15 The point P in the Argand diagram represents the complex number z and the point Q represents the complex number w, where $w = \dfrac{1}{z+1}$.

i) Find w when

a) $z = -\mathrm{i}$ **b)** $z = \mathrm{i}$

expressing your answers in the form $u + \mathrm{i}v$.

ii) Find z in terms of w.

iii) Given that P lies on the circle with centre the origin and radius 1, prove that $|w| = |w - 1|$.

iv) Sketch the locus represented by $|w| = |w - 1|$. (OCR)

16 a) The point P in the complex plane represents the complex number z. Describe the locus of P in each of the following cases:

i) $|z - 2| = 1$ **ii)** $\arg(z - 2) = \dfrac{2\pi}{3}$

On the same diagram of the complex plane, draw each of the loci defined in parts **i)** and **ii)** above.

b) i) The point A in the complex plane represents the complex number $w = a + ib$ (where a and b are real), and is such that $|w - 2| = 1$ **and** $\arg(w - 2) = \dfrac{2\pi}{3}$. Determine the value of a and the value of b, giving each answer in an exact form.

ii) Write down the value of $\arg(w)$, and hence find the least positive integer n for which $\arg(w^n) > 2.5$. (AEB 98)

17 The complex numbers z_1 and z_2 are such that $z_1 = 1 + a\mathrm{i}$ and $z_2 = a + \mathrm{i}$, for some integer $a \geqslant 0$.

a) Given that $w = z_1 + z_2$, show that $|w| = (1 + a)\sqrt{2}$ and write down $\arg(w)$, the argument of w.

Hence find, in terms of a, the value of the complex number w^4.

b) In the case when $a = 2$, the complex numbers z_1 and z_2 are represented in the complex plane by the points P_1 and P_2 respectively.

Determine a cartesian equation of the locus of the point P, which represents the complex number z, given that $|z - z_1| = |z - z_2|$.

c) In the case when $a = 0$, the complex numbers z_1 and z_2 are represented in the complex plane by the points Q_1 and Q_2 respectively.

Describe fully, and sketch, the locus of the point Q, which represents the complex number z, given that $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{2}$. (AEB 97)

2 Further trigonometry with calculus

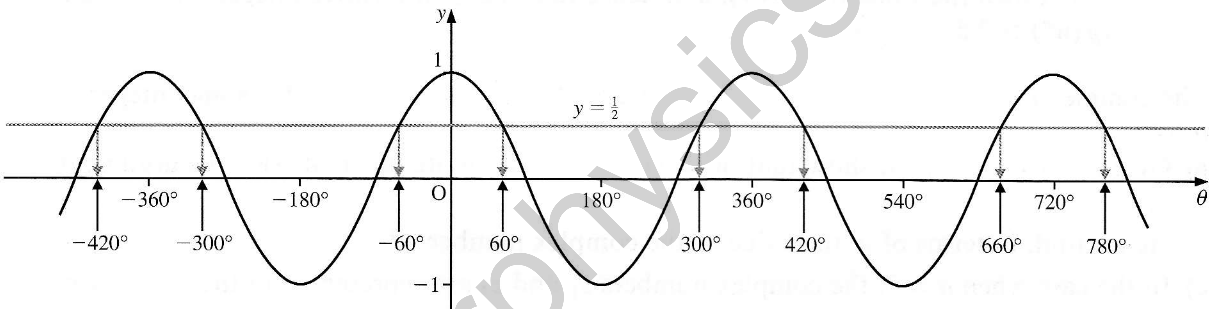
If the triangles were to make a God they would give him three sides.
MONTESQUIEU

General solutions of trigonometric equations

In *Introducing Pure Mathematics* (page 341), we solved the trigonometric equation $\cos\theta = \frac{1}{2}$ by obtaining the solution of 60° from a calculator and using the graphs of $y = \cos\theta$ and $y = \frac{1}{2}$ to obtain the other solutions. When we have several solutions to find, this method is very time-consuming and tends to induce errors.

The usual method of finding more than one solution of such trigonometric equations is to use the **general solution**.

General solutions for cosine curves



When $\cos\theta = \frac{1}{2}$, we will find from the graph of $y = \cos\theta$ (above) that the solutions for θ are

$$\ldots, -300^\circ, -60^\circ, 60^\circ, 300^\circ, 420^\circ, 660^\circ, 780^\circ, 1020^\circ, 1140^\circ, \ldots$$

or $360n^\circ \pm 60^\circ$ for any integer, n.

Hence, the general solution of $\cos\theta = \cos\alpha$ is given by

$$\theta = 360n^\circ \pm \alpha \quad \text{for any integer, } n$$

where θ and α are measured in degrees.

If θ and α are measured in radians, this general solution would be

$$\theta = 2n\pi \pm \alpha$$

Example 1 Find the values of θ from $0°$ to $720°$ for which $\cos\theta = \dfrac{1}{\sqrt{2}}$.

SOLUTION

The calculator gives $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ as $45°$. Hence, the first solution, or α, is $45°$.

Putting $\alpha = 45°$ into the general solution, $\theta = 360n° \pm \alpha$, we get the following solutions:

When $n = 0$ $\quad\theta = 45°$

When $n = 1$ $\quad\theta = 315°$ or $405°$

When $n = 2$ $\quad\theta = 675°$

Example 2 Find the values of θ from $0°$ to $360°$ for which $\cos 5\theta = \dfrac{\sqrt{3}}{2}$.

SOLUTION

After removing the cos term, we apply the general solution, using different values of n until we have a full range of solutions.

The calculator gives $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ as $30°$. Hence, the first solution, or α, is $30°$.

In this case, the general solution is an equation in $\boldsymbol{5\theta}$. So, with $\alpha = 30°$, we have

$$5\theta = 360n° \pm 30°$$
$$\Rightarrow \quad \theta = 72n° \pm 6°$$

Therefore, the solutions are as follows:

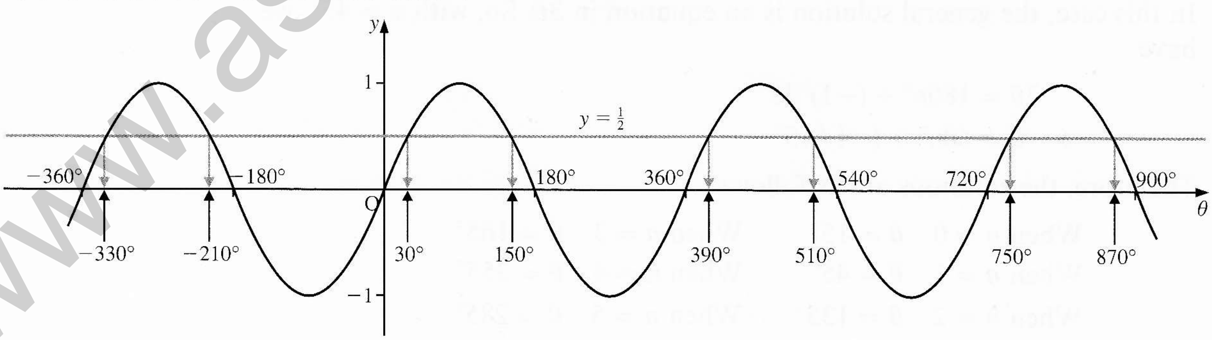
When $n = 0$ $\quad\theta = 6°$ $\qquad$ When $n = 3$ $\quad\theta = 210°$ or $222°$

When $n = 1$ $\quad\theta = 66°$ or $78°$ $\qquad$ When $n = 4$ $\quad\theta = 282°$ or $294°$

When $n = 2$ $\quad\theta = 138°$ or $150°$ $\qquad$ When $n = 5$ $\quad\theta = 354°$

Note We can always check these values on a graphics calculator, after having selected the correct **range** or **view window**.

General solutions for sine curves



When $\sin\theta = \frac{1}{2}$, we will find from the graph of $y = \sin\theta$ (above) that the solutions for θ are

$$\ldots, -330°, -210°, 30°, 150°, 390°, 510°, 750°, \ldots$$

which can be written as

$$\ldots, -360° + 30°, -180° - 30°, 30°, 180° - 30°, 360° + 30°, 540° - 30°, 720° + 30°, \ldots$$

Hence, the general solution of $\sin\theta = \sin\alpha$ is given by

$$\theta = 180n° + (-1)^n\alpha \quad \text{for any integer, } n$$

where θ and α are measured in degrees.

If θ and α are measured in radians, this general solution would be

$$\theta = n\pi + (-1)^n\alpha$$

Example 3 Find the values of θ between 0° and 720° for which $\sin\theta = \dfrac{\sqrt{3}}{2}$.

SOLUTION

The calculator gives $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ as 60°. Hence, the first solution, or α, is 60°.

From the general solution, $\theta = 180n° + (-1)^n\alpha$, we have

$$\theta = 180n° + (-1)^n 60°$$

Therefore, the solutions are as follows:

When $n = 0$ $\quad\theta = 60°$
When $n = 1$ $\quad\theta = 180° - 60° = 120°$
When $n = 2$ $\quad\theta = 360° + 60° = 420°$
When $n = 3$ $\quad\theta = 540° - 60° = 480°$
When $n = 4$ $\quad\theta = 720° + 60° = 780°$

But $\theta = 780°$ is out of the required range. Therefore, there are four solutions: $\theta = 60°, 120°, 420°$ and $480°$.

Example 4 Find the values of θ between 0° and 360° for which $\sin 3\theta = \dfrac{1}{\sqrt{2}}$.

