Example 16 Let T be reflection in the x-axis. Find the transformation matrix of T with respect to these bases:

a) $\mathbf{e}_1 = \mathbf{i}$ and $\mathbf{e}_2 = \mathbf{j}$

b) $\mathbf{e}_1 = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ and $\mathbf{e}_2 = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

SOLUTION

a) When $\mathbf{e}_1 = \mathbf{i}$ and $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{e}_1$ is unchanged by reflection in the line $y = 0$, and $\mathbf{e}_2$ is moved to $-\mathbf{e}_2$. So, we have

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b) When $\mathbf{e}_1 = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, $\mathbf{e}_1$ is reflected to $\mathbf{e}_1' = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$. We have to write this in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$, so we solve the equation

$$\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} = a\mathbf{e}_1 + b\mathbf{e}_2 = a\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) + b\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$$

Equating components of $\mathbf{i}$ and $\mathbf{j}$, we find that

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}a - \frac{1}{2}b$$

$$-\frac{1}{2} = \frac{1}{2}a + \frac{\sqrt{3}}{2}b$$

Solving these, we obtain $a = \frac{1}{2}, b = -\frac{\sqrt{3}}{2}$.

Therefore, we have

$$\mathrm{T}(\mathbf{e}_1) = \frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2$$

Similarly, we can find

$$\mathrm{T}(\mathbf{e}_2) = -\frac{\sqrt{3}}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2$$

Therefore, the transformation matrix is

$$\mathbf{T} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Linear transformations of the set of all complex numbers

The complex conjugate mapping $z \to z^*$ is a linear transformation of the set of all complex numbers regarded as a real vector space. This is because $z \to z^*$ corresponds to a reflection in the real axis of an Argand diagram, which corresponds to a reflection in the line $y = 0$ in $\mathbb{R}^2$.

Example 17 Find the transformation matrix for the mapping

$$f : z \to (1+i)z + (2-i)z^*$$

with respect to the basis $\{1, i\}$.

SOLUTION

We put $\mathbf{e}_1 = \mathbf{1}$ and $\mathbf{e}_2 = \mathbf{i}$. (This reminds us to think of 1 and i as **basis vectors**.)

We now calculate the effect of the transformation on each vector.

$$f(\mathbf{e}_1) = f(\mathbf{1}) = (\mathbf{1}+\mathbf{i}) + (\mathbf{2}-\mathbf{i}) = \mathbf{3} = 3\mathbf{e}_1$$

$$f(\mathbf{e}_2) = f(\mathbf{i}) = (\mathbf{1}+\mathbf{i})\mathbf{i} + (\mathbf{2}-\mathbf{i})(-\mathbf{i}) = -\mathbf{2}-\mathbf{i} = -2\mathbf{e}_1 - \mathbf{e}_2$$

Therefore, with respect to the basis $\{1, i\}$, the transformation matrix is

$$\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix}$$

Exercise 17C

1 The vectors **a**, **b** and **c** are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -2 \\ -8 \end{pmatrix}$$

Prove that **a**, **b** and **c** are linearly dependent.

2 What is the span of the three vectors $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^3$?

Find two vectors with the same span.

3 The vectors **a**, **b** and **c** are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 \\ 9 \\ 13 \end{pmatrix}$$

Prove that **a**, **b** and **c** are linearly independent.

4 The vectors **a**, **b** and **c** in $\mathbb{R}^3$ are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Prove that **a**, **b** and **c** are linearly dependent. Find a vector **d** which, together with **a**, **b** and **c**, spans $\mathbb{R}^3$.

5 Find the transformation matrix (with respect to the standard basis) of the linear transformation which takes $(1, 0)$ to $(2, 1)$ and $(1, 1)$ to $(3, 4)$.

Now find the transformation matrix with respect to the basis $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

6 The linear transformation $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

a) Describe all the points in the image of T.
b) Write down the dimension of the image of T.
c) Find a spanning set for the image of T.

7 Certain sets of functions can be viewed as real vector spaces. For example, consider the set

$$T := \left\{ \text{functions} \,|\, \mathrm{f} : \mathbb{R} \to \mathbb{R}, \mathrm{f}(x) = a_0 + \sum_{n=0}^{\infty} [b_n \sin nx + a_n \cos nx] \right\}$$

'Vectors' in the vector space T are really functions.

a) Prove that T forms a real vector space.
b) Prove that the vectors $\sin x$ and $\sin 2x$ are linearly independent.

8 Let V be a real vector space with operations $+$ and $.$. A set U, with operations $+$ and $.$, is said to be a subspace of V if the following conditions hold:

- U contains 0.
- U is closed under addition.
- U is closed under scalar multiplication.

a) Prove that if U is a subspace of V, then U is also a vector space.
b) What is the smallest possible subspace of $\mathbb{R}^3$?

9 The set $\mathbb{C}$ of complex numbers can be regarded as a real vector space, where addition of complex numbers, and multiplication of a complex number by a real scalar, are defined in the usual way.

i) Show that $\{1, \mathrm{j}\}$ is a basis for this vector space.

Let $u = a + b\mathrm{j}$ and $v = c + d\mathrm{j}$ (where a, b, c and d are real) be fixed complex numbers.
A mapping $\mathrm{T} : \mathbb{C} \to \mathbb{C}$ is defined by $\mathrm{T}(z) = uz + vz^*$ (where z^* is the complex conjugate of z).

ii) Show that T is a linear mapping.
iii) Find the matrix $\mathbf{M}$ associated with T and the basis $\{1, \mathrm{j}\}$.
iv) Given that $\mathbf{M} = \begin{pmatrix} 6 & 1 \\ 11 & 2 \end{pmatrix}$, find the complex number z for which $uz + vz^* = 1 + 4\mathrm{j}$.

(MEI)

Answers

Exercise 1A

1 a) $-\mathrm{i}$ **b)** 1 **c)** -1 **d)** i **2 a)** $3+2\mathrm{i}$ **b)** $6-3\mathrm{i}$ **c)** $-4+3\mathrm{i}$ **d)** $-2+2\sqrt{2}\mathrm{i}$ **e)** $2\mathrm{i}$ **3 a)** $3-4\mathrm{i}$ **b)** $2+6\mathrm{i}$ **c)** $-4+3\mathrm{i}$ **d)** $-8-5\mathrm{i}$ **4 a)** $-1\pm\sqrt{3}\mathrm{i}$ **b)** $\frac{3}{2}\pm\frac{\sqrt{15}}{2}\mathrm{i}$ **c)** $\frac{1}{4}(-1\pm\sqrt{7}\mathrm{i})$ **d)** $1\pm\frac{\sqrt{2}}{2}\mathrm{i}$ **5 a)** $10-2\mathrm{i}$ **b)** $1-\mathrm{i}$ **c)** $-1+2\mathrm{i}$ **d)** $2+55\mathrm{i}$ **e)** $1+\mathrm{i}$ **f)** $3+8\mathrm{i}$ **g)** $10+18\mathrm{i}$ **h)** $18+13\mathrm{i}$ **6 a)** $3+11\mathrm{i}$ **b)** $26+2\mathrm{i}$ **c)** $74+7\mathrm{i}$ **d)** $42-24\mathrm{i}$ **e)** $10+11\mathrm{i}$ **f)** $11-29\mathrm{i}$ **7 a)** $\frac{1}{17}(5+14\mathrm{i})$ **b)** $\frac{1}{26}(23+11\mathrm{i})$ **c)** $1-2\mathrm{i}$ **d)** $\frac{1}{13}(4-19\mathrm{i})$ **8 a)** $x=4, y=-2$ **b)** $x=-11, y=22$ **c)** $x=8, y=-1$ **d)** $x=17, y=-17$ **e)** $x=\frac{13}{5}, y=\frac{9}{5}$ **f)** $x=-5, y=-12$ **9** $3.3+0.9\mathrm{i}$ **10 a)** $-2+\sqrt{3}\mathrm{i}$ **b)** $-1\pm\sqrt{5}\mathrm{i}$ **c)** $\frac{1}{2}(-3\pm3\mathrm{i})$ **d)** $\frac{5}{2}(1\pm\sqrt{3}\mathrm{i})$

Exercise 1B

2 a) $2\sqrt{2}, \pi/4$ **b)** $3\sqrt{2}, 3\pi/4$ **c)** $4, 2\pi/3$ **d)** $\sqrt{2}, -3\pi/4$ **e)** $4, \pi/2$ **f)** $13, \tan^{-1}\left(\frac{12}{5}\right)$ **g)** $4, \pi$ **h)** $7, \tan^{-1}\left(\frac{\sqrt{13}}{6}\right)$

3 a) i) $-7+24\mathrm{i}$ **ii)** $-117+44\mathrm{i}$ **b) i)** 5 **ii)** 25 **iii)** 125 **c) i)** 0.9273 **ii)** 1.8546 **iii)** 2.7819 **4 a)** $1+\sqrt{3}\mathrm{i}$ **b)** $2\sqrt{2}+2\sqrt{2}\mathrm{i}$

4 c) $-\mathrm{i}$ **d)** $-2\sqrt{2}+2\sqrt{2}\mathrm{i}$ **e)** $-\sqrt{3}+\mathrm{i}$ **f)** $-3\sqrt{3}-3\mathrm{i}$ **5 a)** $-\frac{1}{5}+\frac{2}{5}\mathrm{i}$ **b)** $13, \pi-\tan^{-1}\left(\frac{12}{5}\right)$ or 1.9656 **6** $\frac{5}{13}, 2.1033$

7 a) $-\frac{1}{5}+\frac{3}{5}\mathrm{i}$ **b)** $\frac{1}{5}\sqrt{10}, \pi-\tan^{-1}3$ or 1.8925 **8 i) b)** $3\pi/4$ **ii)** $\pm(5-3\mathrm{i})$ **9 i)** $\sec\alpha$ **ii)** $4\sec\alpha$ **iii)** $\pi/2-\alpha$ **iv)** $2\pi/5-\alpha$
10 i) $\sin\alpha$ **ii)** $\pi/2-\alpha$ **11 a)** $3-4\mathrm{i}$ **c)** 1.95 **d)** $6+14\mathrm{i}$ **12 a)** $z=-2\pm\sqrt{3}\mathrm{i}$ **c) i)** 2.65 **ii)** ±2.43 **13** $2-3\mathrm{i}$ **14** $\frac{6}{5}+\frac{12}{5}\mathrm{i}$
15 b) 5 **c)** $\frac{1}{13}(7+17\mathrm{i})$ **d)** 1.18 **e)** $-7, -5$ **16 a)** $-11-2\mathrm{i}, 1+2\mathrm{i}$ **c)** 2.2 **d)** $-1, -2$ **17 a)** $2\sqrt{2}, 3\pi/4$ **b)** $1/2\sqrt{2}, -3\pi/4$
17 d) $90°$ **18 a)** 2.68 rad **b)** $6, 4$ **d)** 28 **19 i)** $\pm(3+2\mathrm{i})$ **ii)** $-\frac{119}{169}$ **20 a)** $14+2\mathrm{i}, 1+\mathrm{i}$ **b) i)** $2\sqrt{5}$, 0.464 rad; $\sqrt{10}, -0.3218$ rad
20 b) ii) $\sqrt{10}$

Exercise 1C

6 a) -1 **b)** 2 **c)** 0 **7 a)** 0 or 3 **b)** 2 **c)** w^2 **d)** -1 or 2 **8** $3, \frac{3}{2}+\frac{1}{2}\sqrt{3}\mathrm{i}, \frac{3}{2}-\frac{1}{2}\sqrt{3}\mathrm{i}$ **10 b)** $5\mathrm{i}, 3\sqrt{2}$ **11 i)** 2 **ii)** $5\pi/6$
14 2 **15 i) a)** $\frac{1}{2}(1+\mathrm{i})$ **b)** $\frac{1}{2}(1-\mathrm{i})$ **ii)** $(1-w)/w$ **16 b) i)** $3/2, \sqrt{3}/2$ **ii)** $\pi/6$ **17 a)** $\pi/4, -4(1+a)^4$ **b)** $y=x$

Exercise 2A

1 $n\pi+(-1)^n\frac{\pi}{4}, 180n°+(-1)^n45°$ **2** $2n\pi\pm\frac{2\pi}{3}, 360n°\pm120°$ **3** $n\frac{\pi}{2}+(-1)^n\frac{\pi}{12}, 90n°+(-1)^n15°$ **4** $n\frac{\pi}{3}+\frac{\pi}{12}, 60n°+15°$

5 $n\pi+\frac{\pi}{8}$ **6** $\frac{2}{3}n\pi+\frac{2\pi}{9}, \frac{2}{3}n\pi$ **7** $n\frac{\pi}{2}-\frac{\pi}{24}$ **8** $n\frac{\pi}{4}+\frac{\pi}{16}, 45n°+11.25°$ **9** $\frac{2}{3}n\pi+\frac{\pi}{3}, 120n°+60°$

10 $n\pi+\frac{\pi}{2}, 2n\pi\pm\frac{\pi}{3}; 180n°+90°, 360n°\pm60°$ **11** $n\pi+\frac{\pi}{2}, 180n°+90°$ **12** $n\frac{\pi}{2}, \frac{2}{3}n\pi\pm\frac{\pi}{9}; 90n°, 120n°\pm20°$ **13** $n\pi, 180n°$

