Using triangle PTQ, we have

$$\text{Gradient of tangent} = \frac{PQ}{QT}$$

$$\Rightarrow \quad f'(\alpha) = \frac{f(\alpha)}{QT}$$

$$\Rightarrow \quad QT = \frac{f(\alpha)}{f'(\alpha)}$$

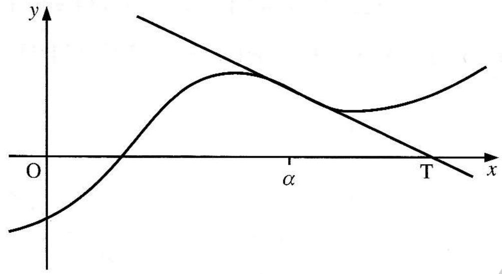


Figure A

The x-value of the point T is

$$\alpha - QT = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$

which is a better approximation to the root of $f(x) = 0$. When the root of $f(x) = 0$ is not close to α, the method may fail. For example, in Figure A, the next x-value found is at T, which is further from the root than α is. And in Figure B, $f'(\alpha) = 0$, which is unhelpful.

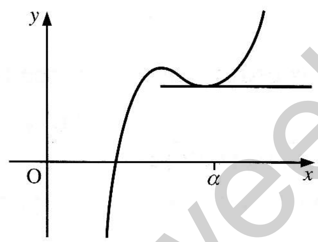


Figure B

In its iterative form, the Newton–Raphson method gives

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

Example 3 Use the Newton–Raphson method, with an initial value of $x = 1$, to find a root of $f(x) \equiv x^3 + 5x - 9$ to three significant figures.

SOLUTION

Let α be the required root.

Differentiating $f(x) \equiv x^3 + 5x - 9$, we have

$$f'(x) = 3x^2 + 5$$

Putting $\alpha_1 = 1$, we obtain

$$f(\alpha_1) = 1 + 5 - 9 = -3$$

$$f'(\alpha_1) = 3 + 5 = 8$$

Using Newton–Raphson, we have

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

which gives

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} = 1 + \frac{3}{8} = 1.375$$

Hence, we have

$$\alpha_3 = 1.375 - \frac{f(1.375)}{f'(1.375)} = 1.375 - \frac{0.474\,609\,375}{10.671\,875} = 1.330\,5271$$

which then gives

$$\alpha_4 = 1.330\,5271 - \frac{f(1.330\,5271)}{f'(1.530\,5271)}$$

$$= 1.330\,5271 - \frac{0.008\,070\,770\,27}{10.310\,907} = 1.329\,744\,36$$

α_3 and α_4 are now so close together that we can say that the root is 1.33 to three significant figures. (The root is actually 1.329 744 to seven significant figures. Were we to repeat the procedure a few more times, we would find that the root is 1.329 744 122, correct to ten significant figures.)

So, the root of $f(x) \equiv x^3 + 5x - 9$ is 1.33, correct to three significant figures.

Iteration

An iterative process is one which is repeated several times, following exactly the same procedure each time.

It provides yet another way of obtaining the solution to an equation $f(x) = 0$, in which we rearrange the equation to create an **iterative formula** of the form

$$x_{n+1} = g(x_n)$$

where x_{n+1} is a closer approximation than x_n to the solution of $f(x) = 0$.

For example, we can rearrange $x^3 + 5x - 9 = 0$ as

$$x^3 = 9 - 5x \quad \Rightarrow \quad x = \sqrt[3]{9 - 5x}$$

from which we can obtain the iterative formula

$$x_{n+1} = (9 - 5x_n)^{\frac{1}{3}}$$

Alternatively, we can rearrange $x^3 + 5x - 9 = 0$ as

$$x^3 = 9 - 5x \quad \Rightarrow \quad x^2 = \frac{9}{x} - 5 \quad \Rightarrow \quad x = \sqrt{\frac{9}{x} - 5}$$

from which we can obtain the iterative formula

$$x_{n+1} = \sqrt{\frac{9}{x_n} - 5}$$

Naturally, some iterative procedures produce an accurate solution more quickly than others, and some iterative procedures fail quickly.

For example, using $x_{n+1} = \sqrt{\dfrac{9}{x_n} - 5}$, with $x_1 = 1$, we obtain

$$x_2 = \sqrt{\frac{9}{1} - 5} = 2$$

$$x_3 = \sqrt{\frac{9}{2} - 5} = \sqrt{-0.5} \quad \text{which does not exist.}$$

How to decide which iterative formula to use is beyond the scope of the A-level syllabuses, and therefore of this book. All the iterations you will meet at this level result in one of the patterns shown next.

Example 4 Use $x_{n+1} = \frac{1}{6}(x_n^3 + 1)$ to find a solution to $f(x) = x^3 - 6x + 1 = 0$.

SOLUTION

As in the previous methods, we need to determine an interval in which the root lies.

Putting $x = 2$ and $x = 3$ in $f(x) = x^3 - 6x + 1$, we obtain

$$f(2) = -3 \quad \text{and} \quad f(3) = 10$$

Therefore, there is a root of $f(x) = 0$ for a value of x between 2 and 3.

(Also, since $f(0) = 1$ and $f(1) = -4$, there is another root of $f(x) = 0$ for a value of x between 0 and 1.)

Using $x_{n+1} = \frac{1}{6}({x_n}^3 + 1)$, with $x_1 = 2$, we have

$$x_2 = \frac{1}{6}(2^3 + 1) = 1.5$$

which gives

$$x_3 = \frac{1}{6}(1.5^3 + 1) = 0.729\,1666$$

$$x_4 = 0.231\,28$$

We see that these values of x are not converging to the required root.

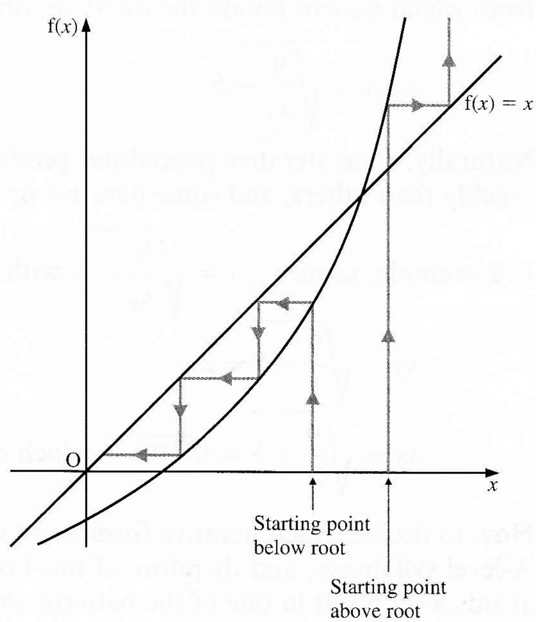
Alternatively, starting at $x_1 = 3$, we have

$$x_2 = \frac{1}{6}(3^3 + 1) = \frac{28}{6} = 4\tfrac{2}{3}$$

$$x_3 = \frac{1}{6}[(4\tfrac{2}{3})^3 + 1] = 17.1049$$

We see that these values of x are not converging to the root either.

