We can take the factor k out of each column. Hence, we obtain

$$\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix} = k^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Transpose of a determinant

The **transpose** of a determinant is obtained by reflecting the determinant in its **leading diagonal**. (This is the diagonal from the top left corner to the bottom right corner. It is also known as the **principal diagonal**.)

The value of the transpose of a determinant is the **same** as the determinant's **original value**. For example, we have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

Example 5 Evaluate $\begin{vmatrix} 2 & 8 & 9 \\ 0 & -1 & 3 \\ 0 & 4 & 1 \end{vmatrix}$

SOLUTION

To simplify the calculation, we replace the given determinant by its transpose:

$$\begin{vmatrix} 2 & 8 & 9 \\ 0 & -1 & 3 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 8 & -1 & 4 \\ 9 & 3 & 1 \end{vmatrix}$$

which gives

$$\begin{vmatrix} 2 & 0 & 0 \\ 8 & -1 & 4 \\ 9 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = 2(-1 - 12) = -26$$

Factorisation of determinants

The easier way to find the factors of a determinant is to use the rules for manipulating determinants. Rarely, if ever, do we multiply out the determinant and then factorise the result.

In Example 6, the factors are obtained by subtracting, in turn, one column from another. In Example 7, a factor is obtained by first adding **all** three rows.

Example 6 Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

SOLUTION

First, we take out the factors a, b and c, which gives

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Next, we subtract column 1 from column 2, and take out a fourth factor:

$$abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc \begin{vmatrix} 1 & 0 & 1 \\ a & b-a & c \\ a^2 & b^2-a^2 & c^2 \end{vmatrix}$$

$$= abc(b-a) \begin{vmatrix} 1 & 0 & 1 \\ a & 1 & c \\ a^2 & b+a & c^2 \end{vmatrix}$$

Then, we subtract column 1 from column 3, and complete the factorisation:

$$abc(b-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & c-a \\ a^2 & b+a & c^2-a^2 \end{vmatrix} = abc(b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= abc(b-a)(c-a)[(c+a)-(b+a)]$$
$$= abc(b-a)(c-a)(c-b)$$
$$= abc(a-b)(b-c)(c-a)$$

Example 7

a) Factorise $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

b) Hence, find the factors of $a^3 + b^3 + c^3 - 3abc$.

SOLUTION

a) First, we add rows 2 and 3 to row 1, which gives

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Next, we take out the factor $(a+b+c)$, which gives

$$(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Then, we subtract column 1 from columns 2 and 3, and complete the factorisation:

$$(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$$

$$= (a+b+c)[(c-b)(b-c)-(a-b)(a-c)]$$
$$= (a+b+c)(bc+ac+ab-a^2-b^2-c^2)$$

b) Expanding the determinant, we obtain

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(cb - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= 3abc - a^3 - b^3 - c^3$$
$$= -(a^3 + b^3 + c^3 - 3abc)$$

Hence, we have

$$a^3 + b^3 + c^3 - 3abc = - \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

That is,

$$a^3 + b^3 + c^3 - 3abc = -(a + b + c)(bc + ca + ab - a^2 - b^2 - c^2)$$
$$= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$

Exercise 5A

1 Find the value of each of these determinants.

a) $\begin{vmatrix} 3 & 8 & 5 \\ 9 & 2 & -2 \\ 2 & 5 & 1 \end{vmatrix}$
b) $\begin{vmatrix} 3 & 3 & 3 \\ 1 & -4 & 1 \\ 6 & -7 & 5 \end{vmatrix}$
c) $\begin{vmatrix} 2 & 5 & 1 \\ 6 & 3 & 3 \\ 8 & -2 & 4 \end{vmatrix}$
d) $\begin{vmatrix} 4 & 3 & 1 \\ 1 & -5 & 2 \\ -5 & -1 & 9 \end{vmatrix}$

2 Factorise each of these determinants.

a) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$
b) $\begin{vmatrix} 3p & 3q & 3r \\ 2p & 2q & r \\ 5p & -3q & 2r \end{vmatrix}$
c) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c3 \end{vmatrix}$

d) $\begin{vmatrix} 0 & x - y & x^2 - y^2 \\ x - y & x & x^2 + 2xy + y^2 \\ y - x & y & 0 \end{vmatrix}$

3 Express the determinant

$$D = \begin{vmatrix} a^3 + a^2 & a & 1 \\ b^3 + b^2 & b & 1 \\ c^3 + c^2 & c & 1 \end{vmatrix}$$

as the product of four linear factors.

Given that no two of a, b and c are equal and that $D = 0$, find the value of $a + b + c$. (NEAB)

4 Show that

$$\det \begin{pmatrix} 2 & 2k & 1 \\ 1 & k - 1 & 1 \\ 2 & 1 & k + 1 \end{pmatrix}$$

has the same value for all values of k. (SQA/CSYS)

Solution of three equations in three unknowns

Consider the three equations

$$a_1 x + b_1 y + c_1 z + d_1 = 0$$
$$a_2 x + b_2 y + c_2 z + d_2 = 0$$
$$a_3 x + b_3 y + c_3 z + d_3 = 0$$

It can be shown by algebraic elimination that their general solution is given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = -\frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Note the following five important facts:

- The determinant under x does not include any of the x-coefficients.
- The determinant under y does not include any of the y-coefficients.
- The determinant under z does not include any of the z-coefficients.
- The y-fraction and the unit fraction carry a minus sign. (The minus sign alternates as in the expansion of a determinant.)
- If one of the determinants is zero, the corresponding unknown is also zero.

  For example,

  $$\text{if } \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} = 0, \text{ then } x = 0$$

From the equations above, we have

$$x = -\frac{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Hence, these three equations **have a unique solution unless** $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Conversely, they **do not have a unique solution if** $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

## Geometric interpretation of three equations in three unknowns

Each of the equations $a_i x + b_i y + c_i z + d_i = 0$ ($i = 1, 2, 3$) may be considered as the equation of a plane in three-dimensional space.

With three planes, there are seven possible configurations.

