b) **i)** Show that the integrating factor for the above differential equation is $\frac{1}{x}$.

ii) Solve the differential equation to find y in terms of x, and use it to show that $y(1.2) = 1.638$, correct to three decimal places.

c) Hence find, correct to one decimal place, the percentage error in using Euler's formula in the evaluation of $y(1.2)$. (NEAB)

16 The variable y satisfies the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$, and $y = 0$ at $x = 0.5$.

Use the approximation $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_0}{h}$ with step length $h = 0.01$ to estimate the values of y at $x = 0.51$, $x = 0.52$ and $x = 0.53$, giving your answers to four decimal places. (EDEXCEL)

17 **a)** The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0$$

can be written as two simultaneous first-order differential equations.

i) If one of these equations is $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, write down the other equation.

ii) Use a step-by-step method with two steps of $\mathrm{d}t = 0.05$ to estimate the value of x at $t = 0.1$, given that at $t = 0$, $x = 0$ and $v = 2$.

b) **i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0$$

ii) Find the particular solution if $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$ at $t = 0$. Hence calculate the value of x when $t = 0.1$, giving your answer to two decimal places. (NEAB/SMP 16–19)

18 The equation $\mathrm{f}(x) = 0$ has a root at $x = a$, which is known to be close to $x = x_0$. Use the Taylor series expansion of $\mathrm{f}(x)$ about $x = x_0$ to derive the formula for the Newton–Raphson method of solution of $\mathrm{f}(x) = 0$

It is known that the equation $\mathrm{f}(x) = 0$, where

$$\mathrm{f}(x) = x^4 - 6x^2 + 2x + 1$$

has four distinct roots of which two are positive.

Show that exactly one root of the equation lies in the interval $[2, 3]$.

Use the Newton–Raphson method to determine this root correct to two decimal places.

It is proposed to determine the other positive root using simple iteration. Show that the equation can be rearranged to give the iterative scheme

$$x_{n+1} = \frac{x_n^3}{6} + \frac{1}{3} + \frac{1}{6x_n}$$

and that this **may** be suitable to obtain a solution in the interval $[0.5, 1]$.

Using $x_0 = 0.5$ as a starting value, and recording the successive iterates to three decimal places, use simple iteration to determine this root to two decimal places.

State the order of convergence of the iterative scheme used and explain how the data from the iterative process can be seen to agree with this. (SQA/CSYS)

19 Derive Euler's method for the approximate solution of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

subject to the initial condition $y(x_0) = y_0$.

The differential equation $\frac{dy}{dx} = (x^2 + y)e^{-2x}$ with $y(1) = 2$ is to be solved.

Use Euler's method with step lengths of 0.1 and 0.05 to obtain two approximations to the solution of this equation at $x = 1.2$. Perform the calculations using four decimal place accuracy.

Assuming that the difference in the two estimates for $y(1.2)$ is due entirely to the truncation error, estimate the size of this error in the calculation with step size 0.05. Hence give a better estimate of $y(1.2)$ to an appropriate degree of accuracy.

The predictor–corrector method of solution where Euler's method is used as the predictor and the trapezium rule as the corrector (with **one** corrector application on each step) is to be used to approximate the solution of the above equation at $x = 1.2$. Use step length 0.1 and perform the calculation using four decimal place accuracy. (SQA/CSYS)

20 The solution of the differential equation

$$x\frac{dy}{dx} = (y + 1)^2 - \cos x \qquad y(1) = 0$$

is required at $x = 1.15$. Obtain an approximation to this solution using Euler's method with step size 0.05. Perform the calculation using three decimal place accuracy.

If a step size 0.01 had been used in this calculation, by what factor would you expect the truncation error to be reduced? (SQA/CSYS)

14 Matrices

Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack.
JAMES JOSEPH SYLVESTER

A matrix stores mathematical information in a concise way. The information is written down in a rectangular array of rows and columns of terms, called **elements** or **entries**, each of which has its own precise position in the array.

$\begin{pmatrix} 4 \\ 8 \\ 7 \end{pmatrix}$ is a matrix, but its meaning depends on the context.

As in Chapter 6, it could represent a vector, meaning $4\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$. In football, it could represent the number of goals scored by three different clubs. In a shop, it could represent the number of packets of three different items bought.

Notation

We normally represent matrices by bold capital letters. For example,

$$\mathbf{M} = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Example 1, on page 300, illustrates an application of this notation.

The order of a matrix

The order of a matrix is its shape. For example, the matrix $\begin{pmatrix} 6 & -2 & 7 \\ 4 & 3 & -5 \end{pmatrix}$ has order 2×3, since its elements are arranged in two rows and three columns.

When stating the order of a matrix, we must **always give first the number of rows**, followed by the number of columns.

$\begin{pmatrix} 4 \\ 8 \\ 7 \end{pmatrix}$ is a **column matrix** and has order 3×1, since its elements are arranged in three rows and only one column.

The matrix $(4 \quad 8 \quad 7)$ has order 1×3 and is a **row matrix**.

When the number of rows and the number of columns are equal, the matrix is called a **square matrix**.

Note (4, 8, 7) with the numbers separated by commas is a point. $\begin{pmatrix} 4 & 8 & 7 \end{pmatrix}$ with no commas is a matrix.

Addition and subtraction of matrices

Only when two matrices are of the **same order** can we add them or subtract them.

