$  approaches  $-3$  from values of  $x$  less than  $-3$ .
- We can now complete the curve of  $y = \frac{4x - 8}{x + 3}$ .



**Example 1** Sketch  $y = \frac{2x - 6}{x - 5}$ .

**SOLUTION**

First, we find the asymptotes.

As  $x \rightarrow \pm\infty$ ,  $y \rightarrow 2$ . That is, the horizontal asymptote is  $y = 2$ .

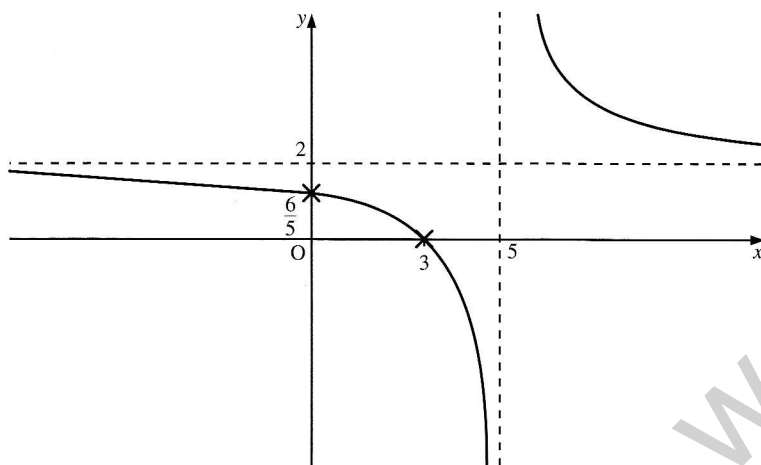
As  $y \rightarrow \pm\infty$ ,  $x - 5 \rightarrow 0$ . Hence,  $x = 5$  is the vertical asymptote.

Next, we find where the curve crosses the axes:

$$\text{When } x = 0: y = \frac{-6}{-5} = \frac{6}{5}$$

$$\text{When } y = 0: 2x - 6 = 0 \Rightarrow x = 3$$

We now complete the sketch of  $y = \frac{2x - 6}{x - 5}$ , which is shown below.



### Curves with an oblique asymptote

For most curves, the value of  $y$  as  $x \rightarrow \pm\infty$  will not be finite.

Consider, for example, the curve

$$y = x + \frac{1}{x}$$

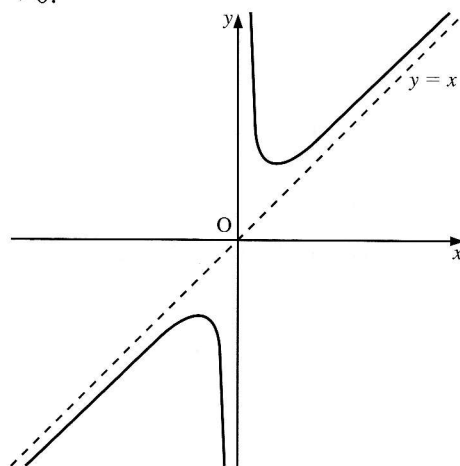
As  $x \rightarrow \pm\infty$ ,  $\frac{1}{x} \rightarrow 0$  and thus  $y \rightarrow x$ . Therefore,  $y = x$  is an asymptote to the curve.

This is called an **oblique asymptote** (sometimes an **inclined asymptote**), as  $y = x$  is not parallel to either axis.

The other asymptote is  $x = 0$  (the  $y$ -axis), as  $\frac{1}{x} \rightarrow \infty$ , when  $x \rightarrow 0$ .

When  $x = 0$ ,  $y$  is not defined, thus the curve does not cross the  $y$ -axis. Thus, the  $y$ -axis is a vertical asymptote, as already shown.

We can now sketch the curve of  $y = x + \frac{1}{x}$ , as shown on the right.