- The three planes intersect in a single point. In this case, the three equations have a **unique solution**.
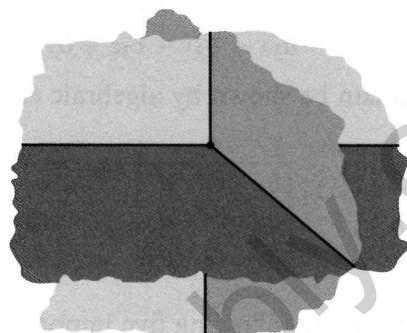
- The three planes form a triangular prism. In this case, there is no point where all three planes intersect. Hence, the equations are said to be **inconsistent**, as they have no solutions.
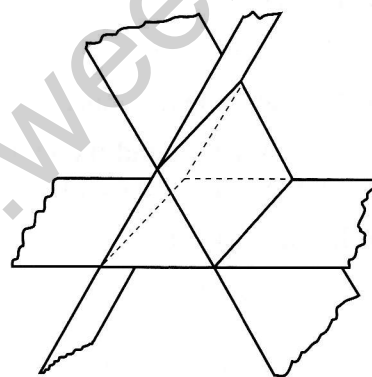
- Two of the planes are parallel and separate, and are intersected by the third plane. Again, there is no point where all three planes intersect, and so the equations are inconsistent in this case, too.
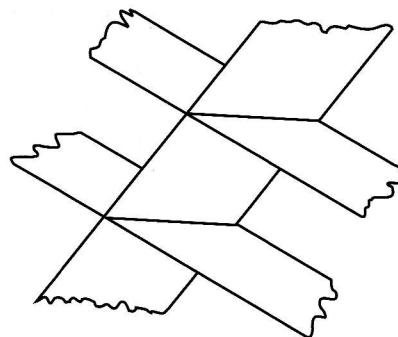
- Two other configurations in which the planes have no common point and therefore their equations are inconsistent are:

  - All three planes are parallel and separate.
  - Two of the planes are coincident and the third plane is parallel but separate.

  The two remaining configurations correspond to the three equations having infinitely many solutions.

- The three planes have a common line, giving an infinite number of points (x, y, z) which satisfy all three equations. In this case, the equations are said to be **linearly dependent**, and the configuration is called a **sheaf of planes** or a **pencil of planes**.
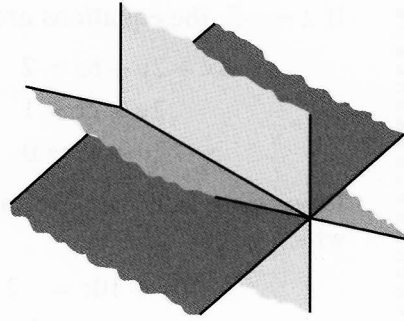
- All three planes coincide, giving an infinite number of points which satisfy all three equations.

Example 8 How many solutions are there to these three equations?

$$4x - \lambda y + 6z = 2$$
$$2y + \lambda z = 1$$
$$x - 2y + 4z = 0$$

SOLUTION

First, we find the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 4 & -\lambda & 6 \\ 0 & 2 & \lambda \\ 1 & -2 & 4 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & \lambda \\ -2 & 4 \end{vmatrix} + \lambda \begin{vmatrix} 0 & \lambda \\ 1 & 4 \end{vmatrix} + 6 \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 4(8 + 2\lambda) + \lambda(-\lambda) + 6(-2)$$

$$= -\lambda^2 + 8\lambda + 20$$

Therefore, there is a unique solution unless

$$-\lambda^2 + 8\lambda + 20 = 0$$
$$\Rightarrow \quad \lambda^2 - 8\lambda - 20 = 0$$
$$\Rightarrow \quad (\lambda - 10)(\lambda + 2) = 0$$

That is, there is a unique solution unless $\lambda = 10$ or $\lambda = -2$.

If $\lambda = 10$, the equations are

$$4x - 10y + 6z = 2 \qquad [1]$$
$$2y + 10z = 1 \qquad [2]$$
$$x - 2y + 4z = 0 \qquad [3]$$

We now use equations [1] and [3] to get the same expression as that on the left-hand side of equation [2]. Subtracting $4 \times$ equation [3] from equation [1], we have

$$-2y - 10z = 2$$
$$\Rightarrow \quad 2y + 10z = -2$$

This contradicts equation [2], and so the equations have no solution. That is, the three equations are **inconsistent**.

If $\lambda = -2$, the equations are

$$4x + 2y + 6z = 2 \qquad [4]$$
$$2y - 2z = 1 \qquad [5]$$
$$x - 2y + 4z = 0 \qquad [6]$$

Proceeding as before, we subtract equation [4] from $4 \times$ equation [6], which gives

$$-10y + 10z = -2$$
$$\Rightarrow \quad 2y - 2z = \frac{1}{5}$$

This contradicts equation [5], and so the equations have no solution. That is, the three equations are inconsistent.

Example 9 Solve the equations

$$2x - 3y + 4z = 1 \qquad [1]$$
$$3x - y = 2 \qquad [2]$$
$$x + 2y - 4z = 1 \qquad [3]$$

SOLUTION

First, we calculate the determinant $\begin{vmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & -4 \end{vmatrix}$ and find that its value is zero.

Therefore, there is not a unique solution to the three equations, and so we cannot use the general formula for the solution of three equations.

Adding equations [1] and [3], we obtain

$$3x - y = 2$$

which is equation [2].

Since one equation is a combination of the other two, the equations are said to be **linearly dependent**.

We cannot find a unique solution for two equations in three unknowns.

To solve these equations, we let $x = t$. Hence, x is no longer an unknown. We thereby have only two unknowns in these two equations, and so we can solve them.

Using equation [2], we obtain $y = 3t - 2$. Substituting this in equation [3], we get

$$4z = t + 2(3t - 2) - 1$$
$$\Rightarrow \quad z = \frac{7t - 5}{4}$$

So, the solution is $\left(t, 3t - 2, \dfrac{7t - 5}{4} \right)$.

Each value of the parameter t gives a different point. Since there is only one parameter, this solution represents a line.

Exercise 5B

1 Express the determinant

$$\begin{vmatrix} a & bc & b+c \\ b & ca & c+a \\ c & ab & a+b \end{vmatrix}$$

as the product of four linear factors.

Hence, or otherwise, find the values of a for which the simultaneous equations

$$ax + 2y + 3z = 0$$
$$2x + ay + (1+a)z = 0$$
$$x + 2ay + (2+a)z = 0$$

have a solution other than $x = y = z = 0$.

Solve the equations when $a = -3$.　(NEAB)

2 Consider the following system of simultaneous equations

$$x - y + 2z = 6$$
$$2x + 3y - z = 7$$
$$x + 9y - 8z = -4$$

i) By evaluating an appropriate determinant, show that this system does not have a unique solution.

ii) Solve this system of simultaneous equations.　(NICCEA)

3 Consider the system of simultaneous equations

$$3x + y - 2z = -4$$
$$x + 2y + 3z = 11$$
$$3x - 4y - 13z = -41$$

i) Solve this system of equations.

ii) Hence show in a sketch how the planes defined by the above equations are arranged so that the solution is of the form found in part **i**.　(NICCEA)

4 Show that the equations

$$x + \lambda y + z = 2a$$
$$x + y + \lambda z = 2b$$
$$\lambda x + y + \lambda z = 2c$$

where $a, b, c \in \mathbb{R}$, have a unique solution for x, y, z provided that $\lambda \neq 1$ and $\lambda \neq -1$.

a) In the case when $\lambda = 1$, state the condition to be satisfied by a, b and c for the equations to be consistent.

b) In the case when $\lambda = -1$, show that for the equations to be consistent

$$a + c = 0$$

Solve the equations in this case.