SOLUTION

The calculator gives $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ as 45°. Hence, the first solution, or α, is 45°.

In this case, the general solution is an equation in $\boldsymbol{3\theta}$. So, with $\alpha = 45°$, we have

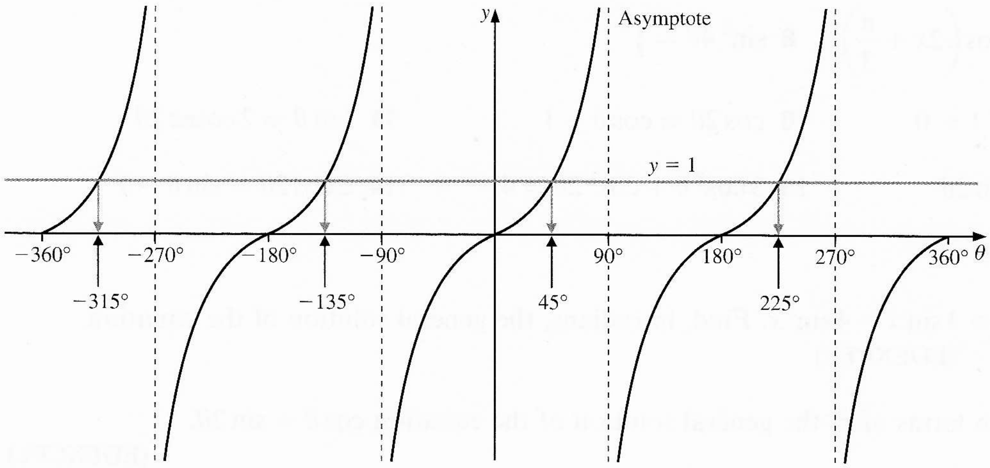
$$3\theta = 180n° + (-1)^n 45°$$
$$\Rightarrow \quad \theta = 60n° + (-1)^n 15°$$

Therefore, the solutions are as follows:

When $n = 0$ $\quad\theta = 15°$ $\qquad$ When $n = 3$ $\quad\theta = 165°$
When $n = 1$ $\quad\theta = 45°$ $\qquad$ When $n = 4$ $\quad\theta = 255°$
When $n = 2$ $\quad\theta = 135°$ $\qquad$ When $n = 5$ $\quad\theta = 285°$

That is, there are six solutions: $\theta = 15°, 45°, 135°, 165°, 255°$ and $285°$.

General solutions for tangent curves



When $\tan\theta = 1$, we will find from the graph of $y = \tan\theta$ (above) that the solutions for θ are

$$\ldots, -135°, 45°, 225°, 405°, \ldots$$

or $180n° + 45°$ for any integer, n.

Hence, the general solution of $\tan\theta = \tan\alpha$ is given by

$$\theta = 180n° + \alpha \quad \text{for any integer, } n$$

where θ and α are measured in degrees.

If θ and α are measured in radians, this general solution would be

$$\theta = n\pi + \alpha$$

Example 5 Find the values between 0° and 360° for which $\tan 4\theta = -\sqrt{3}$.

SOLUTION

The calculator gives $\tan^{-1}(-\sqrt{3})$ as $-60°$. Hence, the first solution, or α, is $-60°$.

In this case, the general solution is an equation in $\boldsymbol{4\theta}$. So, with $\alpha = -60°$, we have

$$4\theta = 180n° - 60°$$
$$\Rightarrow \quad \theta = 45n° - 15°$$

Therefore, the solutions are 30°, 75°, 120°, 165°, 210°, 255°, 300°, 345°.

Exercise 2A

In Questions **1** to **4** and **8** to **15**, find the general solution of each equation in **a)** radians, and **b)** degrees. In Questions **5** to **7**, find the general solution of each equation in radians only.

1 $\sin\theta = \dfrac{1}{\sqrt{2}}$ **2** $\cos\theta = -\frac{1}{2}$ **3** $\sin 2\theta = \frac{1}{2}$

4 $\tan 3\theta = 1$

5 $\sin\left(2x + \frac{\pi}{4}\right) = 1$

6 $\cos\left(3x - \frac{\pi}{3}\right) = \frac{1}{2}$

7 $\sin\left(2x + \frac{\pi}{3}\right) = \cos\left(2x + \frac{\pi}{3}\right)$

8 $\sin^2 4\theta = \frac{1}{2}$

9 $\sin^2 3\theta + \cos 3\theta + 1 = 0$

10 $\cos 2\theta = \cos\theta - 1$

11 $\tan\theta = 2\operatorname{cosec} 2\theta$

12 $\sin 5\theta - \sin\theta = \sin 2\theta$

13 $3\cos^2\theta + \cos^2 2\theta = 4$

14 $2\cos 2\theta = \sin\theta - 1$

15 $\sin 7\theta + \cos 3\theta = 0$

16 Show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. Find, in radians, the general solution of the equation $\sin 3x = 2\sin x$. (EDEXCEL)

17 Find, in radians in terms of π, the general solution of the equation $\cos\theta = \sin 2\theta$. (EDEXCEL)

18 Find the general solution of the equation $\cos 2x = \cos\left(x + \frac{\pi}{3}\right)$, giving your answer in terms of π. (EDEXCEL)

19 Given that $t = \tan x$, write down an expression for $\tan 2x$ in terms of t. Hence, or otherwise, find the general solution, in radians, of the equation $\tan x + \tan 2x = 0$. (AEB 97)

20 Show that the general solution of the equation $\tan\left(3x - \frac{\pi}{4}\right) = \tan x$ is $x = \frac{(4n+1)\pi}{8}$, where n is an integer. (NICCEA)

Harmonic form

As explained on page 374 of *Introducing Pure Mathematics*, the harmonic form is

$$R\cos(\theta \pm \alpha) \quad \text{or} \quad R\sin(\theta \pm \alpha)$$

where $R > 0$ is a constant.

Turning $a\cos\theta + b\sin\theta$ into $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

This is used when solving trigonometric equations, when finding the maximum and minimum of trigonometric expressions, and sometimes when solving problems in simple harmonic motion, where R is the amplitude of the motion.

- $\boldsymbol{a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)}$

Expanding $R\cos(\theta - \alpha)$, we obtain

$$a\cos\theta + b\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

Equating the coefficients of $\cos\theta$ gives: $a = R\cos\alpha$

Equating the coefficients of $\sin\theta$ gives: $b = R\sin\alpha$

Therefore, we have

$$\begin{aligned} a^2 + b^2 &= R^2\cos^2\alpha + R^2\sin^2\alpha \\ &= R^2(\cos^2\alpha + \sin^2\alpha) \\ &= R^2 \\ \Rightarrow \quad R &= \sqrt{a^2+b^2} \end{aligned}$$

Note that R is **always** taken to be **positive**.

To find α, we use

$$\cos\alpha = \frac{a}{R} \quad \text{and} \quad \sin\alpha = \frac{b}{R}$$

which give

$$\tan\alpha = \frac{b}{a}$$

but we need to take care when either a or b is **negative**.

When using $a\cos\theta$ or $b\sin\theta$, if either a or b is negative, always use

$$\cos\alpha = \frac{a}{R} \quad \textbf{and} \quad \sin\alpha = \frac{b}{R},$$

and ensure that both give the same value for α. If they do not, use the value of α which is **not** between 0° and 90°.

Example 6 Turn $3\cos\theta - 4\sin\theta$ into $R\cos(\theta - \alpha)$.

SOLUTION

We have

$$R = \sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$$

which gives

$$\cos\alpha = \frac{3}{5} \qquad \sin\alpha = -\frac{4}{5}$$

$$\alpha = 53.1^\circ \qquad \alpha = -53.1^\circ \quad \text{(from the calculator)}$$

Note -53.1° is also a solution of $\cos\alpha = \frac{3}{5}$, but a calculator always gives the angle between 0° and 90° wherever possible.

Therefore, we use $\alpha = -53.1^\circ$, as this is the value found from both $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = -\frac{4}{5}$ which is **not** between 0° and 90°.

Hence, we get

$$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^\circ)$$

Example 7 Find the general solution of $5\cos\theta - 12\sin\theta = 6.5$. Hence find the solutions which lie between 0° and 360°.

SOLUTION

Using $5\cos\theta - 12\sin\theta = R\cos(\theta+\alpha)$, we find

$$R = \sqrt{5^2+12^2} = 13$$

which gives

$$\cos\alpha = \frac{5}{13} \quad \Rightarrow \quad \alpha = 67.4^\circ$$

Therefore, we have

$$5\cos\theta - 12\sin\theta = 13\cos(\theta + 67.4^\circ)$$

So, $5\cos\theta - 12\sin\theta = 6.5$ becomes

$$13\cos(\theta + 67.4^\circ) = 6.5$$

$$\Rightarrow \quad \cos(\theta + 67.4^\circ) = \frac{6.5}{13} = 0.5$$

which gives

$$\theta + 67.4^\circ = 360n^\circ \pm 60^\circ$$

Therefore, the general solution is

$$\theta = 360n^\circ \pm 60^\circ - 67.4^\circ$$

When $n = 0$, both solutions are negative and are outside the required range. Therefore, the solutions required are 232.6° and 352.6° (when $n = 1$).