**Example 2** Sketch  $y = \frac{x^2 + 3x}{x + 1}$ .

**SOLUTION**

Dividing  $x^2 + 3x$  by  $x + 1$ , we obtain

$$y = x + 2 - \frac{2}{x + 1}$$

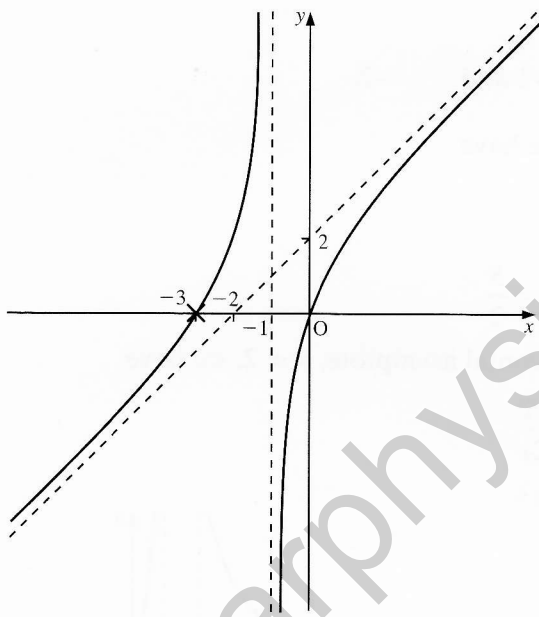
which gives the asymptotes as  $x = -1$  and  $y = x + 2$ .

We now find where the curve crosses the axes:

$$\text{When } y = 0: x^2 + 3x = 0 \Rightarrow x = 0 \text{ and } -3$$

$$\text{When } x = 0: y = 0$$

We now complete the sketch of  $y = \frac{x^2 + 3x}{x + 1}$ .



## Sketching rational functions with a quadratic denominator

### Curves with two vertical asymptotes

Take, for example, the curve  $y = \frac{(x - 3)(2x - 5)}{(x + 1)(x + 2)}$ .

When the denominator is a quadratic expression,

- there are **always two** vertical asymptotes, and
- the curve will normally cross the horizontal asymptote.

Hence, in addition to finding the asymptotes and the points where the curve crosses the axes, we need to establish where the curve crosses the horizontal asymptote.

**Note** The two vertical asymptotes could coincide, as in Example 4, on page 135.

Hence, there are four stages to sketching the given function.

- 1 To find the horizontal asymptote of  $y = \frac{(x-3)(2x-5)}{(x+1)(x+2)}$ , we express the function as

$$y = \frac{\left(1 - \frac{3}{x}\right)\left(2 - \frac{5}{x}\right)}{\left(1 + \frac{1}{x}\right)\left(1 + \frac{2}{x}\right)}$$

As  $x \rightarrow \pm\infty$ ,  $\frac{1}{x} \rightarrow 0$ , and  $y \rightarrow 2$ . Therefore, the horizontal asymptote is  $y = 2$ .

- 2 To find the vertical asymptotes, we equate the denominator to zero, which gives

$$(x+1)(x+2) = 0$$

Hence, the vertical asymptotes are  $x = -1$  and  $x = -2$ .

- 3 To find where the curve cuts the axes, we have

$$\text{When } x = 0: y = \frac{15}{2}$$

$$\text{When } y = 0: x = 3 \text{ and } x = \frac{5}{2}$$

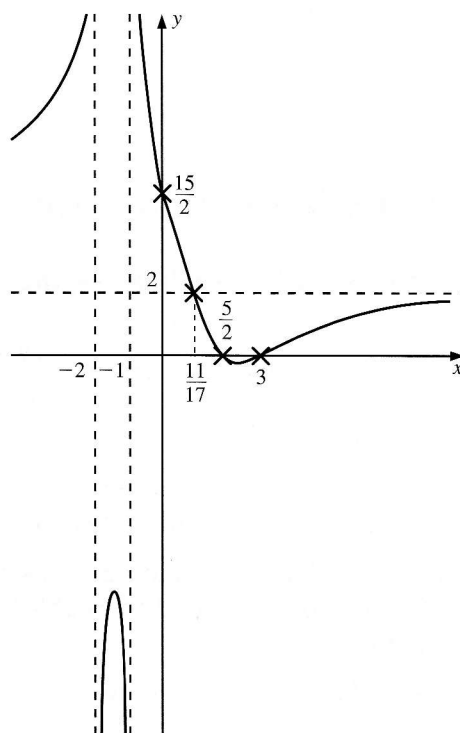
- 4 To find where the curve crosses the horizontal asymptote,  $y = 2$ , we have