Give a geometrical description of the configuration of the three planes represented by the equations in the cases:

i) $\lambda = -1$ and $a + c = 0$.

ii) $\lambda = -1$ and $a + c \neq 0$.　(NEAB)

5 Find the values of k for which the simultaneous equations

$$kx + 2y + z = 0$$
$$3x - 2z = 4$$
$$3x - 6ky - 4z = 14$$

do not have a unique solution for x, y and z.

Show that, when $k = -2$, the equations are inconsistent, and give a geometrical interpretation of the situation in this case. (OCR)

6 Show that if $a \neq 3$ then the system of equations

$$x + 3y + 4z = -5$$
$$2x + 5y - z = 5a$$
$$3x + 8y + az = b$$

has a unique solution.

Given that $a = 3$, find the value of b for which the equations are consistent. (OCR)

7 Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & -1 \end{pmatrix}$$

find $\det \mathbf{M}$ in terms of k.

Determine the values of k for which the simultaneous equations

$$x + y - z = 1$$
$$x + 2y - kz = 0$$
$$x - ky - z = 1$$

have a unique solution.

i) Solve these equations in the case when $k = 2$.
ii) Show that the equations have no solution when $k = 1$.
iii) Find the general solution when $k = -1$.

Give a geometrical interpretation of the equations in each of the three cases $k = 2$, $k = 1$ and $k = -1$. (NEAB)

8 a) Express the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

as the product of four linear factors.
b) Two points, A and B, have coordinates $(1, 2, 8)$ and $(1, 3, 27)$, respectively. A third point, C, which is distinct from A and B, has coordinates $(1, c, c^3)$. Given that the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly dependent, find the value of c. (NEAB)

9 i) Show that the system of equations

$$x + 4y + 12z = 5$$
$$x + ay + 6z = a - 0.5$$
$$3x + 12y + 4az = b - 3$$

has a unique solution provided $a \neq 4$ and $a \neq 9$

ii) Find the solution in the case where $a = 3$ and $b = 42$.

iii) Show that when $a = 9$ the equations do not have a solution unless $b = 18$.

iv) Give a geometrical interpretation of the system in the case where $a = 9$ and $b = 13$.

(OCR)

10 It is given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & p & -3 \\ 1 & -1 & q \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -5 \\ 13 \end{pmatrix}$$

i) Find the determinant of $\mathbf{A}$ in terms of p and q.

ii) Hence show that if $p \neq -1$ and $q \neq 2$ then the system of equations defined by $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

iii) Show that if $p = -1$ then the system does not have a solution unless q has a particular value, q_1, which is to be found.

iv) Give a geometrical interpretation of the system in the case where $p = -1$ and $q = q_1$.

(OCR)

11 Show that the only real value of λ for which the simultaneous equations

$$(2 + \lambda)x - y + z = 0$$
$$x - 2\lambda y - z = 0$$
$$4x - y - (\lambda - 1)z = 0$$

have a solution other than $x = y = z = 0$ is -1.

Solve the equations in the case when $\lambda = -1$, and interpret your result geometrically. (NEAB)

12 Consider the system of equations x, y and z,

$$2x + 3y - z = p$$
$$x - 2z = -5$$
$$qx + 9y + 5z = 8$$

where p and q are real.

Find the values of p and q for which this system has:

i) a unique solution

ii) an infinite number of solutions

iii) no solution. (NICCEA)

6 Vector geometry

The Great Bear is looking so geometrical.
One would think that something or other could be proved.
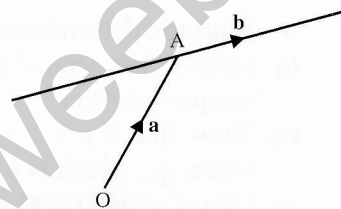CHRISTOPHER FRY

Vector equation of a line

In *Introducing Pure Mathematics* (page 506), we found the vector equation of a line, AB.

From this, it follows that the general equation of a line through the point A and in the direction of **b** is

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

where **a** is the position vector of A, and each value of the parameter t corresponds to a point on the line.

Example 1

a) Find the equation of the line through the point $(2, 4, 5)$ in the direction $-2\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$.

b) Find p and q so that the point $(p, 10, q)$ lies on this line.

SOLUTION

a) The equation of the line is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}$

b) If the point $(p, 10, q)$ lies on the line, then for some t we have

$$\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} p \\ 10 \\ q \end{pmatrix}$$

Considering these coordinates, we have

For **i**: $2 - 2t = p$ [1]
For **j**: $4 + 3t = 10$ [2]
For **k**: $5 + 8t = q$ [3]

From [2], we get $t = 2$.

Substituting $t = 2$ in [1] and [3], we get

$$p = -2 \quad \text{and} \quad q = 21$$

Cartesian equation of a line

To find the three-dimensional cartesian equation of a line which passes through the point (x_1, y_1, z_1) in the direction $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$, we use the vector equation

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

Hence, we obtain the vector equation of this line as

$$\mathbf{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

Let the general vector $\mathbf{r}$ be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

Using the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components, we have

$$x = x_1 + tl$$
$$y = y_1 + tm$$
$$z = z_1 + tn$$

Finding t from each of these equations, we get

$$t = \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Hence, the three-dimensional cartesian equation of a straight line which passes through the point (x_1, y_1, z_1) in the direction $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Example 2 Find the cartesian equation of the line PQ, where P is $(2, 1, 7)$ and Q is $(3, 8, 4)$.

SOLUTION

Let $\mathbf{p}$ and $\mathbf{q}$ be the position vectors of P and Q respectively. Then the direction of the line PQ is

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 3 \\ 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$$

Hence, given that the line passes through P(2, 1, 7), its **vector** equation is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$$

Therefore, its cartesian equation is

$$\frac{x-2}{1} = \frac{y-1}{7} = \frac{z-7}{-3}$$

Note In Example 2, we could have used Q as the point on the line, in which case we would have obtained

$$\mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$$

leading to

$$\frac{x-3}{1} = \frac{y-8}{7} = \frac{z-4}{-3}$$

Example 3 For the line through (4, 7, −1) in the direction $2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, find

a) its vector equation

b) its cartesian equation.