To add two matrices of the same order, we proceed as follows, element by element:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} = \begin{pmatrix} a+p & b+q & c+r \\ d+s & e+t & f+u \\ g+v & h+w & i+x \end{pmatrix}$$

We subtract two matrices of the same order in a similar way.

We **cannot** evaluate $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ because the matrices are **not of the same order**.

Multiplication of matrices

Multiplying a matrix by a number

To multiply a matrix by, for example, k, we multiply **every** element of the matrix by k. Hence, we have

$$k\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{pmatrix}$$

Example 1 Find $3\mathbf{A} + 2\mathbf{B}$ when $\mathbf{A} = \begin{pmatrix} 4 & 7 & -1 \\ 8 & 1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 2 \end{pmatrix}$.

SOLUTION

We have

$$3\mathbf{A} + 2\mathbf{B} = 3\begin{pmatrix} 4 & 7 & -1 \\ 8 & 1 & 5 \end{pmatrix} + 2\begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 2 \end{pmatrix}$$

Multiplying out the RHS, we obtain

$$3\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 12 & 21 & -3 \\ 24 & 3 & 15 \end{pmatrix} + \begin{pmatrix} 6 & 4 & 8 \\ -2 & -6 & 4 \end{pmatrix}$$

which gives

$$3\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 18 & 25 & 5 \\ 22 & -3 & 19 \end{pmatrix}$$

Multiplying one matrix by another

We **cannot** multiply **any** matrix by **any other** matrix.

To allow multiplication, the orders of the two matrices concerned must **conform** to the following rule:

The number of columns in the first matrix must be the same as the number of rows in the second matrix.

For example, if the first matrix has order 3×3, the second must have order $3 \times$ something, as in the case of **A** and **B** below, which we will multiply together:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 3 \\ 0 & 2 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

To multiply **A** by **B**, we start by taking the first row of matrix **A**, $(2 \quad 3 \quad 1)$,

and the first column of matrix **B**, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

We then multiply the first element of the row by the first element of the column, the second element of the row by the second element of the column, and the third element of the row by the last element of the column. We then add up these three products.

This gives the element in the top left-hand corner of the matrix **AB**, which is

$$2 \times 1 + 3 \times 1 + 1 \times 0 = 5$$

So, we have

$$\mathbf{AB} = \begin{pmatrix} 5 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Next, we take the second row of matrix **A**, $(0 \quad -2 \quad 3)$, and the first column

of matrix **B**, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Again, we multiply each element of the row by the corresponding element of the column and add up the products.

This gives the second element of the first column of matrix **AB**, which is

$$0 \times 1 - 2 \times 1 + 3 \times 0 = -2$$

So, now we have

$$\mathbf{AB} = \begin{pmatrix} 5 & ? & ? \\ -2 & ? & ? \\ ? & ? & ? \end{pmatrix}$$

We repeat the procedure on the second and third columns of matrix **B**, eventually obtaining

$$\mathbf{AB} = \begin{pmatrix} 5 & 0 & 4 \\ -2 & 10 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

(Notice that at each stage it looks as if we are finding a scalar dot product of two vectors.)

Generally, the product **PQ** produces a matrix which has the **same number of rows** as **P**, and the **same number of columns** as **Q**. Hence, if **P** has order $p \times t$ and **Q** has order $t \times q$, then **PQ** has order $p \times q$.

Multiplication is not commutative

It is important to note that the multiplication of two matrices is **not commutative**. That is,

$$\mathbf{AB} \neq \mathbf{BA}$$

Therefore, we must ensure that we write the matrices in the **correct sequence**. (See Exercise 14A, Question 1, page 306.)

Also, to avoid ambiguity when referring to the product of **A** and **B**, we must **specify their sequence**. For example, in the case of **AB**, we say either that **A premultiplies B** or that **B postmultiplies A**.

There are, however, three exceptions to the non-commutative law:

- Multiplication of a zero matrix by a non-zero matrix of the same order (see page 304).
- Multiplication of a square matrix by its inverse (see page 304).
- Multiplication of a square matrix by the identity matrix of the same order (see page 303).

We also note the following:

- If **AB** exists, **BA** does not necessarily exist.
- The matrix $\mathbf{A}^2$ is $\mathbf{A} \times \mathbf{A}$, which can only exist if **A** is a square matrix.

Multiplication is associative

We find that for **any** matrices **A**, **B** and **C**, which are conformable for multiplication,

$$\mathbf{A(BC)} = \mathbf{(AB)C}$$

provided their **sequence is not changed**.

Known as the **associative law**, this allows us to decide whether we start the multiplication with the first pair of matrices or the second pair. Consequently, we can refer to the product **ABC** without ambiguity.

Determinant of a matrix

As stated on page 81, determinants always consist of a square array of elements. It follows, therefore, that **only a square matrix** can have a determinant.

From our definition of a determinant, we see that it is the scalar representation of its originating square matrix, and gives the value associated with that matrix.

If **A** is a square matrix, we can find the determinant of **A**, denoted by det **A** or $|\mathbf{A}|$, by the method shown on pages 80–1.

Determinant of the product of two matrices

The determinant of the product **AB** is the same as the product of the determinant of **A** and that of **B**:

$$\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$$

Identity matrices and zero matrices

An **identity matrix** is any square matrix all of whose elements in the leading diagonal are 1, and all of whose other elements are zeros. It is denoted by **I**. Hence,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is known as the 2×2 identity matrix, and

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is known as the 3×3 identity matrix.