- **$a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$**

Expanding $R\sin(\theta + \alpha)$, we obtain

$$a\sin\theta + b\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

Equating the coefficients of $\sin\theta$ gives: $a = R\cos\alpha$

Equating the coefficients of $\cos\theta$ gives: $b = R\sin\alpha$

Therefore, we again have

$$R = \sqrt{a^2 + b^2}$$

$$\cos\alpha = \frac{a}{R} \quad \text{and} \quad \sin\alpha = \frac{b}{R}$$

Example 8 Turn $24\sin\theta + 7\cos\theta$ into $R\sin(\theta + \alpha)$.

SOLUTION

We have

$$R = \sqrt{24^2 + 7^2} = 25$$

which gives

$$\cos\alpha = \frac{24}{25} \quad \Rightarrow \quad \alpha = 16.3^\circ$$

Hence, we get

$$24\sin\theta + 7\cos\theta = 25\sin(\theta + 16.3^\circ)$$

Note To avoid the problem of possibly obtaining two different values for α, we select whichever one of $R\cos(\theta-\alpha)$, $R\cos(\theta+\alpha)$, $R\sin(\theta-\alpha)$ or $R\sin(\theta+\alpha)$ contains the same sign as the expression being simplified.

Thus, we would convert $3\sin\theta - 4\cos\theta$ into the form $R\sin(\theta-\alpha)$, which is the only trigonometric formula giving $a\sin\theta - b\cos\theta$. In this case, we have

$$R = \sqrt{3^2+4^2} = 5$$

$$\cos\alpha = \frac{3}{5} \quad\Rightarrow\quad \alpha = 53.1^\circ$$

which give

$$3\sin\theta - 4\cos\theta = 5\sin(\theta - 53.1^\circ)$$

Example 9 For each of

a) $f(x) = 24\cos\theta + 7\sin\theta$ **b)** $f(x) = \dfrac{1}{2+4\sin\theta - 3\cos\theta}$

find

i) the range of values for $f(x)$ **ii)** a maximum point **iii)** a minimum point

SOLUTION

a) i) Using $24\cos\theta + 7\sin\theta = R\cos(\theta-\alpha)$, we have

$$R = \sqrt{24^2+7^2} = 25$$

$$\cos\alpha = \frac{24}{25} \quad\Rightarrow\quad \alpha = 16.3^\circ$$

which give

$$24\cos\theta + 7\sin\theta = 25\cos(\theta - 16.3^\circ)$$

Now, $\cos\theta$ has a maximum of $+1$ and a minimum of -1. Therefore, the range of values of $\cos(\theta - 16.3^\circ)$ is -1 to $+1$, which gives the range of values of $25\cos(\theta - 16.3^\circ)$ as -25 to $+25$. That is,

$$-25 \leqslant f(x) \leqslant 25$$

ii) For the maximum point, we have $\cos(\theta - 16.3^\circ) = 1$. Therefore,

$$\theta - 16.3^\circ = 0 \quad\Rightarrow\quad \theta = 16.3^\circ$$

Hence, the maximum point is $(16.3^\circ, 25)$.

iii) For the minimum point, we have $\cos(\theta - 16.3^\circ) = -1$. Therefore,

$$\theta - 16.3^\circ = 180^\circ \quad\Rightarrow\quad \theta = 196.3^\circ$$

Hence, the minimum point is $(196.3^\circ, -25)$.

b) i)

$$f(x) = \frac{1}{2+4\sin\theta - 3\cos\theta}$$

We found in Example 6 that $3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^\circ)$. Therefore, we have

$$f(x) = \frac{1}{2 - 5\cos(\theta + 53.1^\circ)}$$

The **denominator** has a range from -3 to 7. (Remember that $1 \div 0 = \infty$.) Therefore, $f(x)$ has a range

$$f(x) \geqslant \tfrac{1}{7} \quad \text{and} \quad f(x) \leqslant -\tfrac{1}{3}$$

ii) The maximum point is found where $\cos(\theta + 53.1^\circ)$ is $+1$. That is,

$$\theta + 53.1^\circ = 360^\circ \quad \Rightarrow \quad \theta = 306.9^\circ$$

Therefore, the maximum point is $(306.9^\circ, -\tfrac{1}{3})$.

iii) The minimum point is found where $\cos(\theta + 53.1^\circ)$ is -1. That is,

$$\theta + 53.1^\circ = 180^\circ \quad \Rightarrow \quad \theta = 126.9^\circ$$

Therefore, the minimum point is $(126.9^\circ, \tfrac{1}{7})$.

Exercise 2B

1 Find the value of R and of α in each of the following identities.

a) $5\cos\theta + 12\sin\theta \equiv R\cos(\theta - \alpha)$

b) $3\cos\theta - 4\sin\theta \equiv R\cos(\theta + \alpha)$

c) $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$

d) $\cos 2\theta + \sin 2\theta \equiv R\cos(2\theta - \alpha)$

e) $6\sin 3\theta + 8\cos 3\theta \equiv R\sin(3\theta + \alpha)$

2 For each of the following expressions, find

i) the maximum and minimum values
ii) the smallest non-negative value of x for which this occurs.

Where necessary, give your answer correct to one decimal place.

a) $12\cos\theta - 9\sin\theta$

b) $8\cos 2\theta + 6\sin 2\theta$

c) $\dfrac{4}{8 - 3\cos\theta - 4\sin\theta}$

d) $\dfrac{6}{8 + 4\sin\theta - 2\cos\theta}$

e) $\dfrac{2}{1 + 3\cos\theta + 4\sin\theta}$

f) $\dfrac{3}{8 + 8\sin\theta + 6\cos\theta}$

3 Find the general solution, in degrees, of each of these equations.

a) $3\cos\theta + 4\sin\theta = 2.5$

b) $12\cos\theta - 5\sin\theta = 6.5$

c) $\cos 2\theta + \sin 2\theta = \dfrac{1}{\sqrt{2}}$

d) $\sin 3\theta - \cos 3\theta = \dfrac{1}{\sqrt{2}}$

e) $2\sin 6\theta + 3\sin^2 3\theta = 0$

Inverse trigonometric functions

The inverse function $\sin^{-1}x$, or $\arcsin x$, is defined as the angle whose sine is x. For example,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \Rightarrow \quad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Hence, if $\theta = \sin^{-1}x$, then $\sin\theta = x$.

Sketching inverse trigonometric functions

Inverse sine graph

The graph of $y = \sin^{-1}x$ is obtained by reflecting the graph of $y = \sin x$ in the line $y = x$.

To enable the sketch to be drawn to an acceptable degree of accuracy, we need to find the gradient of the sine curve at the origin. So, we differentiate $y = \sin x$, which gives

$$\frac{dy}{dx} = \cos x$$

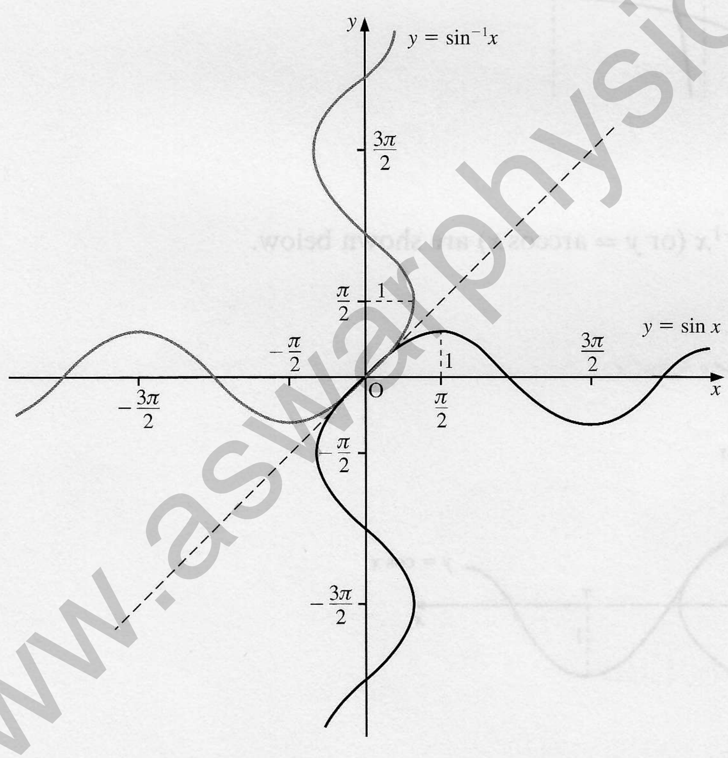
At the origin, where $x = 0$, we have

$$\frac{dy}{dx} = \cos 0 = 1$$

So, the gradient of $y = \sin x$ at the origin is 1.

We then proceed as follows:

- First, draw the line $y = x$. Show this as a dashed line.
- Next, carefully sketch the graph of $y = \sin x$, remembering that $y = x$ is a tangent to $y = \sin x$ at the origin.
- Finally, carefully sketch the reflection of $y = \sin x$ in the line $y = x$, to give the graph shown below.



The graphs of other inverse trigonometric functions are found similarly: that is, by reflecting the graph of the relevant trigonometric function in the line $y = x$. If the curve of the function passes through the origin, start by finding its gradient at that point.