SOLUTION

a) The vector equation is

$$\mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$$

b) The cartesian equation is

$$\frac{x-4}{2} = \frac{y-7}{-3} = \frac{z+1}{-5}$$

which could be written as

$$\frac{x-4}{2} = \frac{7-y}{3} = \frac{-(1+z)}{5}$$

Example 4 Find the vector equation of the line

$$\frac{x-3}{4} = \frac{1-y}{2} = \frac{2z+7}{5}$$

SOLUTION

We **always start** by rearranging the cartesian equation in the form

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

which in this case gives

$$\frac{x-3}{4} = \frac{y-1}{-2} = \frac{z+\frac{7}{2}}{\frac{5}{2}}$$

Therefore, the vector equation of the line is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -\frac{7}{2} \end{pmatrix} + t \begin{pmatrix} 4 \\ -2 \\ \frac{5}{2} \end{pmatrix}$$

Note

- The direction of a line is normally expressed in terms of integers. Hence, the vector equation in Example 4 would be given as

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -\frac{7}{2} \end{pmatrix} + s \begin{pmatrix} 8 \\ -4 \\ 5 \end{pmatrix}$$

where $s = \dfrac{t}{2}$ is also a parameter.

- It is neater to use a point on the line with integer coordinates, whereby this equation could be given as

$$\mathbf{r} = \begin{pmatrix} 7 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -4 \\ 5 \end{pmatrix}$$

However, this further manipulation is not required in A-level examinations.

Example 5 Find the angle between the two lines

$$\frac{x-3}{4} = \frac{y-5}{2} = \frac{z-8}{-1} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + t \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix}$$

SOLUTION

The required angle is between the **directions** of the two lines, which are

$$\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix}$$

Using the scalar product in the form $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \, |\mathbf{b}|}$, where θ is the required angle, we have

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix}}{\sqrt{4^2 + 2^2 + (-1)^2} \times \sqrt{(-7)^2 + 4^2 + 3^2}}$$

$$= \frac{-28 + 8 - 3}{\sqrt{21} \times \sqrt{74}} = -\frac{23}{\sqrt{1554}}$$

The minus sign indicates that the angle between the two directions is obtuse. However, the angle between two lines would normally be taken to be acute. Therefore, the angle between the two lines is

$$\cos^{-1}\left(\frac{23}{\sqrt{1554}}\right) = 54.3° \quad \text{or} \quad 0.95 \text{ radians}$$

Note The scalar product of two vectors **a** and **b** is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**. (See *Introducing Pure Mathematics*, pages 502–4, where examples are given of its application.)

Resolved part of a vector

We may consider the vector **a** to be composed of two parts: one in the direction of a vector **b**, and the other perpendicular to the direction of vector **b**.

In the diagram on the right, we have

$$\overrightarrow{OA} = \overrightarrow{OT} + \overrightarrow{TA}$$

The magnitude of $\overrightarrow{OT}$ is the **resolved part** of vector **a** in the direction of vector **b**. That is,
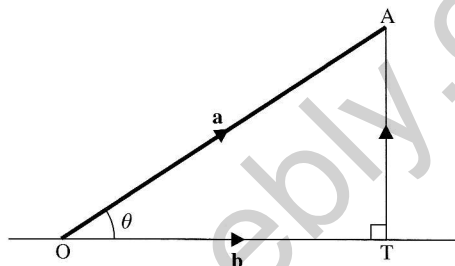
$$OT = a\cos\theta$$

Using the scalar product $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, we have

$$\frac{\mathbf{a} \cdot \mathbf{b}}{b} = a\cos\theta = OT$$

Therefore, the resolved part of vector **a** in the direction of vector **b** is $\dfrac{\mathbf{a} \cdot \mathbf{b}}{b}$.

Note The resolved part is a **scalar**. The vector $\overrightarrow{OT}$ is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{b}\frac{\mathbf{b}}{b} = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{b^2}$$

Example 6 Find the resolved part of the vector of $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ in the direction of $3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$.

SOLUTION

The resolved part is

$$\frac{\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}}{\sqrt{3^2 + 1^2 + (-7)^2}} = \frac{6 - 3 - 42}{\sqrt{9 + 1 + 49}} = -\frac{39}{\sqrt{59}}$$

Direction ratios

When one vector is a scalar multiple of another vector, the two vectors are parallel. For example, vector $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ is parallel to vector $\mathbf{b} = \begin{pmatrix} -15 \\ -20 \\ 25 \end{pmatrix}$ since $\mathbf{b} = -5\mathbf{a}$.

The direction of a vector is specified by the ratios of the components in the **i**, **j**

and **k** directions. These are called the **direction ratios** of the vector, and are normally expressed as integers.

For example, the direction ratios of the vector $28\mathbf{i} - 21\mathbf{j} - 14\mathbf{k}$ are $4 : -3 : -2$. Usually, these would be changed to $-4 : 3 : 2$.

Note   Two lines which do not intersect and are not parallel are said to be **skew**.
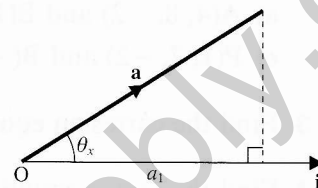
Direction cosines

The angle which vector **a** makes with the **i**-axis is given by $\cos^{-1}\left(\dfrac{a_1}{a}\right)$, where a is the magnitude of vector **a**, and a_1 is the component of **a** in the **i**-direction. If θ_x is the angle which vector **a** makes with the **i**-axis, we have

$$\cos\theta_x = \frac{a_1}{a}$$

Likewise for θ_y and θ_z, we have

$$\cos\theta_y = \frac{a_2}{a} \quad \text{and} \quad \cos\theta_z = \frac{a_3}{a}$$

These three values, $\dfrac{a_1}{a}$, $\dfrac{a_2}{a}$ and $\dfrac{a_3}{a}$, are known as the **direction cosines** of vector **a**. They represent another way of specifying the vector's direction.

Example 7 Find the direction cosines of the vector $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and find the angle which the vector makes with the z-axis.

SOLUTION

The direction cosines are given by $\dfrac{a_1}{a}$, $\dfrac{a_2}{a}$ and $\dfrac{a_3}{a}$, where $a_1 = 3$, $a_2 = -4$, $a_3 = 5$ and a represents the magnitude of the vector.

The magnitude of $\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ is

$$\sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

Hence, the direction cosines are respectively

$$\frac{3}{5\sqrt{2}} \qquad -\frac{4}{5\sqrt{2}} \qquad \frac{5}{5\sqrt{2}}$$

If θ is the angle which the vector makes with the z-axis, we have

$$\cos\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Therefore, the angle which the vector makes with the z-axis is $\dfrac{\pi}{4}$.