When we multiply **I** by any square matrix **M** of the same order as **I**, **I** behaves as unity. That is,

$$\mathbf{IM} = \mathbf{MI} = \mathbf{M}$$

Zero matrices

When all the elements of a matrix are zeros, it is known as a **zero matrix**, and is denoted by **0**.

A zero matrix may have **any** order and therefore is not unique. For example,

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can multiply any non-zero matrix by a zero matrix provided the zero matrix is **conformable for multiplication**. For example,

$$\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Generally, we have

$$\mathbf{0M} = \mathbf{0} \quad \text{and} \quad \mathbf{N0} = \mathbf{0}$$

Also, from the second example, we note that when $\mathbf{0}$ and $\mathbf{M}$ have the **same order**

$$\mathbf{0M} = \mathbf{0} = \mathbf{M0}$$

which is one of the three exceptions to the non-commutative laws discussed on page 302.

When we multiply together two **non-zero matrices**, we can get a **zero matrix** as the result. For example,

$$\begin{pmatrix} 5 & 2 \\ 10 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -5 & -10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverse matrices

If $\mathbf{M}$ is a square matrix, its **inverse**, denoted by $\mathbf{M}^{-1}$, is defined by

$$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

Contrary to the non-commutative law discussed on page 302, we note that the order in which we multiply $\mathbf{M}$ and $\mathbf{M}^{-1}$ does not matter, which means that $\mathbf{M}^{-1}$, if it exists, is unique.

The inverse of a square matrix, $\mathbf{M}$, exists when $\det \mathbf{M} \neq 0$. That is, when $\mathbf{M}$ is said to be **non-singular**. When $\det \mathbf{M} = 0$, $\mathbf{M}$ is said to be **singular**.

The minor determinant

The **minor determinant** of an element of a matrix is the determinant of the matrix formed by deleting the row and column containing that element.

For example, the minor determinant of the middle element, 2, of the matrix $\begin{pmatrix} 5 & 6 & 9 \\ 7 & 2 & 1 \\ 3 & 4 & 8 \end{pmatrix}$ is the determinant of the matrix $\begin{pmatrix} 5 & 9 \\ 3 & 8 \end{pmatrix}$, which is

$$\begin{vmatrix} 5 & 9 \\ 3 & 8 \end{vmatrix} = 13$$

Finding the inverse of a 3×3 matrix

We proceed in the following order:

1 Find the value of the determinant, Δ, of the matrix.

2 Find the value of the minor determinant of each of the elements.

3 Form a new matrix from the minor values, inserting them in the positions corresponding to the elements from which they were derived. Also insert a minus sign at each odd-numbered place, counting on from the top left entry of the matrix. These minor values with their associated signs (+ or −) are called the **cofactors** of the elements of the original matrix.

4 Find the transpose of the result.

Hence, we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} A & -B & C \\ -D & E & -F \\ G & -H & I \end{pmatrix}^{\mathrm{T}}$$

where $A, B, C, \ldots$ are the minor determinants of the elements $a, b, c, \ldots$ respectively.

Example 2 Find the inverse of $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{pmatrix}$

SOLUTION

First, we calculate $\det \mathbf{M}$, which gives

$$\det \mathbf{M} = 1(6-4) - 2(4-4) + 5(2-3) = -3$$

Next, we calculate the minor determinants, obtaining

$$\begin{matrix} 2 & 0 & -1 \\ -1 & -3 & -1 \\ -7 & -6 & -1 \end{matrix}$$

Then, we insert those minor values in their appropriate positions, together with their associated signs (+ or −), to form the matrix to be transposed.

(For example, the minor value of element 4 is $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$. This is inserted three places from the top left corner of the matrix, for which the associated sign is minus, giving $-(-1) = +1$. Thus, +1 is the cofactor of element 4.)

Hence, we have

$$\mathbf{M}^{-1} = \frac{1}{-3}\begin{pmatrix} 2 & -0 & -1 \\ +1 & -3 & +1 \\ -7 & +6 & -1 \end{pmatrix}^{\mathrm{T}}$$

We obtain the transpose by reflecting the matrix in its leading diagonal (see page 84), giving

$$\mathbf{M}^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & 1 & -7 \\ 0 & -3 & 6 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ 0 & 1 & -2 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise 14A

1 Evaluate **PQ** and **QP**, where

$$\mathbf{P} = \begin{pmatrix} 6 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$$

What do you conclude from your results, and why has it happened?

2 Find the inverse of each of the following.

a) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & -2 & 1 \\ 3 & -1 & 5 \\ -1 & 4 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 4 & -2 \\ 2 & -1 & 5 \\ -3 & 4 & 1 \end{pmatrix}$

3 Find the inverse of the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ k & -1 & 0 \end{pmatrix}$ in terms of k. (NICCEA)

4 Given the matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, show by induction that

$$\mathbf{A}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for all positive integers n. (WJEC)

5 a) Calculate the inverse of the matrix

$$\mathbf{A}(x) = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \qquad x \neq \frac{5}{2}$$

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ is the vector $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

b) Find the values of a, b and c. (EDEXCEL)

6 Given that the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and that the determinant of $\mathbf{A} = 20$, find $\mathbf{A}^{-1}$.