Inverse tan graph

Differentiating $y = \tan x$ to find the gradient, we get

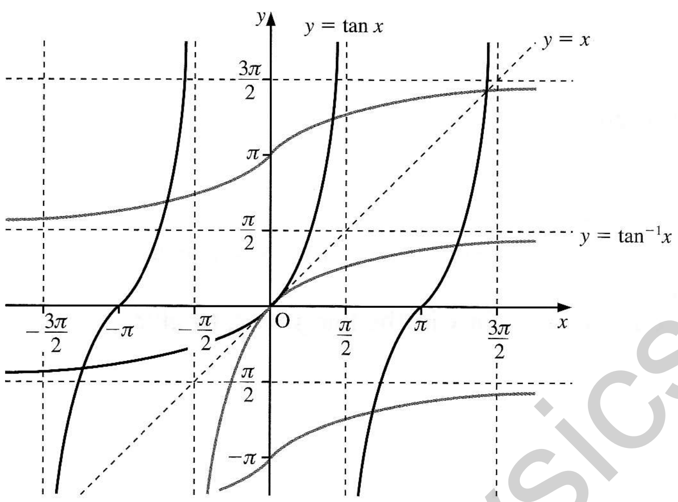
$$\frac{dy}{dx} = \sec^2 x$$

At the origin, where $x = 0$, we have

$$\frac{dy}{dx} = \sec^2 0 = \frac{1}{\cos^2 0} = 1$$

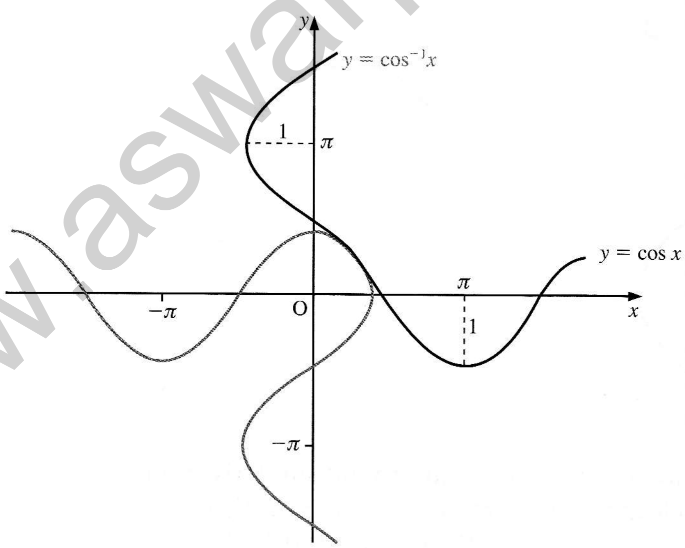
Thus, the gradient of $y = \tan x$ at the origin is 1.

The graphs of $y = \tan x$ and $y = \tan^{-1} x$ (or $\arctan x$) are shown below.



Inverse cosine graph

The graphs of $y = \cos x$ and $y = \cos^{-1} x$ (or $y = \arccos x$) are shown below.



Exercise 2C

1 Find the value of each of these inverse functions.

a) $\sin^{-1}0.5$ **b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **c)** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d) $\tan^{-1}1$ **e)** $\sec^{-1}\sqrt{2}$ **f)** $\cot^{-1}3$

2 Sketch the graph of each of these inverse functions.

a) $\sec^{-1}x$ **b)** $\operatorname{cosec}^{-1}x$ **c)** $\cot^{-1}x$

3 If $\cos^{-1}x = \frac{2\pi}{5}$, find $\sin^{-1}x$.

4 Prove that $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1}x$

5 Find the general solution of the equation

$$3\cos\theta - 7\sin\theta = -6$$

Give your answer in degrees correct to two decimal places. (NICCEA)

6 $$5\cos x - 12\sin x \equiv R\cos(x+\alpha)$$

where $R > 0$ and α is acute and measured in degrees.

a) Find the value of R.
b) Find the value of α to one decimal place.
c) Hence, or otherwise, find the general solution of the equation

$$5\cos x - 12\sin x = 4 \qquad \text{(EDEXCEL)}$$

7 i) Write $f(\theta) = 7\cos\theta - 3\sin\theta$ in the form $R\cos(\theta+\alpha)$, where R is positive and α is acute.
ii) Find the maximum and minimum values of $f(\theta)$.
iii) Solve $7\cos\theta - 3\sin\theta = 1$, giving the general solution in degrees. (NICCEA)

8 a) Find all values of x between $0°$ and $360°$ satisfying

$$3\cos x + \sin x = -1$$

b) Find the general solution of the equation

$$\sin 2x + \sin 4x = \cos 2x + \cos 4x \qquad \text{(WJEC)}$$

9 Given that

$$7\cos\theta + 24\sin\theta \equiv R\cos(\theta - \alpha)$$

where $R > 0$, $0 \leqslant \alpha \leqslant 90°$,

a) find the values of the constants R and α.

Hence find

b) the general solution of the equation $7\cos\theta + 24\sin\theta = 15$
c) the range of the function $f(\theta)$ where

$$f(\theta) \equiv \frac{1}{5 + (7\cos\theta + 24\sin\theta)^2} \qquad 0 \leqslant \theta < 360° \qquad \text{(EDEXCEL)}$$

10 Find, in degrees, the value of the acute angle α for which

$$\cos\theta - (\sqrt{3})\sin\theta \equiv 2\cos(\theta+\alpha)$$

for all values of θ.

Solve the equation

$$\cos x - (\sqrt{3})\sin x = \sqrt{2} \qquad 0^\circ \leqslant x \leqslant 360^\circ$$ (EDEXCEL)

11 Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

Hence find the general solution of the equation

$$\cos\theta + \sqrt{3}\sin\theta = 2\cos 40^\circ$$

giving your answers in degrees. (AEB 96)

12 The angle α is such that $0 < \alpha < \dfrac{\pi}{2}$ and $R\cos(\theta+\alpha) \equiv 84\cos\theta - 13\sin\theta$, where R is some positive real number.

a) State the value of R and find α, in radians, correct to three decimal places.
b) Hence determine the general solution, in radians, of the equation

$$84\cos\theta - 13\sin\theta = 17$$ (AEB 98)

13 $$f(x) \equiv 7\cos x - 24\sin x$$

Given that $f(x) \equiv R\cos(x+\alpha)$, where $R \geqslant 0$, $0 \leqslant \alpha \leqslant \dfrac{\pi}{2}$, and x and α are measured in radians,

a) find R and show that $\alpha = 1.29$ to two decimal places.

Hence write down

b) the minimum value of $f(x)$
c) the value of x in the interval $0 \leqslant x \leqslant 2\pi$ which gives this minimum value.
d) Find the smallest two positive values of x for which

$$7\cos x - 24\sin x = 10$$ (EDEXCEL)

14 i) $$f(\theta) \equiv 9\sin\theta + 12\cos\theta$$

Given that $f(\theta) \equiv R\sin(\theta+\alpha)$ where $R > 0$, $0 \leqslant \alpha \leqslant 90^\circ$,

a) find the values of the constants R and α.
b) Hence find the values of θ, $0 \leqslant \theta < 360^\circ$, for which

$$9\sin\theta + 12\cos\theta = -7.5$$

giving your answers to the nearest tenth of a degree.

ii) Find, in radians in terms of π, the general solution to the equation

$$\sqrt{3}\sin(\theta - \tfrac{1}{6}\pi) = \sin\theta$$ (EDEXCEL)

Differentiation of inverse trigonometric functions

$\sin^{-1}x$ or arcsin x

If $y = \sin^{-1}x$, then $\sin y = x$.

Differentiating $\sin y = x$, we obtain

$$\cos y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Therefore, we have

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}x + c$$

Similarly, if $y = \sin^{-1}\left(\frac{x}{a}\right)$, then $\sin y = \frac{x}{a}$.

Differentiating, we get

$$\cos y \frac{dy}{dx} = \frac{1}{a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a \cos y} = \frac{1}{a\sqrt{1 - \sin^2 y}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{a\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \frac{1}{\sqrt{a^2 - x^2}}$$

Therefore, we have

$$\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

which gives

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

If $y = \cos^{-1}x$, we can show that

$$\frac{d}{dx} \cos^{-1}x = \frac{-1}{\sqrt{1 - x^2}}$$

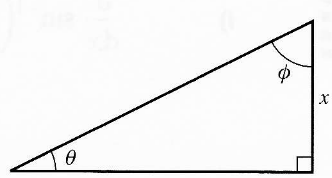
which gives

$$\int \frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1}x + c$$

In the diagram on the right, $\sin\theta = x$, $\cos\phi = x$ and $\phi = \frac{\pi}{2} - \theta$.

Therefore, we have $\theta = \sin^{-1}x$ and $\phi = \cos^{-1}x$, giving

$$\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$$



So, we get

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$
$$= -\cos^{-1}x + c'$$

where $c' = \frac{\pi}{2} + c$.

Hence, it is unusual to use a function in $\cos^{-1}x$ in differentiation or in integration, as it is simply an alternative to $\sin^{-1}x$.

$\tan^{-1}x$ or arctan x

If $y = \tan^{-1}\left(\frac{x}{a}\right)$, then $\tan y = \frac{x}{a}$.