Exercise 6A

1 Find the vector equation of the line

    a) through A(2, −7, 5) in the direction $3\mathbf{i} + 4\mathbf{j} − 7\mathbf{k}$
    b) through B(4, 8, −6) in the direction $−2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$
    c) through P(7, 4, −1) in the direction $2\mathbf{i} − \mathbf{j} − 3\mathbf{k}$
    d) through Q(−8, 1, −3) in the direction $\mathbf{i} + 3\mathbf{j} − 7\mathbf{k}$

2 Find the vector equation of the line through each pair of points.

    a) A(4, 8, −2) and B(1, −3, 4)   **b)** C(−1, 8, 3) and D(2, −3, 9)
    c) P(1, 7, −2) and B(−3, 4, 8)   **d)** R(3, −5, −9) and S(−2, −3, 7)

3 Find the cartesian equation of each line in Question **1**.

4 Find the vector equation of each of these lines.

    a) $\dfrac{x − 3}{4} = \dfrac{y + 2}{3} = \dfrac{z − 4}{−5}$ **b)** $\dfrac{x + 2}{5} = \dfrac{y − 1}{−7} = \dfrac{z + 3}{−2}$

    c) $\dfrac{x + 5}{1} = \dfrac{2 − y}{3} = \dfrac{z + 4}{2}$ **d)** $\dfrac{2x − 3}{4} = \dfrac{y − 5}{3} = \dfrac{2 − z}{1}$

    e) $\dfrac{3x − 5}{6} = \dfrac{y + 2}{4} = \dfrac{2 − z}{3}$

5 Find the acute angles between the lines with equations

    a) $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} − 7\mathbf{k} + t(2\mathbf{i} − \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = −2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} + 5\mathbf{j} − 3\mathbf{k})$

    b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 7 \\ −2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} −2 \\ 3 \\ 11 \end{pmatrix} + s\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

6 Find the equation of the line AB where:

    a) A is (2, 1, 4) and B is (4, 7, 5).
    b) A is (−1, −4, 3) and B is (2, 8, 4).
    c) A is (4, 1, −5) and B is (3, 2, −6).

7 Find the resolved part of $3\mathbf{i} − \mathbf{j} + 2\mathbf{k}$ in the direction $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

8 Find the resolved part of $4\mathbf{i} + 5\mathbf{j} − 2\mathbf{k}$ in the direction $\mathbf{j} − 7\mathbf{k}$.

9 Give for each vector **i)** its direction ratios and **ii)** its direction cosines.

    a) $6\mathbf{i} + 12\mathbf{j} − 12\mathbf{k}$ **b)** $3\mathbf{i} − 4\mathbf{j} − 5\mathbf{k}$
    c) $12\mathbf{i} + 8\mathbf{j} − 20\mathbf{k}$ **d)** $9\mathbf{i} − 18\mathbf{j} − 27\mathbf{k}$

10 Referred to a fixed origin O, the points P, Q and R have position vectors $(2\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(5\mathbf{j} + 3\mathbf{k})$ and $(5\mathbf{i} − 4\mathbf{j} + 2\mathbf{k})$ respectively.

    a) Find in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation of the line PQ.
    b) Show that the point S with position vector $(4\mathbf{i} − 3\mathbf{j} − \mathbf{k})$ lies on PQ.
    c) Show that the lines PQ and RS are perpendicular.
    d) Find the size of ∠PQR, giving your answer to 0.1°.   (EDEXCEL)

11 The lines l_1, l_2 and l_3 are given by

$$l_1: \quad \mathbf{r} = 10\mathbf{i} + \mathbf{j} + 9\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$l_2: \quad x = \frac{y+9}{2} = \frac{z-13}{-3}$$

$$l_3: \quad \mathbf{r} = -3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k} + \lambda(4\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

where μ and λ are parameters.

a) Show that the point A(4, −1, 1) lies on both l_1 and l_2.

b) Rewrite the equation for l_2 in the form $\mathbf{r} = \mathbf{a} + v\mathbf{b}$, where v is a parameter.

c) Show that l_2 and l_3 intersect and find the coordinates of B, the point of intersection.

The lines l_1 and l_3 intersect at the point C(1, −2, −3).

d) Show that AC = BC.

e) Find the size of angle ACB, giving your answer in degrees to the nearest degree.

f) Write down the coordinates of the point D on AB such that CD is perpendicular to AB.

(EDEXCEL)

12 With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1: \quad \mathbf{r} = (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \lambda(-2\mathbf{i} + 4\mathbf{j} + \mathbf{k})$$

$$l_2: \quad \mathbf{r} = (-6\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 meet and find the position vector of their point of intersection.

b) Find, to the nearest 0.1°, the acute angle between l_1 and l_2.   (EDEXCEL)

13 The line l passes through the points with position vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 6\mathbf{j}$ relative to an origin O.

a) Find an equation for l in vector form.

The line m has equation $\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$.

b) Find the acute angle between l and m, giving your answer to the nearest degree.

(EDEXCEL)

14 Two lines have vector equations

$$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) + \lambda(4\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$

and

$$\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} - 13\mathbf{k}) + \mu(-3\mathbf{i} + \mathbf{j} + a\mathbf{k})$$

where λ and μ are scalar parameters and a is a constant.

Given that these two lines intersect, find the position vector of the point of intersection and the value of a.   (AEB 98)

15 With respect to an origin O, the position vectors of the points L and M are $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ respectively.

a) Write down the vector $\overrightarrow{LM}$.

b) Show that $|\overrightarrow{OL}| = |\overrightarrow{LM}|$.

c) Find $\angle OLM$, giving your answer to the nearest tenth of a degree.   (EDEXCEL)

16 Two lines have equations

$$\mathbf{r} = (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) + s(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = (-\mathbf{i} - \mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

  i) Show that the lines meet, and find the point of intersection.
  ii) Calculate the acute angle between the lines.   (OCR)

17 a) Find the angle between the vectors $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$, giving your answer in radians.
  b) The vectors $\mathbf{a}$ and $\mathbf{b}$ are non-zero.
    i) Given that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$, prove that $|\mathbf{a}| = |\mathbf{b}|$.
    ii) Given instead that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, prove that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.   (OCR)

18 The two lines

$$\frac{x+11}{4} = \frac{y+2}{1} = \frac{z+6}{-2} \quad \text{and} \quad \frac{x-6}{5} = \frac{y-5}{4} = \frac{z+20}{-8}$$

intersect. Find the coordinates of the point of intersection.   (OCR)

19 The points A and B have position vectors $7\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ and $4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ respectively, and O is the origin.

  i) Find, in vector form, an equation for the line passing through A and B.
  ii) Find the position vector of the point P on the line AB such that OP is perpendicular to AB.
  iii) Show that the line $\mathbf{r} = (8\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} - 10\mathbf{j} + 4\mathbf{k})$ does not intersect the line AB.