(WJEC)

7 The matrices $\mathbf{A}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the matrix $\mathbf{B}$ satisfying $\mathbf{BA} = \mathbf{C}$. (WJEC)

8 Let matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ and $\mathbf{I}$ be the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

i) Show that $\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}$.
ii) By writing $\mathbf{A}^3 = \mathbf{A} \times \mathbf{A}^2$ and using part **i**, show that $\mathbf{A}^3 = 7\mathbf{A} - 6\mathbf{I}$.
iii) For positive n, use the method of induction to prove that

$$\mathbf{A}^n = (2^n - 1)\mathbf{A} + (2 - 2^n)\mathbf{I} \qquad \text{(NICCEA)}$$

9 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & a & 0 \\ -1 & 1 & 0 \\ a & 5 & 1 \end{pmatrix}$$

where $a \neq -1$.

i) Find $\mathbf{A}^{-1}$.
ii) Given that $a = 2$, find the coordinates of the point which is mapped onto the point with coordinates $(1, 2, 3)$ by the transformation represented by $\mathbf{A}$. (OCR)

10 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & a \end{pmatrix}$$

where $a \neq 1$. Find the inverse of $\mathbf{A}$.

Hence, or otherwise, find the point of intersection of the three planes with equations

$$2x - y + z = 0$$
$$3y + z = 0$$
$$x + y + az = 3 \qquad \text{(OCR)}$$

11 Matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & a \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

where $a \neq 0$.

i) Find the inverse of $\mathbf{A}$.

ii) Given that

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

find the matrix $\mathbf{C}$ such that $\mathbf{ABC} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. (OCR)

12 It is given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ a & 0 & 5 \end{pmatrix}$$

where $a \neq 2$.

i) Show that $\mathbf{A}$ has an inverse, and find it.

ii) It is given that

$$\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{B}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find x_2 in terms of y_1, y_2, y_3 and a. (OCR)

13 Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

a) Determine whether or not $\mathbf{AB} = \mathbf{BA}$.

b) Verify that $\mathbf{A}^2 + 3\mathbf{B}^2 = 12\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix.

c) Find $\mathbf{AB}$, $\mathbf{AB}^2$ and $\mathbf{AB}^3$ as multiples of $\mathbf{A}$, and make a conjecture about a general result for $\mathbf{AB}^n$. Use induction to prove your conjecture.

d) It is given that $\mathbf{B}$ is invertible, with inverse of the form

$$\mathbf{B}^{-1} = \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}$$

Write down a system of linear equations which x, y and z must satisfy, and hence find the values of x, y and z.

e) Verify that $\mathbf{B}^2 - \mathbf{B}$ is a multiple of $\mathbf{I}$, and hence find $\mathbf{B}^{-1}$ in the form $r\mathbf{B} + s\mathbf{I}$ where r, s are real numbers. Hence check your answer to part **d**. (SQA/CSYS)

14 a) Given that $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$, find $\mathbf{A}^2$.

b) Using $\mathbf{A}^3 = \begin{pmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{pmatrix}$, show that $\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} - \mathbf{I} = 0$.

c) Deduce that $\mathbf{A}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I}) = \mathbf{I}$.

d) Hence find $\mathbf{A}^{-1}$. (EDEXCEL)

15 Given that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, use matrix multiplication to find

a) $\mathbf{A}^2$ **b)** $\mathbf{A}^3$

c) Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n - 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad n \geqslant 1$$

d) Find the inverse of $\mathbf{A}^n$. (EDEXCEL)

Transformations

A number of transformations of a two-dimensional plane onto a two-dimensional plane, $\mathbb{R}^2$, and of a three-dimensional space onto a three-dimensional space, $\mathbb{R}^3$, may be represented by a matrix $\mathbf{M}$, where

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

means that the image of (x, y, z) under the transformation, T, is (x_1, y_1, z_1).

Linear transformations

T is described as a **linear transformation** of n-dimensional space (where $n = 2, 3, \ldots$) when it has the properties

$$\mathrm{T}(\lambda\mathbf{x}) = \lambda\mathrm{T}(\mathbf{x}) \quad \text{and} \quad \mathrm{T}(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\mathrm{T}(\mathbf{x}) + \mu\mathrm{T}(\mathbf{y})$$

where λ and μ are arbitrary constants.

We may represent a linear transformation by a matrix. For example, in three dimensions, we might represent T by the matrix

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Hence, to find, under T, the image of the point with position vector **i**, we calculate

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix}$$

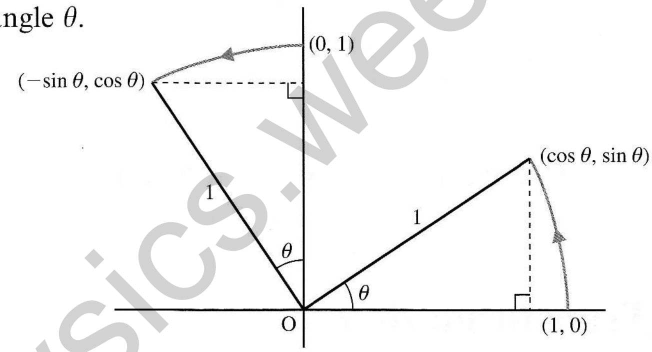
So, under T, the image of the point $(1, 0, 0)$ is (a, d, g), which we can see is the first column of **M**.

To find which type of transformation is represented by a matrix, we find the images of the vectors $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. Common linear transformations are rotations about the origin, reflections in lines through the origin, stretches and shears.

We can represent a linear transformation of $\mathbb{R}^2$, which is the xy-plane, by a matrix. Such a matrix will have order 2×2.

For example, let T be an anticlockwise rotation of two-dimensional space.

The rotation, centred at the origin, is through angle θ.