Differentiating $\tan y = \frac{x}{a}$, we obtain

$$\sec^2 y \frac{dy}{dx} = \frac{1}{a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a\sec^2 y} = \frac{1}{a(1+\tan^2 y)} = \frac{1}{a\left[1+\left(\frac{x}{a}\right)^2\right]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a^2+x^2}$$

Therefore, we have

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$$

which gives

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Note $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ and $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

Example 10 Differentiate each of the following inverse functions.

a) i) $\sin^{-1}\left(\frac{x}{3}\right)$ **ii)** $\sin^{-1}4x$ **b)** $\tan^{-1}\left(\frac{x}{5}\right)$

SOLUTION

a) Using $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$, we have

i) $$\frac{d}{dx}\sin^{-1}\left(\frac{x}{3}\right) = \frac{1}{\sqrt{9-x^2}}$$

ii) $$\frac{d}{dx}\sin^{-1}4x = \frac{d}{dx}\sin^{-1}\left(\frac{x}{\frac{1}{4}}\right) = \frac{1}{\sqrt{\frac{1}{16} - x^2}}$$

$$\Rightarrow \frac{d}{dx}\sin^{-1}4x = \frac{4}{\sqrt{1 - 16x^2}}$$

b) Using $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$, we have

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{5}\right) = \frac{5}{25 + x^2}$$

Example 11 Evaluate **a)** $\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx$ **b)** $\int_0^1 \frac{1}{\sqrt{4 - 3x^2}}\,dx$

SOLUTION

a) $$\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} - 0$$

Therefore, we have

$$\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx = \frac{\pi}{2}$$

b) For integrals in this form, we **always** reduce the coefficient of x^2 to unity before integrating. Hence, in this case, we have

$$\int_0^1 \frac{1}{\sqrt{4 - 3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_0^1 \frac{1}{\sqrt{\frac{4}{3} - x^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^1 \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - x^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{x}{\frac{2}{\sqrt{3}}}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}0\right) = \frac{1}{\sqrt{3}}\,\frac{\pi}{3}$$

Hence, we obtain

$$\int_0^1 \frac{1}{\sqrt{4 - 3x^2}}\,dx = \frac{\pi}{3\sqrt{3}} \quad \text{or} \quad \frac{\pi\sqrt{3}}{9}$$

Example 12 Evaluate $\displaystyle\int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x$.

SOLUTION

$$\int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\tan^{-1}1 - \frac{1}{3}\tan^{-1}0 = \frac{1}{3}\frac{\pi}{4}$$

Therefore, we have

$$\int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x = \frac{\pi}{12}$$

Example 13 Find $\displaystyle\int \frac{1}{16+25x^2}\,\mathrm{d}x$.

SOLUTION

Remember Reduce the coefficient of x^2 to unity before integrating. (See Example 11.)

Hence, we have

$$\int \frac{1}{16+25x^2}\,\mathrm{d}x = \frac{1}{25}\int \frac{1}{\frac{16}{25}+x^2}\,\mathrm{d}x$$

$$= \frac{1}{25}\int \frac{1}{\left(\frac{4}{5}\right)^2+x^2}\,\mathrm{d}x$$

$$= \frac{1}{25}\frac{1}{\frac{4}{5}}\tan^{-1}\left(\frac{x}{\frac{4}{5}}\right) + c$$

Therefore, we have

$$\int \frac{1}{16+25x^2}\,\mathrm{d}x = \frac{1}{20}\tan^{-1}\left(\frac{5x}{4}\right) + c$$

Example 14 Find $\displaystyle\int \frac{\mathrm{d}x}{x^2+6x+25}$.

SOLUTION

When it is anticipated that the integral will be an inverse trigonometric function, we start by using the method of completing the square to turn the quadratic denominator into the form $a(x+b)^2+c$. Then we reduce the coefficient of $(x+b)^2$ to unity so that we can use the standard integration formula with $(x+b)$ replacing x.

Hence, we have

$$x^2+6x+25 = (x+3)^2+16$$

which gives

$$\int \frac{\mathrm{d}x}{x^2+6x+25} = \int \frac{\mathrm{d}x}{(x+3)^2+16}$$

The integral we have obtained is now in the same form as $\int \frac{dx}{x^2 + a^2}$, with $(x+3)$ replacing x and 4 replacing a. Thus, we have

$$\int \frac{dx}{(x+3)^2 + 16} = \frac{1}{4} \tan^{-1}\left(\frac{x+3}{4}\right) + c$$

Example 15 Find $\int \frac{dx}{\sqrt{11 - 8x - 4x^2}}$.

SOLUTION

To convert $11 - 8x - 4x^2$ into the form $a(x+b)^2 + c$, it is easier first to factorise out the minus sign, and then take the sign back inside when the square is completed.

Note The minus sign must be kept **within the square root**.

So, factorising out the minus sign, we have

$$\sqrt{11 - 8x - 4x^2} = \sqrt{-(4x^2 + 8x - 11)}$$

$$= \sqrt{-4\left(x^2 + 2x - \frac{11}{4}\right)}$$

Then, completing the square, we get

$$\sqrt{11 - 8x - 4x^2} = \sqrt{-4\left[(x+1)^2 - \frac{15}{4}\right]}$$

$$= \sqrt{15 - 4(x+1)^2}$$

$$= 2\sqrt{\frac{15}{4} - (x+1)^2}$$

Substituting this into the given integral, we have

$$\int \frac{dx}{\sqrt{11 - 8x - 4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{15}{4} - (x+1)^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x+1}{\sqrt{\frac{15}{4}}}\right) + c$$

which gives

$$\int \frac{dx}{\sqrt{11 - 8x - 4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2(x+1)}{\sqrt{15}}\right) + c$$

Exercise 2D

1 Differentiate each of the following with respect to x.

a) $\sin^{-1}5x$ **b)** $\tan^{-1}3x$ **c)** $\sin^{-1}\sqrt{2}x$ **d)** $\tan^{-1}\frac{3}{4}x$

e) $\sin^{-1}x^2$ **f)** $\tan^{-1}\left(\dfrac{x}{1+x^2}\right)$ **g)** $(\sin^{-1}2x)^3$ **h)** $(3\tan^{-1}5x)^4$

i) $\sec^{-1}x$ **j)** $\cot^{-1}x$

2 Find each of the following integrals.

a) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{4-x^2}}$ **b)** $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{9-x^2}}$ **c)** $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{25-4x^2}}$

d) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{16-9x^2}}$ **e)** $\displaystyle\int\frac{\mathrm{d}x}{9+x^2}$ **f)** $\displaystyle\int\frac{\mathrm{d}x}{16+x^2}$

g) $\displaystyle\int\frac{\mathrm{d}x}{25+16x^2}$ **h)** $\displaystyle\int\frac{\mathrm{d}x}{9+25x^2}$

3 Evaluate each of the following definite integrals, giving the exact value of your answer.

a) $\displaystyle\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^2}}$ **b)** $\displaystyle\int_0^2\frac{\mathrm{d}x}{4+x^2}$ **c)** $\displaystyle\int_0^3\frac{\mathrm{d}x}{\sqrt{9-x^2}}$ **d)** $\displaystyle\int_0^2\frac{\mathrm{d}x}{4+3x^2}$

e) $\displaystyle\int_{-\frac{1}{5}}^{\frac{1}{5}}\frac{\mathrm{d}x}{\sqrt{1-25x^2}}$

4 Evaluate each of the following definite integrals, giving your answer correct to three significant figures.

a) $\displaystyle\int_0^{0.1}\frac{\mathrm{d}x}{\sqrt{4-25x^2}}$ **b)** $\displaystyle\int_1^2\frac{\mathrm{d}x}{4+9x^2}$ **c)** $\displaystyle\int_1^2\frac{\mathrm{d}x}{\sqrt{3-(x-1)^2}}$

d) $\displaystyle\int_0^1\frac{\mathrm{d}x}{4(x+1)^2+5}$ **e)** $\displaystyle\int_0^2\frac{\mathrm{d}x}{\sqrt{20-8x-x^2}}$ **f)** $\displaystyle\int_0^1\frac{\mathrm{d}x}{16x^2+20x+35}$

5 Find the exact value of $\displaystyle\int_0^{\frac{5}{8}}\frac{1}{\sqrt{(25-16x^2)}}\,\mathrm{d}x$. (OCR)

6 Express $5+4x-x^2$ in the form $a-(x-b)^2$, where a and b are positive constants. Hence find the exact value of

$$\int_{\frac{7}{2}}^{5}\frac{1}{\sqrt{(5+4x-x^2)}}\,\mathrm{d}x \qquad \text{(OCR)}$$

7 Express $\dfrac{2x^3+5x^2+11x+13}{(x+1)(x^2+4)}$ in partial fractions.

Show that

$$\int_0^1\frac{2x^3+5x^2+11x+13}{(x+1)(x^2+4)}\,\mathrm{d}x = 2+\ln\left(\frac{5}{2}\right)+\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \qquad \text{(OCR)}$$

8 Given that $y = x - \sqrt{(1 - x^2)}\sin^{-1}x$, show that

$$\frac{dy}{dx} = \frac{x\sin^{-1}x}{\sqrt{(1 - x^2)}}$$

Hence, or otherwise, evaluate

$$\int_0^{\frac{1}{4}\sqrt{3}} \frac{2x\sin^{-1}(2x)}{\sqrt{(1 - 4x^2)}}\,dx$$

giving your answer in terms of π and $\sqrt{3}$. (OCR)

9 Given that $z = \tan^{-1}x$, derive the result $\dfrac{dz}{dx} = \dfrac{1}{1 + x^2}$.

[No credit will be given for merely quoting the result from the *List of Formulae*.]

Hence express $\dfrac{d}{dx}(\tan^{-1}(xy))$ in terms of x, y and $\dfrac{dy}{dx}$.

Given that x and y satisfy the equation

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}(xy) = \tfrac{11}{12}\pi$$

prove that, when $x = 1$, $\dfrac{dy}{dx} = -1 - \tfrac{1}{2}\sqrt{3}$. (OCR)

10 i) Given that $y = \sin^{-1}x$, derive the result $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1 - x^2)}}$.