In Example 4, we note that starting below the root sends the iteration to the smaller root, whereas starting above the root sends the iteration off to infinity. We can graphically represent these results by a **staircase diagram**, as shown on the right.



Example 5 Starting with $x = -1$, use the iteration $x_{n+1} = \frac{1}{6}x_n^{\,2} - 2$ to find the root to three significant figures.

SOLUTION

Using $x_{n+1} = \frac{1}{6}x_n^{\,2} - 2$ with $x_1 = -1$, we have

$$x_2 = \tfrac{1}{6} - 2 = -1\tfrac{5}{6} = -1.833\,33$$

$$x_3 = \tfrac{1}{6}(-1.833\,33)^2 - 2 = -1\tfrac{95}{216} = -1.4398$$

$$x_4 = -1.654\,488\,883 \qquad x_5 = -1.543\,777\,756$$

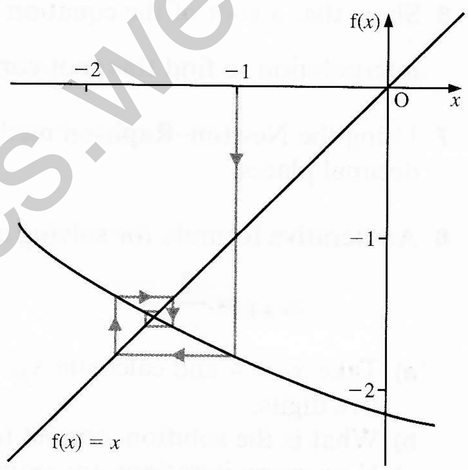
$$x_6 = -1.602\,791\,707 \qquad x_7 = -1.571\,843\,124$$

$$x_8 = -1.588\,218\,199 \qquad x_9 = -1.579\,593\,825$$

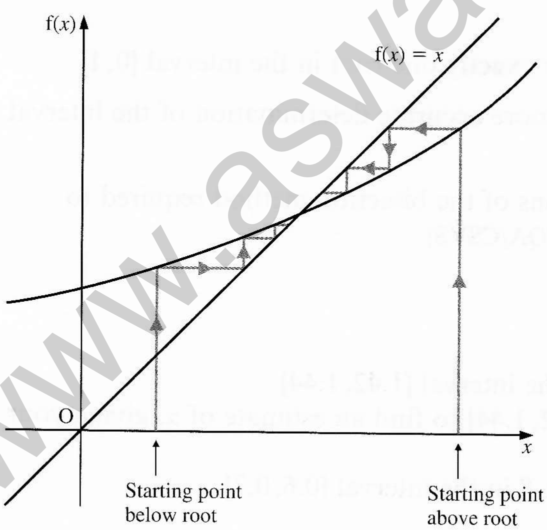
Therefore, the root is -1.58, correct to three significant figures.

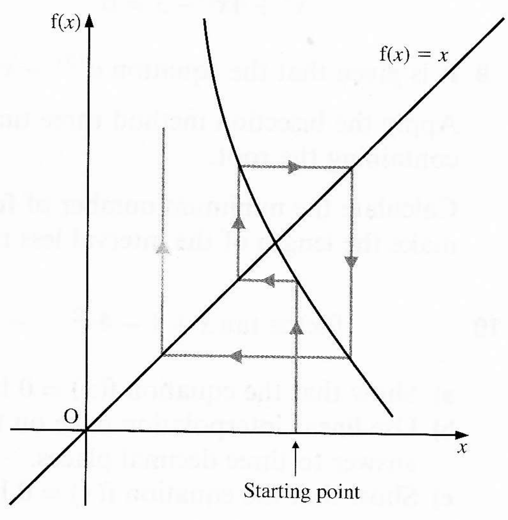
Eventually, we would find that the root is $-1.582\,575\,695$.

The result in Example 5 is represented graphically by a pattern which spirals **into** the root, as shown on the right. Hence, it is called a **cobweb diagram**.



Two other patterns which you are likely to meet are shown below.





In the pattern on the left, the iterative values step directly **into** the root from above (or below). The pattern on the right spirals **out from** the root.

Exercise 13A

1 Show that a root of the equation $x^3 = 7 - 5x$ lies in the interval $1 < x < 2$. Use linear interpolation to find this root correct to two decimal places.

2 Show that a root of the equation $xe^{3x} = 12$ lies in the interval $0 < x < 1$. Use linear interpolation to find this root correct to two decimal places.

3 Show that a root of the equation $x^3 - 4x = 5$ lies in the interval $2 < x < 3$. Use interval bisection to find this root correct to two decimal places.

4 Show that a root of the equation $3x^3 - e^x = 0$ lies in the interval $0 < x < 1$. Use interval bisection to find this root correct to two decimal places.

5 Show that a root of the equation $2x^3 - e^{\frac{1}{2}x} = 0$ lies in the interval $0 < x < 1$. Use linear interpolation to find this root correct to two decimal places.

6 Show that a root of the equation $\sin \dfrac{\pi x}{2} = 3x - 1$ lies in the interval $0 < x < 1$. Use linear interpolation to find this root correct to two decimal places.

7 Using the Newton–Raphson method, find the real root of $x^3 + 3x - 7 = 0$ correct to two decimal places.

8 An iterative formula for solving a cubic is

$$x_{n+1} = \frac{3}{{x_n}^2} - 4$$

a) Take $x_1 = 4$ and calculate $x_2, x_3, x_4, x_5, x_6, x_7$ and x_8. For each iteration, write down the first five digits.
b) What is the solution, correct to three dp?
c) How many iterations are required to find this solution?
d) By replacing x_{n+1} and x_n with x, show that this value is a solution of the equation

$$x^3 + 4x^2 - 3 = 0$$

9 It is given that the equation $e^{0.5x} + x^2 - 3.5x = 0$ has **exactly** one root in the interval $[0, 1]$.

Apply the bisection method three times to obtain a more accurate determination of the interval containing the root.

Calculate the minimum number of **further** applications of the bisection method required to make the length of the interval less than 10^{-5}. (SQA/CSYS)

10 $$f(x) \equiv \tan x + 1 - 4x^2 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

a) Show that the equation $f(x) = 0$ has a root α in the interval $[1.42, 1.44]$
b) Use linear interpolation once on the interval $[1.42, 1.44]$ to find an estimate of α, giving your answer to three decimal places.
c) Show that the equation $f(x) = 0$ has another root β in the interval $[0.6, 0.7]$.

d) Use the iteration

$$x_{n+1} = \tfrac{1}{2}(1 + \tan x_n)^{\frac{1}{2}} \qquad x_0 = 0.65$$

to find β to three decimal places. (EDEXCEL)

11 $$f(x) \equiv 2^x - x^3$$

a) Show that a root, α, of the equation $f(x) = 0$ lies in the interval $1.3 < \alpha < 1.4$.
b) Taking 1.37 as your starting value, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation to this root. Give your answer to three decimal places. (EDEXCEL)

12 Show that the equation

$$e^x + x - 3 = 0$$

has a root between 0 and 1. Use the Newton–Raphson method to solve the equation, giving your answers correct to five decimal places. Record your values of $x_0, x_1, x_2, \ldots$ to as many decimal places as your calculator will allow. (WJEC)

13 Given that x is measured in radians and $f(x) \equiv \sin x - 0.4x$,

a) find the values of $f(2)$ and $f(2.5)$ and deduce that the equation $f(x) = 0$ has a root α in the interval $[2, 2.5]$
b) use linear interpolation once on the interval $[2, 2.5]$ to estimate a value for α, giving your answer to two decimal places
c) using 2.1 as a first approximation to α, use the Newton–Raphson process once to find a second approximation to α, giving your answer to two decimal places. (EDEXCEL)

14 The equation $x^3 + 3x^2 - 1 = 0$ has a root between 0 and 1. Use the Newton–Raphson method, with initial approximation 0.5, to find this root correct to two decimal places.