Then the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is **i**, transforms to the vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which is **j**, transforms to the vector $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

The matrix for T is then given by

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So, to find the matrix representing this transformation, we find the images of $(1, 0)$ and $(0, 1)$, which become the two columns of the matrix.

In three dimensions, we find the images of the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, which are the vertices of the unit cube. In vector form, these are the images of the vectors **i**, **j** and **k**. As seen below and on page 311, these become the columns of the matrix representing the transformation.

Example 3 Find the matrix **M** representing an enlargement, scale factor 2, with the origin as the centre of enlargement.

SOLUTION

The images of the vertices of the unit cube are

$$(1, 0, 0) \rightarrow (2, 0, 0)$$

$$(0, 1, 0) \rightarrow (0, 2, 0)$$

$$(0, 0, 1) \rightarrow (0, 0, 2)$$

Hence, we have

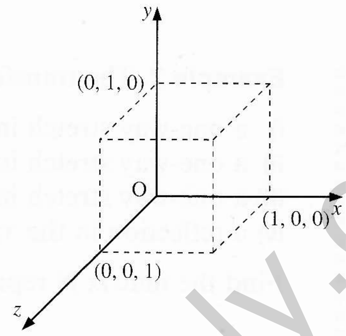
$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 4 Find the matrix $\mathbf{M}$ representing a reflection in the line $y = x$ in the xy-plane.

SOLUTION

The images of the vertices of the unit cube are

$$(1,0,0) \to (0,1,0)$$
$$(0,1,0) \to (1,0,0)$$
$$(0,0,1) \to (0,0,1)$$



Hence, we have

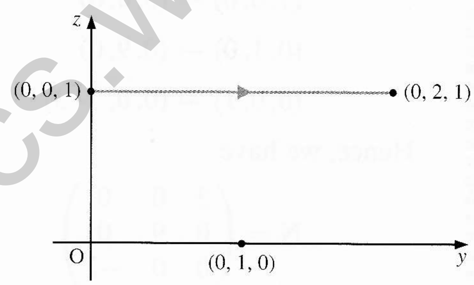
$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 5 Find the matrix $\mathbf{M}$ representing a shear in the yz-plane, in which $(0, 1, 0)$ is invariant and $(0, 0, 1)$ moves to $(0, 2, 1)$.

SOLUTION

The images of the vertices of the unit cube are

$$(1,0,0) \to (1,0,0)$$
$$(0,1,0) \to (0,1,0)$$
$$(0,0,1) \to (0,2,1)$$



Hence, we have

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 6 Find the image of the line $y = 7x$ under the transformation whose matrix is $\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}$.

SOLUTION

To find the image of a line (or a plane), we first obtain the general point on the line (or plane), and then obtain the image of this general point.

The general point on the line $y = 7x$ is $(t, 7t)$.

The image of this point is given by

$$\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} t \\ 7t \end{pmatrix} = \begin{pmatrix} -3t \\ 37t \end{pmatrix}$$

Hence, we have $x = -3t$, $y = 37t$.

So, to find the desired line, we eliminate t, obtaining

$$37x + 3y = 0$$

Therefore, the image of the line $y = 7x$ is $37x + 3y = 0$.

Example 7 The transformation T is the composite transformation of

i) a one-way stretch in the x-direction, scale factor 3
ii) a one-way stretch in the y-direction, scale factor 9
iii) a one-way stretch in the z-direction, scale factor 3
iv) a reflection in the xy-plane.

Find the matrix **N** representing the composite transformation.

SOLUTION

Transformations **iii** and **iv** can be combined to give a one-way stretch in the z-direction of scale factor -3.

After all four transformations have taken place, the images of the vertices of the unit cube are

$$(1, 0, 0) \to (3, 0, 0)$$

$$(0, 1, 0) \to (0, 9, 0)$$

$$(0, 0, 1) \to (0, 0, -3)$$

Hence, we have

$$\mathbf{N} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Invariant points and lines

An **invariant point** of the transformation T is a point which is unchanged by that transformation. That is, $\mathrm{T}(\mathbf{x}) = \mathbf{x}$.

For example, the only points which are unchanged by reflection in the line $y = x$ are the points on the line $y = x$ itself. Therefore, the **only** invariant points in this transformation are on the line $y = x$.

Reflection in the line $y = x$ does not affect the line $y = x$. In addition, the line $y = -x$ maps onto itself. These are the **only** two lines which map onto themselves. Both lines pass through the origin.

We say that these are the **invariant lines** of the transformation which is a reflection in the line $y = x$. The **only** invariant lines are $y = x$ and $y = -x$.

We notice that some points which are not invariant points are on an invariant line. For example, the point $(1, -1)$, which is on the line $y = -x$, is reflected to the point $(-1, 1)$, which is still on the same invariant line $y = -x$.

All invariant lines of a transformation which can be represented by a matrix, other than those with an invariant plane, pass through the origin. If the transformation is represented by the identity matrix, all lines are invariant.

Example 8 Find **a)** the invariant points and **b)** the invariant lines of the transformation whose matrix is $\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}$.

SOLUTION

a) The invariant points are the points (x, y) which satisfy

$$\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

From this, we obtain the following simultaneous equations:

$$4x - y = x \quad \Rightarrow \quad 3x = y \qquad [1]$$

$$2x + 5y = y \quad \Rightarrow \quad 2x = -4y \qquad [2]$$

Substituting [1] into [2], we have

$$2x = -4(3x)$$

$$\Rightarrow \quad 2x = -12x$$

$$\Rightarrow \quad x = 0$$

The only solution to equations [1] and [2] is $x = y = 0$.