14 $2n\pi+\frac{3\pi}{2}, n\pi+(-1)^n0.848; 360n°+270°, 180n°+(-1)^n48.59°$ **15** $n\frac{\pi}{2}-\frac{\pi}{8}, n\frac{\pi}{5}-\frac{\pi}{20}; 90n°-22.5°, 36n°-9°$ **16** $n\pi, n\pi\pm\frac{\pi}{6}$

17 $n\pi+\frac{\pi}{2}, 2n\pi+(-1)^n\frac{\pi}{6}$ **18** $\frac{\pi}{3}+2n\pi, \frac{2}{3}n\pi-\frac{\pi}{9}$ **19** $\frac{2t}{1-t^2}; n\pi, n\pi\pm\frac{\pi}{3}$

Exercise 2B

1 a) $13, 67.3°$ **b)** $5, 53.13°$ **c)** $5, 53.13°$ **d)** $\sqrt{2}, 45°$ **e)** $10, 53.13°$ **2 a) i)** $15, -15$ **ii)** $306.87°, 143.13°$

2 b) i) $10, -10$ **ii)** $18.43°, 108.43°$ **c) i)** $\frac{4}{3}, \frac{4}{13}$ **ii)** $53.13°, 233, 13°$ **d) i)** $\frac{3}{4-\sqrt{5}}, \frac{3}{4+\sqrt{5}}$ **ii)** $296.56°, 116.56°$

2 e) i) $-\frac{1}{2}$ (max), $\frac{1}{3}$ (min) **ii)** 233.13°, 53.13° **f) i)** $-\frac{3}{2}$ (max), $\frac{1}{6}$ (min) **ii)** 216.87°, 36.87° **3 a)** $360n° + 53.13° \pm 60°$
3 b) $360n° - 22.62° \pm 60°$ **c)** $180n° - 7.5°, 180n° + 52.5°$ **d)** $60n° + 15° + (-1)^n 10°$ **e)** $60n°, 60n° - 17.7°$

Exercise 2C

1 a) $-\pi/6$ **b)** $\pi/6$ **c)** $5\pi/6$ **d)** $\pi/4$ **e)** $\pi/4$ **f)** 18.4° or 0.3218 rad **3** $\pi/10$ **5** $360n° - 66.8° \pm 142.0°$ **6 a)** 13 **b)** 67.4°
6 c) $360n° - 67.4° \pm 72.1°$ **7 i)** $\sqrt{58}\cos(\theta + 23.2°)$ **ii)** $\sqrt{58}, -\sqrt{58}$ **iii)** $360n° - 23.2° \pm 82.5°$ **8 a)** 126.9°, 270°
8 b) $180n° + 90°, 60n° + 15°$ **9 a)** 25, 73.7° **b)** $360n° + 20.6°$ or $360n° + 126.8°$ **c)** $\frac{1}{630} \leqslant f(x) \leqslant \frac{1}{5}$ **10** 60°; 195°, 345°
11 $2\cos(\theta - 60°), 360n° + 60° \pm 40°$ **12 a)** 85, 0.154 rad **b)** $2\pi n - 0.154 \pm 1.369$ **13 a)** 25 **b)** −25 **c)** 1.85 rad
13 d) 3.84 rad, 6.16 rad **14 i) a)** 15, 53.13° **b)** 156.9°, 276.9° **ii)** $n\pi + \pi/3$

Exercise 2D

1 (a) $\dfrac{5}{\sqrt{1-25x^2}}$ **b)** $\dfrac{3}{1+9x^2}$ **c)** $\dfrac{\sqrt{2}}{\sqrt{1-2x^2}}$ **d)** $\dfrac{12}{16+9x^2}$ **e)** $\dfrac{2x}{\sqrt{1-x^4}}$ **f)** $\dfrac{1-x^2}{1+3x^2+x^4}$ **g)** $\dfrac{6(\sin^{-1}2x)^2}{\sqrt{1-4x^2}}$ **h)** $\dfrac{1620(\tan^{-1}5x)^3}{1+25x^2}$

1 i) $\dfrac{1}{x\sqrt{x^2-1}}$ **j)** $-\dfrac{1}{x^2+1}$ **2 a)** $\sin^{-1}\left(\dfrac{x}{2}\right) + c$ **b)** $\sin^{-1}\left(\dfrac{x}{3}\right) + c$ **c)** $\dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{5}\right) + c$ **d)** $\dfrac{1}{9}\sin^{-1}\left(\dfrac{3x}{4}\right) + c$ **e)** $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x}{3}\right) + c$

2 f) $\dfrac{1}{4}\tan^{-1}\left(\dfrac{x}{4}\right) + c$ **g)** $\dfrac{1}{20}\tan^{-1}\left(\dfrac{4x}{5}\right) + c$ **h)** $\dfrac{1}{15}\tan^{-1}\left(\dfrac{5x}{3}\right) + c$ **3 a)** $\pi/2$ **b)** $\pi/8$ **c)** $\pi/2$ **d)** $\pi/6\sqrt{3}$ **e)** $\pi/5$

4 a) 0.0505 **b)** 0.0444 **c)** 0.615 **d)** 0.0741 **e)** 0.841 **f)** 0.0207 **5** $\pi/24$ **6** $9 - (x-2)^2, \pi/3$ **7** $2 + \dfrac{1}{1+x} + \dfrac{2x+1}{x^2+4}$

8 $\dfrac{\sqrt{3}}{3} - \dfrac{\pi}{12}$ **9** $\dfrac{y + x\dfrac{dy}{dx}}{1+x^2y^2}$ **10 ii)** $-\dfrac{x}{\sqrt{1-x^2}}$ **iii)** $\dfrac{\pi}{2} - 1$ **11** $-\sin^{-1}\left(\dfrac{1}{x}\right) + c$ **12 ii)** $\frac{1}{2}t\sqrt{1-t^2}$

Exercise 3B

1 a) $x^2 + y^2 = 16$ **b)** $x = 3$ **c)** $y = 7$ **d)** $x^2 + y^2 = ax + a\sqrt{x^2+y^2}$ **e)** $x^2 + y^2 + ax = a\sqrt{x^2+y^2}$ **f)** $y^2 = 4 - 4x$

2 a) $r = 3$ **b)** $r^2\sin 2\theta = 32$ **c)** $\dfrac{r^2\cos^2\theta}{9} + \dfrac{r^2\sin^2\theta}{16} = 1$ **d)** $r = 6\cos\theta$ **e)** $r^2 + 8r\sin\theta = 16$ **f)** $r^2 = \cos 2\theta$

Exercise 3D

1 $\dfrac{7\pi^3a^2}{48}$ **2 a)** $\dfrac{\pi a^2}{8}$ **b)** $\dfrac{\pi a^2}{8}$ **c)** $\dfrac{\pi a^2}{16}$ **3** $\dfrac{\pi a^2}{4}$ **4** $5\sqrt{5} + \dfrac{17}{2}\cos^{-1}\left(-\dfrac{2}{3}\right)$ **5 a)** $r^2 = \sin^4\theta$ **b) ii)** $\dfrac{3\pi}{8}$

6 $\left(1, \dfrac{\pi}{6}\right), \left(1, \dfrac{5\pi}{6}\right); \dfrac{7\pi}{3} - 4\sqrt{3}$ **7 b)** $\dfrac{\pi a^2}{12}$ **8 b)** $\dfrac{9\pi a^2}{2}$

Exercise 3E

1 $y = 0.185a, y = \pm 0.88a$ **2 a)** $y = \dfrac{e^{-\pi/4}}{\sqrt{2}}, y = \dfrac{e^{3\pi/4}}{\sqrt{2}}, \ldots$ **b)** $x = \dfrac{e^{-3\pi/4}}{\sqrt{2}}, x = \dfrac{e^{\pi/4}}{\sqrt{2}}, \ldots$ **3 a)** $y = \pm\dfrac{2}{3\sqrt{6}}$ **b)** $x = \pm a$

4 b) $\dfrac{\pi}{16}$ **c) iii)** $\frac{16}{27}$ **5** $\sqrt{3}\cos\theta - \cos^2\theta$; $0, \dfrac{\pi}{6}, -\dfrac{\pi}{6}, \pi$ **6 b)** (0.667, 0.421), (0.667, −0.421) **7 b)** $a^2\left(2 - \dfrac{\pi}{4}\right)$

8 b) $(0, 0), \left(3, \dfrac{\pi}{3}\right)$ **c)** 1.33 **9 a)** $r = 2$ **b)** $r\cos\theta = 3$ **c)** $r\sin\left(\theta + \dfrac{\pi}{3}\right) = 2\sqrt{3}$

Exercise 4A

1 a) x^2 **b)** $\sqrt{x^2+1}$ **c)** $\dfrac{1}{x^3}$ **d)** $\sec x$ **e)** $\sqrt{x^2-1}$ **f)** 2^{3x} **2** $y = \frac{1}{3}x - \frac{1}{9} + ce^{-3x}$ **3** $y = -\frac{1}{3}e^{2x} + ce^{5x}$ **4** $y = \dfrac{x^2}{3} + \dfrac{c}{x}$

5 $y = x^3 + cx^2$ **6** $y = 5(x-1)^4\ln(x-1) + c(x-1)^4$ **7** $y\sin x = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + c$ **8 a)** $ye^x = x + c$
8 b) i) $y = (x-1)e^{-x}$ **ii)** $y = 0$ **9** $y = (c + \frac{1}{2}x^2)e^{x^3}, y = (1 + \frac{1}{2}x^2)e^{x^3}$ **10** $y = x\cos x + c\cos x$

11 $y = \frac{1}{5}x + cx^{-4}, y = \frac{1}{5}(x + 4x^{-4})$ **12** $y = x^2 - x + cx^{-4}$ **13 i)** $v = \dfrac{1}{\alpha+\beta}e^{\beta t} + ce^{\alpha t}$ **14** $v = 200\tan 0.2 + \sec 0.2$

15 i) $y\sec x = x + c$ **ii)** $y = (x+2)\cos x$ **16** $y = (3x^4+5)e^{-x^2-x}$ **17** $N = \frac{\mu}{\lambda}t + \frac{\mu}{\lambda^2} + \left(N_0 - \frac{\mu}{\lambda^2}\right)e^{\lambda t}$ **18 i)** $y = ce^{kx} - x - \frac{1}{k}$

18 ii) a) $y = -x - \frac{1}{4} - \frac{3}{4}e^{4(x-1)}$ **19 ii)** $y\sin^2 x = -\frac{1}{2}\cos 2x + c$ **20** $s = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$, $s = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{\pi}{8}$

21 a) $y = cx - xe^{-x}$ **b) i)** $y = -xe^{-x}$ **ii)** 0

Exercise 4B

1 $y = e^{(3+\sqrt{17})x} + Be^{(3-\sqrt{17})x}$ **2** $y = Ae^{-x} + Be^{-2x}$ **3** $y = Ae^{2x} + Be^{-3/2x}$ **4** $y = Ae^{x} + Be^{-7x/3}$ **5** $x = Ae^{8t} + Be^{-t}$

6 $x = Ae^{7t} + Be^{4t}$ **7** $y = (A + Bx)e^{2x}$ **8** $y = (A + Bx)e^{-3x}$ **9** $y = e^{-x/2}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

10 $y = e^{-2x}(A\cos 2x + B\sin 2x)$ **11** $x = Ae^{(3+\sqrt{2})t} + Be^{(3-\sqrt{2})t}$ **12** $x = e^{-t}(A\cos 2\sqrt{3}t + B\sin 2\sqrt{3}t)$

Exercise 4C

1 $y = Ae^{-8x} + Be^{x} - 2x - \frac{7}{4}$ **2** $y = Ae^{-x} + Be^{-3x} - 4e^{-2x}$ **3** $y = Ae^{-x} + Be^{5x/2} - 2x^2 + \frac{12}{5}x - \frac{81}{25}$

4 $y = Ae^{-x} + Be^{x/3} - \frac{304}{5876}\sin 5x - \frac{40}{5876}\cos 5x$ **5** $x = Ae^{5t} + Be^{-t} - \frac{3}{8}e^{3t}$

6 $s = Ae^{5t} + Be^{3t} + \frac{55}{377}\cos 2t - \frac{80}{377}\sin 2t$ **7** $y = Ae^{-4x} + Be^{-x} + \frac{2}{3}xe^{-x}$ **8** $y = (A + Bx + \frac{5}{2}x^2)e^{3x}$

9 $y = e^{x}(A\cos\sqrt{2}x + B\sin\sqrt{2}x) + 2e^{4x}$ **10** $y = e^{-3x}(A\cos x + B\sin x) + \frac{3}{2}e^{-4x}$ **11** $x = (A + Bt + 2t^2)e^{t}$

12 $x = A\cos 4t + B\sin 4t + \frac{3}{8}t\sin 4t$ **13** $x = (2\sin 2t - 3\cos 2t)e^{t}$ **14 a)** $y = Ae^{2x} + Be^{-2x} + 2e^{3x}$ **b)** $y = e^{-2x} - 5e^{2x} + 2e^{3x}$

15 $y = A + Bx + \frac{1}{4}e^{2x} - 4\cos\frac{1}{2}x$; $y(0)$ and $y'(0)$, or y for two values of x **16 i)** $y = e^{-2x}(4\cos 3x + 3\sin 3x)$ **ii)** $\frac{3}{2} - \frac{\pi}{24}$

17 $y = e^{2x}(A\cos x + B\sin x) + \frac{1}{65}(\sin 2x + 8\cos 2x)$ **18 i)** $x = A\cos 4t + B\sin 4t$ **iii)** 1.424 **19** $x = 2t + 3 + 4e^{-t/10}\sin\left(\frac{t}{5}\right)$