$$\begin{aligned} 2 &= \frac{(x-3)(2x-5)}{(x+1)(x+2)} \\ 2(x^2 + 3x + 2) &= 2x^2 - 11x + 15 \\ \Rightarrow x &= \frac{11}{17} \end{aligned}$$

To sketch the curve, we need to insert all four points, as well as the three asymptotes.

#### Note

- The curve can cross an axis or an asymptote **only** at the points found.
- If one branch of the curve goes to  $+\infty$ , the next branch must return from  $-\infty$ . The exception to this is when the two vertical asymptotes coincide as the result of a squared factor in the denominator. See Example 4 on page 135.



**Example 3** Sketch  $y = \frac{(x+1)(x-4)}{(x-2)(x-5)}$ .

**SOLUTION**

The horizontal asymptote is  $y = 1$ .

The vertical asymptotes are  $x = 2$  and  $x = 5$ .

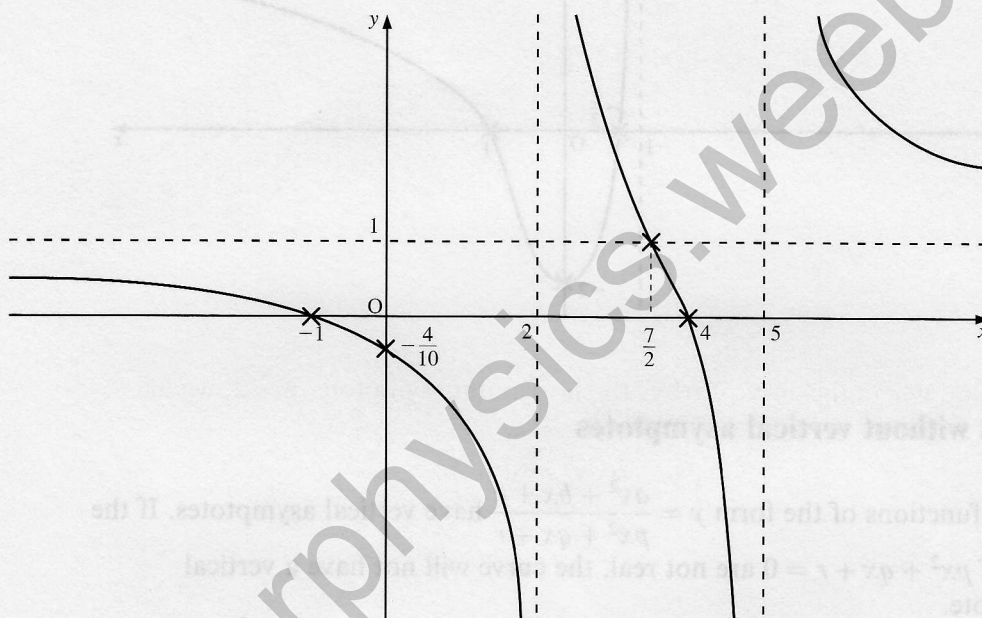
The curve crosses the axes at  $x = 0, y = -\frac{4}{10}$ , and at  $y = 0, x = -1, 4$ .

The curve crosses the horizontal asymptote when  $y = 1$ , which gives

$$x^2 - 7x + 10 = x^2 - 3x - 4$$

$$\Rightarrow x = \frac{7}{2}$$

We can now sketch the curve.



**Example 4** Sketch the curve  $y = \frac{(x-1)(3x+2)}{(x+1)^2}$ .

**SOLUTION**

The horizontal asymptote is  $y = 3$ .

The vertical asymptotes are  $x = -1$  (twice).