Therefore, the origin $(0, 0)$ is the only invariant point under this transformation.

b) The line $y = mx + c$ is invariant if points on it map onto points on the same line, but not necessarily onto the same points.

Thus, the general point, (t, mt), on the line $y = mx$ should map onto another point, (T, mT), on the line. So, we must solve the equation

$$\begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$$

Multiplying out the LHS, we obtain

$$\begin{pmatrix} (4-m)t \\ (2+5m)t \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$$

Therefore, we have the simultaneous equations

$$(4-m)t = T$$

$$(2+5m)t = mT$$

which give

$$\frac{4-m}{2+5m} = \frac{1}{m}$$

Cross-multiplying, we obtain

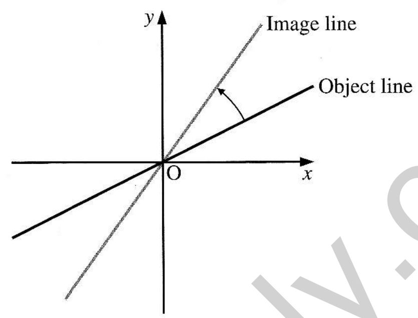
$$4m - m^2 = 2 + 5m$$

$$\Rightarrow \quad m^2 + m + 2 = 0$$

This equation has no real roots, and so the transformation has no invariant line.

Consider the anticlockwise rotation by $\frac{\pi}{2}$ about the origin in $\mathbb{R}^2$. Every line is rotated, and so there are no invariant lines. Also, there is only one invariant point, namely $(0, 0)$.

Any rotation (except by the angle 0° or 180°) in two-dimensional space has no invariant lines. For example, we can see from the figure on the right that the image line can never lie along the object line, unless $\theta = 0°$ or $180°$.



However, in three-dimensional space, a rotation must have an invariant line, namely the line about which the rotation occurs. In three-dimensional space, a plane always maps onto a plane unless the matrix is singular (that is, $\det \mathbf{M} = 0$). When the matrix is singular, a plane sometimes maps onto a line or a point. Similarly, a line always maps onto a line unless the matrix is singular, in which case the line might map onto a point.

Eigenvectors and eigenvalues

An **eigenvector** of a linear transformation T is a vector pointing in the direction of an invariant line under the transformation T.

For example, let T be a reflection in the line $y = x$. Then $(1, -1)$ is on the invariant line $y = -x$, but it maps onto $(-1, 1)$.

The **eigenvalue** for the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is -1, since all the points on the line $y = -x$ map onto points whose coordinates are -1 times the original coordinates.

To summarise, if $\mathbf{M}$ is the matrix for a transformation T, then

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

means that $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of T, and λ is the eigenvalue of T associated with $\begin{pmatrix} x \\ y \end{pmatrix}$.

In this case, we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$$

In three-dimensional space,

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

means that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector of T, and that λ is the eigenvalue of T associated with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Finding eigenvectors and eigenvalues

To find the eigenvalues of a transformation whose matrix is

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

we solve the equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ for λ, which we will now prove.

We have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which gives

$$ax + by + c = \lambda x$$

$$dx + ey + fz = \lambda y$$

$$gx + hy + iz = \lambda z$$

from which we obtain

$$(a - \lambda)\,x + by + cz = 0$$

$$dx + (e - \lambda)\,y + fz = 0$$

$$gx + hy + (i - \lambda)\,z = 0$$

For the eigenvectors to be non-zero, these three equations must have non-unique solutions (see page 87). Hence, we have

$$\begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} = 0$$

which is

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

To find the eigenvectors, we solve

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for each value of λ.

Example 9 Find **a)** the eigenvalues and **b)** the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

SOLUTION

a) We solve the equation

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

which gives

$$(1-\lambda)[(2-\lambda)(3-\lambda)-2]-1(2)+2(2-\lambda)=0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2-5\lambda+4)-2+4-2\lambda=0$$

$$\Rightarrow \quad \lambda^3-6\lambda^2+11\lambda-6=0$$

This equation is known as the **characteristic equation** of the matrix (see page 323).

Factorising the LHS, we obtain

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow \quad \lambda = 1, 2, 3$$

Therefore, the eigenvalues are 1, 2 and 3.

b) The eigenvector for the eigenvalue 1 is given by a solution to the equation

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x+y+2z \\ 2y+2z \\ -x+y+3z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

from which we obtain the simultaneous equations

$$x+y+2z=x \qquad [1]$$

$$2y+2z=y \qquad [2]$$

$$-x+y+3z=z \qquad [3]$$

We note that there are only two different equations from which to solve for three unknowns. Therefore, we cannot obtain a unique solution to such a set of equations (see page 87). Hence, we will let one of the unknowns be t. We also note that subtracting [3] from [1] gives $x=0$.

So, we let $z=t$ and solve the simultaneous equations for y:

$$x+y+2t=x \qquad [4]$$

$$2y+2t=y \qquad [5]$$

$$-x+y+3t=t \qquad [6]$$

From [4], we obtain $y = -2t$.

Therefore, the direction of the eigenvector is $\begin{pmatrix} 0 \\ -2t \\ t \end{pmatrix}$.

Hence, $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue 1.

The eigenvector for the eigenvalue 2 is given by a solution to the equation

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

from which we obtain the simultaneous equations

$$x + y - 2z = 2x \qquad [7]$$

$$2y + 2z = 2y \qquad [8]$$

$$-x + y + 3z = 2z \qquad [9]$$

This time, we do not let $z = t$, since [8] immediately gives $z = 0$.