    (OCR)

Vector product

The product of two vectors can be formed in two distinct ways. One of these, the **scalar product**, we have already met in *Introducing Pure Mathematics* (page 502), and in the present book on pages 97 and 98. The other is called the **vector product** (or sometimes the **cross product**).

The **vector product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$, and is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\,\hat{\mathbf{n}}$$

where θ is the angle measured in the anticlockwise sense between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is a unit vector, such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ (in that order) form a right-handed set (see the diagram below).
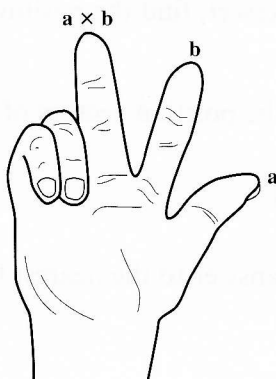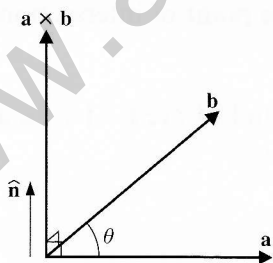
Some important properties of the vector product

The vector product is not commutative

Since $\mathbf{a} \times \mathbf{b} = ab \sin \theta \, \hat{\mathbf{n}}$, it follows that

$$\mathbf{b} \times \mathbf{a} = ab \sin (-\theta) \, \hat{\mathbf{n}} = -ab \sin \theta \, \hat{\mathbf{n}}$$

Therefore, we have

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

which is known as the **anticommutative rule**.

The vector product of parallel vectors is zero

The angle, θ, between two parallel vectors, $\mathbf{a}$ and $\mathbf{b}$, is $0°$ or $180°$. Therefore, $\sin \theta = 0$, which gives

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

0 is called the **zero vector**. It is usually represented by an ordinary zero, 0, as below.

Likewise, $\mathbf{a} \times \mathbf{a} = 0$, since the angle between $\mathbf{a}$ and $\mathbf{a}$ is zero. Hence, we have the following important result:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

Remember The scalar product $\mathbf{a} \cdot \mathbf{a} = a^2$.

The vector product of perpendicular vectors

Considering the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we have

$$\mathbf{i} \times \mathbf{j} = 1 \times 1 \sin 90° \, \hat{\mathbf{n}} = \hat{\mathbf{n}}$$

Therefore, $\mathbf{i}$, $\mathbf{j}$, $\hat{\mathbf{n}}$ form a right-handed set.

But, by definition, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a right-handed set. Therefore, $\hat{\mathbf{n}} = \mathbf{k}$. Hence, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

Similarly, we have

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
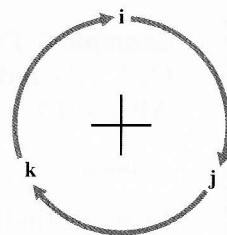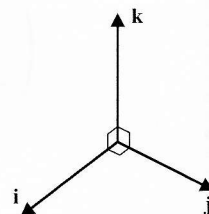
We notice (see diagram on the right) that these vector products are **positive** when the alphabetical order in which $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are taken is **clockwise**, but **negative** when this order is **anticlockwise**.

Remember For perpendicular vectors $\mathbf{a}$ and $\mathbf{b}$, the scalar product $\mathbf{a} \cdot \mathbf{b} = 0$.

The vector product in component form

Expressing $\mathbf{a}$ and $\mathbf{b}$ in their component form, we have $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Therefore,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times b_1\mathbf{i} + a_2\mathbf{j} \times b_1\mathbf{i} + a_3\mathbf{k} \times b_1\mathbf{i} + a_1\mathbf{i} \times b_2\mathbf{j} + a_2\mathbf{j} \times b_2\mathbf{j} + a_3\mathbf{k} \times b_2\mathbf{j} + \\
&\quad + a_1\mathbf{i} \times b_3\mathbf{k} + a_2\mathbf{j} \times b_3\mathbf{k} + a_3\mathbf{k} \times b_3\mathbf{k}
\end{aligned}$$

$$\Rightarrow \quad \mathbf{a} \times \mathbf{b} = a_1 b_2 \mathbf{k} - a_2 b_1 \mathbf{k} + a_3 b_1 \mathbf{j} - a_1 b_3 \mathbf{j} + a_2 b_3 \mathbf{i} - a_3 b_2 \mathbf{i}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - b_1 a_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \qquad [1]$$

From the definition of a 3×3 determinant on page 80, we obtain

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - b_1 a_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \qquad [2]$$

We note that the RHS of [1] and [2] are identical. Therefore, we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 8 Evaluate $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (7\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$.

SOLUTION

We use the result

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which gives

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 7 & 4 & 2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 7 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix}$$

Therefore, we have

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} = 10\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$$

Example 9 Evaluate $|\overrightarrow{AB} \times \overrightarrow{CD}|$, where A is $(6, -3, 0)$, B is $(3, -7, 1)$, C is $(3, 7, -1)$ and D is $(4, 5, -3)$. Hence find the shortest distance between AB and CD.

SOLUTION

First, we find $\overrightarrow{AB}$ and $\overrightarrow{CD}$:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Then, we find their vector product:

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -4 & 1 \\ 1 & -2 & -2 \end{vmatrix}$$

which gives

$$\overrightarrow{AB} \times \overrightarrow{CD} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$$

Therefore, we have

$$|\overrightarrow{AB} \times \overrightarrow{CD}| = \sqrt{10^2 + (-5)^2 + 10^2} = \sqrt{15}$$

The line which is the shortest distance between AB and CD is perpendicular to both AB and CD. So, if P and Q are general points on AB and CD respectively, and PQ is perpendicular to both AB and CD, we have

$$\overrightarrow{PQ} = k(\overrightarrow{AB} \times \overrightarrow{CD})$$

which gives

$$\begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right] = k\begin{pmatrix} 10 \\ -5 \\ 10 \end{pmatrix}$$

Hence, we have

$$2 - 3t - s = 10k$$
$$-8 - 4t + 2s = 5k$$
$$3 + t + 2s = 10k$$

Solving these simultaneous equations, we obtain $k = 0.4$, $s = 1$ and $t = -1$. Therefore, we have

$$\text{Shortest distance between AB and CD} = 0.4(\overrightarrow{AB} \times \overrightarrow{CD})$$
$$= 0.4 \times 15 = 6$$

Area of a triangle

Consider the triangle ABC whose sides are **a**, **b** and **c**, as shown in the diagram. From the definition of the vector product, we have

$$|\mathbf{a} \times \mathbf{b}| = |ab\sin\theta\,\hat{\mathbf{n}}|$$

where θ is the angle between **a** and **b**.