The curve crosses the axes at  $x = 0, y = -2$ , and at  $y = 0, x = 1, -\frac{2}{3}$ .

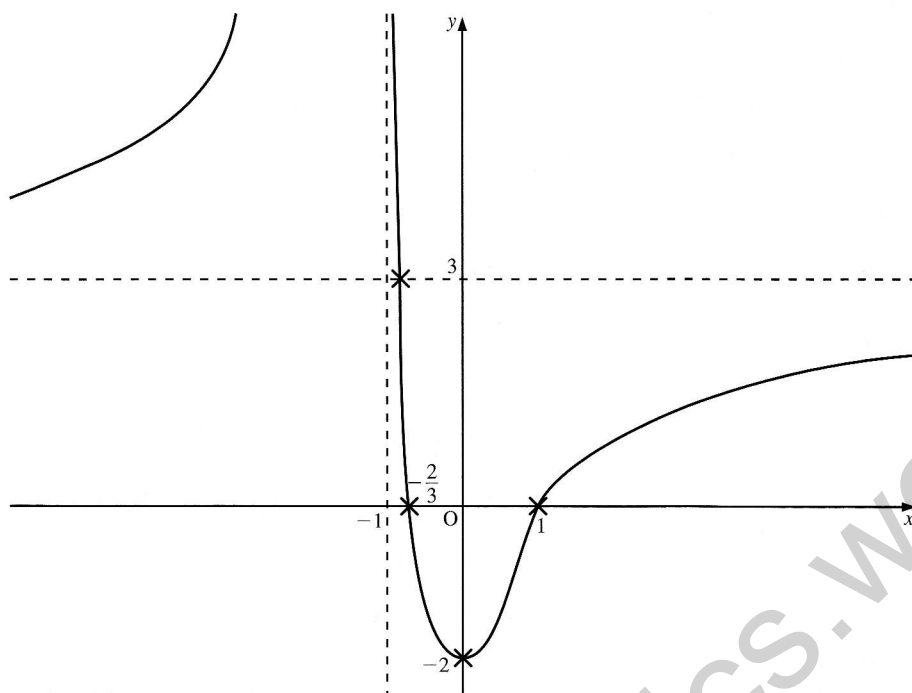
The curve crosses the horizontal asymptote when  $y = 3$ , which gives

$$3 = \frac{3x^2 - x - 2}{x^2 + 2x + 1}$$

$$3(x^2 + 2x + 1) = 3x^2 - x - 2$$

$$\Rightarrow x = -\frac{5}{7}$$

**Note** Since  $x = -1$  is a repeat asymptote, and the curve tends to  $+\infty$  as  $x$  approaches the value of  $-1$  from the right (that is,  $x$  tends to  $-1$  from above), it also tends to  $+\infty$  as  $x$  approaches the value of  $-1$  from the left (that is, from below).



### Curves without vertical asymptotes

Not all functions of the form  $y = \frac{ax^2 + bx + c}{px^2 + qx + r}$  have vertical asymptotes. If the roots of  $px^2 + qx + r = 0$  are not real, the curve will not have a vertical asymptote.

**Example 5** Sketch the curve  $y = \frac{2x^2 + 5x + 3}{4x^2 + 5x + 3}$ , and find the range of possible values for  $y$ .

**SOLUTION**

To find the horizontal asymptote of  $y = \frac{2x^2 + 5x + 3}{4x^2 + 5x + 3}$ , we express the function as

$$y = \frac{2 + \frac{5}{x} + \frac{3}{x^2}}{4 + \frac{5}{x} + \frac{3}{x^2}}$$

As  $x \rightarrow \infty$ ,  $y \rightarrow \frac{1}{2}$ . Therefore,  $y = \frac{1}{2}$  is the horizontal asymptote.

For the vertical asymptotes, we have  $4x^2 + 5x + 3 = 0$ , which gives

$$x = \frac{-5 \pm \sqrt{-23}}{8}$$

These are not real. Therefore, the curve does not have a vertical asymptote.