Give a clear reason why it would be impossible to use the Newton–Raphson method with initial approximation 0. (OCR)

15 Use the Newton–Raphson method to find, correct to three decimal places, the root of the equation $x^3 - 10x = 25$ which is close to 4. (OCR)

16 $$f(x) \equiv \cosh x - x^3$$

a) Show that the equation $f(x) = 0$ has one root, α, between 1 and 2.

A second root, β, of the equation $f(x) = 0$ lies close to 6.14.

b) Apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation to β, giving your answer to three decimal places. (EDEXCEL)

17 $$f(x) \equiv e^x - 2x^2$$

a) Show that the equation $f(x) = 0$ has a root α in the interval $[-1, 0]$ and a root β in the interval $[1, 2]$.
b) Use linear interpolation once on the interval $[1, 2]$ to find an approximation to β, giving your answer to two decimal places.
c) Apply the Newton–Raphson process twice to $f(x)$, starting with -0.5, to find an approximation to α, giving your final answer as accurately as you think is appropriate. (EDEXCEL)

18 **a)** Solve $x = 0.5 + \sin x$ by **each** of the following two methods.

i) An iterative method, other than the Newton–Raphson method, starting with $x_1 = 1.5$. Give a solution which is correct to five significant figures.

ii) The Newton–Raphson method, **applied once only**, starting from $x_1 = 1.5$.

b) Calculate the gradient of $0.5 + \sin x$, where $x = 1.5$. Comment on its relevance to **one** of the methods used in part **a**. (NEAB/SMP 16–19)

19 Given that $f(\theta) = \theta - \sqrt{(\sin\theta)}$, $0 < \theta < \frac{1}{2}\pi$, show that

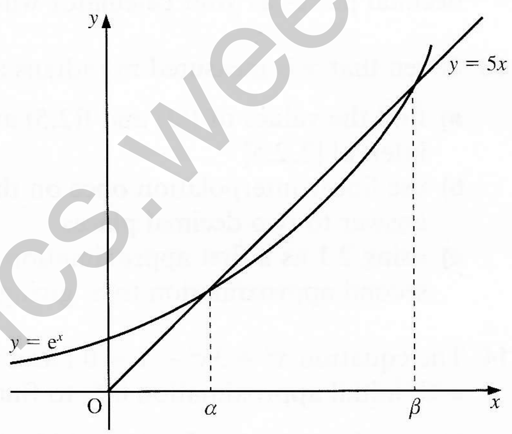
a) the equation $f(\theta) = 0$ has a root lying between $\frac{1}{4}\pi$ and $\frac{3}{10}\pi$

b) $f'(\theta) = 1 - \dfrac{\cos\theta}{2\sqrt{(\sin\theta)}}$

c) Taking $\frac{3}{10}\pi$ as a first approximation to this root of the equation $f(\theta) = 0$, use the Newton–Raphson procedure once to find a second approximation, giving your answer to two decimal places.

d) Show that $f'(\theta) = 0$ when $\sin\theta = \sqrt{5} - 2$. (EDEXCEL)

20 The figure shows the line with equation $y = 5x$ and the curve with equation $y = e^x$. They meet where $x = \alpha$ and $x = \beta$. Approximate values for α and β are 0.2 and 2.5 respectively.



a) The iterative formula $a_{n+1} = \frac{1}{5}e^{a_n}$ is used to find a more accurate approximation for α. Taking $a_1 = 0.2$ use the iterative formula to obtain a_2, a_3, a_4 and a_5, giving your answers to four decimal places.

The Newton–Raphson process is used to find a more accurate approximation for β.

b) Taking $f(x) \equiv e^x - 5x$ and a first approximation to β of 2.5, apply the Newton–Raphson process once to obtain a second approximation, giving your answer to three decimal places.

c) Explain, with the aid of a diagram, why the Newton–Raphson process fails if the first approximation used for β is $\ln 5$. (EDEXCEL)

21 **a)** The cubic equation

$$x^3 - 9x + 3 = 0$$

has a root that lies between 0 and 1. Use the Newton–Raphson method with starting value $x_0 = 0.5$ to find this root, giving your answer correct to six decimal places.

b) A rearrangement of the equation

$$x + 3 = 2\tan x$$

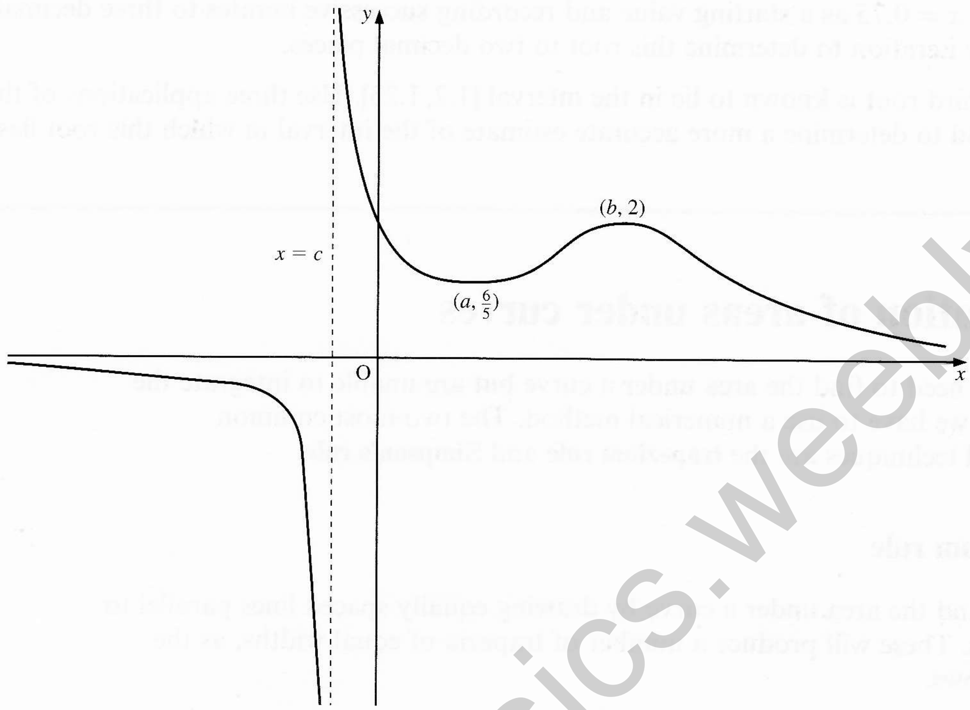
gives the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{x_n + 3}{2}\right)$$

By considering the condition for convergence, show that this iterative formula can be used to find any root of the equation. (WJEC)

22 The diagram below shows part of the graph of the function f, where

$$f(x) = \frac{6}{4x^3 - 12x^2 + 9x + 3}$$



a) The graph of f has a minimum turning point at $(a, \frac{6}{5})$ and a maximum turning point at $(b, 2)$. Use calculus to obtain the values of a and b.

b) The line $x = c$ is a vertical asymptote to the graph of f.

i) Write down an equation which c must satisfy.

ii) Use Newton's method, with $x_0 = -0.2$, to find an approximation to the value of c correct to four decimal places.