20 $x = Ae^{-4t} + Be^{-t} + \sin 3t$ **21** $y = Ae^{4x} + Be^{-x} + 3\cos 2x - 4\sin 2x$, $y = 3\cos 2x - 4\sin 2x - 3e^{-x}$

22 i) $x = e^{2t}(A\cos 5t + B\sin 5t) + 2\sin 2t$ **ii)** $x = 3e^{2t}\cos 5t + 2\sin 2t$ **23 a)** $\ln(1+y) = c + \frac{1}{2}x^2$

23 b) $y = Ce^{3x}$; $y = -\frac{2}{9} - \frac{2}{3}x + e^{4x}$, $y = Ce^{3x} - \frac{2}{9} - \frac{2}{3}x + e^{4x}$ **24 a)** $y = e^{-2x}(A\cos 3x + B\sin 3x)$ **b)** $\frac{3}{4}$, $\frac{1}{4}$

24 d) $y = \frac{3}{4}\cos 3x + \frac{1}{4}\sin 3x - \frac{3}{4}e^{-2x}(\cos 3x + \sin 3x)$ **25 a)** $y = Ae^{4x} + Be^{-x/2} + \sin x + 2\cos x$ **b)** $y = 2e^{4x} - 4e^{-x/2} + \sin x + 2\cos x$

26 i) $x = e^{-t}\left[A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right] + 2\cos t$ **ii)** \$9 550 000 **27** $p = 0, q = -\frac{1}{4}$, $y = A\sin 2x + B\cos 2x - \frac{1}{4}x\cos 2x$, $y \approx \frac{1}{4}(n + \frac{1}{2})\pi$

28 a) $\frac{1}{2}$ **b)** $y = (1+t)e^{-t} + \frac{1}{2}t^2e^{-t}$

Exercise 4D

1 a) $x^3(x - 4y) = c$ **b)** $y^2 = 2x^2\ln x + cx^2$ **c)** $\ln\left[\frac{x^4(x-y)}{x+y}\right] = \frac{2x}{x+y} + c$ **d)** $\frac{3}{8}\ln\left[\frac{(y-2x)(y+2x)}{y^2}\right] = \ln x + c$

2 $x + y - \frac{1}{4}\ln(4x + 4y + 5) = 4x + c$ **3** $2x + 3y + \frac{21}{8}\ln(16x + 24y - 13) = 8x + c$ **4 a)** $y = Ax + \frac{B}{x^2}$ **b)** $y = Ax^{3+\sqrt{15}} + Bx^{3-\sqrt{15}}$

4 c) $y = Ax^2 + Bx^2\ln x$ **d)** $y = x^{\frac{1}{2}}\left[A\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$ **5** $\frac{dy}{dx} = 2\sqrt{t}\frac{dy}{dt}$, $y = Ae^{x^2} + Be^{-4x^2} + \frac{1}{6}e^{2x^2}$

6 a) $z = \frac{1}{2}e^{x} + ce^{-x}$ **b)** $y = \frac{1}{2}xe^{x} + cxe^{-x}$ **7 b)** 2 **c)** $xy = 2e^4(1-x)e^{-2x} + 2e^{2x}$ **d)** Infinite

8 $x = (A + Bt)e^{-t}$, $y = (A + B + Bt)e^{-t}$; $x = (1-t)e^{-t}$, $y = -te^{-t}$ **9 a)** $y = \frac{(\cosh^{-1}x + c)}{\sqrt{x^2 - 1}}$ **b) ii)** $y = \frac{1}{x}(A\sin 5x + B\cos 5x)$

Exercise 5A

1 a) 177 **b)** 15 **c)** 0 **d)** −255 **2 a)** $(a+b+c)(a-b)(b-c)(c-a)$ **b)** $-24pqr$ **c)** $(ab+bc+ca)(b-c)(c-a)$

d) 0 **3** $(a-b)(b-c)(c-a)(a+b+c+1)$, −1

Exercise 5B

1 $(a-b)(b-c)(c-a)(a+b+c)$; 1, 2, −3; $(t, 0, t)$ **2 ii)** $(5-t, t-1, t)$ **3 i)** $\left(\frac{7t-19}{5}, \frac{37-11t}{5}, t\right)$ **4 a)** $a = b = c$

4 b) $x = a + b$, $y = b - a + t$, $z = t$ **i)** Planes intersect in a line **ii)** Two planes parallel **5** $\frac{1}{2}$, −2; planes form triangular prism

6 10 **7** $1 - k^2$; $k \neq 1, -1$ **i)** 2, 1, 0

7 iii) $2+3t, -1-2t, t$ Three planes meet at point; planes form triangular prism; two planes coincident

8 a) $(a-b)(b-c)(c-a)(a+b+c)$ **b)** -5 **9 ii)** $x=-17, y=\frac{17}{2}, z=-1$ **iv)** Two parallel planes, intersecting third plane

10 i) $(p+1)(q-2)$ **iii)** 7 **iv)** Sheaf of planes **11** $(-\frac{1}{3}t, \frac{2}{3}t, t)$; three planes intersect in line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{-3}$ **12 i)** $q \neq 2$

12 ii) $p=-4, q=2$ **iii)** $q=2, p \neq -4$

Exercise 6A

1 a) $\mathbf{r}=\begin{pmatrix}2\\-7\\5\end{pmatrix}+t\begin{pmatrix}3\\4\\-7\end{pmatrix}$ **b)** $\mathbf{r}=\begin{pmatrix}4\\8\\-6\end{pmatrix}+t\begin{pmatrix}-2\\3\\6\end{pmatrix}$ **c)** $\mathbf{r}=\begin{pmatrix}7\\4\\-1\end{pmatrix}+t\begin{pmatrix}2\\-1\\-3\end{pmatrix}$ **d)** $\mathbf{r}=\begin{pmatrix}-8\\1\\-3\end{pmatrix}+t\begin{pmatrix}1\\3\\-7\end{pmatrix}$

2 a) $\mathbf{r}=\begin{pmatrix}4\\8\\-2\end{pmatrix}+t\begin{pmatrix}3\\11\\-6\end{pmatrix}$ **b)** $\mathbf{r}=\begin{pmatrix}-1\\8\\3\end{pmatrix}+t\begin{pmatrix}3\\-11\\6\end{pmatrix}$ **c)** $\mathbf{r}=\begin{pmatrix}1\\7\\-2\end{pmatrix}+t\begin{pmatrix}4\\3\\-10\end{pmatrix}$ **d)** $\mathbf{r}=\begin{pmatrix}3\\-5\\-9\end{pmatrix}+t\begin{pmatrix}-5\\2\\16\end{pmatrix}$

3 a) $\frac{x-2}{3}=\frac{y+7}{4}=\frac{z-5}{-7}$ **b)** $\frac{x-4}{-2}=\frac{y-8}{3}=\frac{z+6}{6}$ **c)** $\frac{x-7}{2}=\frac{y-4}{-1}=\frac{z+1}{-3}$ **d)** $\frac{x+8}{1}=\frac{y-1}{3}=\frac{z+3}{-7}$

4 a) $\mathbf{r}=\begin{pmatrix}3\\-2\\4\end{pmatrix}+t\begin{pmatrix}4\\3\\-5\end{pmatrix}$ **b)** $\mathbf{r}=\begin{pmatrix}-2\\1\\-3\end{pmatrix}+t\begin{pmatrix}5\\-7\\-2\end{pmatrix}$ **c)** $\mathbf{r}=\begin{pmatrix}-5\\2\\-4\end{pmatrix}+t\begin{pmatrix}1\\-3\\2\end{pmatrix}$ **d)** $\mathbf{r}=\begin{pmatrix}\frac{3}{2}\\5\\2\end{pmatrix}+t\begin{pmatrix}2\\3\\-1\end{pmatrix}$

4 e) $\mathbf{r}=\begin{pmatrix}\frac{5}{3}\\-2\\2\end{pmatrix}+t\begin{pmatrix}2\\4\\-3\end{pmatrix}$ **5 a)** $109°$ **b)** $93.3°$ **6 a)** $\mathbf{r}=\begin{pmatrix}2\\1\\4\end{pmatrix}+t\begin{pmatrix}2\\6\\1\end{pmatrix}$ **b)** $\mathbf{r}=\begin{pmatrix}-1\\-4\\3\end{pmatrix}+t\begin{pmatrix}3\\12\\1\end{pmatrix}$

6 c) $\mathbf{r}=\begin{pmatrix}4\\1\\-5\end{pmatrix}+t\begin{pmatrix}-1\\1\\-1\end{pmatrix}$ **7** $2\sqrt{2}$ **8** $19/5\sqrt{2}$ **9 a) i)** $1:2:-2$ **ii)** $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$ **b) i)** $3:-4:-5$ **ii)** $3/5\sqrt{2}, -4/5\sqrt{2}, -1/\sqrt{2}$

9 c) i) $3:2:-5$ **ii)** $3/\sqrt{38}, 2/\sqrt{38}, -5/\sqrt{38}$ **d) i)** $1:-2:-3$ **ii)** $1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14}$

10 a) $\mathbf{r}=2\mathbf{i}+\mathbf{j}+\mathbf{k}+t(-2\mathbf{i}+4\mathbf{j}+2\mathbf{k})$ **d)** $18.7°$ **11 b)** $\mathbf{r}=-9\mathbf{j}+13\mathbf{k}+t(\mathbf{i}+2\mathbf{j}-3\mathbf{k})$ **c)** $(5, 1, -2)$ **e)** $43°$ **f)** $(4.5, 0, -0.5)$

12 a) $(4, -1, -3)$ **b)** $71.4°$ **13 a)** $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}0\\4\\-3\end{pmatrix}$ **b)** $21°$ **14** $47\mathbf{i}-9\mathbf{j}+62\mathbf{k}, a=-5$ **15 a)** $\mathbf{i}-3\mathbf{j}-\mathbf{k}$ **c)** $95.2°$

16 i) $(2, 3, 5)$ **ii)** $40.9°$ **17 a)** 0.148 rad **18** $(1, 1, -12)$ **19 i)** $\mathbf{r}=\begin{pmatrix}7\\-8\\7\end{pmatrix}+t\begin{pmatrix}1\\-5\\1\end{pmatrix}$ **ii)** $5\mathbf{i}+2\mathbf{j}+5\mathbf{k}$

Exercise 6B

1 a) $-5\mathbf{i}+7\mathbf{j}+11\mathbf{k}$ **b)** $31\mathbf{i}+22\mathbf{j}+\mathbf{k}$ **c)** $22\mathbf{i}+14\mathbf{j}+16\mathbf{k}$ **d)** $-32\mathbf{i}+23\mathbf{j}-10\mathbf{k}$ **2 a)** $\mathbf{r}.\begin{pmatrix}3\\-5\\4\end{pmatrix}=-13$ **b)** $\mathbf{r}.\begin{pmatrix}9\\7\\-2\end{pmatrix}=47$

2 c) $\mathbf{r}.\begin{pmatrix}28\\-17\\18\end{pmatrix}=41$ **3 a)** $3x+y+7z=4$ **b)** $2x+4y+3z=8$ **c)** $-x+5y+3z+7=0$ **4 a)** $68.5°$ **b)** $34.1°$ **c)** $28.1°$

4 d) $48.5°$ **5** $29.1°$ **6** $0°$ **7** $\mathbf{r}.\begin{pmatrix}3/5\sqrt{2}\\4/5\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}=2\sqrt{2}, 2\sqrt{2}$ **8 a) i)** -13 **ii)** $-12\mathbf{i}+8\mathbf{j}+8\mathbf{k}$ **b) i)** $+\frac{13}{21}$ **ii)** $2\sqrt{17}$

8 b) iii) $\mathbf{r}.\begin{pmatrix}-3\\2\\2\end{pmatrix}=3$ **9 a)** $-\mathbf{i}+8\mathbf{j}-4\mathbf{k}$ **b)** $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ **d)** $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}4\\13\\25\end{pmatrix}$ **10 a)** $\mathbf{r}=24\mathbf{i}+6\mathbf{j}+t(\mathbf{i}+\mathbf{j}+2\mathbf{k})$

10 c) $(35, 17, 32)$ **d)** $18\sqrt{101}$ **11 a)** $36\mathbf{i}+12\mathbf{j}+9\mathbf{k}$ **b)** $\mathbf{r}.\begin{pmatrix}36\\12\\9\end{pmatrix}=9$ **c)** $(1, -3, 1)$ **12 i)** $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ **13** $\mathbf{r}.\begin{pmatrix}11\\5\\-7\end{pmatrix}=14$

14 $\mathbf{r}=\begin{pmatrix}1\\-8\\7\end{pmatrix}+t\begin{pmatrix}1\\2\\-1\end{pmatrix}$ **i)** $-x+5y+3z=46$ **iii)** $12\mathbf{i}+14\mathbf{j}-4\mathbf{k}$ **15 i)** $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ **iv)** $31.8°$ **16** $3\mathbf{i}+8\mathbf{k}-15\mathbf{j}$, $14.4°$

17 ii) $3x+y+2z=15$ **iii)** $5-x=y=z$ **18 a)** $2\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ **b)** $\frac{1}{2}\sqrt{17}$ **c)** $\mathbf{r}.(2\mathbf{i}-3\mathbf{j}-2\mathbf{k})=-7$ **d)** $2x-3y-2z=-7$

18 e) $\frac{7}{\sqrt{17}}$ **f)** $3.2°$ **19** $\mathbf{i}+3\mathbf{j}-3\mathbf{k}$ **20 i)** $(2, 1, 1)$ **ii)** $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ **iii)** $8\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ **iv)** $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}2\\-47\\22\end{pmatrix}$