To find where the curve cuts the axes, we have

$$\begin{aligned}\text{When } y = 0: \quad 2x^2 + 5x + 3 &= 0 \\ (2x + 3)(x + 1) &= 0 \\ \Rightarrow x &= -1 \quad \text{and} \quad -\frac{3}{2}\end{aligned}$$

$$\text{When } x = 0: \quad y = 1$$

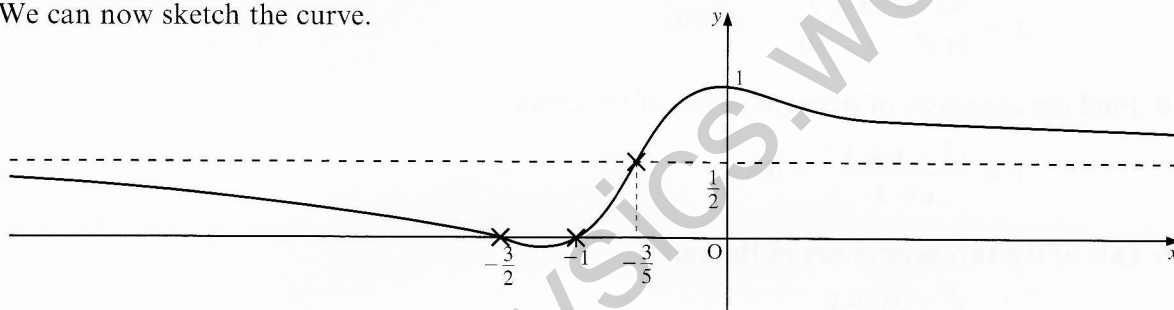
The curve crosses the horizontal asymptote  $y = \frac{1}{2}$  when

$$\frac{1}{2} = \frac{2x^2 + 5x + 3}{4x^2 + 5x + 3}$$

which gives

$$\begin{aligned}4x^2 + 5x + 3 &= 4x^2 + 10x + 6 \\ \Rightarrow x &= -\frac{3}{5}\end{aligned}$$

We can now sketch the curve.



To find the range of values of  $y$ , we need to find the values for which

$$y = \frac{2x^2 + 5x + 3}{4x^2 + 5x + 3} \text{ has real roots for } x.$$

Cross-multiplying  $y = \frac{2x^2 + 5x + 3}{4x^2 + 5x + 3}$ , we obtain

$$\begin{aligned}4yx^2 + 5yx + 3y &= 2x^2 + 5x + 3 \\ \Rightarrow (4y - 2)x^2 + (5y - 5)x + 3y - 3 &= 0\end{aligned}$$

From the quadratic formula, we know that  $b^2 - 4ac \geq 0$  for the roots of  $x$  to be real. Therefore, we have

$$\begin{aligned}(5y - 5)^2 - 4(4y - 2)(3y - 3) &\geq 0 \\ \Rightarrow 23y^2 - 22y - 1 &\leq 0 \\ \Rightarrow (23y + 1)(y - 1) &\leq 0 \\ \Rightarrow -\frac{1}{23} &\leq y \leq 1\end{aligned}$$

Therefore, the range of possible values of  $y$  is  $-\frac{1}{23} \leq y \leq 1$ .

Hence, the maximum value of  $y$  is 1, and the minimum value is  $-\frac{1}{23}$ .

**Note** We could have used calculus to find these two stationary points.

## Exercise 7A

Sketch the graph of each of the following functions.