[Newton's method uses the iteration $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$ to produce successive approximations to a solution of the equation $p(x) = 0$.] (SQA/CSYS)

23 The equation $f(x) = 0$ has a root at $x = a$, which is known to be close to $x = x_0$. By drawing a suitable graph to illustrate this situation, derive the formula for the first iteration of the Newton–Raphson method of solution of $f(x) = 0$. Hence explain how the general formula is obtained.

It is known that the equation $f(x) = 0$, where

$$f(x) = 3x^5 - 8x^2 + 4$$

has three distinct real roots of which two are positive.

Use the Newton–Raphson method with starting value -1 to determine the negative root correct to three decimal places.

It is known that the other two roots lie in the narrow interval [0.75, 1.25]. Use a diagram to explain why the Newton–Raphson method may be difficult to use in the determination of these roots.

It is proposed to determine the root near $x = 0.75$ using simple iteration with the iterative scheme

$$x_{n+1} = \frac{3x_n^4}{8} + \frac{1}{2x_n}$$

Show that this **may** be suitable to obtain a solution in the neighbourhood of $x = 0.75$.

Using $x = 0.75$ as a starting value and recording successive iterates to three decimal places, use simple iteration to determine this root to two decimal places.

The third root is known to lie in the interval [1.2, 1.25]. Use three applications of the bisection method to determine a more accurate estimate of the interval in which this root lies.

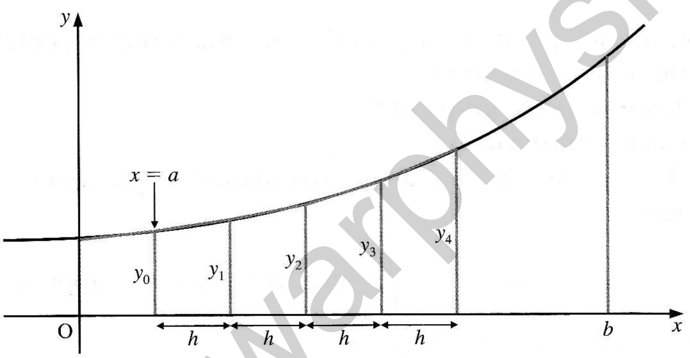
(SQA/CSYS)

Evaluation of areas under curves

When we need to find the area under a curve but are unable to integrate the function, we have to use a numerical method. The two most common numerical techniques are the **trapezium rule** and **Simpson's rule**.

Trapezium rule

We can find the area under a curve by drawing equally spaced lines parallel to the y-axis. These will produce a number of trapezia of equal widths, as the figure shows.



If we divide the x-axis from $x = a$ to $x = b$ into n equal intervals, then we will obtain n trapezia.

Let the y-values of the curve at these x-values be $y_0, y_1, \ldots, y_n$, as shown.

The area of the first trapezium is $\frac{1}{2}h(y_0 + y_1)$, where h is the width of each strip.

The area of the second trapezium is $\frac{1}{2}h(y_1 + y_2)$.

Hence, the total area of the trapezia is

$$\tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \ldots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

By collecting like terms, we obtain the **trapezium rule**, which is

$$\text{Area} \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

where h is the width of a strip and y_0 and y_n are the first and last ordinates.

Example 6 Find, by the trapezium rule, an approximate value for $\int_1^7 e^x \, dx$. Use six intervals.

SOLUTION

First, we divide the x-axis from $x = 1$ to $x = 7$ (the limits of the integral) into six strips (as requested).

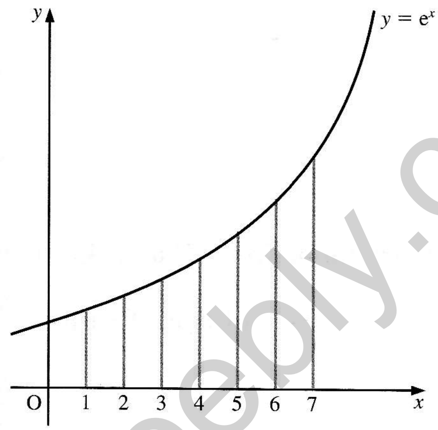
Hence, the x-values of these points are $x = 1, 2, 3, 4, 5, 6, 7$, as the figure shows.

The corresponding y-values are $e^1, e^2, e^3, e^4, e^5, e^6, e^7$.

Therefore, using the trapezium rule, we have

$$\text{Area} \approx \frac{h}{2}[y_0 + y_n + 2(y + y_2 + \ldots + y_{n-1})]$$

$$\approx \frac{1}{2}[e^1 + e^7 + 2(e^2 + e^3 + e^4 + e^5 + e^6)]$$



which gives

$$\text{Area} = 1183.590\,416 \quad \text{or} \quad 1183.6 \quad \text{to 1 dp}$$

Note

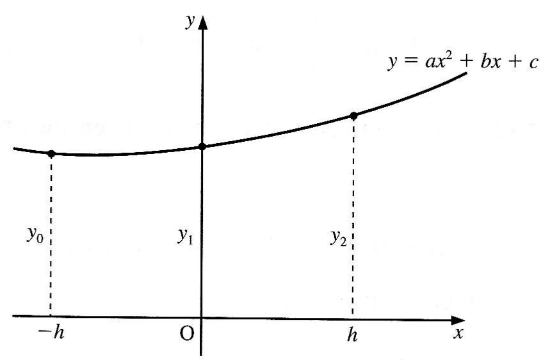
- The accurate answer to Example 6 is $e^7 - e^1$, which is 1093.914 877 or 1093.9 to one decimal place.
- The answer obtained by the trapezium rule can be made more accurate by using more strips of smaller width.

Simpson's rule

The trapezium rule is rarely very accurate because we usually use too small a number of trapezia to approximate the area to be found.

We obtain a better approximation by imposing a known, integrable quadratic curve which passes through points on the original curve.

Simpson's rule is based upon the use of a quadratic curve which passes through three consecutive points. Thus, Simpson's rule finds the approximate value for a **pair** of strips.



Consider the quadratic curve $y = ax^2 + bx + c$, passing through three consecutive points, (h, y_2), $(0, y_1)$ and $(-h, y_0)$, as shown on the right.