[No credit will be given for merely quoting the result from the *List of Formulae*.]

ii) Find $\dfrac{d}{dx}\sqrt{(1 - x^2)}$.

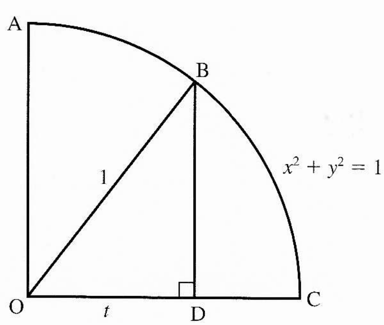
iii) Using the above results, find $\displaystyle\int_0^1 \sin^{-1}x\,dx$. (OCR)

11 Given that $x = \dfrac{1}{y}$, show that

$$\int \frac{1}{x\sqrt{(x^2 - 1)}}\,dx = -\int \frac{1}{\sqrt{(1 - y^2)}}\,dy$$

Find $\displaystyle\int \frac{1}{x\sqrt{(x^2 - 1)}}\,dx$. (OCR)

12 OAC is a quadrant of the circle whose equation is $x^2 + y^2 = 1$. From B, a point on the circumference, a perpendicular is dropped to D, a point on the radius OC, so that the x-coordinate of D is t. This is shown in the figure on the right.



i) Show that the area of the sector AOB is $\tfrac{1}{2}\sin^{-1}t$.
ii) Find the area of the triangle OBD in terms of t.
iii) Hence show that

$$\sin^{-1}t = 2\int_0^t \sqrt{1 - x^2}\,dx - t\sqrt{1 - t^2}$$

iv) By using integration by parts, show that

$$\int_0^t \sqrt{1-x^2}\,\mathrm{d}x = t\sqrt{1-t^2} + \int_0^t \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$$

v) By using parts **iii** and **iv**, prove that

$$\sin^{-1} t = \int_0^t \frac{\mathrm{d}x}{\sqrt{1-x^2}} \qquad \text{(NICCEA)}$$

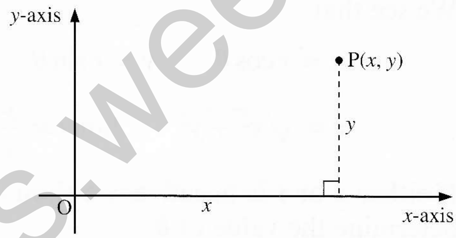
3 Polar coordinates

All places are distant from Heaven alike.
ROBERT BURTON

Position of a point

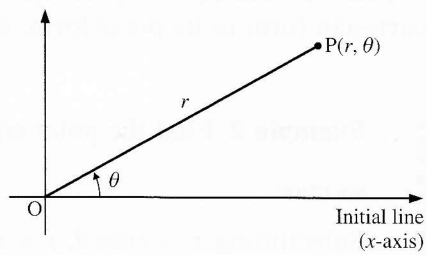
The position of a point, P, in a plane may be given in terms of its distance from a fixed point, O, called the **pole**, and the angle which OP makes with a fixed line, called the **initial line**. When the position of a point is given in this way, we have the **polar coordinates** of the point.

In the diagram on the right, the cartesian coordinates of point P would be given as (x, y).



Its position in polar coordinates would be given as (r, θ), where $r(\geqslant 0)$ is the distance of P from the origin, O, and θ is the **anticlockwise angle** which OP makes with the x-axis, which is normally taken as the initial line.

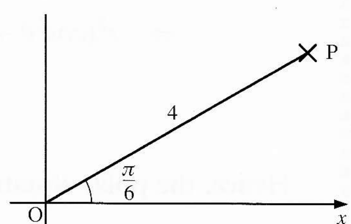
θ is normally measured in radians and its **principal value** is taken to be between $-\pi$ and π.



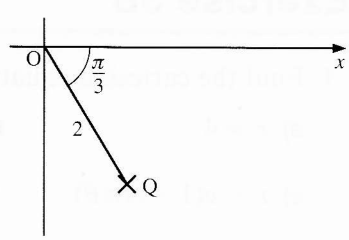
Example 1 Plot the point P with coordinates $\left(4, \frac{\pi}{6}\right)$ and the point Q with coordinates $\left(2, -\frac{\pi}{3}\right)$.

SOLUTION

a) Draw the line OP at $\frac{\pi}{6}$ radians to the x-axis.
Make OP $= 4$ units.
Then P is the point identified.



b) Draw the line OQ at $-\frac{\pi}{3}$ radians to the x-axis. The negative value of the angle means that $\frac{\pi}{3}$ is measured in a **clockwise** direction from the x-axis.
Make OQ $= 2$ units.
Then Q is the point identified.



Exercise 3A

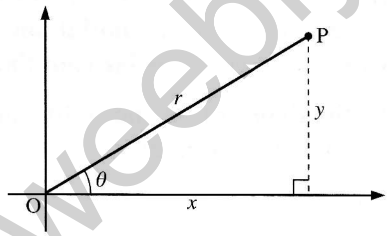
Plot the points with the following polar coordinates.

1 $\left(3, \frac{\pi}{4}\right)$ **2** $\left(2, \frac{2\pi}{3}\right)$ **3** $\left(3, -\frac{\pi}{3}\right)$

4 $\left(2, \frac{3\pi}{2}\right)$ **5** $\left(4, -\frac{\pi}{4}\right)$

Connection between polar and cartesian coordinates

In the diagram on the right, the point P is (x, y) in cartesian coordinates and (r, θ) in polar coordinates.



We see that

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

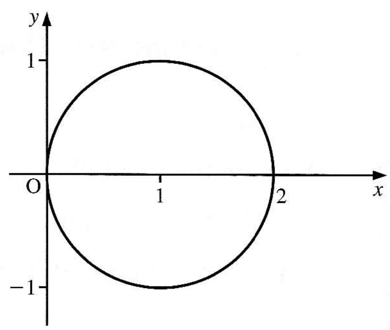
If either x or y is negative, we should refer to the position of the point to determine the value of θ.

We can use the above equations to convert the equation of a curve from its cartesian form to its polar form, or vice versa.

Example 2 Find the polar equation of the curve $x^2 + y^2 = 2x$.

SOLUTION

Substituting $x = r\cos\theta$, $y = r\sin\theta$ into $x^2 + y^2 = 2x$ (shown on the right), we have



$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$$\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = 2r\cos\theta$$

$$\Rightarrow \quad r^2 = 2r\cos\theta$$

$$\Rightarrow \quad r = 2\cos\theta \quad (\text{since } r \neq 0)$$

Hence, the polar equation of the given curve is $r = 2\cos\theta$.

Exercise 3B

1 Find the cartesian equation of each of these curves.

a) $r = 4$ **b)** $r\cos\theta = 3$ **c)** $r\sin\theta = 7$ **d)** $r = a(1 + \cos\theta)$

e) $r = a(1 - \cos\theta)$ **f)** $\frac{2}{r} = 1 + \cos\theta$

2 Find the polar equation of each of these curves.

a) $x^2 + y^2 = 9$ **b)** $xy = 16$ **c)** $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ **d)** $x^2 + y^2 = 6x$

e) $x^2 + y^2 + 8y = 16$ **f)** $(x^2 + y^2)^2 = x^2 - y^2$

Sketching curves given in polar coordinates

The normal way to sketch a curve expressed in polar coordinates is to plot points roughly using simple values of θ.

Example 3 Sketch $r = a\cos 3\theta$.

SOLUTION

Part of a table giving values for r is shown below.

$\boldsymbol{\theta}$	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{9}$	$\frac{11\pi}{18}$	$\frac{2\pi}{3}$	$\frac{13\pi}{18}$	$\frac{7\pi}{9}$	$\frac{5\pi}{6}$
$\boldsymbol{r}$	a	$\frac{\sqrt{3}}{2}a$	$\frac{1}{2}a$	0	0	$\frac{1}{2}a$	$\frac{\sqrt{3}}{2}a$	a	$\frac{\sqrt{3}}{2}a$	$\frac{1}{2}a$	0

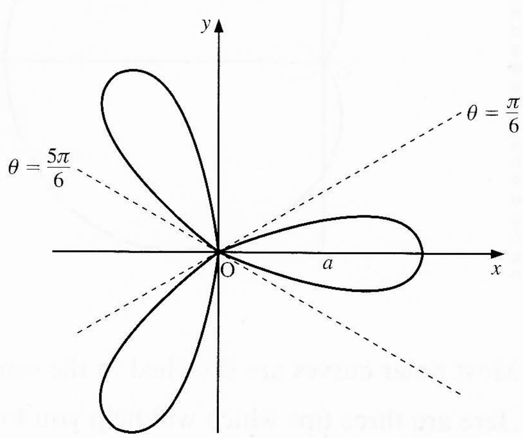
Note When $\theta = \dfrac{2\pi}{9}$, $r = a\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}a$. Therefore, since r must always be positive, the curve does not exist when $\theta = \dfrac{2\pi}{9}$.

Similarly, the curve does not exist for any value of θ between

$$\frac{\pi}{6} \text{ and } \frac{\pi}{2},\ \frac{5\pi}{6} \text{ and } \frac{7\pi}{6},\ \frac{9\pi}{6} \text{ and } \frac{11\pi}{6},\ \frac{13\pi}{6} \text{ and } \frac{15\pi}{6}$$

Plotting the values given in the table and joining the points gives a curve with three loops or lobes.