So, we put $y = t$. Then from [7], we obtain $x = t$.

Therefore, the direction of the eigenvector is $\begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$.

Hence, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2.

The eigenvector for the eigenvalue 3 is given by a solution to the equation

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

from which we obtain the simultaneous equations

$$x + y + 2z = 3x$$

$$2y + 2z = 3y$$

$$-x + y + 3z = 3z$$

which give

$$-2x + y + 2z = 0 \qquad [10]$$

$$2z = y \qquad [11]$$

$$-x + y = 0 \qquad [12]$$

We let $x = t$. Then from [12] and [11], we have $y = t$ and $z = \frac{t}{2}$.

Therefore, the direction of the eigenvector is $\begin{pmatrix} t \\ t \\ \frac{1}{2}t \end{pmatrix}$.

Hence, $\begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$ is an eigenvector for the eigenvalue 3.

Since any scalar multiple of an eigenvector is also an eigenvector,

we can write the eigenvector for 3 as $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

Example 10 Show that $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & 4 \end{pmatrix}$$

Find the associated eigenvalue.

SOLUTION

If $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, then we have

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

where λ is the eigenvalue associated with $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

Hence, we obtain

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$$

$$\Rightarrow \mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

We note that

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

has the same form as $\mathbf{Ax} = \lambda\mathbf{x}$, therefore $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and its associated eigenvalue is 3.

Diagonalisation

If $\mathbf{M}$ is a **symmetric matrix**, then $\mathbf{M}^{\mathrm{T}} = \mathbf{M}$. That is, the transpose of the matrix $\mathbf{M}$ is the same as the original matrix $\mathbf{M}$.

For example, $\begin{pmatrix} 3 & 4 & -2 \\ 4 & 1 & 7 \\ -2 & 7 & 4 \end{pmatrix}$ is a symmetric matrix, since we have

$$\begin{pmatrix} 3 & 4 & -2 \\ 4 & 1 & 7 \\ -2 & 7 & 4 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 3 & 4 & -2 \\ 4 & 1 & 7 \\ -2 & 7 & 4 \end{pmatrix}$$

Note that the eigenvectors of a symmetric matrix which have non-equal eigenvalues are **mutually perpendicular**,

A **diagonal matrix** is one in which every element is 0 except those in the leading diagonal.

For example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a diagonal matrix.

If $\mathbf{P}$ is the matrix with the eigenvectors of $\mathbf{M}$ as each of its columns, we have

$$\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$$

where $\mathbf{D}$ is a **diagonal matrix**, the diagonal elements of which are the eigenvalues.

Since $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$, we have

$$\mathbf{PP}^{-1}\mathbf{MP} = \mathbf{PD} \quad \Rightarrow \quad \mathbf{MP} = \mathbf{PD}$$

$$\Rightarrow \quad \mathbf{MPP}^{-1} = \mathbf{PDP}^{-1} \quad \Rightarrow \quad \mathbf{M} = \mathbf{PDP}^{-1}$$

Hence, $\mathbf{M}$ representing the transformation may be expressed in terms of a diagonal matrix when the eigenvectors are used as axes. (See page 323 for an example of this.)

If $\mathbf{P}$ is the matrix with the eigenvectors of the **symmetric matrix** $\mathbf{M}$ as each of its columns, we have

$$\mathbf{P}^{\mathrm{T}}\mathbf{MP} = \mathbf{D}_1$$

where $\mathbf{D}_1$ is also a diagonal matrix.

If the eigenvectors used in $\mathbf{P}$ are **normalised** (that is, converted to unit vectors), then the elements of $\mathbf{D}_1$ are also the eigenvalues.

However, if the eigenvectors of the symmetric matrix $\mathbf{P}$ are not normalised, then each element in the leading diagonal is the product of an eigenvalue and the square of the modulus of the associated eigenvector.

In Example 9 (pages 316–18), we found that the eigenvectors of $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

are $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. Hence, we have

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 2 & -1 & -2 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

which gives the diagonal matrix, $\mathbf{P}^{-1}\mathbf{MP}$, as

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 2 & -1 & -2 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 4 & -2 & -4 \\ -\frac{3}{2} & \frac{3}{2} & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

That is, we have

$$\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$$

The diagonalisation of a symmetric matrix is given in Example 11.

Example 11 The transformation T is represented by

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

Find

a) the eigenvalues of **M**
b) their associated eigenvectors
c) a matrix, **P**, so that $\mathbf{P}^{\mathrm{T}}\mathbf{MP} = \mathbf{D}$, where **D** is a diagonal matrix whose diagonal elements are the eigenvalues.

SOLUTION

a) To find the eigenvalues, we have

$$\mathbf{Mx} = \lambda\mathbf{x}$$
$$\Rightarrow \quad (\mathbf{M} - \lambda\mathbf{I})\,\mathbf{x} = 0$$
$$\Rightarrow \quad |\mathbf{M} - \lambda\mathbf{I}| = 0$$

which gives

$$\begin{vmatrix} 3-\lambda & 4 & -4 \\ 4 & 5-\lambda & 0 \\ -4 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(5-\lambda)(1-\lambda) - 4 \times 4(1-\lambda) - 4 \times 4(5-\lambda) = 0$$
$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$$

Factorising, we obtain

$$(\lambda - 3)(\lambda + 3)(\lambda - 9) = 0$$
$$\Rightarrow \quad \lambda = 3, 9, -3$$

Therefore, the eigenvalues of **M** are $3, 9, -3$.

b) When $\lambda = 3$, we find the associated eigenvector from $\mathbf{Mx} = 3\mathbf{x}$, which gives

$$\begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} 3x + 4y - 4z = 3x \quad &\Rightarrow \quad 4y - 4z = 0 \qquad [1] \\ 4x + 5y = 3y \quad &\Rightarrow \quad 4x + 2y = 0 \qquad [2] \\ -4x + z = 3z \quad &\Rightarrow \quad -4x = 2z \qquad [3] \end{aligned}$$

Putting $x = t$, we obtain, from [2] and [3], $y = -2t$ and $z = -2t$.