When $x = 0$, $y = y_1 \quad \Rightarrow \quad c = y_1$ [1]

When $x = h$, $y = y_2 \quad \Rightarrow \quad y_2 = ah^2 + bh + y_1$ [2]

When $x = -h$, $y = y_0 \quad \Rightarrow \quad y_0 = ah^2 - bh + y_1$ [3]

Adding [2] and [3], we obtain

$$y_0 + y_2 = 2ah^2 + 2y_1 \qquad [4]$$

Using integration to find the area under the quadratic curve, we have

$$\text{Area of pair of strips} = \int_{-h}^{h} (ax^2 + bx + c)\,dx$$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \frac{ah^3}{3} + \frac{bh^2}{2} + ch - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right)$$

$$= \frac{2ah^3}{3} + 2ch$$

Substituting from [1] and [4], we obtain

$$\text{Area of pair of strips} = \frac{h(y_0 + y_2 - 2y_1)}{3} + 2y_1 h$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Using a number of such pairs of strips, we have

$$\text{Total area} \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \frac{h}{3}(y_4 + 4y_5 + y_6) + \ldots$$

$$\approx \frac{h}{3}(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \ldots)$$

By factorising, we obtain Simpson's rule, which is

$$\text{Area} \approx \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + y_6 + \ldots)]$$

Or

$$\text{Area} \approx \frac{1}{3} \times \text{Strip width (First + Last + 4} \times \text{Sum of odds + 2} \times \text{Sum of evens)}$$

Note There **must** always be an **even** number of strips. That is, ***n* must be even**.

Example 7 Find, by Simpson's rule, an approximate value for $\int_1^7 e^x\,dx$. Use six intervals.

SOLUTION

First, we divide the x-axis from $x = 1$ to $x = 7$ (the limits of the integral) into six strips (as requested).

Note Since we are using Simpson's rule, we ensure that we use an **even** number of strips.

Hence, the x-values of these points are $x = 1, 2, 3, 4, 5, 6, 7$. (See top figure on page 281.)

The corresponding y-values are $e^1, e^2, e^3, e^4, e^5, e^6, e^7$.

Therefore, using Simpson's rule, we have

$$\text{Area} \approx \frac{1}{3}[e^1 + e^7 + 4(e^2 + e^4 + e^6) + 2(e^3 + e^5)]$$

which gives

$$\text{Area} = 1099.337\,61 \quad \text{or} \quad 1099.3 \quad \text{to 1 dp}$$

Note On page 281, we gave the accurate value of this area as 1093.9 (to 1 dp). Hence, the value obtained by Simpson's rule gives a better approximation than that obtained by the trapezium rule.

Exercise 13B

1 Using five equally spaced ordinates, estimate the value of each of the following to four decimal places by means of **i)** the trapezium rule, and **ii)** Simpson's rule.

a) $\int_1^5 x^2 \, dx$ **b)** $\int_2^6 x^3 \, dx$ **c)** $\int_0^6 \sin \frac{x}{4} \, dx$ **d)** $\int_1^4 e^{\sin x} \, dx$

2 Using six strips, find an estimate for $\int_2^5 x^x \, dx$ by means of **i)** Simpson's rule, and **ii)** the trapezium rule.

3 **a)** Show that the length of the arc, s, of the curve with equation $y = \cosh x$ between $x = 0$ and $x = 2$ is given by

$$s = \int_0^2 \cosh x \, dx$$

b) Obtain an estimate to this integral by using Simpson's rule with five equally spaced ordinates, giving your answer to four decimal places.

c) Find the exact value of s.

d) Determine the percentage error which results from using the estimate for s calculated in part **b** rather than the exact value obtained in part **c**, giving your answer to one significant figure. (EDEXCEL)

4 $$I_n = \int_0^1 x^{\frac{1}{2}n} e^{-\frac{1}{2}x} \, dx \qquad n \geqslant 0$$

a) Show that $I_n = nI_{n-2} - 2e^{-\frac{1}{2}}$, $n \geqslant 2$.

b) Evaluate I_0 in terms of e.

c) Find, using the results of parts **a** and **b**, the value of I_4 in terms of e.

d) Show that the approximate value for I_1 using Simpson's rule with three equally spaced ordinates is

$$\frac{1}{6}(2\sqrt{2}e^{-\frac{1}{4}} + e^{-\frac{1}{2}}) \qquad \text{(EDEXCEL)}$$

5 $$A = \int_2^4 \frac{1}{\sqrt{(4x^2 - 9)}}\,dx$$

a) Using five equally spaced ordinates, obtain estimates for A, to four decimal places, by means of

i) the trapezium rule

ii) Simpson's rule.

b) Find

$$\int \frac{1}{\sqrt{(4x^2 - 9)}}\,dx$$

and hence evaluate A, giving your answer to four decimal places.

c) Which of your estimates in part **a** is the more accurate? Give a reason for your answer.

(EDEXCEL)

6 $$I_n = \int \frac{x^n}{\sqrt{(1 + x^2)}}\,dx$$

a) Show that $nI_n = x^{n-1}\sqrt{(1 + x^2)} - (n - 1)I_{n-2},\ n \geqslant 2.$

The curve C has equation

$$y^2 = \frac{x^2}{\sqrt{(1 + x^2)}} \qquad y \geqslant 0$$

The finite region R is bounded by C, the x-axis and the lines with equations $x = 0$ and $x = 2$. The region R is rotated through 2π radians about the x-axis.

b) Find the volume of the solid so formed, giving your answer in terms of π, surds and natural logarithms.

An estimate for the volume obtained in part **b** is found using Simpson's rule with three ordinates.

c) Find the percentage error resulting from using this estimate, giving your answer to three decimal places. (EDEXCEL)

7 For $0 < x < \pi$, the curve C has equation $y = \ln(\sin x)$. The region of the plane bounded by C, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

Show that the surface area of the solid generated in this way is given by S, where

$$S = 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left| \frac{\ln(\sin x)}{\sin x} \right| dx$$

Use the trapezium rule with four ordinates (three strips) to find an approximate value for S, giving your answer to three decimal places. (AEB 97)

8 Use the trapezium rule, with six intervals, to estimate the value of

$$\int_0^3 \ln(1 + x)\,dx$$

showing your working. Give your answer correct to three significant figures.

Hence write down an approximate value for

$$\int_0^3 \ln\sqrt{(1 + x)}\,dx \qquad \text{(OCR)}$$

9 Use the trapezium rule with five intervals to estimate the value of

$$\int_0^{0.5} \sqrt{(1+x^2)}\,dx$$

showing your working. Give your answer correct to two decimal places.

By expanding $(1+x^2)^{\frac{1}{2}}$ in powers of x as far as the term in x^4, and integrating term by term, obtain a second estimate for the value of

$$\int_0^{0.5} \sqrt{(1+x^2)}\,dx$$

giving this answer also correct to two decimal places. (OCR)

10 Derive Simpson's rule with two strips for evaluating an approximation to $\int_{-h}^{h} f(x)\,dx$.

Use Simpson's composite rule with **four** strips to obtain an estimate of $\int_2^3 \cos(x-2)\ln x\,dx$.