Notice that the lines $\theta = \dfrac{\pi}{6}$, $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{5\pi}{6}$ are all **tangents** to the loops. The tangents meet at the origin or pole. All three loops are **congruent**.



Example 4 Sketch $r = 1 + 2\cos\theta$.

SOLUTION

Part of a table giving values for r is shown below.

$\boldsymbol{\theta}$	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	$-\dfrac{2\pi}{3}$	$-\dfrac{\pi}{2}$	$-\dfrac{\pi}{3}$	$-\dfrac{\pi}{6}$	0
$\boldsymbol{r}$	3	$1+\sqrt{3}$	2	1	0	0	1	2	$1+\sqrt{3}$	3

To find when r is negative, we solve $r = 0$:

$$1 + 2\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = -\tfrac{1}{2}$$

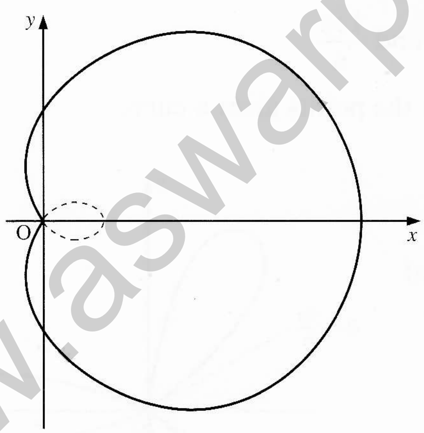
$$\Rightarrow \quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \left(\text{or } -\frac{2\pi}{3}\right)$$

We note that $1 + 2\cos\theta$ is negative for $\dfrac{2\pi}{3} < \theta < \dfrac{4\pi}{3}$. However, in A-level examinations **only positive values** of r are required, which means that, as far as you are concerned, the curve does not exist for values of θ between $\dfrac{2\pi}{3}$ and $\dfrac{4\pi}{3}$, and should not be shown.

When most models of the graphics calculator display this curve, they include negative values of r, which you should ignore.

The sketch of $r = 1 + 2\cos\theta$ is shown below.

The dashed part represents the negative values of r which are commonly displayed by graphics calculators.



Most polar curves are sketched in the same way.

Here are three tips which will help you to sketch curves given in polar coordinates

- Look for any **symmetry**. If r is a function of $\cos\theta$ only, there is symmetry about the initial line. If r is a function of $\sin\theta$ only, there is symmetry about the line $\theta = \dfrac{\pi}{2}$.
- The equations $r = a\sin\theta$ and $r = a\cos\theta$ are **circles**.

Example 5 Find the cartesian equation of the curve $r = a\cos\theta$.

SOLUTION

Multiplying $r = a\cos\theta$ by r, we get

$$r^2 = ar\cos\theta$$

Substituting $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we have

$$x^2 + y^2 = ax$$

which gives

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

This is a circle with centre $\left(\dfrac{a}{2}, 0\right)$ and radius a.

- When a polar equation contains $\sec\theta$ or $\operatorname{cosec}\theta$, it is often easier to **use its cartesian equation**.

Example 6 Sketch $r = a\sec\theta$.

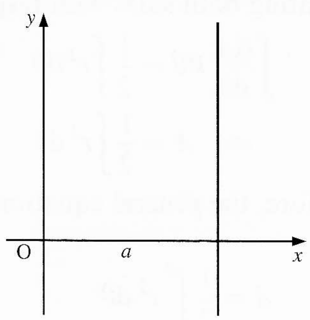
SOLUTION

$$r = a\sec\theta = \frac{a}{\cos\theta}$$

$$\Rightarrow \quad r\cos\theta = a$$

$$\Rightarrow \quad x = a$$

This is the straight line shown on the right.



Example 7 Sketch $r = a\sec(\alpha - \theta)$.

SOLUTION

Transposing terms, we have

$$r\cos(\alpha - \theta) = a$$

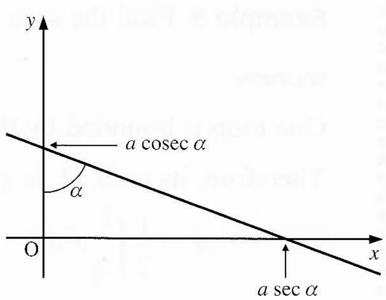
which gives

$$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = a$$

Replacing $r\cos\theta$ with x and $r\sin\theta$ with y, we get

$$x\cos\alpha + y\sin\alpha = a.$$

which is the straight line shown on the right.



Exercise 3C

1 Sketch each of the curves given in Question **1** of Exercise 3B.

2 Sketch each of the following curves.

a) $r = a\sin 2\theta, \quad 0 < \theta < 2\pi$

b) $r = a\cos 4\theta, \quad 0 < \theta < 2\pi$

c) $r = 2 + 3\cos\theta, \quad -\pi < \theta < \pi$

d) $r = a\theta, \quad 0 < \theta < 2\pi$

e) $r = 4\sec\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Area of a sector of a curve

Let A be the area bounded by the curve $r = \mathrm{f}(\theta)$ and the two radii at α and at θ. As θ increases by $\delta\theta$, the increase in area, δA, shown shaded, is given by

$$\tfrac{1}{2}r^2\delta\theta \leqslant \delta A \leqslant \tfrac{1}{2}(r+\delta r)^2\delta\theta \quad \text{(using areas of sectors)}$$

Dividing throughout by $\delta\theta$, we obtain

$$\tfrac{1}{2}r^2 \leqslant \frac{\delta A}{\delta\theta} \leqslant \tfrac{1}{2}(r+\delta r)^2$$

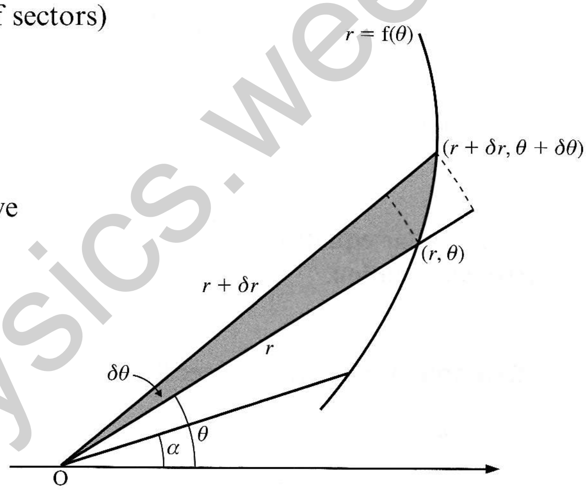
As $\delta\theta \to 0$, $\dfrac{\delta A}{\delta\theta} \to \dfrac{\mathrm{d}A}{\mathrm{d}\theta}$ and $\delta r \to 0$. Therefore, we have

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}r^2$$

Integrating both sides with respect to θ, we obtain

$$\int \frac{\mathrm{d}A}{\mathrm{d}\theta}\,\mathrm{d}\theta = \frac{1}{2}\int r^2\,\mathrm{d}\theta$$

$$\Rightarrow \quad A = \frac{1}{2}\int r^2\,\mathrm{d}\theta$$



Therefore, the general equation for the area of a sector of a curve is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$$

when the area is bounded by the radii $\theta = \alpha$ and $\theta = \beta$.

Example 8 Find the area of one loop of the curve $r = a\cos 3\theta$.

SOLUTION

One loop is bounded by the tangent lines $\theta = \dfrac{\pi}{6}$ and $\theta = -\dfrac{\pi}{6}$ (see page 46).

Therefore, its area, A, is given by

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2\mathrm{d}\theta \quad \Rightarrow \quad A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2\cos^2 3\theta\,\mathrm{d}\theta$$

Using the double-angle formula to integrate, we have

$$A = \frac{1}{2}a^2 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(\cos 6\theta + 1)\,d\theta$$

$$= \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{a^2\pi}{12}$$

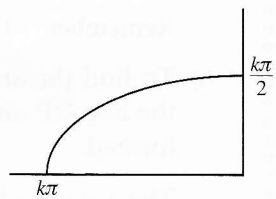
So, the area of one loop of $r = a\cos 3\theta$ is $\dfrac{a^2\pi}{12}$.

Note It is often preferable to use only the area in the first quadrant when a curve is symmetrical in other quadrants. Thus, in Example 8, instead of using $\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2\cos^2 3\theta\,d\theta$, we could have used $2 \times \frac{1}{2}\int_0^{\frac{\pi}{6}} a^2\cos^2 3\theta\,d\theta$.

Example 9 Find the area bounded by the curve $r = k\theta$ and the lines $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$.

SOLUTION

The curve $r = k\theta$ is shown for $\dfrac{\pi}{2} \leqslant \theta \leqslant \pi$.



The area, A, required is given by

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} k^2\theta^2\,d\theta$$

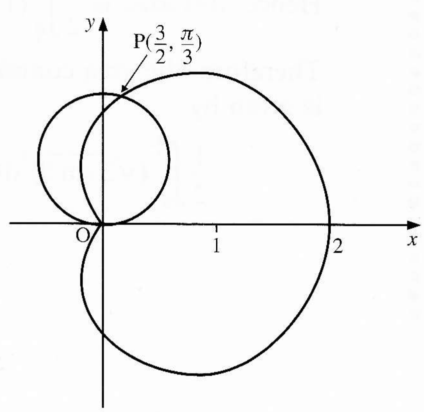
$$= \frac{k^2}{2}\left[\frac{\theta^3}{3}\right]_{\frac{\pi}{2}}^{\pi} = \frac{k^2}{2}\left(\frac{\pi^3}{3} - \frac{\pi^3}{24}\right) = \frac{7k^2\pi^3}{48}$$

Hence, the area required is $\dfrac{7k^2\pi^3}{48}$.