21 $-2\mathbf{a}\times\mathbf{b}$; $0, \pi$ **22 b)** $7\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ **c)** $72.8°$ **d)** $(-3, \frac{1}{2}, 4\frac{1}{2})$ **e)** $\frac{1}{2}\sqrt{26}$ **23 a)** $\frac{x}{1}=\frac{y-2}{-1}=\frac{z+3}{-1}$ **b)** $x+z=0$

24 a) $(1, -2, 3)$ **b)** $2x+y-3z=-9$ **c)** $70.9°$ **25 ii)** $7+\frac{2}{3}k$ **iii)** $-\frac{21}{2}$ **iv)** $\mathbf{r}=\begin{pmatrix}4\\12\\5\end{pmatrix}+t\begin{pmatrix}2\\10\\11\end{pmatrix}$ **26 i)** $2\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

26 ii) $x-y-z=12$ **27 a)** $(-3, 0, 1)$ **b)** $-3, -1$ **c)** $x-3y-z+4=0$ **28** $1:-2:4$, $2x-y-z=0$ **29 a) i)** $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$

29 a) iii) $\mathbf{r}.\begin{pmatrix}2\\-3\\6\end{pmatrix}=14$ **b) i)** $\mathbf{r}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$ **iii)** Opposite side of plane to origin; distance 1 from Π

30 a) ii) $\mathbf{r}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=1$ **b) ii)** $3x+5y+3z+1=0$ **31 a)** $\mathbf{r}=\mathbf{i}+2\mathbf{j}+\mathbf{k}+t(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$ **b)** $(3, 3, 4)$ **c)** $5\mathbf{i}-\mathbf{j}-3\mathbf{k}$ **d)** $\sqrt{\frac{35}{34}}$

31 e) $(5, 4, 7)$ **32 i)** $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+t\begin{pmatrix}2\\2\\-1\end{pmatrix}$, $\frac{\sqrt{65}}{3}$ **ii)** $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\3\end{pmatrix}$, $50.8°$ **iii)** $\frac{3}{\sqrt{10}}$ **33 i)** 3 **ii)** $\begin{pmatrix}-17\\-10\\14\end{pmatrix}$ **iii)** $\frac{21}{\sqrt{65}}$

34 $3\mathbf{i}-5\mathbf{j}-7\mathbf{k}$, $3x-5y-7z=1$ **35** $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$, $3x-2y-2z=-1$ **36 b) i)** $(4, 0, 0)$ **iv)** $(2, 1, 4)$ **v)** 8

Exercise 6C

1 -73 **2** 177 **3** 21 **4** 8 **5 i)** 0 **ii)** $\overrightarrow{EF}$ **iii)** $\sqrt{\frac{2}{3}}$ **6 a)** $-30\mathbf{i}-15\mathbf{j}+45\mathbf{k}$ **b)** $\mathbf{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+t\begin{pmatrix}2\\1\\-3\end{pmatrix}$ **d)** 35

7 a) $5\mathbf{i}-3\mathbf{j}-4\mathbf{k}$ **b)** 100 **c)** 50 **8 a)** $-6\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ **b)** $\frac{1}{2}\sqrt{65}$, $\frac{1}{6}$ **c)** $\mathbf{r}.(6\mathbf{i}+2\mathbf{j}-5\mathbf{k})=1$ **d)** $(\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$ **e)** $9.52°$

Exercise 7A

7 a) $y\leqslant 1, y\geqslant \frac{49}{25}$ **b)** $y\geqslant 1, y\leqslant \frac{5}{13}$ **8** $x=-\frac{2}{3}, x=\frac{5}{7}, y=\frac{4}{21}$ **9** $x=-1, y=x-2$ **10 i)** $x=-2$ **ii)** 7

11 i) $x=2, x=-2, y=10$ **ii)** $\frac{16}{(x-2)^3}-\frac{54}{(x+2)^3}$ **iii)** $(10, 8\frac{3}{4})$ **12 a)** $y=1, x=2\sqrt{3}-1, x=-1-2\sqrt{3}$ **b)** $(5, \frac{5}{6}), (1, \frac{1}{2})$

13 a) i) $x=-\frac{1}{2}, y=\frac{x}{2}$ **ii)** $(0, 0), (-1, -1)$ **14 a)** $\frac{5}{4}$ **b) i)** $x=2$ **ii)** $y=2x+1$ **c)** $(3, 8)$ min **15 i)** $y=2+\frac{3}{x-1}+\frac{1}{x+2}$

15 iii) $y=2, x=1, x=-2$ **iv)** $(-\frac{5}{4}, 2)$ **16 a)** $\frac{2t}{(1+t)^2}$ **c)** $y=\frac{x}{2}, x=0$ **d) ii)** $(1, 1), (-1, -1)$

Exercise 7B

1 a) $x>-1, x<-2$ **b)** $x>3$ **c)** $-10<x<-3$ **d)** $x>5, x<-14$ **e)** $-\frac{3}{10}>x>-\frac{1}{2}$ **f)** $\frac{1}{5}<x<\frac{6}{11}$

2 a) $x<-2, -1<x<0$ **b)** $x>8, 2<x<3$ **c)** $5<x<\frac{1}{2}(3+\sqrt{65}), \frac{1}{2}(3-\sqrt{65})<x<-1$ **d)** $-7<x, 3<x<\frac{44}{13}$

2 e) $-2<x<-1\frac{1}{2}$ **3 a)** $-\frac{5}{2}<x\,(x\neq -2)$ **b)** $-5<x<-1\,(x\neq -2)$ **c)** $\frac{5}{3}<x<11\,(x\neq 4)$ **d)** $x>6, -\frac{4}{3}>x\,(x\neq -5)$

3 e) $x>5, x<-\frac{3}{5}\,(x\neq -2)$ **f)** $x>-2\frac{1}{2}\,(x\neq -3)$ **4 a)** $1>x>-2$ **b)** $x>1, x<-\frac{3}{2}$ **c)** $x<-2\,(x\neq -1)$

5 $x>4, -3<x<0$ **6** $x>1, x<-1$ **7** $x>2, -1<x<1$ **8** $-2<x<-1, 1<x<4$ **9** $2<x<5, x<-1$

11 a) $f(x)\leqslant \frac{1}{6}$, $f(x)\geqslant \frac{3}{2}$; $(-4, \frac{3}{2})(6, \frac{1}{6})$ **b)** One **12** $x<-\frac{3}{2}, x>\frac{7}{4}$ **13** $x>-1, x<-\frac{5}{2}$ **14** 1, 2, 4; $y=x+2$

15 a) $(0, 3), (-\frac{3}{2}, 0)$ **b)** $x>-\frac{13}{6}$

Exercise 8A

1 a) $-3, -7$ **b)** 11, 5 **c)** $-5, -4$ **d)** $-\frac{11}{3}, \frac{2}{3}$ **e)** $-2, -5$ **f)** $-2, -\frac{7}{2}$ **2 a)** $x^2-7x+15=0$ **b)** $x^2+3x+5=0$

2 c) $x^2+2x-4=0$ **d)** $x^2+5x-11=0$ **3 a)** 0 **b)** -10 **c)** -9 **4 i)** $9+\mathrm{i}$ **ii)** $\frac{5}{2}-2\mathrm{i}$ **5 i)** $-10-37\mathrm{i}$ **ii)** $-\frac{8}{3}-\frac{7}{3}\mathrm{i}$

6 $x^3-4x^2-4x-25=0$ **7** 6, -6 **8** $3x^2-11x+10=0$ **9 a)** 15

Exercise 8B

1 $x^2+14x+44=0$ **2** $x^2-45x+63=0$ **3** $3x^3-8x^2+32x-56=0$ **4** $8x^3-12x^2-22x+5=0$ **5** $4x^2+59x+289=0$
6 $9x^2+41x+225=0$ **7 a)** $x^2+7x+6=0$ **b)** $6x^2+7x+1=0$ **c)** $4x^2-37x+9=0$ **d)** $2x^2-x-3=0$
8 a) $3x^2+36x-32=0$ **b)** $6x^2+9x-1=0$ **c)** $9x^2-93x+4=0$ **d)** $3x^2+27x+52=0$ **9 a)** $x^3+9x^2+45x+189=0$
9 b) $x^3+x^2-17x-49=0$ **c)** $x^3-6x^2+14x-8=0$ **10** $x^4+9x^3+63x^2-297x+81=0$ **11** $x^2+10x+75=0$
12 $4x^2-3x+1=0$

Exercise 8C

1 $\mathrm{i}, -\mathrm{i}, \frac{5}{2}\pm\sqrt{\frac{21}{2}}$ **2** $2\mathrm{i}, -2\mathrm{i}, 2, -\frac{5}{3}$ **3** 1 **4** 3 **5** $0<k<4$ **6** $1-\mathrm{i}, \frac{3}{2}\pm\frac{\mathrm{i}\sqrt{11}}{2}$ **7 b)** $3-\mathrm{i}, -11$ **8 b) i)** $2+3\mathrm{i}, -1$ (twice)
8 b) ii) $(z^2-4z+13)(z^2+2z+1)$ **9 b)** $-\mathrm{i}, 1+\mathrm{i}, 1-\mathrm{i}$ **10** $2-\mathrm{i}, \frac{2}{3}$ **11 i)** $1+3\mathrm{i}$ **ii)** $5, -\frac{1}{2}$ **12** $-3\mathrm{i}, \frac{5}{3}$
13 b) $(z-2)(z^2-6z+10)$; $2, 3+\mathrm{i}, 3-\mathrm{i}$ **14 i)** $1-3\mathrm{i}$ **ii)** $z^3-4z^2+14z-20=0$ **15** $1-\mathrm{i}$; 2, 18 **16 a)** $\frac{8}{7}$ **b)** $1-2\mathrm{i}, -\frac{6}{7}$
17 i) $3p^2$ **iii)** $z^3-3pz^2+3p^2z-(p^3+q^3)=0$ **18** $5+4\mathrm{i}, 5-4\mathrm{i}, \sqrt{2}(1+\mathrm{i}), \sqrt{2}(1-\mathrm{i}), -\sqrt{2}(1+\mathrm{i}), -\sqrt{2}(1-\mathrm{i})$

Exercise 9A

6 4 **11** $\frac{n}{3}(4n^2-1)$ **14 a)** $\frac{n^2}{(2n-1)(2n+1)}$ **16 a)** $\frac{n}{4}(n+1)(n+2)(n-1)$

18 Counter-example: $\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$ **23** $\frac{n}{6}(n+1)(2n+7)\ln 2$ **25 b)** Not convergent. $(-2)^n$ does not tend to 0

27 Yes **28** $\frac{1}{6}$ **29 i)** Not true: for example, $u=2, v=3, w=6$ **ii)** True
29 iii) True. Converse of **ii** is 'If u divides $v+w$, then u divides both u and w'. This is not true.

Exercise 9B

1 $\frac{2n}{3}(n+1)(n+2)$ **2** $\frac{n}{2}(n+1)(n^2+n+1)$ **3** $\frac{n}{3}(n^2-7)$ **4** $\frac{n}{6}(4n^2+33n-1)$ **5 b)** 61 907 **6** $n^2(2n^2-1)$ **7** 18 760
8 $N^2(2N+1)^2, -N^2(4N+3)$

Exercise 9C

1 $\frac{3}{4}$ **2** $\frac{n^2+3n}{4(n+1)(n+2)}$ **3 a)** $\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ **b)** $a=b=1$ **c)** 0 **4** 8 **5** $\frac{1}{2}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right), \frac{1}{6}-\frac{1}{2(2n+3)}, \frac{1}{6}$
6 2 **7 i)** $1-\mathrm{e}^{Nx}$ **ii)** $x>0, 1$ **8** $\frac{1}{7}-\frac{1}{\sqrt{2N+1}}, \frac{1}{7}$ **9 i)** $1-\frac{1}{(n+1)!}$ **ii)** $2\mathrm{e}-3$ **10 b)** $\frac{n}{2(n+2)}$

Exercise 9D

1 b) $\frac{25}{8}$ **c)** 0.110 **2 i)** $x-\frac{x^2}{2}+\frac{x^3}{3}$ **ii)** $\frac{5}{6}, -\frac{3}{5}$

3 a) $2(1+x)\cos x-(1+x)^2\sin x$, $2\cos x-4(1+x)\sin x-(1+x)^2\cos x$; $-6\sin x-6(1+x)\cos x+(1+x)^2\sin x$

3 b) $1+2x+\frac{x^2}{2}-x^3$ **4 i)** $x-\frac{x^3}{3!}+\frac{x^5}{5!}$ **5 b)** 0.029 565 **6** $(1+\sin x+\cos x)\mathrm{e}^x$; $\frac{1}{2}, 4$ **7 i)** $A=0, B=1, C=0, D=-\frac{1}{3}$

7 ii) $1-u^2+u^4$ **iii)** $x-\frac{x^3}{2}+\frac{x^5}{5}$ **iv)** $\tan^{-1}x$ **8** $1, \frac{1}{2}, \frac{1}{8}$ **9 a)** $-\frac{6}{(1+x)^4}$ **c)** $\frac{(-1)^r x^r}{r(r-1)}$ **10 a) i)** $\frac{2x}{2+x^2}$ **ii)** $\ln 2+\frac{x^2}{2}$