(Use five decimal place arithmetic in your calculation.) (SQA/CSYS)

11 Use the composite trapezium rule with **four** sub-intervals to obtain an approximation to the definite integral

$$\int_0^{\frac{1}{2}} x\sin(\pi x)\,dx$$

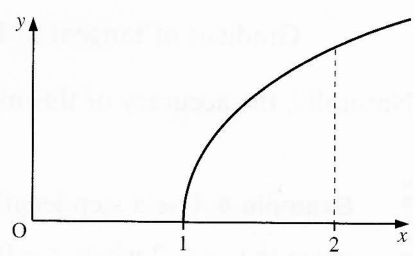
(Give your final answer to four decimal places.) (SQA/CSYS)

12 Use the trapezium rule, with four intervals, to estimate the value of

$$\int_1^2 \sqrt{\left(x-\frac{1}{x}\right)}\,dx$$

showing your working and giving your answer correct to two decimal places.

The diagram shows part of the graph of $y=\sqrt{\left(x-\frac{1}{x}\right)}$.

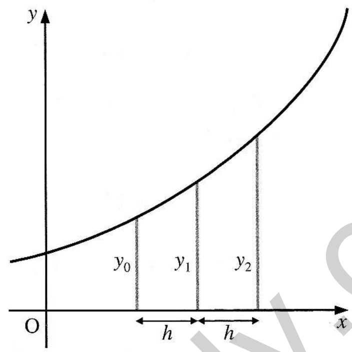


i) State, with a reason, whether this use of the trapezium rule gives an underestimate or an overestimate of the value of $\int_1^2 \sqrt{\left(x-\frac{1}{x}\right)}\,dx$.

ii) State, without further calculation, whether increasing the number of intervals in the trapezium rule from four to eight would lead to a larger or a smaller estimate for $\int_1^2 \sqrt{\left(x-\frac{1}{x}\right)}\,dx$. Give a reason for your answer. (OCR)

Step-by-step solution of differential equations

First-order differential equations



Most differential equations cannot be solved exactly, but need a step-by-step approach.

These methods depend on drawing lines parallel to the y-axis, distance h apart.

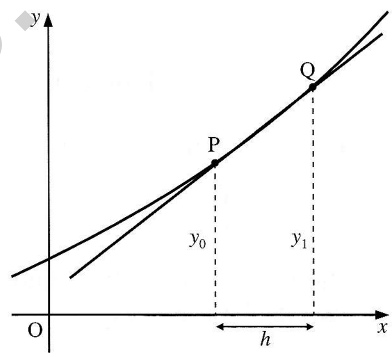
h is called the **step length**.

Single-step approximation

The linear approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

is commonly used in the step-by-step solution of first-order differential equation. It is known as **Euler's method**, after Léonard Euler (1707–83), the prolific Swiss mathematician. We derive it as follows.



With reference to the figure on the right, $P(x_0, y_0)$ is a point on the curve $y = f(x)$ and $Q(x_1, y_1)$ is another point on the curve close to P, where $x_1 - x_0 = h$ and h is small.

We see that the gradient of the chord PQ is approximately the same as the gradient of the tangent at P. Hence, we have

$$\text{Gradient of PQ} = \frac{y_1 - y_0}{h}$$

which gives

$$\text{Gradient of tangent at P} = \left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

Naturally, the accuracy of this method depends on the size of the step length, h.

Example 8 Use a step length of 0.1 to find $y(0.3)$ for $\frac{dy}{dx} = \ln(x + y)$, given that $y = 2$ when $x = 0$.

SOLUTION

Using $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$, we obtain

$$y_1 \approx y_0 + h\left(\frac{dy}{dx}\right)_0$$

which means that

y at new value of x (i.e. when x is 0.1) =

$$= y \text{ at original value of } x + h \times \frac{dy}{dx} \text{ at original value of } x$$

Hence, we have

$$y(0.1) \approx 2 + 0.1 \ln(0 + 2)$$

$$\Rightarrow \quad y(0.1) \approx 2.0693$$

We repeat this procedure with the values obtained for y and $\frac{dy}{dx}$ when $x = 0.1$ now being treated as the original values, and the new value for y being found for $x = 0.2$. Thus, we obtain

$$y(0.2) \approx y(0.1) + h\left(\frac{dy}{dx}\right)_{x=0.1}$$

$$\Rightarrow \quad y(0.2) \approx 2.0693 + 0.1 \ln(0.1 + 2.0693)$$

$$\Rightarrow \quad y(0.2) \approx 2.1467$$

Repeating again, we have

$$y(0.3) \approx y(0.2) + h\left(\frac{dy}{dx}\right)_{x=0.2}$$

$$\Rightarrow \quad y(0.3) \approx 2.1467 + 0.1 \ln(0.2 + 2.1467)$$

$$\Rightarrow \quad y(0.3) \approx 2.2320$$

Example 9 Use a step length of 0.2 to find $y(1.4)$ for $\frac{dy}{dx} = e^{\cos x}$ given that $y = 3$ when $x = 1$.

SOLUTION

Using $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$, we obtain

$$y_1 \approx y_0 + h\left(\frac{dy}{dx}\right)_0$$

which means that

y at new value of x (i.e. when x is 1.2) =

$$= y \text{ at original value of } x + h \times \frac{dy}{dx} \text{ at original value of } x$$

Hence, we have

$$y(1.2) \approx 3 + 0.2\, e^{\cos 1}$$

$$\Rightarrow \quad y(1.2) \approx 3.3433$$

We repeat this procedure with the values obtained for y and $\frac{dy}{dx}$ when $x = 1.2$ now being treated as the original values, and the new value for y being found for $x = 1.4$. Thus, we obtain

$$y(1.4) \approx y(1.2) + h\left(\frac{dy}{dx}\right)_{x=1.2}$$

$$\Rightarrow \quad y(1.4) \approx 3.3433 + 0.2\, e^{\cos 1.2}$$

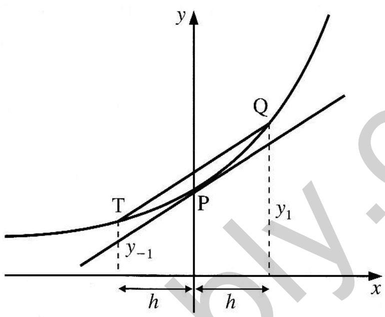
$$\Rightarrow \quad y(1.4) \approx 3.6307$$

Double-step approximation

A better approximation is given by

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$

which uses a double step, as shown in the figure on the right.



We see that the gradient of the chord TQ is a better approximation to the gradient of the tangent at P than that obtained with the single step.

We have

$$\text{Gradient of chord TQ} = \frac{y_1 - y_{-1}}{2h}$$

which gives

$$\text{Gradient of tangent at P} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$

Example 10 Using $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$, and using a step length of 0.1, find y when $x = 1.2$ for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y^2}{2xy}$$

given that $y = 2$ at $x = 1$.

SOLUTION

Since we are required to use the double-step approximation, we need to know the values of y at two values of x.