$$1 \quad y = \frac{(x-3)(x-1)}{(x+2)(x-2)}$$

$$2 \quad y = \frac{(2x-1)(x+4)}{(x-1)(x-2)}$$

$$3 \quad y = \frac{(x+4)(x-5)}{(x-2)(x-3)}$$

$$4 \quad y = \frac{(x+1)(2x+5)}{(x+2)(x-5)}$$

$$5 \quad y = \frac{2x^2 + 3x - 5}{x^2 - x - 2}$$

$$6 \quad y = \frac{3x^2 + 4x + 4}{x^2 - 2x - 3}$$

7 Find the range of values of

$$a) \quad y = \frac{4x^2 - x - 3}{2x^2 - x - 3}$$

$$b) \quad y = \frac{x^2 + x - 1}{x^2 + x - 3}$$

8 Find the equations of the three asymptotes of the curve

$$y = \frac{4x^2 - 5x + 7}{21x^2 - x - 10} \quad (\text{OCR})$$

9 Find the equations of the asymptotes of the curve

$$y = \frac{x^2 - x + 1}{x + 1} \quad (\text{OCR})$$

10 One of the two asymptotes of the curve

$$y = \frac{x^2 + \lambda x + 1}{x + 2}$$

where  $\lambda$  is a constant, is  $y = x + 5$ .

i) State the equation of the other asymptote.

ii) Find the value of  $\lambda$ . (OCR)

11 The curve  $C$  has equation

$$y = 10 + \frac{8}{x-2} - \frac{27}{x+2}$$

i) Write down the equations of the asymptotes of  $C$ .

ii) Find  $\frac{d^2y}{dx^2}$ .

iii) Show that  $C$  has one point of inflexion, and find the coordinates of this point. (OCR)

12 A curve has equation  $y = \frac{x^2 - 5}{x^2 + 2x - 11}$ .

a) Determine the equations of the three asymptotes to the curve, giving each answer in an exact form.

b) Prove algebraically that there are no values of  $x$  for which  $\frac{1}{2} < y < \frac{5}{6}$ .

Hence, or otherwise, calculate the coordinates of the turning points on the curve. (AEB 98)

13 A curve has equation  $y = \frac{x^2}{2x+1}$ .

- a) i) Write down the equation of the vertical asymptote to the curve, and determine the equation of the oblique asymptote.
- ii) Use differentiation to determine the coordinates of the stationary points on the curve.
- b) The region bounded by the curve, the  $x$ -axis between  $x = 0$  and  $x = 1$ , and the line  $x = 1$  is rotated through one revolution about the  $x$ -axis to form a solid with volume  $V$ .

Using the substitution  $u = 2x + 1$ , or otherwise, show that

$$V = \frac{\pi}{24} (4 - 3 \ln 3) \quad (\text{AEB 98})$$

14 Let the function  $f$  be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2} \quad x \neq 2$$

- a) The graph of  $y = f(x)$  crosses the  $y$ -axis at  $(0, a)$ . State the value of  $a$ .
- b) For the graph of  $f(x)$ 
  - i) write down the equation of the vertical asymptote,
  - ii) show algebraically that there is a non-vertical asymptote and state its equation.
- c) Find the coordinates and nature of the stationary point of  $f(x)$ .
- d) Show that  $f(x) = 0$  has a root in the interval  $-2 < x < 0$ .
- e) Sketch the graph of  $y = f(x)$ . (You must include on your sketch the results obtained in the first four parts of this question.) (SQA/CSYS)

15 The curve  $C$  has equation

$$y = \frac{2x^2 + 6x + 1}{(x-1)(x+2)}$$

- i) Express  $y$  in partial fractions.
- ii) Deduce that
  - a) at every point of  $C$ , the gradient is negative
  - b)  $y > 2$  for all  $x > 1$ .
- iii) Write down the equations of the asymptotes of  $C$ .
- iv) One of the asymptotes has a point in common with  $C$ . Determine the coordinates of this point. (OCR)

16 A curve  $C$  is defined by the equations

$$x = \frac{1+t}{1-t} \quad y = \frac{1+t^2}{1-t^2}$$

where  $t$  is a real parameter,  $t \neq \pm 1$ .

- a) Find an expression for  $\frac{dy}{dx}$  in terms of  $t$ , simplifying your answer as much as possible.
- b) By eliminating  $t$ , prove that  $C$  has cartesian equation  $y = \frac{x^2 + 1}{2x}$ .
- c) Write down the equations of the two asymptotes of  $C$ .
- d) i) Prove algebraically that there are no values of  $x$  for which  $-1 < y < 1$ .
- ii) Hence, or otherwise, determine the coordinates of the turning points of  $C$ . (AEB 96)