10 b) $-\frac{x^4}{8}$ **11 i)** $1+rx+\frac{r(r-1)}{2!}x^2+\frac{r(r-1)(r-2)}{3!}x^3+\frac{r(r-1)(r-2)(r-3)}{4!}x^4$ **ii)** $\sqrt[3]{\frac{3}{2}}$ **12 i)** $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$

12 ii) $\frac{(3n-2)}{n!}$ **iii)** $\mathrm{e}+2$

Exercise 9E

1 a) $2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\dots$ **b)** $5x-\frac{125}{6}x^3+\frac{625}{24}x^5$ **c)** $1+8x+32x^2+\frac{256}{3}x^3+\dots$ **d)** $x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\dots$

1 e) $-\left(2x+2x^2+\frac{8}{3}x^3+\dots\right)$ **2 a)** $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\dots+(-1)^n\frac{x^{(4n+2)}}{(2n+1)!}+\dots$ **b)** $1+4x+\frac{15}{2}x^2+9x^3+\frac{63}{8}x^4$

2 c) $2-8x^2+\frac{9}{4}x^4$ **d)** $e\left(1-\frac{x^2}{2}+\frac{x^4}{6}\right)$ **e)** $\ln 2-\frac{x^2}{4}-\frac{x^4}{96}$ **3 a)** Converge **b)** Converge **c)** Converge

4 $1-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\dots+(-1)^n\frac{x^{6n}}{(2n)!}+\dots$, all values of x **5** $1+2x^2+2x^4+\frac{4}{3}x^6+\dots$ **6** $|x|<3$ **7 a)** $2-\frac{x}{4}-\frac{x^2}{64}-\frac{x^3}{512}$

7 b) $6x-\frac{3}{4}x^2-\frac{579}{64}x^3$ **8 a)** $\frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3$ **b)** $x-\frac{3}{2}x^2+\frac{19}{12}x^3-\frac{19}{8}x^4$ **9 a)** $\frac{1}{2}, -\frac{1}{2}\sqrt{3}$

10 a) $a-\frac{1}{3}b,\ \frac{1}{2}a^2+\frac{1}{9}b^2,\ \frac{1}{6}a^3-\frac{5}{81}b^3$ **b) i)** 1, 3

Exercise 10A

1 a) i) $\frac{1}{2}(e^2+e^{-2})$ **ii)** 3.76 **b) i)** $\frac{1}{2}(e^3-e^{-3})$ **ii)** 10.0 **c) i)** $\frac{e^8-1}{e^8+1}$ **ii)** 0.999 **3 a)** $2\sinh 2x$ **b)** $5\cosh 5x$ **c)** $3\,\mathrm{sech}^2 3x$

3 d) $8\sinh 4x-15\cosh 3x$ **e)** $6\sinh 2x+30\cosh 5x$ **f)** $-\mathrm{cosech}^2 x$ **g)** $-\mathrm{sech}\,x\tanh x$ **h)** $45\sinh 3x\cosh^4 3x$

3 i) $64\cosh 8x\sinh^3 8x$ **j)** $\tanh x$ **k)** $2\cosh 2x\,e^{\sinh 2x}$ **l)** $5\,\mathrm{cosech}\,5x\,\mathrm{sech}\,5x$ **4 a)** $\frac{1}{3}\cosh 3x$ **b)** $\frac{1}{4}\sinh 4x$ **c)** $3\cosh\left(\frac{x}{3}\right)$

4 d) $10\sinh\left(\frac{x}{5}\right)$ **e)** $\frac{3}{5}\sinh 5x-4\cosh\left(\frac{x}{2}\right)$ **f)** $\frac{1}{4}\ln\cosh 4x$ **5 a)** 0.540 **b)** 0.693 **c)** 0.457 **d)** 0.191 **e)** 1.10, −0.625

5 f) 0.514 **6** $\ln\frac{1}{2}, \ln\frac{2}{3}$ **7 a) i)** $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ **b)** $\ln 2$ **8** $(-\ln 2, 4)$, min **10 i)** $x\sinh x+4\cosh x$ **ii), iii)** $x\sinh x+2n\cosh x$

11 $-\ln 3, 0$ **16 b) i)** (1.32, 2.15) **ii)** 3.3000 **c)** −1, 4

Exercise 10B

1 a) $\frac{5}{\sqrt{1+25x^2}}$ **b)** $\frac{3}{\sqrt{9x^2-1}}$ **c)** $\frac{\sqrt{2}}{\sqrt{1+2x^2}}$ **d)** $\frac{3}{\sqrt{9x^2-16}}$ **e)** $\frac{2x}{\sqrt{1+x^4}}$ **f)** $-\frac{1}{x\sqrt{1-x^2}}$ **g)** $\frac{1}{1-x^2}$ **2 a)** $\cosh^{-1}\left(\frac{x}{2}\right)+c$

2 b) $\cosh^{-1}\left(\frac{x}{3}\right)+c$ **c)** $\frac{1}{2}\cosh^{-1}\left(\frac{2x}{5}\right)+c$ **d)** $\frac{1}{3}\cosh^{-1}\left(\frac{3x}{4}\right)+c$ **e)** $\sinh^{-1}\left(\frac{x}{3}\right)+c$ **f)** $\sinh^{-1}\left(\frac{x}{4}\right)+c$ **g)** $\frac{1}{4}\sinh^{-1}\left(\frac{4x}{5}\right)+c$

2 h) $\frac{1}{5}\sinh^{-1}\left(\frac{5x}{3}\right)+c$ **3 a)** $\ln(1+\sqrt{2})$ **b)** $\ln(1+\sqrt{2})$ **c)** $\ln(2+\sqrt{3})$ **d)** $\frac{1}{\sqrt{3}}\ln(2+\sqrt{3})$ **e)** $\frac{1}{5}\ln(5+\sqrt{24})$

4 a) $\frac{1}{5}\ln\left(\frac{10+4\sqrt{6}}{5+\sqrt{21}}\right)$ **b)** $\frac{1}{3}\ln\left(\frac{6+2\sqrt{10}}{3+\sqrt{13}}\right)$ **c)** $\ln\left(1+\sqrt{\frac{2}{3}}\right)$ **d)** $\frac{1}{2}\ln\left(\frac{4+\sqrt{21}}{5}\right)$ **e)** $\ln\left(1+\sqrt{\frac{2}{3}}\right)$ **f)** $\frac{1}{4}\ln\left(\frac{13+2\sqrt{71}}{5+2\sqrt{35}}\right)$

5 a) $\cosh^{-1}\left(\frac{x+2}{4}\right)+c$ **b)** $\ln\left(\frac{3+\sqrt{8}}{2+\sqrt{3}}\right)$ **6 b)** $\ln(1+\sqrt{2})$ **7 a)** $x\tanh x-\ln\cosh x+c$

7 b) $y=x\sinh x-\cosh x\ln\cosh x+c\cosh x$ **8 a)** 2, 1, 4 **b)** $\frac{1}{4}\tanh^{-1}\left(\frac{2x+1}{2}\right)+c$ **9 b)** 0.2763 **10 a)** $(2x+1)^2+25$

10 b) $\frac{1}{2}\sinh\left(\frac{2x+1}{5}\right)+c$ **11 i)** 3, 4, −25 **12** $(x-3)^2-1^2$, $\ln(2+\sqrt{3})$ **13 c)** $x\sinh^{-1}x-\sqrt{1+x^2}+c$

17 iii) $\frac{1}{3}\ln(6+\sqrt{37})$ **18 a)** $x\geqslant 1$ **b) ii)** $\sqrt{2}-\ln(1+\sqrt{2})$; $\sqrt{2}, 1+\sqrt{2}$ **19 a)** $2\tan^{-1}e^x+c$ **c)** 2.604 **20** $\pi\ln(2+\sqrt{5})$

21 c) $\frac{3}{x^2-9}$ **d)** $-\frac{3}{x}-\frac{9}{x^3}-\frac{243}{5x^5}-\frac{2187}{7x^7}, -\frac{3^{2n+1}}{2n+1}$ **e)** $\frac{3}{x^2}+\frac{27}{x^4}+\frac{243}{x^6}+\frac{2187}{x^8}, 3^{2n-1}$ **22 a)** $x+\frac{x^3}{3}+\frac{x^5}{5}, \frac{1}{2n+1}$

22 b) $\frac{1}{2}\ln\left(\frac{1}{5}\right)$ **c)** $\frac{1}{4}\ln\left(\frac{27}{2}\right)$ **23 b ii)** $2\ln 2-1$

Exercise 10C

1 a) $1+2x+\frac{2}{3}x^4$ **b)** $3x+\frac{9}{32}x^3$ **c)** $1+x+\frac{25}{2}x^2+\frac{25}{2}x^3+\frac{625}{24}x^4$ **d)** $6x+12x^2+36x^3+72x^4$

2 $\frac{1}{2}x\sqrt{x^2-9}-\frac{9}{2}\cosh^{-1}\left(\frac{x}{3}\right)+c$ **3** $\frac{1}{2}x\sqrt{x^2+16}+8\sinh^{-1}\left(\frac{x}{4}\right)+c$ **4** $\frac{1}{2}x\sqrt{x^2+25}+\frac{25}{2}\sinh^{-1}\left(\frac{x}{5}\right)+c$

5 $\frac{1}{2}x\sqrt{x^2-25}-\frac{25}{2}\cosh^{-1}\left(\frac{x}{5}\right)+c$ **6** $\frac{1}{2}x\sqrt{x^2-4}+2\cosh^{-1}\left(\frac{x}{2}\right)+c$ **7** $\frac{1}{8}x\sqrt{x^2+9}-\frac{5}{2}\sinh^{-1}\left(\frac{x}{3}\right)+c$

8 $\frac{x}{2}\sqrt{x^2+4}+2\sinh^{-1}\left(\frac{x}{2}\right)+c$ **9 i)** 0; $\frac{1}{2}\left[\frac{1-(n+1)\cosh nx+n\cosh(n+1)x}{\cosh x-1}\right]$ **12** $y=\frac{1}{2}\coth x-\frac{1}{2}x\operatorname{cosech}x+c\operatorname{cosech}^2x$

14 $\frac{x}{\sqrt{x^2-1}}$; $\frac{5}{4}, -\frac{3}{4}$ **15 c) ii)** 1.76 **17 d)** $\frac{1}{2}\ln(2+\sqrt{3})-\frac{1}{4}\sqrt{3}$; $\frac{1}{2}, -\frac{1}{4}$ **18 b)** $1, -\frac{1}{2}, \frac{13}{120}$ **c)** $x^3+\frac{1}{6}x^5$

Exercise 11A

1 a) $(4, 0), x=-4$ **b)** $(7, 0), x=-7$ **c)** $(0, 2), y=-2$ **d)** $(0, -4), y=4$ **e)** $(-3, 0), x=3$ **f)** $(8, -1), x=-8$ **g)** $(5, 2), x=1$

2 a) $y^2=12x$ **b)** $y^2=16x$ **c)** $x^2=8y$ **d)** $x^2=-20y$ **3 a)** $ty=x+5t^2$ **b)** $py=x+5p^2$ **c)** $y=5+x$ **d)** $2y=x+20$

4 a) $x+y=6$ **b)** $(18, -12)$

Exercise 11B

1 a) $\frac{\sqrt{7}}{4}, (\pm\sqrt{7}, 0), x=\pm\frac{16}{\sqrt{7}}$ **b)** $\frac{\sqrt{33}}{7}, (\pm\sqrt{33}, 0), x=\pm\frac{49}{\sqrt{33}}$ **c)** $\frac{3}{5}; (\pm 3, 0), x=\pm\frac{25}{3}$ **d)** $\frac{\sqrt{5}}{3}, (0, \pm 2\sqrt{5}), y=\pm\frac{18}{5}$

1 e) $\frac{4}{5}, (5, -2), x=\frac{29}{4}, x=-\frac{21}{4}$ **2 a)** $\frac{x^2}{36}+\frac{y^2}{27}=1$ **b)** $\frac{x^2}{36}+\frac{y^2}{32}=1$ **c)** $\frac{x^2}{16}+\frac{y^2}{32}=1$ **d)** $\frac{x^2}{36}+\frac{y^2}{45}=1$

3 a) $4x\cos\theta+5y\sin\theta=20$ **b)** $4y\cos\theta=5x\sin\theta-9\sin\theta\cos\theta$ **4** $(-\frac{12}{5}a, 0), (-\frac{6}{5}a, 0)$

Exercise 11C

1 a) $\frac{5}{4}, (\pm 5, 0), x=\pm\frac{16}{5}$ **b)** $\frac{\sqrt{65}}{7}, (\pm\sqrt{65}, 0), x=\pm\frac{49}{\sqrt{65}}$ **c)** $\frac{\sqrt{41}}{5}, (\pm\sqrt{41}, 0), x=\pm\frac{25}{\sqrt{41}}$ **d)** $\frac{\sqrt{13}}{2}, (\pm\sqrt{13}, 0), x=\pm\frac{4}{\sqrt{13}}$

1 e) $\frac{\sqrt{34}}{5}, (1\pm\sqrt{34}, -2), x=1\pm\frac{25}{\sqrt{34}}$ **2 a)** $\frac{x^2}{36}-\frac{y^2}{108}=1$ **b)** $\frac{x^2}{36}-\frac{y^2}{288}=1$ **c)** $\frac{y^2}{32}-\frac{x^2}{32}=1$ **d)** $\frac{y^2}{45}-\frac{x^2}{180}=1$

3 a) $4x\sec\theta-5y\tan\theta=20$ **b)** $5x\sin\theta+4y=41\tan\theta$

Exercise 11D

2 a) $(0, 2t)$ **b)** $y^2=9x$ **3 b)** $(apq, a(p+q))$ **c)** $pq=-1$ **4 b)** $3x+7y=37, y+3x=13$ **5 a)** $\frac{x}{a}\cos t+\frac{y}{b}\sin t=1$

5 b) $ax\sin t-by\cos t=(a^2-b^2)\sin t\cos t$ **c)** $\left(\frac{a^2-b^2}{2a}\right)\cos t, \frac{b}{2\sin t}$ **6 a)** $3x\cos\theta+2y\sin\theta=6$

6 c) $\theta=0°, 233.1°, 360°; (-1.2, -2.4)$ **8 c)** 1 **9 a)** $\frac{(x-2)^2}{9}+\frac{y^2}{4}=1, \frac{\sqrt{5}}{3}$ **b)** $(\pm\sqrt{5}, 0)$ **d)** 33.84

10 c) C_1: $(\pm 3a, 0), y=\pm\frac{x}{3}$; C_2: $\left(\pm\frac{10a}{3}, 0\right), y=\pm 3x$ **d)** C_1: $\frac{\sqrt{10}}{3}, (\pm\sqrt{10}a, 0), x=\pm\frac{9}{10}$; C_2: $\sqrt{10}, \left(\pm\frac{100a}{3}, 0\right), x=\pm\frac{a}{3}$

12 iv) $(-1, 2t)$

Exercise 12A

The constant of integration is omitted from these answers.