To find the second value of y, we use the single-step method. As we are given the y-value when $x = 1$, the original value of x is 1, and the new value of x is 1.1. Hence, we have

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_0}{h} \quad \Rightarrow \quad y_1 \approx y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$$

which gives

$$y(1.1) \approx y(1) + 0.1\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=1}$$

When $x = 1$ and $y = 2$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y^2}{2xy} = \frac{3 - 4}{2 \times 1 \times 2} = -\frac{1}{4}$$

which gives

$$y(1.1) \approx 2 + 0.1 \times -\frac{1}{4}$$

$$\Rightarrow \quad y(1.1) \approx 1.975$$

Now we have two values for y, we can use the double-step approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$

$$\Rightarrow \quad y_1 \approx y_{-1} + 2h\left(\frac{dy}{dx}\right)_0$$

which gives

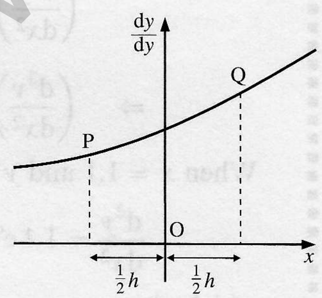
$$y(1.2) \approx y(1) + 2h\left(\frac{dy}{dx}\right)_{x=1.1}$$

$$\Rightarrow \quad y(1.2) \approx 2 + 2 \times 0.1 \times \frac{3 \times 1.1^2 - 1.975^2}{2 \times 1.1 \times 1.975}$$

$$\Rightarrow \quad y(1.2) \approx 2 + 2 \times 0.1 \times -0.062\,284$$

Therefore, when $x = 1.2$, $y = 1.9875$, correct to 4 dp.

Second-order differential equations of the form $\frac{d^2y}{dx^2} = f(x, y)$



With reference to the figure on the right, P is a point at $x = -\frac{1}{2}h$ on the curve of $\frac{dy}{dx}$ against x, and Q is a point at $x = \frac{1}{2}h$ on the same curve, where h is small.

We see that the gradient of the chord PQ is approximately the same as the gradient of the tangent at $x = 0$. Hence, we have

$$\text{Gradient of PQ} = \frac{\left(\frac{dy}{dx}\right)_{\frac{1}{2}h} - \left(\frac{dy}{dx}\right)_{-\frac{1}{2}h}}{h}$$

That is, we have

$$\text{Gradient of tangent} = \left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{\left(\frac{dy}{dx}\right)_{\frac{1}{2}h} - \left(\frac{dy}{dx}\right)_{-\frac{1}{2}h}}{h}$$

which gives

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{\frac{y_1 - y_0}{h} - \frac{y_0 - y_{-1}}{h}}{h}$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

To solve numerically a second-order differential equation, we need either the values of y at two different values of x, or one value of y and one value of $\frac{dy}{dx}$.

Example 11

$$\frac{d^2y}{dx^2} = xe^{\cos y}$$

Using a step length of 0.1, find y when $x = 1.3$, given that $y = 1$ when $x = 1$ and $y = 1.2$ when $x = 1.1$.

SOLUTION

We use

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

with

y_0 as the value when $x = 1.1$

y_1 as the value when $x = 1.1 + h = 1.2$

y_{-1} as the value when $x = 1.1 - h = 1$

Hence, we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.1} \approx \frac{y(1.2) - 2y(1.1) + y(1)}{0.1^2}$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_{x=1.1} \approx \frac{y(1.2) - 2.4 + 1}{0.01}$$

When $x = 1.1$ and $y = 1.2$, we have

$$\frac{d^2y}{dx^2} = 1.1\,e^{\cos 1.2} = 1.580\,38$$

which gives

$$\frac{y(1.2) - 1.4}{0.01} \approx 1.580\,38$$

$$\Rightarrow \quad y(1.2) \approx 2.4 - 1 + 0.015\,8038$$

$$\Rightarrow y(1.2) \approx 1.4158$$

We repeat this procedure, using

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

with

y_0 as the value when $x = 1.2$

y_1 as the value when $x = 1.2 + h = 1.3$

y_{-1} as the value when $x = 1.2 - h = 1.1$

Hence, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} \approx \frac{y(1.3) - 2y(1.2) + y(1.1)}{0.1^2}$$

For $x = 1.2$ and $y = 1.4158$, this gives

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_{x=1.2} = 1.4003 \approx \frac{y(1.3) - 2.8316 + 1.2}{0.01}$$

$$\Rightarrow \quad y(1.3) \approx 2.8316 - 1.2 + 0.014\,003$$

$$\Rightarrow \quad y(1.3) \approx 1.6456$$

Therefore, when $x = 1.3$, $y = 1.6456$, correct to 4 dp.

Example 12

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + x\cos y + \sin y \cos y$$

Using a step length of 0.05, find y when $x = 1.1$, given that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ and $y = 0$ when $x = 1$.

SOLUTION

Because we are given y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at **only one** value of x, we need to use a first-order step-by-step approximation to find a second value for y.

We know the value of y when $x = 1$, so $x = 1$ becomes the original value for x. We require a step length of 0.05, hence we use

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

with

y_0 as the value when $x = 1$

y_1 as the value when $x = 1 + h = 1.05$

y_{-1} as the value when $x = 1 - h = 0.95$

The most accurate first-order step-by-step method is the double-step approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$

which gives

$$1 \approx \frac{y(1.05) - y(0.95)}{0.1}$$

$$\Rightarrow \quad 0.1 \approx y(1.05) - y(0.95) \qquad [1]$$

Using $\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 \approx \dfrac{y_1 - 2y_0 + y_{-1}}{h^2}$, with $x = 1$ and $y = 0$, we obtain

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_{x=1} \approx \frac{y(1.05) - 2 \times 0 + y(0.95)}{h^2}$$

When $x = 1$ and $y = 0$, we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 1 + 1\cos 0 + \sin 0 \cos 0 = 2$$

which gives

$$2 \approx \frac{y(1.05) + y(0.95)}{0.0025}$$

$$\Rightarrow \quad 0.005 \approx y(1.05) + y(0.95) \qquad [2]$$

Hence, adding [1] and [2], we obtain $y(1.05) \approx 0.0525$.

We have now two values of y, namely $y(1)$ and $y(1.05)$, so we are able to use

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

to find y when $x = 1.1$

Thus, we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.05} \approx \frac{y(1.1) - 2y(1.05) + y(1)}{0.05^2}$$

When $x = 1.05$ and $y = 0.0525$, we also have

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.05} = 1 + 1.05\cos 0.0525 + \sin 0.0525 \cos 0.0525$$
$$= 2.100\,957$$

which gives

$$2.100\,957 \approx \frac{y(1.1) - 2 \times 0.0525 + 0}{0.05^2}$$

$$\Rightarrow \quad y(1.1) \approx 0.0025 \times 2.1009\,57 + 0.105$$

$$\Rightarrow \quad y(1.1) \approx 0.1103$$

Therefore, when $x = 1.1$, $y = 0.1103$, correct to 4 dp.