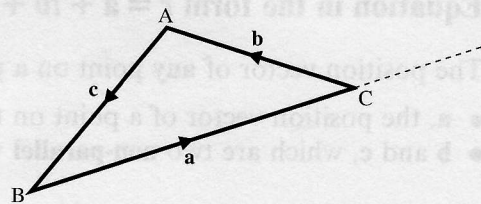However, the angle between **a** and **b** is $180° - C$, and $\sin(180° - C) = \sin C$. Therefore, we obtain

$$|\mathbf{a} \times \mathbf{b}| = |ab\sin(180° - C)\,\hat{\mathbf{n}}| = |ab\sin C\,\hat{\mathbf{n}}|$$

Since $\hat{\mathbf{n}}$ is a unit vector, $|\mathbf{a} \times \mathbf{b}| = ab\sin C$. Hence, we have

$$\text{Area of triangle ABC} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

Similarly, we can show that the area of triangle ABC is given by

$$\tfrac{1}{2}bc\sin A = \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| \quad \text{and} \quad \tfrac{1}{2}ac\sin B = \tfrac{1}{2}|\mathbf{a} \times \mathbf{c}|$$

Generally, we have

$$\text{Area of a triangle} = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| \quad \text{or} \quad \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| \quad \text{or} \quad \tfrac{1}{2}|\mathbf{a} \times \mathbf{c}|$$

Example 10 Find the area of triangle PQR where P is (4, 2, 5), Q is (3, −1, 6) and R is (1, 4, 2).

SOLUTION

First, we find any two sides (see diagram):

$$\overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$
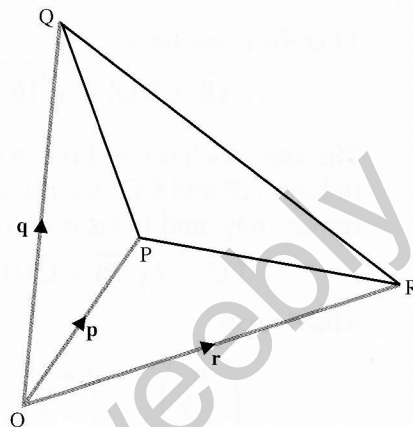
Then, we find their vector product:

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -3 \\ -1 & -3 & 1 \end{vmatrix}$$

which gives

$$\overrightarrow{PR} \times \overrightarrow{PQ} = -7\mathbf{i} + 6\mathbf{j} + 11\mathbf{k}$$
$$\Rightarrow \quad |\overrightarrow{PR} \times \overrightarrow{PQ}| = \sqrt{49 + 36 + 121} = \sqrt{206}$$

Therefore, we have

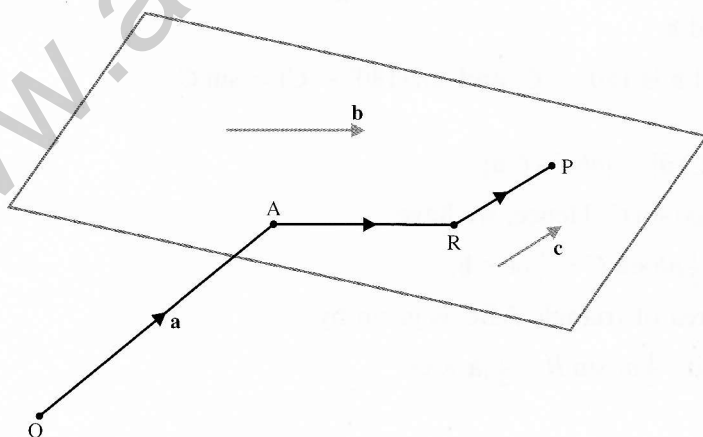$$\text{Area of triangle PQR} = \tfrac{1}{2}\sqrt{206}$$

Equation of a plane

Equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b} + s\mathbf{c}$

The position vector of **any** point on a plane can be expressed in terms of:

- **a**, the position vector of a point on the plane, and
- **b** and **c**, which are two **non-parallel** vectors **in** the plane.

From the diagram, we see that the position vector of a point P on the plane is given by

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AR} + \overrightarrow{RP}$$

where $\overrightarrow{AR}$ is parallel to vector **b**, and $\overrightarrow{RP}$ is parallel to vector **c**.

Hence, $\overrightarrow{AR} = t\mathbf{b}$ and $\overrightarrow{RP} = s\mathbf{c}$, for some parameters t and s.

The vector equation of a plane through point A is, therefore,

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} + s\mathbf{c}$$

where **b** and **c** are non-parallel vectors in the plane, and t and s are parameters.

Equation in the form $\mathbf{r} \cdot \mathbf{n} = d$

Given **n** is a vector perpendicular to the plane, we have

$$\mathbf{r} \cdot \mathbf{n} = (\mathbf{a} + t\mathbf{b} + s\mathbf{c}) \cdot \mathbf{n}$$
$$= \mathbf{a} \cdot \mathbf{n} + t\mathbf{b} \cdot \mathbf{n} + s\mathbf{c} \cdot \mathbf{n}$$

Since **b** and **c** are perpendicular to **n**, $\mathbf{b} \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{n} = 0$. Hence, we have

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Therefore the vector equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = d$$

where d is a constant which determines the position of the plane.

Note

- If **n** is a unit vector, then d is the perpendicular distance of the plane from the origin.

- When d has the **same sign** for two planes, these planes are on the **same side** of the origin.

- When d has **opposite signs** for two planes, these planes are on **opposite sides** of the origin.

Cartesian form

In a similar way to finding the cartesian equation of a line, we take $\mathbf{r} \cdot \mathbf{n} = d$ and replace **r** by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, which gives the equation of a plane as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \mathbf{n} = d$$

Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the cartesian equation becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \quad \text{or} \quad ax + by + cz = d$$

Example 11 Find the equation of the plane through (3, 2, 7) which is perpendicular to the vector $\begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix}$, giving its equation **a)** in vector form, and **b)** in cartesian form.

SOLUTION

a) Using $\mathbf{r \cdot n = a \cdot n}$, we have

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix}$$

Hence, the equation of the plane is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = 49$.

b) Replacing $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = 49$$

Therefore, the cartesian equation is $x - 5y + 8z = 49$.

Note A plane is identified by

- a vector perpendicular to the plane, and
- a point on the plane.

Example 12 Find the unit vector perpendicular to the plane $2x + 3y - 7z = 11$.

SOLUTION

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the plane $ax + by + cz = d$.

Therefore, the vector perpendicular to the given plane is $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$.

The magnitude of this vector is $\sqrt{2^2 + 3^2 + (-7)^2} = \sqrt{62}$.

Now, the unit vector perpendicular to the given plane must be of magnitude 1.

Therefore, the unit vector perpendicular to the given plane is $\begin{pmatrix} \dfrac{2}{\sqrt{62}} \\ \dfrac{3}{\sqrt{62}} \\ -\dfrac{7}{\sqrt{62}} \end{pmatrix}$.