1 a) $\frac{1}{6}(x^2+1)^6$ **b)** $\frac{1}{10}(x^2-1)^5$ **c)** $\frac{1}{32}(x^4-1)^8$ **d)** $-\frac{1}{15}(1-x^3)^5$ **e)** $-\frac{1}{6}\cos^6 x$ **f)** $\frac{1}{5}\sinh^5 x$ **g)** $\frac{1}{15}\cosh^5 3x$ **h)** $\frac{1}{12}\sin^6 2x$

2 a) $\frac{1}{2}e^x(\cos x+\sin x)$ **b)** $\frac{1}{5}e^x(\cos 2x+2\sin 2x)$ **c)** $\frac{1}{5}e^{2x}(\sin x+2\cos x)$ **d)** $\frac{1}{34}e^{3x}(5\sin 5x+3\cos 5x)$

2 e) $\frac{1}{10}e^{4x}(2\cosh 2x-\sinh 2x)$ **f)** $-\frac{1}{40}e^{-7x}(3\cosh 3x+7\sinh 3x)$ **3 a)** $x-\tan^{-1}x$ **b)** $x-5\tan^{-1}\left(\frac{x}{4}\right)$ **c)** $\frac{1}{4}x-\frac{23}{32}\ln(8x+3)$

3 d) $-\frac{7}{4}x-\frac{47}{16}\ln(5-4x)$ **e)** $\ln(x^2+2x+3)-\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$ **f)** $\frac{1}{2}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

3 g) $\sqrt{x^2+x-1}-\frac{3}{2}\cosh^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$ **h)** $\sqrt{2x^2-4x+5}-\frac{5}{\sqrt{2}}\sinh^{-1}\left(\sqrt{\frac{2}{3}}(x-1)\right)$ **i)** $-2\sqrt{1-4x-x^2}+\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)$

3 j) $-\sqrt{2-5x-3x^2}-\frac{19}{2\sqrt{3}}\sin^{-1}\left(\frac{6x+5}{7}\right)$ **4 a)** $-\sqrt{1-x^2}+\sin^{-1}x$ **b)** $1-\frac{\sqrt{3}}{2}+\frac{\pi}{6}$ **5** $\sin^{-1}\left(\frac{x+3}{5}\right)$

6 i) $1+\frac{2}{1+x}-\frac{1}{9+x^2}$ **7** $\frac{1}{3x+4}+\frac{2x+1}{x^2+9}$

Exercise 12B

1 a) $5\pi/32$ **b)** $\frac{16}{35}$ **2** $I_n = x^n e^n - n I_{n-1}$ **3** $120 - 326/e$ **4 a)** $\frac{1}{4}\pi - \frac{2}{3}$ **b)** $\frac{5}{12} - \frac{1}{2}\ln 2$

6 $\frac{1}{5}\cosh^4 1 \sinh 1 + \frac{4}{15}\cosh^2 1 \sinh 1 + \frac{8}{15}\sinh 1$ **11 b)** $\frac{316}{81}$ **12** $\frac{1}{16}\pi^4 - 3\pi^2 + 24$ **13** $\frac{8}{15}$ **14 b)** $2 - \frac{2}{\sqrt{e}}$ **c)** $16 - \frac{26}{\sqrt{e}}$

15 ii) $\frac{26}{15}$ **16** $\frac{1}{2}\ln 2 - \frac{1}{4}$ **17 b)** $\sqrt{3} + \frac{1}{2}\ln(2+\sqrt{3})$ **18** $\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}$; $\frac{1}{3}\ln 2 + \frac{\pi}{3\sqrt{3}}$; $\frac{1}{6} + \frac{2\pi}{9\sqrt{3}} + \frac{2}{9}\ln 2$ **21** $\frac{4\pi}{3} - 2\sqrt{3}$ **24** $\frac{1}{8}e^2 + \frac{3}{8}$

26 b) $-\frac{17}{15}\sqrt{2} + \frac{4}{5}\sqrt{3}$ **27 a)** $\frac{5}{4}, \frac{3}{4}$ **b) ii)** $\frac{57}{64}$ **28 a)** π **c)** $\frac{5}{16}\pi$ **d)** $\frac{5a^2\pi}{32}$ **29 b) i)** $\ln 2$ **iv)** $\frac{3}{5}\ln 2$ **30 i)** $-\frac{1}{n+1}\cos^{n+1}\theta$

31 b) $\pi\sqrt{5} - \frac{\pi}{2}\ln(2+\sqrt{5})$ **c)** 1.641%

Exercise 12C

1 $\frac{1}{27}(31\sqrt{31} - 8)$ **2** $\frac{(14\sqrt{14} - 11\sqrt{11})}{9\sqrt{2}}$ **3** $\frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln(t + \sqrt{1+t^2})$ **4** $4a$ **5** $c\sinh 1$ **6 a)** $\frac{12\pi}{5}(2 + 782\sqrt{17})$

6 b) $\frac{8\pi}{3}(5\sqrt{5} - 1)$ **c)** $\frac{12\pi a^2}{5}$ **d)** $\frac{5\pi}{6}(61^{3/2} - 41^{3/2})$ **e)** $\pi(\frac{1}{2}\sinh 2 + 1)$

6 f) $\frac{\pi}{3}e^{12}\sqrt{1+9e^{24}} + \frac{\pi}{9}\sinh^{-1}(3e^{12}) - \frac{\pi}{3}e^3\sqrt{1+9e^6} - \frac{\pi}{9}\sinh^{-1}(3e^3)$ **7** $8a$ **8 a)** $-1, 2$ **b) i)** $\frac{2x}{1-x^2}$ **10** $\frac{1}{4}(e^{2\pi} - 1)$ **11 i)** 3

12 $\frac{64\pi a^2}{3}$ **13** $2 - 2e^{-\pi/2}$, $2\int_0^{\pi/2}[4 + e^{-t}(\cos t - \sin t)]2e^{-t}\,dt$ **14** 6 **16 i)** $3\sqrt{2}$ **ii)** $\frac{24\sqrt{2}}{5}\pi$ **17 b) i)** 12 **ii)** $\frac{576}{5}\pi$

19 $\frac{\pi}{27}(10\sqrt{10} - 1)$ **20 b)** $(\frac{1}{5}e^{\pi} - \frac{2}{5})2\sqrt{2}\pi$ **22 a)** $\frac{1}{3}\cosh^3 t + c$ **c)** 438

Exercise 12D

1 $\frac{3}{2}$ **2** Does not exist **3** Does not exist **4** Does not exist **5** Does not exist **6** Does not exist **7** Does not exist **8** $\pi/2a$

9 Does not exist **10** Does not exist **11 a)** $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ **b)** $-\frac{1}{4}$ **12 a)** $\frac{1}{2}$ **b)** $\ln\left(\frac{3}{2}\right)$ **13 a)** $-\frac{k\pi}{2}$ **b) ii)** $\frac{1}{2}\ln 2$

14 i) $\ln 2$ **iii)** $\ln 2$ **iv)** $\pi/4$ **15** $\frac{1}{4}$ **16 iii)** -4

Exercise 13A

1 1.12 **2** 0.87 **3** 2.46 **4** 0.95 **5** 0.93 **6** 0.60 **7** 1.41 **8 a)** $-3.8125, -3.7936, -3.7915, -3.7913, -3.7912, -3.7912, -3.791$

8 b) -3.791 **c)** Five **9** [0.375, 0.5], fourteen **10 b)** 1.432 **d)** 0.669 **11 b)** 1.373

12 Starting with $x_0 = 0$: $x_1 = 1$, $x_2 = 0.806\,824\,2641$, $x_3 = 0.792\,134\,9597$, $x_4 = 0.792\,059\,9704$, $x_5 = 0.792\,059\,9684$, $x_6 = 0.792\,059\,9684$

13 a) 0.109, -0.402 **b)** 2.11 **c)** 2.13 **14** 0.53 Tangent parallel to x-axis **15** 4.026 **16 b)** 6.135 **17 b)** 1.54 **c)** -0.54

18 a) i) 1.4973 **ii)** 1.497 3043 **19 c)** 1.07 **20 a)** 0.2443, 0.2553, 0.2582, 0.2589 **b)** 2.544 **21 a)** 0.337 609 **22 a)** $\frac{1}{2}, \frac{3}{2}$

22 b) i) $4x^3 - 12x^2 + 9x + 3 = 0$ **ii)** -0.2460 **23** -0.670, 0.78, [1.2, 1.206 25]

Exercise 13B

1 a) i) 42.0000 **ii)** 41.3333 **b) i)** 328.0000 **ii)** 320.0000 **c) i)** 3.6734 **ii)** 3.7175 **d) i)** 5.0898 **ii)** 5.1795 **2 i)** 1255.81

2 ii) 1403.734 **3 b)** 3.6281 **c)** $\sinh 2$ **d)** 0.03% **4 b)** $2 - \frac{2}{\sqrt{e}}$ **c)** $16 - \frac{26}{\sqrt{e}}$ **5 a) i)** 0.4285 **ii)** 0.4217 **b)** 0.4207

6 b) $\frac{\pi}{4}\left[2\sqrt{5} - \ln(2+\sqrt{5})\right]$ **c)** 1.641% **7** 0.579 **8** 2.53, 1.26 **9** 0.52, 0.52 **10** 0.749 88 **11** 0.1026

12 0.82 **i)** Underestimate **ii)** Larger estimate and better approximation

Exercise 13C

1 $y = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{35x^4}{8}$ **2** $y = 2 + 2x^2 + \frac{2x^3}{3} + 2x^4$ **3** $y = x - \frac{5x^3}{6}$ **4** $y = 1 - \frac{x^4}{6}$, 0.999 999 998

5 $y = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \ldots$ **a)** 0.1055 **b)** -0.0955 **6** $y = 2x + \frac{8x^3}{3}$

7 $\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2$, 0.719 448

8 $f(0.5 + h) = -2 - 28h - 784h^2$ Sensitive because derivative large near $x = 0.5$ **9** $y = 1 + 2x - \frac{3x^2}{2} - \frac{4x^3}{3}$

10 a) $3\left(1 + \frac{dy}{dx}\right)(x+y)^2$, $3\frac{d^2y}{dx^2}(x+y)^2 + 6\left(1 + \frac{dy}{dx}\right)^2(x+y)$ **b)** $y = 1 + x + 3x^2 + 7x^3$ **c)** 0.9 **11** $y = 3 + 2x + 3x^2 - \frac{7x^3}{3}$

12 a) 0.9 **b)** 0.8362 **c)** 0.7607 **13 a)** 1.1 **b)** 1.221 **14 a)** $x = 2.27, y = -0.33$ **b)** $\frac{d^2x}{dt^2} + 16x = 20$ **15 a)** 1.628

15 b) $y = x(2\ln x + 1)$ **c)** 0.6% **16** 0.0025, 0.0051, 0.0078 **17 a) i)** $\frac{dv}{dt} = 4 - \frac{3x}{v}$ **ii)** 0.22 **b) i)** $x = Ae^{3t} + Be^t$

17 b) ii) $x = e^{3t} - e^t$, 0.245 **18** 2.21, 0.64 **19** 2.0766, 2.0743 **20** 0.049, $\frac{1}{5}$

Exercise 14A

1 $\begin{pmatrix} 14 & 0 \\ 8 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 18 & 17 \end{pmatrix}$, $\mathbf{PQ} \neq \mathbf{QP}$ **2 a)** $\begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}$ **b)** $\begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}$ **c)** $\begin{pmatrix} -20 & 4 & -9 \\ -5 & 1 & -2 \\ 11 & -2 & 5 \end{pmatrix}$ **d)** $\begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & -3 \\ -2 & 3 & 5 \end{pmatrix}$

2 e) $-\frac{1}{141}\begin{pmatrix} -21 & -12 & 18 \\ -17 & -3 & -19 \\ 5 & -24 & -11 \end{pmatrix}$ **3** $\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{k}{3} & \frac{k}{3} & -1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$ **5 a)** $\frac{1}{2x-5}\begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$ **b)** 19, −14, −27