Example 10 Sketch the curves $r = 1 + \cos\theta$ and $r = \sqrt{3}\sin\theta$. Find

a) the points where the curves meet
b) the area contained between the curves.

SOLUTION



Before sketching the two curves, we note that

- $r = 1 + \cos\theta$ is similar to $1 + 2\cos\theta$ (page 46)
- $r = \sqrt{3}\sin\theta$ is similar to $r = a\cos\theta$ (page 47).

a) The two curves meet when

$$1 + \cos\theta = \sqrt{3}\sin\theta$$

Using

$$\sin\theta \equiv \sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right) \equiv 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

and

$$\cos\theta \equiv 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

we express $1 + \cos\theta = \sqrt{3}\sin\theta$ as

$$1 + 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\sqrt{3}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right) = 2\sqrt{3}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

which gives

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{3}\sin\left(\frac{\theta}{2}\right) \quad \text{or} \quad \cos\left(\frac{\theta}{2}\right) = 0$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{3}} \qquad\qquad \Rightarrow \quad \theta = \pi$$

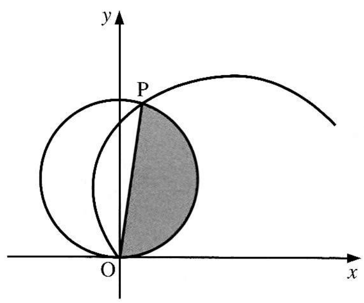
$$\Rightarrow \quad \frac{\theta}{2} = \frac{\pi}{6} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

Therefore, the curves meet at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $(0, \pi)$

Remember These are polar coordinates (r, θ).

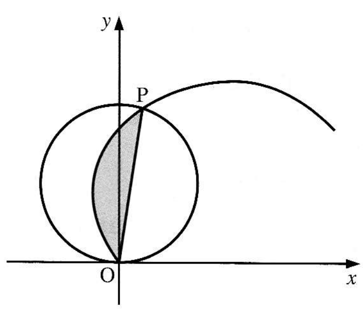
b) To find the area contained between the curves, we draw the line OP and consider separately the two areas so formed.

The area shaded in the diagram immediately right is bounded by the curve $r = \sqrt{3}\sin\theta$ and the two radii $\theta = 0$ and $\theta = \frac{\pi}{3}$.



Hence, this area is $\frac{1}{2}\int_0^{\frac{\pi}{3}} (\sqrt{3}\sin\theta)^2\, d\theta$.

The area shaded in the lower diagram on the right is bounded by the curve $r = 1 + \cos\theta$ and the two radii $\theta = \frac{\pi}{3}$ and π.



Hence, this area is $\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1 + \cos\theta)^2\, d\theta$.

Therefore, the area contained between the two curves is given by

$$\frac{1}{2}\int_0^{\frac{\pi}{3}} (\sqrt{3}\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} 3\sin^2\theta\, d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \frac{1}{2}(1-\cos 2\theta)\,d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}\left[1+2\cos\theta+\frac{1}{2}(\cos 2\theta+1)\right]d\theta$$

$$= \frac{3}{4}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{3}{4}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)+\frac{1}{2}\left(\frac{3\pi}{2}-\frac{3\pi}{6}-\sqrt{3}-\frac{\sqrt{3}}{8}\right)$$

$$= \frac{3\pi}{4}-\frac{3\sqrt{3}}{4}$$

Therefore, the area contained within the curves is $\frac{3\pi}{4}-\frac{3\sqrt{3}}{4}$.

Exercise 3D

1 Find the area bounded by the curve $r = a\theta$ and the radii $\theta = \frac{\pi}{2}$, $\theta = \pi$.

2 For each of the following curves, find the area enclosed by one loop.

a) $r = a\cos 2\theta$ **b)** $r = a\sin 2\theta$ **c)** $r = a\cos 4\theta$

3 Find the area enclosed by the curve $r = a\cos\theta$.

4 Find the area enclosed by the curve $r = 2 + 3\cos\theta$.

5 **a)** Find the polar equation of the curve $(x^2+y^2)^3 = y^4$.
b) Hence, **i)** sketch the curve, and **ii)** find the area enclosed by the curve.

6 Find where the following two curves intersect.

$$r = 2\sin\theta \qquad 0 \leqslant \theta < \pi$$

and $$r = 2(1-\sin\theta) \qquad -\pi < \theta < \pi$$

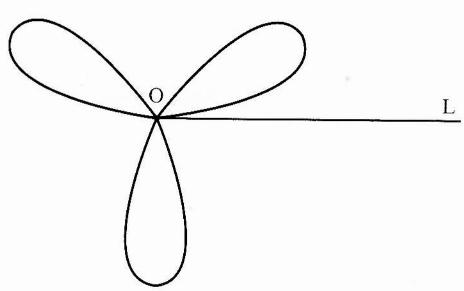
Hence, find the area which is between the two curves.

7 In this question you may use the identity $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$. The cartesian equation of a curve C is

$$(x^2+y^2)(x^2+y^2-3ay)+4ay^3 = 0$$

where $a > 0$.

a) Show that, in terms of polar coordinates (r, θ), the equation of C is $r = a\sin 3\theta$.
b) The curve consists of three equal loops, as shown in the diagram. The point O is the pole, and OL is the initial line.



Find, in terms of a, the exact value of the area of one of these loops. (NEAB)

8 **a)** Sketch the curve with polar equation

$$r = a(2 + \cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

and a is a positive constant.

Mark on your sketch the polar coordinates of the points where the curve meets the half-lines $\theta = 0$, $\theta = \pi$, $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$.

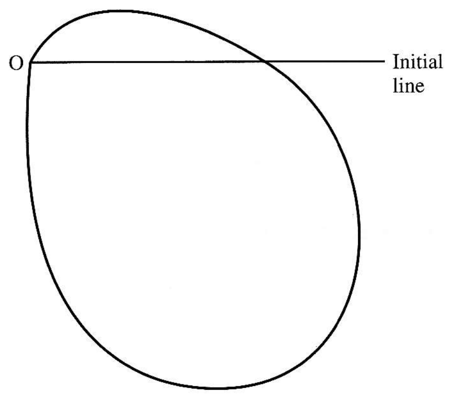
b) Find the area of the region enclosed by this curve, giving your answer in terms of π and a.

(EDEXCEL)

9 The diagram shows a sketch of the loop whose polar equation is

$$r = 2(1 - \sin\theta)\sqrt{(\cos\theta)} \qquad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi$$

where O is the pole.

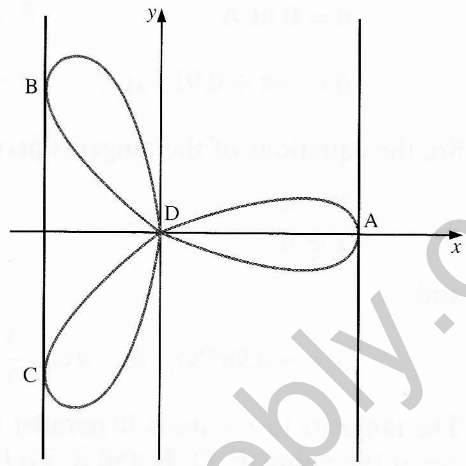


a) Show that the area enclosed by the loop is $\dfrac{16}{3}$.

b) Show that the initial line divides the area enclosed by the loop in the ratio 1 : 7. (NEAB)

Equations of the tangents to a curve

The tangents to $r = a\cos 3\theta$ **perpendicular** to the initial line are shown on the right.



These are at points A, where x is at a maximum, B and C where x is at a minimum, and D, where x has a point of inflexion.

We note that $x = r\cos\theta$. Therefore, to find the maximum and minimum values of x, we find the maximum and the minimum values of $r\cos\theta$.

Since $r = a\cos 3\theta$, we have

$$x = a\cos 3\theta \cos\theta$$

which gives

$$\frac{dx}{d\theta} = -3a\sin 3\theta\cos\theta - a\cos 3\theta\sin\theta$$

The maximum and the minimum values occur when $\dfrac{dx}{d\theta} = 0$. That is, when

$$-3a\sin 3\theta\cos\theta - a\cos 3\theta\sin\theta = 0$$

We simplify this expression using the factor formulae:

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

and

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

which give $\dfrac{dx}{d\theta} = 0$ when

$$\frac{3}{2}[\sin 4\theta + \sin 2\theta] + \frac{1}{2}[\sin 4\theta - \sin 2\theta] = 0$$

$$\Rightarrow \quad 2\sin 4\theta + \sin 2\theta = 0$$

Applying the double-angle formula, we get

$$4\sin 2\theta\cos 2\theta + \sin 2\theta = 0$$

$$\sin 2\theta(4\cos 2\theta + 1) = 0$$

which gives

$$\sin 2\theta = 0 \quad \text{or} \quad \cos 2\theta = -\tfrac{1}{4}$$

$$\sin 2\theta = 0 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}, \pi, \ldots$$

$$\cos 2\theta = -\tfrac{1}{4} \quad \Rightarrow \quad \theta = n\pi \pm 0.912$$

We have to ensure that the curve exists at these points. For example, when $\theta = 0.912$, $\cos 3\theta$ is negative, thus r is negative and the curve does not exist.