Example 13 Find the equation of a plane through A(1, 4, 6), B(2, 7, 5) and C(−3, 8, 7).

SOLUTION

First, we find two vectors in the plane ABC: for example,

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} -3 \\ 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$$

Result 1 To find the equation of the plane in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b} + s\mathbf{c}$, we need to identify **one** point on the plane.

If we choose A(1, 4, 6), the equation of the plane ABC is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$$

Note

• Instead of choosing A, we could have chosen B(2, 7, 5) or C(−3, 8, 7).

• Instead of $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, we could have used $\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$.

• Instead of $\begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$, we could have used $\begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}$.

Result 2 To find the equation of the plane in the form $\mathbf{r} \cdot \mathbf{n} = d$, we need to find a vector perpendicular to the plane ABC.

A vector perpendicular to the plane ABC is given by $\overrightarrow{AB} \times \overrightarrow{AC}$, or any similar vector product of two vectors **in** the plane. (This follows from the definition of the vector product, page 102.)

Therefore, we have

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ -4 & 4 & 1 \end{vmatrix}$$

which gives the vector perpendicular to the plane as $7\mathbf{i} + 3\mathbf{j} + 16\mathbf{k}$.

Hence, the vector equation of the plane ABC is

$$\mathbf{r} \cdot \begin{pmatrix} 7 \\ 3 \\ 16 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 3 \\ 16 \end{pmatrix} = 14 + 21 + 80$$

$$\Rightarrow \quad \mathbf{r} \cdot \begin{pmatrix} 7 \\ 3 \\ 16 \end{pmatrix} = 115$$

Therefore, the cartesian equation is $7x + 3y + 16z = 115$.

Example 14 Find the angle between the planes $3x + 4y + 5z = 7$ and $x + 2y - 2z = 11$.

SOLUTION

The angle between the planes is the angle between the vectors perpendicular to the planes. That is, the angle between the two vectors

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Using $\cos \theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{ab}$, where θ is the required angle, we have

$$\cos \theta = \frac{3 + 8 - 10}{5\sqrt{2} \times 3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{15\sqrt{2}}\right) = 87.3° \quad \text{(correct to 1 dp)}$$

Example 15 Find where the line from A(2, 7, 4), perpendicular to the plane Π, $3x - 5y + 2z + 2 = 0$, meets Π.

SOLUTION

Let T be the point where the line from A(2, 7, 4), perpendicular to the plane Π, $3x - 5y + 2z + 2 = 0$, meets Π.

The equation of AT is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

Hence, T is the point where $\mathbf{r} = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$ meets Π.

So, putting $x = (2 + 3t)$, $y = (7 - 5t)$ and $z = (4 + 2t)$ into the equation of the plane Π, we have

$$3(2 + 3t) - 5(7 - 5t) + 2(4 + 2t) + 2 = 0$$

$$\Rightarrow \quad 38t = 19 \quad \Rightarrow \quad t = \tfrac{1}{2}$$

Therefore, the point T is $(3\tfrac{1}{2}, 4\tfrac{1}{2}, 5)$

Example 16 Find the angle between the plane $3x + 4y - 5z = 6$ and the line

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

SOLUTION

The required angle is $90° - \theta$, where θ is the angle between the line and the vector perpendicular to the plane. That is,
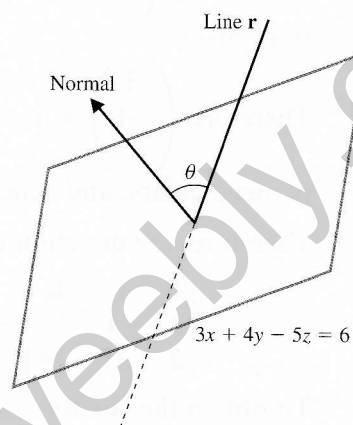
Required angle =

$$90° - \text{Angle between} \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

Using $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{ab}$, we have

$$\cos\theta = \frac{3 + 20 + 15}{5\sqrt{2} \times \sqrt{35}}$$

which gives

$$\text{Required angle} = 90° - \cos^{-1}\left(\frac{38}{\sqrt{35} \times 5\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{38}{5\sqrt{70}}\right) = 65.3° \quad \text{(correct to 1 dp)}$$

Example 17 Find the equation of the plane containing the two lines

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

SOLUTION

The two vectors in the plane are the directions of the two lines, which are

$$\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Therefore, the vector perpendicular to this plane is

$$\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -4 \\ 2 & -1 & 5 \end{vmatrix} = 11\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

Hence, the equation of the plane is $11x - 3y - 5z = d$.

From the first line, we know that the point $(3, 1, 2)$ is in the plane. So, we have $d = 11 \times 3 - 3 \times 1 - 5 \times 2 = 20$.

Therefore, the equation of the plane containing the two lines is $11x - 3y - 5z = 20$.

Example 18 Find the equation of the common line (line of intersection) of the two planes

$$\Pi_1, \; 3x - y - 5z = 7 \quad \text{and} \quad \Pi_2, \; 2x + 3y - 4z = -2$$

SOLUTION

The vectors $\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ are perpendicular to Π_1 and Π_2 respectively.

Therefore, $\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ is perpendicular to both of these

perpendiculars, and hence is in the direction of the common line.

Therefore, the direction of the common line is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -5 \\ 2 & 3 & -4 \end{vmatrix} = 19\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

To obtain the equation of the common line, we need to find a point on it.

Solving Π_1 and Π_2 will give only two equations to solve for three unknowns. So, we let $x = 0$ and solve the equations for the remaining two unknowns. However, if letting $x = 0$ causes problems because of the particular equations given, we may let either $y = 0$ or $z = 0$.

$$\Pi_1 \text{ is } 3x - y - 5z = 7 \qquad \text{When } x = 0, \Pi_1 \text{ gives } \quad -y - 5z = 7$$
$$\Pi_2 \text{ is } 2x + 3y - 4z = -2 \quad \text{When } x = 0, \Pi_2 \text{ gives } \quad 3y - 4z = -2$$

Solving these simultaneous equations, we find $z = -1$, $y = -2$.

Therefore, the point $(0, -2, -1)$ lies on the common line, giving its equation as

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 19 \\ 2 \\ 11 \end{pmatrix}$$

Distance of a plane from the origin

Consider the equation of a plane in the form $\mathbf{r} \cdot \mathbf{n} = d$.

If $\mathbf{n}$ is a unit vector (usually denoted by $\hat{\mathbf{n}}$), then d is the perpendicular distance of the plane from the origin.

Example 19 Find the distance to the plane $3x + 4y - 5z = 21$ from the origin.

SOLUTION

The equation of the plane is

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = 21$$