6 $\frac{1}{20}\begin{pmatrix} -1 & 11 & -4 \\ -4 & 4 & 4 \\ 11 & -21 & 4 \end{pmatrix}$ **7** $\begin{pmatrix} -1 & 0 & 1 \\ 6\frac{1}{2} & -2\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ **9 i)** $\frac{1}{1+a}\begin{pmatrix} 1 & -a & 0 \\ 1 & 1 & 0 \\ -5-a & a^2-5 & 1+a \end{pmatrix}$ **ii)** $(-1, 1, 0)$

10 $\frac{1}{6a-6}\begin{pmatrix} 3a-1 & a+1 & -4 \\ 1 & 2a-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}, \left(\frac{2}{1-a}, \frac{1}{1-a}, \frac{3}{1-a}\right)$ **11 i)** $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix}$ **ii)** $\begin{pmatrix} \frac{1}{a} & 1 & -\frac{1}{a} \\ 1 - \frac{1}{2a} & -1 & \frac{1}{2a} \\ -\frac{1}{2a} & 0 & \frac{1}{2a} \end{pmatrix}$

12 i) $\frac{1}{10-a}\begin{pmatrix} 5 & 5 & -5 \\ 3a-5 & 5-2a & -1 \\ -a & -a & 2 \end{pmatrix}$ **ii)** $\frac{2}{5}y_1 - \frac{3}{5}y_2 + \left(\frac{a}{10-5a}\right)y_3$ **13 c)** $-\mathbf{A}, \mathbf{A}, -\mathbf{A}; (-1)^n\mathbf{A}$

13 d) $x - y - z = 1, -x + y - z = 0, -x - y + z = 0; x = 0, y = z = -\frac{1}{2}$ **e)** $\frac{1}{2}, -\frac{1}{2}$ **14 a)** $\begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix}$ **d)** $\begin{pmatrix} 4 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}$

15 a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 7 \\ 0 & 0 & 1 \end{pmatrix}$ **d)** $\frac{1}{2^n}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$

Exercise 14B

1 Rotation clockwise about O through $2\pi/3$

2 Reflection in y-axis, followed by one-way stretch in y-direction, scale factor 2 $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

3 Shear in x-direction moving (0, 1) to (−1, 1) **i)** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ **ii)** Rotation clockwise about O through $\pi/3$

4 i) (0, 0), (3, 0), (2, 4) **ii)** 6, 150 **iii)** 6 **5** 0, 1 **6 a)** 3, −1 **b)** $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **7 a) i)** 5, −1 **ii)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **b)** 6, 5

8 a) 4, −3 **c)** $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}$ **9 a)** 9, −3 **b)** $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$ **10 a) i)** 5, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **ii)** $x + 2y = 0$

10 b) 25 **11 a)** −2, 1 **b)** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **d)** $\frac{1}{3}(a - 2b + c)\mathbf{v}_1 + \frac{1}{3}(a + b - 2c)\mathbf{v}_2 + \frac{1}{3}(a + b + c)\mathbf{v}_3$ **12 a)** 4, −2; $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

12 b) ii) $\frac{1}{8}, -\frac{1}{4}$ **13 a)** $2, -2, 3$ **c)** 5 **d)** $\begin{pmatrix}3\\1\\0\end{pmatrix}, 10$ **14 a) i)** $(0, -7)$ **ii)** $k = 7, \mathbf{A} = \begin{pmatrix}4 & 2\\2 & 7\end{pmatrix}$ **b) i)** $3, 8; \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$

14 b) ii) $x + 2y = 0, y = 2x$ **14 c)** One-way stretch centred at $(0, -7)$, scale factor 3, in direction of $x + 2y = 0$, followed by one-way stretch centred at $(0, -7)$, scale factor 8, in direction of $y = 2x$ **15 i)** $3, \begin{pmatrix}\frac{2}{3} & \frac{1}{3} & -1\\ \frac{5}{3} & \frac{4}{3} & -4\\ -3 & -2 & 7\end{pmatrix}$ **ii)** $\begin{pmatrix}-6 & -4 & 12\\ -29 & -17 & 66\\ -13 & -8 & 29\end{pmatrix}$

15 iii) $\lambda^3 - 6\lambda^2 + 3\lambda + 10 = 0$ **iv)** $-\frac{1}{10}, \frac{3}{5}, -\frac{3}{10}$ **16 b)** $2x + 2y - z = 0$ **17** $2, -3, 5$ **18 a)** $k^2 - 2k + 1$ **b)** $3, -1$

19 b) $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & -1\end{pmatrix}$ **c) i)** $x = y, z = 0$ **ii)** $90°$ **d)** $\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$ **20 c) iii)** $\frac{16\pi}{3}$ **21 i)** $(-3, 1, 2)$ **ii)** $x = y = z$

21 iii) $x + y + z = 0$ **iv)** $\frac{2\pi}{3}$ $\begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ Rotation about L through $\frac{4\pi}{3}$; identity **22** $1, 27; \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}5\\-1\end{pmatrix}$

22 a) $5x = y$ **c)** 27 Scale factor of area enlargement **d)** One-way stretch in $\begin{pmatrix}5\\-1\end{pmatrix}$ direction, scale factor 27

23 i) $\frac{1}{1-2k}\begin{pmatrix}k-2 & 4-3k & 1\\-1-k & k+2 & 1\\3 & -5 & -2\end{pmatrix}$ **ii)** $(-1, -1, 3)$ **iv)** $\frac{x}{3} = \frac{y}{2} = -\frac{z}{3}$

Exercise 15A

1 a) $\cos 6\theta + \mathrm{i}\sin 6\theta$ **b)** $\cos 8\theta + \mathrm{i}\sin 8\theta$ **c)** -1 **d)** $-\mathrm{i}$ **e)** $\cos 8\theta - \mathrm{i}\sin 8\theta$ **f)** -1 **g)** 1 **h)** $-\mathrm{i}$ **2 a)** $\cos 10\theta + \mathrm{i}\sin 10\theta$

2 b) $\cos\theta - \mathrm{i}\sin\theta$ **c)** -1 **d)** -8 **3 a)** 16 **b)** $512\mathrm{i}$ **c)** $24\sqrt{3}$ **d)** -4 **e)** 2^{12} **f)** $-512\mathrm{i}$ **4 a)** $\cos 5\theta - \mathrm{i}\sin 5\theta$

4 b) $\cos 4\theta + \mathrm{i}\sin 4\theta$ **c)** $-\cos 6\theta + \mathrm{i}\sin 6\theta$ **d)** -1 **5** $\cos 11x + \mathrm{i}\sin 11x$

Exercise 15B

1 a) i) $\pm\sqrt{2}(1 \pm \mathrm{i})$ **ii)** $2\mathrm{e}^{\mathrm{i}\pi/4}, 2\mathrm{e}^{-\mathrm{i}\pi/4}, 2\mathrm{e}^{3\mathrm{i}\pi/4}, 2\mathrm{e}^{-3\mathrm{i}\pi/4}$ **b) i)** $2^{\frac{2}{3}}(1 + \mathrm{i}), 2^{\frac{7}{6}}\left[\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right], 2^{\frac{7}{6}}\left[\cos\left(\frac{5\pi}{12}\right) - \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right]$

1 b) ii) $2^{\frac{7}{6}}\mathrm{e}^{\mathrm{i}\pi/4}, 2^{\frac{7}{6}}\mathrm{e}^{11\mathrm{i}\pi/12}, 2^{\frac{7}{6}}\mathrm{e}^{-5\mathrm{i}\pi/12}$ **c) i)** $\frac{3}{2}(\sqrt{3} + \mathrm{i}), \frac{3}{2}(-\sqrt{3} + \mathrm{i}), -3\mathrm{i}$ **ii)** $3\mathrm{e}^{\mathrm{i}\pi/6}, 3\mathrm{e}^{5\mathrm{i}\pi/6}, 3\mathrm{e}^{-\mathrm{i}\pi/2}$ **d) i)** $\pm 2\sqrt{2}(1 + \mathrm{i})$

1 d) ii) $4\mathrm{e}^{\mathrm{i}\pi/4}, 4\mathrm{e}^{-3\mathrm{i}\pi/4}$ **e) i)** $\frac{5}{\sqrt{2}}(-1 + \mathrm{i}), \frac{5}{\sqrt{2}}(1 - \mathrm{i})$ **ii)** $5\mathrm{e}^{3\mathrm{i}\pi/4}, 5\mathrm{e}^{-\mathrm{i}\pi/4}$ **f) i)** $-2, 2\left[\cos\left(\frac{\pi}{5}\right) \pm \mathrm{i}\sin\left(\frac{\pi}{5}\right)\right], 2\left[\cos\left(\frac{3\pi}{5}\right) \pm \mathrm{i}\sin\left(\frac{3\pi}{5}\right)\right]$

1 f) ii) $2\mathrm{e}^{\mathrm{i}\pi/5}, 2\mathrm{e}^{3\mathrm{i}\pi/5}, 2\mathrm{e}^{\mathrm{i}\pi}, 2\mathrm{e}^{-\mathrm{i}\pi/5}, 2\mathrm{e}^{-3\mathrm{i}\pi/5}$ **2** $\mathrm{e}^{\pm\mathrm{i}\pi/3}, \mathrm{e}^{\pm 2\mathrm{i}\pi/3}, \mathrm{e}^{\mathrm{i}\pi}, \mathrm{e}^{\mathrm{i}0}$ or $1, -1, \pm\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right)$ **3 a)** $2 - 2\mathrm{i}, -2 - 2\mathrm{i}$ **b)** $3, \pm\sqrt{3}\mathrm{i}$

3 c) $-\frac{1}{2}$ **d)** $1 + 3\mathrm{i}, \frac{1}{3} - \mathrm{i}$ **e)** $-3 - \frac{1}{2}\mathrm{i}, -\frac{3}{2} + \frac{1}{4}\mathrm{i}, -\frac{21}{10} + \frac{7}{10}\mathrm{i}, -\frac{3}{2} - \frac{1}{2}\mathrm{i}$ **4** $\mathrm{e}^{\mathrm{i}0}, \mathrm{e}^{2\mathrm{i}\pi/7}, \mathrm{e}^{4\mathrm{i}\pi/7}, \mathrm{e}^{6\mathrm{i}\pi/7}, \mathrm{e}^{-2\mathrm{i}\pi/7}, \mathrm{e}^{-4\mathrm{i}\pi/7}, \mathrm{e}^{-6\mathrm{i}\pi/7}$

5 $2\mathrm{e}^{\mathrm{i}\pi/10}, 2\mathrm{e}^{\mathrm{i}\pi/2}, 2\mathrm{e}^{9\mathrm{i}\pi/10}, 2\mathrm{e}^{-3\mathrm{i}\pi/10}, 2\mathrm{e}^{-7\mathrm{i}\pi/10}$ **9 a)** $\frac{1}{41}(4\cos 5x + 5\sin 5x)\mathrm{e}^{4x} + c$ **b)** $\frac{1}{58}(3\sin 7x - 7\cos 7x)\mathrm{e}^{3x} + c$

9 c) $-\frac{1}{20}(2\sin 4x + 4\cos 4x)\mathrm{e}^{-2x} + c$ **d)** $\frac{1}{25}(3\sin 3x - 4\cos 3x) + c$ **10** $2\mathrm{e}^{-\mathrm{i}\pi/6}, 2\mathrm{e}^{-2\mathrm{i}\pi/3}, 2\mathrm{e}^{5\mathrm{i}\pi/6}, 2\mathrm{e}^{\mathrm{i}\pi/3}$ **11** $3 - \mathrm{i}, -3 + 3\mathrm{i}$

12 i) $8, -5\pi/6$ **ii)** $2\mathrm{e}^{-5\mathrm{i}\pi/18}, 2\mathrm{e}^{7\mathrm{i}\pi/18}, 2\mathrm{e}^{-17\mathrm{i}\pi/18}$ **13** $16\left[\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right], 2\left[\cos\left(\frac{\pi}{16}\right) + \mathrm{i}\sin\left(\frac{\pi}{16}\right)\right],$

$2\left[\cos\left(\frac{9\pi}{16}\right) + \mathrm{i}\sin\left(\frac{9\pi}{16}\right)\right], 2\left[\cos\left(-\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{16}\right)\right], 2\left[\cos\left(-\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{15\pi}{16}\right)\right]$ **14** $2\mathrm{e}^{-\mathrm{i}\pi/3}, 2\mathrm{e}^{\mathrm{i}\pi/6}$

15 b) $2^{\frac{1}{6}}, \pi/12; 2^{\frac{1}{6}}, 3\pi/4; 2^{\frac{1}{6}}, -7\pi/12$.