Therefore, one eigenvector is $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

Similarly, we find the other eigenvectors are $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

c) From part **b**, we have

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$$

We find that the magnitude of each of the eigenvectors $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is 3.

Therefore, normalising the eigenvectors, we obtain respectively

$$\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \quad \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

which give

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Hence, we have

$$\mathbf{P}^{\mathrm{T}}\mathbf{MP} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

which gives

$$\mathbf{P}^{\mathrm{T}}\mathbf{PM} = \begin{pmatrix} 1 & -2 & -2 \\ 6 & 6 & -3 \\ -2 & 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

which is a diagonal matrix with the eigenvalues of $\mathbf{M}$ as its elements.

We noticed in Example 7 (page 312) that the transformation composed of

i) a one-way stretch in the x-direction, scale factor 3

ii) a one-way stretch in the y-direction, scale factor 9

iii) a one-way stretch in the z-direction, scale factor 3

iv) a reflection in the xy-plane

was represented by

$$\mathbf{N} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

By geometrical consideration of the actual transformation, we can deduce that the eigenvectors of this transformation are the three mutually perpendicular vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with associated eigenvalues $3, 9, -3$.

We have just found that the transformation represented by

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

also has three mutually perpendicular eigenvectors with associated eigenvalues $3, 9, -3$. Thus, these two transformations (Example 7, page 312, and Example 11, page 320) are the same transformation but about different axes: that represented by $\mathbf{N}$ has its one-way stretches in each of the three mutually perpendicular directions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, whereas that represented by $\mathbf{M}$ has its one-way stretches of the same scale factors in the three mutually perpendicular directions $\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$, $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$, $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

Naturally, both matrices have determinant -81, being the scale factor of the volume of the enlargement, which is the volume of the image of the unit cube.

Hence, the transformation $\mathbf{x}' = \mathbf{Mx}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

with respect to axes in the direction of the eigenvectors becomes the transformation $\mathbf{X}' = \mathbf{DX}$, where

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

which is the diagonalised form of **M**.

The characteristic equation

On page 316, we mentioned that the **characteristic equation** of the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

where the values of λ are the eigenvalues of **M**.

M also satisfies this characteristic equation. Hence, we have

$$\mathbf{M}^3 - 6\mathbf{M}^2 + 11\mathbf{M} - 6\mathbf{I} = 0$$

From this equation, we can find $\mathbf{M}^{-1}$.

Postmultiplying by $\mathbf{M}^{-1}$, we obtain

$$\mathbf{M}^3\mathbf{M}^{-1} - 6\mathbf{M}^2\mathbf{M}^{-1} + 11\mathbf{M}\mathbf{M}^{-1} - 6\mathbf{M}^{-1} = 0$$

$$\Rightarrow \quad \mathbf{M}^2 - 6\mathbf{M} + 11\mathbf{I} - 6\mathbf{M}^{-1} = 0$$

which gives

$$\mathbf{M}^{-1} = \frac{1}{6}\mathbf{M}^2 - \mathbf{M} + \frac{11}{6}\mathbf{I}$$

Exercise 14B

1 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

Give a full description of the geometrical transformation represented by $\mathbf{A}^4$. (OCR)

2 The matrix $\mathbf{C}$ is $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$. The geometrical transformation represented by $\mathbf{C}$ may be considered as the result of a reflection followed by a stretch. By considering the effect on the unit square, or otherwise, describe fully the reflection and the stretch.

Find the matrices $\mathbf{A}$ and $\mathbf{B}$ which represent the reflection and the stretch respectively. (OCR)

3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Describe fully the geometrical transformation represented by $\mathbf{M}$.

The matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(\sqrt{3}-1) \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}(\sqrt{3}+1) \end{pmatrix}$$

$\mathbf{C}$ represents the combined effect of the transformation represented by $\mathbf{M}$ followed by the transformation represented by a matrix $\mathbf{B}$.

i) Find the matrix $\mathbf{B}$.
ii) Describe fully the geometrical transformation represented by $\mathbf{B}$. (OCR)

4 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Under the transformation represented by $\mathbf{AB}$, a triangle P maps onto the triangle Q whose vertices are $(0, 0)$, $(9, 12)$ and $(22, -4)$.

i) Find the coordinates of the vertices of P.
ii) State the area of P and hence find the area of Q.
iii) Find the area of the image of P under the transformation represented by $\mathbf{ABA}^{-1}$. (OCR)

5 Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. Write down the matrix $\mathbf{A} - \lambda\mathbf{I}$, where $\lambda \in \mathbb{R}$ and $\mathbf{I}$ is the 3×3 identity matrix.

Find the values of λ for which the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is zero. (SQA/CSYS)

6 The matrix $\mathbf{P}$ is defined by

$$\mathbf{P} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$