Taylor's series

The other main method for solving differential equations numerically is to use Taylor's series (the derivation of which is beyond the scope of this book):

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

We use this series to find values of $f(x)$, or y, near a given value of $f(x)$ (see Example 12). Its most common application is in the special case when $a = 0$, which gives

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

Notice that this is the same as Maclaurin's series, which we studied on pages 177–9. In the numerical solution of differential equations, when we refer to a series we always mean Taylor's series, though it is rarely seen in its full form.

Example 13 Expand f(x) up to terms in x^4, where

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

given that $y=1$ and $\frac{dy}{dx}=0$ at $x=0$. Hence find y when $x=0.01$, giving your answer to 11 decimal places.

SOLUTION

Differentiating $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$, we obtain

$$\frac{d^3y}{dx^3}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Differentiating again, we obtain

$$\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}+x\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \quad \frac{d^4y}{dx^4}+x\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}=0$$

But f(0) = 1 and f′(0) = 0 (given), so we have

$$\text{From } \frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0: \quad f''(0)=-1$$

$$\text{From } \frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0: \quad f'''(0)=0$$

$$\text{From } \frac{d^4y}{dx^4}+x\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}=0: \quad f''''(0)=3$$

which give

$$f(x)=1-\frac{x^2}{2!}+\frac{3x^4}{4!}$$

Therefore, substituting $x=0.01$, we obtain

$$f(0.01)=1-0.000\,05+\frac{1}{8}\times 0.000\,000\,01$$

That is, $y=0.999\,950\,001\,25$, correct to 11 dp as the next term is 10^{-12}.

Example 14 Expand y up to terms in $(x-1)^3$, where

$$\frac{d^2y}{dx^2}+y\frac{dy}{dx}=x$$

given that $y=0$ and $\frac{dy}{dx}=1$ at $x=1$. Hence find y when $x=1.01$, giving your answer to six decimal places.

SOLUTION

As the values of y and $\frac{dy}{dx}$ are given when $x = 1$, we must use the full version of Taylor's series and obtain a solution for y in powers of $x - 1$.

Differentiating $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$, we obtain

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} - 1 = 0$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} - 1 = 0$$

But $f(1) = 0$ and $f'(1) = 1$ (given), so we have

$$\text{From } \frac{d^2y}{dx^2} + y\frac{dy}{dx} = x\text{:} \quad f''(1) = 1$$

$$\text{From } \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} - 1 = 0\text{:} \quad f'''(1) = 0$$

which give

$$f(x) = (x-1) + \frac{(x-1)^2}{2!}$$

(Note that since $f'''(1) = 0$, there is no term in $(x-1)^3$.)

When $x = 1.01$, we obtain

$$f(0.1) = 0.01 + 0.000\,05$$

Therefore, $y = 0.010\,050$, correct to 6 dp as the next term is 10^{-8}.

Exercise 13C

In Questions **1** to **4**, find the Taylor's series solution for y up to and including terms in x^4:

1 $\frac{dy}{dx} = y^3 + x^8$, for which $y = 1$, when $x = 0$.

2 $\frac{dy}{dx} = x^2y + xy^2$, for which $y = 2$, when $x = 0$.

3 $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = 0$, for which $\frac{dy}{dx} = 1$ and $y = 0$ when $x = 0$.

4 $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 2x^2y = 0$, for which $\frac{dy}{dx} = 0$ and $y = 1$ when $x = 0$.
Hence find y correct to nine decimal places when $x = 0.01$.

5 Given that

$$\frac{d^2y}{dx^2} = x^3 + 2y^3$$

and that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$, expand y as a power series in $(x - 1)$. Hence find y, correct to four decimal places, when **a)** $x = 1.1$, and **b)** $x = 0.9$.

6 Given that y satisfies the differential equation

$$\frac{d^2y}{dx^2} - 4y\frac{dy}{dx} = 0$$

and that $y = 0$ at $x = 0$, and $\frac{dy}{dx} = 2$ at $x = 0$, use the Taylor series method to find a series for y in ascending powers of x up to, and including, the term in x^3. (EDEXCEL)

7 Obtain the Taylor polynomial of degree two for the function $\sin x$ near $x = \frac{\pi}{4}$. Estimate the value of $\sin 46°$ using the first-degree approximation. (SQA/CSYS)

8 Obtain the Taylor polynomial of degree two, in the form $f(0.5 + h) = c_0 + c_1h + c_2h^2$ for the function $f(x) = \frac{1}{7x - 4}$ near $x = 0.5$.

State, with a reason, whether $f(x)$ is sensitive to small changes in the value of x in the neighbourhood of $x = 0.5$. (SQA/CSYS)

9
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 3y = 0$$

where $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 2$ at $x = 0$.

Find y as a series in ascending powers of x, up to and including the term in x^3. (EDEXCEL)

10 Given that y satisfies the differential equation $\frac{dy}{dx} = (x + y)^3$, and $y = 1$ at $x = 0$,

a) find expressions for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b) Hence, or otherwise, find y as a series in ascending powers of x up to and including the term in x^3.

c) Use your series to estimate the value of y at $x = -0.1$, giving your answer to one decimal place. (EDEXCEL)

11 Obtain the series solution in ascending powers of x, up to and including the term in x^3, of the differential equation

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} - 4y = 0$$

given that $y = 3$ and $\frac{dy}{dx} = 2$ at $x = 0$. (EDEXCEL)

12 $$\frac{dy}{dx} = y(xy - 1), \ y = 1 \text{ at } x = 0$$

a) Use the approximation of $\dfrac{y_1 - y_0}{h} \approx \left(\dfrac{dy}{dx}\right)_0$ to estimate the value of y at $x = 0.1$.

b) Using a step length of 0.1 with the approximation $\dfrac{y_2 - y_0}{2h} \approx \left(\dfrac{dy}{dx}\right)_1$ and your answer from part **a**, estimate the value of y at $x = 0.2$.

c) Using a step length of 0.1 again and by repeating the application of the approximation used in part **b**, estimate the value of y at $x = 0.3$. (EDEXCEL)

13 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$, and $y(0) = 1$.

a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r,\ y_r)$$

with $h = 0.1$ to obtain an approximation to $y\ (0.1)$.

b) Use the improved Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

together with your answer to part **a** to obtain an approximation to $y(0.2)$, giving your answer correct to three decimal places. (NEAB)

14 The motion of one point of a turbine blade is given by

$$\frac{dx}{dt} = 4y + 3 \qquad \frac{dy}{dt} = 5 - 4x$$

Initially, $x = 2$, $y = 0$.

a) Use a step-by-step method with $dt = 0.05$ to estimate its position one tenth of a second later.

b) Find a second-order equation, in x and t only, which gives the displacement x at any time t.

c) Write down a first-order differential equation in x and y only. Solve this equation by an exact method, leaving your solution in the form $f(y) = g(x)$. (NEAB/SMP I6–19)

15 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = 2 + \dfrac{y}{x}$ and $y(1) = 1$.

a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.05$ to obtain an approximate value for $y(1.2)$, giving your answer correct to three decimal places.