16 $64, \pi; 2\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right], 2\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right], 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right], 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right];$

$2 + 2\mathrm{i}, -2 + 2\mathrm{i}, -2 - 2\mathrm{i}, 2 - 2\mathrm{i}$ **17 a)** $\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/20}, \sqrt{2}\mathrm{e}^{9\mathrm{i}\pi/20}, \sqrt{2}\mathrm{e}^{17\mathrm{i}\pi/20}$ **18 ii)** $\frac{1}{13}\mathrm{e}^{3x}(3\cos 2x + 2\sin 2x), \frac{1}{13}\mathrm{e}^{3x}(3\sin 2x - 2\cos 2x)$

18 iii) $\frac{1}{13}\mathrm{e}^{3x}(3\cos 2x + 2\sin 2x) + c, \frac{1}{13}\mathrm{e}^{3x}(3\sin 2x - 2\cos 2x) + c$ **19 b)** $1 + \mathrm{e}^{3\mathrm{i}\pi/5}, 1 + \mathrm{e}^{\mathrm{i}\pi}$ (or 0), $1 + \mathrm{e}^{-3\mathrm{i}\pi/5}, 1 + \mathrm{e}^{-\mathrm{i}\pi/5}, 1 + \mathrm{e}^{\mathrm{i}\pi/5}$

19 c) $(1, 0), 1$ **d) i)** $\frac{\pi}{10}$ **ii)** $2\cos\left(\frac{\pi}{10}\right)$ **20 i)** $4 - 2\mathrm{i}, 4 + \sqrt{3} + \mathrm{i}, 4 - \sqrt{3} + \mathrm{i}; 3\sqrt{3}$ **ii)** $-\mathrm{i}; -11, -50$

21 $\cos\left(\frac{2r\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2r\pi}{5}\right)$ $(r = 0, 1, 2, 3, 4)$ **i)** $\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{10}\right), \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right), \cos\left(\frac{9\pi}{10}\right) + \mathrm{i}\sin\left(\frac{9\pi}{10}\right),$

$\cos\left(\frac{13\pi}{10}\right) + \mathrm{i}\sin\left(\frac{13\pi}{10}\right), \cos\left(\frac{17\pi}{10}\right) + \mathrm{i}\sin\left(\frac{17\pi}{10}\right)$ **22 i)** $\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}, \sqrt{2}\mathrm{e}^{7\mathrm{i}\pi/12}, \sqrt{2}\mathrm{e}^{11\mathrm{i}\pi/12}, \sqrt{2}\mathrm{e}^{-3\mathrm{i}\pi/4}, \sqrt{2}\mathrm{e}^{-5\mathrm{i}\pi/12}, \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/12}$

Exercise 15C

1 a) $2i\sin\theta$ **b)** $2\cos 4\theta$ **c)** $2\cos 5\theta$ **d)** $2i\sin 2\theta - 2i\sin\theta$ **2 a)** $\frac{1}{2}\left(z^6+\frac{1}{z^6}\right)$ **b)** $\frac{1}{2i}\left(z^5-\frac{1}{z^5}\right)$ **c)** $\frac{1}{2}\left(z^4+\frac{1}{z^4}\right)$

2 d) $\frac{1}{2i}\left(z^3-\frac{1}{z^3}\right)$ **e)** $-\frac{1}{4}\left(z^5-\frac{1}{z^5}\right)^2$ **f)** $\frac{1}{16}\left(z^3+\frac{1}{z^3}\right)^4$ **3 a)** $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ **b)** $8\cos^4\theta - 8\cos^2\theta + 1$

3 c) $8\cos^3\theta - 4\cos\theta$ **d)** $32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ **4 a)** $3\sin\theta - 4\sin^3\theta$ **b)** $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

4 c) $-64\sin^6\theta + 80\sin^4\theta - 24\sin^2\theta + 1$ **d)** $16\sin^4\theta - 12\sin^2\theta + 1$ **5 a)** $\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ **b)** $\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

5 c) $\frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$ **d)** $\frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ **e)** $\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos\theta + \frac{5}{16}$

7 $\tan\left(\frac{\pi}{12}\right), \tan\left(\frac{5\pi}{12}\right), \tan\left(\frac{3\pi}{4}\right)$ **8** $\frac{27}{8}, \frac{7}{8}$ **9 vi)** $-1+2i$ **13 a)** $4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ **b)** $z^4 = \cos 4\theta + i\sin 4\theta$

13 c) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ **d)** 8, 4, 2, 1 **14** $5, -10, 1; \pm 2\cos\left(\frac{\pi}{5}\right), \pm 2\cos\left(\frac{2\pi}{5}\right)$

15 $\cos\theta + i\sin\theta,\ \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right),\ \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),\ \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right),\ \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$

15 b) $(z-1)\left[z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right]\left[z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right]$ **c)** $\frac{1}{4}(\sqrt{5}-1)$

17 $e^{5i\pi/6}, e^{-i\pi/2}, e^{-i\pi/6}, e^{-5i\pi/6}, e^{i\pi/2}; (z^2+\sqrt{3}z+1)(z^2-\sqrt{3}z+1)(z^2+1)$

Exercise 15D

1 a) w lies on circle, centre O, radius 25 **b)** w lies on that part of real axis with argument 0

1 c) w lies on that part of real axis with argument π **2 a)** w lies on circle, centre O, radius $\sqrt{5}$

2 b) w lies on circle, centre O, radius $\sqrt{2}$ **c)** w lies on line $u = v$ **3** $v^2 = 4k^2(u+k^2)$ **4 a)** Circle, centre $-\frac{1}{4}i$, radius $\frac{3}{4}$

4 b) Imaginary axis **c)** $\pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ **5** $|w+5-2i| = 12$ or circle, centre $(-5, 2i)$, radius 12 **6 a)** 4, 9, −4 **b)** (4, 0), 5

7 Circle $|w - \frac{9}{5} - \frac{21}{20}i| = \frac{3}{4}\sqrt{17}$ **8** Ellipse: $\left(\frac{4u}{31}\right)^2 + \left(\frac{3v}{23}\right)^2 = 1$ **9** Straight line: $w = 3$ **11 i)** $2e^{i\pi/2}, 2e^{-i\pi/6}, 2e^{-5i\pi/6}$

11 ii) b) Circle, centre −3, radius 2; circle, centre 9i, radius 6; circle, centre −3i, radius 2 **12** $w = 0$ **14 b) ii)** $\pi/8, 5\pi/8$

15 $1 + \frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}$

Exercise 16

1 $\frac{125}{78}$ **2** $\frac{(1+e^2)^{3/2}}{e}$ **3** $\frac{5\sqrt{5}}{4\sqrt{3}}$ **4** $2\sqrt{2}$ **5** $\frac{13\sqrt{13}}{6}$ **6** $\frac{17\sqrt{17}c}{16}$ **7** ∞ **8** $\frac{\sqrt{3}}{2}a$ **9 a)** $3\psi^2 - \sin\psi$ **b)** $3 + 4\sin\psi + 4\psi\cos\psi$

9 c) $\cos\psi - \psi\sin\psi + 2\psi$ **10** $s = \ln|\sec\psi + \tan\psi|$ **11 a)** $-a\sin\psi$ **12 b)** 3.81 **c)** $\frac{9\pi}{2}(e^2 + 4 - e^{-2})$ **13 ii)** $(42, 26\frac{1}{4})$

13 iii) $16\sqrt{3}$ **iv)** $\left(384 - \frac{192\sqrt{3}}{5}\right)\pi$ **14 i)** $5e^{\alpha} - 5$ **iii)** $\frac{5}{2}e^{\alpha}$

Exercise 17A

5 $3 \times 3 = 9$, set is not closed **6** No inverse of −1

Exercise 17B

1 i) G_1:1, G_2:0 **ii)** Because 0 does not have an inverse under × **iii)** $f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$

2 i) G_1

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

G_2

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

G_3

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

G_2 and G_3 isomorphic. Isomorphism G_2 to G_3: $1 \to 1, 5 \to 3, 7 \to 5, 11 \to 7$

2 ii) G_1: $x = 1, 9$; G_2: $x = 1, 5, 7, 11$; G_3: $x = 1, 3, 5, 7$

3 Shear in x-direction moving $(0, 1)$ to $(n, 1)$

4 i)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

ii) $\{1\}, \{1, 3\}, \{1, 5\}, \{1, 7\}$

4 iii)

	f	g	h	k
f	f	g	h	k
g	g	f	k	h
h	h	k	f	g
k	k	h	g	f

iv) Yes. Composition table of {f, g, h, k} obtained from composition table of {1, 3, 5, 7} by replacing 1 with f, 3 with g, 5 with h, and 7 with k.

5 a)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

b) Not closed, $2 \times 4 = 0 \pmod 8$ **c)** Whenever n is not a prime

6 H subgroup of G

8 i)

	I	**A**	**B**	**C**	**D**	**E**
I	**I**	**A**	**B**	**C**	**D**	**E**
A	**A**	**B**	**I**	**E**	**C**	**D**
B	**B**	**I**	**A**	**D**	**E**	**C**
C	**C**	**D**	**E**	**I**	**A**	**B**
D	**D**	**E**	**C**	**B**	**I**	**A**
E	**E**	**C**	**D**	**A**	**B**	**I**

iii) {**I**}, {**I**, **C**}, {**I**, **D**}, {**I**, **E**}, {**I**, **A**, **B**}

iv) a) No **b)** Yes **c)** No

9 i) 2, 2 **ii)** $\{e, a, b, c\}$, $\{e, a, bc, abc\}$, $\{e, b, ac, abc\}$ and $\{e, c, ab, abc\}$ **iii)** Lagrange's theorem, 8 not divisible by 3

9 iv)

Element	0	1	2	3	4	5	6	7
Order	1	8	4	8	2	8	4	8

v) No. Three elements of order 2 in G, only one in H.

10 i)

Element	e	g	g^2	g^3	g^4	g^5
Order	1	6	3	2	3	6

ii)

Element	i	h_1	h_2	h_3	h_4	h_5
Order	1	3	3	2	2	2

Proper subgroups of H: $\{i\}$, $\{i, h_1, h_2\}$, $\{i, h_3\}$, $\{i, h_4\}$, $\{i, h_5\}$

10 iii) M isomorphic to G. Correspondence is: $\begin{matrix} e & g & g^2 & g^3 & g^4 & g^5 \\ 1 & 4 & 3 & 12 & 9 & 10 \end{matrix}$ or $\begin{matrix} e & g & g^2 & g^3 & g^4 & g^5 \\ 1 & 10 & 9 & 12 & 3 & 4 \end{matrix}$

11 i) 4, 4

11 ii) $\{e, p_1, p_2, p_3\}$ and $\{e, p_2, q_1, q_2\}$ **iii)** No, 8 not divisible by 6 **v)** G isomorphic to K

12 b)

8	10	14	16
4	14	16	8
2	16	8	4

12 c) 10; {10, 8}, {10, 4, 16} **d)** Yes

13 i) $\mathbf{E}_1$ one-way stretch in x-direction, scale factor $\frac{1}{3}$. $\mathbf{E}_2$ shear

13 ii) $\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & \frac{2}{3} \\ 1 & 4 \end{pmatrix}$, $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$, $\mathbf{E}_3(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$

13 iii) $\mathbf{E}_1^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\mathbf{E}_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{10}{3} \end{pmatrix}$, $\mathbf{E}_4^{-1} = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$ **iv)** $\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}$

14 i)

	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

6 self-inverse **ii)** -1, $\frac{17}{3}$ **15 ii)** $\mathbf{B}$, $\mathbf{I}$, $\mathbf{A}^2\mathbf{B}$

15 iii) $\mathbf{I}$, 1; $\mathbf{A}$, 4; $\mathbf{A}^2$, 2; $\mathbf{A}^3$, 4; $\mathbf{B}$, 2; $\mathbf{AB}$, 2; $\mathbf{A}^2\mathbf{B}$, 2; $\mathbf{A}^3\mathbf{B}$, 2 **v)** $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3\}$ and $\{\mathbf{I}, \mathbf{AB}, \mathbf{A}^3\mathbf{B}, \mathbf{A}^2\}$ **vi)** Only $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3\}$ cyclic

16 a) By Lagrange's theorem: if $\mathbf{X}$ has n elements, then 2 divides n and so does 4. Minimum 8

17 b) $\mathbf{M}(\frac{1}{3}\mathrm{i})$ has order 4. Required groups are: group of order 4: $\{\mathbf{M}(\frac{1}{3}\mathrm{i}), \mathbf{M}(-\frac{1}{3}), \mathbf{M}(-\frac{1}{3}\mathrm{i}), \mathbf{M}(\frac{1}{3})\}$; group of order 2: $\mathbf{M}(-\frac{1}{3}), \mathbf{M}(\frac{1}{3})\}$

17 d) Inverse of $\mathbf{M}(\frac{2}{3})$ is $\mathbf{M}(\frac{1}{6})$, which is not in S. Hence, S is not a group, and so cannot be a subgroup of G.

18 a) For A, let $\theta = 0$; for B, let $\theta = \dfrac{\pi}{2}$. Then $A^2 = B^2 = I$, but $(AB)^2 = -I$ **d) i)** D_8 not abelian since $q * a \neq a * q$

18 d) ii) Set has 6 elements. Because 8 not divisible by 6, there is no subgroup of D_8 with 6 elements.

19 iii)

Second ○ First	π_1	π_2	π_3	π_4	π_5	π_6
π_1	π_1	π_2	π_3	π_4	π_5	π_6
π_2	π_2	π_3	π_1	π_6	π_4	π_5
π_3	π_3	π_1	π_2	π_5	π_6	π_4
π_4	π_4	π_6	π_5	π_1	π_2	π_3
π_5	π_5	π_4	π_6	π_3	π_1	π_2
π_6	π_6	π_5	π_4	π_2	π_3	π_1

iv) Symmetries of equilateral triangle

20 iii) 0 **iv)** No inverse of -1 **v)** Delete -1

Exercise 17C

2 $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = 0$, $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **4** For example: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **5** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$

6 a) Points on line $2y = x$ **b)** 1 **c)** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **8 b)** Point O **9 iii)** $\mathbf{M} = \begin{pmatrix} a+c & d-b \\ b+d & a-c \end{pmatrix}$ **iv)** $z = -2 + 13\mathrm{j}$

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